

Black holes and modularity

Sergei Alexandrov

Laboratoire Charles Coulomb, CNRS, Montpellier

Journée "Gravité quantique"
November, 2022

Black hole entropy

$$S_{\text{BH}} = \frac{A_{\text{hor}}}{4\ell_{\text{Pl}}^2}$$



$$N = \log S_{\text{BH}}$$

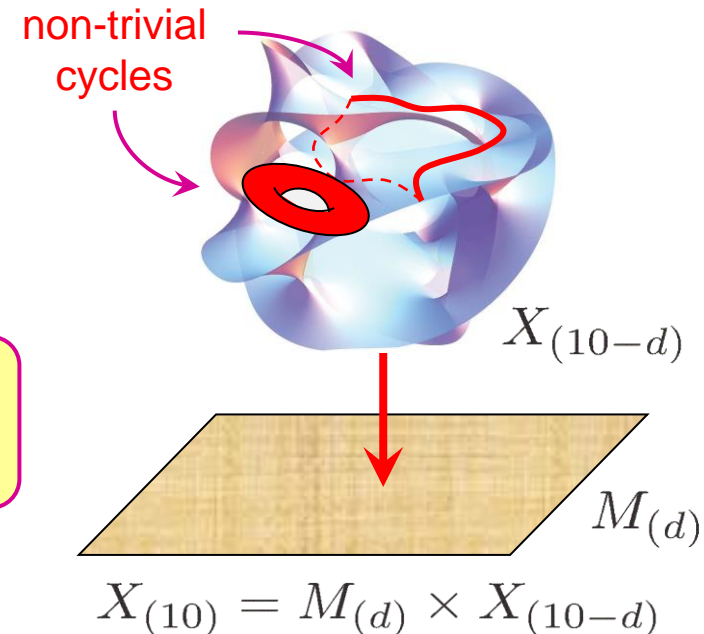
number of black hole states

What are the microstates
counted by black hole
entropy?

String theory solution

Black holes in string theory at small string coupling become *solitonic* objects constructed from D-branes wrapping *non-contractible cycles* on the internal manifold

BH microstates — D-brane configurations



BPS black holes in N=8 SUGRA

In some simple situations one can count BH entropy *exactly*!

Simple \rightarrow sufficiently many SUSY \rightarrow sufficiently simple internal manifold

The simplest case $X_{(6)} = T^6$ — 6d torus



Effective theory in 4d — N=8 SUGRA

This theory has black hole solutions preserving a part of SUSY — *BPS black holes*
 $\frac{1}{8}$ BPS black holes are characterized by one integer valued charge

It is possible to find a generating function of their degeneracies

[Moore, Maldacena, Strominger '99, Pioline '05, Shih, Strominger, Yin '05]

$$\underbrace{\frac{\theta_3(2\tau)}{\eta^6(4\tau)}}_{\text{theta function}} = \sum_{n \geq -1} \Omega(n) q^n = q^{-1} + 2 + 8q^3 + 12q^4 + 39q^7 + \dots \quad q = e^{2\pi i \tau}$$

$\underbrace{\hspace{10em}}_{\text{Dedekind eta function}}$

For large charge, it reproduces the area law

$$\Omega(n) \sim e^{\pi\sqrt{n}} \sim e^{A_{hor}(n)/4}$$

modular functions

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$$

**Modular symmetry governs
BH entropy**

Modular forms

$$h_{\mu}\left(\frac{a\tau + b}{c\tau + d}\right) = \sum_{\nu} \rho_{\mu\nu}(g) h_{\nu}(\tau)$$

w — modular weight

multiplier system
(phase factor)

Asymptotic properties of Fourier coefficients:

$$q = e^{2\pi i \tau}$$

$$h_{\mu}(\tau) = \sum_{n \geq n_{\min}} \Omega_{\mu}(n) q^n \quad \rightarrow$$

$$n_{\min} > 0 \Rightarrow \Omega_{\mu}(n) \sim n^{w/2}$$

$$n_{\min} = 0 \Rightarrow \Omega_{\mu}(n) \sim n^{w-1}$$

$$n_{\min} < 0 \Rightarrow \Omega_{\mu}(n) \sim e^{C\sqrt{n}}$$

relevant for
black holes

In fact, there is an *exact* formula for Fourier coefficients with positive n in terms of ones with negative n (*polar coefficients*)

Rademacher expansion:

$$\Omega_{\mu}(n) = 2\pi \sum_{\nu} \sum_{n' < 0} \Omega_{\nu}(n') \sum_{c=1}^{\infty} \frac{1}{c} K_{\mu}^{\nu}(n, n') \left(\frac{|n'|}{n}\right)^{(1-w)/2} I_{1-w}\left(\frac{4\pi}{c} \sqrt{-nn'}\right)$$

Kloosterman sum

Bessel function

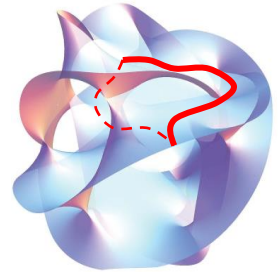
$$K_{\mu}^{\nu}(x, y) = i^{-w} \sum_{\substack{-c \leq d < 0 \\ (c, d) = 1}} (\rho^{-1})_{\mu}^{\nu} e^{2\pi i \left(\frac{a}{c} y + \frac{d}{c} x\right)}$$

Exactly the same structure
emerges from localization of quantum
gravity path integral!

D4-D2-D0 black holes in Type IIA/CY

More complicated setup:

Type IIA string theory on
a Calabi-Yau threefold



Effective theory in 4d — N=2 SUGRA

Important class of $\frac{1}{2}$ BPS black holes is constructed as bound states of D4, D2 and D0-branes characterized by electro-magnetic charge

$$\gamma = (0, p^a, q_a, q_0) \quad a = 1, \dots, b_2(CY)$$

no D6 \nearrow \nwarrow label 4- and 2-dim cycles wrapped by D4 and D2-branes

BPS index $\Omega(\gamma)$ — black hole degeneracy = generalized Donaldson-Thomas invariant of CY

Generating
function

$$h_{p,\mu}(\tau) = \sum_{\hat{q}_0 \leq \hat{q}_0^{\max}} \Omega(\gamma) e^{-2\pi i \hat{q}_0 \tau}$$

Mock modular forms

From *string dualities* and analysis of D-instanton effects, one can show that

SA, Banerjee, Manschot, Pioline '16,
SA, Pioline '18

The functions $h_p(\tau)$ are (higher depth) *mock* modular forms

First examples of *mock theta functions* appeared in the last letter of Ramanujan to Hardy 100 years ago, but remained mysterious objects until recently....



Zwegers. '02

Mock modular form — a *holomorphic* function which is “almost” modular with an anomaly controlled by another modular form (*shadow*)


$$h(\tau) \mapsto (c\tau + d)^w \left(h(\tau) - \int_{-d/c}^{-i\infty} \frac{\overline{g(\bar{z})}}{(\tau - z)^w} dz \right)$$

modular form of weight $2 - w$

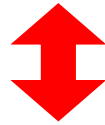
and (*non-holomorphic*) modular *completion* given by

$$\hat{h}(\tau, \bar{\tau}) = h(\tau) - \int_{\bar{\tau}}^{-i\infty} \frac{\overline{g(\bar{z})}}{(\tau - z)^w} dz$$

Applications

- Mock modularity and knowledge of the shadow determine uniquely the Fourier coefficients in terms of polar terms  Rademacher expansion

Explicit expressions for generating functions of black hole degeneracies



Donaldson-Thomas invariants for compact CY

- Non-compact CY (zoom in around a singularity)  geometric engineering of SUSY gauge theories



SU(N) Vafa-Witten invariants of complex surfaces
(count instantons in a topological theory)

- New results in pure mathematics



Explicit construction of non-holomorphic completions of indefinite theta series



Development of the theory of higher depth mock modular forms

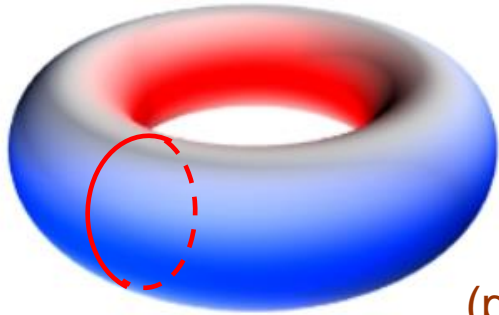
Conclusions

- Modular symmetry is ubiquitous in string theory and is a powerful tool to obtain *exact* results
- It controls generating functions of degeneracies of BPS black holes appearing in string compactifications
- For some classes of black holes, modularity is replaced by its *mock* version
- These results have numerous applications ranging from calculation of exact black hole degeneracies to pure mathematics



Thank you!

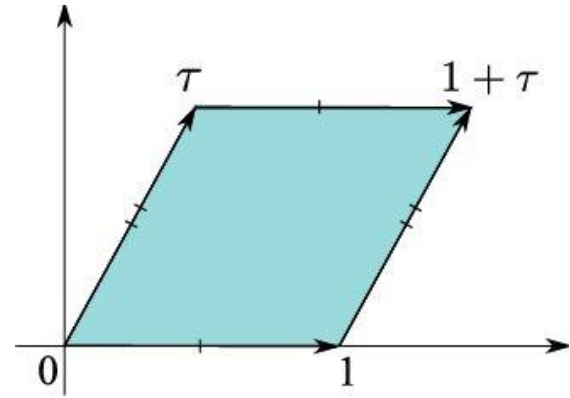
Modular symmetry: torus



$$T^2 = \mathbb{C} / \{z \sim z + n + m\tau\}$$

$$n, m \in \mathbb{Z} \quad \text{Im}\tau > 0$$

τ — *torus modulus*
(parametrizes its *complex structure*)



Infinitesimal transformations of τ change the torus,
but global can leave it invariant:

$$\tau \mapsto \frac{a\tau + b}{c\tau + d} \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \quad \longrightarrow \quad \begin{aligned} am + cn &= m' \\ bm + dn &= n' \end{aligned}$$



**All well-defined quantities on torus
must be *modular invariant*!**

$$h\left(\frac{a\tau + b}{c\tau + d}\right) = h(\tau)$$

Wall-crossing

multi-centered black holes (bound states of black holes) are stable only in a region of the moduli space and decay in another



$\Omega(\gamma)$ *is only piecewise constant*

wall of marginal stability

bound state

$$\gamma_1 + \gamma_2$$

does not exist

bound state

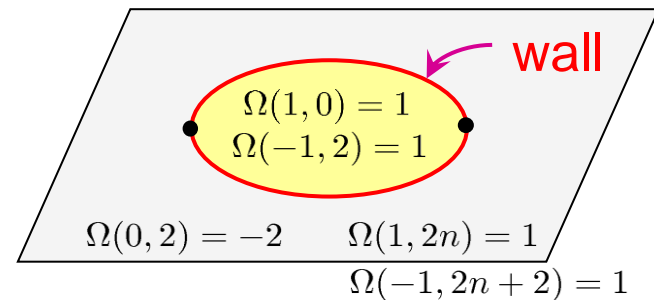
$$\gamma_1 + \gamma_2$$

exists

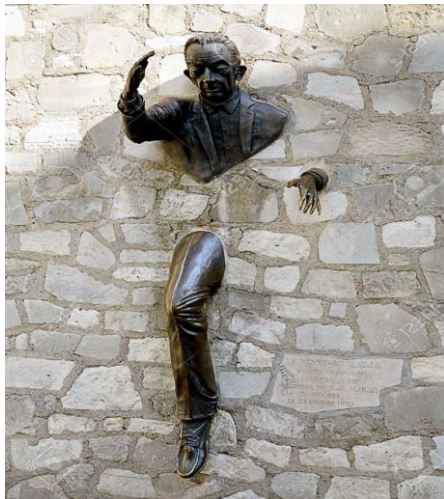
position
of walls

$$M_{\gamma_1} + M_{\gamma_2} = M_{\gamma_1 + \gamma_2}$$

Example: pure SU(2) $\mathcal{N}=2$ SUYM



Seiberg, Witten '94



How to cross a wall?

Kontsevich-Soibelman formula

allows to evaluate $\Omega(\gamma)$ on one side of a wall from their values on the other side



It is enough to find $\Omega(\gamma)$ in one chamber

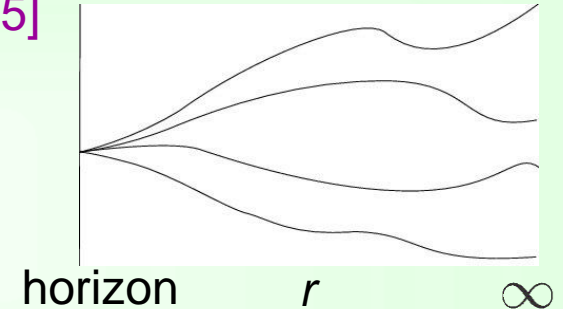
Attractor chamber

There is one special chamber in the moduli space: *attractor chamber*

Attractor mechanism [Ferrara,Kallosch,Strominger '95]

CY moduli — scalar fields in 4d which depend on the radial coordinate in the black hole background

SUGRA equations fix their values at the horizon in terms of the black hole charges and *independently* of their values at infinity



$$z_{\star}^a(\gamma)$$

In the attractor chamber all multi-centered black holes are unstable (up to “*scaling solutions*”)

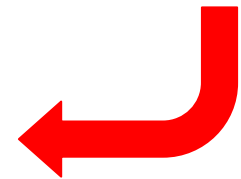
Maldacena, Strominger, Witten '97

$$\Omega_{\gamma}^{\text{MSW}} = \Omega(\gamma, z_{\star}^a(\gamma)) = \Omega_p(\hat{q}_0)$$

$$\hat{q}_0 \equiv q_0 - \frac{1}{2} \kappa^{ab} q_a q_b \text{ — invariant charge bounded from above}$$

generating function of MSW invariants

$$h_p(\tau) = \sum_{\hat{q}_0 \leq \hat{q}_0^{\text{max}}} \Omega_p(\hat{q}_0) e^{-2\pi i \hat{q}_0 \tau}$$



Modularity in N=2 black holes

Our setup: $\text{Type IIA} / \text{CY} \times S^1 = \text{M-theory} / \text{CY} \times \overbrace{S^1 \times S^1}^{T^2}$ ← modular group

Effective theory in 3d — SUSY non-linear sigma model
with target space \mathcal{M}

After compactification on the circle

$\text{D4-brane} / \Gamma_4 \times S^1$ — pointlike object
(*instanton*)

Affects the metric on \mathcal{M} by instanton corrections

Each instanton is weighted by the BPS index $\Omega(\gamma)$

\mathcal{M} carries an isometric action
of the modular group

Restriction on (the generating function of) BPS indices $\Omega(\gamma)$

The functions $h_p(\tau)$ are (higher depth) *mock* modular forms