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► **To cite this version:**

Isabeau Birindelli, Ariela Briani, Hitoshi Ishii. Fully nonlinear elliptic PDEs in thin domains with oblique boundary condition. 2024. hal-04823183

HAL Id: hal-04823183

<https://hal.science/hal-04823183v1>

Preprint submitted on 6 Dec 2024

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FULLY NONLINEAR ELLIPTIC PDES IN THIN DOMAINS WITH OBLIQUE BOUNDARY CONDITION

ISABEAU BIRINDELLI, ARIELA BRIANI, AND HITOSHI ISHII

ABSTRACT. This note is the natural continuation of what was started in [8] i.e. the extension to fully nonlinear operators of the well known result on thin domains of Hale and Raugel [13]. Here we consider oblique boundary condition, and find some new phenomena, in particular the limit equations contain "new terms" in the second, first and zeroth order terms which don't have an equivalent in the Neumann case.

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1. INTRODUCTION

Let Ω_ε be a cylindrical domain, which is thin in the direction of the axe i.e.

$$\Omega_\varepsilon = \{(x, y) \in \Omega \times \mathbb{R} : \varepsilon g^-(x) < y < \varepsilon g^+(x)\}$$

with $\Omega \subset \mathbb{R}^N$ a bounded open domain, $g^\pm \in C(\overline{\Omega}, \mathbb{R})$, and $g^-(x) < g^+(x)$. We think of $\partial\Omega_\varepsilon$, as divided in three parts the top, the bottom and the lateral part:

$$\begin{aligned} \partial_T\Omega_\varepsilon &= \{(x, y) : x \in \overline{\Omega}, y = \varepsilon g^+(x)\}, & \partial_B\Omega_\varepsilon &= \{(x, y) : x \in \overline{\Omega}, y = \varepsilon g^-(x)\}, \\ \partial_L\Omega_\varepsilon &= \{(x, y) : x \in \partial\Omega, \varepsilon g^-(x) \leq y \leq \varepsilon g^+(x)\}. \end{aligned}$$

We consider the elliptic PDE problem with oblique boundary conditions on the top and bottom of the domain:

$$(1) \quad \begin{cases} F(D^2u^\varepsilon, Du^\varepsilon, u^\varepsilon, x, y) = 0 & \text{in } \Omega_\varepsilon, \\ \gamma^+ \cdot Du^\varepsilon = \beta^+ & \text{on } \partial_T\Omega_\varepsilon, \quad \gamma^- \cdot Du^\varepsilon = \beta^- & \text{on } \partial_B\Omega_\varepsilon, \\ B(Du^\varepsilon, u^\varepsilon, x) = 0 & \text{on } \partial_L\Omega_\varepsilon, \end{cases}$$

2020 *Mathematics Subject Classification.* 35B40, 35D40, 35J25 49L25 .

Key words and phrases. asymptotic behavior of solutions, thin domains.

A. Briani was partially supported by l'Agence Nationale de la Recherche (ANR), project ANR-22-CE40-0010 COSS; I. Birindelli was partially supported by project Leoni 2023 GNAMPA-INDAM and project "At the Edge of Reaction-diffusion equations" Sapienza Università di Roma. H. Ishii was partially supported by the JSPS KAKENHI Grant Nos. JP20K03688, JP23K20224, and JP23K20604.

where $\gamma^\pm \in C(\overline{\Omega} \times [-1, 1], \mathbb{R}^{N+1})$ and $\beta^\pm \in C(\overline{\Omega} \times [-1, 1], \mathbb{R})$.

Our purpose in this paper is to investigate the asymptotic behaviour of u^ε when the positive parameter ε is sent to zero. In particular we want to determine the equation satisfied by the limit function in Ω . This paper is the natural continuation of [8]. In that work we had generalized to a non variational setting the famous result of Hale and Raugel [13] using the so called test function approach à la Evans. We shall see here that, choosing the oblique boundary condition, completely changes the limit equation even in the case of the Laplacian, introducing new first and second order terms.

On the other hand, the choice of the boundary condition on the lateral boundary $\partial_L \Omega_\varepsilon$ does not play any crucial role. Hence we shall prove the results with either Neumann, or oblique or Dirichlet boundary condition on the lateral boundary i.e. either $B(Du^\varepsilon, u^\varepsilon, x) := \gamma \cdot Du^\varepsilon - \beta(x)$ or $B(u^\varepsilon) := u^\varepsilon - \beta(x)$. For the clarity of the discussion, in the introduction and in the formal expansion of the next section, we concentrate mainly on the standard Neumann condition:

$$(2) \quad B(Du^\varepsilon, u^\varepsilon, x) := \nu \cdot D_x u^\varepsilon = 0 \quad \text{on } \partial_L \Omega_\varepsilon,$$

where $\nu = \nu(x)$ denotes the outward unit normal vector at $x \in \partial\Omega$. Notice that $\nu_\varepsilon(x, y) = (\nu(x), 0) \in \mathbb{R}^{N+1}$ is the outward unit normal vector at $(x, y) \in \partial_L \Omega_\varepsilon$.

Regarding the obliqueness of γ^\pm , we assume that, when we write $\gamma^\pm = (\gamma_1^\pm, \gamma_2^\pm)$, with $\gamma_1^\pm \in C(\overline{\Omega} \times [-1, 1], \mathbb{R}^N)$ and $\gamma_2^\pm \in C(\overline{\Omega} \times [-1, 1], \mathbb{R})$,

$$(3) \quad \pm \gamma_2^\pm > 0 \quad \text{on } \overline{\Omega} \times [-1, 1].$$

Hence, without loss of generality, we can suppose that

$$(4) \quad \gamma_2^+ = 1 \quad \text{and} \quad \gamma_2^- = -1.$$

So we will consider the oblique boundary condition

$$(5) \quad \gamma_1^+ \cdot D_x u^\varepsilon + u_y^\varepsilon = \beta^+ \quad \text{on } \partial_T \Omega_\varepsilon \quad \text{and} \quad \gamma_1^- \cdot D_x u^\varepsilon - u_y^\varepsilon = \beta^- \quad \text{on } \partial_B \Omega_\varepsilon.$$

If $g^\pm \in C^1(\overline{\Omega}, \mathbb{R})$, ν_ε denotes the outward unit normal vector on $\partial_T \Omega_\varepsilon \cup \partial_B \Omega_\varepsilon$, and $\varepsilon > 0$ is sufficiently small, then $\nu_\varepsilon \cdot (\gamma_1^+, 1) > 0$ on $\partial_T \Omega_\varepsilon$ and $\nu_\varepsilon \cdot (\gamma_1^-, -1) > 0$ on $\partial_B \Omega_\varepsilon$.

To present the results, we need some hypotheses and notation. As basic hypotheses, we assume throughout this paper the following (H1) and (H2):

(H1) $F \in C(\mathcal{S}(N+1) \times \mathbb{R}^{N+1} \times \mathbb{R} \times \overline{\Omega} \times [-1, 1], \mathbb{R})$, and for every $(p, x, y) \in \mathbb{R}^{N+1} \times \overline{\Omega} \times [-1, 1]$ and some constant $\alpha > 0$, if $X, Y \in \mathcal{S}(N+1)$, $r, s \in \mathbb{R}$, $X \geq Y$ in the semi-definite sense, and $r \leq s$ then $F(X, p, r, x, y) - \alpha r \leq F(Y, p, s, x, y) - \alpha s$.

(H2) The set Ω is a bounded, open domain with C^1 boundary, $\gamma^\pm \in C(\overline{\Omega} \times [-1, 1], \mathbb{R}^{N+1})$, and $\beta^\pm \in C(\overline{\Omega} \times [-1, 1], \mathbb{R})$.

Here and below, $\mathcal{S}(k)$ denotes the space of $k \times k$ real symmetric matrices.

We always assume that

$$(6) \quad g^\pm \in C^1(\overline{\Omega}), \quad g^-(x) < g^+(x) \quad \text{for } x \in \overline{\Omega}.$$

Since we are interested in the asymptotic behaviour, as $\varepsilon \rightarrow 0^+$, of the solutions to the boundary value problem (1), we may focus our considerations only on small ε , say $0 < \varepsilon < \varepsilon_0$, which allows us to assume always that

$$(7) \quad \begin{cases} \overline{\Omega}_\varepsilon \subset \overline{\Omega} \times [-1, 1], \\ \nu_\varepsilon \cdot (\gamma_1^+, 1) > 0 \quad \text{on } \partial_T \Omega_\varepsilon, \\ \nu_\varepsilon \cdot (\gamma_1^-, -1) > 0 \quad \text{on } \partial_B \Omega_\varepsilon. \end{cases}$$

A crucial assumption on γ^\pm and β^\pm is the following:

$$(8) \quad \beta^+(x, 0) = -\beta^-(x, 0) =: \beta_o(x) \quad \text{and} \quad \gamma_1^+(x, 0) = -\gamma_1^-(x, 0) =: \gamma_o(x) \quad \text{for } x \in \overline{\Omega}.$$

The necessity of this assumption will be evident in Section 2 where we perform a formal expansion for the clarity of the argument.

We assume that

$$(9) \quad \beta_o \in C^1(\overline{\Omega}, \mathbb{R}) \quad \text{and} \quad \gamma_o \in C^1(\overline{\Omega}, \mathbb{R}^N).$$

In this paper, our notation that $f \in C^i(K, \mathbb{R}^k)$ for a compact subset K of \mathbb{R}^j indicates that $f : K \rightarrow \mathbb{R}^k$ is the restriction of a function $g \in C^i(\mathbb{R}^j, \mathbb{R}^k)$ to the set K .

We need to assume that for some $k^\pm \in C(\overline{\Omega}, \mathbb{R}^N)$ and $l^\pm \in C(\overline{\Omega}, \mathbb{R})$,

$$(10) \quad \begin{cases} \gamma_1^\pm(x, y) = \pm\gamma_o(x) + k^\pm(x)y + o(|y|) \\ \beta^\pm(x, y) = \pm\beta_o(x) + l^\pm(x)y + o(|y|), \end{cases}$$

as $y \rightarrow 0$, where $o(|y|)/|y| \rightarrow 0$ uniformly on $\overline{\Omega}$ as $y \rightarrow 0$.

Having introduced all the terms in the first order expansion in y , we are in a position to write down the limit equation.

To simplify the notations we introduce the terms $b \in C(\overline{\Omega}, \mathbb{R}^N)$, $c \in C(\overline{\Omega}, \mathbb{R})$:

$$(11) \quad b(x) = \gamma_o(D\gamma_o)^\top - \frac{1}{g^+ - g^-} (g^+ k^+ + g^- k^-),$$

$$(12) \quad c(x) = -\gamma_o \cdot D\beta_o + \frac{1}{g^+ - g^-} (g^+ l^+ + g^- l^-),$$

where $D\gamma_o(x)$ denotes the matrix, whose (i, j) th entry is given by $\partial\gamma_{oi}(x)/\partial x_j$ and $D\beta_o(x)$ denotes the row vector, whose i th entry is $\partial\beta_o(x)/\partial x_i$. From now on, as in (11) and (12), in order to simplify notation, when a function f is given and no confusion is expected, the value $f(x)$ of the function f at x is often written as f .

We define $G : \mathcal{S}(N) \times \mathbb{R}^N \times \mathbb{R} \times \overline{\Omega} \rightarrow \mathbb{R}$ by

$$(13) \quad G(X, p, r, x) = F(A + B + C, (p, \beta_o - \gamma_o \cdot p), r, (x, 0)),$$

where

$$(14) \quad \begin{aligned} A &= \begin{pmatrix} X & -X\gamma_o(x)^\top \\ -\gamma_o(x)X & \gamma_o(x)X\gamma_o(x)^\top \end{pmatrix}, \\ B &= \begin{pmatrix} 0 & -(pD\gamma_o(x))^\top \\ -pD\gamma_o(x) & b(x) \cdot p \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & D\beta_o(x)^\top \\ D\beta_o(x) & c(x) \end{pmatrix}. \end{aligned}$$

We remark that if $A(X)$ denotes the above matrix A and $X \geq 0$, then

$$A(X) = \begin{pmatrix} I_N \\ -\gamma_o(x) \end{pmatrix} X \begin{pmatrix} I_N & -\gamma_o(x)^\top \end{pmatrix} \geq 0,$$

which ensures that G is degenerate elliptic, i.e., if $X, Y \in \mathcal{S}(N)$ and $X \leq Y$, then $G(X, p, r, x) \geq G(Y, p, r, x)$ for every $(p, r, x) \in \mathbb{R}^N \times \mathbb{R} \times \overline{\Omega}$. In the above and below, I_N denotes the $N \times N$ identity matrix.

The limit problem for (1) with (2), is the boundary value problem

$$(15) \quad G(D^2u, Du, u, x) = 0 \quad \text{in } \Omega,$$

$$(16) \quad \nu \cdot Du = 0 \quad \text{on } \partial\Omega.$$

When the lateral condition is either oblique or Dirichlet then the limit equation inherits on $\partial\Omega$ the lateral condition.

In the basic case $F(D^2u) = -\Delta u$, we get that the limit equation is given by

$$(17) \quad G(D^2u, Du, u, x) = -(\Delta u + \gamma_o(x)D^2u\gamma_o(x)^T + b(x) \cdot Du + c(x)u).$$

When the boundary data on the top and bottom is not oblique but it is just the Neumann boundary condition, and when $g^-(x) = 0$ as in our previous paper [8], the previous quantities γ_o , β_o , k^- and l^\pm are zero while $k^+(x) = -\frac{Dg^+}{g^+}$ and hence we recover the limit equation:

$$F\left(\begin{pmatrix} X & 0 \\ 0 & \frac{Dg^+ \cdot Du}{g^+(x)} \end{pmatrix}, (Du, 0), u, (x, 0)\right) = 0 \text{ in } \Omega, \quad \nu \cdot Du = 0 \text{ on } \partial\Omega.$$

As discussed also in [8], the presence of corners in Ω_ε requires a little care in the definition of sub and super viscosity solution. Hence we give a definition of viscosity solution to (1), which, contrary to [10], does not require the continuity of the solution. The precise definition of sub and super viscosity solution up to the boundary is given in Section 3 below but let us anticipate that we call a bounded function u on $\overline{\Omega_\varepsilon}$ a viscosity solution to (1) if the upper and lower semicontinuous envelopes of u are viscosity sub and super solutions to (1) in the sense of [10], respectively. The definition of viscosity solution to (15)–(16) is made in the same way.

The first result we want to quote is an existence result for ε fixed small. We shall state it when, on the lateral boundary, the Neumann condition holds:

Proposition 1. *Assume (H1), (H2), (4), (6), (8), (9), and (10). Then, there exist positive constants $\varepsilon_1 < \varepsilon_0$ and C_0 such that for each $0 < \varepsilon < \varepsilon_1$, there is a viscosity solution u^ε to (1) with (2) furthermore any solution u^ε will satisfy*

$$\sup_{\overline{\Omega_\varepsilon}} |u^\varepsilon| \leq C_0.$$

We can now state the convergence theorem. Let S_ε denote the set of viscosity solutions to (1) with (2) for given $\varepsilon > 0$. Under the hypotheses of Proposition 1, we define the half relaxed limits u^\pm by

$$u^+(x) = \lim_{r \rightarrow 0^+} \sup \{u(\xi, \eta) : u \in S_\varepsilon, (\xi, \eta) \in \overline{\Omega_\varepsilon}, 0 < \varepsilon < r, |\xi - x| < r\},$$

$$u^-(x) = \lim_{r \rightarrow 0^+} \inf \{u(\xi, \eta) : u \in S_\varepsilon, (\xi, \eta) \in \overline{\Omega_\varepsilon}, 0 < \varepsilon < r, |\xi - x| < r\},$$

which are bounded functions on $\overline{\Omega}$ by Proposition 1.

Theorem 1. *Under the hypotheses of Proposition 1, the functions u^+ and u^- are a viscosity sub and super solutions to (15)–(16), respectively.*

Clearly, if G , as defined in (13) and (14), satisfies the comparison principle then $u_o := u^+ = u^-$ and it is a solution of (15)–(16). Furthermore u^ε converges uniformly to u_o , see Corollary 5.

Remark that the assumption (8) is indeed crucial. Consider problem:

$$\begin{cases} -u_{yy} + u = 1 & \text{in } (0, 1) \times (0, \varepsilon), \\ u_y(x, \varepsilon) = 1, \quad -u_y(x, 0) = 0, \quad u_x(0, y) = 0, \quad u_x(1, y) = 0 \end{cases}$$

for which all hypotheses, but (8), of Proposition 1 are satisfied. Clearly its solution is $u^\varepsilon(x, y) = \frac{1}{e^\varepsilon - e^{-\varepsilon}} (e^y + e^{-y}) + 1$, which is not bounded as ε goes to zero.

As already mentioned, in the paper the existence results and the convergence results will be stated and proved for a more general case of lateral conditions, see Proposition 2 and Theorem 4.

After the work of Hale and Raugel, there has been a number of interesting results that generalise it in many contexts see e.g. the works of Arrieta, Nakasato, Nogueira, Perreira, Villanueva-Pesqueira [3, 5, 4, 6] with for example oscillating boundary or quasi linear operators. On the other hand there is also a very large number of papers that treat the “variational” convergence of functionals F_ε to a new functional defined in a lower dimensional space, where the main tool is Γ convergence, it is impossible to mention all the works in that realm, but we can mention e.g. [1, 2, 9]. But, even if the problems are not so far mathematically, we believe that the point of view are so different that it would require quite some work to make the connection. And even if this would be very interesting it is certainly behind the scope of this introduction.

The paper is organised in the following way: In the next section we perform the formal expansion that will allow us to guess the limit equation and upload the test function approach. In the third section we prove the existence and bounds at ε fixed. In the fourth section we prove the main asymptotic results and finally in the last section we prove that when F is a Bellman-Isaacs operator under the right general condition the limit operator satisfies the comparison principle i.e. the convergence is uniform.

2. FORMAL EXPANSION

To gain an intuitive understanding of (15), we perform the following formal calculations. First, we assume that the following expansion, as $\varepsilon \rightarrow 0^+$, is valid in an appropriate sense:

$$u^\varepsilon(x, y) = u^0(x) + \varepsilon u^1(x, y/\varepsilon) + \varepsilon^2 u^2(x, y/\varepsilon) + \dots,$$

where $u^0 : \bar{\Omega} \rightarrow \mathbb{R}$, $u^1, u^2 : \bar{\Omega} \times [-1, 1] \rightarrow \mathbb{R}$, and every ingredients appearing above and elsewhere are supposed to be smooth enough. At this point, we do not require condition (8) yet, but consider the expansion formula (10). That is, we set it as follows.

$$(18) \quad \begin{cases} \gamma_1^\pm(x, y) = \gamma_1^\pm(x, 0) + k^\pm(x)y + o(|y|) \\ \beta^\pm(x, y) = \beta^\pm(x, 0) + l^\pm(x)y + o(|y|). \end{cases}$$

Insert the above expansions into the boundary condition (5), on the respective boundaries $\partial_T \Omega_\varepsilon$ and $\partial_B \Omega_\varepsilon$,

$$\begin{aligned} 0 &= \gamma_1^\pm \cdot D_x u^\varepsilon \pm u_y^\varepsilon - \beta^\pm \\ &= \gamma_1^\pm \cdot D u^0 + \varepsilon \gamma_1^\pm \cdot D_x u^1 + \varepsilon^2 \gamma_1^\pm \cdot D_x u^2 \pm u_y^1 \pm \varepsilon u_y^2 - \beta^\pm + \dots \\ &= \gamma_1^\pm(x, 0) \cdot D u^0(x) + \varepsilon g^\pm k^\pm(x) \cdot D u^0(x) + \varepsilon \gamma_1^\pm(x, 0) \cdot D_x u^1(x, g^\pm) \\ &\quad \pm u_y^1(x, g^\pm) \pm \varepsilon u_y^2(x, g^\pm) - \beta^\pm - \varepsilon g^\pm l^\pm(x) + \dots \end{aligned}$$

Here and below, for function $u = u(x, y)$, u_y denotes the y -derivative $\partial u / \partial y = D_y u$ of u . Equating to zero the coefficients of ε^0 and ε^1 on the right-hand side of the above, we obtain

$$(19) \quad 0 = \gamma_1^\pm(x, 0) \cdot D u^0(x) \pm u_y^1(x, g^\pm(x)) - \beta^\pm(x, 0),$$

$$(20) \quad \begin{aligned} 0 &= g^\pm k^\pm(x) \cdot D u^0(x) + \gamma_1^\pm(x, 0) \cdot D_x u^1(x, g^\pm(x)) \\ &\quad \pm u_y^2(x, g^\pm(x)) - g^\pm(x) l^\pm(x). \end{aligned}$$

In the domain Ω_ε , we have

$$(21) \quad 0 = F(D^2u^\varepsilon, Du^\varepsilon, u^\varepsilon, x, y),$$

with

$$(22) \quad \begin{cases} D^2u^\varepsilon = \begin{pmatrix} D_x^2u^0 & (D_xu_y^1)^T \\ D_xu_y^1 & \varepsilon^{-1}u_{yy}^1 + u_{yy}^2 \end{pmatrix} + o(1), \\ Du^\varepsilon = (D_xu^0, u_y^1) + o(1), \quad u^\varepsilon = u^0(x) + o(1). \end{cases}$$

A natural ansatz here, to obtain a PDE for $u^0 = \lim_{\varepsilon \rightarrow 0^+} u^\varepsilon$, is to impose that

$$u_{yy}^1(x, y) = 0 \quad \text{for } x \in \bar{\Omega}, \quad g^-(x) < y < g^+(x).$$

To achieve this, we assume that there is a function $v : \bar{\Omega} \rightarrow \mathbb{R}$ such that

$$(23) \quad u^1(x, y) = v(x)y \quad \text{for } y \in \mathbb{R}.$$

With this ansatz, (19) and (20) read

$$(24) \quad 0 = \gamma_1^\pm(x, 0) \cdot Du^0(x) \pm v(x) - \beta^\pm(x, 0),$$

$$(25) \quad 0 = g^\pm k^\pm(x) \cdot Du^0(x) + g^\pm \gamma_1^\pm(x, 0) \cdot Dv(x) \pm u_y^2(x, g^\pm) - g^\pm l^\pm(x).$$

The solution of (24) with respect to v is

$$v(x) = \beta^+(x, 0) - \gamma_1^+ \cdot Du^0(x) = -\beta^-(x, 0) + \gamma_1^- \cdot Du^0(x).$$

This is well defined if condition (8) holds, and it determines the value for v as

$$(26) \quad v(x) := \beta_o - \gamma_o \cdot Du^0(x),$$

and the expansion formula (18) turns out to be identical to (10).

With this choice of v , from (25), we obtain

$$(27) \quad u_y^2(x, g^\pm(x)) = \pm g^\pm (l^\pm(x) - k^\pm(x) \cdot Du^0(x) \mp \gamma_o(x) \cdot Dv).$$

For each x , we look for a quadratic function $Q(y)$ with slope $u_y^2(x, g^-(x))$ and $u_y^2(x, g^+(x))$ at $y = g^-(x)$ and $y = g^+(x)$, respectively, and choose:

$$Q(y) = \frac{(y - g^-)^2}{2(g^+ - g^-)} u_y^2(x, g^+) - \frac{(y - g^+)^2}{2(g^+ - g^-)} u_y^2(x, g^-).$$

Thus, our choice of u^2 is:

$$(28) \quad \begin{aligned} u^2(x, y) &= \frac{(y - g^-)^2 g^+}{2(g^+ - g^-)} \left(l^+(x) - k^+(x) \cdot Du^0(x) - \gamma_o \cdot Dv \right) \\ &+ \frac{(y - g^+)^2 g^-}{2(g^+ - g^-)} \left(l^-(x) - k^-(x) \cdot Du^0(x) + \gamma_o \cdot Dv \right). \end{aligned}$$

We set

$$(29) \quad w^\pm(x) = \frac{g^\pm}{(g^+ - g^-)} (l^\pm(x) - k^\pm(x) \cdot Du^0(x) \mp \gamma_o \cdot Dv),$$

so that (28) can be written as

$$(30) \quad u^2(x, y) = \frac{1}{2}(y - g^-(x))^2 w^+(x) + \frac{1}{2}(y - g^+(x))^2 w^-(x).$$

In view of (22), we wish to give convenient formulas for Du^1 and u_{yy}^2 , which involve explicitly neither functions u^1 nor u^2 . We compute that

$$(31) \quad Du_y^1 = Dv = D(\beta_o - \gamma_o \cdot Du^0(x))$$

$$\begin{aligned}
&= D\beta_o - Du^0 D\gamma_o - \gamma_o D^2 u^0, \\
(32) \quad \gamma_o \cdot Dv &= \gamma_o \cdot D(\beta_o - \gamma_o \cdot Du^0) \\
&= \gamma_o \cdot D\beta_o - \gamma_o \cdot Du^0 D\gamma_o - \gamma_o D^2 u^0 \gamma_o^T \\
&= \gamma_o \cdot D\beta_o - [\gamma_o (D\gamma_o)^T] \cdot Du^0 - \gamma_o D^2 u^0 \gamma_o^T,
\end{aligned}$$

and

$$\begin{aligned}
(33) \quad u_{yy}^2(x, y) &= w^+(x) + w^-(x) \\
&= \frac{1}{g^+ - g^-} (g^+ l^+ + g^- l^-) - \frac{1}{g^+ - g^-} (g^+ k^+ + g^- k^-) \cdot Du^0 - \gamma_o \cdot Dv \\
&= \frac{1}{g^+ - g^-} (g^+ l^+ + g^- l^-) - \frac{1}{g^+ - g^-} (g^+ k^+ + g^- k^-) \cdot Du^0 \\
&\quad - \gamma_o \cdot D\beta_o + [\gamma_o (D\gamma_o)^T] \cdot Du^0 + \gamma_o D^2 u^0 \gamma_o^T.
\end{aligned}$$

Recalling (11) and (12), we obtain

$$(34) \quad u_{yy}^2 = \gamma_o D^2 u^0 \gamma_o^T + b \cdot Du^0 + c.$$

We go back to (22), to find that as $\varepsilon \rightarrow 0^+$,

$$\begin{aligned}
D^2 u^\varepsilon &= \begin{pmatrix} D^2 u^0 & (D\beta_o - Du^0 D\gamma_o - \gamma_o D^2 u^0)^T \\ D\beta_o - Du^0 D\gamma_o - \gamma_o D^2 u^0 & \gamma_o D^2 u^0 \gamma_o^T + b \cdot Du^0 + c \end{pmatrix} + o(1) \\
&= \begin{pmatrix} D^2 u^0 & -D^2 u^0 \gamma_o^T \\ -\gamma_o D^2 u^0 & \gamma_o D^2 u^0 \gamma_o^T \end{pmatrix} + \begin{pmatrix} 0 & -(Du^0 D\gamma_o)^T \\ -Du^0 D\gamma_o & b \cdot Du^0 \end{pmatrix} + \begin{pmatrix} 0 & D\beta_o^T \\ D\beta_o & c \end{pmatrix} + o(1),
\end{aligned}$$

and

$$Du^\varepsilon = (Du^0, v) + o(1) = (Du^0, \beta_o - \gamma_o \cdot Du^0) + o(1).$$

Accordingly, in the limit as $\varepsilon \rightarrow 0^+$, we deduce that (15) is the equation for the limit function u^0 of the solutions u^ε to (1) and (5).

3. EXISTENCE AND BOUNDS

As mentioned in the introduction and as seen in [8], the presence of corners in Ω_ε requires a little care in the definition of viscosity sub and super solution. Instead since the choice of lateral condition does not affect the proofs of the main results we now introduce the more general case.

But first some notations: In the following proofs, we adopt the $o(\varepsilon^p)$ and $O(\varepsilon^p)$ symbols, to indicate, respectively, functions $q(x, \varepsilon)$ and $r(x, \varepsilon)$ on $\overline{\Omega}$ such that $|q(x, \varepsilon)|/\varepsilon^p$ converges uniformly to 0 on $\overline{\Omega}$ and $r(x, \varepsilon)/\varepsilon^p$ is uniformly bounded on $\overline{\Omega}$ as $\varepsilon \rightarrow 0^+$. Similarly, when $q(x, y, \varepsilon)$ and $r(x, y, \varepsilon)$ are functions of $\overline{\Omega_\varepsilon}$, $q = o(\varepsilon^p)$ and $r = O(\varepsilon)$ mean that, as $\varepsilon \rightarrow 0^+$, $\sup_{\overline{\Omega_\varepsilon}} |q(x, y, \varepsilon)|/\varepsilon^p \rightarrow 0$ and $\sup_{\overline{\Omega_\varepsilon}} |r(x, y, \varepsilon)|/\varepsilon^p$ is bounded by a constant.

Recall that $\varepsilon_0 > 0$ has been chosen so that (7) holds, that is, for $\varepsilon \in (0, \varepsilon_0)$, we have

$$\begin{aligned}
&\overline{\Omega_\varepsilon} \subset \overline{\Omega} \times [-1, 1], \\
(35) \quad -\varepsilon Dg^+ \cdot \gamma_1^+ + 1 &> 0 \quad \text{on } \partial_T \Omega_\varepsilon \quad \text{and} \quad \varepsilon Dg^- \cdot \gamma_1^- + 1 > 0 \quad \text{on } \partial_B \Omega_\varepsilon.
\end{aligned}$$

Let $\beta \in C(\partial\Omega \times [-1, 1], \mathbb{R})$ and $\gamma \in C(\partial\Omega \times [-1, 1], \mathbb{R}^{N+1})$ be given. We are interested in the boundary value problems for (1), with the boundary condition (5) on the top and bottom of the boundary and either of the conditions on the lateral boundary $\partial_L \Omega_\varepsilon$:

$$(36) \quad \gamma \cdot Du = \beta,$$

or

$$(37) \quad u = \beta.$$

In the case of (36) we need the following conditions

$$(38) \quad \begin{cases} \gamma = (\gamma_1, \gamma_2) \in C(\partial\Omega \times [-1, 1], \mathbb{R}^{N+1}), & \beta \in C(\partial\Omega \times [-1, 1], \mathbb{R}), \\ \gamma_1 \cdot \nu > 0 & \text{on } \partial\Omega \times [-1, 1], \quad \gamma_2(x, 0) = 0 \quad \text{for } x \in \partial\Omega. \end{cases}$$

For notational convenience, given $\varepsilon > 0$ we call problem (O_ε) (resp., problem (D_ε)) problem (1), with boundary condition B as in (36) (resp., problem (1), with boundary condition B as in (37)).

Notice that problem (1)–(2) is a particular case of problem (O_ε) , with $\gamma = \nu_L$ and $\beta = 0$, which we call (N_ε) .

In the sequel when we focus on either problem (N_ε) , (O_ε) , or (D_ε) we call it (P_ε) .

Viscosity solutions to these problems are defined as follows. When u is a bounded function on $\overline{\Omega_\varepsilon}$, we call u a viscosity subsolution of problem (O_ε) (resp., problem (D_ε)) if the following condition holds for the upper semicontinuous envelope v of u : whenever $\phi \in C^2(\overline{\Omega_\varepsilon})$, $\hat{z} = (\hat{x}, \hat{y}) \in \overline{\Omega_\varepsilon}$ and $\max_{\overline{\Omega_\varepsilon}}(v - \phi) = (v - \phi)(\hat{z})$, we have

$$(39) \quad F(D^2\phi(\hat{z}), D\phi(\hat{z}), v(\hat{z}), \hat{z}) \leq 0$$

if $\hat{z} \in \Omega_\varepsilon$, we have either (39) or

$$(40) \quad \gamma(\hat{z}) \cdot D\phi(\hat{z}) \leq \beta(\hat{z}) \quad (\text{resp., } v(\hat{z}) \leq \beta(\hat{z})),$$

if $\hat{z} \in \partial_L \Omega_\varepsilon \setminus (\partial_B \Omega_\varepsilon \cup \partial_T \Omega_\varepsilon)$, we have either (39) or

$$(41) \quad \gamma^-(\hat{z}) \cdot D\phi(\hat{z}) \leq \beta^-(\hat{z})$$

if $\hat{z} \in \partial_B \Omega_\varepsilon \setminus \partial_L \Omega_\varepsilon$, we have either (39) or

$$(42) \quad \gamma^+(\hat{z}) \cdot D\phi(\hat{z}) \leq \beta^+(\hat{z})$$

if $\hat{z} \in \partial_T \Omega_\varepsilon \setminus \partial_L \Omega_\varepsilon$, we have either (39), (40), or (41) if $\hat{z} \in \partial_L \Omega_\varepsilon \cap \partial_B \Omega_\varepsilon$, and we have either (39), (40), or (42) if $\hat{z} \in \partial_L \Omega_\varepsilon \cap \partial_T \Omega_\varepsilon$. Replacing “max”, “ \leq ”, and “upper semicontinuous envelope”, with “min”, “ \geq ”, and “lower semicontinuous envelope” respectively, in the above condition yields the right definition of viscosity supersolution. Viscosity solutions are functions which are both viscosity sub and super solutions.

The main result of this section is the following:

Proposition 2. *Assume (H1), (H2), (4), (6), (8), (9), and (10). Then, there exist positive constants $\varepsilon_1 < \varepsilon_0$ and C_0 such that for each $0 < \varepsilon < \varepsilon_1$, there is a viscosity solution u^ε to :*

$$(1), (5), \text{ and either (36) with (38) or (37) , on } \partial_L \Omega_\varepsilon.$$

Furthermore any solution u^ε will satisfy $\sup_{\overline{\Omega_\varepsilon}} |u^\varepsilon| \leq C_0$.

Before starting the proof we need a few technical results.

Lemma 2. *Let $u \in C(\overline{\Omega}, \mathbb{R})$. There is a family $\{u_\varepsilon : \varepsilon > 0\}$ of functions in $C^\infty(\overline{\Omega}, \mathbb{R})$ such that as $\varepsilon \rightarrow 0^+$,*

$$u_\varepsilon = u + o(1) \quad \text{and} \quad Du^\varepsilon = o(\varepsilon^{-1}).$$

Proof. By extending the domain of definition of u , we may assume that u is defined, bounded, uniformly continuous in \mathbb{R}^N . Let $\lambda \in C^\infty(\mathbb{R}^N, \mathbb{R})$ be a standard mollification kernel, with $\text{supp } \lambda$ being contained in the unit ball B_1 centered at the origin. Let λ_ε be defined by $\lambda_\varepsilon(x) = \varepsilon^{-N} \lambda(x/\varepsilon)$. Let m be the modulus of continuity of u . Set $u_\varepsilon = u * \lambda_\varepsilon$ for $\varepsilon > 0$, and observe that

$$\begin{aligned} |u_\varepsilon(x) - u(x)| &\leq \int_{B_\varepsilon} \lambda_\varepsilon(y) |u(x-y) - u(x)| dy \leq m(\varepsilon) \int_{B_\varepsilon} \lambda_\varepsilon(y) dy = m(\varepsilon), \\ |Du_\varepsilon(x)| &= \left| \int_{B_\varepsilon} u(x-y) D\lambda_\varepsilon(y) dy \right| = \left| \int_{B_\varepsilon} (u(x-y) - u(x)) D\lambda_\varepsilon(y) dy \right| \\ &\leq m(\varepsilon) \int_{B_\varepsilon} |D\lambda(y/\varepsilon)| \varepsilon^{-N-1} dy = m(\varepsilon) \varepsilon^{-1} \int_{B_1} |D\lambda(y)| dy. \end{aligned}$$

□

Remark 3. The above proof extends to a general case where $u \in C^k(\overline{\Omega}, \mathbb{R})$, with nonnegative integer k . Let u_ε be the function defined as above. Fix any $j \in \mathbb{N}$. The conclusion is: for multi-index $\sigma = (\sigma_1, \dots, \sigma_N)$, if $|\sigma| < k$, then

$$D^\sigma u_\varepsilon = D^\sigma u + O(\varepsilon),$$

if $|\sigma| = k$, then

$$D^\sigma u_\varepsilon = D^\sigma u + o(1)$$

and, if $k < |\sigma| \leq k + j$, then

$$D^\sigma u_\varepsilon = o(\varepsilon^{k-|\sigma|}).$$

Notice that if $|\sigma| < k$, then $D^\sigma u$ is Lipschitz continuous on Ω .

It is well known that, to obtain an existence result like e.g. Proposition 2, one can apply the Perron method to the boundary value problem (P_ε) (see, for instance, [10, Remark 4.5]). A crucial property in this construction is that classical solutions (twice continuously differentiable solutions in the pointwise sense) are also solution in the viscosity sense.

A result in this direction is the following.

Proposition 4. *Assume $g^\pm \in C^1(\overline{\Omega}, \mathbb{R})$, (H1), (H2) and (38). There exists $\varepsilon_* \in (0, \varepsilon_0)$ such that for $0 < \varepsilon < \varepsilon_*$, if $u \in C^2(\overline{\Omega}_\varepsilon)$ is a classical sub (resp., super) solution of problem (O_ε) ,*

then it is a viscosity sub (resp., super) solution of (O_ε) .

This proposition is a straightforward consequence of the following Lemma (see also [8, Remark 4]).

Lemma 3. *Assume $g^\pm \in C^1(\overline{\Omega}, \mathbb{R})$, (H2), and (38). There exist constants $\varepsilon_* \in (0, \varepsilon_0)$ and $\theta > 0$ such that if $0 < \varepsilon < \varepsilon_*$, $\psi \in C^1(\overline{\Omega}_\varepsilon)$, and ψ takes a maximum at $c \in \partial_L \Omega_\varepsilon \cap \partial_T \Omega_\varepsilon$ (resp., $c \in \partial_L \Omega_\varepsilon \cap \partial_B \Omega_\varepsilon$), then*

$$(\gamma + \theta\gamma^+) \cdot D\psi(c) \geq 0 \quad (\text{resp., } (\gamma + \theta\gamma^-) \cdot D\psi(c) \geq 0).$$

We present a brief proof of Proposition 4, based on Lemma 3, for completeness.

Proof of Proposition 4. We only consider the claim about the subsolution property. Let $u \in C^2(\overline{\Omega}_\varepsilon)$ be a classical subsolution of (O_ε) . Let $\phi \in C^2(\overline{\Omega}_\varepsilon)$ and $c \in \overline{\Omega}_\varepsilon$. Assume that $u - \phi$ takes a maximum at c . If $c \in \Omega_\varepsilon$, then the degenerate ellipticity of F , a consequence of (H1), assures that $F(D^2\phi(c), D\phi(c), u(c), c) \leq F(D^2u(c), Du(c), u(c), c)$, which yields $F(D^2\phi(c), D\phi(c), u(c), c) \leq 0$. If $c \in \partial_T \Omega_\varepsilon \setminus \partial_L \Omega_\varepsilon$, then, by (35), γ^+ is oblique to $\partial_T \Omega_\varepsilon$,

which implies that $\gamma^+ \cdot D(\phi - u)(c) \leq 0$, and hence $\gamma^+ \cdot D\phi(c) \leq \beta^+(c)$. Similarly, γ^- and γ are oblique to $\partial_B \Omega_\varepsilon$ at $c \in \partial_B \Omega_\varepsilon \setminus \partial_L \Omega_\varepsilon$ and to $\partial_L \Omega_\varepsilon$ at $c \in \partial_L \Omega_\varepsilon \setminus (\partial_T \Omega_\varepsilon \cup \partial_B \Omega_\varepsilon)$, respectively, and hence, if $c \in \partial_B \Omega_\varepsilon \setminus \partial_L \Omega_\varepsilon$, then $\gamma^- \cdot D\phi(c) \leq \beta^-(c)$ and if $c \in \partial_L \Omega_\varepsilon \setminus (\partial_T \Omega_\varepsilon \cup \partial_B \Omega_\varepsilon)$, then $\gamma \cdot D\phi(c) \leq \beta(c)$. Finally, we consider the case $c \in (\partial_T \Omega_\varepsilon \cup \partial_B \Omega_\varepsilon) \cap \partial_L \Omega_\varepsilon$. If $c \in \partial_T \Omega_\varepsilon \cap \partial_L \Omega_\varepsilon$, then, by Lemma 3, we have $(\gamma + \theta\gamma^+) \cdot D(u - \phi)(c) \geq 0$ for some $\theta > 0$. This implies either $\gamma \cdot D(u - \phi)(c) \geq 0$ or $\gamma^+ \cdot D(u - \phi)(c) \geq 0$. Here, the former and the latter yield $\gamma \cdot D\phi(c) \leq \beta(c)$ and $\gamma^+ \cdot D\phi(c) \leq \beta^+(c)$, respectively. If, instead, $c \in \partial_L \Omega_\varepsilon \cap \partial_B \Omega_\varepsilon$, then we have $(\gamma + \theta\gamma^-) \cdot D(u - \phi)(c) \geq 0$ for some $\theta > 0$ by Lemma 3. Hence, similarly to the above, we have either $\gamma \cdot D\phi(c) \leq \beta(c)$ or $\gamma^- \cdot D\phi(c) \leq \beta^-(c)$. Thus, we conclude that u is a viscosity subsolution of problem (O_ε) . \square

Proof of Lemma 3. We use the notation that, given a vector $\zeta = (\bar{\zeta}_1, \dots, \bar{\zeta}_{N+1}) \in \mathbb{R}^{N+1}$, we write $\zeta_1 = (\bar{\zeta}_1, \dots, \bar{\zeta}_N) \in \mathbb{R}^N$ and $\zeta_2 = \bar{\zeta}_{N+1} \in \mathbb{R}$, so that $\zeta = (\zeta_1, \zeta_2)$.

It is enough to prove the following

Claim: *There exist constants $\varepsilon_* \in (0, \varepsilon_0)$ and $\theta > 0$ such that if $0 < \varepsilon < \varepsilon_*$, then*

- (i) *for any $c \in \partial_L \Omega_\varepsilon \cap \partial_T \Omega_\varepsilon$, there is $t_c > 0$ such that $c - t(\gamma + \theta\gamma^+) \in \Omega_\varepsilon$ for $0 < t < t_c$,*
- (ii) *for any $c \in \partial_L \Omega_\varepsilon \cap \partial_B \Omega_\varepsilon$, there is $t_c > 0$ such that $c - t(\gamma + \theta\gamma^-) \in \Omega_\varepsilon$ for $0 < t < t_c$.*

Choose $\rho \in C^1(\mathbb{R}^N, \mathbb{R})$ so that

$$\rho < 0 \quad \text{in } \Omega, \quad \rho > 0 \quad \text{in } \mathbb{R}^N \setminus \bar{\Omega}, \quad \text{and} \quad D\rho \neq 0 \quad \text{on } \partial\Omega.$$

We may regard ρ as a function of $(x, y) \in \mathbb{R}^{N+1}$ independent of y . Define $\rho^\pm(x, y) = \pm(y - \varepsilon g^\pm(x))$ on \mathbb{R}^{N+1} . We only need to show that for a choice of $\varepsilon_* \in (0, \varepsilon_0)$ and $\theta > 0$,

- (iii) $(\gamma + \theta\gamma^+) \cdot D\rho(c) > 0$ and $(\gamma + \theta\gamma^+) \cdot D\rho^+(c) > 0$ for $c \in \partial_L \Omega_\varepsilon \cap \partial_T \Omega_\varepsilon$ and $0 < \varepsilon < \varepsilon_*$,
- (iv) $(\gamma + \theta\gamma^-) \cdot D\rho(c) > 0$ and $(\gamma + \theta\gamma^-) \cdot D\rho^-(c) > 0$ for $c \in \partial_L \Omega_\varepsilon \cap \partial_B \Omega_\varepsilon$ and $0 < \varepsilon < \varepsilon_*$.

Fix $\varepsilon \in (0, \varepsilon_0)$ and $\theta > 0$. For $c = (c_1, c_2) \in \partial_L \Omega_\varepsilon \cap \partial_T \Omega_\varepsilon$, we compute that

$$(43) \quad \begin{aligned} (\gamma + \theta\gamma^+) \cdot D\rho(c) &= (\gamma_1 + \theta\gamma_1^+, \gamma_2 + \theta) \cdot (|D\rho(c_1)|\nu(c_1), 0) \\ &= |D\rho(c_1)|(\gamma_1(c) \cdot \nu(c_1) + \theta\gamma_1^+(c) \cdot \nu(c_1)), \end{aligned}$$

and

$$(44) \quad \begin{aligned} (\gamma + \theta\gamma^+) \cdot D\rho^+(c) &= (\gamma + \theta\gamma^+) \cdot (-\varepsilon Dg^+(c_1), 1) \\ &= -\varepsilon(\gamma_1 + \theta\gamma_1^+) \cdot Dg^+(c_1) + (\gamma_2 + \theta)(c). \end{aligned}$$

Similarly, for $c \in \partial_L \Omega_\varepsilon \cap \partial_B \Omega_\varepsilon$, we compute that

$$(45) \quad \begin{aligned} (\gamma + \theta\gamma^-) \cdot D\rho(c) &= (\gamma_1 + \theta\gamma_1^-, \gamma_2 - \theta) \cdot (|D\rho(c_1)|\nu(c_1), 0) \\ &= |D\rho(c_1)|(\gamma_1(c) \cdot \nu(c_1) + \theta\gamma_1^-(c) \cdot \nu(c_1)), \end{aligned}$$

and

$$(46) \quad \begin{aligned} (\gamma + \theta\gamma^-) \cdot D\rho^-(c) &= (\gamma + \theta\gamma^-) \cdot (\varepsilon Dg^-(c_1), -1) \\ &= \varepsilon(\gamma_1 + \theta\gamma_1^-) \cdot Dg^-(c_1) - (\gamma_2 - \theta)(c). \end{aligned}$$

Since

$$\min_{(x,y) \in \partial\Omega \times [-1,1]} \gamma_1(x, y) \cdot \nu(x) > 0 \quad \text{by (38),}$$

we may choose $\theta > 0$ small enough so that

$$\min_{(x,y) \in \partial\Omega \times [-1,1]} \min\{(\gamma_1(x,y) + \theta\gamma_1^+(x,y)) \cdot \nu(x), (\gamma_1(x,y) + \theta\gamma_1^-(x,y)) \cdot \nu(x)\} > 0.$$

Consequently, from (43) and (45), we find that

$$(\gamma + \theta\gamma^+) \cdot D\rho(c) > 0 \quad \text{for } c \in \partial_L\Omega_\varepsilon \cap \partial_T\Omega_\varepsilon,$$

and

$$(\gamma + \theta\gamma^-) \cdot D\rho(c) > 0 \quad \text{for } c \in \partial_L\Omega_\varepsilon \cap \partial_B\Omega_\varepsilon.$$

Next, thanks to (38), we can choose $\delta \in (0, 1)$ so that

$$\max_{\partial\Omega \times [-\delta, \delta]} |\gamma_2(c)| \leq \frac{\theta}{2}.$$

We choose $\varepsilon_1 \in (0, \varepsilon_0)$ so that

$$\varepsilon_1 \max\{|g^+(x)|, |g^-(x)|\} \leq \delta \quad \text{for } x \in \bar{\Omega},$$

which implies that for $0 < \varepsilon \leq \varepsilon_1$,

$$\bar{\Omega}_\varepsilon \subset \bar{\Omega} \times [-\delta, \delta],$$

and moreover,

$$|\gamma_2(c)| \leq \frac{\theta}{2} \quad \text{for } c \in (\partial_L\Omega_\varepsilon \cap \partial_T\Omega_\varepsilon) \cup (\partial_L\Omega_\varepsilon \cap \partial_B\Omega_\varepsilon) \quad \text{and } 0 < \varepsilon \leq \varepsilon_1.$$

Now, from (44) and (46), we find that for $0 < \varepsilon < \varepsilon_1$, if $c \in \partial_L\Omega_\varepsilon \cap \partial_T\Omega_\varepsilon$, then

$$(\gamma + \theta\gamma^+) \cdot D\rho^+(c) \geq \frac{\theta}{2} - \varepsilon(\gamma_1 + \theta\gamma_1^+) \cdot Dg^+(c_1)$$

and if $c \in \partial_L\Omega_\varepsilon \cap \partial_B\Omega_\varepsilon$,

$$(\gamma + \theta\gamma^-) \cdot D\rho^-(c) \geq \frac{\theta}{2} + \varepsilon(\gamma_1 + \theta\gamma_1^-) \cdot Dg^-(c_1).$$

We select $\varepsilon_* \in (0, \varepsilon_1)$ so that

$$\varepsilon_* \max_{c \in \partial\Omega \times [-\delta, \delta]} [(\gamma_1 + \theta\gamma_1^+)(c) \cdot Dg^+(c_1)]^+ < \frac{\theta}{2},$$

and

$$\varepsilon_* \max_{c \in \partial\Omega \times [-\delta, \delta]} [(\gamma_1 + \theta\gamma_1^-)(c) \cdot Dg^-(c_1)]^- < \frac{\theta}{2},$$

to obtain for $0 < \varepsilon < \varepsilon_*$,

$$(\gamma + \theta\gamma^+) \cdot D\rho^+(c) > 0 \quad \text{for } c \in \partial_L\Omega_\varepsilon \cap \partial_T\Omega_\varepsilon,$$

and

$$(\gamma + \theta\gamma^-) \cdot D\rho^-(c) > 0 \quad \text{for } c \in \partial_L\Omega_\varepsilon \cap \partial_B\Omega_\varepsilon.$$

This proves the claim and hence the lemma. \square

In the case of the Dirichlet boundary condition (36), as the next proposition states, classical sub and super solutions are also viscosity sub and super solutions, respectively. The corners of the domain do not matter.

Proposition 5. *Assume $g^\pm \in C^1(\bar{\Omega}, \mathbb{R})$, (H1), (H2). Let $\varepsilon \in (0, \varepsilon_0)$. If $u \in C^2(\bar{\Omega}_\varepsilon)$ is a classical sub (resp., super) solution of problem (D_ε) , then it is a viscosity sub (resp., super) solution of (D_ε) .*

Proof. We only treat the claim about the subsolution property. Let $u \in C^2(\overline{\Omega_\varepsilon})$ be a classical subsolution to (D_ε) . Assume that for some $\phi \in C^2(\overline{\Omega_\varepsilon})$ and $c \in \overline{\Omega_\varepsilon}$, $u - \phi$ takes a maximum at c . As in the proof of Proposition 4, thanks to the degenerate ellipticity of F and the oblique property of γ^\pm , as stated as (35), we easily see that if $c \in \Omega_\varepsilon$, then $F(D^2\phi(c), D\phi(c), u(c), c) \leq 0$, if $c \in \partial_T\Omega_\varepsilon \setminus \partial_L\Omega_\varepsilon$, then $\gamma^+ \cdot D\phi(c) \leq \beta^+(c)$, and if $c \in \partial_B\Omega_\varepsilon \setminus \partial_L\Omega_\varepsilon$, then $\gamma^- \cdot D\phi(c) \leq \beta^-(c)$. It remains to check the case when $c \in \partial_L\Omega_\varepsilon$. However, if $c \in \partial_L\Omega_\varepsilon$, then we have $u(c) \leq \beta(c)$, which is enough to conclude that u is a viscosity subsolution to (D_ε) . \square

Proof of Proposition 2. First, we build classical strict sub and super solutions to (P_ε) . Choose functions $\rho, h \in C^2(\overline{\Omega}, \mathbb{R})$ such that h satisfies

$$g^-(x) < h(x) < g^+(x) \quad \text{for } x \in \overline{\Omega}.$$

while ρ will be chosen depending on the lateral boundary condition. We choose a constant $\delta_0 > 0$ so that

$$(47) \quad \pm(g^\pm(x) - h(x)) > \delta_0 \quad \text{for } x \in \overline{\Omega}.$$

As suggested by the formal expansion argument in the previous section, we set

$$\bar{v}(x) = \beta_o(x) - \gamma_o(x) \cdot D\rho(x), \quad \underline{v}(x) = \beta_o(x) + \gamma_o(x) \cdot D\rho(x) \quad \text{for } x \in \overline{\Omega},$$

where β_o, γ_o are defined by (8) and we choose families $\{\bar{v}_\varepsilon : 0 < \varepsilon < \varepsilon_0\}$ and $\{\underline{v}_\varepsilon : 0 < \varepsilon < \varepsilon_0\}$ of C^2 functions, approximating \bar{v} and \underline{v} , respectively. According to Lemma 2 and Remark 3, we can choose $\{\bar{v}_\varepsilon\}$ and $\{\underline{v}_\varepsilon\}$ so that as $\varepsilon \rightarrow 0+$,

$$\begin{aligned} \bar{v}_\varepsilon &= \bar{v} + O(\varepsilon), & D\bar{v}_\varepsilon &= D\bar{v} + o(1), & \text{and} & & D^2\bar{v}_\varepsilon &= o(\varepsilon^{-1}). \\ \underline{v}_\varepsilon &= \underline{v} + O(\varepsilon), & D\underline{v}_\varepsilon &= D\underline{v} + o(1), & \text{and} & & D^2\underline{v}_\varepsilon &= o(\varepsilon^{-1}). \end{aligned}$$

For a constant $\Lambda > 0$, to be fixed later, we can define $\bar{\psi}_\varepsilon, \underline{\psi}_\varepsilon \in C^2(\overline{\Omega} \times [-1, 1])$ by putting

$$\bar{\psi}_\varepsilon(x, y) = \rho(x) + \bar{v}_\varepsilon(x)y + \frac{\Lambda}{2}(y - \varepsilon h(x))^2 \quad \text{for } (x, y) \in \overline{\Omega} \times [-1, 1]$$

and

$$\underline{\psi}_\varepsilon(x, y) = -\rho(x) + \underline{v}_\varepsilon(x)y - \frac{\Lambda}{2}(y - \varepsilon h(x))^2 \quad \text{for } (x, y) \in \overline{\Omega} \times [-1, 1].$$

The function ρ will be chosen later in order that $\bar{\psi}_\varepsilon$ and $\underline{\psi}_\varepsilon$ satisfy the lateral boundary condition. When ever we need to treat both $\bar{\psi}_\varepsilon$ and $\underline{\psi}_\varepsilon$ we will use ψ_ε idem for \bar{v}_ε and $\underline{v}_\varepsilon$.

Note by the definition of \bar{v} that

$$\begin{aligned} (\gamma_1^+ \cdot D\rho + \bar{v}_\varepsilon - \beta^+)(x, \varepsilon g^+(x)) &= (\gamma_1^+ \cdot D\rho + \bar{v} - \beta^+)(x, \varepsilon g^+(x)) + O(\varepsilon) \\ &= (\gamma_o \cdot D\rho + \bar{v} - \beta_o)(x) + O(\varepsilon) = O(\varepsilon), \end{aligned}$$

and

$$\begin{aligned} (\gamma_1^- \cdot D\rho - \bar{v}_\varepsilon - \beta^-)(x, \varepsilon g^-(x)) &= (\gamma_1^- \cdot D\rho - \bar{v} - \beta^-)(x, \varepsilon g^-(x)) + O(\varepsilon) \\ &= (-\gamma_o \cdot D\rho - \bar{v} + \beta_o)(x) + O(\varepsilon) = O(\varepsilon), \end{aligned}$$

where we have used that by (10), $(\gamma_1^\pm, \beta^\pm)(x, \varepsilon g(x)) = (\pm\gamma_o, \pm\beta_o)(x) + O(\varepsilon)$, (for $g = g^+$ and $g = g^-$). Similarly, we see that

$$(-\gamma_1^+ \cdot D\rho + \underline{v}_\varepsilon - \beta^+)(x, \varepsilon g^+(x)) = (-\gamma_o \cdot D\rho + \underline{v} - \beta_o)(x) + O(\varepsilon) = O(\varepsilon),$$

and

$$(-\gamma_1^- \cdot D\rho - \underline{v}_\varepsilon - \beta^-)(x, \varepsilon g^-(x)) = (\gamma_o \cdot D\rho - \underline{v} + \beta_o)(x) + O(\varepsilon) = O(\varepsilon).$$

We now compute

$$D\bar{\psi}_\varepsilon(x, y) = (D\rho(x) + yD\bar{v}_\varepsilon(x) + \Lambda\varepsilon(\varepsilon h(x) - y)Dh(x), \bar{v}_\varepsilon(x) + \Lambda(y - \varepsilon h(x))).$$

Hence,

$$\begin{aligned} (\gamma^\pm \cdot D\bar{\psi}_\varepsilon - \beta^\pm)(x, \varepsilon g^\pm(x)) &= (\gamma_1^\pm \cdot D\rho \pm \bar{v}_\varepsilon - \beta^\pm)(x, \varepsilon g^\pm(x)) \\ &\quad + \gamma_1^\pm(x, \varepsilon g^\pm(x)) \cdot (\varepsilon g^\pm(x)D\bar{v}_\varepsilon(x) + \Lambda\varepsilon(\varepsilon h(x) - \varepsilon g^\pm(x))Dh(x)) \\ &\quad \pm \Lambda(\varepsilon g^\pm(x) - \varepsilon h(x)) \\ &= \pm\varepsilon\Lambda(g^\pm(x) - h(x)) + O(\varepsilon). \end{aligned}$$

Similarly,

$$(\gamma^\pm \cdot D\underline{\psi}_\varepsilon - \beta^\pm)(x, \varepsilon g^\pm(x)) = \mp\varepsilon\Lambda(g^\pm(x) - h(x)) + O(\varepsilon).$$

These, combined with (47), assure that, for some constant $C_1 > 0$ independent of ε ,

$$(\gamma^\pm \cdot D\bar{\psi}_\varepsilon - \beta^\pm)(x, \varepsilon g^\pm(x)) \geq \varepsilon\Lambda\delta_0 - C_1\varepsilon,$$

and

$$(\gamma^\pm \cdot D\underline{\psi}_\varepsilon - \beta^\pm)(x, \varepsilon g^\pm(x)) \leq -\varepsilon\Lambda\delta_0 + C_1\varepsilon.$$

Selecting $\Lambda > 0$ so that $\delta_0\Lambda > C_1$, we find that

$$(48) \quad \begin{cases} \gamma^+ \cdot D\bar{\psi}_\varepsilon > \beta^+ & \text{on } \partial_T\Omega_\varepsilon, & \gamma^- \cdot D\bar{\psi}_\varepsilon > \beta^- & \text{on } \partial_B\Omega_\varepsilon, \\ \gamma^+ \cdot D\underline{\psi}_\varepsilon < \beta^+ & \text{on } \partial_T\Omega_\varepsilon, & \text{and } \gamma^- \cdot D\underline{\psi}_\varepsilon < \beta^- & \text{on } \partial_B\Omega_\varepsilon. \end{cases}$$

Next, we need to choose ρ in order to satisfy the lateral condition. Hence we need to treat two different cases:

For problem (O_ε) we select $\rho \in C^1(\bar{\Omega})$ so that $D\rho(x)/|D\rho(x)| = \nu(x)$ for $x \in \partial\Omega$. Since $\gamma_1 \cdot \nu > 0$ on $\partial\Omega \times [-1, 1]$, we have $\gamma_1 \cdot D\rho > 0$ on $\partial\Omega \times [-1, 1]$. By approximating ρ by a smooth function, we may assume that $\rho \in C^2(\bar{\Omega})$ and $\gamma_1 \cdot D\rho > 0$ on $\partial\Omega \times [-1, 1]$. Multiplying ρ by a large constant and replacing ρ with the resulting function, we may assume that

$$\gamma_1 \cdot D\rho > |\beta| + 1 \quad \text{on } \partial\Omega \times [-1, 1].$$

We compute that for any $x \in \partial\Omega$,

$$\begin{aligned} \gamma \cdot D\bar{\psi}_\varepsilon &= \gamma_1 \cdot D\rho + y\gamma_1 \cdot D\bar{v}_\varepsilon(x) + \varepsilon\Lambda(\varepsilon h(x) - y)\gamma_1 \cdot Dh(x) + \gamma_2\bar{v}_\varepsilon + O(\varepsilon) \\ &= \gamma_1 \cdot D\rho + o(1) > \beta \end{aligned}$$

Similarly

$$\gamma \cdot D\underline{\psi}_\varepsilon = -\gamma_1 \cdot D\rho + o(1) < \beta.$$

For problem (D_ε) instead we choose $\rho \in C^2(\bar{\Omega})$ such that

$$\rho > |\beta| + 1$$

and the lateral Dirichlet condition is satisfied.

Observe that $\bar{\psi}_\varepsilon$ and $\underline{\psi}_\varepsilon$ have the same regularity, as functions on $\bar{\Omega}_\varepsilon$. We need to prove that

$$\psi_\varepsilon = O(1), \quad D\psi_\varepsilon = O(1), \quad \text{and} \quad D^2\psi_\varepsilon = O(1).$$

In this regard, the only terms in ψ_ε that need to be taken care of are $f_\varepsilon(x, y) := y\bar{v}_\varepsilon(x)$ or $f_\varepsilon(x, y) := y\underline{v}_\varepsilon(x)$ but in both cases

$$\begin{cases} f_\varepsilon(x, y) = O(\varepsilon), & Df_\varepsilon(x, y) = (yDv_\varepsilon(x), v_\varepsilon(x)) = O(1), \\ D_y Df_\varepsilon(x, y) = (Dv_\varepsilon, 0) = O(1), & D_y^2 f_\varepsilon(x, y) = 0, \\ D_x^2 f_\varepsilon(x, y) = yD^2v_\varepsilon(x) = O(\varepsilon)o(\varepsilon^{-1}) = o(1). \end{cases}$$

Consequently,

$$F(D^2\psi_\varepsilon, D\psi_\varepsilon, \psi_\varepsilon, x, y) = O(1)$$

as functions on $\overline{\Omega_\varepsilon}$, and there is a constant $C_3 > 0$ such that

$$|F(D^2\psi_\varepsilon, D\psi_\varepsilon, \psi_\varepsilon, x, y)| \leq C_3 \quad \text{for } (x, y) \in \overline{\Omega_\varepsilon} \text{ and } 0 < \varepsilon < \varepsilon_1.$$

For any $M > 0$, by (H1), we have for $(x, y) \in \overline{\Omega_\varepsilon}$,

$$\begin{aligned} F(D^2\overline{\psi}_\varepsilon, D\overline{\psi}_\varepsilon, \overline{\psi}_\varepsilon + M, x, y) &\geq -C_3 + \alpha M, \\ F(D^2\underline{\psi}_\varepsilon, D\underline{\psi}_\varepsilon, \underline{\psi}_\varepsilon - M, x, y) &\leq C_3 - \alpha M. \end{aligned}$$

By selecting $M > 0$ so that $\alpha M > C_3$, recalling (48) and the appropriate choice of ρ that implies the lateral condition, we find that for every $\varepsilon \in (0, \varepsilon_1)$, the functions $\overline{\psi}_\varepsilon + M$ and $\underline{\psi}_\varepsilon - M$ are classical, strict, sub and supersolutions to (P_ε) , respectively.

Now, let $\varepsilon \in (0, \varepsilon_1)$. It follows from the classical argument for the maximum principle that if u^ε is a viscosity solution to (P_ε) , then $\underline{\psi}_\varepsilon - M \leq u^\varepsilon \leq \overline{\psi}_\varepsilon + M$ on $\overline{\Omega_\varepsilon}$. This shows that the family of all viscosity solutions u^ε to (P_ε) , with $\varepsilon \in (0, \varepsilon_1)$, is uniformly bounded on $\overline{\Omega_\varepsilon}$. That is, there is a constant $C > 0$, independent of $\varepsilon > 0$, such that if u^ε is a viscosity solution to (P_ε) , then $|u^\varepsilon(x, y)| \leq C$ for all $(x, y) \in \overline{\Omega_\varepsilon}$ and $\varepsilon \in (0, \varepsilon_1)$.

Let $\varepsilon_* > 0$ be a constant from Proposition 4 or 5 depending on the lateral condition. We may assume that $\varepsilon_1 \leq \varepsilon_*$. Let $\varepsilon \in (0, \varepsilon_1)$. By Proposition 4 or 5, the functions $\overline{\psi}_\varepsilon + M$ and $\underline{\psi}_\varepsilon - M$ are viscosity sub and super solutions of (P_ε) , respectively. Thus, for any $\varepsilon \in (0, \varepsilon_1)$, Perron's method readily produces a viscosity solution u^ε to (P_ε) such that $\underline{\psi}_\varepsilon - M \leq u^\varepsilon \leq \overline{\psi}_\varepsilon + M$ on $\overline{\Omega_\varepsilon}$. Indeed, if we set

$$u^\varepsilon(z) = \sup\{v(z) : v \text{ is a viscosity subsolution of } (1) - (2), \underline{\psi}_\varepsilon - M \leq v \leq \overline{\psi}_\varepsilon + M\}$$

then the function u^ε is a viscosity solution to (P_ε) . Note that Proposition 4 or 5 are crucial when one checks that the lower semicontinuous envelope of u^ε is a supersolution to (P_ε) . \square

4. PROOF OF MAIN RESULT

We begin by stating Theorem 1 in the general case of problem (P_ε) . Let S_ε denote the set of viscosity solutions to (P_ε) , with given $\varepsilon > 0$. Under the hypotheses of Proposition 2, we define the half relaxed limits u^\pm by

$$(49) \quad \begin{aligned} u^+(x) &= \lim_{r \rightarrow 0^+} \sup\{u(\xi, \eta) : u \in S_\varepsilon, (\xi, \eta) \in \overline{\Omega_\varepsilon}, 0 < \varepsilon < r, |\xi - x| < r\}, \\ u^-(x) &= \lim_{r \rightarrow 0^+} \inf\{u(\xi, \eta) : u \in S_\varepsilon, (\xi, \eta) \in \overline{\Omega_\varepsilon}, 0 < \varepsilon < r, |\xi - x| < r\}, \end{aligned}$$

which are bounded functions on $\overline{\Omega}$ by Proposition 2. Let us consider the following problems in Ω

$$(50) \quad \begin{cases} G(D^2u, Du, u, x) = 0 & \text{in } \Omega, \\ \gamma \cdot Du = \beta(x, 0) & \text{on } \partial\Omega \end{cases}$$

or

$$(51) \quad \begin{cases} G(D^2u, Du, u, x) = 0 & \text{in } \Omega, \\ u = \beta(x, 0) & \text{on } \partial\Omega \end{cases}$$

which we also call respectively (O_0) and (D_0) .

Theorem 4. *Under the hypotheses of Proposition 2, if S_ε are the solutions of problem (O_ε) , the functions u^+ and u^- are a viscosity sub and super solutions to (50), respectively; while if S_ε are the solutions of problem (D_ε) , the functions u^+ and u^- are a viscosity sub and super solutions to (51), respectively.*

Proof of Theorem 4. We show first that the relaxed limit u^+ is a viscosity subsolution of (50). Let $\phi \in C^\infty(\overline{\Omega})$ and assume that $u^+ - \phi$ has a strict maximum at a point $\hat{x} \in \overline{\Omega}$.

We adapt the calculations in the previous section, to modify the test function ϕ , so that the resulting function ψ satisfies $\gamma_1^+ \cdot D_x \psi + D_y \psi > \beta^+$ on $\partial_T \Omega_\varepsilon$ and $\gamma_1^- \cdot D_x \psi - D_y \psi > \beta^-$ on $\partial_B \Omega_\varepsilon$. Referring (23), (26), (29), and (30), with ϕ in place of u^0 , we define

$$\begin{aligned} v(x) &= \beta_o(x) - \gamma_o(x) \cdot D\phi(x), \\ w^\pm(x) &= \frac{g^\pm}{(g^+ - g^-)} (l^\pm - k^\pm \cdot D\phi \mp \gamma_o(x) \cdot Dv), \\ \psi_\varepsilon(x, y) &:= \phi(x) + \varepsilon v(x)(y/\varepsilon) + \frac{\varepsilon^2}{2} \left\{ (y/\varepsilon - g^-)^2 w^+(x) + (y/\varepsilon - g^+)^2 w^-(x) \right. \\ &\quad \left. + \delta(y/\varepsilon - h)^2 \right\} \\ &= \phi(x) + v(x)y + \frac{1}{2} \left\{ (y - \varepsilon g^-)^2 w^+(x) + (y - \varepsilon g^+)^2 w^-(x) + \delta(y - \varepsilon h)^2 \right\}, \end{aligned}$$

where $h \in C^2(\overline{\Omega})$ is a function chosen to satisfy

$$g^-(x) < h(x) < g^+(x) \quad \text{for } x \in \overline{\Omega},$$

and δ is an arbitrarily fixed positive constant. Comparison of ψ_ε above to the formal expansion in the previous section indicates that ψ_ε adds the last term as an additional one, which will allow us to make the boundary inequalities strict.

By assumption, we have $v \in C^1(\overline{\Omega}, \mathbb{R})$, $w^\pm \in C(\overline{\Omega}, \mathbb{R})$, and $\psi_\varepsilon \in C(\overline{\Omega} \times [-1, 1], \mathbb{R})$. To enhance the smoothness of ψ_ε , we introduce families $\{v_\varepsilon\}$, $\{w_\varepsilon^\pm\}$, and $\{g_\varepsilon^\pm\}$, with $\varepsilon > 0$, of functions on $\overline{\Omega}$ approximating v , w^\pm , and g^\pm , respectively, such that

$$(52) \quad \begin{cases} v_\varepsilon, w_\varepsilon^\pm, g_\varepsilon^\pm \in C^\infty(\overline{\Omega}), \\ v_\varepsilon = v + O(\varepsilon), \quad Dv_\varepsilon = Dv + o(1), \quad D^2v_\varepsilon = o(\varepsilon^{-1}), \\ g_\varepsilon^\pm = g^\pm + O(\varepsilon), \quad Dg_\varepsilon^\pm = Dg^\pm + o(1), \quad D^2g_\varepsilon^\pm = o(\varepsilon^{-1}), \\ w_\varepsilon^\pm = w^\pm + o(1), \quad Dw_\varepsilon^\pm = o(\varepsilon^{-1}), \quad D^2w_\varepsilon^\pm = o(\varepsilon^{-2}). \end{cases}$$

Such a choice of v_ε , w_ε^\pm , and g_ε^\pm is possible thanks to Lemma 2 and Remark 3. Moreover, reparametrizing v_ε , we may assume that

$$(53) \quad v_\varepsilon = v + o(\varepsilon), \quad Dv_\varepsilon = Dv + o(1), \quad D^2v_\varepsilon = o(\varepsilon^{-1}).$$

Select $\varepsilon_1 \in (0, \varepsilon_0)$ sufficiently small so that for $0 < \varepsilon < \varepsilon_1$,

$$g_\varepsilon^- + \varepsilon_1 < h < g_\varepsilon^+ - \varepsilon_1 \quad \text{on } \overline{\Omega}.$$

In what follows we consider only those with parameter $\varepsilon \in (0, \varepsilon_1)$. We define $\Psi_\varepsilon \in C^\infty(\overline{\Omega} \times [-1, 1], \mathbb{R})$ by

$$\Psi_\varepsilon(x, y) = \phi(x) + v_\varepsilon(x)y + \frac{1}{2} \left\{ (y - \varepsilon g_\varepsilon^-)^2 w_\varepsilon^+ + (y - \varepsilon g_\varepsilon^+)^2 w_\varepsilon^- + \delta(y - \varepsilon h)^2 \right\}.$$

Compute that

$$(54) \quad \begin{cases} D_x \Psi_\varepsilon = D\phi + yDv_\varepsilon + \frac{1}{2}\{(y - \varepsilon g_\varepsilon^-)^2 Dw_\varepsilon^+ + (y - \varepsilon g_\varepsilon^+)^2 Dw_\varepsilon^-\} \\ \quad - \varepsilon\{(y - \varepsilon g_\varepsilon^-)w_\varepsilon^+ Dg_\varepsilon^- + (y - \varepsilon g_\varepsilon^+)w_\varepsilon^- Dg_\varepsilon^+ + \delta(y - \varepsilon h)Dh\}, \\ D_y \Psi_\varepsilon = v_\varepsilon + (y - \varepsilon g_\varepsilon^-)w_\varepsilon^+ + (y - \varepsilon g_\varepsilon^+)w_\varepsilon^- + \delta(y - \varepsilon h), \end{cases}$$

and observe by (52) and (53) that for $g = g^+$ and $g = g^-$,

$$(55) \quad \begin{cases} D_x \Psi_\varepsilon(x, \varepsilon g(x)) = D\phi + \varepsilon g Dv + o(\varepsilon), \\ D_y \Psi_\varepsilon(x, \varepsilon g(x)) = v + \varepsilon(g - g^-)w^+ + \varepsilon(g - g^+)w^- + \varepsilon\delta(g - h) + o(\varepsilon). \end{cases}$$

Hence,

$$\begin{aligned} (\gamma_1^+ \cdot D_x \Psi_\varepsilon + D_y \Psi_\varepsilon - \beta^+)(x, \varepsilon g^+(x)) &= \gamma_1^+(x, \varepsilon g^+) \cdot (D\phi + \varepsilon g^+ Dv) \\ &\quad + v + \varepsilon(g^+ - g^-)w^+ + \varepsilon\delta(g^+ - h) \\ &\quad - \beta^+(x, \varepsilon g^+) + o(\varepsilon) \\ &= \gamma_o(x) \cdot (D\phi + \varepsilon g^+ Dv) + \varepsilon g^+ k^+ \cdot D\phi \\ &\quad + v + \varepsilon(g^+ - g^-)w^+ + \varepsilon\delta(g^+ - h) \\ &\quad - \beta_o(x) - \varepsilon g^+ l^+ + o(\varepsilon). \end{aligned}$$

By the definition of v , we have

$$\gamma_o(x) \cdot D\phi + v = \beta_o(x),$$

and by the definition of w^+ , we have

$$(g^+ - g^-)w^+ = g^+(l^+ - k^+ \cdot D\phi - \gamma_o(x) \cdot Dv).$$

Combining these together yields

$$(\gamma_1^+ \cdot D_x \Psi_\varepsilon + D_y \Psi_\varepsilon - \beta^+)(x, \varepsilon g^+(x)) = \varepsilon\delta(g^+ - h) + o(\varepsilon).$$

A parallel calculation shows that

$$(\gamma_1^- \cdot D_x \Psi_\varepsilon - D_y \Psi_\varepsilon - \beta^-)(x, \varepsilon g^-(x)) = \varepsilon\delta(h - g^-) + o(\varepsilon).$$

Consequently, reselecting ε_1 small enough, we may assume that for any $0 < \varepsilon < \varepsilon_1$,

$$(56) \quad \gamma_1^+ \cdot D_x \Psi_\varepsilon + D_y \Psi_\varepsilon > \beta^+ \quad \text{on } \partial_T \Omega_\varepsilon \quad \text{and} \quad \gamma_1^- \cdot D_x \Psi_\varepsilon - D_y \Psi_\varepsilon > \beta^- \quad \text{on } \partial_B \Omega_\varepsilon.$$

By the definition of u^+ , there exists sequences $\{\varepsilon_j \in (0, \varepsilon_*) : j \in \mathbb{N}\}$, $\{u_j \in \text{USC}(\overline{\Omega_{\varepsilon_j}}, \mathbb{R}) : j \in \mathbb{N}\}$, and $\{(\xi_j, \eta_j) \in \overline{\Omega_{\varepsilon_j}}\}$ such that u_j is a viscosity subsolution to (O_ε) , with $\varepsilon = \varepsilon_j$, $\lim_{j \rightarrow \infty} u_j(\xi_j, \eta_j) = u^+(\hat{x})$, and $\lim_{j \rightarrow \infty} (\xi_j, \eta_j) = (\hat{x}, 0)$. For every $j \in \mathbb{N}$, choose a maximum point $(x_j, y_j) \in \overline{\Omega_{\varepsilon_j}}$ of $u_j - \Psi_{\varepsilon_j}$. We claim that $\lim_{j \rightarrow \infty} (x_j, y_j) = (\hat{x}, 0)$. To see this, by passing to a subsequence and concentrating on the subsequence, we may assume that for some $\bar{x} \in \overline{\Omega}$, $\lim_{j \rightarrow \infty} (x_j, y_j) = (\bar{x}, 0)$. Noting that

$$\lim_{\varepsilon \rightarrow 0^+} \max_{(x,y) \in \overline{\Omega_\varepsilon}} |\Psi_\varepsilon(x, y) - \phi(x)| = 0,$$

we observe that

$$\begin{aligned} \limsup_{j \rightarrow \infty} (u_j - \Psi_{\varepsilon_j})(x_j, y_j) &\leq (u^+ - \phi)(\bar{x}), \\ \liminf_{j \rightarrow \infty} (u_j - \Psi_{\varepsilon_j})(x_j, y_j) &\geq \liminf_{j \rightarrow \infty} (u_j - \Psi_{\varepsilon_j})(\xi_j, \eta_j) = (u^+ - \phi)(\hat{x}), \end{aligned}$$

which assures that $\bar{x} = \hat{x}$ and our claim is valid. The argument above can be used to show that $\lim_{j \rightarrow \infty} u_j(x_j, y_j) = u^+(\hat{x})$.

Since u_j is a subsolution to (O_ε) , with $\varepsilon = \varepsilon_j$, we find in view of (56) that if $(x_j, y_j) \in \overline{\Omega_{\varepsilon_j}} \setminus \partial_L \Omega_{\varepsilon_j}$, then

$$(57) \quad F(D^2\Psi_{\varepsilon_j}(x_j, y_j), D\Psi_{\varepsilon_j}(x_j, y_j), u_j(x_j, y_j), x_j, y_j) \leq 0,$$

and if $(x_j, y_j) \in \partial_L \Omega_{\varepsilon_j}$, then either (57) or the following holds

$$(58) \quad \gamma \cdot D_x \Psi_{\varepsilon_j}(x_j, y_j) \leq \beta(x_j, y_j).$$

From (54), together with (52), we obtain on $\overline{\Omega_\varepsilon}$,

$$\begin{aligned} D_x^2 \Psi_\varepsilon &= D^2 \phi + o(1), \\ D_x D_y \Psi_\varepsilon &= Dv + o(1) \\ D_y^2 \Psi_\varepsilon &= w_\varepsilon^+ + w_\varepsilon^- + \delta \\ &= w^+ + w^- + \delta + o(1). \end{aligned}$$

Similarly to (31), we have

$$Dv = D\beta_o - D\phi D\gamma_o - \gamma_o D^2 \phi.$$

From these together, we obtain

$$\begin{aligned} D^2 \Psi_\varepsilon &= \begin{pmatrix} D^2 \phi & Dv^T \\ Dv & w^+ + w^- + \delta \end{pmatrix} + o(1) \\ &= \begin{pmatrix} D^2 \phi & (D\beta_o - D\phi D\gamma_o - \gamma_o D^2 \phi)^T \\ D\beta_o - D\phi D\gamma_o - \gamma_o D^2 \phi & w^+ + w^- + \delta \end{pmatrix} + o(1), \end{aligned}$$

and, similarly to (55),

$$D\Psi_\varepsilon = (D\phi, v) + O(\varepsilon) = (D\phi, \beta_o - \gamma_o \cdot D\phi) + O(\varepsilon).$$

Recalling the definition of b and c and following the computation from (33) to (34), we find that

$$(w^+ + w^-)(x) = c(x) + b(x) \cdot D\phi(x) + \gamma_o(x) D^2 \phi(x) \gamma_o(x)^T.$$

Thus, we have

$$D^2 \Psi_\varepsilon = \begin{pmatrix} D^2 \phi & (D\beta_o - D\phi D\gamma_o - \gamma_o D^2 \phi)^T \\ D\beta_o - D\phi D\gamma_o - \gamma_o D^2 \phi & c + b \cdot D\phi + \gamma_o D^2 \phi \gamma_o^T + \delta \end{pmatrix} + o(1),$$

and we infer from (57) and (58) that if $\hat{x} \in \Omega$, then

$$(59) \quad F(A + B + C, \hat{p}, u^+(\hat{x}), \hat{x}, 0) \leq 0,$$

where $\hat{p} = (D\phi(\hat{x}), \beta_o(\hat{x}) - \gamma_o(\hat{x}) \cdot D\phi(\hat{x}))$,

$$\begin{aligned} A &= \begin{pmatrix} D^2 \phi(\hat{x}) & -D^2 \phi(\hat{x}) \gamma_o(\hat{x})^T \\ -\gamma_o(\hat{x}) D^2 \phi(\hat{x}) & \gamma_o(\hat{x}) D^2 \phi(\hat{x}) \gamma_o(\hat{x})^T \end{pmatrix}, \\ B &= \begin{pmatrix} 0 & -(D\phi(\hat{x}) \gamma_o(\hat{x}))^T \\ -D\phi(\hat{x}) \gamma_o(\hat{x}) & b(\hat{x}) \cdot D\phi(\hat{x}) \end{pmatrix}, \\ C &= \begin{pmatrix} 0 & D\beta_o(\hat{x})^T \\ D\beta_o(\hat{x}) & c(\hat{x}) + \delta \end{pmatrix}, \end{aligned}$$

and if $\hat{x} \in \partial\Omega$, then either (59) or the following holds

$$\gamma(\hat{x}, 0) \cdot D\phi(\hat{x}) \leq \beta(\hat{x}, 0).$$

Since $\delta > 0$ is arbitrary, we conclude that u^+ is a viscosity subsolution to (50).

An argument parallel to the above shows that u^- is a viscosity supersolution to (50), the detail of which is left to the interested reader.

The case of problem (D_ε) is also similar. \square

Once the comparison principle is established for (P_0) , the above theorem assures the uniform convergence of solutions u^ε to (P_ε) on $\bar{\Omega}$ as $\varepsilon \rightarrow 0^+$.

Corollary 5. *Assume the same hypotheses as Proposition 2 and, moreover, the comparison principle for (P_0) stated as follows:*

(H4) *If v and w are viscosity sub and super solutions to (P_0) , respectively, then $v \leq w$ on $\bar{\Omega}$.*

Let u^ε be a viscosity solutions to (P_ε) such that $\sup_{\bar{\Omega}_\varepsilon} |u^\varepsilon| \leq C_0$ for every $0 < \varepsilon < \varepsilon_1$ and some $\varepsilon_1 \in (0, \varepsilon_0)$ and $C_0 > 0$. Then, for a (unique) viscosity solution u^0 to (P_0) , we have

$$(60) \quad \lim_{\varepsilon \rightarrow 0^+} \sup_{(x,y) \in \bar{\Omega}_\varepsilon} |u^\varepsilon(x,y) - u^0(x)| = 0.$$

Proof. Let u^+ and u^- be the functions defined by (49). By Theorem 4, u^+ and u^- are viscosity sub and super solutions to (P_0) , respectively. The comparison principle (H4) implies that $u^+ = u^-$, $u^0 := u^+ = u^- \in C(\bar{\Omega}, \mathbb{R})$, and u^0 is a viscosity solution of (P_0) . By the definition of u^\pm , we easily deduce that (60) holds. \square

5. COMPARISON PRINCIPLES AND EXAMPLES

In this section, we are concerned with the operators F , for which the uniform convergence (60) is valid i.e. such that the limit equation enjoys the condition for the validity of the comparison principle. Standard and classical requirements for G are:

$$(61) \quad \left\{ \begin{array}{l} \text{for any } (X, p, x) \in \mathcal{S}(N) \times \mathbb{R}^N \times \bar{\Omega}, r, s \in \mathbb{R} \text{ and some constant } \alpha > 0, \\ G(X, p, r, x) - G(X, p, s, x) \geq \alpha(r - s) \quad \text{if } r \geq s \end{array} \right.$$

and

$$(62) \quad \left\{ \begin{array}{l} \text{for each } R > 0, \text{ there is a function } \omega_R : [0, \infty) \rightarrow [0, \infty), \text{ with } \omega_R(0) = 0, \\ \text{such that} \\ G(Y, p, r, y) - G(X, p, r, x) \leq \omega_R(\theta|x - y|^2 + |x - y|(|p| + 1)), \\ \text{whenever } \theta > 0, p \in \mathbb{R}^N, r \in [-R, R], x, y \in \mathbb{R}^N, \text{ and } X, Y \in \mathcal{S}(N) \text{ satisfy} \\ -\theta I_{2N} \leq \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} \leq \theta \begin{pmatrix} I_N & -I_N \\ -I_N & I_N \end{pmatrix}. \end{array} \right.$$

5.1. Comparison principle. Regarding the validity of the comparison principle (H4), we follow our previous work [8] and need more conditions on the boundary of Ω . We always assume:

$$(63) \quad \left\{ \begin{array}{l} \text{There are a neighborhood } V \text{ of } \partial\Omega, \text{ relative to } \bar{\Omega}, \text{ and a function } \omega : [0, \infty) \rightarrow \\ [0, \infty), \text{ with } \omega(0^+) = 0, \text{ such that} \\ G(X, p, r, x) - G(Y, q, r, x) \leq \omega(|X - Y| + |p - q|) \\ \text{for } X, Y \in \mathcal{S}(N), p, q \in \mathbb{R}^N, r \in \mathbb{R}, \text{ and } x \in V. \end{array} \right.$$

Moreover, if we are considering on the lateral boundary the general oblique conditions, in addition to (38), we assume either

$$(64) \quad \text{the boundary } \partial\Omega \text{ is of class } C^{2,1} \text{ and } \gamma_1(\cdot, 0) \in C^{0,1}(\partial\Omega),$$

or

$$(65) \quad \partial\Omega \text{ is of class } C^1 \text{ and } \gamma_1(\cdot, 0), \beta(\cdot, 0) \in C^{1,1}(\partial\Omega),$$

while, in the case of standard Neumann conditions on the lateral boundary, the following suffices:

$$(66) \quad \left\{ \begin{array}{l} \text{In addition to the condition that } \partial\Omega \text{ is } C^1, \text{ there is a constant } r_0 > 0 \text{ such} \\ \text{that} \\ B_{r_0}(x + r_0\nu(x)) \cap \Omega = \emptyset \quad \text{for } x \in \partial\Omega. \end{array} \right.$$

The precise result, according to [7, Theorem I.2], [14, Theorem 2.1], and [10], is the following:

Lemma 6. *Assume (61)–(63).*

(O) *Assume (38), and either (64) or (65). Then, comparison principle (H4) holds for problem (O_0) , i.e. (50).*

(N) *Under the assumption (66), the comparison principle (H4) holds for (N_0) , i.e. (15)–(16).*

5.2. Examples. We seek for some conditions on F so that that G , defined by (13), satisfies (61)–(66). To do this, we restrict ourselves to the Bellman-Isaacs type operators and consider function F given by

$$(67) \quad F(X, p, r, z) = \inf_{\lambda \in L} \sup_{\mu \in M} F_{\lambda\mu}(X, p, r, z) \quad \text{on } \mathcal{S}(N+1) \times \mathbb{R}^{N+1} \times \mathbb{R} \times \bar{\Omega} \times [-1, 1],$$

where L, M are given nonempty sets and

$$F_{\lambda\mu}(X, p, r, z) = -\text{tr } \sigma_{\lambda\mu}^T \sigma_{\lambda\mu} X - b_{\lambda\mu} \cdot p + c_{\lambda\mu} r - f_{\lambda\mu},$$

with

$$\begin{aligned} \sigma_{\lambda\mu} &\in C^{0,1}(\bar{\Omega} \times [-1, 1], \mathcal{M}(k, N+1)), \quad b_{\lambda\mu} \in C^{0,1}(\bar{\Omega} \times [-1, 1], \mathbb{R}^{N+1}), \\ c_{\lambda\mu}, f_{\lambda\mu} &\in C(\bar{\Omega} \times [-1, 1], \mathbb{R}). \end{aligned}$$

In the above, $\mathcal{M}(k, N+1)$ denotes the set of real $k \times (N+1)$ matrices and $k \in \mathbb{N}$ is a given fixed number. We assume the following uniform boundedness and equi-continuity of $\sigma_{\lambda\mu}, b_{\lambda\mu}, c_{\lambda\mu}, f_{\lambda\mu}$: for any $(\lambda, \mu) \in L \times M$ and $z, z' \in \bar{\Omega} \times [-1, 1]$,

$$(68) \quad \left\{ \begin{array}{l} \max\{|\sigma_{\lambda\mu}(z)|, |b_{\lambda\mu}(z)|, |c_{\lambda\mu}(z)|, |f_{\lambda\mu}(z)|\} \leq C_F, \\ \max\{|\sigma_{\lambda\mu}(z) - \sigma_{\lambda\mu}(z')|, |b_{\lambda\mu}(z) - b_{\lambda\mu}(z')|\} \leq C_F |z - z'|, \\ \max\{|c_{\lambda\mu}(z) - c_{\lambda\mu}(z')|, |f_{\lambda\mu}(z) - f_{\lambda\mu}(z')|\} \leq \omega_F(|z - z'|), \end{array} \right.$$

where C_F is a positive constant and $\omega_F : [0, \infty) \rightarrow [0, \infty)$ is a function satisfying $\omega_F(0^+) = 0$. Also, assume that for some constant $\alpha > 0$,

$$(69) \quad c_{\lambda\mu}(z) \geq \alpha \quad \text{for } (\lambda, \mu) \in L \times M \text{ and } z \in \bar{\Omega} \times [-1, 1].$$

With all the structural conditions on F stated above, it is easily seen that F satisfies (H1) and that

$$(70) \quad |F(X, p, r, z) - F(Y, q, r, z)| \leq C_F^2 |X - Y| + C_F |p - q|$$

for $X, Y \in \mathcal{S}(N+1)$, $p, q \in \mathbb{R}^{N+1}$, and $(r, z) \in \mathbb{R} \times \bar{\Omega} \times [-1, 1]$. Moreover, if we write $A(X, x), B(p, x), C(x)$ for $A, B, C \in \mathcal{S}(N+1)$ in (14), respectively, it follows that for some constant $C_1 > 0$,

$$\begin{aligned} & |(A(X, x) + B(p, x) + C(x)) - (A(Y, x) + B(q, x) + C(x))| \\ & + |(p, \beta_0 - \gamma_0 \cdot p) - (q, \beta_0 - \gamma_0 \cdot q)| \leq C_1(|X - Y| + |p - q|) \end{aligned}$$

for $X, Y \in \mathcal{S}(N)$ and $p, q \in \mathbb{R}^N$. The constant C_1 can be chosen to depend only on $\|(\gamma_0, \beta_0, D\gamma_0, D\beta_0)\|_{L^\infty(\Omega)}$. It is now clear that (70) implies (63).

Since

$$A(X, x) = \begin{pmatrix} I_N \\ -\gamma_0(x) \end{pmatrix} X \begin{pmatrix} I_N & -\gamma_0(x)^T \end{pmatrix},$$

setting

$$\tilde{\sigma}_{\lambda\mu}(x) := \sigma_{\lambda\mu}(x, 0) \begin{pmatrix} I_N \\ -\gamma_0(x) \end{pmatrix} \in \mathcal{M}(k, N),$$

we have

$$\text{tr}[(\sigma_{\lambda\mu}^T \sigma_{\lambda\mu})(x, 0)A(X, x)] = \text{tr}[(\tilde{\sigma}_{\lambda\mu}^T \tilde{\sigma}_{\lambda\mu})(x)X].$$

Under assumption (9), we see that $\tilde{\sigma}_{\lambda\mu} \in C^{0,1}(\bar{\Omega}, \mathcal{M}(k, N))$ and the family $\{\tilde{\sigma}_{\lambda\mu} : (\lambda, \mu) \in L \times M\}$ is equi-Lipschitz continuous on $\bar{\Omega}$. Let e_1, \dots, e_N denote the standard basis of \mathbb{R}^N . Define $\tilde{b}_{\lambda\mu} = (\tilde{b}_{\lambda\mu,1}, \dots, \tilde{b}_{\lambda\mu,N}) \in C(\bar{\Omega}, \mathbb{R}^N)$ by

$$\tilde{b}_{\lambda\mu,i}(x) = \text{tr}[(\sigma_{\lambda\mu}^T \sigma_{\lambda\mu})(x, 0)B(e_i, x)] + b_{\lambda\mu,i}(x, 0) - b_{\lambda\mu,N+1}(x, 0)\gamma_0(x) \cdot e_i$$

where $b_{\lambda\mu,i}$ denotes the i^{th} entry of the $(N+1)$ -vector $b_{\lambda\mu}$. In addition to the regularity assumptions (6) and (9), if we assume that

$$(71) \quad \gamma_0 \in C^{1,1}(\bar{\Omega}, \mathbb{R}^N) \quad \text{and} \quad k^\pm \in C^{0,1}(\bar{\Omega}, \mathbb{R}^N),$$

then $\tilde{b}_{\lambda\mu} \in C^{0,1}(\bar{\Omega}, \mathbb{R}^N)$ and the family $\{\tilde{b}_{\lambda\mu}\}$ is equi-Lipschitz continuous on $\bar{\Omega}$. Set

$$\tilde{f}_{\lambda\mu}(x) = f_{\lambda\mu}(x, 0) + \text{tr}[(\sigma_{\lambda\mu}^T \sigma_{\lambda\mu})(x, 0)C(x)] + b_{\lambda\mu,N+1}(x, 0)\beta_0(x) \quad \text{for } x \in \bar{\Omega},$$

and note that the family $\{\tilde{f}_{\lambda\mu}\}$ is equicontinuous on $\bar{\Omega}$. Note also that the family $\{(\tilde{\sigma}_{\lambda\mu}, \tilde{b}_{\lambda\mu}, \tilde{f}_{\lambda\mu}) : (\lambda, \mu) \in L \times M\}$ is uniformly bounded on $\bar{\Omega}$. Observe that

$$(72) \quad G(X, p, r, x) = \inf_{\lambda \in L} \sup_{\mu \in M} (-\text{tr}(\tilde{\sigma}_{\lambda\mu}^T \tilde{\sigma}_{\lambda\mu} X) - \tilde{b}_{\lambda\mu} \cdot p + c_{\lambda\mu} r - \tilde{f}_{\lambda\mu}),$$

and, as is well-known (see e.g., [10]), this function satisfies (62), provided that the family $\{(\tilde{\sigma}_{\lambda\mu}, \tilde{b}_{\lambda\mu}, c_{\lambda\mu}, \tilde{f}_{\lambda\mu})\}$ is uniformly bounded, $\{(\tilde{\sigma}_{\lambda\mu}, \tilde{b}_{\lambda\mu})\}$ is equi-Lipschitz continuous, $\{c_{\lambda\mu}, \tilde{f}_{\lambda\mu}\}$ is equi-continuous on $\bar{\Omega}$, which is the case under assumptions (68) and (71).

To simplify the presentation, we introduce the following condition which combines several conditions:

- (H5) The function F takes the form (67) and the conditions (68) and (69) are satisfied. Also, the conditions (H2), (4), (6), (8), (9), (10), and (71) are in effect.

Theorem 7. *Assume (H5).*

- (O) *Assume (38), and either (64) or (65). For $\varepsilon > 0$ sufficiently small, let u^ε be a viscosity solution to (O_ε) i.e. (1), (5). Let u^0 be a unique viscosity solution to (O_0) i.e. (50). Then, u^ε converges to u^0 uniformly on $\bar{\Omega}$:*

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{(x,y) \in \bar{\Omega}_\varepsilon} |u^\varepsilon(x, y) - u^0(x)| = 0.$$

(N) Assume (66). For $\varepsilon > 0$ sufficiently small, let u^ε be a viscosity solution to (N_ε) i.e. (1)–(2). Let u^0 be a unique viscosity solution to (N_0) i.e. (15)–(16). Then, u^ε converges to u^0 uniformly on $\overline{\Omega}$:

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{(x,y) \in \overline{\Omega_\varepsilon}} |u^\varepsilon(x,y) - u^0(x)| = 0.$$

Proof. Thanks to Lemma 6, this theorem is just an application of Corollary 5, since, as noted above, F satisfies (H1), which, together with (H5), completes the hypotheses of Proposition 2. \square

5.2.1. Dirichlet boundary condition. We consider the case where (P_ε) (resp., (P_0)) corresponds to (D_ε) (resp., (D_0)), in other words, problem (1), (5), and (37) (resp., (51)). If the Dirichlet condition in (51) is consistent with the classical one, the comparison principle (H4) for (P_0) is well understood.

As in the previous subsections, we restrict ourselves to the Bellman-Isaacs type operators and thus keep the structure condition that (67), (68), (69), and (71) are valid, and seek for some cases when the Dirichlet condition in (51) is consistent with the classical one.

Lemma 8. Assume (66), (67), (68), (69), and (71). Furthermore, assume that for any $x \in \partial\Omega$,

$$(73) \quad \inf_{(\lambda,\mu) \in L \times M} |\nu(x) (I_N - \gamma_0^T(x)) \sigma_{\lambda,\mu}^T(x,0)| > 0.$$

Then, the Dirichlet condition in (51) can be understood in the classical sense.

Notice that $\nu(x) (I_N - \gamma_0^T(x)) \sigma_{\lambda,\mu}^T(x,0)$ is a $1 \times k$ matrix, i.e., a k -vector. Recall that we are always assuming that $\beta \in C(\partial\Omega \times [-1,1], \mathbb{R})$.

Remark 6. Condition (73) reads

$$\inf_{(\lambda,\mu) \in L \times M} \text{tr}[\tilde{\sigma}_{\lambda\mu}^T \tilde{\sigma}_{\lambda\mu}(x) \nu(x) \otimes \nu(x)] > 0 \quad \text{for all } x \in \partial\Omega.$$

(See also (75) below.) This condition can be thought of as a strict ellipticity of the operator G in the normal direction on the boundary $\partial\Omega$.

Proof. The conclusion of the lemma states precisely that if u is an upper semicontinuous subsolution (resp., lower semicontinuous supersolution) to (D_0) , then $u(x) \leq \beta(x,0)$ (resp., $u(x) \geq \beta(x,0)$) for any $x \in \partial\Omega$. We give only the proof in the case when u is such a subsolution to (D_0) , the other case is treated similarly.

We fix any $z \in \partial\Omega$. Thanks to (66) (the exterior sphere condition), by reselecting $r_0 > 0$ small enough if necessary, we have

$$(74) \quad \overline{B_{r_0}(z + r_0\nu(z))} \cap \overline{\Omega} = \{z\}.$$

Due to (73), we infer by continuity that for some $\delta > 0$ and all $(\xi, x) \in B_\delta(\nu(z)) \times (B_\delta(z) \setminus \overline{\Omega})$,

$$\inf_{(\lambda,\mu) \in L \times M} |\xi (I_N - \gamma_0^T(x)) \sigma_{\lambda\mu}(x,0)| > \delta,$$

which reads

$$(75) \quad \inf_{(\lambda,\mu) \in L \times M} \text{tr}[\tilde{\sigma}_{\lambda\mu}^T \tilde{\sigma}_{\lambda\mu}(x) \xi \otimes \xi] > \delta^2.$$

For the brevity of notation, we set $y := z + r_0\nu(z)$. Choose $\varepsilon \in (0, \delta)$ so that

$$\frac{y-x}{r_0} \in B_\delta(\nu(z)) \quad \text{for } x \in B_\varepsilon(z).$$

By (74), we can choose $r_1 > r_0$, close enough to r_0 , so that

$$\overline{B_{r_1}(y)} \cap \overline{\Omega} \subset B_\varepsilon(z).$$

Under this choice of r_1 , we have by (75),

$$(76) \quad \inf_{(\lambda, \mu) \in L \times M} \text{tr}[\tilde{\sigma}_{\lambda\mu} \tilde{\sigma}_{\lambda\mu}(x)(x-y) \otimes (x-y)] > r_0^2 \delta^2 \quad \text{for all } x \in B_{r_1}(y) \cap \overline{\Omega}.$$

Now, we fix any $b > \beta(z, 0)$ and will show that $u(z) \leq b$, which is enough to complete the proof. We may assume, by reselecting $\varepsilon > 0$ small enough if necessary, that $b > \beta(x, 0)$ for all $x \in B_\varepsilon(z) \cap \partial\Omega$. For $t > 0$, we define the smooth function ψ_t in \mathbb{R}^N by

$$\psi_t(x) = b + C_t(1 - e^{t(r_0^2 - |x-y|^2)}), \quad \text{with } C_t := e^{t(r_1^2 - r_0^2)}.$$

Note that, by (74),

$$\min_{\overline{\Omega}} \psi_t = \psi_t(z) = b,$$

and

$$\min_{\overline{\Omega} \setminus B_{r_1}(y)} \psi_t = b + C_t(1 - e^{t(r_0^2 - r_1^2)}) \rightarrow +\infty \quad \text{as } t \rightarrow +\infty.$$

Since u is bounded above in $\overline{\Omega}$, this last observation allows us to choose $T > 0$ so that for $t \geq T$,

$$(77) \quad \max_{\overline{\Omega}} u < \min_{\overline{\Omega} \setminus B_{r_1}(y)} \psi_t.$$

Henceforth, we consider only those ψ_t with $t \geq T$.

We consider the case where $u \leq \psi_t$ in $\overline{\Omega}$ for some $t \geq T$. Then we have $u(z) \leq \psi_t(z) = b$, and we finish the proof. This is only the case we have. Indeed, otherwise, we get a contradiction. To check this, suppose

$$\max_{\overline{\Omega}}(u - \psi_t) > 0 \quad \text{for all } t \geq T.$$

This and (77) imply

$$\max_{\overline{\Omega}}(u - \psi_t) = \max_{B_{r_1}(y) \cap \overline{\Omega}}(u - \psi_t) > 0.$$

Let $x_t \in \overline{B_{r_1}(y)} \cap \overline{\Omega}$ be a maximum point of $u - \psi_t$. Since $x_t \in B_\varepsilon(z)$, if $x_t \in \partial\Omega$, then

$$u(x_t) > \psi_t(x_t) \geq b > \beta(x_t, 0).$$

Hence, the subsolution property of u yields

$$(78) \quad 0 \geq G(D^2\psi_t(x_t), D\psi_t(x_t), u(x_t), x_t) \geq G(D^2\psi_t(x_t), D\psi_t(x_t), b, x_t).$$

Computing, for any $x \in \mathbb{R}^N$,

$$\begin{aligned} D\psi_t(x) &= 2tC_t e^{t(r_0^2 - |x-y|^2)}(x-y), \\ D^2\psi_t(x) &= C_t e^{t(r_0^2 - |x-y|^2)}(2tI_N - 4t^2(x-y) \otimes (x-y)), \end{aligned}$$

and moreover, using (76) and that $|x_t - y| \leq r_1$:

$$\begin{aligned} & -\operatorname{tr}[\tilde{\sigma}_{\lambda\mu}^T \tilde{\sigma}_{\lambda\mu}(x_t) D^2 \psi_t(x_t)] - \tilde{b}_{\lambda\mu}(x_t) \cdot D \psi_t(x_t) \\ & = t C_t e^{t(r_0^2 - |x_t - y|^2)} \left(-2 \operatorname{tr}[\tilde{\sigma}_{\lambda\mu}^T \tilde{\sigma}_{\lambda\mu}(x_t)] + 4t \operatorname{tr}[\tilde{\sigma}_{\lambda\mu}^T \tilde{\sigma}_{\lambda\mu}(x_t)(x_t - y) \otimes (x_t - y)] \right. \\ & \quad \left. - 2 \tilde{b}_{\lambda\mu}(x_t) \cdot (x_t - y) \right) \\ & \geq t C_t e^{t(r_0^2 - |x_t - y|^2)} \left(-2B + 4r_0^2 \delta^2 t - 2r_1 B \right), \end{aligned}$$

where $B > 0$ is a constant chosen so that

$$\sup_{(x, \lambda, \mu) \in \bar{\Omega} \times L \times M} \max\{|\tilde{b}_{\lambda\mu}(x)|, \operatorname{tr}[\tilde{\sigma}_{\lambda\mu}^T \tilde{\sigma}_{\lambda\mu}(x)]\} \leq B.$$

Observe that for t sufficiently large, $4r_0^2 \delta^2 t - 2(1 + r_1)B \geq 1$ and

$$\begin{aligned} & -\operatorname{tr}[\tilde{\sigma}_{\lambda\mu}^T \tilde{\sigma}_{\lambda\mu}(x_t) D^2 \psi_t(x_t)] - \tilde{b}_{\lambda\mu}(x_t) \cdot D \psi_t(x_t) \\ & \geq t C_t e^{t(r_0^2 - |x_t - y|^2)} \geq t C_t e^{t(r_0^2 - r_1^2)} = t. \end{aligned}$$

This implies that, as $t \rightarrow +\infty$,

$$G(D^2 \psi_t(x_t), D \psi_t(x_t), b, x_t) \rightarrow +\infty,$$

which contradicts (78). \square

Theorem 9. *Assume (H5), (66), and (73). For $\varepsilon > 0$ sufficiently small, let u^ε be a viscosity solution to (D_ε) . Then u^ε converges to u^0 uniformly on $\bar{\Omega}$:*

$$\lim_{\varepsilon \rightarrow 0^+} \sup_{(x, y) \in \bar{\Omega}_\varepsilon} |u^\varepsilon(x, y) - u^0(x)| = 0.$$

Proof. Since G satisfies (61) and (62) and the Dirichlet boundary condition in (51) can be considered in the classical sense thanks to Lemma 8, it is well-known (see, for instance, [10]) that the comparison principle (H4) holds for (D_0) . So we just need to apply Corollary 5. \square

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