Second order BSDEs with jumps by measurable selection argument

Laurent Denis, Anis Matoussi, Chao Zhou

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Abstract

We prove existence and uniqueness for solution of second order BSDEs with jumps (2BSDEJs). More precisely, our problem of interest consists in the optimization, over a set of possibly non-dominated probability measures, of solutions of backward stochastic differential equations with jumps (BSDEJs). After proving a dynamic programming principle for this control problem in an abstract setting, we obtain a wellposedness result for second order BSDEJs (as introduced in Kazi-Tani, Possamaï, and Zhou [10]) which does not require any regularity assumption on the terminal condition and the generator.

Key words: Stochastic control, measurable selection, second order BSDEs with jumps

MSC 2000 subject classifications: 60H10; 60H30

1 Introduction

Notations: Throughout this paper, we fix a constant $p > 1$. Let $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$ and let $\mathbb{R}_+^*$ be the set of real positive numbers. For every $d-$dimensional vector b with $d \in \mathbb{N}^*$, we denote by $b^1, \dots, b^d$ its coordinates and for $\alpha, \beta \in \mathbb{R}^d$ we denote by $\alpha \cdot \beta$ the usual inner product, with associated norm $\|\cdot\|$, which we simplify to $|\cdot|$ when d is equal to 1. We also let $\mathbf{1}_d$ be the vector whose coordinates are all equal to 1. For any $(l, c) \in \mathbb{N}^* \times \mathbb{N}^*$, $\mathcal{M}_{l,c}(\mathbb{R})$ will denote the space of $l \times c$ matrices with real entries. Elements of the matrix $M \in \mathcal{M}_{l,c}$ will be denoted by $(M^{i,j})_{1 \le i \le l,\ 1 \le j \le c}$, and the transpose of M will be denoted by $M^\top$. When $l = c$, we let

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$\mathcal{M}_l(\mathbb{R}) := \mathcal{M}_{l,l}(\mathbb{R})$. We also identify $\mathcal{M}_{l,1}(\mathbb{R})$ and $\mathbb{R}^l$. Let $\mathbb{S}_d^{\geq 0}$ denote the set of all symmetric positive semi-definite $d \times d$ matrices. We fix a map $\psi : \mathbb{S}_d^{\geq 0} \longrightarrow \mathcal{M}_d(\mathbb{R})$ which is (Borel) measurable and satisfies $\psi(a)(\psi(a))^\top = a$ for all $a \in \mathbb{S}_d^{\geq 0}$, and denote $a^{\frac{1}{2}} := \psi(a)$.

2 Existence and uniqueness of 2BSDEJs

2.1 A primer on 2BSDEJs and main difficulties

Before giving all notations in detail and precise definition of 2BSDEJs, we would like to start by presenting the main object of interest in this paper, as well as the main difficulties we need to address in our framework.

First, as mentioned in the introduction, we shall consider the following 2BSDEJ, for $0 \leq t \leq T$ and $\mathbb{P} - a.s.$,

$$Y_t = \xi + \int_t^T \widehat{F}_s^{\mathbb{P}}(Y_s, Z_s, U_s)ds - \int_t^T Z_s dB_s^{\mathbb{P},c} - \int_t^T \int_E U_s(x)\tilde{\mu}_B^{\mathbb{P}}(dx, ds) + K_T^{\mathbb{P}} - K_t^{\mathbb{P}}.$$

for every $\mathbb{P} \in \mathcal{P}_H^\kappa$, which is a family, not necessarily dominated, of local martingale probability measures. These different probability measures represent the model uncertainty. $B^{\mathbb{P},c}$ and $\tilde{\mu}_B^{\mathbb{P}}$ denote respectively the continuous local martingale part and the compensated jump measure associated to the purely discontinuous local martingale part of the canonical process B under any local martingale measure $\mathbb{P}$. Here $K^{\mathbb{P}}$ is a non-decreasing process, we will show that solutions to 2BSDEJs have to be understood as suprema of families of classical BSDEJs.

Let us now highlight the new difficulties in our framework compared to the continuous 2BSDEs as considered in [19, 17]. In our setting, in general, it's not possible to aggregate the drift and the compensated jump measure. That is the reason why the generator $\widehat{F}^{\mathbb{P}}$ and the compensated jump measure $\widetilde{\mu}^{\mathbb{P}}$ above depend explicitly on the probability measure, through the jump compensator defined under each $\mathbb{P}$. However, we can still prove that the solution of a 2BSDEJ, (Y, Z, U), can be constructed in such a way that it is defined for all ω, independently of probability measures (we refer the reader to KPZ [9, 10] for more details). This is important from the point of view of financial applications, since, if we look for instance at classical problems of portfolio optimization in finance, the process Z is usually related to the corresponding optimal investment strategy. Therefore, in a context of uncertainty, one will definitely need an optimal strategy which works for every possible model, that is to say for every measure $\mathbb{P}$.

Another crucial point in the definition of 2BSDEs in [19, 18, 20], is that they work under a set of measure corresponding to the so-called strong formulation of stochastic control. Roughly speaking, this corresponds to considering the laws under the Wiener measure of stochastic integrals with respect to the canonical process B, with the constraint that these integrands have to take values in the space of symmetric definite positive matrices. Such a choice has several extremely important advantages: first of all, it allows them to define their measures through a unique reference measure (i.e. the Wiener measure), and even more importantly, they showed that all the measures thus constructed satisfy the martingale representation property and the Blumenthal $0-1$ law, which are known to be fundamental properties for the wellposedness of classical BSDEs (which, as recalled in the introduction are a kind of nonlinear martingales).

In KPZ [9, 10], they also work with the strong formulation by considering a whole family of reference measures. in their setting, they need to consider special jumps compensators with some restrictions.

2.2 Probabilistic framework

2.2.1 Canonical space

Let $d \in \mathbb{N}^*$ and $\Omega := \mathbb{D}([0,T], \mathbb{R}^d)$ be the space of càdlàg paths defined on $[0,T]$ with values in $\mathbb{R}^d$ and such that $w(0) = 0$, equipped with the Skorohod topology, so that it is a complete, separable metric space (see [1] for instance).
We denote X the canonical process, *i.e.* $X_t(\omega) := \omega_t$, for all $\omega \in \Omega$. Denote by $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ the canonical filtration generated by X, and by $\mathbb{F}_+ = (\mathcal{F}_t^+)_{0\le t\le T}$ the right limit of $\mathbb{F}$ with $\mathcal{F}_t^+ := \cap_{s>t}\mathcal{F}_s$ for all $t \in [0,T)$ and $\mathcal{F}_T^+ := \mathcal{F}_T$.
Let $\mathbb{M}_1$ denote the collection of all probability measures on $(\Omega, \mathcal{F}_T)$. Notice that $\mathbb{M}_1$ is a Polish space equipped with the weak convergence topology. We denote by $\mathfrak{B}$ its Borel $\sigma-$field. Then for any $\mathbb{P} \in \mathbb{M}_1$, denote by $\mathcal{F}_t^{\mathbb{P}}$ the completed $\sigma-$field of $\mathcal{F}_t$ under $\mathbb{P}$. Denote also the completed filtration by $\mathbb{F}^{\mathbb{P}} = \left(\mathcal{F}_t^{\mathbb{P}}\right)_{t\in[0,T]}$ and $\mathbb{F}_+^{\mathbb{P}}$ the right limit of $\mathbb{F}^{\mathbb{P}}$, so that $\mathbb{F}_+^{\mathbb{P}}$ satisfies the usual conditions. Moreover, for $\mathcal{P} \subset \mathbb{M}_1$, we introduce the universally completed filtration $\mathbb{F}^U := \left(\mathcal{F}_t^U\right)_{0\le t\le T}$, $\mathbb{F}^{\mathcal{P}} := \left(\mathcal{F}_t^{\mathcal{P}}\right)_{0\le t\le T}$, and $\mathbb{F}^{\mathcal{P}+} := \left(\mathcal{F}_t^{\mathcal{P}+}\right)_{0\le t\le T}$, defined as follows

$$\mathcal{F}_t^U := \bigcap_{\mathbb{P}\in\mathbb{M}_1} \mathcal{F}_t^{\mathbb{P}},\ \mathcal{F}_t^{\mathcal{P}} := \bigcap_{\mathbb{P}\in\mathcal{P}} \mathcal{F}_t^{\mathbb{P}},\ t\in[0,T],\ \mathcal{F}_t^{\mathcal{P}+} := \mathcal{F}_{t+}^{\mathcal{P}},\ t\in[0,T),\ \text{and } \mathcal{F}_T^{\mathcal{P}+} := \mathcal{F}_T^{\mathcal{P}}.$$

We also introduce an enlarged canonical space $\overline{\Omega} := \Omega \times \Omega'$, where Ω' is identical to Ω. By abuse of notation, we denote by (X, B) its canonical process, *i.e.* $X_t(\bar\omega) := \omega_t$, $B_t(\bar\omega) := \omega'_t$ for all $\bar\omega := (\omega, \omega') \in \overline{\Omega}$, by $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{0\le t\le T}$ the canonical filtration generated by (X, B), and by $\overline{\mathbb{F}}^X = (\overline{\mathcal{F}}_t^X)_{0\le t\le T}$ the filtration generated by X. Similarly, we denote the corresponding right-continuous filtrations by $\overline{\mathbb{F}}_+^X$ and $\overline{\mathbb{F}}_+$, and the augmented filtration by $\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}$ and $\overline{\mathbb{F}}_+^{\overline{\mathbb{P}}}$, given a probability measure $\overline{\mathbb{P}}$ on $\overline{\Omega}$.

2.2.2 The models space: the semi-martingale measures

We then define a semi-martingale measure $\mathbb{P}$ as a probability measure such that X is a $\mathbb{P}$-semi-martingale. We then associate to the jumps of X a counting measure μ_X, which is a random measure on $\mathbb{R}^+ \times E$ equipped with its Borel σ-field $\mathcal{B}(\mathbb{R}^+) \times \mathcal{B}(E)$ (where $E := \mathbb{R}^d \backslash \{0\}$), defined pathwise by

$$\mu_X(A, [0,t]) := \sum_{0<s\le t} \mathbf{1}_{\{\Delta X_s \in A\}},\ \forall t \ge 0,\ \forall A \in \mathcal{B}(E). \tag{2.1}$$

We recall that (see for instance Theorem I.4.18 in [6]) under any semi-martingale measure $\mathbb{P}$, X admits the canonical decomposition. We emphasize that such a decomposition depends on the underlying probability measure. Then, we define $\overline{\mathcal{P}}_W$ as the set of all semi-martingale measures $\mathbb{P}$, such that $\mathbb{P}$-a.s.:

(i) $(X_s)_{s\in[t,T]}$ is a $(\mathbb{P},\mathbb{F})-$semi-martingale admitting the canonical decomposition (see e.g. [6, Theorem I.4.18])

$$X_s = \int_t^s b_r^{\mathbb{P}} dr + X_s^{c,\mathbb{P}} + X_s^{\mathbb{P},d},\ s\in[t,T],\ \mathbb{P}-a.s.,$$

where $b^{\mathbb{P}}$ is a $\mathbb{F}^{\mathbb{P}}$-predictable $\mathbb{R}^d$-valued process, $X^{c,\mathbb{P}}$ is the continuous local martingale part of X and $X^{d,\mathbb{P}}$ is the purely discontinuous local martingale part of X under $\mathbb{P}$.

(ii) The quadratic variation of $X^{\mathbb{P},c}$ is absolutely continuous with respect to the Lebesgue measure dt and its density takes values in $\mathbb{S}_d^{\geq 0}$, which is the space of all $d\times d$ real valued positive semi-definite matrices.

(iii) The compensator $\lambda_t^{\mathbb{P}}(dx,dt)$ of the jump measure μ_X exists under $\mathbb{P}$ and can be decomposed, for some $\mathbb{F}$-predictable random measure $\nu^{\mathbb{P}}$ on E, as follows

$$\lambda_t^{\mathbb{P}}(dx,dt) = \nu_t^{\mathbb{P}}(dx)dt.$$

We will denote by $\widetilde{\mu}_X^{\mathbb{P}}(dx,dt)$ the corresponding compensated measure, and for simplicity, we will often call $\nu^{\mathbb{P}}$ the compensator of the jump measure associated to X.

In this discontinuous setting, we will say that a probability measure $\mathbb{P}\in\overline{\mathcal{P}}_W$ satisfies the martingale representation property if for any $(\overline{\mathbb{F}}^{\mathbb{P}},\mathbb{P})$-local martingale M, there exists a unique $\overline{\mathbb{F}}^{\mathbb{P}}$-predictable processes H and a unique $\overline{\mathbb{F}}^{\mathbb{P}}$-predictable function U such that $(H,U)\in\mathbb{H}^2_{loc}(\mathbb{P})\times\mathbb{J}^2_{loc}(\mathbb{P})$ (these spaces are defined later in Section **??**) and

$$M_t = M_0 + \int_0^t H_s dX_s^{\mathbb{P},c} + \int_0^t\int_E U_s(x)\widetilde{\mu}_X^{\mathbb{P}}(dx,ds),\ \mathbb{P}-a.s.$$

Bichteler [**?**], Karandikar [7], or more recently Nutz [15] all showed in different contexts and under different assumptions, that it is possible to find an aggregator for the Itô stochastic integrals ${}^{(\mathbb{P})}\!\int_0^t H_s dX_s$. A direct consequence of this result is the possibility to define the quadratic variation process $\{[X,X]_t,\ t\geq 0\}$ pathwisely. Indeed, using Itô's formula, we can write for any semi-martingale measure $\mathbb{P}$:

$$[X,X]_t = |X_t|^2 - 2\int_0^t X_{s^-}dX_s,\ \mathbb{P}-a.s.,$$

and the aggregation of the stochastic integrals automatically yields the aggregation of the bracket $\{[X,X]_t,\ t\geq 0\}$.

Next, since $[X,X]$ has finite variation, we can define its path-by-path continuous part $[X,X]^c$ (by subtracting the sum of the jumps) and finally the corresponding density

$$\widehat{a}_t := \overline{\lim_{\varepsilon\downarrow 0}}\ \frac{[X,X]_t^c - [X,X]_{t-\varepsilon}^c}{\varepsilon}.$$

Notice that since for any semi-martingale measure $\mathbb{P}$,

$$[X,X]^c = \langle X^{\mathbb{P},c}\rangle,\ \mathbb{P}-a.s.,$$

then $\widehat{a}$ coincides with the density of quadratic variation of $X^{\mathbb{P},c}$, $\mathbb{P}-a.s.$ Therefore $\widehat{a}$ takes values in $\mathbb{S}_d^{\geq 0}$, $dt\times d\mathbb{P}-a.e.$, and

$$\widehat{a}_t = \frac{d\langle X^{\mathbb{P},c}\rangle_t}{dt},\ \mathbb{P}-a.s..$$

2.2.3 Conditioning and concatenation of probability measures

We also recall that for every probability measure $\mathbb{P}$ on Ω and $\mathbb{F}-$stopping time τ taking value in $[0,T]$, there exists a family of regular conditional probability distribution (r.c.p.d. for short) $(\mathbb{P}^\tau_\omega)_{\omega\in\Omega}$ (see e.g. Stroock and Varadhan [**?**]), satisfying:

(i) For every $\omega\in\Omega$, $\mathbb{P}^\tau_\omega$ is a probability measure on $(\Omega,\mathcal{F}_T)$.

(ii) For every $E\in\mathcal{F}_T$, the mapping $\omega\longmapsto\mathbb{P}^\tau_\omega(E)$ is $\mathcal{F}_\tau-$measurable.

(iii) The family $(\mathbb{P}^\tau_\omega)_{\omega\in\Omega}$ is a version of the conditional probability measure of $\mathbb{P}$ on $\mathcal{F}_\tau$, *i.e.*, for every integrable $\mathcal{F}_T-$measurable random variable ξ we have $\mathbb{E}^{\mathbb{P}}[\xi|\mathcal{F}_\tau](\omega)=\mathbb{E}^{\mathbb{P}^\tau_\omega}\big[\xi\big]$, for $\mathbb{P}-a.e.\ \omega\in\Omega$.

(iv) For every $\omega\in\Omega$, $\mathbb{P}^\tau_\omega(\Omega^\omega_\tau)=1$, where $\Omega^\omega_\tau:=\big\{\overline{\omega}\in\Omega:\ \overline{\omega}(s)=\omega(s),\ 0\le s\le\tau(\omega)\big\}$.

Furthermore, given some $\mathbb{P}$ and a family $(\mathbb{Q}_\omega)_{\omega\in\Omega}$ such that $\omega\longmapsto\mathbb{Q}_\omega$ is $\mathcal{F}_\tau-$measurable and $\mathbb{Q}_\omega(\Omega^\omega_\tau)=1$ for all $\omega\in\Omega$, one can then define a concatenated probability measure $\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot$ by

$$\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot\big[A\big]:=\int_\Omega\mathbb{Q}_\omega\big[A\big]\ \mathbb{P}(d\omega),\ \forall A\in\mathcal{F}_T.$$

2.2.4 Hypotheses

Let $\mathcal{L}$ be the set of all Lévy measures on $\mathcal{B}(E)$ i.e. the set of non-negative measures, ν, on $(E,\mathcal{B}(E))$ such that

$$\int_E 1\wedge|x|^2\,\nu(dx)<+\infty.$$

Following [13], Lemma 2.3, we can build a metric on it such that it is separable.
Let $\mathcal{N}$ be the set of $\mathbb{F}$-predictable random measures ν on $\mathcal{B}(E)$ satisfying

$$\int_0^T\int_E(1\wedge|x|^2)\nu_s(\omega,dx)ds<+\infty,\ \text{for all }\omega\in\Omega \tag{2.2}$$

Let us define the following spaces for $p\ge1$

$$\hat{L}^p:=\{\psi,\ \mathcal{B}(E)\text{-measurable, s.t. }\psi\in L^p(\nu),\ \text{for every }\nu\in\mathcal{L}\}.$$

We shall consider a random variable $\xi:\Omega\longrightarrow\mathbb{R}$ and a generator function

$$f:(t,\omega,y,z,u,a,b,\nu)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^d\times\hat{L}^2\times\mathbb{S}_d^{\ge0}\times\mathbb{R}^d\times\mathcal{L}\longrightarrow\mathbb{R}.$$

Define for simplicity

$$\widehat{f}^{\mathbb{P}}_s(y,z,u):=f(s,X_{\cdot\wedge s},y,z,u,\widehat{a}_s,b^{\mathbb{P}}_s,\nu^{\mathbb{P}}_s)\ \text{ and }\ \widehat{f}^{\mathbb{P},0}_s:=f(s,X_{\cdot\wedge s},0,0,0,\widehat{a}_s,b^{\mathbb{P}}_s,\nu^{\mathbb{P}}_s). \tag{2.3}$$

Moreover, we are given a family $(\mathcal{P}(t,\omega))_{(t,\omega)\in[0,T]\times\Omega}$ of sets of probability measures on $(\Omega,\mathcal{F}_T)$, where $\mathcal{P}(t,\omega)\subset\mathcal{P}^W_t$ for all $(t,\omega)\in[0,T]\times\Omega$. Denote also $\mathcal{P}_t:=\cup_{\omega\in\Omega}\mathcal{P}(t,\omega)$. We make the following assumption on ξ, f and the family $(\mathcal{P}(t,\omega))_{(t,\omega)\in[0,T]\times\Omega}$.

Assumption 2.1. (i) *The random variable ξ is $\mathcal{F}_T-$measurable, the generator function f is jointly Borel measurable and such that for every $(t,\omega,y,y',z,z',u,u'a,b,\nu)\in[0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{R}^d\times\hat{L}^2\times\mathbb{S}_d^{\geq 0}\times\mathbb{R}^d\times\mathcal{L}$,*

$$\left|f(t,\omega,y,z,u,a,b,\nu)-f(t,\omega,y',z',u,a,b,\nu)\right|\leq C\left(\left|y-y'\right|+\left|a^{1/2}(z-z')\right|\right),$$

and for every fixed (y,z,u,a,b,ν), the map $(t,\omega)\longmapsto f(t,\omega,y,z,u,a,b,\ nu)$ is $\mathbb{F}-$progressively measurable.

(ii) *For all $(t,\omega,y,z,u^1,u^2,a,b,\nu)$, there exist two processes γ and γ' such that*

$$\int_E\delta^{1,2}u(x)\gamma_t'(x)\nu(dx)\leq f_t(\omega,y,z,u^1,a,b,\nu)-f_t(\omega,y,z,u^2,a,b,\nu)\leq\int_E\delta^{1,2}u(x)\gamma_t(x)\nu(dx),$$

where $\delta^{1,2}u:=u^1-u^2$ and $c_1(1\wedge|x|)\leq\gamma_t(x)\leq c_2(1\wedge|x|)$ with $-1+\delta\leq c_1\leq 0,\ c_2\geq 0$, and $c_1'(1\wedge|x|)\leq\gamma_t'(x)\leq c_2'(1\wedge|x|)$ with $-1+\delta\leq c_1'\leq 0,\ c_2'\geq 0$, for some $\delta>0$.

(iii) *For the fixed constant $p>1$, one has for every $(t,\omega)\in[0,T]\times\Omega$,*

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}^{\mathbb{P}}\left[|\xi|^p+\int_t^T\left|f(s,X_{\cdot\wedge s},0,0,0,\widehat{a}_s,b_s^{\mathbb{P}},\nu_s^{\mathbb{P}})\right|^p ds\right]<+\infty. \tag{2.4}$$

(iv) *For every $(t,\omega)\in[0,T]\times\Omega$, one has $\mathcal{P}(t,\omega)=\mathcal{P}(t,\omega_{\cdot\wedge t})$ and $\mathbb{P}(\Omega_t^\omega)=1$ whenever $\mathbb{P}\in\mathcal{P}(t,\omega)$. The graph $[[\mathcal{P}]]$ of $\mathcal{P}$, defined by $[[\mathcal{P}]]:=\{(t,\omega,\mathbb{P}):\mathbb{P}\in\mathcal{P}(t,\omega)\}$, is upper semi-analytic in $[0,T]\times\Omega\times\mathbb{M}_1$.*

(v) *$\mathcal{P}$ is stable under conditioning, i.e. for every $(t,\omega)\in[0,T]\times\Omega$ and every $\mathbb{P}\in\mathcal{P}(t,\omega)$ together with an $\mathbb{F}-$stopping time τ taking values in $[t,T]$, there is a family of r.c.p.d. $(\mathbb{P}_{\mathsf{w}})_{\mathsf{w}\in\Omega}$ such that $\mathbb{P}_{\mathsf{w}}\in\mathcal{P}(\tau(\mathsf{w}),\mathsf{w})$, for $\mathbb{P}-a.e.\ \mathsf{w}\in\Omega$.*

(vi) *$\mathcal{P}$ is stable under concatenation, i.e. for every $(t,\omega)\in[0,T]\times\Omega$ and $\mathbb{P}\in\mathcal{P}(t,\omega)$ together with a $\mathbb{F}-$stopping time τ taking values in $[t,T]$, let $(\mathbb{Q}_{\mathsf{w}})_{\mathsf{w}\in\Omega}$ be a family of probability measures such that $\mathbb{Q}_{\mathsf{w}}\in\mathcal{P}(\tau(\mathsf{w}),\mathsf{w})$ for all $\mathsf{w}\in\Omega$ and $\mathsf{w}\longmapsto\mathbb{Q}_{\mathsf{w}}$ is $\mathcal{F}_\tau-$measurable, then the concatenated probability measure $\mathbb{P}\otimes_\tau\mathbb{Q}_\cdot\in\mathcal{P}(t,\omega)$.*

We notice that for $t=0$, we have $\mathcal{P}_0:=\mathcal{P}(0,\omega)$ for any $\omega\in\Omega$.

Remark 2.1. *Let us comment on the above assumptions.*

1. *The first assumption except the jointly Borel measurability are quite standard in the classical RBSDE literature. The second one is classic assumptions to derive time consistence property for a family of nonlinear operators(see [5], [17]). The last two assumptions are related to regular conditional probability and cumulate to the second assumption, this allow us to establish the measurability of a value function of the stochastic control problem over a family of probability measures. The jointly measurable assumption is introduced in [14] in a measurability with respect to a probability measure. Unlike in [10, 9] , we do not need regularity conditions under the terminal value, the generator and the obstacle. These conditions were necessary in [9] to establish the existence of the solution through the dynamic programming principle. In this work, we use measurable selection theorem to provide the measurability of the value function.*

2.3 Spaces and norms

We now give the spaces and norms which will be needed in the rest of the paper. Fix some $t \in [0,T]$ and some $\omega \in \Omega$. In what follows, $\mathbb{X} := (\mathcal{X}_s)_{t \leq s \leq T}$ will denote an arbitrary filtration on $(\Omega, \mathcal{F}_T)$, and $\mathbb{P}$ an arbitrary element in $\mathcal{P}(t,\omega)$. Denote also by $\mathbb{X}_\mathbb{P}$ the $\mathbb{P}-$augmented filtration associated to $\mathbb{X}$.

For $p \geq 1$, $\mathbb{L}^p_{t,\omega}(\mathbb{X})$ (resp. $\mathbb{L}^p_{t,\omega}(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathcal{X}_T-$measurable scalar random variable ξ with

$$\|\xi\|^p_{\mathbb{L}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[|\xi|^p\right] < +\infty, \ \left(\text{resp. } \|\xi\|^p_{\mathbb{L}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[|\xi|^p\right] < +\infty\right).$$

$\mathbb{H}^p_{t,\omega}(\mathbb{X})$ (resp. $\mathbb{H}^p_{t,\omega}(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathbb{X}-$predictable $\mathbb{R}^d-$valued processes Z, which are defined $\widehat{a}_s ds - a.e.$ on $[t,T]$, with

$$\|Z\|^p_{\mathbb{H}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left\|\widehat{a}_s^{1/2} Z_s\right\|^2 ds\right)^{\frac{p}{2}}\right] < +\infty,$$

$$\left(\text{resp. } \|Z\|^p_{\mathbb{H}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \left\|\widehat{a}_s^{1/2} Z_s\right\|^2 ds\right)^{\frac{p}{2}}\right] < +\infty\right).$$

$\mathbb{J}^p_{t,\omega}(\mathbb{X})$ (resp. $\mathbb{J}^p_{t,\omega}(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathbb{X}$-predictable functions U with

$$\|U\|^p_{\mathbb{J}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \int_E |U_s(x)|^2 \nu_s^{\mathbb{P}}(dx)ds\right)^{\frac{p}{2}}\right] < +\infty.$$

$$\left(\text{resp. } \|U\|^p_{\mathbb{J}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[\left(\int_t^T \int_E |U_s(x)|^2 \nu_s^{\mathbb{P}}(dx)ds\right)^{\frac{p}{2}}\right] < +\infty.\right)$$

$\mathbb{M}^p_{t,\omega}(\mathbb{X},\mathbb{P})$ denotes the space of all $(\mathbb{X},\mathbb{P})-$optional martingales M with $\mathbb{P}-a.s.$ càdlàg paths on $[t,T]$, with $M_t = 0$, $\mathbb{P}-a.s.$, and

$$\|M\|^p_{\mathbb{M}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[[M]_T^{\frac{p}{2}}\right] < +\infty.$$

Furthermore, we will say that a family $(M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)}$ belongs to $\mathbb{M}^p_{t,\omega}((\mathbb{X}_\mathbb{P})_{\mathbb{P}\in\mathcal{P}(t,\omega)})$ if, for any $\mathbb{P} \in \mathbb{P}(t,\omega)$, $M^{\mathbb{P}} \in \mathbb{M}^p_{t,\omega}(\mathbb{X}_\mathbb{P},\mathbb{P})$ and

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left\|M^{\mathbb{P}}\right\|_{\mathbb{M}^p_{t,\omega}(\mathbb{P})} < +\infty.$$

$\mathbb{I}^p_{t,\omega}(\mathbb{X},\mathbb{P})$ (resp. $\mathbb{I}^{o,p}_{t,\omega}(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathbb{X}-$predictable (resp. $\mathbb{X}-$optional) processes K with $\mathbb{P}-a.s.$ càdlàg and non-decreasing paths on $[t,T]$, with $K_t = 0$, $\mathbb{P}-a.s.$, and

$$\|K\|^p_{\mathbb{I}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[K_T^p\right] < +\infty \ \ (\text{resp. } \|K\|^p_{\mathbb{I}^{o,p}_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[K_T^p\right] < +\infty).$$

We will say that a family $(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}(t,\omega)}$ belongs to $\mathbb{I}^p_{t,\omega}((\mathbb{X}_\mathbb{P})_{\mathbb{P}\in\mathcal{P}(t,\omega)})$ (resp. $\mathbb{I}^{o,p}_{t,\omega}((\mathbb{X}_\mathbb{P})_{\mathbb{P}\in\mathcal{P}(t,\omega)})$) if, for any $\mathbb{P} \in \mathcal{P}(t,\omega)$, $K^{\mathbb{P}} \in \mathbb{I}^p_{t,\omega}(\mathbb{X}_\mathbb{P},\mathbb{P})$ (resp. $K^{\mathbb{P}} \in \mathbb{I}^{o,p}_{t,\omega}(\mathbb{X}_\mathbb{P},\mathbb{P})$) and

$$\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left\|K^{\mathbb{P}}\right\|_{\mathbb{I}^p_{t,\omega}(\mathbb{P})} < +\infty \ \ \left(\text{resp. } \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \left\|K^{\mathbb{P}}\right\|_{\mathbb{I}^{o,p}_{t,\omega}(\mathbb{P})} < +\infty\right).$$

$\mathbb{D}^p_{t,\omega}(\mathbb{X})$ (resp. $\mathbb{D}^p_{t,\omega}(\mathbb{X},\mathbb{P})$) denotes the space of all $\mathbb{X}-$progressively measurable $\mathbb{R}-$valued processes Y with $\mathcal{P}(t,\omega)-q.s.$ (resp. $\mathbb{P}-a.s.$) càdlàg paths on $[t,T]$, with

$$\|Y\|^p_{\mathbb{D}^p_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\sup_{t\le s\le T}|Y_s|^p\right] < +\infty, \ \left(\text{resp. } \|Y\|^p_{\mathbb{D}^p_{t,\omega}(\mathbb{P})} := \mathbb{E}^{\mathbb{P}}\left[\sup_{t\le s\le T}|Y_s|^p\right] < +\infty\right).$$

For each $\xi \in \mathbb{L}^1_{t,\omega}(\mathbb{X})$ and $s\in[t,T]$ denote

$$\mathbb{E}^{\mathbb{P},t,\omega,\mathbb{X}}_s[\xi] := \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_{t,\omega}(s,\mathbb{P},\mathbb{X})} \mathbb{E}^{\mathbb{P}'}[\xi|\mathcal{X}_s] \text{ where } \mathcal{P}_{t,\omega}(s,\mathbb{P},\mathbb{X}) := \left\{\mathbb{P}'\in\mathcal{P}(t,\omega),\ \mathbb{P}' = \mathbb{P} \text{ on } \mathcal{X}_s\right\}.$$

Then we define for each $p\ge\kappa\ge 1$,

$$\mathbb{L}^{p,\kappa}_{t,\omega}(\mathbb{X}) := \left\{\xi\in L^p_{t,\omega}(\mathbb{X}),\ \|\xi\|_{\mathbb{L}^{p,\kappa}_{t,\omega}} < +\infty\right\},$$

where

$$\|\xi\|^p_{\mathbb{L}^{p,\kappa}_{t,\omega}} := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}^{\mathbb{P}}_{t\le s\le T}\left(\mathbb{E}^{\mathbb{P},t,\omega,\mathbb{F}^+}_s[|\xi|^\kappa]\right)^{\frac{p}{\kappa}}\right].$$

Similarly, given a probability measure $\overline{\mathbb{P}}$ and a filtration $\overline{\mathbb{X}}$ on the enlarged canonical space $\overline{\Omega}$, we denote the corresponding spaces by $\mathbb{D}^p_{t,\omega}(\overline{\mathbb{X}},\overline{\mathbb{P}})$, $\mathbb{H}^p_{t,\omega}(\overline{\mathbb{X}},\overline{\mathbb{P}})$, $\mathbb{M}^p_{t,\omega}(\overline{\mathbb{X}},\overline{\mathbb{P}})$, ... Furthermore, when $t=0$, there is no longer any dependence on ω, since $\omega_0=0$, so that we simplify the notations by suppressing the $\omega-$dependence and write $\mathbb{H}^p_0(\mathbb{X})$, $\mathbb{H}^p_0(\mathbb{X},\mathbb{P})$,... Similar notations are used on the enlarged canonical space.

2.4 Formulation of second order BSDEs with jumps

The following formulation is an extension of the one introduced by Possamai, Tan and Zhou [17] in the jumps case. Unlike in [19], we work with the filtration $\mathbb{F}^{\mathcal{P}_0}_+$. Since every $\mathbb{P}\in\mathcal{P}_0$ doesn't a priori satisfy the martingale representation property, then for every $\mathbb{P}\in\mathcal{P}_0$, we consider a 2RBSDE driven by the $\mathbb{P}$-martingale part $X^{c,\mathbb{P}}$ of X. Following the definition of BSDEs in general filtration studied in [3] and the wellposedness of 2BSDE of [17], we formulate BSDE with respect to filtration $\mathbb{F}^{\mathcal{P}_0}_+$. We consider the following 2BSDEJ :

$$Y_t = \xi + \int_t^T \widehat{f}^{\mathbb{P}}_s(Y_s, \widehat{a}^{1/2}_s Z_s, U^{\mathbb{P}}_s)ds - \int_t^T Z_s\cdot dX^{c,\mathbb{P}}_s - \int_s^T\int_E U^{\mathbb{P}}_r(e)\tilde{\mu}^{\mathbb{P}}(de,dr) - \int_t^T dM^{\mathbb{P}}_t + K^{\mathbb{P}}_T - K^{\mathbb{P}}_t, \tag{2.5}$$

$$0\le t\le T,\ \mathbb{P}\text{-a.s.},\ \forall\mathbb{P}\in\mathcal{P}_0, \tag{2.6}$$

where for every $\mathbb{P}\in\mathcal{P}_0$, $M^{\mathbb{P}}$ is a $\mathbb{P}$-martingale null at 0 orthogonal to $\mathbb{X}^{c,\mathbb{P}}$ and $K^{\mathbb{P}}$ is a non decreasing process null at 0.

For any $\mathbb{P}\in\mathcal{P}_0$, $\mathbb{F}$-stopping time τ and $\mathcal{F}_\tau$-measurable random $\xi\in\mathbb{L}^p_0(\mathbb{F})$, let $(y^{\mathbb{P}},z^{\mathbb{P}},u^{\mathbb{P}},m^{\mathbb{P}}) := (y^{\mathbb{P}}(\tau,\xi),z^{\mathbb{P}}(\tau,\xi),u^{\mathbb{P}}(\tau,\xi),m^{\mathbb{P}}(\tau,\xi)$ denote the solution to the following standard BSDEJ:

$$y^{\mathbb{P}}_t = \xi + \int_t^\tau \widehat{f}^{\mathbb{P}}_s(y^{\mathbb{P}}_s, \widehat{a}^{1/2}_s z^{\mathbb{P}}_s, u^{\mathbb{P}}_s)ds - \int_t^\tau z^{\mathbb{P}}_s\cdot dX^{c,\mathbb{P}}_s - \int_s^T\int_E u^{\mathbb{P}}_r(e)\tilde{\mu}^{\mathbb{P}}(de,dr) - \int_t^\tau dm^{\mathbb{P}}_s,\ 0\le t\le\tau,\ \mathbb{P}\text{-a.s.} \tag{2.7}$$

where $m^{\mathbb{P}}$ is a martingale orthogonal to X under $\mathbb{P}$. Kruse and Popier in [11] (see also [12]) have proved existence and uniqueness of a solution to the BSDEJs (2.7) with $(y_t^{\mathbb{P}}, z_t^{\mathbb{P}}, u_t^{\mathbb{P}}, m_t^{\mathbb{P}})_{t\in[0,T]}$ belongs to $\mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})\times\mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})\times\mathbb{J}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})\times\mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})$ satisfying equation (2.7) under each $\mathbb{P}$. This equation gives the classical formulation to a BSDEJ on Ω in a general filtration. Following [10], we give the definition of a 2BSDEJ for fixed $p>1$ as following:

Definition 2.1. *We say that $(Y,Z,U)\in\mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0})\times\mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0})\times\mathbb{J}_0^p(\mathbb{F}_+^{\mathcal{P}_0})$ is a solution to the 2BSDEJ (2.5) if :*

(i) $Y_T=\xi,\ \ t\in[0,T],\mathcal{P}_0$*-q.s.;*

(ii) $\forall\mathbb{P}\in\mathcal{P}_0$*, the process $K^{\mathbb{P}}$ defined below has non-decreasing paths $\mathbb{P}$-a.s., $\forall t\in[0,T]$,*

$$K_t^{\mathbb{P}}:=Y_0-Y_t-\int_0^t\widehat{f}_s^{\mathbb{P}}(Y_s,\widehat{a}_s^{1/2}Z_s,U_s)ds+\int_s^T Z_s\cdot dX_s^{c,\mathbb{P}}+\int_s^T\int_E U_r^{\mathbb{P}}(e)\tilde{\mu}^{\mathbb{P}}(de,dr)+\int_s^T dM_s^{\mathbb{P}}. \tag{2.8}$$

(iii) We have the following minimality condition:

$$K_t^{\mathbb{P}}=\operatorname*{ess\,inf}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)}\mathbb{E}^{\mathbb{P}'}\Big[K_T^{\mathbb{P}'}\big|\mathcal{F}_{t+}^{\mathbb{P}}\Big],\quad 0\le t\le T,\ \mathbb{P}\text{-a.s., }\forall\mathbb{P}\in\mathcal{P}_0. \tag{2.9}$$

Remark 2.2. *1. Rigorously, the solution is $(Y,Z,U,(M^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0},(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})\in\mathbb{D}_0^p(\mathbb{F}_+^{\mathcal{P}_0})\times\mathbb{H}_0^p(\mathbb{F}_+^{\mathcal{P}_0})\times\mathbb{J}_0^p(\mathbb{F}_+^{\mathcal{P}_0})\times\mathbb{M}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})\times\mathbb{I}_0^p((\mathbb{F}_+^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0})$ and through misuse of language, we denote (Y,Z,U), given the dependence in $\mathbb{P}$ of $K^{\mathbb{P}}$ and $M^{\mathbb{P}}$.*

2. Using the above definition, then for any $\mathbb{P}\in\mathcal{P}_0$, $K^{\mathbb{P}}-M^{\mathbb{P}}$ is a semimartingale defined by

$$K_t^{\mathbb{P}}-M_t^{\mathbb{P}}:=Y_0-Y_t-\int_0^t\widehat{f}_s^{\mathbb{P}}(Y_s,\widehat{a}_s^{1/2}Z_s,U_s)ds+\int_0^t Z_s\cdot dX_s^{c,\mathbb{P}}+\int_s^T\int_E U_r^{\mathbb{P}}(e)\tilde{\mu}^{\mathbb{P}}(de,dr),\ t\in[0,T],$$

Using results of Nutz [15], under additional assumptions (related to axiomatic set theory) the family of semimartingales $(K^{\mathbb{P}}-M^{\mathbb{P}})_{\mathbb{P}}$ can not be aggregated into a universal semimartingale $K-M$ because the coefficient $\widehat{f}^{\mathbb{P}}$ defined in (2.3) depend of the compensator measure $\nu^{\mathbb{P}}$.

Following [17] in addition to Assumption 2.1, we will always assume the following in order to prove uniqueness of the solution to the 2BSDEJ (2.5).

Assumption 2.2. *For fixed $p>1$, there is some $\kappa\in(1,p]$ such that the following integrability conditions are satisfies:*

$$\phi_f^{p,\kappa}:=\sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\left[\operatorname*{ess\,sup}^{\mathbb{P}}_{0\le t\le T}\left(\mathbb{E}^{\mathbb{P}}\left[\int_0^T|\widehat{f}_s^{\mathbb{P},0}|^{\kappa}ds\big|\mathcal{F}_{t+}^{\mathbb{P}}\right]\right)^{\frac{p}{\kappa}}\right]<+\infty. \tag{2.10}$$

(2.11)

One can consider the case $\kappa=p=2$.

Remark 2.3. *If $\widehat{a}_s$ is non-degenerate $\mathbb{P}$-a.s., for all $s \in [0,T]$, the we can construct a brownian motion $W^{\mathbb{P}}$ on Ω by*

$$W_t^{\mathbb{P}} := \int_0^t \widehat{a}_s^{-1/2} dX_s^{c,\mathbb{P}},\ t \in [0,T],\ \mathbb{P}\text{-a.s.} \tag{2.12}$$

Otherwise we define the enlarged canonical space $\overline{\Omega} := \Omega \times \Omega'$, where Ω' is identical to Ω and set (X,B) its canonical process, i.e. $X_t(\overline{\omega}) := \omega_t$, $B_t(\overline{\omega}) := \omega'_t$ for all $\bar{\omega} := (\omega,\omega') \in \overline{\Omega}$. The extension from Ω to $\overline{\Omega}$ of a random variable or a process λ is defined by

$$\lambda(\overline{\omega}) := \lambda(\omega),\ \forall \overline{\omega} = (\omega,\omega') \in \overline{\Omega}.$$

In particular $\widehat{a}$ can be extended on $\overline{\Omega}$. For $\mathbb{P} \in \mathcal{P}_t^W$, a probability measure on the enlarged space $\overline{\Omega}$ is denote by $\overline{\mathbb{P}}$ with $\overline{\mathbb{P}} := \mathbb{P} \otimes \mathbb{P}_0$. We also consider like in [?], the canonical filtration $\overline{F}$ generated by (X,B), the filtration $\overline{F}^X$ generated by X, the right-continuous filtrations $\overline{\mathbb{F}}_+^X$ and $\overline{\mathbb{F}}_+$, and the augmented filtration $\overline{\mathbb{F}}_+^{X,\overline{\mathbb{P}}}$ and $\overline{\mathbb{F}}_+^{\overline{\mathbb{P}}}$ given a probability measure $\overline{\mathbb{P}}$ on $\overline{\Omega}$.
From the above it follows that X in $(\overline{\Omega}, \overline{\mathcal{F}}_T, \overline{\mathbb{P}}, \overline{\mathbb{F}})$ is a semimartingale measure with the same triplet of characteristics as X in $(\Omega, \mathcal{F}_T, \mathbb{P}, \mathbb{F})$, B is a $\overline{\mathbb{F}}$-Brownian motion and X is independent of B. Then for every $\mathbb{P} \in \mathcal{P}_t^W$, there is some $\mathbb{R}^d$-valued $\overline{\mathbb{F}}$-Brownian motion $W^{\mathbb{P}}$ such that(see Theorem 4.5.2 of Stroock and Varadhan [21])

$$X_s = \int_t^s b_r^{\mathbb{P}} dr + \int_t^s \widehat{a}_r^{\frac{1}{2}} dW_+^{\mathbb{P}} + X_s^{\mathbb{P},d},\ \ s \in [t,T],\ \overline{\mathbb{P}}\text{-a.s.} \tag{2.13}$$

where the defintion of $b^{\mathbb{P}}$ and $\widehat{a}$ are extended on $\overline{\Omega}$.

Remark 2.4. *The decomposition (2.13) of the canonical process X with a Brownian motion $W^{\mathbb{P}}$ is on the enlarged space $\overline{\Omega}$. The interest of this decomposition lies in the fact that in calculations, we can apply some known results related to Brownian Motion like Girsanov Theorem, linearization arguments and others. To have the same decomposition on Ω, $\widehat{a}$ needs to be non-degenerate. Since we cannot ensure that this condition will be satisfy, we will extend some equivalence introduced in [17] in our work.*

Throughout the rest of the paper for the sake of simplicity, we consider the case where $\widehat{a}$ is non-degenerate and then there exists the Brownian motion $W^{\mathbb{P}}$ on Ω under $\mathbb{P}$ satisfying (2.12). Therefore the BSDEJs $(y^{\mathbb{P}}, z^{\mathbb{P}}, u^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ associated to the 2BSDEJ (2.5) will be considered on $(\Omega, \mathcal{F}_T, \mathbb{P})$ w.r.t. the filtration $\mathbb{F}_+^{\mathbb{P}}$:

$$y_t^{\mathbb{P}} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}, u_s^{\mathbb{P}}) ds - \int_t^T z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_s^T \int_E u_r^{\mathbb{P}}(e) \tilde{\mu}^{\mathbb{P}}(de,dr) - \int_t^T dm_s^{\mathbb{P}},\ \mathbb{P}\text{-a.s.} \tag{2.14}$$

The case where $\widehat{a}$ is degenerate can be easily adapted by working in the enlarged space with equivalence of BSDEJs. We refer the reader to Lemma 2.1 (p. 563) of [17] for the equivalence between BSDEs on the enlarged space. For the BSDEJs case, the equivalence is established based on the same principle.

2.4.1 Representation and uniqueness of the solution

We have similarly as in [17] and [10], the following representation result:

Theorem 2.1. *Let Assumptions 2.1 and 2.2 hold. Let $\xi \in \mathbb{L}_0^{p,\kappa}$ and (Y,Z) be a solution to the 2BSDEJ (2.5). For any $\mathbb{P} \in \mathcal{P}_0$, let $(y_s^{\mathbb{P}}, z_s^{\mathbb{P}}, m_s^{\mathbb{P}}, u_s^{\mathbb{P}}, k_s^{\mathbb{P}})_{s\in[t,T]} \in \mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P}) \times \mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P}) \times \mathbb{I}_0^p(\mathbb{F}_+^{\mathbb{P}},\mathbb{P})$ be the solutions to the corresponding BSDEJs (2.7). Then for any $\mathbb{P} \in \mathcal{P}_0$ and $0 \le t_1 \le t_2 \le T$,*

$$Y_{t_1} = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}'\in\mathcal{P}_0(t_1,\mathbb{P},\mathbb{F}_+)} y_{t_1}^{\mathbb{P}'}(t_2, Y_{t_2}),\ \mathbb{P}\text{-}a.s. \tag{2.15}$$

Thus, the 2BSDEJ (2.5) has at most one solution in $\mathbb{D}_0^p(\mathbb{F}_+^{\mathbb{P}}) \times \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}})$.

Proof. The proof is similar as for Theorem 4.1 in [10] (p. 2890-2893), so you omit it.

2.4.2 A priori estimates

Proposition 2.1. *Let Assumptions 2.1 and 2.2 hold. Assume $\xi \in \mathbb{L}_0^{p,\kappa}$ and (Y,Z) is the solution to the $2BSDEJ(2.5)$. Let $\{(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})\}_{\mathbb{P}\in\mathcal{P}_0}$ be the solutions of the corresponding BSDEJs (2.14). Then there exists a constant C depending only on κ, T, p and the Lipschitz constant L_f of f such that*

$$\sup_{\mathbb{P}\in\mathcal{P}_0} \|y^{\mathbb{P}}\|^p_{\mathbb{D}_0^p(\mathbb{P})} + \sup_{\mathbb{P}\in\mathcal{P}_0} \Big\{ \|z^{\mathbb{P}}\|^p_{\mathbb{H}_0^p(\mathbb{P})} + \mathbb{E}^{\mathbb{P}}\Big[[m^{\mathbb{P}}]_T^{\frac{p}{2}}\Big] + \mathbb{E}^{\mathbb{P}}\Big[(k_T^{\mathbb{P}})^p\Big] \Big\} \le C\Big(\|\xi\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi_f^{p,\kappa} \Big)$$

and

$$\|Y\|^p_{\mathbb{D}_0^p} + \|Z\|^p_{\mathbb{H}_0^p} + \|U\|^p_{\mathbb{J}_0^p} + \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[[M^{\mathbb{P}}]_T^{\frac{p}{2}}\Big] + \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[(K_T^{\mathbb{P}})^p\Big] \le C\Big(\|\xi\|^p_{\mathbb{L}_0^{p,\kappa}} + \phi_f^{p,\kappa} \Big).$$

Proof. A priori estimates for solution of BSDEJs (2.14) are given in [11] (see also [12]) and a priori estimates for solution of 2BSDEJs are similar as in [10, 9] (see also [12]).

We have also the following stability result:

Theorem 2.2. *Let Assumption 2.1 holds, and consider two generators f^1 , f^2 such that Assumption 2.2 holds. For $i = 1, 2$, let $((Y^i, Z^i, U^i)$ be a solution to the 2BSDEJ (2.5) with the generator f^i, the terminal condition ξ^i. Define*

$$\phi^{p,\kappa}_{f^1,f^2} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\operatorname*{ess\,sup}_{0\le t\le T}{}^{\mathbb{P}} \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\widehat{f}_s^{1,\mathbb{P}} - \widehat{f}_s^{2,\mathbb{P}}|^\kappa (y_s^{1,\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{1,\mathbb{P}}) ds \Big)^{\frac{p}{\kappa}} \Big| \mathcal{F}_{t+}^{\mathbb{P}} \Big] \Big] < +\infty,$$

$$\varphi^{p,\kappa}_{f^1,f^2} := \sup_{\mathbb{P}\in\mathcal{P}_0} \mathbb{E}^{\mathbb{P}}\Big[\Big(\int_0^T |\widehat{f}_s^{1,\mathbb{P}} - \widehat{f}_s^{2,\mathbb{P}}|(Y_s^1, \widehat{a}_s^{1/2} Z_s^1) \Big)^p \Big] + \infty,$$

Then, there exists a constant C depending only on κ, T and the Lipschitz constant of f^1 and f^2 such that

$$\begin{aligned}
&\big\|Y^1 - Y^2\big\|^p_{\mathbb{D}^p_0} \leq C\Big(\big\|\xi^1 - \xi^2\big\|^p_{\mathbb{L}^{p,\kappa}_0} + \varphi^{p,\kappa}_{f^1,f^2}\Big),\\
&\big\|Z^1 - Z^2\big\|^p_{\mathbb{H}^p_0} + \big\|U^1 - U^2\big\|^p_{\mathbb{J}^p_0} + \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[\big[M^{1,\mathbb{P}} - M^{2,\mathbb{P}}\big]^{\frac{p}{2}}_T\Big] + \sup_{\mathbb{P}\in\mathcal{P}_0}\mathbb{E}^{\mathbb{P}}\Big[(K^{1,\mathbb{P}}_T - K^{2,\mathbb{P}}_T)^p\Big]\\
&\qquad\qquad \leq C\Big(\big\|\xi^1 - \xi^2\big\|^p_{\mathbb{L}^{p,\kappa}_0} + \phi^{p,\kappa}_{f^1,f^2}\Big)\\
&+ C\Big(\big\|\xi^1 - \xi^2\big\|^{p/2}_{\mathbb{L}^{p,\kappa}_0} + \big(\phi^{p,\kappa}_{f^1,f^2}\big)^{p/2}\Big) \times \Big(\big\|\xi^1\big\|^p_{\mathbb{L}^{p,\kappa}_0} + \big\|\xi^2\big\|^p_{\mathbb{L}^{p,\kappa}_0} + \phi^{p,\kappa}_{f^1} + \phi^{p,\kappa}_{f^2}\Big)^{1/2}.
\end{aligned}$$

2.5 Existence of the solution

The key idea to prove existence of a solution is the dynamic programming principle and the selection measurable theorem. The value function is defined pathwise as a supremum of the conditional expectation of Picard iteration of solution to BSDEJs over a set of probability measures. After proved the DPP for the value function, the solution is obtained from this one. Following the representation formula of 2BSDEJs (2.15), a natural candidate to the solution of 2BSDEJs could be: For every $(t,\omega) \in [0,T]\times\Omega$,

$$V_t(\omega) := \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}[y^{\mathbb{P}}_t],$$

where $y^{\mathbb{P}}$ is the first component of the solution of BSDEJ (2.14). The proof of existence of a solution of 2BSDEJ (2.5) will be divided in four steps:

Step one. In order to establish the dynamic programming principle for the above value function V, we need a jointly measurable (with respect to time, space and probability $\mathbb{P}$) version of $y^{\mathbb{P}}$ solution of BSDEJs. We recall that for every $\mathbb{P}\in\mathcal{P}_0$, these BSDEJs already have a unique solution then our goal is to construct a jointly measurable version of the solutions. Thereby, we use the Picard iteration of the solution to the BSDEJs and prove the converge of the iterations.

Step two. After the converge of approximations, the resulting solutions $y^{\mathbb{P}}_t$ can be interpreted as a function of t,ω and $\mathbb{P}$. We now show that $y^{\mathbb{P}}$ is jointly measurable: that means $(t,\omega,\mathbb{P}) \mapsto y^{\mathbb{P}}_t$ is a measurable function.

Step three. This step is dedicated to establish the following dynamic programming principle for the value function.

$$V_t(\omega) = \sup_{\mathbb{P}\in\mathcal{P}(t,\omega)} \mathbb{E}^{\mathbb{P}}\big[y^{\mathbb{P}}_t(\tau, V_\tau)\big]$$

where τ is a stopping time taking value in $[t,T]$

Step four. This last step consists of path modification of the value function in order to obtain a càdlàg process and deduce the solutions by the Doob decomposition.

2.5.1 An iterative construction of the solution to BSDEJ (2.14)

Let us first recall the classical construction of the $y^{\mathbb{P}}$ part of the solution to the BSDEJ (2.14) under some probability $\mathbb{P} \in \mathcal{P}(t,\omega)$ using Picard's iteration. Let us first define for any $m \geq 0$

$$\xi^m := (\xi \vee m) \wedge (-m), \quad f^m(t,\omega,y,z,a,b,\nu) := (f(t,\omega,y,z,a,b,\nu) \vee m) \wedge (-m).$$

First, let $\mathcal{Y}_s^{\mathbb{P},0,m} \equiv 0$, $\mathcal{Z}_s^{\mathbb{P},0,m} \equiv 0$ and $\mathcal{U}_s^{\mathbb{P},0,m} \equiv 0$, for all $s \in [t,T]$. We define by induction the following BSDEJ:

$$\begin{aligned}\mathcal{Y}_s^{\mathbb{P},n+1,m} = \xi^m - \int_s^T \widehat{f}_r^m(\mathcal{Y}_r^{\mathbb{P},n,m}, \widehat{a}_r^{1/2}\mathcal{Z}_r^{\mathbb{P},n,m}, \mathcal{U}_r^{\mathbb{P},n,m})dr - \int_s^T \mathcal{Z}_r^{\mathbb{P},n+1,m} \cdot dX_r^{c,\mathbb{P}} \\ - \int_s^T \int_E \mathcal{U}_r^{\mathbb{P},n+1,m}(x)\tilde{\mu}_r^{\mathbb{P}}(dx,dr) - \int_s^T dM_r^{\mathbb{P},n+1,m}.\end{aligned}$$

(i) First, let $\mathcal{Y}_s^{\mathbb{P},0,m} \equiv 0$, $\mathcal{Z}_s^{\mathbb{P},0,m} \equiv 0$ and $\mathcal{U}_s^{\mathbb{P},0,m} \equiv 0$, for all $s \in [t,T]$.

(ii) Given a family of $\mathbb{F}_+-$progressively measurable processes $\left(\mathcal{Y}_s^{\mathbb{P},n,m}, \mathcal{Z}_s^{\mathbb{P},n,m}, \mathcal{U}_s^{\mathbb{P},n,m}\right)_{s\in[t,T]}$, we define

$$\overline{\mathcal{Y}}_s^{\mathbb{P},n+1,m} := \mathbb{E}^{\mathbb{P}}\left[\xi - \int_s^T \widehat{f}_r^m(\mathcal{Y}_r^{\mathbb{P},n,m}, \widehat{a}_r^{1/2}\mathcal{Z}_r^{\mathbb{P},n,m}, \mathcal{U}_r^{\mathbb{P},n,m})dr \middle| \mathcal{F}_s\right], \ \mathbb{P}-a.s. \tag{2.16}$$

(iii) Let $\mathcal{Y}^{\mathbb{P},n+1,m}$ be a right-continuous modification of $\overline{\mathcal{Y}}^{\mathbb{P},n+1,m}$ defined by

$$\mathcal{Y}_s^{\mathbb{P},n+1,m} := \limsup_{\mathbb{Q}\ni r\downarrow s} \overline{\mathcal{Y}}_r^{\mathbb{P},n+1,m}, \ \mathbb{P}-a.s. \tag{2.17}$$

(iv) Notice that $\mathcal{Y}^{\mathbb{P},n+1,m}$ is a semi-martingale under $\mathbb{P}$. Let $\langle \mathcal{Y}^{\mathbb{P},n+1,m}, X\rangle^{\mathbb{P}}$ be the predictable quadratic covariation of the process $\mathcal{Y}^{\mathbb{P},n+1,m}$ and X under $\mathbb{P}$. Define

$$\widehat{a}_s^{1/2}\mathcal{Z}_s^{\mathbb{P},n+1,m} := \limsup_{\mathbb{Q}\ni\varepsilon\downarrow 0} \frac{\langle \mathcal{Y}^{\mathbb{P},n+1,m}, X\rangle_s^{\mathbb{P}} - \langle \mathcal{Y}^{\mathbb{P},n+1,m}, X\rangle_{s-\varepsilon}^{\mathbb{P}}}{\varepsilon}. \tag{2.18}$$

(v) The jump of $\mathcal{Y}^{\mathbb{P},n+1,m}$ is given by

$$\Delta[\mathcal{Y}^{\mathbb{P},n+1,m}, X]_s = \mathcal{U}_s^{\mathbb{P},n+1,m}(\Delta X_s)\Delta X_s,$$

(v) Notice that the sequence $(\mathcal{Y}^{\mathbb{P},n,m})_{n\geq 0}$ is a Cauchy sequence for the norm

$$\|(Y,Z,U)\|_\alpha^2 := \mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{\alpha s}\,|Y_s|^2\,ds\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^T e^{\alpha s}\left\|\widehat{a}_s^{1/2}Z_s\right\|^2 ds\right] + \mathbb{E}^{\mathbb{P}}\left[\int_t^T\int_E e^{\alpha s}\ |U_s(x)|^2\,\nu_s^{\mathbb{P}}(dx)ds\right],$$

for α large enough, see for instance [11] (see also [12]). Indeed, this is a consequence of the classical estimates for BSDEJs recaled in Proposition . Then by taking some suitable sub-sequence $(n_k^{\mathbb{P},m})_{k\geq 1}$, we can define

$$\mathcal{Y}_s^{\mathbb{P},m} := \limsup_{k\to\infty} \mathcal{Y}_s^{\mathbb{P},n_k^{\mathbb{P},m},m}.$$

(vi) Finally, we can again use the estimates given in Proposition 2.5.1 (see also [11, 12]) to show that the sequence $(\mathcal{Y}^{\mathbb{P},m})_{m\geq 0}$ is a Cauchy sequence in $\mathbb{D}_0^p(\overline{\mathbb{F}}_+^{\mathbb{P}},\mathbb{P})$, so that by taking once more a suitable subsequence $(m_k^{\mathbb{P}})_{k\geq 1}$, we can define the solution to the BSDE as

$$\mathcal{Y}_s^{\mathbb{P}} := \limsup_{k\to\infty} \mathcal{Y}_s^{\mathbb{P},m_k^{\mathbb{P}}}. \tag{2.19}$$

2.5.2 Construction of measurable version of solution

Here we show that the iteration in Section 2.5.1 can be taken in a measurable way w.r.t. the reference probability measure $\mathbb{P}$, which allows us to use the measurable selection theorem to derive the dynamic programming principle.

Lemma 2.1. *Let $\mathcal{P}$ be a measurable set in $\mathbb{M}_1$, $(\mathbb{P},\omega,t)\longmapsto H_t^{\mathbb{P}}(\omega)$ be a measurable function such that for all $\mathbb{P}\in\mathcal{P}$, $H^{\mathbb{P}}$ is right-continuous, $\mathbb{F}_+-$adapted and a $(\mathbb{P},\mathbb{F}_+^{\mathbb{P}})-$semi-martingale. Then there is a measurable function $(\mathbb{P},\omega,t)\longmapsto \langle H\rangle_t^{\mathbb{P}}(\omega)$ such that for all $\mathbb{P}\in\mathcal{P}$, $\langle H\rangle^{\mathbb{P}}$ is right-continuous, $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$predictable, and*

$\langle H\rangle_\cdot^{\mathbb{P}}$ is the predictable quadratic variation of the semi-martingale $H^{\mathbb{P}}$ under $\mathbb{P}$.

Proof. (i) For every $n\geq 1$, we define the following sequence of random times

$$\begin{cases} \tau_0^{\mathbb{P},n}(\omega) := 0,\ \omega\in\Omega, \\ \tau_{i+1}^{\mathbb{P},n}(\omega) := \inf\left\{t\geq \tau_i^n(\omega),\ \left|H_t^{\mathbb{P}}(\omega) - H_{\tau_i^n}^{\mathbb{P}}(\omega)\right|\geq 2^{-n}\right\}\wedge 1,\ \omega\in\Omega,\ i\geq 1. \end{cases} \tag{2.20}$$

We notice that the $\tau_i^{\mathbb{P},n}$ are all $\mathbb{F}_+-$stopping times since the $H^{\mathbb{P}}$ are right-continuous and $\mathbb{F}_+-$adapted. We then define

$$[H^{\mathbb{P}}]_\cdot(\omega) := \limsup_{n\to+\infty}\sum_{i\geq 0}\left(H^{\mathbb{P}}_{\tau_{i+1}^{\mathbb{P},n}\wedge\cdot}(\omega) - H^{\mathbb{P}}_{\tau_i^{\mathbb{P},n}\wedge\cdot}(\omega)\right)^2. \tag{2.21}$$

It is clear that $(\mathbb{P},\omega,t)\longmapsto [H^{\mathbb{P}}]_t(\omega)$ is a measurable function, and for all $\mathbb{P}\in\mathcal{P}$, $[H^{\mathbb{P}}]$ is non-decreasing, $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$optional. Then, it follows by Karandikar [**?**] that $[H^{\mathbb{P}}]$ coincides with the quadratic variation of the semi-martingale $H^{\mathbb{P}}$ under $\mathbb{P}$. Moreover, by taking its right limit over rational time instants, we can choose $[H^{\mathbb{P}}]$ to be right continuous.

(ii) Finally, using Proposition 5.1 of Neufeld and Nutz [13], we can then construct a process $\langle H\rangle_t^{\mathbb{P}}(\omega)$ satisfying the required conditions. □

Notice that the construction above can also be carried out for the predictable quadratic covariation $\langle H^{\mathbb{P},1},H^{\mathbb{P},2}\rangle^{\mathbb{P}}$, by defining it through the polarization identity

$$\langle H^{\mathbb{P},1},H^{\mathbb{P},2}\rangle^{\mathbb{P}} := \frac{1}{4}\left(\langle H^{\mathbb{P},1}+H^{\mathbb{P},2}\rangle^{\mathbb{P}} - \langle H^{\mathbb{P},1}-H^{\mathbb{P},2}\rangle^{\mathbb{P}}\right), \tag{2.22}$$

for all measurable functions $H_t^{\mathbb{P},1}(\omega)$ and $H_t^{\mathbb{P},2}(\omega)$ satisfying the conditions in Lemma 2.1.

We now show that the iteration in Section 2.5.1 can be taken in a measurable way w.r.t. $\mathbb{P}$, which provides a key step for the proof of the DPP result.

Lemma 2.2. *Let $m > 0$ be a fixed constant, $(s, \omega, \mathbb{P}) \longmapsto (\mathcal{Y}_s^{\mathbb{P},n,m}(\omega), \mathcal{Z}_s^{\mathbb{P},n,m}(\omega))$ be a measurable map such that for every $\mathbb{P} \in \mathcal{P}_t$, $\mathcal{Y}^{\mathbb{P},n,m}$ is right-continuous, $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$optional, $\mathcal{Z}^{\mathbb{P},n,m}$ is $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$predictable. Then we can choose a measurable map $(s, \omega, \mathbb{P}) \longmapsto \left(\mathcal{Y}_s^{\mathbb{P},n,m}(\omega), \mathcal{Z}_s^{\mathbb{P},n,m}(\omega)\right)$ such that for every $\mathbb{P} \in \mathcal{P}_t$, $\mathcal{Y}^{\mathbb{P},n+1,m}$ is right-continuous, $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$optional, $\mathcal{Z}^{\mathbb{P},n+1,m}$ is $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$predictable.*

Proof. (i) First, using Lemma 3.1 of Neufeld and Nutz [13], there is a version of $(\overline{\mathcal{Y}}^{\mathbb{P},n+1,m})$ defined by (2.16), such that $(\mathbb{P}, \omega) \longmapsto \overline{\mathcal{Y}}_s^{\mathbb{P},n+1,m}$ is $\mathfrak{B} \otimes \mathcal{F}_s-$measurable for every $s \in [t, T]$.

(ii) Next, we notice that the measurability is not lost by taking the limit along a countable sequence. Then with the above version of $(\overline{\mathcal{Y}}^{\mathbb{P},n+1,m})$, it is clear that the family $(\mathcal{Y}_s^{\mathbb{P},n+1,m}(\omega))$ defined by (2.17) is measurable in $(s, \omega, \mathbb{P})$, and for all $\mathbb{P} \in \mathcal{P}_t$, $\mathcal{Y}^{\mathbb{P},n+1,m}$ is $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$optional.

(iii) Then using Lemma 2.1 as well as the definition of the quadratic covariation in (2.22), it follows that there is a measurable function

$$(s, \omega, \mathbb{P}) \longmapsto \langle \mathcal{Y}^{\mathbb{P},n+1,m}, X \rangle_s^{\mathbb{P}}(\omega),$$

such that for every $\mathbb{P} \in \mathcal{P}_t$, $\langle \mathcal{Y}^{\mathbb{P},n+1,m}, X \rangle^{\mathbb{P}}$ is right-continuous, $\mathbb{F}_+-$adapted and coincides with the predictable quadratic covariation of $\mathcal{Y}^{\mathbb{P},n+1,m}$ and X under $\mathbb{P}$.

(iv) Finally, with the above version of $\left(\langle \mathcal{Y}^{\mathbb{P},n+1,m}, X \rangle^{\mathbb{P}}\right)$, it is clear that the family $(\mathcal{Z}_s^{\mathbb{P},n+1,m}(\omega))$ defined by (2.18) is measurable in $(s, \omega, \mathbb{P})$ and for every $\mathbb{P} \in \mathcal{P}_t$, $\mathcal{Z}^{\mathbb{P},n+1,m}$ is $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$predictable. □

Lemma 2.3. *Let $m > 0$ be a fixed constant, $(s, \omega, \mathbb{P}) \longmapsto (\mathcal{Y}_s^{\mathbb{P},n,m}(\omega), \mathcal{Z}_s^{\mathbb{P},n,m}(\omega), \mathcal{U}_s^{\mathbb{P},n,m}, \mathcal{M}_s^{\mathbb{P},n,m})$ be a measurable map such that for every $\mathbb{P} \in \mathcal{P}_t$, $\mathcal{Y}^{\mathbb{P},n,m}$ is right-continuous, $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$optional, $\mathcal{Z}^{\mathbb{P},n,m}$ is $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$predictable, $\mathcal{U}^{\mathbb{P},n,m}$ is $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$predictable, and $\mathcal{M}^{\mathbb{P},n,m}$ is right-continuous, $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$ optional. Then we can choose a measurable map $(s, \omega, \mathbb{P}) \longmapsto \left(\mathcal{U}_s^{\mathbb{P},n+1,m}(\omega), \mathcal{M}_s^{\mathbb{P},n+1,m}(\omega)\right)$ such that for every $\mathbb{P} \in \mathcal{P}_t$, $\mathcal{U}^{\mathbb{P},n+1,m}$ is $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$predictable, $\mathcal{M}^{\mathbb{P},n+1,m}$ is right-continuous, $\mathbb{F}_+-$adapted and $\mathbb{F}_+^{\mathbb{P}}-$optional,*

Proof. For all $\mathbb{P}$, we introduce the non-negative measure on $([0, T] \times \Omega \times E, \mathcal{B}([0, T]) \otimes \mathcal{F}_T \otimes \mathcal{B}(E))$

$$\Theta^{\mathbb{P}}(\mathcal{U}) = \mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_E \mathcal{U}(s, x) \mu_X^{\mathbb{P}}(ds, dx)\right].$$

We denote by $\mathcal{H}$ the tensor product of the predictable σ-field on $[0, T] \times \Omega$ and of $\mathcal{B}(E)$. Then, following [6] (Theorem III.4.20), we know that $\mathcal{U}^{\mathbb{P},n+1,m}$ is expressed in term of "generalized" conditional expectation by the following formula (see [6], Section III.3c, p.170):

$$\mathcal{U}^{\mathbb{P},n+1,m} = \Theta^{\mathbb{P}}[\Delta \mathcal{Y}^{\mathbb{P},n+1,m} | \mathcal{H}],$$

indeed with notations of [6], Section III.4b, we have

$$\int_0^t \int_E \Delta \mathcal{Y}^{\mathbb{P},n+1,m}(s, x) \nu_s^{\mathbb{P}}(dx) ds = 0.$$

Since any element of $\mathcal{H}$ is separable (i.e. generated by a countable family of sets) we can adapt the proof of Lemma 3.1 in [13] and we get that there exist a version of $\mathcal{U}^{\mathbb{P},n+1,m}$, which we still denote by $\mathcal{U}^{\mathbb{P},n+1,m}$, such that $(\mathbb{P},\omega)\mapsto\mathcal{U}^{\mathbb{P},n+1,m}(\omega,\cdot,\cdot)$ is measurable and such that for all $\mathbb{P}$, $\mathcal{U}^{\mathbb{P},n+1,m}$ is $\mathcal{H}$ measurable, which is exactly the desired property.
Then for any $t\in[0,T]\cap\mathbb{Q}$, we have:

$$\mathcal{M}_t^{\mathbb{P},n+1,m}=\mathcal{Y}_t^{\mathbb{P},n+1,m}-\mathcal{Y}_0^{\mathbb{P},n+1,m}-\int_0^t f(r,X_{\cdot\wedge r},\mathcal{Y}_r^{\mathbb{P},n,m},\widehat{a}_r^{1/2}\mathcal{Z}_r^{\mathbb{P},n,m},\mathcal{U}_r^{\mathbb{P},n,m},\widehat{a}_r,b_r^{\mathbb{P}},\nu_r^{\mathbb{P}})dr$$
$$-\int_0^t\mathcal{Z}_r^{\mathbb{P},n+1,m}\cdot dX_r^{c,\mathbb{P}}-\int_0^t\int_E\mathcal{U}_r^{\mathbb{P},n+1,m}(x)\tilde{\mu}_r^{\mathbb{P}}(dx,dr)$$

Now, we just have to put for any $t\in[0,T]$

$$\bar{\mathcal{M}}_t^{\mathbb{P},n+1,m}=\limsup_{s>t,s\in\mathbb{Q}}\mathcal{M}_s^{\mathbb{P},n+1,m}$$

to get a right continuous modification of $\mathcal{M}_t^{\mathbb{P},n+1,m}$.

□

Lemma 2.4. *There are families of subsequences $(n_k^{\mathbb{P},m},\ k\geq1)$ and $(m_i^{\mathbb{P}},\ i\geq1)$ such that the limit $\mathcal{Y}_s^{\mathbb{P}}(\omega)=\lim_{i\to\infty}\lim_{k\to\infty}\mathcal{Y}_s^{\mathbb{P},n_k^{\mathbb{P},m},m_i^{\mathbb{P}}}$ exists for all $s\in[t,T]$, $\mathbb{P}-$almost surely, for every $\mathbb{P}\in\mathcal{P}_t$, and $(s,\omega,\mathbb{P})\longmapsto\mathcal{Y}_s^{\mathbb{P}}(\omega)$ is a measurable function. Moreover, $\mathcal{Y}^{\mathbb{P}}$ provides a solution to the BSDEJ (2.14) for every $\mathbb{P}\in\mathcal{P}_t$.*

Proof. By integrability conditions in (2.4), $(\mathcal{Y}^{\mathbb{P},n,m},\mathcal{Z}^{\mathbb{P},n,m},\mathcal{U}^{\mathbb{P},n,m})_{n\geq1}$ provides a Picard iteration under the $(\mathbb{P},\beta)$-norm, for $\beta>0$ large enough (see [11, 12]), defined by

$$||\varphi||_{\mathbb{P},\beta}^2:=\mathbb{E}^{\mathbb{P}}\left[\sup_{t\leq s\leq T}e^{\beta s}|\varphi_s|^2\right].$$

Hence, $\mathcal{Y}^{\mathbb{P},n,m}$ converges (under the $(\mathbb{P},\beta)$-norm) to some process $\mathcal{Y}^{\mathbb{P},m}$ as $n\longrightarrow\infty$, which solves the BSDEJ (2.14) with the truncated terminal condition ξ^m and truncated generator f^m. Moreover, by Theorem 2.2 $(\mathcal{Y}^{\mathbb{P},m})_{m\geq1}$ is a Cauchy sequence in $\mathbb{D}_{t,\omega}^p(\overline{\mathbb{F}}_+^{\mathbb{P}},\mathbb{P})$. Then using Lemma 3.2 of [13], we can find two families of subsequences $(n_k^{\mathbb{P},m},k\geq1,\mathbb{P}\in\mathcal{P}_t)$ and $(m_i^{\mathbb{P}},i\geq1,\mathbb{P}\in\mathcal{P}_t)$ satisfying the required properties. □

2.5.3 Dynamic programming principle

The dynamic programming principle here is principally based on universally selection measurable theorem. The following result extends Theorem 2.1 of [17] in the case of BSDEJs.

Theorem 2.3. *Suppose that Assumption 2.1 holds true. Then for all $(t,\omega)\in[0,T]\times\Omega$, one has $V_t(\omega)=V_t(\omega_{\cdot\wedge t})$, and $(t,\omega)\to V_t(\omega)$ is $\mathcal{B}([0,T])\otimes\mathcal{F}_T$-universally measurable. Moreover, for all $(t,\omega)\in[0,T]\times\Omega$ and $\mathbb{F}$-stopping time τ taking values in $[t,T]$, we have*

$$V_t(\omega)=\sup_{\mathbb{P}\in\mathcal{P}(t,\omega)}\mathbb{E}^{\mathbb{P}}\big[y_t^{\mathbb{P}}(\tau,V_\tau)\big],$$

where $y_t^{\mathbb{P}}(\tau, V_\tau)$ is obtained from the solution to the following BSDEJ with terminal time τ and terminal condition V_τ,

$$y_t^{\mathbb{P}} = V_\tau + \int_t^\tau \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}, u_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_s^T \int_E u_r^{\mathbb{P}}(e)\tilde{\mu}^{\mathbb{P}}(de, dr) - \int_t^\tau dm_s^{\mathbb{P}},\ \mathbb{P}\text{-}a.s. \tag{2.23}$$

The first step of the proof is to establih the dynamic programming principle of our BSDEJ associated to the 2BSDEJ.

Lemma 2.5. *Let $t \in [0,T], \mathbb{P} \in \mathcal{P}_0, \tau$ be an $\mathbb{F}$-stopping time taking values in $[t,T]$ and $(y^{\mathbb{P}}, z^{\mathbb{P}}, m^{\mathbb{P}}, k^{\mathbb{P}})$ be a solution to the BSDEJ (2.14) under $\mathbb{P}$. Then one has*

$$y_t^{\mathbb{P}}(T, \xi) = y_t^{\mathbb{P}}(\tau, y_\tau^{\mathbb{P}}) = y_t^{\mathbb{P}}(\tau, \mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}]),\ \mathbb{P}\text{-}a.s.$$

Proof.
First, we consider a solution $(y^{\mathbb{P}}, z^{\mathbb{P}}, u^{\mathbb{P}}, m^{\mathbb{P}})$ to the BSDEJ (2.14) associated to (ξ, f) under $\mathbb{P}$ w.r.t. the filtration $\mathbb{F}_+^{\mathbb{P}} = (\mathcal{F}_{s+}^{\mathbb{P}})_{0 \le s \le T}$, then

$$y_t^{\mathbb{P}} = y_\tau^{\mathbb{P}} + \int_t^\tau \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}}, u_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_s^T \int_E u_r^{\mathbb{P}}(e)\tilde{\mu}^{\mathbb{P}}(de, dr) - \int_t^\tau dm_s^{\mathbb{P}},\ \mathbb{P}\text{-a.s.}$$

Taking conditional expectation w.r.t. $\mathcal{F}_\tau^{\mathbb{P}}$ under $\mathbb{P}$, we get $\mathbb{P}$-a.s. that

$$y_t^{\mathbb{P}} = \mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}] + \int_t^\tau \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2} z_s^{\mathbb{P}})ds - \int_t^\tau z_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_s^T \int_E u_r^{\mathbb{P}}(e)\tilde{\mu}^{\mathbb{P}}(de, dr) - \int_t^\tau d\widehat{m}_s^{\mathbb{P}},\ \mathbb{P}\text{-a.s.}$$

where $\widehat{m}_\tau^{\mathbb{P}} := \mathbb{E}^{\mathbb{P}}[m_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}]$, and $\widehat{m}_s^{\mathbb{P}} := m_s^{\mathbb{P}}$ when $s < \tau$. It is apparent that $\widehat{m}^{\mathbb{P}} \in \mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$ and by identification, we deduce that

$$\widehat{m}_\tau^{\mathbb{P}} = m_\tau^{\mathbb{P}} + \mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}] - y_\tau^{\mathbb{P}}.$$

and then $\widehat{m}^{\mathbb{P}}$ is orthogonal to $W^{\mathbb{P}}$ and $\tilde{\mu}^{\mathbb{P}}$ under $\mathbb{P}$. Let consider a BSDEJ associated to $(\mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}], f)$ on $[0, \tau]$, by uniqueness of this solution associated with the properties verifies by $\widehat{m}^{\mathbb{P}}$ and $\widehat{k}^{\mathbb{P}}$, it follows that

$$y_t^{\mathbb{P}}(\tau, y_\tau^{\mathbb{P}}) = y_t^{\mathbb{P}}(\tau, \mathbb{E}^{\mathbb{P}}[y_\tau^{\mathbb{P}} | \mathcal{F}_\tau^{\mathbb{P}}]),\ \mathbb{P}\text{-a.s.}$$

Finally, by definition of the BSDEJ (2.14) it is clear that $y_t^{\mathbb{P}}(T, \xi) = y_t^{\mathbb{P}}(\tau, y_\tau^{\mathbb{P}})$.
We now back to the proof of Theorem 2.3. The proof is exactly the same to Theorem 2.1 of [17] since we have prove the previous Lemma.

2.5.4 Path regularization of the value function

After proving the DPP, we are interested in the right-continuity property that the first component of the solution of the 2BSDEJ (2.5) should verify. The first step is to represent the right-continuity modification of V as a semi-martingale under any $\mathbb{P} \in \mathcal{P}_0$ and then give its decomposition. We define for all $(t, \omega) \in [0, T) \times \Omega$

$$V_t^+ := \overline{\lim_{r \in \mathbb{Q} \cap [0,T], r \downarrow t}} V_t,\ \text{ and } V_T^+ := V_T.$$

Our first objective is to show that V^+ admits right-and left-limits outside a $\mathcal{P}_0$-polar set. Since for all $t \in (0,T], V_t^+$ is by definition $\mathcal{F}_t^{U+}$-measurable, we can deduce that V^+ is in fact $\mathbb{F}^{\mathcal{P}_0+}$-optionnal.
Let $J := (\tau_n)_{n\in\mathbb{N}}$ be a countable family of $\mathbb{F}$-stopping times taking values in $[0,T]$ such that for any $(i,j) \in \mathbb{N}^2$, one has either $\tau_i \leq \tau_j$, or $\tau_i \geq \tau_j$, for every $\omega \in \Omega$. Let $a > b$ and $J_n \subset J$ be a finite subset ($J_n = \{0 \leq \tau_1 \leq \cdots \tau_n \leq T\}$). We denote by $D_a^b(V, J_n)$ the number of downcrossings of the process $(V_{\tau_k})_{1\leq k\leq n}$ from b to a. We then define

$$D_a^b(V,J) := \sup\Big\{ D_a^b(V,J_n) : J_n \subset J, \text{ and } J_n \text{ is a finite set } \Big\}$$

The following lemma follows very closely the related result proved in Lemma $A.1$ of [2].

Lemma 2.6. *Fix some $\mathbb{P} \in \mathcal{P}_0$. Let Assumptions 2.1 and 2.2 hold. Denote by L_f the Lipschitz constant of the generator f. Then, for all $a < b$, there exists a probability measures $\mathbb{Q}$, equivalent to $\mathbb{P}$, such that*

$$\mathbb{E}^{\mathbb{Q}}\Big[D_a^b(V,J)\Big] \leq \frac{e^{L_f T}}{b-a}\mathbb{E}^{\mathbb{Q}}\Big[e^{L_f T}(V_0 \wedge b - a) - e^{-L_f T}(V_T \wedge b - a)^+ \\ + e^{L_f T}(V_T \wedge b - a)^- + e^{L_f T}\int_0^T \big|\widehat{f}^{\mathbb{P}}(a,0)\big| ds\Big]$$

Moreover, outside a $\mathcal{P}_0$-polar set, we have

$$\overline{\lim_{r\in\mathbb{Q}\cap(t,T],r\downarrow t}} V_t(\omega) := \lim_{r\in\mathbb{Q}\cap(t,T],r\downarrow t} V_t(\omega), \text{ and } \overline{\lim_{r\in\mathbb{Q}\cap(t,T],r\uparrow t}} V_t(\omega) := \lim_{r\in\mathbb{Q}\cap(t,T],r\uparrow t} V_t(\omega).$$

To prove the above result, we need to recall some property verifed by V defined at $\mathbb{F}$-stopping times. For any stopping $\mathbb{F}$-stopping times $\tau \geq \sigma$, we have from Theorem 2.3 that

$$V_{\sigma(\omega)}(\omega) = \sup_{\mathbb{P}\in\mathcal{P}(\sigma(\omega),\omega)} \mathbb{E}^{\mathbb{P}}\Big[y^{\mathbb{P}}_{\sigma(\omega)}(\tau, V_\tau)\Big], \tag{2.24}$$

We refer the reader to [4] for the precise details about the proof of this result.

Lemma 2.7. *For any $\mathbb{P} \in \mathcal{P}_0$, for any $\mathbb{F}$-stopping times $0 \leq \sigma \leq \tau \leq T$, we have*

$$\mathbb{E}^{\mathbb{P}^{\sigma(\omega)}_\omega}\Big[y^{\mathbb{P}^{\sigma(\omega)}_\omega}_{\sigma(\omega)}(\tau, V_\tau)\Big] = \mathbb{E}^{\mathbb{P}}\Big[y^{\mathbb{P}}_\sigma(\tau, V_\tau)\Big|\mathcal{F}_\sigma\Big](\omega), \quad \text{for } \mathbb{P}\text{-a.e. } \omega \in \Omega.$$

The following inequality is the consequence of the above equation.

$$V_{\sigma(\omega)} \geq \mathbb{E}^{\mathbb{P}}\Big[y^{\mathbb{P}}_{\sigma(\omega)}(\tau, V_\tau)\Big], \quad \text{for any } \mathbb{P} \in \mathcal{P}(\sigma(\omega),\omega). \tag{2.25}$$

These inequalities allow one to prove Lemma 2.6.

Proof of Lemma 2.6. Without loss of generality, we suppose that $a = 0$. Let $J_n = \{\tau_0, \tau_1, \cdots, \tau_n\}$ with $0 = \tau_0 < \tau_1 < \cdots < \tau_n = T$. For any $i = 1, \ldots, n$, and $\omega \in \Omega$, let the following BSDEJ under $\mathbb{P}_\omega^{\tau_{i-1}(\omega)}$ on $[\tau_{i-1}, \tau_i]$

$$\begin{aligned} y_t^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} &:= V_{\tau_i} + \int_t^{\tau_i} \Big(\widehat{f}_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0} + \lambda_s^i y_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} + \eta_s^i \cdot \widehat{a}_s^{1/2} z_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} + \int_E \gamma_s^i u_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} d\nu^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}(de) \Big) ds \\ &\quad - \int_t^{\tau_i} z_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} - \int_t^{\tau_i} \int_E u_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}(e) \tilde{\mu}^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}(de, dr) - \int_t^{\tau_i} dm_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}, \ \mathbb{P}_\omega^{\tau_{i-1}(\omega)}\text{-a.s.}, \end{aligned}$$

where λ^i and η^i are two bounded processes (by the the Lipschitz constant L_f of f) appearing in the linearization of f due to the Lipschitz property of f. Define the linear BSDEJ,

$$\begin{aligned} \bar{y}_t^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} &:= V_{\tau_i} + \int_t^{\tau_i} \Big(- |\widehat{f}_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0}| + \lambda_s^i \bar{y}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} + \eta_s^i \cdot \widehat{a}_s^{1/2} \bar{z}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} + \int_E \gamma_s^i \bar{u}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} d\nu^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}(de) \Big) ds \\ &\quad - \int_t^{\tau_i} \bar{z}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} - \int_t^{\tau_i} \int_E \bar{u}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}(e) \tilde{\mu}^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}(de, dr) - \int_t^{\tau_i} d\bar{m}_s^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}}, \ \mathbb{P}_\omega^{\tau_{i-1}(\omega)}\text{-a.s.} \end{aligned}$$

It is easy to get

$$\bar{y}_{\tau_{i-1}}^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} = \mathbb{E}_{\tau_{i-1}}^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \Big[L_{\tau_i} \Big(V_{\tau_i} e^{\int_{\tau_{i-1}}^{\tau_i} \lambda_s^i ds} - \int_{\tau_{i-1}}^{\tau_i} e^{\int_{\tau_{i-1}}^{s} \lambda_r^i dr} \Big| \widehat{f}_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)},0} \Big| ds \Big) \Big| \bar{\mathcal{F}}_{\tau_{i-1}}^+ \Big]$$

where

$$L_t := \mathcal{E}\Big(\int_{\tau_{i-1}}^t \eta_s^i \cdot dW_s^{\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} + \int_E \int_{\tau_{i-1}}^t \gamma_s^i d\tilde{\mu}^{\mathbb{P}^{\tau_{i-1}(\omega)}}(de, ds) \Big), \quad t \in [\tau_{i-1}, \tau_i].$$

By Assumption 2.1(iv), for $\mathbb{P}$-almost every where $\omega \in \Omega$, we have $\mathbb{P}_\omega^{\tau_{i-1}(\omega)} \in \mathcal{P}(\tau_{i-1}(\omega), \omega)$. Therefore on one side by comparison principle for supersolution of BSDEs and on another side by (2.24), we have

$$\bar{y}_{\tau_{i-1}}^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \leq y_{\tau_{i-1}}^{i,\mathbb{P}_\omega^{\tau_{i-1}(\omega)}} \leq V_{\tau_{i-1}}(\omega). \tag{2.26}$$

then the we can finish the proof of downcrossing inequality and the second part of le Lemma as in the proof of Lemma 3.2 in [17] (p. 574-575). □

Using Lemma 2.6, we obtain

$$V_t^+ := \lim_{r \in \mathbb{Q} \cap [0,T], r \downarrow t} V_t, \ \text{ outside a } \mathcal{P}_0\text{-polar set},$$

and from this we deduced that V^+ is right-continuous outside a $\mathcal{P}_0$-polar set.

2.5.5 Representation formula

We begin by extend the inequality (2.25) to V^+.

Lemma 2.8. *For any $0 \leq s \leq t \leq T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_s^+ \geq y_s^{\mathbb{P}}(t, V_t^+), \ \mathbb{P}\text{-a.s.}$$

The proof of this Lemma is established in [17] (p. 576). The slight difference is that we need a convergence of the solution y of BSDEJ along a sequence of stopping times which converges. This result is easily establish using the stability result about BSDEJs.
The next result is an extension of the previous result to stopping times and the prove is the same as in [17].

Lemma 2.9. *For any $\mathbb{F}$-stopping times $0 \leq \sigma \leq \tau \leq T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_\sigma^+ \geq y_\sigma^{\mathbb{P}}(\tau, V_\tau^+),\ \mathbb{P}\text{-a.s.}$$

In particular V^+ is càdlàg, $\mathcal{P}_0$-q.s.

Similarly to Lemma 3.5. in [17], we have the following representations.

Lemma 2.10. *For any $\mathbb{F}$-stopping times $0 \leq \sigma \leq \tau \leq T$, for any $0 \leq t \leq T$, for any $\mathbb{P} \in \mathcal{P}_0$, we have*

$$V_\sigma = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(\sigma,\mathbb{P},\mathbb{F})} \mathbb{E}^{\mathbb{P}'}\left[y_\sigma^{\mathbb{P}'}(\tau, y_\tau^{\mathbb{P}'}) \middle| \mathcal{F}_\sigma\right],\ \mathbb{P}\text{-a.s. and } V_t^+ = \operatorname*{ess\,sup}^{\mathbb{P}}_{\mathbb{P}' \in \mathcal{P}_0(t,\mathbb{P},\mathbb{F}_+)} y_t^{\mathbb{P}'}(T, \xi),\ \mathbb{P}\text{-a.s.}$$

where $\mathcal{P}_0(\sigma, \mathbb{P}, \mathbb{F})$ is defined in Section 2.3. In particular, if Assumption 2.2 holds, one has $V^+ \in \mathbb{D}_0^p(\mathbb{F}^{\mathcal{P}_0+})$.

Proof. The proof for the representations is the same as the proof of Proposition 4.2 in [10] and Lemme 3.5 in [17].

The next result shows that V^+ is actually a semi-martingale under any $\mathbb{P} \in \mathcal{P}_0$, gives its decomposition and deduce the existence of a solution to the 2BSDEJ (2.5).

Lemma 2.11. *Let Assumptions 2.1 and 2.2 hold. For any $\mathbb{P} \in \mathcal{P}_0$, there exists $(Z^{\mathbb{P}}, U_s^{\mathbb{P}}, M^{\mathbb{P}}, K^{\mathbb{P}}) \in \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{I}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$ such that*

$$V_t^+ = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(V_s^+, \widehat{a}^{1/2} Z_s^{\mathbb{P}}, U_s^{\mathbb{P}})ds - \int_t^T Z_s^{\mathbb{P}} \cdot dX_s^{c,\mathbb{P}} - \int_E \int_t^T U_s^{\mathbb{P}}(e) d\tilde{\mu}^{\mathbb{P}}(de, ds)$$
$$- \int_t^T dM_s^{\mathbb{P}} + \int_t^T dK_s^{\mathbb{P}},\ t \in [0,T], \mathbb{P}\text{-a.s.}$$

Moreover, there is $\mathbb{F}^{\mathcal{P}_0}$-predictable processes (Z, U) which aggregates the family $(Z^{\mathbb{P}}, U^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0}$ and the quadruple $(V^+, Z, U, (M^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0}, (K^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0})$ is solution to the 2BSDEJ (2.5).

Proof.
The proof will be divided into three steps. The first step will be devote to the semi-martingale decomposition of V^+, in the second one, we will justify the aggregation of the family $(Z^{\mathbb{P}})_{\mathbb{P} \in \mathcal{P}_0}$ and finally we show that the quadruple is solution to the 2BSDEJ (2.5)

(i) Fix some $\mathbb{P} \in \mathcal{P}_0$. Consider the following reflected BSDE on the enlarged space. For $0 \leq t \leq T, \mathbb{P}$-a.s.

$$\check{y}_t^{\mathbb{P}} = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(\check{y}_s^{\mathbb{P}}, \widehat{a}_s^{1/2} \check{z}_s^{\mathbb{P}}, \check{u}_s^{\mathbb{P}})ds - \int_t^T \int_E \check{u}_s^{\mathbb{P}}(e) d\check{\tilde{\mu}}^{\mathbb{P}}(de, ds) - \int_t^T \check{z}_s^{\mathbb{P}} \cdot \widehat{a}_s^{1/2} dW_s^{\mathbb{P}} - \int_t^T d\check{m}_s^{\mathbb{P}} + \int_t^T d\check{k}_s^{\mathbb{P}}$$
$$\check{y}_t^{\mathbb{P}} \geq V_t^+,$$
$$\int_0^T (\check{y}_{t^-}^{\mathbb{P}} - V_{t^-}^+) d\check{k}_t^{\mathbb{P}} = 0.$$

By [11], this BSDEJ is wellposended and $\check{y}^{\mathbb{P}}$ is càdlàg. We claim that $\check{y}^{\mathbb{P}} = V^+$, $\mathbb{P}\otimes\mathbb{P}_0$-a.s. Indeed, we argue by contradiction, and assume without loss of generality that $\check{y}^{\mathbb{P}} > V_0^+$. For each $\epsilon > 0$, denote $\tau_\epsilon := \inf\{t : \check{y}_t^{\mathbb{P}} \le V_t^+ + \epsilon\}$. Then τ_ϵ is an $\check{\mathbb{F}}_+$-stopping time and $\check{y}_{t^-}^{\mathbb{P}} \ge V_{t^-}^+ + \epsilon > V_{t^-}^+$ for all $t \le \tau_\epsilon$. Thus $\check{k}_t^{\mathbb{P}} = \check{k}_{\tau_\epsilon}^{\mathbb{P}}$, $\mathbb{P}$-a.s. for $0 \le t \le \tau_\epsilon$ and thus

$$\check{y}_t^{\mathbb{P}} = \check{y}_{\tau_\epsilon}^{\mathbb{P}} + \int_t^{\tau_\epsilon} \widehat{f}_s^{\mathbb{P}}(\check{y}_s^{\mathbb{P}}, \widehat{a}_s^{1/2}\check{z}_s^{\mathbb{P}}, \check{u}_s^{\mathbb{P}}))ds - \int_t^{\tau_\epsilon} \check{z}_s^{\mathbb{P}}\cdot\widehat{a}_s^{1/2}dW_s^{\mathbb{P}} - \int_t^T\int_E \check{u}_s^{\mathbb{P}}(e)d\check{\tilde{\mu}}^{\mathbb{P}}(de,ds) - \int_t^{\tau_\epsilon} d\check{m}_s^{\mathbb{P}},\ \mathbb{P}\otimes\mathbb{P}_0\text{-a.s.}$$

Therefore,

$$\begin{aligned}\check{y}_t^{\mathbb{P}} - y_t^{\mathbb{P}}(\tau_\epsilon, V_{\tau_\epsilon}^+) = \check{y}_{\tau_\epsilon}^{\mathbb{P}} - \widehat{Y}_{\tau_\epsilon}^+ + \int_t^{\tau_\epsilon}\{\widehat{f}_s^{\mathbb{P}}(\check{y}_s^{\mathbb{P}}, \widehat{a}_s^{1/2}\check{z}_s^{\mathbb{P}}, \check{u}_s^{\mathbb{P}}) - \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2}z_s^{\mathbb{P}}, u_s^{\mathbb{P}})\}ds\\ - \int_t^{\tau_\epsilon}(\check{z}_s^{\mathbb{P}} - z_s^{\mathbb{P}})\cdot\widehat{a}_s^{1/2}dW_s^{\mathbb{P}} - \int_t^T\int_E(\check{u}_s^{\mathbb{P}}(e) - u_s^{\mathbb{P}}(e))d\check{\tilde{\mu}}^{\mathbb{P}}(de,ds) - \int_t^{\tau_\epsilon} d(\check{m}_s^{\mathbb{P}} - m_s^{\mathbb{P}}).\end{aligned}$$

where $(y^{\mathbb{P}}, z^{\mathbb{P}}, u^{\mathbb{P}}, m^{\mathbb{P}})$ is a solution to the BSDEJ (2.14). Using linearization argument implies that there exists two processes λ and η such that

$$\widehat{f}_s^{\mathbb{P}}(\check{y}_s^{\mathbb{P}}, \widehat{a}_s^{1/2}\check{z}_s^{\mathbb{P}}, \check{u}_s^{\mathbb{P}})) - \widehat{f}_s^{\mathbb{P}}(y_s^{\mathbb{P}}, \widehat{a}_s^{1/2}z_s^{\mathbb{P}}, u_s^{\mathbb{P}})) = \lambda_s(\check{y}_s^{\mathbb{P}} - y_s^{\mathbb{P}}) + \eta_s\cdot\widehat{a}_s^{1/2}(\check{z}_s^{\mathbb{P}} - z_s^{\mathbb{P}}) + \int_E \gamma_s(\check{u}_s^{\mathbb{P}}(e) - u_s^{\mathbb{P}}(e))d\nu^{\mathbb{P}}(de)$$

and

$$\begin{aligned}\check{y}_0^{\mathbb{P}} - y_0^{\mathbb{P}} = e^{\int_0^{\tau_\epsilon}\lambda_s ds}(\check{y}_{\tau_\epsilon}^{\mathbb{P}} - V_{\tau_\epsilon}^+) - \int_0^{\tau_\epsilon} e^{\int_0^s\lambda_s ds}(\check{z}_s^{\mathbb{P}} - z_s^{\mathbb{P}})\cdot\widehat{a}_s^{1/2}\{dW_s^{\mathbb{P}} - \eta_s ds\}\\ - \int_0^{\tau_\epsilon} e^{\int_0^s\lambda_s ds}d(\check{m}_s^{\mathbb{P}} - m_s^{\mathbb{P}}) - \int_0^{\tau_\epsilon} e^{\int_0^s\lambda_s ds}dk_s^{\mathbb{P}}.\end{aligned}$$

Then, there exists a probability measure $\mathbb{Q}_{\mathbb{P}}$ equivalent to $\mathbb{P}$ such that

$$\begin{aligned}\check{y}_0^{\mathbb{P}} - y_0^{\mathbb{P}} &= \mathbb{E}^{\mathbb{Q}_{\mathbb{P}}}\Big[e^{\int_0^{\tau_\epsilon}\lambda_s ds}(\check{y}_{\tau_\epsilon}^{\mathbb{P}} - V_{\tau_\epsilon}^+) - \int_0^{\tau_\epsilon} e^{\int_0^s\lambda_s ds}d\tilde{k}_s^{\mathbb{P}}\Big]\\ &\le \mathbb{E}^{\mathbb{Q}_{\mathbb{P}}}\big[e^{\int_0^{\tau_\epsilon}\lambda_s ds}(\check{y}_{\tau_\epsilon}^{\mathbb{P}} - V_{\tau_\epsilon}^+)\big]\\ &\le C\mathbb{E}^{\mathbb{Q}_{\mathbb{P}}}\big[\check{y}_{\tau_\epsilon}^{\mathbb{P}} - V_{\tau_\epsilon}^+\big] \le C\epsilon.\end{aligned}$$

where $C > 0$ is a constant only depending on the Lipschitz constant of f. Note that this equivalent to $\check{y}_0^{\mathbb{P}} \le y_0^{\mathbb{P}}(\tau_\epsilon, V_{\tau_\epsilon}^+) + C\epsilon$. However by Lemma 2.8, we know that $y_0^{\mathbb{P}}(\tau_\epsilon, V_{\tau_\epsilon}^+) \le V_0^+$, which contradicts the fact that $\check{y}_0^{\mathbb{P}} > V_0^+$.
For some $(Z^{\mathbb{P}}, U^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0} \subset \mathbb{H}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{J}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$, and $(M^{\mathbb{P}}, K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0} \subset \mathbb{M}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P}) \times \mathbb{I}_0^p(\mathbb{F}_+^{\mathbb{P}}, \mathbb{P})$

$$\begin{aligned}V_t^+ = \xi + \int_t^T \widehat{f}_s^{\mathbb{P}}(V_s^+, \widehat{a}^{1/2}Z_s^{\mathbb{P}}, U_s^{\mathbb{P}})ds - \int_t^T Z_s^{\mathbb{P}}\cdot dX_s^{c,\mathbb{P}} - \int_E\int_t^T U_s^{\mathbb{P}}(e)d\tilde{\mu}^{\mathbb{P}}(de,ds)\\ - \int_t^T dM_s^{\mathbb{P}} + \int_t^T dK_s^{\mathbb{P}},\ t\in[0,T],\mathbb{P}\text{-a.s.}\end{aligned}$$

(ii) By Karandikar [8], since V^+ is a càdlàg semi-martingale, we can define a universal process denote by $\langle V^+, X\rangle$ which coincides with the quadratic co-variation of V^+ and X under each probability $\mathbb{P}\in\mathcal{P}_0$. In particular, the process $\langle V^+, X\rangle$ is $\mathcal{P}_0$-quasi-surely continuous and hence

is $\mathbb{F}^{\mathcal{P}_0+}$-predictable (or equivalently $\mathbb{F}^{\mathcal{P}_0}$-predictable). Similar to the proof of Theorem 2.4 in [16], we can then define a universal $\mathbb{F}^{\mathcal{P}_0}$-predictable process Z and $\mathbb{F}^{\mathcal{P}_0}$-predictable process U such that

$$Z_t := \widehat{a}_t^{\oplus} \frac{d\langle V^+, X\rangle_t}{dt}$$
$$\Delta[V^+, X]_t = U_t(\Delta X_t)\Delta X_t$$

where $\widehat{a}_t^{\oplus}$ represents the Moore-Penrose pseudoinverse of $\widehat{a}_t$. In particular, Z (respectively U) aggregates the family $(Z^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ (respectively $(U^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$).

(*iii*) Fix $t \in [0,T]$ and $\mathbb{P} \in \mathcal{P}_0$. According to the two previous steps, it remains to show that the families $(K^{\mathbb{P}})_{\mathbb{P}\in\mathcal{P}_0}$ satisfies the minimality condition (2.9). But the proof is the same as the proof of Proposition 4.9 in [9] (p. 13-14) so we omit it.

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