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Tight (Double) Exponential Bounds for Identification Problems: Locating-Dominating Set and Test Cover

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Abstract

Foucaud et al. [ICALP 2024] demonstrated that some problems in NP can admit (tight) double-exponential lower bounds when parameterized by treewidth or vertex cover number. They showed these first-of-their-kind results by proving conditional lower bounds for certain graph problems, in particular, the metric-based identification problems (STRONG) METRIC DIMENSION. We continue this line of research and highlight the usefulness of this type of problems, to prove relatively rare types of (tight) lower bounds. We investigate fine-grained algorithmic aspects of classical (non-metric based) identification problems in graphs, namely LOCATING-DOMINATING SET, and in set systems, namely TEST COVER. In the first problem, an input is a graph G on n vertices and an integer k , and the objective is to decide whether there is a subset S of k vertices such that any two distinct vertices not in S are dominated by distinct subsets of S . In the second problem, an input is a set of items U , a collection of subsets \mathcal{F} of U called *tests*, and an integer k , and the objective is to select a set S of at most k tests such that any two distinct items are contained in a distinct subset of tests of S .

For our first result, we adapt the techniques introduced by Foucaud et al. [ICALP 2024] to prove similar (tight) lower bounds for these two problems.

- LOCATING-DOMINATING SET (respectively, TEST COVER) parameterized by the treewidth of the input graph (respectively, the natural auxiliary graph) does not admit an algorithm running in time $2^{2^{\sigma(\text{tw})}} \cdot \text{poly}(n)$ (respectively, $2^{2^{\sigma(\text{tw})}} \cdot \text{poly}(|U| + |\mathcal{F}|)$), unless the ETH fails.

This augments the short list of NP-Complete problems that admit tight double-exponential lower bounds when parameterized by treewidth, and shows that “local” (non-metric-based) problems can also admit such bounds. We show that these lower bounds are tight by designing treewidth-based dynamic programming schemes with matching running times.

Next, we prove that these two problems also admit ‘exotic’ (and tight) lower bounds, when parameterized by the solution size k . We prove that unless the ETH fails,

- LOCATING-DOMINATING SET does not admit an algorithm running in time $2^{\sigma(k^2)} \cdot \text{poly}(n)$, nor a polynomial-time kernelization algorithm that reduces the solution size and outputs a kernel with $2^{\sigma(k)}$ vertices, and
- TEST COVER does not admit an algorithm running in time $2^{2^{\sigma(k)}} \cdot \text{poly}(|U| + |\mathcal{F}|)$ nor a kernel with $2^{2^{\sigma(k)}}$ vertices.

Again, we show that these lower bounds are tight by designing (kernelization) algorithms with matching running times. To the best of our knowledge, LOCATING-DOMINATING SET is the first known problem which is FPT when parameterized by solution size k , where the optimal running time has a quadratic function in the exponent. These results also extend the (very) small list of problems that admit an ETH-based lower bound on the number of vertices in a kernel, and (for



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TEST COVER) a double-exponential lower bound when parameterized by the solution size. Whereas it is the first example, to the best of our knowledge, that admit a double exponential lower bound for the number of vertices.

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1 Introduction

The article aims to study the algorithmic properties of certain identification problems in discrete structures. In identification problems, one wishes to select a solution substructure of an input structure (a subset of vertices, the coloring of a graph, etc.) so that the solution substructure uniquely identifies each element. Some well-studied examples are, for example, the problems TEST COVER for set systems and METRIC DIMENSION for graphs (Problems [SP6] and [GT61] in the book by Garey and Johnson [39], respectively). This type of problem has been studied since the 1960s both in the combinatorics community (see e.g. Rényi [60] or Bondy [9]), and in the algorithms community since the 1970s [7, 10, 25, 55]. They have multiple practical and theoretical applications, such as network monitoring [59], medical diagnosis [55], bioinformatics [7], coin-weighing problems [62], graph isomorphism [4], games [19], machine learning [18] etc. An online bibliography on the topic with over 500 entries as of 2024 is maintained at [46].

In this article, we investigate fine-grained algorithmic aspects of identification problems in graphs, namely LOCATING-DOMINATING SET, and in set systems, namely TEST COVER. Like most other interesting and practically motivated computational problems, identification problems also turned out to be NP-hard, even in very restricted settings. See, for example, [20] and [39], respectively. We refer the reader to ‘Related Work’ towards the end of this section for a more detailed overview on their algorithmic complexity.

To cope with this hardness, these problems have been studied through the lens of parameterized complexity. In this paradigm, we associate each instance I with a parameter ℓ , and are interested to know whether the problem admits a *fixed parameter tractable* (FPT) algorithm, i.e., an algorithm with the running time $f(\ell) \cdot |I|^{\mathcal{O}(1)}$, for some computable function f . A parameter can either originate from the formulation of the problem itself or can be a property of the input. If a parameter originates from the formulation of the problem itself, then that is called a *natural parameter*. Otherwise, the parameters that are properties of the input graph are called the *structural parameters*. One of the most well-studied structural parameters is ‘treewidth’ (which, informally, quantifies how close the input graph is to a

tree, and is denoted by tw). We refer readers to [26, Chapter 7] for a formal definition. Courcelle’s celebrated theorem [21] states that the class of graph problems expressible in Monadic Second-Order Logic (MSOL) of constant size admit an algorithm running in time $f(tw) \cdot \text{poly}(n)$. Hence, a large class of problems admit an FPT algorithm when parameterized by the treewidth. Unfortunately, the function f is a tower of exponents whose height depends roughly on the size of the MSOL formula. Hence, this result serves as a starting point to obtain an (usually impractical) FPT algorithm.

Over the years, researchers have searched for more efficient problem-specific algorithms when parameterized by the treewidth. There is a rich collection of problems that admit an FPT algorithm with single- or almost-single-exponential dependency with respect to treewidth, i.e., of the form $2^{\mathcal{O}(tw)} \cdot n^{\mathcal{O}(1)}$ or $2^{\mathcal{O}(tw \log(tw))} \cdot n^{\mathcal{O}(1)}$, (see, for example, [26, Chapter 7]). There are a handful of graph problems that only admit FPT algorithms with double- or triple-exponential dependence in the treewidth [8, 30, 31, 32, 42, 54]. In the respective articles, the authors prove that this double- (respectively, triple-) dependence in the treewidth cannot be improved unless the Exponential Time Hypothesis (ETH)¹ fails.

All the double- (or triple-) exponential lower bounds in treewidth mentioned in the previous paragraph are for problems that are $\#\text{NP}$ -complete, Σ_2^p -complete, or Π_2^p -complete. Indeed, until recently, this type of lower bounds were known only for problems that are complete for levels that are higher than NP in the polynomial hierarchy. Foucaud et al. [36] recently proved for the first time, that it is not necessary to go to higher levels of the polynomial hierarchy to achieve double-exponential lower bounds in the treewidth. The authors studied three NP -complete *metric-based graph problems* viz METRIC DIMENSION, STRONG METRIC DIMENSION, and GEODETIC SET. They proved that these problems admit double-exponential lower bounds in tw (and, in fact in the size of minimum vertex cover size vc for the second problem) under the ETH. The first two of these three problems are identification problems.

In this article, we continue this line of research and highlight the usefulness of identification problems to prove relatively rare types of lower bounds, by investigating fine-grained algorithmic aspects of LOCATING-DOMINATING SET and TEST COVER, two classical (non-metric-based) identification problems. This also shows that this type of bounds can hold for “local” (i.e., non-metric-based) problems (the problems studied in [36] were all metric-based). Apart from serving as examples for double-exponential dependence on treewidth, these problems are of interest in their own right, and possess a rich literature both in the algorithms and discrete mathematics communities, as highlighted in ‘Related Work’.

LOCATING-DOMINATING SET

Input: A graph G on n vertices and an integer k .

Question: Does there exist a locating-dominating set of size k in G , that is, a set S of $V(G)$ of size at most k such that for any two different vertices $u, v \in V(G) \setminus S$, their neighbourhoods in S are different, i.e., $N(u) \cap S \neq N(v) \cap S$ and non-empty?

TEST COVER

Input: A set of items U , a collection \mathcal{F} of subsets of U called *tests*, and an integer k .

Question: Does there exist a collection of at most k tests such that for each pair of items, there is a test that contains exactly one of the two items?

¹ The ETH roughly states that n -variable 3-SAT cannot be solved in time $2^{o(n)} n^{\mathcal{O}(1)}$. See [26, Chapter 14].

As TEST COVER is defined over set systems, for structural parameters, we define an *auxiliary graph* in the natural way: A bipartite graph G on n vertices with bipartition $\langle R, B \rangle$ of $V(G)$ such that sets R and B contain a vertex for every set in \mathcal{F} and for every item in U , respectively, and $r \in R$ and $b \in B$ are adjacent if and only if the set corresponding to r contains the element corresponding to b .

The LOCATING-DOMINATING SET problem is also a *graph domination problem*. In the classical DOMINATING SET problem, an input is an undirected graph G and an integer k , and the objective is to decide whether there is a subset $S \subseteq V(G)$ of size k such that for any vertex $u \in V(G) \setminus S$, at least one of its neighbours is in S . It can also be seen as a local version of METRIC DIMENSION² in which the input is the same and the objective is to determine a set S of $V(G)$ such that for any two vertices $u, v \in V(G) \setminus S$, there exists a vertex $s \in S$ such that $\text{dist}(u, s) \neq \text{dist}(v, s)$.

We demonstrate the applicability of the techniques from [36] to LOCATING-DOMINATING SET and TEST COVER. We adopt the main technique developed in [36] to our setting, namely, the *bit-representation gadgets* and *set representation gadget* to prove the following result.

► **Theorem 1.** *Unless the ETH fails, LOCATING-DOMINATING SET (respectively, TEST COVER) parameterized by the treewidth of the input graph (respectively, the natural auxiliary graph) does not admit an algorithm running in time $2^{2^{o(\text{tw})}} \cdot \text{poly}(n)$.*

We remark that the algorithmic lower bound of Theorem 1 holds true even with respect to treedepth (and hence with respect to pathwidth), a parameter larger than treewidth. In contrast, DOMINATING SET admits an algorithm running in time $\mathcal{O}(3^{\text{tw}} \cdot n^2)$ [67, 52]. In the full version of the paper, we prove that both LOCATING-DOMINATING SET and TEST COVER admit an algorithm with matching running time, by nontrivial dynamic programming schemes on tree decompositions.

Theorem 1 adds LOCATING-DOMINATING SET and TEST COVER to the short list of NP-Complete problems that admit (tight) double-exponential lower bounds for treewidth. Using the techniques mentioned in [36], two more problems, viz. NON-CLASHING TEACHING MAP and NON-CLASHING TEACHING DIMENSION, from learning theory were recently shown in [14] to admit similar lower bounds.

Next, we prove that LOCATING-DOMINATING SET and TEST COVER also admit ‘exotic’ lower bounds, when parameterized by the solution size k . First, note that both problems are trivially FPT when parameterized by the solution size. Indeed, as any solution must have size at least logarithmic in the number of elements/vertices (assuming no redundancy in the input), the whole instance is a trivial single-exponential kernel for LOCATING-DOMINATING SET, and double-exponential in the case of TEST COVER. To see this, note that in both problems, any two vertices/items must be assigned a distinct subset from the solution set. Hence, if there are more than 2^k of them, we can safely reject the instance. Thus, for LOCATING-DOMINATING SET, we can assume that the graph has at most $2^k + k$ vertices, and for TEST COVER, at most 2^k items. Moreover, for TEST COVER, one can also assume that every test is unique (otherwise, delete any redundant test), in which case there are at most 2^{2^k} tests. Hence, LOCATING-DOMINATING SET admits a kernel with size $\mathcal{O}(2^k)$, and an FPT algorithm running in time $2^{\mathcal{O}(k^2)}$ (See Proposition 7). We prove that both of these bounds are optimal.

² Note that METRIC DIMENSION is also an identification problem, but it is inherently non-local in nature, and indeed was studied together with two other non-local problems in [36], where the similarities between these non-local problems were noticed.

► **Theorem 2.** *Unless the ETH fails, LOCATING-DOMINATING SET, parameterized by the solution size k , does not admit*

- *an algorithm running in time $2^{o(k^2)} \cdot n^{\mathcal{O}(1)}$, nor*
- *a polynomial time kernelization algorithm that reduces the solution size and outputs a kernel with $2^{o(k)}$ vertices.*

To the best of our knowledge, LOCATING-DOMINATING SET is the first known problem to admit such an algorithmic lower bound, with a matching upper bound, when parameterized by the solution size. The only other problems known to us, admitting similar lower bounds, are for structural parameterizations like vertex cover [1, 14, 36] or pathwidth [58, 61]. The second result is also quite rare in the literature. The only results known to us about ETH-based conditional lower bounds on the number of vertices in a kernel when parameterized by the solution size are for EDGE CLIQUE COVER [27] and BICLIQUE COVER [15]³. Theorem 2 also improves upon a “no $2^{\mathcal{O}(k)} n^{\mathcal{O}(1)}$ algorithm” bound from [5] (under $W[2] \neq \text{FPT}$) and a $2^{o(k \log k)}$ ETH-based lower bound recently proved in [11].

Now, consider the case of TEST COVER. As mentioned before, it is safe to assume that $|\mathcal{F}| \leq 2^{|U|}$ and $|U| \leq 2^k$. By Bondy’s celebrated theorem [9], which asserts that in any feasible instance of TEST COVER, there is always a solution of size at most $|U| - 1$, we can also assume that $k \leq |U| - 1$. Hence, the brute-force algorithm that enumerates all the sub-collections of tests of size at most k runs in time $|\mathcal{F}|^{\mathcal{O}(|U|)} = 2^{\mathcal{O}(|U|^2)} = 2^{2^{\mathcal{O}(k)}}$. Our next result proves that this simple algorithm is again optimal.

► **Theorem 3.** *Unless the ETH fails, TEST COVER does not admit*

- *an algorithm running in time $2^{2^{o(k)}} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, nor*
- *a polynomial time kernelization algorithm that reduces the solution size and outputs a kernel with $2^{2^{o(k)}}$ vertices.*

This result adds TEST COVER to the relatively rare list of NP-complete problems that admit such double-exponential lower bounds when parameterized by the solution size and the matching algorithm. The only other examples that we know of are EDGE CLIQUE COVER [27], DISTINCT VECTORS PROBLEM [57], and TELEPHONE BROADCAST [66]. For double-exponential algorithmic lower bounds with respect to structural parameters, please see [33, 42, 45, 47, 48, 50, 51, 53].

The second result in the theorem is a simple corollary of the first result. Assume that the problem admits a kernel with $2^{2^{o(k)}}$ vertices. Then, the brute-force enumerating all the possible solutions works in time $\binom{2^{2^{o(k)}}}{k} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, which is $2^{k \cdot 2^{o(k)}} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, which is $2^{2^{o(k)}} \cdot (|U| + |\mathcal{F}|)^{\mathcal{O}(1)}$, contradicting the first result. To the best of our knowledge, TEST COVER is the first problem that admit a double exponential kernelization lower bound for the number of vertices when parameterized by solution size, or by any natural parameter.

Related Work. LOCATING-DOMINATING SET was introduced by Slater in the 1980s [63, 64]. The problem is NP-complete [20], even for special graph classes such as planar unit disk graphs [56], planar bipartite subcubic graphs, chordal bipartite graphs, split graphs and co-bipartite graphs [35], interval and permutation graphs of diameter 2 [38]. By a straightforward application of Courcelle’s theorem [22], LOCATING-DOMINATING SET is FPT for parameter treewidth and even cliquewidth [23]. Explicit polynomial-time algorithms were given for

³ Additionally, POINT LINE COVER does not admit a kernel with $\mathcal{O}(k^{2-\epsilon})$ vertices, for any $\epsilon > 0$, unless $\text{NP} \subseteq \text{coNP}/\text{poly}$ [49].

trees [63], block graphs [3], series-parallel graphs [20], and cographs [37]. Regarding the approximation complexity of LOCATING-DOMINATING SET, see [35, 40, 65].

It was shown in [5] that LOCATING-DOMINATING SET cannot be solved in time $2^{o(n)}$ on bipartite graphs, nor in time $2^{o(\sqrt{n})}$ on planar bipartite graphs or on apex graphs, assuming the ETH. Moreover, they also showed that LOCATING-DOMINATING SET cannot be solved in time $2^{O(k)}n^{O(1)}$ on bipartite graphs, unless $W[2] = \text{FPT}$. Note that the authors of [5] have designed a complex framework with the goal of studying a large class of identification problems related to LOCATING-DOMINATING SET and similar problems.

In [12], structural parameterizations of LOCATING-DOMINATING SET were studied. It was shown that the problem admits a linear kernel for the parameter max-leaf number, however (under standard complexity assumptions) no polynomial kernel exists for the solution size, combined with either the vertex cover number or the distance to a clique. They also provide a double-exponential kernel for the parameter distance to the cluster. In the full version [11] of [12], the same authors show that LOCATING-DOMINATING SET does neither admit a $2^{o(k \log k)}n^{O(1)}$ -time nor an $n^{o(k)}$ -time algorithm, assuming the ETH.

TEST COVER was shown to be NP-complete by Garey and Johnson [39, Problem SP6] and it is also hard to approximate within a ratio of $(1 - \epsilon) \ln n$ [10] (an approximation algorithm with ratio $1 + \ln n$ exists by reduction to SET COVER [7]). As any solution has size at least $\log_2(n)$, the problem admits a trivial kernel of size 2^{2^k} , and thus TEST COVER is FPT parameterized by solution size k . TEST COVER was studied within the framework of “above/below guarantee” parameterizations in [6, 24, 25, 41] and kernelization in [6, 24, 41]. These results have shown an intriguing behavior for TEST COVER, with some nontrivial techniques being developed to solve the problem [6, 25]. TEST COVER is FPT for parameters $n - k$, but $W[1]$ -hard for parameters $m - k$ and $k - \log_2(n)$ [25]. However, assuming standard assumptions, there is no polynomial kernel for the parameterizations by k and $n - k$ [41], although there exists a “partially polynomial kernel” for parameter $n - k$ [6] (i.e. one with $O((n - k)^7)$ elements, but potentially exponentially many tests). When the tests have all a fixed upper bound r on their size, the parameterizations by k , $n - k$ and $m - k$ all become FPT with a polynomial kernel [24, 41].

The problem DISCRIMINATING CODE [16] is very similar to TEST COVER (with the distinction that the input is presented as a bipartite graph, one part representing the elements and the other, the tests, and that every element has to be covered by some solution test), and has been shown to be NP-complete even for planar instances [17].

Organization. Due to the space constraints, we present overviews of the reductions in this extended abstract. Formal proofs for the arguments can be found in the full version of the paper. We use the LOCATING-DOMINATING SET problem to demonstrate key technical concepts regarding our lower bounds and algorithms. We present an overview of the arguments about LOCATING-DOMINATING SET in Sections 3 and 4. The arguments regarding TEST COVER follow the identical line, and an overview is presented in Section 5. We conclude with an open problem in Section 6.

2 Preliminaries

For a positive integer q , we denote the set $\{1, 2, \dots, q\}$ by $[q]$. We use \mathbb{N} to denote the collection of all non-negative integers.

Graph theory We use standard graph-theoretic notation, and we refer the reader to [28] for any undefined notation. For an undirected graph G , sets $V(G)$ and $E(G)$ denote its set of vertices and edges, respectively. We denote an edge with two endpoints u, v as uv . Unless otherwise specified, we use n to denote the number of vertices in the input graph G of the problem under consideration. Two vertices u, v in $V(G)$ are *adjacent* if there is an edge uv in G . The *open neighborhood* of a vertex v , denoted by $N_G(v)$, is the set of vertices adjacent to v . The *closed neighborhood* of a vertex v , denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. We say that a vertex u is a *pendant vertex* if $|N_G(v)| = 1$. We omit the subscript in the notation for neighborhood if the graph under consideration is clear. For a subset S of $V(G)$, we define $N[S] = \bigcup_{v \in S} N[v]$ and $N(S) = N[S] \setminus S$. For a subset S of $V(G)$, we denote the graph obtained by deleting S from G by $G - S$. We denote the subgraph of G induced on the set S by $G[S]$.

Locating-Dominating Sets A subset of vertices S in graph G is called its *dominating set* if $N[S] = V(G)$. A dominating set S is said to be a *locating-dominating set* if for any two different vertices $u, v \in V(G) \setminus S$, we have $N(u) \cap S \neq N(v) \cap S$. In this case, we say vertices u and v are *distinguished* by the set S . We say a vertex u is *located* by set S if for any vertex $v \in V(G) \setminus \{u\}$, u and v are distinguished by S (equivalently $N(u) \cap S \neq N(v) \cap S$). Note that, if S locates u , then any superset $S' \supset S$ also locates u . By extension, a set X is *located* by S if all vertices in X are located by S . We note the following simple observation (see also [13, Lemma 5]).

► **Observation 4.** *If S is a locating-dominating set of a graph G , then there exists a locating-dominating set S' of G such that $|S'| \leq |S|$ and that contains all vertices that are adjacent with a pendant vertices (i.e. vertices of degree 1) in G .*

Proof. Let u be a pendant vertex which is adjacent with a vertex v of G . We now look for a locating dominating set S' of G such that $|S'| \leq |S|$ and contains the vertex v . As S is a (locating) dominating set, we have $\{u, v\} \cap S \neq \emptyset$. If $v \in S$, then take $S' = S$. Therefore, let us assume that $u \in S$ and $v \notin S$. Define $S' = (S \cup \{v\}) \setminus \{u\}$. It is easy to see that S' is a dominating set. If S' is not a locating-dominating set, then there exists w , apart from u , in the neighbourhood of v such that both u and w are adjacent with *only* v in S' . As u is a pendant vertex and v its unique neighbour, w is not adjacent to u . Hence, w was not adjacent with any vertex in $S' \setminus \{v\} = S \setminus \{u\}$. This, however, contradicts the fact that S is a (locating) dominating set. Hence, S' is a locating-dominating set and $|S'| = |S|$. Thus, the result follows from repeating this argument for each vertex of G adjacent to a pendant vertex. ◀

Parameterized complexity An instance of a parameterized problem Π consists of an input I , which is an input of the non-parameterized version of the problem, and an integer k , which is called the *parameter*. Formally, $\Pi \subseteq \Sigma^* \times \mathbb{N}$. A problem Π is said to be *fixed-parameter tractable*, or FPT, if given an instance (I, k) of Π , we can decide whether (I, k) is a YES-instance of Π in time $f(k) \cdot |I|^{\mathcal{O}(1)}$. Here, $f : \mathbb{N} \mapsto \mathbb{N}$ is some computable function depending only on k . A parameterized problem Π is said to admit a *kernelization* if given an instance (I, k) of Π , there is an algorithm that runs in time polynomial in $|I| + k$ and constructs an instance (I', k') of Π such that (i) $(I, k) \in \Pi$ if and only if $(I', k') \in \Pi$, and (ii) $|I'| + k' \leq g(k)$ for some computable function $g : \mathbb{N} \mapsto \mathbb{N}$ depending only on k . If $g(\cdot)$ is a polynomial function, then Π is said to admit a *polynomial kernelization*. For a detailed introduction to parameterized complexity and related terminologies, we refer the reader to the recent books by Cygan et al. [26] and Fomin et al. [34].

3 Locating-Dominating Set Parameterized by Treewidth

We first present a bird's eye overview of the dynamic programming algorithm. Let $\mathcal{T} = (T, \{X_t\}_{t \in V(T)})$ be a nice tree decomposition of G . For every node $t \in V(T)$, consider the subtree T_t of T rooted at t . Let G_t denote the subgraph of G that is induced by the vertices that are present in the bags of T_t . For every node $t \in T$, we define a subproblem (or DP-state) using a tuple $[t, (Y, W), (\mathcal{A}, \mathcal{D}), \mathcal{B}]$. Consider a partition $(Y, W, X_t \setminus (Y \cup W))$ of X_t . The first part denotes the vertices in the partial solution S . The second part denotes the vertices in X_t that are dominated (but need not be located) by the solution vertices in G_t but that are outside X_t . To extend this partial solution, we need to keep track of vertices that are adjacent to a unique subset in Y . For example, suppose there is vertex $u \in V(G_t) \setminus (S \cup X_t)$ such that $N_{G_t}(u) \cap S = A$ for some subset $A \subseteq Y$. Then u still needs to be located by S . It means that there should not be a vertex, say v , in $V(G) \setminus V(G_t)$ such that $N_{G_t}(v) \cap S' = A$, where S' is an extension of the partial solution S . Hence, we need to keep track of all such vertices by keeping track of the neighbourhood of all such vertices. We define \mathcal{A} , which is a subset of the power set of Y , to store all such sets that are the neighborhoods of vertices in $V(G_t) \setminus X_t$. Similarly, we define \mathcal{D} to store all such sets with respect to vertices that are in X_t . Finally, we define \mathcal{B} to store the pairs of vertices that need to be resolved by the extension of the partial solution. We formalise these ideas in the full version of the paper to prove the following theorem.

► **Theorem 5.** *LOCATING-DOMINATING SET, parameterized by the treewidth tw of the input graph admits an algorithm running in time $2^{2^{\mathcal{O}(tw)}} \cdot n^{\mathcal{O}(1)}$.*

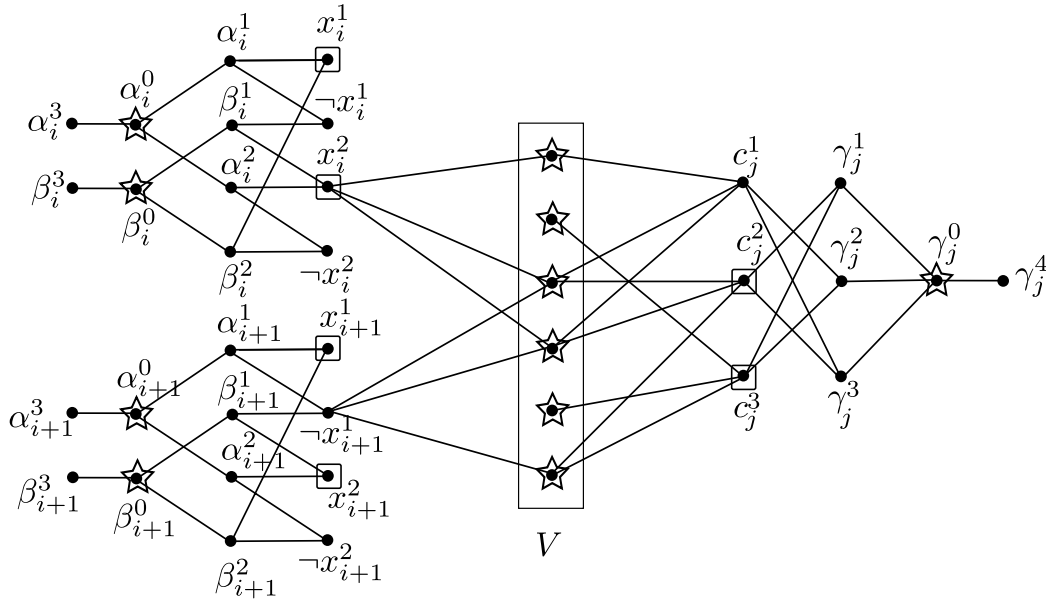
In the remainder of this section, we prove the lower bound mentioned in Theorem 1 by presenting a reduction from a variant of 3-SAT called (3,3)-SAT. In this variation, an input is a boolean satisfiability formula ψ in conjunctive normal form such that each clause contains *at most* 3 variables, and each variable appears at most 3 times. Using the ETH [43], the sparsification lemma [44], and a simple reduction from 3-SAT, we have the following result.

► **Proposition 6.** *(3,3)-SAT, with n variables and m clauses, does not admit an algorithm running in time $2^{o(m+n)}$, unless the ETH fails.*

We highlight that every variable appears positively and negatively at least once. Otherwise, if a variable appears only positively (respectively, only negatively) then we can assign it **True** (respectively, **False**) and safely reduce the instance by removing the clauses containing this variable. Hence, instead of the first, second, or third appearance of the variable, we use the first positive, first negative, second positive, or second negative appearance of the variable.

Reduction The reduction takes as input an instance ψ of (3,3)-SAT with n variables and outputs an instance (G, k) of LOCATING-DOMINATING SET such that $\text{tw}(G) = \mathcal{O}(\log(n))$. Suppose $X = \{x_1, \dots, x_n\}$ is the collection of variables and $C = \{C_1, \dots, C_m\}$ is the collection of clauses in ψ . Here, we consider $\langle x_1, \dots, x_n \rangle$ and $\langle C_1, \dots, C_m \rangle$ to be arbitrary but fixed orderings of variables and clauses in ψ . For a particular clause, the first order specifies the first, second, or third (if it exists) variable in the clause in a natural way. The second ordering specifies the first/second positive/negative appearance of variables in X in a natural way. The reduction constructs a graph G as follows.

- To construct a variable gadget for x_i , it starts with two claws $\{\alpha_i^0, \alpha_i^1, \alpha_i^2, \alpha_i^3\}$ and $\{\beta_i^0, \beta_i^1, \beta_i^2, \beta_i^3\}$ centered at α_i^0 and β_i^0 , respectively. It then adds four vertices



■ **Figure 1** For the sake of clarity, we do not explicitly show the pendant vertices adjacent to vertices in V . The variable and clause gadgets are on the left-side and right-side of V , respectively. In this example, we consider a clause $C_j = x_i \vee \neg x_{i+1} \vee x_{i+2}$. Moreover, suppose this is the second positive appearance of x_i and the first negative appearance of x_{i+1} , and x_i corresponds to c_j^1 and x_{i+1} corresponds to c_j^2 . Suppose V contains 6 vertices indexed from top to bottom, and the set corresponding to these two appearances are $\{1, 3, 4\}$ and $\{3, 4, 6\}$ respectively. The star boundary denote the vertices that we can assume to be in any locating-dominating set, without loss of generality. The square boundary corresponds to selection of other vertices in S . In the above example, it corresponds to setting both x_i and x_{i+1} to **True**. On the clause side, the selection corresponds to selecting x_i to satisfy the clause C_j .

- $x_i^1, \neg x_i^1, x_i^2, \neg x_i^2$, and the corresponding edges, as shown in Figure 1. Let A_i be the collection of these twelve vertices and we define $A = \cup_{i=1}^n A_i$. Define $X_i := \{x_i^1, \neg x_i^1, x_i^2, \neg x_i^2\}$.
- To construct a clause gadget for C_j , the reduction starts with a star graph centered at γ_j^0 and with four leaves $\{\gamma_j^1, \gamma_j^2, \gamma_j^3, \gamma_j^4\}$. It then adds three vertices c_j^1, c_j^2, c_j^3 and the corresponding edges shown in Figure 1. Let B_j be the collection of these eight vertices and $B = \cup_{j=1}^m B_j$.
 - Let p be the smallest positive integer such that $4n \leq \binom{2p}{p}$. Define \mathcal{F}_p as the collection of subsets of $[2p]$ that contains exactly p integers (such a collection \mathcal{F}_p is called a *Sperner family*). Define **set-rep** : $\cup_{i=1}^n X_i \rightarrow \mathcal{F}_p$ as an injective function by arbitrarily assigning a set in \mathcal{F}_p to a vertex x_i^ℓ or $\neg x_i^\ell$, for every $i \in [n]$ and $\ell \in [2]$. In other words, every appearance of a literal is assigned a distinct subset in \mathcal{F}_p .
 - The reduction adds a *connection portal* V , which is a clique on $2p$ vertices v_1, v_2, \dots, v_{2p} . For every vertex v_q in V , the reduction adds a pendant vertex u_q adjacent to v_q .
 - For each vertex $x_i^\ell \in X$ where $i \in [n]$ and $\ell \in [2]$, the reduction adds edges (x_i^ℓ, v_q) for every $q \in \mathbf{set-rep}(x_i^\ell)$. Similarly, it adds edges $(\neg x_i^\ell, v_q)$ for every $q \in \mathbf{set-rep}(\neg x_i^\ell)$.
 - For a clause C_j , suppose variable x_i appears positively for the ℓ^{th} time as the r^{th} variable in C_j . For example, x_i appears positively for the second time as the third variable in C_j . Then, the reduction adds edges across B and V such that the vertices c_j^r and x_i^ℓ have the same neighbourhood in V , namely, the set $\{v_q : q \in \mathbf{set-rep}(x_i^\ell)\}$. Similarly, it adds edges for the negative appearance of the variables.

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This concludes the construction of G . The reduction sets $k = 4n + 3m + 2p$ and returns (G, k) as the reduced instance of LOCATING-DOMINATING SET.

We now provide an overview of the proof of correctness in the reverse direction. The crux of the correctness is: Without loss of generality, all the vertices in the connection portal V are present in any locating-dominating set S of G . Consider a vertex, say x_i^1 , on the ‘variable-side’ of S and a vertex, say c_j^1 , on the ‘clause-side’ of S . If both of these vertices have the same neighbors in the connection portal and are not adjacent to the vertices in $S \setminus V$, then at least one of x_i^1 or c_j^1 must be included in S .

More formally, suppose S is a locating-dominating set of G of size at most $k = 4n + 3m + 2p$. Then, we prove that S must have exactly 4 vertices from each variable gadget and exactly 3 vertices from each clause gadget. Further, S contains either $\{\alpha_0^i, \beta_0^i, x_i^1, x_i^2\}$ or $\{\alpha_0^i, \beta_0^i, \neg x_i^1, \neg x_i^2\}$, but no other combination of vertices in the variable gadget corresponding to x_i . For a clause gadget corresponding to C_j , S contains either $\{\gamma_j^0, c_j^2, c_j^3\}$, $\{\gamma_j^0, c_j^1, c_j^3\}$, or $\{\gamma_j^0, c_j^1, c_j^2\}$, but no other combination. These choices imply that c_j^1 , c_j^2 , or c_j^3 are not adjacent to any vertex in $S \setminus V$. Consider the first case and suppose c_j^1 corresponds to the second positive appearance of variable x_i . By the construction, the neighborhoods of x_i^2 and c_j^1 in V are identical. This forces a selection of $\{\alpha_0^i, \beta_0^i, x_i^1, x_i^2\}$ in S from the variable gadget corresponding to x_i , which corresponds to setting x_i to True. Hence, a locating dominating set S of size at most k implies a satisfying assignment of ψ .

Sketch of Proof of Theorem 1. Note that since each component of $G - V$ is of constant order, the tree-width of G is $\mathcal{O}(|V|)$. By the asymptotic estimation of the central binomial coefficient, $\binom{2p}{p} \sim \frac{4^p}{\sqrt{\pi p}}$ [2]. To get the upper bound of $2p$, we scale down the asymptotic function and have $4n \leq \frac{4^p}{2^p} = 2^p$. Since we choose the value of p as small as possible such that $2^p \geq 4n$, we choose $p = \log(n) + 3$. This ensures us that $p = \mathcal{O}(\log(n))$. And hence, $|V| = \mathcal{O}(\log(n))$ which implies $\text{tw}(G) = \mathcal{O}(\log n)$. The remaining arguments are standard for proving the conditional lower bound under ETH. ◀

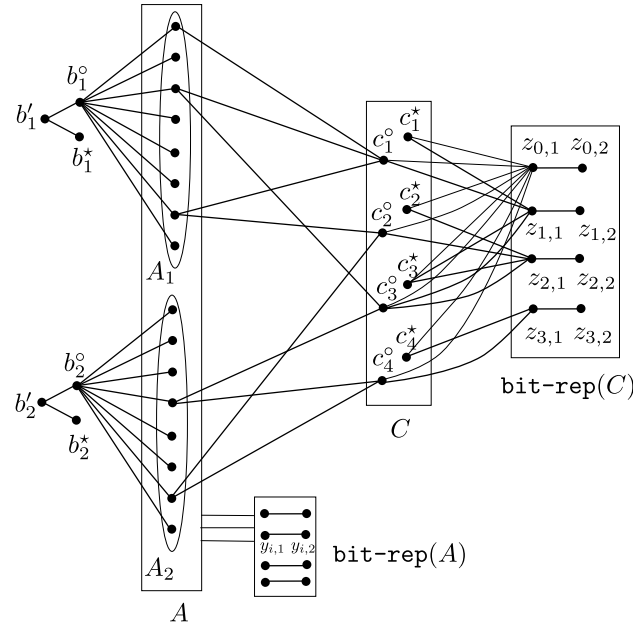
4 Locating-Dominating Set Parameterized by the Solution Size

In this section, we study the parameterized complexity of LOCATING-DOMINATING SET when parameterized by the solution size k . First, we formally prove that the problem admits a kernel with $\mathcal{O}(2^k)$ vertices, and hence a simple FPT algorithm running in time $2^{\mathcal{O}(k^2)}$. Next, we prove that both results mentioned above are optimal under the ETH.

► **Proposition 7.** *LOCATING-DOMINATING SET admits a kernel with $\mathcal{O}(2^k)$ vertices and an algorithm running in time $2^{\mathcal{O}(k^2)} + \mathcal{O}(k \log n)$.*

Proof. Slater proved that for any graph G on n vertices with a locating-dominating set of size k , we have $n \leq 2^k + k - 1$ [64]. Hence, if $n > 2^k + k - 1$, we can return a trivial NO instance (this check takes time $\mathcal{O}(k \log n)$). Otherwise, we have a kernel with $\mathcal{O}(2^k)$ vertices. In this case, we can enumerate all subsets of vertices of size k , and for each of them, check in quadratic time if it is a valid solution. Overall, this takes time $\binom{n}{k} n^2$; since $n \leq 2^k + k - 1$, this is $\binom{2^{\mathcal{O}(k)}}{k} \cdot 2^{\mathcal{O}(k)}$, which is $2^{\mathcal{O}(k^2)}$. ◀

To prove Theorem 2, we present a reduction that takes as input an instance ψ , with n variables, of 3-SAT and returns an instance (G, k) of LOCATING-DOMINATING SET such that $|V(G)| = 2^{\mathcal{O}(\sqrt{n})}$ and $k = \mathcal{O}(\sqrt{n})$. By adding dummy variables in each set, we can assume that \sqrt{n} is an even integer. Suppose the variables are named $x_{i,j}$ for $i, j \in [\sqrt{n}]$. The reduction constructs graph G as follows:



■ **Figure 2** Suppose an instance ψ of 3-SAT has $n = 9$ variables and 4 clauses. We do not show the third variable bucket and explicit edges across A and $\text{bit-rep}(A)$ for brevity.

- It partitions the variables of ψ into \sqrt{n} many *buckets* $X_1, X_2, \dots, X_{\sqrt{n}}$ such that each bucket contains exactly \sqrt{n} many variables. Let $X_i = \{x_{i,j} \mid j \in [\sqrt{n}]\}$ for all $i \in [\sqrt{n}]$.
 - For every X_i , it constructs set A_i of $2^{\sqrt{n}}$ new vertices, $A_i = \{a_{i,\ell} \mid \ell \in [2^{\sqrt{n}}]\}$. Each vertex in A_i corresponds to a unique assignment of variables in X_i . Let A be the collection of all the vertices added in this step.
 - For every X_i , the reduction adds a path on three vertices $b_i^{\circ}, b_i',$ and b_i^* with edges (b_i°, b_i') and (b_i', b_i^*) . Suppose B is the collection of all the vertices added in this step.
 - For every X_i , it makes b_i° adjacent with every vertex in A_i .
- For every clause C_j , the reduction adds a pair of vertices c_j°, c_j^* . For a vertex $a_{i,\ell} \in A_i$ for some $i \in [\sqrt{n}]$, and $\ell \in [2^{\sqrt{n}}]$, if the assignment corresponding to vertex $a_{i,\ell}$ satisfies clause C_j , then it adds edge $(a_{i,\ell}, c_j^{\circ})$.
- The reduction adds a bit-representation gadget⁴ to locate set A . Once again, informally speaking, it adds some supplementary vertices such that it is safe to assume these vertices are present in a locating-dominating set, and they locate every vertex in A . More precisely:
 - First, set $q := \lceil \log(|A|) \rceil + 1$. This value for q allows to uniquely represent each integer in $[|A|]$ by its bit-representation in binary (starting from 1 and not 0).
 - For every $i \in [q]$, the reduction adds two vertices $y_{i,1}$ and $y_{i,2}$ and edge $(y_{i,1}, y_{i,2})$.
 - For every integer $q' \in [|A|]$, let $\text{bit}(q')$ denote the binary representation of q' using q bits. Connect $a_{i,\ell} \in A$ with $y_{i,1}$ if the i^{th} bit in $\text{bit}((i + (\ell - 1) \cdot \sqrt{n}))$ is 1.
 - It adds two vertices $y_{0,1}$ and $y_{0,2}$, and edge $(y_{0,1}, y_{0,2})$. It also makes every vertex in A adjacent with $y_{0,1}$.
 Let $\text{bit-rep}(A)$ be the collection of the vertices $y_{i,1}$ for all $i \in \{0\} \cup [q]$ added in this step.

⁴ With the problem-specific adaptations, the bit-representation gadgets resembles the gadget used in [29].

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- Finally, the reduction adds a bit-representation gadget to locate set C . However, it adds the vertices in such a way that for any pair c_j°, c_j^\star , the supplementary vertices adjacent to them are identical.
 - The reduction sets $p := \lceil \log(|C|/2) \rceil + 1$ and for every $i \in [p]$, it adds two vertices $z_{i,1}$ and $z_{i,2}$ and edge $(z_{i,1}, z_{i,2})$.
 - For every integer $j \in [|C|/2]$, let $\mathbf{bit}(j)$ denote the binary representation of j using q bits. Connect $c_j^\circ, c_j^\star \in C$ with $z_{i,1}$ if the i^{th} bit in $\mathbf{bit}(j)$ is 1.
 - It adds two vertices $z_{0,1}$ and $z_{0,2}$, and edge $(z_{0,1}, z_{0,2})$. It also makes every vertex in C adjacent with $y_{0,1}$.
Let $\mathbf{bit}\text{-rep}(C)$ be the collection of the vertices $z_{i,1}$ for all $i \in \{0\} \cup [p]$ added in this step.

This completes the reduction. See Figure 2 for an illustration. Please check this. The reduction sets

$$k = |B|/3 + (\lceil \log(|A|) \rceil + 1 + 1) + \lceil (\log(|C|/2) \rceil + 1 + 1) + \sqrt{n} = \mathcal{O}(\sqrt{n})$$

as $|B| = 3\sqrt{n}$, $|A| = \sqrt{n} \cdot 2^{\sqrt{n}}$, and $|C| = \mathcal{O}(n^3)$, and returns (G, k) as a reduced instance.

We present a brief overview of the proof of correctness in the reverse direction. Suppose S is a locating-dominating set of graph G of size at most k . Note that $b_i^\star, y_{i,2}$ and $z_{i,2}$ are pendant vertices for appropriate i . We argue that it is safe to consider that vertices $b_i^\circ, y_{i,1}$, and $z_{i,1}$ are in S . This already forces $|B|/2 + \lceil \log(|A|) \rceil + 2 + \lceil \log(|C|/2) \rceil + 2$ many vertices in S . The remaining \sqrt{n} many vertices need to locate vertices in pairs (b_i°, b_i^\star) and (c_j°, c_j^\star) for every $i \in [\sqrt{n}]$ and $j \in [|C|]$. Note that the only vertices that are adjacent with b_i° but are *not* adjacent with b_i^\star are in A_i . Also, the only vertices that are adjacent with c_j° but are *not* adjacent with c_j^\star are in A_j and correspond to an assignment that satisfies C_j . Hence, any locating-dominating set should contain at least one vertex in A_i (which will locate b_i° from b_i^\star) such that the union of these vertices resolves all pairs of the form (c_j°, c_j^\star) , and hence corresponds to a satisfying assignment of ψ .

Proof of Theorem 2. Assume there exists an algorithm, say \mathcal{A} , that takes as input an instance (G, k) of LOCATING-DOMINATING SET and correctly concludes whether it is a YES-instance in time $2^{o(k^2)} \cdot |V(G)|^{\mathcal{O}(1)}$. Consider algorithm \mathcal{B} that takes as input an instance ψ of 3-SAT, uses the reduction above to get an equivalent instance (G, k) of LOCATING-DOMINATING SET, and then uses \mathcal{A} as a subroutine. The correctness of algorithm \mathcal{B} follows from the correctness of the reduction and of algorithm \mathcal{A} . From the description of the reduction and the fact that $k = \sqrt{n}$, the running time of algorithm \mathcal{B} is $2^{\mathcal{O}(\sqrt{n})} + 2^{o(k^2)} \cdot (2^{\mathcal{O}(\sqrt{n})})^{\mathcal{O}(1)} = 2^{o(n)}$. This, however, contradicts the ETH. Hence, LOCATING-DOMINATING SET does not admit an algorithm with running time $2^{o(k^2)} \cdot |V(G)|^{\mathcal{O}(1)}$ unless the ETH fails.

For the second part of Theorem 2, assume that such a kernelization algorithm exists. Consider the following algorithm for 3-SAT. Given a 3-SAT formula on n variables, it uses the above reduction to get an equivalent instance of (G, k) such that $|V(G)| = 2^{\mathcal{O}(\sqrt{n})}$ and $k = \mathcal{O}(\sqrt{n})$. Then, it uses the assumed kernelization algorithm to construct an equivalent instance (H, k') such that H has $2^{o(k)}$ vertices and $k' \leq k$. Finally, it uses a brute-force algorithm, running in time $|V(H)|^{\mathcal{O}(k')}$, to determine whether the reduced instance, equivalently the input instance of 3-SAT, is a YES-instance. The correctness of the algorithm follows from the correctness of the respective algorithms and our assumption. The total running time of the algorithm is $2^{\mathcal{O}(\sqrt{n})} + (|V(G)| + k)^{\mathcal{O}(1)} + |V(H)|^{\mathcal{O}(k')} = 2^{\mathcal{O}(\sqrt{n})} + (2^{\mathcal{O}(\sqrt{n})})^{\mathcal{O}(1)} + (2^{o(\sqrt{n})})^{\mathcal{O}(\sqrt{n})} = 2^{o(n)}$. This, however, contradicts the ETH. ◀

5 Test Cover Parameterization by the Solution Size

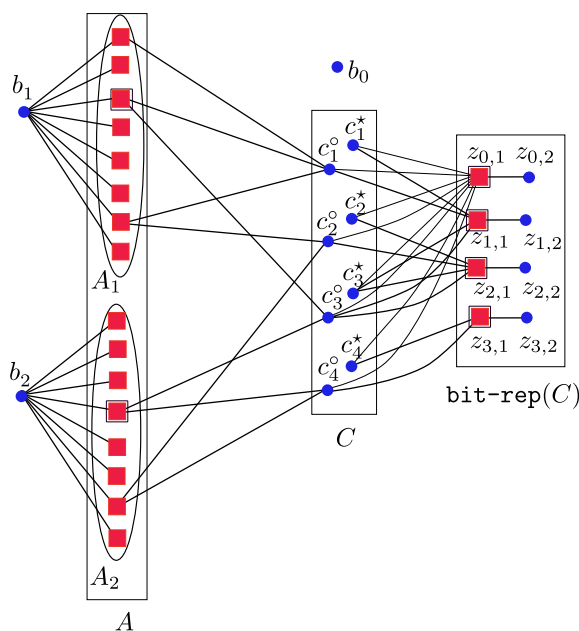
In this section, we present a reduction that is very close to the reduction used in the proof of Theorem 2 to prove Theorem 3.

For notational convenience, instead of specifying an instance of TEST COVER, we specify the auxiliary graph as mentioned in the definition. The reduction takes as input an instance ψ , with n variables and m clauses, of 3-SAT and returns a reduced instance $(G, \langle R, B \rangle, k)$ of TEST COVER and $k = \mathcal{O}(\log(n) + \log(m)) = \mathcal{O}(\log(n))$, using the sparcification lemma [44]. The reduction constructs graph G as follows.

- The reduction adds some dummy variables to ensure that $n = 2^{2^q}$ for some positive integer q which is power of 2. This ensures that $r = \log_2(n) = 2^q$ and $s = \frac{n}{r}$ both are integers. It partitions the variables of ψ into r many *buckets* X_1, X_2, \dots, X_r such that each bucket contains s many variables. Let $X_i = \{x_{i,j} \mid j \in [s]\}$ for all $i \in [r]$.
- For every X_i , the reduction constructs a set A_i of 2^s many red vertices, that is, $A_i = \{a_{i,\ell} \mid \ell \in [2^s]\}$. Each vertex in A_i corresponds to a unique assignment of the variables in X_i . Moreover, let $A = \cup_{i=1}^r A_i$.
- Corresponding to each X_i , let the reduction add a blue vertex b_i and the edges $(b_i, a_{i,\ell})$ for all $i \in [r]$ and $\ell \in [2^s]$. Let $B = \{b_i \mid i \in [r]\}$.
- For every clause C_j , the reduction adds a pair of blue vertices c_j°, c_j^\star . For a vertex $a_{i,\ell} \in A_i$ with $i \in [r]$, and $\ell \in [2^s]$, if the assignment corresponding to vertex $a_{i,\ell}$ satisfies the clause C_j , then the reduction adds the edge $(a_{i,\ell}, c_j^\circ)$. Let $C = \{c_j^\circ, c_j^\star \mid j \in [m]\}$.
- The reduction adds a bit-representation gadget to locate set C . However, it adds the vertices in such a way that for any pair c_j°, c_j^\star , the supplementary vertices adjacent to them are identical.
 - The reduction sets $p := \lceil \log(m) \rceil + 1$ and for every $i \in [p]$, it adds two vertices, a red vertex $z_{i,1}$ and a blue vertex $z_{i,2}$, and edge $(z_{i,1}, z_{i,2})$.
 - For every integer $j \in [m]$, let $\text{bit}(j)$ denote the binary representation of j using p bits. Connect $c_j^\circ, c_j^\star \in C$ with $z_{i,1}$ if the i^{th} bit in $\text{bit}(j)$ is 1.
 - It add two vertices $z_{0,1}$ and $z_{0,2}$, and edge $(z_{0,1}, z_{0,2})$. It also makes every vertex in C adjacent with $z_{0,1}$. Let $\text{bit-rep}(C)$ be the collection of all the vertices added in this step.
- The reduction adds an isolated blue vertex b_0 .

This completes the construction. The reduction sets $k = r + \frac{1}{2} |\text{bit-rep}(C)| = \mathcal{O}(\log(n)) + \mathcal{O}(\log(m)) = \mathcal{O}(\log(n))$, and returns $(G, \langle R, B \rangle, k)$ as an instance of TEST COVER. We refer to Figure 3 for an illustration.

We present a brief overview of the proof of correctness for the backward direction (\Leftarrow). Suppose R' is a set of tests of the graph G of order at most k . Since b_0 is an isolated blue vertex of G , it implies that the set R' dominates and locates every pair of vertices in $B \setminus \{b_0\}$. The blue vertices in $\text{bit-rep}(C)$ are pendant vertices that are adjacent with red vertices in $\text{bit-rep}(C)$. Hence, all the red vertices in $\text{bit-rep}(C)$ are in R' . The remaining r many vertices need to locate vertices in pairs (c_j°, c_j^\star) , for every $j \in [m]$, which have the same neighbourhood in $\text{bit-rep}(C)$. To do so, note that the only vertices adjacent to c_j° and *not* to c_j^\star are in A_i and corresponds to an assignment satisfying clause C_j . Hence, for every $j \in [m]$, the set R' should contain at least one vertex $\{a_{i,\ell}\}$ in order to locate c_j°, c_j^\star , where $(a_{i,\ell}, c_j^\circ)$ is an edge for some $i \in [r]$ and $\ell \in [2^s]$. Moreover, in order to dominate the vertices of B , for each $i \in [r]$, the set R' must have a vertex from each A_i . Hence, the set R' is forced to contain *exactly* one vertex from each A_i . Concatenating the assignments corresponding to each $a_{i,\ell}$ in R' , we thus obtain a satisfying assignment of ψ . Proof of Theorem 3 follows



■ **Figure 3** An illustrative example of the graph constructed by the reduction in Section 5. Red (squared) nodes denote the tests whereas blue (filled circle) nodes the elements.

from the arguments that are standard to proving such lower bounds.

6 Conclusion

We presented several results that advance our understanding of the algorithmic complexity of LOCATING-DOMINATING SET and TEST COVER, which we showed to have very interesting and rare parameterized complexities. Moreover, we believe the techniques used in this article can be applied to other identification problems to obtain relatively rare conditional lower bounds. The process of establishing such lower bounds boils down to designing *bit-representation gadgets* and *set-representation gadgets* for the problem in question.

Apart from the broad question of designing such lower bounds for other identification problems, we mention an interesting problem left open by our work. Can our tight double-exponential lower bound for LOCATING-DOMINATING SET parameterized by treewidth/treedepth be applied to the feedback vertex set number? The question could also be studied for other related parameters.

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