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Maths with Coq in L1, a pedagogical experiment *

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In France, the first year of study at university is usually abbreviated L1 (for *première année de Licence*). At “Sorbonne Paris Nord” University, we have been teaching an 18 hour introductory course in formal proofs to L1 students for 3 years. These students are in a double major mathematics and computer science curriculum. The course is mandatory and consists only of hands-on sessions with the Coq proof assistant.

We present some of the practical sessions worksheets, the methodology we used to write them and some of the pitfalls we encountered. Finally we discuss how this course evolved over the years and will see that there is room for improvement in many different technical and pedagogical aspects.

1 Introduction

In France, in most mathematics and/or computer science curricula, the first semester after high school is often a key step for the students: the transition to the “rigorous stage”¹ of the mathematical activity, where the emphasis shifts from calculus to proofs. This is done with different topics, of various abstract levels: e.g. naive set theory and relations, arithmetics, real analysis, ... Curiously, formal logic is almost never considered, except for the occasional truth table, and the students usually learn what is a proof by contradiction or a proof by induction without knowing what a proof is.

In this context, we wanted to create at fall 2021 a new specific course for double major mathematics and computer science students. We wanted this new course to be challenging, backed by research and at the interface of these two sciences. We chose the Coq proof assistant because we knew we could find local technical expertise. The 18h constraint comes from the fact that the course replaces an 18h methodology course and, in France, most computer science departments are overloaded with work. To make the most of these 18h, we chose to have only hands-on, 3h, sessions. Each year we had two groups of about 25 students each in computer labs.

*This work was partially supported by the Inria Challenge LiberAbaci <https://liberabaci.gitlabpages.inria.fr/>

¹<https://terrytao.wordpress.com/career-advice/theres-more-to-mathematics-than-rigour-and-proofs/>

This pedagogical experiment is not isolated, see [4] for other computer assisted proof writing courses at the beginning of university in France. Our course does not use, at this point, any layer on top of the proof assistant. In this respect, it differs from other courses using GUI-based software like Edukera or DEAduction. Since we expect our students to learn computer science and mathematics, we were not scared to make them write proofs as sequences of tactics in a text editor. “Software Foundations” by Pierce et al. ([7]) has been a key influence. It certainly encouraged us to use plain Coq so that the most ambitious students could follow it after our course, without having to learn basic proof writing a second time.

We decided quite early during the creation of the course that our ultimate goal would be analysis of sequences of real numbers. On a pedagogical level, it is very appealing since it involves many different quantifiers and we knew that students usually struggle to prove correctly, e.g. that the sum of two converging sequences is converging (and even often have no idea what exactly they are expected to prove). In practice this was a bit too demanding for such a small course at the first semester, but this will be discussed later. With such a goal in mind, we settled each year for, more or less, the following plan:

1. Propositional logic
2. Natural numbers and induction
3. Predicate calculus
4. Real numbers and sequences of real numbers

The course, in its latest version, is available (in French) on a dedicated webpage². A read-only git repository with the 2023 edition of this course, which is the one we describe in this paper is available on GitHub³. The course is (almost) completely centered on filling proofs for lemmas which are already stated in the worksheets.

The rest of the paper is organized according to the same plan as the course. For each of these items, we will sum up its content in our course and give some feedback about its reception by the students. We will then explain our assessment methods and will conclude with possible improvements in terms of pedagogical and technical aspects.

Before, we briefly present other courses using a proof assistant at the early years of university and how they differ from our own experiment. Patrick Massot has created such a course at Paris-Saclay university. It also targets double degree students in mathematics and computer science during their first year, but during their second semester. His course uses Lean with a set of custom tactics called “lean-verbose” ([5]) so that the proof scripts written by the students are close to a (very detailed) mathematical proof. His course is also directed towards elementary real analysis and is backed by the community-developed mathlib⁴ Lean library for mathematics. At Université Paris Cité, Antoine Chambert-Loir and Ricardo Brasca teach a purely Lean and mathlib based course (see its Gitlab repository⁵ for its source code) which also covers linear algebra. Heather MacBeth’s course entitled “The Mechanics of Proof⁶” is another course targeted at students learning how to write mathematical proofs, using again Lean with its mathlib. It is oriented towards number theory, instead of real analysis. Still with Lean and the mathlib, Jeremy Avigad’s course “Mathematics in Lean⁷” is of a much higher mathematical level, with

²<https://www.math.univ-paris13.fr/~rousselin/ipf.html>

³<https://github.com/Villeteuse/ipf-2023>

⁴<https://github.com/leanprover-community/mathlib4>

⁵<https://plmlab.math.cnrs.fr/chambert/LeanTeaching>

⁶<https://hrmacbeth.github.io/math2001/>

⁷https://leanprover-community.github.io/mathematics_in_lean/

topics such as linear algebra or measure theory. Another Lean-based experiment has been conducted by Frédéric Tran-Minh ([9]) comparing the difficulties of the students when using “term-mode” Lean *versus* the Edukera graphical proof assistant.

In contrast to the mathlib-based courses, our course uses an almost “bare bones” Coq, which has its pros and cons. Having less abstractions can be appealing for a first course, especially for students learning both computer science and mathematics. One may also be afraid that the various levels of abstraction used in the mathlib “leak”, causing some tactics to have a less predictable behaviour for the students, but we certainly need more insight on this subject. On the other hand, it would be very difficult, for instance, to cover linear algebra in this setting.

2 Propositional logic: how to write an exercise sheet

2.1 Propositional (intuitionistic) logic

We start with intuitionistic logic. The logical connectives `forall`, `->` (for implication), `/\` (for conjunction), `\|` (for disjunction) are described in a mechanical way by how we can use them and how we can prove them.

We always start with a commented example. The students are expected to play these examples in their interactive environment to see how each tactic modifies the proof state. Our first “Hello, World!” example is `imp_refl`:

```
Theorem imp_refl : forall P : Prop, P -> P.
```

```
Proof.
```

```
  (* Let [P] be any proposition. *)
```

```
  intros P.
```

```
  (* To show an implication, one assumes that what is on the left of the arrow
     holds, and then prove what is on the right of the arrow. *)
```

```
  (* We assume (hypothesis ([H])) that [P] holds. *)
```

```
  intros H.
```

```
  (* We need to prove [P], but ([H]) is exactly a proof of [P]. *)
```

```
  exact H.
```

```
Qed. (* Quod erat demonstrandum. What was to be demonstrated. *)
```

Every year, showing this proof, during the first minutes of the first hands-on session in September is a daunting “throw the students in the water” moment. We can’t explain what is the type `Prop`. They will need to get a feeling of what it is by practicing. The students need to learn that `intros` may introduce a variable or an hypothesis. And there is even Latin! It is worth noting that our first iteration was even worse: we thought it would be a good idea to name the hypothesis `HP`, but then many students thought that the name of the hypothesis was meaningful and that calling it, say `HQ` would assume that some *other* proposition `Q` would hold. We also used `assumption` (which scans the context for a proof of the goal) instead of `exact H`, but this was far too magical at this stage.

That said, with practice, the students actually manage to digest a lot more than what one would expect. Following the example is (always) a straightforward exercise in order to ease this digestion process.

And, from there, the students usually keep on working, with the occasional help from the teacher. In one or two sessions, almost all the students have learned the elimination and introduction rules of the logical connectives (except `exists` which is seen later), how to read multiple arrows, e.g. `P -> (Q -> R)` as “if `P` and `Q` hold then `R` also holds”, ...

connective	introduction (prove)	elimination (use)
\rightarrow	<code>intros</code>	<code>apply</code>
\wedge	<code>split</code>	<code>destruct</code>
\vee	<code>left</code> or <code>right</code>	<code>destruct</code>
\forall	<code>intros</code>	specialize, instantiation
\perp		<code>destruct</code>
\exists	<code>exists</code>	<code>destruct</code>

Figure 1: Logical tactics used in our course

As in “Software Foundations”, we have chosen our basic tactics to be as close as possible to natural deduction rules; at the end of this exercise sheet table 1 is mostly complete. The `False` proposition is described by the principle of explosion, and the negation of a proposition, $\sim P$, is defined as $P \rightarrow \text{False}$. We keep away from the excluded middle at this point, because we feel that the natural deduction rules are closer to usual mathematical practice (to prove an implication, start with “assume”). We also restrict ourselves to backwards reasoning for the moment. This is what Coq natively encourages, and we have observed that exposing the students to forwards reasoning too early (with e.g. `apply H in H'`) was a source of confusion. Indeed, during this first exposure to the proof assistant, every tactic should, ideally, fill one simple role. The implication to *apply to the goal* is chosen mostly by comparing the *conclusion* of the implication with the goal. When we introduce `apply H in H'` too early, we add the rule that we then need to compare the *premise* of H with H' , which may be one rule too much at this point. Another possibility, which we did not try yet, would be to start with forward reasoning first, but most tactics modify the goal anyway, so it may be unpractical.

2.2 Sources of difficulties

For the students, the main difficulty in this part is, when proving a disjunction, the premature choice of `left` or `right`. Some of them keep their eyes on their proofs (or worse, on the keyboard) and do not watch closely enough the proof state to figure out that their proof state is a dead end and they need to *go back* in their proof.

The main difficulty as teachers, is to respect two guiding principles when writing exercise sheets. First, the teachers have to honor a *contract*: it should *always* be possible for a student to solve an exercise with the tools (tactics, lemmas) which have been described beforehand. Second, teachers have to do all they can to *make the information flow manageable* by the students. This last part can be very pedagogically challenging at times and requires polishing again and again the exercise sheets. In some cases, it exposed defects in Coq itself and has led to improvements. For instance, the Coq command `Set Printing Parentheses`, which makes Coq display, for instance, $A \rightarrow B \rightarrow C$ as $A \rightarrow (B \rightarrow C)$, works as intended since the recent version 8.19, and a refinement to select the operators for which we want parentheses displayed is under work.

A final important issue is the fact that we work with real-life Coq, and tactics shown in table 1 *go beyond the educational framework* we installed, the biggest culprits being `apply` (which is a lot stronger than what we want it to be in this course) and `destruct` which can actually destruct almost anything in the context. This is an issue when blocked students start trying things at random (and then, when it magically works, share the solution with their classmates). It then becomes the role of the teachers to enforce the rule:

Magic is not allowed in this course.

and to discuss it with the students (what good is a solution you do not understand?). It is tempting to write our own, more restricted, versions of the basic tactics (with the same names to remain compatible with “Software Foundation”). We could then put back the real life versions when we are sure that the basic logical rules are understood. This, however, is not a priority at this point as it is not an insurmountable problem for the students.

3 Natural numbers, computation and induction

3.1 Content of the worksheet

Since we have a specific public and work at the interface of computer science and mathematics, we chose to show how Peano’s natural numbers can be defined in Coq as:

```
Inductive nat : Set := 0 : nat | S : nat -> nat.
```

There is a little bit of hand-waving involved since we do not go in the details of what an `Inductive` type is. Instead, we count on experiments and interactive feedback from Coq itself to make students understand the key concepts of “no junk” (all natural numbers are obtained this way) and “no confusion” (if two natural numbers are written differently they are different). It is also the first time the students work with equality. We first present the `discriminate` tactics which proves `False` from an equality such as `S (S 0) = S (S (S 0))`. Then we show that addition is defined as a recursive program in the following way:

```
Fixpoint add (n m : nat) : nat :=
  match n with
  | 0 => m (* If n = 0, then n + m = m. *)
  | S p => S (add p m) (* (S p) + m = S (p + m). *)
  end.
```

and students are then encouraged to use pen and paper to compute $2 + 2$ using these two rules. This approach, which is closer to computer science than mathematics, has some interests (especially in our context) but it has some drawbacks we will discuss shortly. We then introduce the `reflexivity` tactics which, in our course, is used to prove an equality when both sides are syntactically identical⁸. We show how the `simpl` tactic asks Coq to use the addition program as much as it can in a term. Students then use `destruct` to perform proofs by case on the nullity of an integer and `rewrite` to *use equalities* to replace some terms in the goal with other terms.

At this point, the students are shown that proofs by cases are not sufficient to prove, for instance, that $\forall(n : \mathbb{N}), n + 0 = 0$, which allows us to introduce the `induction` tactic. This tactic, and how it forces the students to prove a base case and an induction step with an induction hypothesis is arguably enough to legitimate this course by itself. With it, students can sharpen a skill which is essential both in computer science and mathematics. From there, the multiplication is defined and students prove basic properties of addition and multiplication such as commutativity, associativity, etc. As a good side effect, it’s also a very concrete exposition to these algebraic notions which are mostly new for the students.

3.2 Activity

At this point, we would like to insist on a key aspect of this course: *the students are constantly active*. Many of them even skip the 15 minutes break and work without interruption for 3 hours. We believe that

⁸This is an oversimplification and `reflexivity` goes beyond our framework for two reasons: first, it checks (computes) that both sides are convertible, second, it is tweaked with typeclass instances, so that it can prove, for instance, that $x \leq x$.

there are three reasons to that.

First, the interactive aspect of the proof assistant, giving *instant feedback* and allowing to experiment. The quick `Qed`. reward certainly also plays a role. Some students compared the course to a video game. Second, the joy of theorem proving itself (with or without a proof assistant) should not be neglected, *it is made easier by the fact that everything here is explicit*. There is no hidden forgotten sine identity trick needed to solve an exercise. Finally, we put a lot of efforts to *make the difficulty as progressive as it could be*. Of course, if students were facing an impossible task from the beginning, we would lose the benefits of the first two points.

3.3 Computational aspects and rewriting

There are still some points of friction and possible improvements (or different choices). First, the computational aspects are not well understood. Some students try to use `simpl` or `discriminate` at random. The `simpl` tactic often gives more than what was expected. For instance, using `simpl` on the goal

$$(1 + n) ^ 2 = n ^ 2 + 2 * n + 1$$

gives

$$S (n * 1 + n * S (n * 1)) = n * (n * 1) + (n + (n + 0)) + 1$$

which then makes the proof almost impossible (and remember that students rarely go back to make different choices). This year, we have restricted the usage of `simpl` to proofs of basic rewriting rules derived from the definition and encouraged the students to use the `rewrite` tactic instead. About the computational aspects themselves, we think we are sitting on a fence between either dropping them completely (and then presenting `nat` in an axiomatic way) or putting more time and effort to it. With our specific public, we would like to try the second approach in the future. In a course for mathematics students only, we would recommend the first one.

Finally, `rewrite` has its own issues. It can be hard to select precisely the part of the goal one wants to use, say, commutativity on. We could introduce `replace` at this point, but this would introduce yet another tactic, and it makes proofs more verbose. Another option, when rewriting becomes very tedious, would be to introduce more automation with tactics such as `ring` and `lia`. We opted out of it at this point. These powerful automation tactics should be introduced only when it is clear that they only offer time saving. Otherwise we risk to encourage magical thinking.

4 Predicate calculus and classical logic

4.1 Content of the worksheets

The predicate calculus worksheet aims at working with `forall` and `exists` predicates. The `specialize` tactic plays the role of an elimination rule for the `forall` predicate. We work both with abstract formulas, for instance:

`Theorem forall_or_forall :`

$$(\text{forall } x, (P \ x)) \ \vee \ (\text{forall } x, (Q \ x)) \ \rightarrow \ \text{forall } x, (P \ x \ \vee \ Q \ x).$$

and concrete predicates, for instance `is_null` and `is_non_null` on `nat` to provide examples and counter-examples.

We believe that spending time on the `exists` connective is very important. At this point, there is still no excluded middle, so the only way to prove an existential formula is to exhibit a witness with the `exists` tactic. The elimination of such a formula is also interesting since it introduces in the context a

new element with only one assumption about it. The students spend a lot of time working on predicate calculus tautologies or false formulas. We make it apparent that `exists` behaves like a generalized \setminus and `forall` behaves like a generalized \wedge .

Next is a small worksheet about classical logic. We add the excluded middle and derive new logical identities which were not provable without it. In particular, we work on negations of existential and universal formulas.

4.2 A very apparent speed decrease

The students usually find these two worksheets a lot more difficult than the previous ones. Everything takes a lot of time. We have (non measurably) identified two causes. First, some students have difficulties to use their mathematical intuition in front of the proof assistant. The relation between formal proving and their usual mathematical activity is not clear to them. We hope to work on this aspect in the next course sessions. Second, they struggle with more abstract, higher-order logic. This might be a mathematical issue, where the proof assistant acts as an amplifier of the students difficulties. There is certainly no easy solution to this pedagogical problem, which is probably as old as mathematics. We intend to try adding intermediate steps, for instance predicates on a finite type.

We were also surprised by the difficulties of the students in classical logic. After all, it's "their" logic in everyday mathematics. It turns out that they do not relate strongly truth tables and proofs. Even when this is not an issue, the excluded middle is very special since, in contrast with other logical rules, it can happen any time with any proposition. On a more technical side, at this point Coq with its standard library does not offer much to work with classical logic. For instance it lacks an equivalent of the `mathlib`'s `push_neg`⁹ tactic which transforms automatically a negated first-order formula with an equivalent one where the negation is at the end. This should be improved if one wants to use it more for mathematical courses.

5 Real numbers

5.1 Content of the last worksheets

For real numbers, as in the rest of the course, we use Coq's standard library. Historically, Coq's real numbers have been introduced axiomatically by Micaela Mayero ([6]) as an ordered field satisfying the least-upper-bound property. They have been constructed more recently by Vincent Semeria ([8]) using Dedekind's cuts¹⁰. We do not want to expose the internal construction to the students, so we present the real numbers in an axiomatic way. This has the good side effects to introduce them to abstract algebra.

The operations and constants are introduced progressively, together with the axioms they satisfy and consequences of these axioms. For instance, we ask them to prove

Theorem `unique_opp` : `forall x y z : R, x + y = 0 \wedge x + z = 0 -> y = z.`

using the fact that $(\mathbb{R}, +)$ is a commutative monoid. From the axioms, we play the "real numbers game" until we prove that $0 < 1$. In passing, the students also work with strict and non-strict orders and compatibility properties. Many proofs involve transitivity and antisymmetry, on top of the other, algebraic, properties of the operations.

⁹https://leanprover-community.github.io/mathlib4_docs/Mathlib/Tactic/PushNeg.html

¹⁰Actually, there are now two flavors of real numbers, one is constructive, the other not; we work with the non-constructive version.

The next exercise sheet deals with the absolute value. It was added in 2022: we saw during the first edition of the course that the students were not familiar enough with this notion to handle subsequent real analysis work.

Finally, the worksheet about real sequences introduces the convergence of real sequences. The example shown is the uniqueness of the limit. Our initial goal for this whole course, was that the students prove the following lemma (given, here, with its solution):

```
Theorem CV_plus (An Bn : nat -> R) (l1 l2 : R) :
  Un_cv An l1 -> Un_cv Bn l2 -> Un_cv (fun n => An n + Bn n) (l1 + l2).
```

Proof.

```
unfold Un_cv.
intros HA HB eps Heps.
destruct (HA (eps / 2)) as [n1 Hn1]. lra.
destruct (HB (eps / 2)) as [n2 Hn2]. lra.
remember (max n1 n2) as n3 eqn: def_n3.
exists n3.
intros n Hn.
replace eps with (eps/2 + eps/2) by lra.
apply (Rle_lt_trans _ ((R_dist (An n) l1) + (R_dist (Bn n) l2))). {
  apply R_dist_plus.
}
apply Rplus_lt_compat.
- apply Hn1. lia.
- apply Hn2. lia.
```

Qed.

Even without prior Coq experience, one can probably see how it closely matches the usual mathematical proof. There are, however, subtle points which need be discussed in order to grasp the difficulties of the students. After the introductions, the hypothesis HA has type

```
HA : forall eps, eps > 0 -> exists N, forall n, n >= N -> (Rdist (An n) l1) < eps
```

What we perform in the first `destruct` tactic is first, to specialize it with `eps / 2`, then extract a witness `n1` satisfying the property `Hn1`, and add the new subgoal that `eps / 2 > 0`. This new subgoal would certainly never be explicit in a mathematics course. To handle such cases, we have allowed the usage of the powerful solvers `lra` (for “Linear Real Arithmetic”) and `lia` (for “Linear Integer Arithmetic”). In our case, `lra` can handle the proof that `eps / 2 > 0` without complaining.

The next step is to consider the maximum `n3` of the two indices `n1` and `n2`, this is done with the tactic `remember`. The most tedious part to write in Coq is certainly the use of the triangle inequality. In a mathematical course, this would probably have been written

$$\begin{aligned} |A_n + B_n - (l_1 + l_2)| &= |A_n - l_1 + l_2 - B_n| \\ &\leq |A_n - l_1| + |B_n - l_2| && \text{(by the triangle inequality)} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon, \end{aligned}$$

where one does not need explicitly stating transitivity results. We would be happier if our formal proof resembled this. It is not possible in our course yet, but Lean has the `calc` tactic¹¹ which does just that and `coq-waterproof`¹² also allows this style, so we can reasonably hope this will be available in the

¹¹<https://leanprover-community.github.io/extras/calc.html>

¹²<https://github.com/impermeable/coq-waterproof>

near future. Notice the usage of the `lia` tactic in the last steps, to show that, indeed, n_3 is greater than n_1 and n_2 .

Now, honesty compels us to say that very few students actually managed to prove this theorem (only 5 students in 2022, none during the two other sessions). In 2021, it was our first time writing and teaching this course, so it was a bit less polished. In 2023, we made a big pedagogical mistake about forward reasoning (more about this later) which made the student lose a lot of time. Anyway, 18h is probably not enough to cover that much mathematical and proof assistant related content during the first semester of university.

5.2 Forward reasoning, rewrite and apply

As mentioned earlier, Coq has a tendency towards backwards reasoning. One possible reason is that there is always one active goal and maybe many hypotheses, so this is a bit simpler to write. Still, it is possible to `apply` an implication, say a proof `HI` of $A \rightarrow B$ in a proof, say `H`, of some proposition `A` to transform `H` into a proof of `B`. It is arguably the more natural way to write a mathematical proof: start with the assumptions, and derive consequences until you reach the goal.

In 2021, we introduced this style in the first exercise sheet, alongside with backwards reasoning and we have observed that this created some confusion for the students, who already had a lot to digest. In 2022, we have introduced this style later, just before working on real numbers. In 2023, we thought that (almost) removing forward reasoning would simplify our exposition, hence the life of our students. In the same movement, we omitted such lemmas as

```
Lemma Rplus_eq_compat_1 : forall r r1 r2 : R, r1 = r2 -> r + r1 = r + r2.
```

which have no mathematical content and serve primarily in forward reasoning. But then, we noticed that they were quite slower in the part about real numbers (without at first, identifying why).

It turns out dropping forward reasoning was a big mistake. *We should not prevent the students to write their proofs in the way they are used to.* On the contrary, the proof assistant should get closer to what is deemed more natural to the user. Still, the risk to create confusion between forward and backward reasoning exists and we do not know properly yet when and how to discuss this distinction in our course.

Another source of difficulty is the confusion between `rewrite` and `apply` when the conclusion of the implication is an equality. For instance, consider the following cancelation lemma:

```
Lemma Rplus_eq_reg_1: forall r r1 r2 : R, r + r1 = r + r2 -> r1 = r2.
```

and assume we want to show:

```
Theorem double_fixpoint_0 : forall (x : R), x + x = x -> x = 0.
```

Proof.

```
intros x H.
```

At this point, using `apply (Rplus_eq_reg_1 x)` unifies `r1` with `x` and `r2` with `0` and changes the goal into the manageable $x + x = x + 0$. This kind of reasoning requires that the user have acquired some reflexes concerning unification (and again, it is probably more natural for our students to transform the hypothesis `H` into $x + x = x + 0$ and then apply `Rplus_eq_reg_1` in `H` without having to instantiate any variable explicitly). However, if the student tries to `rewrite Rplus_eq_reg_1`, Coq will rightfully complain that it has no idea which term to rewrite (`r1`) and into what (`r2`). Even if, after struggling, the user finally manages to instruct `rewrite (Rplus_eq_reg_1 x x 0)`, then Coq will change the goal into $0 = 0$ and ask for a proof that $x + x = x + 0$, which is slightly worse than the `apply` solution.

In practice, we noticed that this distinction between `apply` and `rewrite` is not well understood and we should probably take more time to work on it with the students.

5.3 Solvers

As we mentioned earlier, we introduce the tactics `lia` and `lra` very late in the course. We feel that the students need to be able to prove most easy equalities and inequalities, before giving them these powerful solvers. However, it is clear that they also serve pedagogical purposes: with them it is possible to treat exercises that would have been too tedious otherwise. This year, the last exercise of a homework allowed the students to use about anything at their disposal. Here is the statement, with a part of a student's solution.

```

Lemma polynome2_positive (a b c : R) : a > 0 -> (b2 - 4 * a * c < 0) ->
  forall x, a * x2 + b * x + c > 0.
Proof.
  unfold Rdiv, Rsqr.
  intros H H1 x.
  replace (a*(x * x) + b * x + c) with
    (a*((x+(b/(2*a)))2 + (- (b2-4*a*c))/(4*a2))).
  - apply (Ropp_gt_lt_0_contravar ((b2-4*a*c))) in H1.
    (* 9 lines to prove that a*((x+(b/(2*a)))2 + (- (b2-4*a*c))/(4*a2)) > 0 *)
  - unfold Rsqr. field. lra.
Qed.

```

As one can see, the heart of the proof is the transformation of the polynomial into its canonical form. This was not guided, so the student took this initiative herself, and then, *after writing the equation*, used the `field` tactic to verify it.

6 Assessments

The students have to upload two homework assignments (as Coq files) and take a final 1h30 exam in computer lab.

The homework assignments have easy exercises as well as challenging ones. Except the first year, there has always been a handful of perfect homework assignments. We adopt a very simple grading scheme with `Qed` or nothing except in very specific edge cases. These are quickly corrected with the help of Coq (and a quick human read to watch for edge cases). We should certainly add more homework assignments in the future.

We did not want to make this course too centered around assessments (in part because it is still somewhat experimental), so its weight in the final semester grade is quite small (1 out of 30). Moreover, it is good to tell the students once in a while that they are not here to learn how to pass exams but to study mathematics and computer science. Still, students being students, they take their grades very seriously. We wanted the grades to be “good”, in a French sense, so with a mean of around 14/20, in order to not handicap our double major students (which would certainly get good grades in the methodology course it replaces).

In the end, the heterogeneity of the students is incredible. A few of them still struggle with basic logic, while others are able to prove more than twenty lemmas, some of them not that trivial, during the exam. A strong majority of students solve all basic logic exercises and most easy inductions. We have opted for a diminishing returns scale : the first `Qed` weighs twice more than the tenth which weighs twice more than the twentieth.

7 Conclusion and prospects

We have shown the content of our course and emphasized some of the difficulties we, or the students, have encountered, be them of pedagogical or technical origins. Our course contrasts with other such experiments in its usage of “real life (plain) Coq” instead of a specific layer on top of it, for better and for worse. There is room for improvement in many different directions. We already mentioned that we could work on restricting the use cases of our tactics to their pedagogic ideal, but we are not experts on this subject. The students could also benefit from a rewriting of the `Reals` library, so that they face a more uniform naming scheme. Other, more modern, libraries deal with analysis: `Coquelicot` (see [3]) and `mathcomp-analysis` (see [1]) offer a lot more, but are not really suited for teaching at this level since they use more advanced and abstract notions such as abstract algebraic hierarchies and filters.

However, the biggest improvements, from our perspective, would be about user interfaces in a broad sense : in terms of both the input/output of Coq itself and graphical interface. From Coq, we mentioned the need to ease writing of equalities and inequalities in general. Error messages and the absence of hints are also a concern. As for the graphical interface, we have been using `coqIDE` in 2021 and 2022 and have switched to `jsCoq` in 2023. Using `jsCoq` is a lot better: the exercise sheets are opened in a Web browser and the comments are displayed as HTML. The students actually read the course while, with a text editor, they usually skipped the commentaries, displayed as a greyish heap of ASCII characters. We could have a lot more, for instance a list of authorized lemmas for each exercise, so that the student can browse a manageable list of results at any time, or easy to write mathematical formulas, function plots, ... In short, since Coq is a member of the teaching team, we would like to be able to tweak it, in order to make it a better teacher.

Finally, we need to assess the usefulness of this course on the mathematics side. The course certainly does not harm (the results of our students in mathematics courses are quite good) but it would be useful to have more specific feedback in that respect. Starting next year, we will add pen and paper exercise sessions to work on mathematical proof writing in relation to the work with the proof assistant. This will certainly help in this regard, but this assessment task is certainly difficult. First, it is hard to measure the gap between the activities of typing formal proofs with Coq and writing mathematical pen-and-paper proofs to convince human beings. Then, we would need to understand better which part of this gap is related to Coq and/or our formalization choices for this course. The recent paper [2] presents a first analysis of the differences between proof assistants for education. There is certainly a lot more to study in this direction.

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