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Introduction to topological string theory

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Introduction to topological strings

Topological string theory is not only a simplified model of real strings, but

- captures various corners of real string theory
- has applications to BPS black holes
- and to SUSY gauge theories
- has numerous relations to mathematics (mirror symmetry, computation of topological invariants...)

Plan:

1. Preliminaries
 - a) Calabi-Yau manifolds
 - b) Cohomological field theories
2. Topological twist in $N=(2,2)$ σ -models
3. Correlation functions in topological string theory
4. Holomorphic anomaly and background independence.

Calabi - Yau manifolds

- Calabi-Yau manifold - $2n$ real dimensional with $SU(n)$ holonomy group

(Parallel transport operators $U_\gamma = \mathcal{P} e^{\int \omega}$ form a group
 $\gamma = \gamma_1 \vee \gamma_2 \Rightarrow U_\gamma = U_{\gamma_1} U_{\gamma_2}$ $\underbrace{\gamma_1 \vee \gamma_2}$
 γ - closed curve $\rightarrow U_\gamma$ - holonomy group $\rightarrow \text{Hol}(\omega) \subseteq O(2n)$)

This is not very practical definition. One can show:
 $CY \iff$ Kähler and Ricci flat

\downarrow
 $U(n)$ holonomy

\uparrow
 $c_1(\mathcal{M}) = 0$

$R_{\mu\nu}$ plays the role of the curvature for $U(1)$ in $U(n)$.

Inverse statement - famous theorem of Yau conjectured by Calabi: there is a unique Ricci flat metric for a Kähler manifold with $c_1 = 0$.

- $(1,0)$ -forms belong to $[n]$ - fund. repr. of $SU(n)$
 $\underbrace{[n] \otimes \dots \otimes [n]}_{n \text{ times}} = [1] \oplus \dots \Rightarrow \exists$ covariantly constant
 $\underbrace{}_{\text{singlet} \sim \epsilon_{i_1 \dots i_n}} \quad \Omega \in H^{n,0}(\mathcal{M})$

In fact: $c_1 = 0 \iff \nabla \Omega = 0 \iff \Omega$ is nowhere vanishing

- CY is characterized by 2 compatible structures:

ω
Kähler str.

Ω
complex str.

gives the metric $g(u,v) = \omega(u,v)$

determines holomorphic coordinates $\Omega \sim dz^1 \dots dz^n$

Compatibility: $d\omega = d\Omega = 0$

$$\Omega \wedge \omega = 0, \quad \Omega \wedge \bar{\Omega} \sim \underbrace{\omega \wedge \dots \wedge \omega}_n$$

• Examples

$n=1$: T^2 (since curvature must vanish)

$n=2$: $K3$ - the only 4d CY , realized as $x^4 + y^4 + z^4 + w^4 = 0$ in $\mathbb{C}P^3$

$n=3$: most interesting case

Hodge numbers

• On complex manifolds $d = \partial + \bar{\partial}$, $\partial^2 = \bar{\partial}^2 = 0$

Dolbeault cohomology

$$\Lambda^k = \bigoplus_{p+q=k} \Lambda^{(p,q)} \leftarrow \begin{array}{l} p \text{ holomorphic} \\ q \text{ anti-holomorphic} \\ \text{indices} \end{array}$$

$$H_{\bar{\partial}}^{p,q} = \frac{\text{Ker}(\bar{\partial}(\Lambda^{(p,q)}))}{\text{Im}(\bar{\partial}(\Lambda^{(p,q-1)})}) = \frac{\bar{\partial}\text{-closed}}{\bar{\partial}\text{-exact}}$$

$$h^{p,q} = \dim H_{\bar{\partial}}^{p,q} \text{ - Hodge number} \\ = \{ \text{harmonic forms of } \Delta_{\bar{\partial}} = \bar{\partial}\bar{\partial}^{\dagger} + \bar{\partial}^{\dagger}\bar{\partial} \}$$

For Kähler manifolds $\Delta_{\bar{\partial}} = \Delta_{\partial} = \frac{1}{2} \Delta_d$

$$\Rightarrow H_{\bar{\partial}}^{p,q} = H_{\partial}^{p,q} \equiv H^{p,q}$$

$$H^k = \bigoplus_{p+q=k} H^{p,q}, \quad b_k = \sum_{p=0}^k h^{p,k-p}$$

$$h^{p,q} = h^{q,p} = h^{n-p,n-q}$$

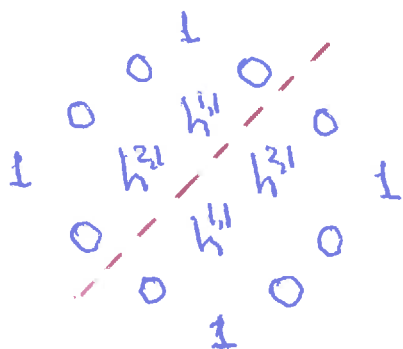
• For CY manifolds

$$h^{n,0} = h^{0,n} = 1 \quad \leftarrow \exists! \Omega$$

$$h^{0,0} = h^{n,n} = 1 \quad \leftarrow \text{for any complex manifold}$$

$$h^{k,0} = h^{0,k} = 0 \quad \text{for } 1 \leq k \leq n-1$$

• Hodge diamond for CY with $n=3$



- only 2 unfixed numbers: $h^{2,1}, h^{1,1}$

- Euler characteristic

$$\chi = 2(h^{1,1} - h^{2,1})$$

- Symmetry $h^{1,1} \leftrightarrow h^{2,1}$ - mirror symmetry

for any CY there is its mirror image (except rigid CYs with $h^{2,1} = 0$)

Moduli spaces of CY

Let us fix topology of a CY

What are its possible continuous deformations?

→ Moduli space

deformations of Kähler str. \times deformations of complex str.

Kähler moduli space \mathcal{M}_K

For a fixed complex structure, $\mathcal{M}_K \subset H^{1,1}(\mathcal{M})$

such that $\int_{\gamma_{2k}} \omega^k > 0$ for all $2k$ -cycles γ_{2k} , $k=1, \dots, n$

⇒ \mathcal{M}_K - Kähler cone

locally these conditions are irrelevant, so

$T(\mathcal{M}_K) = H^{1,1}(\mathcal{M})$
space of infinitesimal deformations

Complex structure moduli space \mathcal{M}_c

Complex structure is determined by $\Omega = f(z) dz^1 \wedge dz^2 \wedge dz^3$

Change of compl. str. \leftrightarrow change of $z^i \rightarrow z^i + \varepsilon_j^i \bar{z}^j$

⇒ $\delta \Omega = \varepsilon_j^i (\bar{z}^j \partial_i f dz^1 \wedge dz^2 \wedge dz^3 + \frac{1}{2} f \varepsilon_{ijk} d\bar{z}^j \wedge dz^k \wedge dz^l)$
(3,0)-form (2,1)-form

⇒ deformations are classified by (2,1)-forms

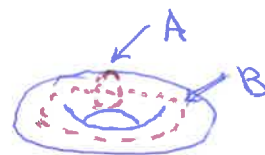
⇒ locally $\mathcal{M}_c \sim H^{2,1}(\mathcal{M})$

On both moduli spaces one can define natural metrics.

Metric on \mathcal{M}_c

- Let (A^α, B_α) - a canonical basis in $H^3(\mathcal{M})$
 $\alpha = 0, \dots, h^{2,1}$

$$\begin{aligned} A^\alpha \cap A^\beta &= B_\alpha \cap B_\beta = 0 \\ A^\alpha \cap B_\beta &= \delta^\alpha_\beta \end{aligned}$$



Let $(\alpha_\alpha, \beta^\alpha) \in H^3$ be a dual basis

$$\int_{A^\alpha} \alpha_\beta = +\delta^\alpha_\beta, \quad \int_{B_\alpha} \beta^\beta = -\delta^\alpha_\beta, \quad \int_{B_\alpha} \alpha_\beta = \int_{A^\alpha} \beta^\beta = 0$$

(The basis can be found by defining symplectic structure on H^3 : $\{\alpha, \beta\} = \int \alpha \wedge \beta$ and bringing it to the canonical Darboux^M form.)

- In this basis the complex structure of CY is captured by $2h^{2,1} + 2$ parameters $\Omega = X^\alpha \alpha_\alpha - F_\alpha \beta^\alpha$ or $\int X^\alpha = \int_{A^\alpha} \Omega$
 But there are only $h^{2,1}$ independent parameters $(F_\alpha = + \int_{B_\alpha} \Omega)$
 \rightarrow there must be relations between X^α and F_α

$$0 = \int_{\mathcal{M}} \Omega \wedge \frac{\partial \Omega}{\partial X^\alpha} = \int_{\mathcal{M}} (X^\beta \alpha_\beta - F_\beta \beta^\beta) \wedge (\alpha_\alpha - \partial_{X^\alpha} F_\beta \beta^\beta)$$

\Downarrow Riemann bilinear identity $\int_{\mathcal{M}} \alpha \wedge \beta = \sum_{\alpha} \left(\int_{A^\alpha} \alpha \int_{B_\alpha} \beta - \int_{B_\alpha} \alpha \int_{A^\alpha} \beta \right)$

$$F_\alpha - X^\beta \partial_{X^\alpha} F_\beta = 0$$

$$F_\alpha = \frac{1}{2} \partial_{X^\alpha} (X^\beta F_\beta) \text{ or } F_\alpha = \partial_{X^\alpha} F \text{ where } F = \frac{1}{2} X^\alpha F_\alpha$$

\Downarrow
F-homogeneous of degree 2.

F-prepotential

X^α -homogeneous coordinates

$z^\alpha = X^\alpha / X^0$ - projective coordinates on \mathcal{M}_c
 $\alpha = 1, \dots, h^{2,1}$

- The metric on \mathcal{M}_c is defined to be Kähler with $K = -\log(i \int_{\mathcal{M}} \Omega \wedge \bar{\Omega}) \Rightarrow K = -\log[i(X^\alpha F_\alpha - X^\alpha F_\alpha)]$
 \rightarrow determined by a single holomorphic function
 $\rightarrow \mathcal{M}_c$ - (projective or local) special Kähler manifold

Metric on M_K

- Denote the Kähler 2-form by γ
 $\rightarrow \gamma_m^n = g^{nk} \gamma_{mk}$ - complex structure

Basis in even cohomology:

$\omega_i \in H^2(M)$, $\tilde{\omega}_i^j \in H^4(M)$, $\omega_{vol} \in H^6(M)$
 $i=1, \dots, h^{1,1}$ volume form

$\omega_i \wedge \tilde{\omega}^j = \delta_i^j \omega_{vol}$
 $\omega_i \wedge \omega_j = R_{ijk} \omega_k$
 triple intersection numbers
 $R_{ijk} = \int_M \omega_i \wedge \omega_j \wedge \omega_k$

- In this basis
 $\gamma = t^i \omega_i$, $t^i \in \mathbb{R}$ - coordinates on M_K

Let us complexify M_K : $t^i \rightarrow v^i \in \mathbb{C}$

In string theory this happens due to the B-field

$\gamma \rightarrow (B + i\gamma) = (b^i + it^i) \omega_i \equiv v^i \omega_i$

- The metric on M_K is defined to be Hermitian (i.e. $g_{i\bar{j}} = g_{\bar{j}i} = 0$)
 with $g_{i\bar{j}} = \frac{1}{R} \int_M \omega_i \wedge \omega_{\bar{j}}$

where $R = \frac{1}{6} \int_M \gamma \wedge \gamma \wedge \gamma$

notations
 $R_i = \partial_i R = \frac{1}{2} \int_M \omega_i \wedge \gamma \wedge \gamma$
 $R_{ij} = \partial_i \partial_{\bar{j}} R = \int_M \omega_i \wedge \omega_{\bar{j}} \wedge \gamma$
 $\partial_i \partial_{\bar{j}} \partial_{\bar{k}} R = R_{ijk}$

One can prove that

$\ast \omega_i = -\left(\gamma \wedge \omega_i - \frac{R_i}{2R} \gamma \wedge \gamma\right)$

\Downarrow
 $g_{i\bar{j}} = -\left(\frac{R_{ij}}{R} - \frac{R_i R_{\bar{j}}}{R^2}\right) = -\partial_{t^i} \partial_{\bar{t}^j} \log(\delta R) = \partial_{v^i} \partial_{\bar{v}^j} K$

where $K = -\log \delta R$ - Kähler potential

Moreover

$K = -\log [i(\bar{X}^I F_I - X^I \bar{F}_I)] \Big|_{\substack{X^i = v^i \\ X^0 = 1}} \quad I=0, \dots, h^{1,1}$

where $F = -\frac{1}{6} \epsilon_{0^i} R_{ijk} X^i X^j X^k$

$\rightarrow M_K$ - projective special Kähler manifold
 with prepotential F

Cohomological field theories

- We are interested in topological theories which do not depend either on metric or some background structures such as complex structure

Note: if the topological theory is invariant under general coordinate transformations, then $\langle O_1(x_1) \dots O_n(x_n) \rangle$ do not depend on x_i (since trans-n of x_i is compensated by tr. of $g_{\mu\nu}$)

- There are 2 types of topological theories

1) Schwarz-type

S and O_i are independent of the metric

Example: Chern-Simons theory $S = \int_{M_3} \text{Tr}(A \wedge dA - \frac{2}{3} A \wedge A \wedge A)$

2) Witten-type or cohomological

Let \exists fermionic operator Q such that it generates a symmetry and 1) nilpotence $Q^2 = 0$

2) physical operators are Q -closed: $[Q, O_i] = 0$

3) Q -symmetry is not spontaneously broken, i.e. $Q|\Psi\rangle = 0$ _{phys}

$\Rightarrow O_i \simeq O_i + \{Q, A_i\}$

\Rightarrow physical operators $\leftrightarrow Q$ -cohomology

2) Energy-momentum tensor is Q -exact: $T_{ab} = \frac{\delta S}{\delta h^{ab}} = \{Q, G_{ab}\}$

the most non-trivial condition, which ensures that (provided O_i are metric independent) correlation functions are invariant under deformations

\rightarrow This generalizes to any structure $\sum \frac{\delta}{\delta h^{ab}} \langle O_1(x_1) \dots O_n(x_n) \rangle = 0$ appearing only in Q -exact terms.

Example: $L = \{Q, \Psi\} \Rightarrow$ independence of $\hbar \Rightarrow$ classical approximation is exact.

Families of operators

$\int O^{(0)}$ - scalar physical operator. Use 1) $\Rightarrow P_a = \{Q, G_a\}$

$\Rightarrow \frac{d}{dx^a} O^{(0)} = i[P_a, O^{(0)}] = i[\{Q, G_a\}, O^{(0)}] = \pm i \{ \{G_a, O^{(0)}\}, Q \} - i \{ O^{(0)}, \{Q, G_a\} \}$
 $= \{Q, O_a^{(1)}\}$ where $O_a^{(1)} = i \{G_a, O^{(0)}\} \Rightarrow dO^{(0)} = \{Q, O^{(1)}\}$ ₀

Similarly $dO^{(k)} = \{Q, O^{(k+1)}\}$ $0 \leq k < n$, $dO^{(n)} = 0$, $n = \dim M$ 1-form

$\Rightarrow \int_{\delta P} O^{(p)}$ - physical operator (non-local) and $\int_M O^{(n)}$ can be added to the action

Topological twist

- Consider a 2d non-linear σ -model with $N=(2,2)$ SUSY (In 2d, the Lorentz group is $SO(2)=U(1) \Rightarrow s=\frac{1}{2}$ -reducible rep.)

$$S = \int d^2z d^4\theta K(\Phi^i, \bar{\Phi}^i) = - \int d^2z \frac{\delta K}{\delta \varphi^i \delta \bar{\varphi}^i} \partial^{\alpha} \varphi^i \partial_{\alpha} \bar{\varphi}^i + \text{fermions}$$

Φ^i -chiral $N=2$ super field ($\bar{\partial}_{\pm} \Phi^i = 0$, $\bar{\partial}_{\pm} = -\frac{\partial}{\partial \theta^{\pm}} + i\theta^{\pm} \partial_{\pm}$)

$$\varphi^i + \psi^i_+ \theta^+ + \psi^i_- \theta^- + F \theta^+ \theta^-$$

$$Q_{\pm} = \partial_{\theta^{\pm}} + i\bar{\theta}^{\pm} \partial_{\pm}$$

$K(\varphi, \bar{\varphi})$ - Kähler potential on target space (required by $(2,2)$ SUSY)

Symmetry algebra

$$\{Q_{\pm}, \bar{Q}_{\pm}\} = -2i \partial_{\pm} = P \pm H$$

M -Lorentz generator
 \pm -its eigenspaces

$$(Q_{+})^* = \bar{Q}_{-}$$

$$[M, Q_{\pm}] = \pm Q_{\pm}, [M, \bar{Q}_{\pm}] = \pm \bar{Q}_{\pm}, [M, H] = 2P, [M, 2P] = 2H$$

vector and axial
 R-symmetries

$$[V, Q_{\pm}] = Q_{\pm}, [V, \bar{Q}_{\pm}] = -\bar{Q}_{\pm}$$

$$[A, Q_{\pm}] = \pm Q_{\pm}, [A, \bar{Q}_{\pm}] = \mp \bar{Q}_{\pm}$$

$$\Rightarrow Q_A = \bar{Q}_+ + Q_- : \{Q_A, Q_+ - \bar{Q}_-\} = 2M, \{Q_A, Q_+ + \bar{Q}_-\} = 2P, Q_A^2 = 0$$

$$Q_B = \bar{Q}_+ + \bar{Q}_- : \{Q_B, Q_+ - Q_-\} = 2H, \{Q_B, Q_+ + Q_-\} = 2P, Q_B^2 = 0$$

$\rightarrow Q_{A/B}$ satisfy all requirements (nilpotence and exactness of T_{ab})

But in the presence of hol, SUSY requires the existence of covariantly constant spinor - does not exist on arbitrary Riemann surface.

- Idea: change transformation properties so that Q be a scalar \leftarrow achieved by twisting
 declare a new Lorentz generator

1) $M_A = M + V$

2) $M_B = M + A$

$$[M_A, Q_+] = 2Q_+, [M_A, Q_-] = [M_A, \bar{Q}_+] = 0, [M_A, \bar{Q}_-] = -2\bar{Q}_-$$

$$[M_B, Q_{\pm}] = \pm 2Q_{\pm}, [M_B, \bar{Q}_{\pm}] = 0$$

\Downarrow
 Q_A -scalar

\Downarrow
 Q_B -scalar

- The twist removes conformal anomaly!

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1) \delta_{m+n}$$

$$[L_m, Y_n] = -n Y_{m+n}$$

$$[Y_m, Y_n] = \frac{c}{3} m \delta_{m+n}$$

$M = 2\pi i(L_0 - \bar{L}_0)$
 where $\bar{\partial}Y = 0$ conserved current (like V or A)

$$\Rightarrow \tilde{T} = T + \frac{1}{2} \partial Y \Rightarrow \bar{\partial} \tilde{T} = 0 \text{ and } \tilde{L}_m = L_m - \frac{1}{2} (m+1) Y_m \Rightarrow [\tilde{L}_m, \tilde{L}_n] = (m-n) \tilde{L}_{m+n}$$

\rightarrow no restriction on the dimension of space (just here)

A-model

- Fermions after twisting

$$\psi_+^i \rightarrow \psi_+^i \in \Lambda^{(1,0)} \otimes \varphi^*(T^{(1,0)}\mathcal{M}), \quad \psi_-^i \rightarrow \chi^i \in \varphi^*(T^{(1,0)}\mathcal{M}) + \text{compl. conj.}$$

↑ pull back of $\varphi: \Sigma \rightarrow \mathcal{M}$

The action

$$S = -2t \int d^2z [g_{i\bar{j}} \partial_z \psi^i \partial_{\bar{z}} \bar{\psi}^{\bar{j}} + g_{i\bar{j}} \partial_{\bar{z}} \psi^i \partial_z \bar{\psi}^{\bar{j}} + i g_{i\bar{j}} \psi^i \nabla_{\bar{z}} \bar{\psi}^{\bar{j}} + i g_{i\bar{j}} \bar{\psi}^{\bar{j}} \nabla_z \psi^i + \frac{t}{2} R_{i\bar{j}k\bar{l}} \psi^i \bar{\psi}^{\bar{j}} \chi^k \bar{\chi}^{\bar{l}}] = -t \int d^2z [i \iota_{Q_A} \psi \bar{\psi} - 2g_{i\bar{j}} (\partial_z \psi^i \partial_{\bar{z}} \bar{\psi}^{\bar{j}} - \partial_{\bar{z}} \psi^i \partial_z \bar{\psi}^{\bar{j}})]$$

where $\psi = g_{i\bar{j}} (\psi^i \partial_{\bar{z}} \bar{\psi}^{\bar{j}} + \partial_z \psi^i \bar{\psi}^{\bar{j}})$

Second term = $t \int_{\Sigma} \varphi^*(\omega) = t \int_{\varphi(\Sigma)} \omega \leftarrow$ Kähler form on \mathcal{M}
 $\downarrow d\omega = 0$

→ dependence on the metric h^{ab} only through the Q_A -exact term

depends only on homology class of $\varphi(\Sigma)$

topological theory on Σ

- Dependence on complex structure is also only in the Q_A -exact term → no dependence on it in correlation functions
- Dependence on t : exact term and e^{-tS}
 → in the fixed homological class, the classical limit $t \rightarrow 0$ is exact
 → path integral localizes on Q_A -exact configurations
 () $Q: \theta \rightarrow \theta + \varepsilon \Rightarrow \int d\theta e^{-S} = 0$ except fixed points $Q\varphi = 0$

In our case $\iota_{Q_A} \psi \bar{\psi} = \chi^i, \iota_{Q_A} \chi^i = 0, \iota_{Q_A} \bar{\psi}^{\bar{j}} = i \bar{\partial} \psi^i - \chi^i \Gamma_{i\bar{j}k}^{\bar{l}} \bar{\psi}^{\bar{k}}$
 $\Rightarrow \partial_{\bar{z}} \psi^i = \partial_z \bar{\psi}^{\bar{i}} = 0 \rightarrow$ φ -holomorphic map

A-model $\rightarrow \sum_{\beta \in H_2^+(\mathcal{M})} e^{-t\omega \cdot \beta} \int \text{moduli space of holomorphic maps } \Sigma \rightarrow \mathcal{M}$

Physical operators

$O_a = a_{i_1 \dots i_p \bar{j}_1 \dots \bar{j}_q}(\varphi) \chi^{i_1} \dots \chi^{i_p} \bar{\chi}^{\bar{j}_1} \dots \bar{\chi}^{\bar{j}_q}$ (ψ 's have indices on Σ)

From the action of $Q_A: \chi^i \leftrightarrow d\psi^i \Rightarrow$ $\{Q_A, O_a\} = -O da$

Thus, physical operators \leftrightarrow de Rham cohomology
 In particular, $\mathcal{O} \in \Lambda^{(p,q)}$ → O_a -deformation of Kähler str.

A model \leftrightarrow theory of Kähler structures

B-model

• Fermions after twisting

$$\psi_{\pm}^i \rightarrow \rho_{\pm}^i \in \Lambda^{(1,0)} \otimes \varphi^*(T^{(1,0)}\mathcal{M}), \quad \bar{\psi}_{\pm}^i \rightarrow \eta^{\bar{i}}, \quad \theta_i \in \varphi^*(T^{(0,1)}\mathcal{M})$$

The action

$$S = -t \int d^2z \left[2g_{i\bar{j}} \partial_z \psi^i \partial_{\bar{z}} \bar{\psi}^{\bar{j}} + 2g_{i\bar{j}} \partial_{\bar{z}} \psi^i \partial_z \bar{\psi}^{\bar{j}} + i g_{i\bar{j}} \eta^{\bar{j}} (\nabla_{\bar{z}} \rho_z^i + \nabla_z \rho_{\bar{z}}^i) + i \theta_i (\nabla_{\bar{z}} \rho_z^i - \nabla_z \rho_{\bar{z}}^i) + \frac{1}{2} R_{i\bar{j}k\bar{l}} \rho_z^i \rho_{\bar{z}}^{\bar{k}} \eta^{\bar{j}} \theta_l \right] =$$

$$= -t \int d^2z \left[i \{Q_B, \Psi\} + i \theta_i (\nabla_{\bar{z}} \rho_z^i - \nabla_z \rho_{\bar{z}}^i) + \frac{1}{2} R_{i\bar{j}k\bar{l}} \rho_z^i \rho_{\bar{z}}^{\bar{k}} \eta^{\bar{j}} \theta_l \right]$$

where $\Psi = g_{i\bar{j}} (\rho_z^i \partial_{\bar{z}} \bar{\psi}^{\bar{j}} + \rho_{\bar{z}}^i \partial_z \bar{\psi}^{\bar{j}})$

antisym-c in (z, \bar{z})

↓
differential form

• This implies (up to Q_B -exact term):

- 1) no dependence on metric → topological
- 2) no dependence on Kähler structure (can be shown)
- 3) dependence on t can be absorbed into θ_i
→ localization on $\partial_z \psi^i = \partial_{\bar{z}} \psi^i = 0 \Rightarrow \psi^i = \text{const}$

$$\left. \begin{aligned} \{Q_B, \psi^i\} &= \{Q_B, \theta_i\} = \{Q_B, \eta^{\bar{i}}\} = 0 \\ \{Q_B, \bar{\psi}^{\bar{i}}\} &= -\eta^{\bar{i}}, \quad \{Q_B, \rho_z^i\} = -\partial_z \psi^i \end{aligned} \right\} \curvearrowright$$

↓
 $\varphi(\bar{z}) = \text{point}$

↓
B-model - local field theory on \mathcal{M}

• Physical operators

$$O_a = a_{i_1 \dots i_p} \bar{a}_{\bar{j}_1 \dots \bar{j}_p} \eta^{\bar{i}_1} \dots \eta^{\bar{i}_p} \theta_{\bar{j}_1} \dots \theta_{\bar{j}_p}$$

From the action of Q_B : $\eta^{\bar{i}} \leftrightarrow \bar{\partial} \psi^{\bar{i}}, \theta_i \leftrightarrow \frac{\partial}{\partial \psi^i} \Rightarrow \{Q_B, O_a\} = -O_{\bar{\partial} a}$

Thus, physical operators $\leftrightarrow (0, q)$ cohomologies valued in $\Lambda^p(T^{(1,0)}\mathcal{M})$

In particular, $a \in \Lambda^{(0,1)} \otimes T^{(1,0)}\mathcal{M} \rightarrow O_a$ -deformation of complex structure ($R_{i\bar{j}k\bar{l}} a_{\bar{e}}^k \in H^{2,1}$)

B-model \leftrightarrow theory of complex structures

Anomalies

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Problem! the axial $U(1)$ symmetry can be anomalous
The anomaly arises due to zero modes of the Dirac operator
→ the measure in the path integral has a non-zero change

$$2K = 2(\dim \text{Ker } \not{D}_+ - \dim \text{Ker } \not{D}_-)$$

This is an index independent of continuous def-ns.
Can be evaluated by Atiyah-Singer theorem

- In the original $N=(2,2)$ σ -model

$$K = \int_{\varphi(\Sigma)} c_1(\mathcal{M}) \Rightarrow \text{anomaly disappears only on } \underline{\mathcal{M}-CY}$$

- A-model: $K = d(1-g) + \int_{\varphi(\Sigma)} c_1(\mathcal{M})$ $d = \dim_{\mathbb{C}} \mathcal{M}$
 g - genus of Σ

Even for $\mathcal{M}-CY$, it seems that

the model exists only on torus ($g=1$)

(and on sphere ~~for~~ correlation functions of operators of charge $-2d \rightarrow 2d$ -forms, i.e. top-forms)

Topological strings and anomaly cancellation

- In string theory we have to integrate over w.s. metrics.

$$\int \mathcal{D}h_{ab} \rightarrow \int \mathcal{D}m \quad \int \mathcal{D}_{(g)}h_{ab}$$

moduli space of Σ_g conformally equivalent metrics

Moduli of Σ_g - changes in complex structure (like τ on torus)

$$\# \text{moduli} \equiv B_g = \begin{cases} 0, & g=0 \\ 1, & g=1 \\ 3g-3, & g>1 \end{cases}$$

$$dz \rightarrow dz + m^\alpha \mu_{\alpha, \bar{z}}^z d\bar{z}$$

Beltrami differentials

- The measure on the moduli space \mathcal{M}_g

$$\int_{\mathcal{M}_g} \prod_{\alpha=1}^{B_g} (dm^\alpha d\bar{m}^\alpha \int_{\Sigma} a_{zz} (\mu_\alpha)^z_{\bar{z}} \int_{\Sigma} a_{\bar{z}\bar{z}} (\mu_\alpha)^{\bar{z}}_z)$$

Analogue in string theory:

$$a_{zz} \leftrightarrow b_{zz} \text{-ghost}$$

$$\{Q_{BRST}, b\} = T$$

needed to get (1,1)-form and allows to keep the theory topological
 $\{Q, a\} = T \Rightarrow \frac{\delta}{\delta h_{ab}}$ - total derivative on \mathcal{M}_g

- $a\bar{c}$ has axial charge +2

\Downarrow

total charge of $\langle O_1, \dots, O_n \rangle$ on CY

$$2d(1-g) + 2B_g + \sum_{k=1}^n \text{deg } O_k = \begin{cases} g=0: & 2d \\ g=1: & 2 \\ g \geq 2: & 2(d-3)(1-g) \end{cases} - \sum_{k=1}^n \text{deg } O_k$$

\Rightarrow for $d=3$, the partition function (i.e. $n=0$) exists for all genera, \rightarrow One should choose

$$\mathcal{M} = \text{CY}_3$$

$$g=0: \quad \sum \text{deg } O_k = 2d = 6$$

$$g=1: \quad \sum \text{deg } O_k = 2$$

$$\Rightarrow \text{deg } O_k = 2 \quad \begin{matrix} n=3 \\ n=1 \end{matrix}$$

\leftarrow the same # points must be fixed to fix conformal symmetry

Correlation functions

• Why $\deg O_k = 2$?

For such choice the corresponding integrated operator $\int O_k^{(2)}$ has vanishing charge and can be added Σ to the action.

\Rightarrow Partition function $Z_g[t] = \int \mathcal{D} \dots e^{-S_A + \sum_k t^k \int O_k^{(2)}}$

Correlation functions $\frac{\delta^n}{\delta t^{k_1} \dots \delta t^{k_n}} Z_g[t] \Big|_{t=0}$

For $g=0/1$ one should insert $3/1$ unintegrated operators

• Example for $g=0$:

$\langle O_1 O_2 O_3 \rangle = \int_{\mathcal{M}} O_1 \wedge O_2 \wedge O_3 + \sum_{\beta \in H_+(M)} d_{\beta}^{(0)} \sum_{m=1}^{\infty} e^{-tm\beta \cdot \omega} \int_{\beta} \int_{\beta} \int_{\beta} O_1 O_2 O_3$

"classical" contr-n
 $\Sigma \rightarrow \text{cpt } \mathbb{S}$
w.s. instantons
 $\Sigma \rightarrow \beta$
 m counts multicoverings

Can be integrated into

$\langle O_1 O_2 O_3 \rangle = \frac{\int_{\mathbb{S}^3} F_0}{\int_{\mathbb{S}^1 \times \mathbb{S}^2 \times \mathbb{S}^1} F_0} \leftarrow d'$ -connected prepotential on $d'_e(\mathbb{C}Y)$

$F_0(x) = -\frac{1}{6x^0} K_{ijk} X^i X^j X^k + \frac{1}{2} A_{IJS} X^I X^J X^S + y_{CY} \frac{S(3)(x^0)^2}{2(2\pi i)^3} - \frac{(x^0)^2}{(2\pi i)^3} \sum_{\beta \in H_+(M)} d_{\beta}^{(0)} Li_3(e^{2\pi i \beta_i x^i / x^0})$

↑ perturb term
↑ genus 0 Gopakumar-Vafa inv.

• F_g - non-trivial generalizations of F_0 containing information about more general topological invariants ($d_{\beta}^{(g)}, N_{\beta}^{(g)}, \Delta T, \dots$)

• In the B-model, F_0 - holomorphic prepotential on $d'_e(\mathbb{C}Y)$ (no quantum corrections)

F-terms

- F_0 determines second derivatives terms in the effective action obtained by compactifying string theory (type IIA or IIB) on M . (CY moduli \rightarrow fields)

What is the role of F_g ?

They determine a class of terms with higher deriv-s.

- 4d $N=2$ SUGRA can be formulated off-shell using superconformal formalism

$$S_{\text{eff}} = \int d^4x d^4\theta d^4\bar{\theta} \mathcal{D}(\Phi, \bar{\Phi}) + \int d^4x d^4\theta \mathcal{W}(\Phi, W^2)$$

where W^2 - scalar chiral superfield \mathcal{D} -term \mathcal{W} -term
constructed from Weyl superfield W

$$W_{\mu\nu}(x, \theta) = F_{\mu\nu} - \frac{1}{2} R_{\mu\nu\sigma\rho} \epsilon_{\alpha\beta} \theta^\alpha \theta^\beta + \dots$$

↑
graviphoton field-strength

$$\mathcal{W}(\Phi, W^2) = \sum_{g=0} W^{2g} F_g(\Phi)$$

- The claim: F_g - free energy of A/B model for compactification of type IIA/IIB strings

The terms determined by F_g :

$$\int d^4x F_g(x) [R^2 F^{2(g-1)} + 2(g-1)(RF)^2 F^{2(g-2)}]$$

gives scattering of 2 gravitons and $2(g-1)$ graviphotons

- Mirror symmetry

A-model on CY	=	B-model on \tilde{CY}
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Holomorphic anomaly

- We want to have a real action $\rightarrow \sum_k (t^k \int_{\Sigma_g} O_k^{(2)} + \bar{t}^k \int_{\Sigma_g} \bar{O}_k^{(2)})$
 But is $\bar{O}^{(2)}$ physical?

Consider the B-model where $\{Q_B, Q_+ Q_-\} = 2M \Rightarrow a_2 \sim Q_+$
 $\{Q_B, Q_+^+ Q_-\} = 2P \Rightarrow a_2 \sim Q_-$

$$\begin{aligned} \bar{O}_{\bar{z}\bar{z}}^{(2)} &= i \{a_2, O_{\bar{z}}^{(1)}\} = - \{a_2, [Q_-, O^{(1)}]\} \sim - \{Q_+, [Q_-, \bar{O}^{(1)}]\} = \\ &= \frac{1}{2} \{ \underbrace{Q_+^+ + Q_-}_{Q_B}, [Q_-, \bar{O}^{(1)}] \} \Rightarrow \bar{O}^{(2)} = Q_B\text{-exact} \end{aligned}$$

- $F_g = \langle e^{\sum_k (t^k \int_{\Sigma_g} O_k^{(2)} + \bar{t}^k \int_{\Sigma_g} \bar{O}_k^{(2)})} \rangle$ seems to be independent of \bar{t}^k
 i.e. holomorphic

The naive argument fails

because of the non-trivial measure on the moduli space.

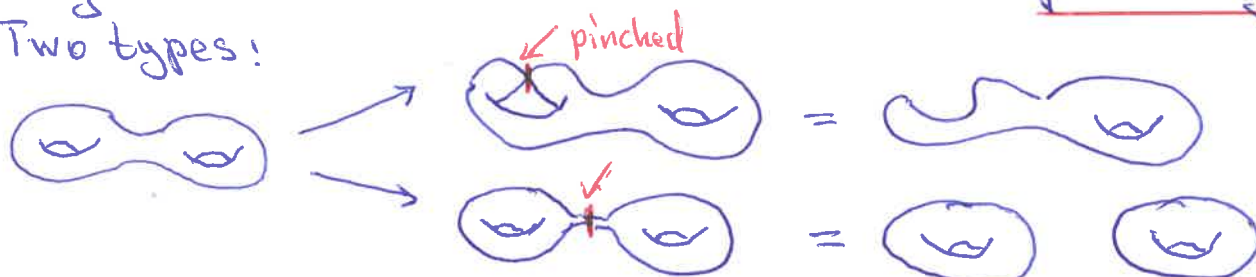
Due to $\{Q_B, Q \cdot \mu_\alpha\} = T \cdot \mu_\alpha \sim \frac{\partial S}{\partial m^\alpha}$, one gets the total derivative:

$$\frac{\partial F_g}{\partial \bar{t}^k} \sim \int_{\mathcal{M}_g} \prod_{\alpha=1}^{2g-3} dm^\alpha d\bar{m}^\alpha \left(\sum_P \frac{\partial}{\partial m^P} \langle (\prod_{x \neq P} \mu_x \cdot Q) (\prod_x \bar{\mu}_x \cdot \bar{Q}) \int_{\Sigma_g} \bar{O}_k^{(2)} \rangle + \text{c.c.} \right)$$

But this is not zero because \mathcal{M}_g has boundaries!

$\partial \mathcal{M}_g$ - degenerate Riemann surfaces with pinched cycles

Two types:



$$\Rightarrow \frac{\partial F_g}{\partial \bar{t}^k} = \frac{1}{2} C_{\bar{k}mn} e^{2k} g^{\bar{m}m} g^{\bar{n}n} \left(\mathcal{D}_m \mathcal{D}_{\bar{n}} F_{g+1} + \sum_{r=1}^{g-1} \mathcal{D}_m \mathcal{F}_r \mathcal{D}_{\bar{n}} F_{g-r} \right)$$

where $\mathcal{D}_m F_g = (\partial_{t^m} - (2-2g)\partial_{t^m k}) F_g$ - covariant der-ve on L^{2-2g} over \mathcal{M}_g

k - Kähler potential on \mathcal{M}_g , $g_{m\bar{n}}$ - Kähler metric

$$C_{k\bar{m}n} = \langle O_k^{(1)} O_{\bar{m}}^{(1)} O_n^{(1)} \rangle_{S^2}$$

equation of holomorphic anomaly

- $Z_{top} = \exp \left[\sum_{g=0}^{\infty} \lambda^{2g-2} F_g \right] \Rightarrow \left[\partial_{\bar{k}} - \lambda^2 C_{\bar{k}mn} e^{2k} g^{\bar{m}m} g^{\bar{n}n} \mathcal{D}_m \mathcal{D}_{\bar{n}} \right] Z_{top} = 0$

The equation allows to find F_g up to "holomorphic ambiguity" which can be fixed by a finite number of conditions.

Background independence

- The holomorphic anomaly can be interpreted as dependence on "background complex structure":
 - take t^k and \bar{t}^k independent
 - \bar{t}^k determines Ω_0 , while variations δt^k around $t^k = (\bar{t}^k)^*$ generate correlators
 - anomaly implies dependence on Ω_0 .
- Idea: find a new interpretation for the same system

Z_{top} - wave function

Choice of Ω_0 - choice of polarization

Background independence holds for the state, but not for its representation!

$$Z_{\text{top}} = \Psi_{\text{top}}(x) = \langle \text{top} | x \rangle_{\Omega_0} \quad x \sim \delta t$$

↑ back.ind. ↑ background

Realization:

Ψ_{top} lives in the Hilbert space obtained by quantization of $H^3(M, \mathbb{R})$ with symplectic structure $\omega(\alpha, \beta) = \int_M \alpha \wedge \beta$.

Ω defining compl. str. allows to introduce a polarization: $\Omega \in H^{3,0}$, $\Omega_i \Omega \in H^{2,1} \Rightarrow \alpha = (\lambda^{-1} \Omega + x^i \partial_i \Omega + \bar{x}^i \bar{\partial}_i \Omega + \bar{\lambda}^{-1} \bar{\Omega})$

$$\omega(\delta \alpha, \delta \beta) = (\delta \lambda^{-1}_\alpha \delta \bar{\lambda}^{-1}_\beta - \delta \bar{\lambda}^{-1}_\alpha \delta \lambda^{-1}_\beta + (\delta x^i_\alpha \delta \bar{x}^j_\beta - \delta \bar{x}^j_\alpha \delta x^i_\beta) \partial_i \bar{\partial}_j) \int_M \alpha \wedge \bar{\beta}$$

Hodge decomposition

$$= i e^{-k} (\delta \lambda^{-1}_\alpha \delta \bar{\lambda}^{-1}_\beta - \delta \bar{\lambda}^{-1}_\alpha \delta \lambda^{-1}_\beta - g_{ij} (\delta x^i_\alpha \delta \bar{x}^j_\beta - \delta \bar{x}^j_\alpha \delta x^i_\beta))$$

$$\Rightarrow \{ \lambda^{-1}, \bar{\lambda}^{-1} \} \sim e^{-k}, \quad \{ x^i, \bar{x}^j \} \sim e^k g^{ij}$$

Then $\Psi_{\text{top}}(x, \lambda^{-1}) = \langle \text{top} | x, \lambda^{-1} \rangle_{\Omega(\mathbb{F})}$ λ - coupling constant of top. string.

The holomorphic anomaly equation describes the change of the wave function under the change of polarization.

- One can choose different representations.

For example: $\alpha = p^a \alpha_a - q_a \beta^a$ defines "real" representation $\Psi_{\mathbb{R}}(p)$ which is background independent, but basis dependent

OSV conjecture

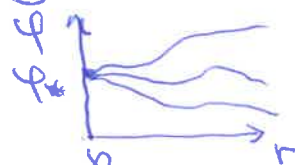
- In 4d $N=2$ SUGRA, obtained as Type IIB/CY, there are solutions describing BPS (extremal) black holes.

Attractor mechanism: values of scalar fields (moduli) are fixed by black hole charges independently of their asymptotic values

$$\text{Re } X^\Lambda = p^\Lambda, \quad \text{Re } F_\Lambda = q_\Lambda$$

$$S_{\text{BH}} = \pi M^2 = \frac{-\beta \dot{c}}{4} \int_{\text{CY}} \mathcal{L}_\Lambda \bar{\mathcal{L}}^\Lambda$$

$M^2 = e^K |Z_\gamma|^2$



$$Z_\gamma = q_\Lambda X^\Lambda - p^\Lambda F_\Lambda - \text{central charge}$$

- This result can be rewritten as Legendre transform:

$$S_{\text{BH}}(q, p) = \mathcal{F}_{\text{BH}}(\psi, p) - \pi \varphi^\Lambda \frac{\partial}{\partial \varphi^\Lambda} \mathcal{F}_{\text{BH}}(\psi, p)$$

where $\mathcal{F}_{\text{BH}}(\psi, p) = -\pi \text{Im} F(p + i\psi)$, $q_\Lambda = -\frac{1}{\pi} \frac{\partial \mathcal{F}_{\text{BH}}}{\partial \varphi^\Lambda}$

Remarkably, this result remains valid after including α' -corrections provided F - full topological free energy
 S_{BH} - Wald's entropy

Conjecture

This is an approximation (of large charges) to exact formula:

$$\Rightarrow \Omega(q, p) = \int dx e^{-\pi i x^\Lambda q_\Lambda} \bar{\Psi}_{\text{top}}(p-x) \Psi_{\text{top}}(p+x)$$

Wigner distribution function

$$Z_{\text{BH}}(q, p) \equiv \sum_{q_\Lambda} \Omega(q, p) e^{\pi i q_\Lambda p^\Lambda} = |Z_{\text{top}}(p + i\varphi)|^2$$

mixed ensemble
 p - micro-canonical
 q - canonical

↑
 probability amplitude for wave function

- What is $\Psi_{\text{top}}(p)$?

This is not $Z_{\text{top}}(x)|_{x^\Lambda=p^\Lambda}$, but $\Psi_{\text{top}}(p) = \langle \text{top} | p \rangle$

Problems:

measure of integration, contour
 hypermultiplet contributions
 non-perturbative effects.

Topological invariants

- Consider the A-model in the holomorphic limit $\hbar \rightarrow 0$

Then $F_g = \sum_{\beta \in H_2^+(U)} N_{\beta}^{(g)} e^{2\pi i \beta \cdot k}$, $N_{\beta}^{(g)}$ - Gromov-Witten inv. rational
count holomorphic maps

- There is another representation

$$F = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g = \sum_{g=0}^{\infty} \sum_{\beta \in H_2^+(U)} \sum_{d=1}^{\infty} n_{\beta}^{(g)} \frac{1}{d} \left(2 \sin\left(\frac{d\lambda}{2}\right) \right)^{2g-2} e^{2\pi i d \beta \cdot k}$$

$n_{\beta}^{(g)}$ - Gopakumar-Vafa invariants
integer

(This arises from the one-loop contribution of M2-brane after lifting to M-theory)

- S-duality implies

$$Z_{\text{top}}(t, \lambda) = M(-e^{i\lambda})^{\frac{1}{2} \chi(U)} Z_{\text{DT}}(e^{2\pi i t / \lambda}, e^{i\lambda})$$

where $M(q) = \prod_{k \geq 0} (1 - q^k)^{-k}$ - MacMahon function

$$Z_{\text{DT}}(y, q) = \sum_{\beta \in H_2^+} \sum_n \text{DT}(\beta, n) y^{\beta} q^n \quad \left| \begin{array}{l} \text{captures contributions of} \\ \text{degenerate maps with } \beta=0 \\ \text{(note: } n_0^{(0)} = -\frac{1}{2} \chi(U) \end{array} \right.$$

DT(β, n) - Donaldson-Thomas invariants

count bound states of D6, D2 on β , and n D0-branes

- MVOP formula:

$$Z_{\text{DT}}(y, q) = M(-q)^{\chi(U)} \prod_{\beta \in H_2^+} \prod_{k \geq 0} (1 - (-q)^k y^{\beta})^{k n_{\beta}^{(0)}} \\ \times \prod_{\beta \in H_2^+} \prod_{g=1}^{g_{\text{max}}(\beta)} \prod_{\ell=0}^{2g-2} (1 - (-q)^{g-\ell-1} y^{\beta})^{(-1)^{g+\ell} \binom{2g-2}{\ell} n_{\beta}^{(g)}}$$

Other applications of topological strings

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1) Mirror symmetry

Solve "classical" B-model, then map to A-model

2) Geometric engineering

Consider a CY with a singularity and zoom around it
(or non-compact CY) \rightarrow gravity decouples

\rightarrow $N=2$ gauge theory

with prepotential given by \mathbb{F}_0

and Seiberg-Witten curve encoded in the mirror CY
(in type IIB)

3) Topological branes

There is also a theory of open topological strings containing 2-branes

They are used to describe:

- geometric transition - open/closed string duality in the limit of large N ($\#$ 2-branes)
- $N=1$ gauge theories

4) Refined topological string

Ω -background, non-commutativity

5) Relations to matrix models

integrable hierarchies
resurgence

6) Topological M-theory

Theory of 3-forms on G_2 manifolds.

Explain various features of top. str.

such as wave function property

and dependence of only half of CY moduli.