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Large deviation principle at speed n for the random metric in first-passage percolation

Julien VERGES*

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Abstract

Consider standard first-passage percolation on \mathbb{Z}^d . We study the lower-tail large deviations of the rescaled random metric $\widehat{\mathbf{T}}_n$ restricted to a box. If all exponential moments are finite, we prove that $\widehat{\mathbf{T}}_n$ follows the large deviation principle at speed n with a rate function J , in a suitable space of metrics. Moreover, we give three expressions for $J(D)$. The first two involve the metric derivative with respect to D of Lipschitz paths and the lower-tail rate function for the point-point passage time. The third is an integral against the 1-dimensional Hausdorff measure of a local cost. Under a much weaker moment assumption, we give an estimate for the probability of events of the type $\{\widehat{\mathbf{T}}_n \leq D\}$.

1 Introduction

1.1 Framework

1.1.1 First-passage percolation

We first present the model of *first-passage percolation* (FPP), introduced in 1966 by Hammersley and Welsh [15]. The reader interested in a summary of the achievements on this topic is invited to consult Auffinger, Damron and Hanson's survey [2]. Let $d \geq 2$ be an integer and \mathbb{E}^d the set of all non-oriented nearest-neighbour edges in \mathbb{Z}^d . A finite sequence $\pi := (x_0, \dots, x_r)$ of elements of \mathbb{Z}^d is called a *discrete path* if for all $i \in \llbracket 0, r-1 \rrbracket$, $(x_i, x_{i+1}) \in \mathbb{E}^d$. We denote by $\|\pi\|$ its number of edges.

Let ν be a probability distribution on $[0, \infty)$ and consider a family $(\tau_e)_{e \in \mathbb{E}^d}$ of i.i.d. random variables with distribution ν . In this paper a will denote the infimum of its support. The variable τ_e is called the *passage time along the edge e* . The *passage time along a discrete path $\pi = (x_0, \dots, x_r)$* is defined as

$$\tau(\pi) := \sum_{i=0}^{r-1} \tau_{(x_i, x_{i+1})}. \quad (1.1)$$

For all $A \subseteq \mathbb{R}^d$ and $x, y \in \mathbb{Z}^d$, the *passage time between x and y restricted in A* is defined as

$$\mathbf{T}_A(x, y) := \inf_{\substack{x \overset{\pi}{\rightsquigarrow} y \\ \pi \subseteq A}} \tau(\pi), \quad (1.2)$$

where the infimum spans over discrete paths included in A , whose endpoints are x and y . The map $\mathbf{T}_A(\cdot, \cdot)$ is a pseudometric on A . We call \mathbf{T}_A -*discrete geodesic* between x and y any minimizer in (1.2). We will write $\mathbf{T} := \mathbf{T}_{\mathbb{Z}^d} = \mathbf{T}_{\mathbb{R}^d}$. A well-known result ([2], Equation 2.4) states that, under a moment

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condition on ν , there exists a homogeneous function μ on \mathbb{R}^d , known as the *time constant*, such that for all $x \in \mathbb{Z}^d$,

$$\frac{\mathbf{T}(0, nx)}{n} \xrightarrow[n \rightarrow \infty]{\text{a.s.}} \mu(x). \quad (1.3)$$

Furthermore, without any moment assumption on ν , one can still define μ , such that for all $x \in \mathbb{Z}^d$,

$$\frac{\mathbf{T}(0, nx)}{n} \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \mu(x). \quad (1.4)$$

See e.g. ([5], Theorem 4) for the formulation (1.4). The time constant is a norm if

$$\nu(\{0\}) < p_c(\mathbb{Z}^d), \quad (\text{SubC})$$

where $p_c(\mathbb{Z}^d)$ is the critical parameter for bond percolation in \mathbb{Z}^d ; otherwise $\mu(x) = 0$ for all $x \in \mathbb{R}^d$. We only study the former case, as the latter is trivial for our purposes.

As a consequence of (1.4) the probability of an event of the form $\{\mathbf{T}(0, ne_1) \leq \zeta n\}$, with $\zeta < \mu(e_1)$ or $\{\mathbf{T}(0, ne_1) \geq \zeta n\}$, with $\zeta > \mu(e_1)$ (the so-called lower-tail and upper-tail large deviation events) converges to 0 as $n \rightarrow \infty$. In 1986, Kesten [17] obtained estimates for the speed of convergence: there exists ([17], Theorem 5.2) a convex decreasing function $J_K : (a, \mu(e_1)) \rightarrow (0, \infty)$ such that for all $a < \zeta < \mu(e_1)$,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(0, ne_1) \leq \zeta n) = J_K(\zeta). \quad (1.5)$$

Besides ([17], Theorem 5.9), under the assumption

$$\forall \lambda > 0, \quad \int_{[0, \infty)} \exp(\lambda t) \nu(dt) < \infty, \quad (\text{Moment})$$

for all $\zeta > \mu(e_1)$,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(0, ne_1) \geq \zeta n) = \infty. \quad (1.6)$$

If ν has a bounded support, we have the stronger estimate

$$\varliminf_{n \rightarrow \infty} -\frac{1}{n^d} \log \mathbb{P}(\mathbf{T}(0, ne_1) \geq \zeta n) > 0.$$

Kesten's proof of (1.6) may be adapted to any direction, i.e. under (Moment), for all $x \in \mathbb{Z}^d$ and $\zeta > \mu(x)$,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(0, nx) \geq \zeta n) = \infty. \quad (1.7)$$

We prove a somewhat stronger version in Appendix B.

1.1.2 Aim of the paper

Our first result, Theorem 1.1, is an extension of (1.5) to any direction, for $\mathbf{T}_{[0, n]^d}$ and \mathbf{T} . Consider

$$\mathcal{X} := \{(x, \zeta) \in \mathbb{R}^d \times (0, \infty) \mid \zeta > a\|x\|_1\}, \quad (1.8)$$

where $\|\cdot\|_1$ is the usual 1-norm, defined by (1.41), and a is the infimum of ν 's support, which may be zero. Define

$$X := [0, 1]^d. \quad (1.9)$$

Theorem 1.1. *There exists a function $J_{\text{pp}} : \mathcal{X} \rightarrow [0, \infty)$, such that for all $x, y \in X$ and $\zeta > a\|x - y\|_1$,*

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n\zeta) = \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathbf{T}_{[0, n]^d}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n\zeta) = J_{\text{pp}}(x - y, \zeta), \quad (1.10)$$

where $\lfloor \cdot \rfloor$ denotes the componentwise floor function. Moreover:

(i) J_{pp} is convex and absolutely homogeneous.

(ii) For all $(x, \zeta) \in \mathcal{X}$,

$$J_{\text{pp}}(x, \zeta) = J_{\text{pp}}(|x|, \zeta), \quad (1.11)$$

where $|x|$ denotes the vector of \mathbb{R}^d whose components are the modules of x 's components.

(iii) For all $(x_1, \zeta_1), (x_2, \zeta_2) \in \mathcal{X}$, if $0 \leq x_1 \leq x_2$ for the componentwise order and $\zeta_1 \geq \zeta_2$, then

$$J_{\text{pp}}(x_1, \zeta_1) \leq J_{\text{pp}}(x_2, \zeta_2). \quad (1.12)$$

(iv) For all $(x, \zeta) \in \mathcal{X}$, $J_{\text{pp}}(x, \zeta) = 0$ if and only if $\zeta \geq \mu(x)$.

For all $x \in \mathbb{R}^d$, we also define

$$J_{\text{pp}}(x, a\|x\|_1) := \lim_{\substack{\zeta \rightarrow a\|x\|_1 \\ \zeta > a\|x\|_1}} \uparrow J_{\text{pp}}(x, \zeta), \quad (1.13)$$

where the notation \uparrow is used to emphasize that $\zeta \mapsto J_{\text{pp}}(x, \zeta)$ is nonincreasing. Note that the extension of J_{pp} on $\overline{\mathcal{X}}$ is also convex and lower semicontinuous. We will call J_{pp} the *elementary rate function*, as it will appear as an integrand in the expression of more sophisticated rate functions.

Remark 1.2. By convexity and (iv), for all $x \in \mathbb{R}^d \setminus \{0\}$, the function $\zeta \mapsto J_{\text{pp}}(x, \zeta)$ is strictly decreasing on $[a\|x\|_1, \mu(x)]$.

We are interested in extending (1.10) for the random metric $\mathbf{T}_{[0, n]^d}$ rather than simply $\mathbf{T}(\lfloor nx \rfloor, \lfloor ny \rfloor)$. We define, for all $n \geq 1$,

$$\begin{aligned} \widehat{\mathbf{T}}_n : X \times X &\longrightarrow [0, \infty) \\ (x, y) &\longmapsto \frac{1}{n} \mathbf{T}_{[0, n]^d}(\lfloor nx \rfloor, \lfloor ny \rfloor). \end{aligned} \quad (1.14)$$

The function $\widehat{\mathbf{T}}_n$ is a pseudometric on X . Our second result, Theorem 1.7, gives an estimate for the probability of $\widehat{\mathbf{T}}_n$ taking values below a target pseudometric D , of the form

$$\mathbb{P}\left(\widehat{\mathbf{T}}_n \lesssim D\right) \approx \exp(-J^-(D)n).$$

Its main assumption is the existence of $\xi > 0$ such that

$$\mathbb{E} \left[\left(\min_{1 \leq i \leq d} \tau_i \right)^{d+\xi} \right] < \infty, \quad (\text{StrongShape})$$

where the τ_i are i.i.d. with distribution ν .

Under Assumption (Moment), Theorem 1.10 goes further and states that $(\widehat{\mathbf{T}}_n)_{n \geq 1}$ follows a *large deviation principle* at speed n , i.e. essentially an approximation of the form

$$\mathbb{P}\left(\widehat{\mathbf{T}}_n \simeq D\right) \approx \exp(-J(D)n).$$

1.2 Main theorems

Sections 1.2.1 and 1.2.2 contain the minimal definitions for our main results, namely Theorems 1.7 and 1.10 to make sense: respectively, the natural limit space for $\widehat{\mathbf{T}}_n$ and usual large deviation objects. Our main theorems are stated in Section 1.2.3.

1.2.1 The limit space

Almost surely, for all $n \geq 1$, $\widehat{\mathbf{T}}_n$ belongs to the space \mathcal{F}_b of bounded functions on X^2 . We endow \mathcal{F}_b with the uniform distance, defined for all $D_1, D_2 \in \mathcal{F}_b$ as

$$\mathfrak{d}_\infty(D_1, D_2) := \max_{(x,y) \in X^2} |D_1(x,y) - D_2(x,y)|, \quad (1.15)$$

For all $D_1, D_2 \in \mathcal{F}_b$ we denote by $D_1 \leq D_2$ the assertion

$$\forall x, y \in X, \quad D_1(x, y) \leq D_2(x, y), \quad (1.16)$$

which defines a partial order on \mathcal{F}_b .

Definition 1.3. Given a subset A of \mathbb{R}^d , a pseudometric D on A and a continuous path $\gamma : [0, T] \rightarrow A$ for the usual topology, we define the D -length of γ as

$$D(\gamma) := \sup_{0=t_0 < \dots < t_r=T} \sum_{i=0}^{r-1} D(\gamma(t_i), \gamma(t_{i+1})). \quad (1.17)$$

In the special case where D is the metric induced by $\|\cdot\|_1$, we will denote by $\|\gamma\|_1$ the D -length of γ . We say that γ is a D -geodesic if it is Lipschitz for $\|\cdot\|_1$, and

$$D(\gamma) = D(\gamma(0), \gamma(T)). \quad (1.18)$$

For all $t \in (0, T)$ such that the following limit exists, we define the *metric derivative* of γ at t , with respect to D as

$$|\dot{\gamma}|_D(t) := \lim_{h \rightarrow 0} \frac{1}{|h|} D(\gamma(t), \gamma(t+h)). \quad (1.19)$$

Note that our definition of geodesic differ from the usual meaning, as we do not require that γ is an isometry with respect to D , but some regularity with respect to $\|\cdot\|_1$. All the pseudometrics D we will consider are dominated by $\|\cdot\|_1$, therefore their geodesics will also be Lipschitz for the metric D .

Definition 1.4. Let $L > 0$ and g be a norm on \mathbb{R}^d . We define \mathcal{D}_g^L as the set of all pseudometrics D on X such that

(i) For all $x, y \in X$,

$$D(x, y) \leq g(x - y). \quad (1.20)$$

(ii) For all $x, y \in X$, there exists a D -geodesic σ from x to y , such that $\|\sigma\|_1 \leq L$.

We define

$$\mathcal{D}_g := \bigcup_{L \geq 0} \mathcal{D}_g^L. \quad (1.21)$$

1.2.2 Large deviations

We give here some general large deviation tools. See Dembo-Zeitouni (2009) [11] for the general theory.

Definition 1.5. Let \mathbb{X} be a Hausdorff topological space. We call *rate function* a lower semicontinuous map $I : \mathbb{X} \rightarrow [0, \infty]$, i.e. a map whose sublevels $\{x \in \mathbb{X} \mid I(x) \leq t\}$, for $t \geq 0$, are closed. We further say that I is a *good rate function* if its sublevels are compact.

We say that a random process $(X_n)_{n \geq 1}$ with values in \mathbb{X} follows the *large deviation principle* (LDP), at speed a_n , with the rate function I if for every Borel set $A \subseteq \mathbb{X}$,

$$\inf_{x \in \bar{A}} I(x) \leq \varliminf_{n \rightarrow \infty} -\frac{1}{a_n} \log \mathbb{P}(X_n \in A) \leq \overline{\lim}_{n \rightarrow \infty} -\frac{1}{a_n} \log \mathbb{P}(X_n \in A) \leq \inf_{x \in \bar{A}} I(x). \quad (1.22)$$

In this article, except when stated otherwise, we will only consider LDPs at speed n . Lemma 1.6 will be of constant use. It is a consequence of arguments given in the proof of Theorem 4.1.11 in [11]. The version here was used in ([22], Lemma 1.2) with the speed n^d instead of n . Apart from this difference, the proof may be copied verbatim.

Lemma 1.6. *Let $(\mathbb{X}, d_{\mathbb{X}})$ be a metric space and $(X_n)_{n \geq 1}$ a random process with values in \mathbb{X} . Define, for all $x \in \mathbb{X}$,*

$$\bar{I}(x) := \lim_{\varepsilon \rightarrow 0} \uparrow \overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(d_{\mathbb{X}}(x, X_n) \leq \varepsilon)$$

and

$$\underline{I}(x) := \lim_{\varepsilon \rightarrow 0} \uparrow \underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(d_{\mathbb{X}}(x, X_n) \leq \varepsilon).$$

Then

(i) \bar{I} and \underline{I} are rate functions on \mathbb{X} .

(ii) For every open set $U \subseteq \mathbb{X}$,

$$\overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(X_n \in U) \leq \inf_{x \in U} \bar{I}(x). \quad (1.23)$$

(iii) For every compact set $K \subseteq \mathbb{X}$,

$$\underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(X_n \in K) \geq \min_{x \in K} \underline{I}(x). \quad (1.24)$$

If $I := \bar{I} = \underline{I}$, we say that $(X_n)_{n \geq 1}$ satisfies the *weak LDP* at speed n , with the rate function I .

1.2.3 Result statements

For all $D \in \mathcal{D}_{\mu}$, $n \in \mathbb{N}^*$ and $\varepsilon > 0$, we introduce the event

$$\text{LD}_n^-(D, \varepsilon) := \left\{ \forall x, y \in X, \quad \widehat{\mathbf{T}}_n(x, y) \leq D(x, y) + \varepsilon \right\}. \quad (1.25)$$

We then define

$$\bar{J}^-(D) := \lim_{\varepsilon \rightarrow 0} \uparrow \overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{LD}_n^-(D, \varepsilon)) \quad (1.26)$$

and

$$\underline{J}^-(D) := \lim_{\varepsilon \rightarrow 0} \uparrow \underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{LD}_n^-(D, \varepsilon)). \quad (1.27)$$

Those functions are both rate functions. Theorem 1.7 states that $J^-(D) := \bar{J}^-(D) = \underline{J}^-(D)$ and provides several expressions of $J^-(D)$, which involve:

- The elementary rate function J_{pp} , defined in Theorem 1.1.
- A *highway network* of D , which is essentially a family of disjoint D -geodesics dense enough so that any two points $x, y \in X$ are linked by a path inside the network with D -length $D(x, y)$ (see Definition 2.4).
- The *gradient by paths* $(\text{P-grad } D)_z(u)$ of D at $z \in X$, in the direction $u \in \mathbb{R}^d$, which describes the D -length of infinitesimal paths originating from D with speed u (see Definition 2.8).

In Equation (1.31), \mathcal{H}^1 denotes the usual 1-dimensional Hausdorff measure on \mathbb{R}^d , whose definition is recalled in Section 1.5, and \mathbb{S}_2 denotes the Euclidean unit sphere.

Theorem 1.7. *Assume (SubC) and (StrongShape). For all $D \in \mathcal{D}_{\mu}$,*

$$J^-(D) := \underline{J}^-(D) = \bar{J}^-(D). \quad (1.28)$$

If $(\sigma_k : [0, L] \rightarrow X)_{k \geq 1}$ is a highway network for D , then

$$J^-(D) = \sum_{k \geq 1} \int_0^L J_{\text{pp}}(\sigma'_k(t), |\sigma_k|_D(t)) dt. \quad (1.29)$$

Moreover,

$$\begin{aligned} J^-(D) &= \sup \left\{ \sum_{k=1}^K \int_0^{T_k} J_{\text{pp}}(\gamma'_k(t), |\gamma_k|_D(t)) dt \mid \begin{array}{l} K \in \mathbb{N}, (\gamma_k : [0, T_k] \rightarrow X)_{1 \leq k \leq K} \text{ 1-Lipschitz,} \\ \text{injective and pairwise disjoint.} \end{array} \right\} \\ &= \sup \left\{ \sum_{k=1}^{\infty} \int_0^{T_k} J_{\text{pp}}(\gamma'_k(t), |\gamma_k|_D(t)) dt \mid \begin{array}{l} (\gamma_k : [0, T_k] \rightarrow X)_{k \geq 1} \text{ 1-Lipschitz,} \\ \text{injective and pairwise disjoint.} \end{array} \right\}, \end{aligned} \quad (1.30)$$

and

$$J^-(D) = \int_X \max_{u \in \mathbb{S}_2} J_{\text{pp}}(u, (\text{P-grad } D)_z(u)) \mathcal{H}^1(dz). \quad (1.31)$$

Besides, for all distinct $D_1, D_2 \in \mathcal{D}_\mu$, if $D_1 \leq D_2$ and $J^-(D_2) < \infty$,

$$J^-(D_1) > J^-(D_2). \quad (1.32)$$

Remark 1.8. Equation (1.31) holds with the unit sphere with respect to any norm instead of \mathbb{S}_2 , provided one also takes the Hausdorff measure associated with the chosen norm. However, we use the Euclidean norm because it is the usual framework.

Remark 1.9. Contrary to (1.29), (1.31) does not depend on an arbitrary choice of highway network for D . Actually, we will prove that if $J^-(D) < \infty$, then for \mathcal{H}^1 -almost all z , for all $u \in \mathbb{S}_2$ except maybe in one direction, $(\text{P-grad } D)_z(u) = \mu(u)$ (see Lemma 4.9). In particular, by Item in Theorem 1.1.(iv), for \mathcal{H}^1 -almost all z , the maximum in (1.31) is either 0 or the only positive value of $J_{\text{pp}}(u, (\text{P-grad } D)_z(u))$.

Under stronger assumptions, the LDP holds with a rate function that coincides with J^- on \mathcal{D}_μ , and is infinite outside.

Theorem 1.10. *Assume (SubC) and (Moment). The process $(\widehat{\mathbf{T}}_n)_{n \geq 1}$ follows the large deviation principle at speed n with the good rate function*

$$\begin{aligned} J : \mathcal{F}_b &\longrightarrow [0, \infty] \\ D &\longmapsto \begin{cases} J^-(D) & \text{if } D \in \mathcal{D}_\mu, \\ \infty & \text{otherwise.} \end{cases} \end{aligned} \quad (1.33)$$

In other words, for all $A \in \mathcal{B}(\mathcal{F}_b)$,

$$\inf_{D \in A} J(D) \leq \varliminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\widehat{\mathbf{T}}_n \in A) \leq \varlimsup_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\widehat{\mathbf{T}}_n \in A) \leq \inf_{D \in \bar{A}} J(D). \quad (1.34)$$

1.3 Open questions and related works

Upper-tail large deviations for the point-point and face-face times. Contrary to the lower-tail, the order of the upper-tail large deviation probability $\mathbb{P}(\mathbf{T}(0, ne_1) \geq n\zeta)$, with $\zeta > e_1$, depends on the distribution. Kesten ([17], Theorem 5.9) proved that if ν has a bounded support then it is of order $\exp(-n^d)$. Basu, Ganguly and Sly [3] later proved the existence of the rate function, in dimension 2,¹ under a regularity assumption on ν . This regularity assumption was recently relaxed by the author (see Corollary 1.6 in [22]).

Concerning distributions with unbounded support, Cranston, Gauthier and Mountford gave ([9], Theorem 1.3) a criterion for $\mathbb{P}(\mathbf{T}(0, ne_1) \geq n\zeta)$ to be of order $\exp(-n^d)$, provided the tail of ν has a

¹Their proof may be adapted in any dimension, though.

specific shape. In a recent article, Cosco and Nakajima [7] proved that for all $0 < r < d$, if the tail of ν decays like $\exp(-t^r)$, then $\mathbb{P}(\mathbf{T}(0, ne_1) \geq n\zeta)$ has order $\exp(-n^r)$, and they gave an expression of the point-point rate function.

It is worth noting that the so-called face-face passage time (i.e. the minimal passage time among paths traversing the box $\llbracket 0, n \rrbracket^d$ from one face to the opposite one) seems to display less diverse orders in its upper-tail large deviations. Indeed Chow and Zhang [6] showed that if ν has one exponential moment, then the probability that the face-face passage time takes abnormally large values is of order $\exp(-n^d)$. They also proved the existence of rate function.

Large deviations at the metric level. Assuming that ν is supported by $[a, b]$, with $0 < a < b < \infty$, the author [22] proved a LDP at speed n^d for $\widehat{\mathbf{T}}_n$, with a good rate function I . An integral expression similar to (1.31) is provided, except the measure is the Lebesgue measure on X . For all possible adherence values D of $\widehat{\mathbf{T}}_n$, $I(D) = 0$ if and only if $D \leq \mu$, and is finite except on marginal cases, meaning that result provides an appropriate estimate for the probability of $\{\widehat{\mathbf{T}}_n \simeq D\}$, whenever D is not bounded by μ . On the other hand, whenever $D \leq \mu$ and $D \neq \mu$, $J(D) > 0$ and is the limit of metrics D_n such that $J(D_n) < \infty$, meaning that Theorem 1.10 provides an appropriate estimate for the probability of $\{\widehat{\mathbf{T}}_n \simeq D\}$, whenever $D \leq \mu$. Consequently, the LDPs at speed n and n^d give a full picture of the large deviations for $(\widehat{\mathbf{T}}_n)_{n \geq 1}$, in the sense that for all positive sequence $(a_n)_{n \geq 1}$ satisfying

$$a_n \ll n, \text{ or } n \ll a_n \ll n^d, \text{ or } n^d \ll a_n, \quad (1.35)$$

for all pseudometrics D on X , either

$$\lim_{\varepsilon \rightarrow 0} \liminf_{n \rightarrow \infty} -\frac{1}{a_n} \log \mathbb{P}(\text{LD}_n(D, \varepsilon)) = \infty, \quad (1.36)$$

or

$$\lim_{\varepsilon \rightarrow 0} \limsup_{n \rightarrow \infty} -\frac{1}{a_n} \log \mathbb{P}(\text{LD}_n(D, \varepsilon)) = 0. \quad (1.37)$$

In other words, there cannot be a third speed a_n , for which $(\widehat{\mathbf{T}}_n)_{n \geq 1}$ satisfies the LDP with a rate function taking at least one positive, finite value.

However, if $\nu([t, \infty))$ decays like $\exp(-t^r)$, with $1 < r < d$ then at least three LDPs, at three different speeds, are required to describe the large deviations of $(\widehat{\mathbf{T}}_n)_{n \geq 1}$. Indeed, in this regime the probability of deviation events studied in the present article have order $\exp(-n)$, while as stated above, the upper-tail deviations of the point-point passage time have order $\exp(-n^r)$ and the upper-tail deviations of the face-face passage time have order $\exp(-n^d)$. It is thus plausible that there exist at least three speeds for which $(\widehat{\mathbf{T}}_n)_{n \geq 1}$ follows a LDP with a nontrivial rate function. The case where $\nu([t, \infty))$ decays like $\exp(-t)$ may also be quite rich, because lower-tail behaviours and some upper-tail behaviours are both of order $\exp(-n)$. In particular the conclusion of Theorem 1.10 fails in this context, because if $(\widehat{\mathbf{T}}_n)_{n \geq 1}$ followed a LDP at speed n , the associated rate function would take positive, finite values for some metric greater than μ .

Large deviations for the chemical distance in bond percolation. In supercritical bond percolation, we call *chemical distance* between two vertices the length of the shortest open path between these vertices. In the framework of FPP, this corresponds to the case where ν is supported by $\{1, \infty\}$. Garet and Marchand [13] showed that the probability of large deviations events for the chemical distance has order $\exp(-n)$. For the upper-tail part, Dembin and Nakajima [10] recently proved the existence of the associated rate function.

1.4 Outline of the proofs

In Section 2, we give some topological preliminaries about \mathcal{D}_g . We show that the sets \mathcal{D}_g^L are compact subsets of \mathcal{F}_b , any pseudometric $D \in \mathcal{D}_\mu$ has a highway network and the D -length of Lipschitz paths is given by an integral involving the gradient by paths. The highway method is a refinement of an argument used by the author in [22].

In Section 3, we prove Theorem 1.1. By the FKG inequality, the probability that two translations of the event in (1.10) occur simultaneously is greater than the product of the individual probabilities. This allows for a subadditive argument, providing the existence and convexity of the elementary rate function.

In Section 4, we prove Theorem 1.7. We first upper bound $\bar{J}^-(D)$ by the right-hand side of (1.29). To do so, we note that prescribing a passage time smaller than D between a large number of milestones scattered along the highways essentially implies prescribing a passage time smaller than D everywhere. Thanks to the FKG inequality, the probability of this scenario is lower bounded by the product of the individual point-point lower-tail deviation events involved. Applying Theorem 1.1 and letting the number of milestones tend to infinity yields the desired bound. We then lower bound $\underline{J}^-(D)$ by the right-hand side of (1.30), using a somehow opposite strategy. Rather than providing an appropriate scenario for the large deviation event $\text{LD}_n^-(D, \varepsilon)$ to occur, we show that for all families (γ_k) as in (1.30), on the large deviation event, there exists a family of abnormally fast *pairwise disjoint* discrete paths that "follow" the paths γ_k . Finally, we prove (1.31) by the so-called area formula (see e.g. Corollary 5.1.13 in [18]). The hard part is to show that $(\text{P-grad } D)$ has \mathcal{H}^1 -almost everywhere a simple expression with respect to the speeds of the highways σ_k .

In Section 5, we prove Theorem 1.10. For all $D \in \mathcal{F}_b$, $n \in \mathbb{N}^*$ and $\varepsilon > 0$, we introduce the event

$$\text{LD}_n(D, \varepsilon) := \left\{ \mathfrak{d}_\infty(\widehat{\mathbf{T}}_n, D) \leq \varepsilon \right\}, \quad (1.38)$$

and the rate functions

$$\bar{J}(D) := \lim_{\varepsilon \rightarrow 0} \uparrow \overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{LD}_n(D, \varepsilon)), \quad (1.39)$$

$$\underline{J}(D) := \lim_{\varepsilon \rightarrow 0} \uparrow \underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{LD}_n(D, \varepsilon)), \quad (1.40)$$

i.e. the functions \bar{I} and \underline{I} involved in Lemma 1.6, in the special case $\mathbb{X} = \mathcal{F}_b$ and $X_n = \widehat{\mathbf{T}}_n$. We first show that if the metric $\widehat{\mathbf{T}}_n^{(b)}$ associated with the truncated passage times $\tau_\varepsilon \wedge b$ is exponentially equivalent to another pseudometric $\widetilde{\mathbf{T}}_n^{(b)}$, which has better tightness properties. We then show that $\widetilde{\mathbf{T}}_n^{(b)}$, thus $\widehat{\mathbf{T}}_n^{(b)}$, follows the LDP with the rate function J , which essentially amounts to showing that $\underline{J} = \bar{J} = J$. In the case where ν has an unbounded support, we use the fact that the truncated passage times are exponentially good approximations of $\widehat{\mathbf{T}}_n$.

1.5 Notations and conventions

Vectors of \mathbb{R}^d and norms on \mathbb{R}^d . Let $(e_i)_{1 \leq i \leq n}$ denote the canonical basis of \mathbb{R}^d . We endow \mathbb{R}^d with the norms defined by

$$\|x\|_1 := \sum_{i=1}^d |x_i|, \text{ and } \|x\|_2 := \left(\sum_{i=1}^d x_i^2 \right)^{1/2}, \quad (1.41)$$

for all $x = (x_1, \dots, x_d) \in \mathbb{R}^d$. We define d as the metric associated with $\|\cdot\|_1$, \mathbb{S}_1 the unit sphere for $\|\cdot\|_1$ and $\text{B}_1(z, r)$ (resp. $\overline{\text{B}}_1(z, r)$) the open (resp. closed) ball of center z and radius r for $\|\cdot\|_1$. Likewise, we denote by \mathbb{S}_2 , $\text{B}_2(z, r)$ and $\overline{\text{B}}_2(z, r)$ their analogues for $\|\cdot\|_2$. Given a norm g on \mathbb{R}^d , we define

$$\|g\|_1 := \sup_{u \in \mathbb{S}_1} g(u) = \sup_{x \in \mathbb{R}^d \setminus \{0\}} \frac{g(x)}{\|x\|_1}. \quad (1.42)$$

For all $\lambda > 0$, we say that a function with values in \mathbb{R}^d is λ -Lipschitz if it is λ -Lipschitz for the norm $\|\cdot\|_1$.

For all $x \in \mathbb{R}^d$, we denote by $|x|$ (resp. $\lfloor x \rfloor$) the element of \mathbb{R}^d whose components are the modules (resp. the floors) of the components of x . We denote by \leq the componentwise order on \mathbb{R}^d .

Edges and paths. We will identify any discrete path $\pi = (x_0, \dots, x_r)$ to the continuous path defined as the piecewise affine function $\pi : [0, r] \rightarrow \mathbb{R}^d$ such that for all $i \in \llbracket 0, r \rrbracket$, $\pi(i) = x_i$. The number of edges $\|\pi\|$ of π , seen as a discrete path, is equal to $\|\pi\|_1$ and $\|\pi\|_2$ as defined by (1.17), so there is no collision of notations. Given a continuous path $\gamma : [0, T] \rightarrow \mathbb{R}^d$, and $x, y \in \mathbb{R}^d$, we will write $x \rightsquigarrow y$ if $\gamma(0) = x$ and $\gamma(T) = y$. For all subsets A of \mathbb{Z}^d , we denote by $E(A)$ the set of edges whose endpoints are both in A . We say that a point $x \in \mathbb{R}^d$ belongs to an edge $e \in E(A)$ if it belongs to the segment with the same endpoints as e .

Whenever a continuous path $\gamma : [0, T] \rightarrow \mathbb{R}^d$ appears inside an expression involving set operators, we identify it with its image. For example, " $x \in \gamma$ " will mean " $x \in \gamma([0, T])$ ".

Hausdorff measure. Consider a Borel subset A of \mathbb{R}^d . For all $R > 0$, we define

$$\phi_R^1(A) := \inf \left\{ \sum_{i \geq 1} \text{diam}(A_i) \mid \{A_i\}_{i \geq 1} \text{ subsets of } \mathbb{R}^d \text{ s.t. } A \subseteq \bigcup_{i \geq 1} A_i \text{ and } \forall i \geq 1, \text{diam}(A_i) \leq R \right\}, \quad (1.43)$$

where $\text{diam}(A_i)$ denotes the diameter of A_i for the Euclidean norm. The *1-dimensional Hausdorff measure* of A is defined as (see e.g. Section 2.1 in [18])

$$\mathcal{H}^1(A) := \lim_{R \rightarrow 0} \uparrow \phi_R^1(A). \quad (1.44)$$

For more details on the Hausdorff measure, see Krantz-Parks (2008) [18].

Miscellaneous We denote by $\#A$ the cardinal of a set A . For all integers $n \leq m$, we define $\llbracket n, m \rrbracket := [n, m] \cap \mathbb{Z}$. The infimum of ν 's support is denoted by a .

2 Topological preliminaries

2.1 Pseudometrics with bounded-length geodesics

In this section we fix a norm g and study the properties of the space \mathcal{D}_g . Up to changing the parametrization, any Lipschitz path on \mathbb{R}^d may be assumed to be 1-Lipschitz. Unless stated otherwise, for all $L > 0$ and $D \in \mathcal{D}_g^L$, our D -geodesics are 1-Lipschitz paths $\sigma : [0, L] \rightarrow X$.

2.1.1 Compactness

Proposition 2.1. *For all $L > 0$, the space \mathcal{D}_g^L is compact for the topology of the uniform convergence.*

Proof. Let $L > 0$. For all $D \in \mathcal{D}_g^L$, $x, x', y, y' \in X$, by triangle inequality,

$$|D(x, y) - D(x', y')| \leq D(x, x') + D(y, y') \leq g(x - x') + g(y - y').$$

In particular, \mathcal{D}_g^L is equicontinuous.

By Arzelà-Ascoli theorem it is thus sufficient to show that it is closed in \mathcal{F}_b . Let $(D_n)_{n \geq 1}$ be a sequence of elements of \mathcal{D}_g^L converging to $D \in \mathcal{F}_b$. Then D is clearly a pseudometric dominated by g . Let $x, y \in X$. Let us show that D has a geodesic from x to y with $\|\cdot\|_1$ -length at most L . For all $n \geq 1$, there exists a D_n -geodesic $\sigma_n : [0, L] \rightarrow X$ from x to y . Since all the σ_n are 1-Lipschitz, the family $(\sigma_n)_{n \geq 1}$ is equicontinuous, hence there exists an extraction $\varphi : \mathbb{N} \rightarrow \mathbb{N}$ and 1-Lipschitz path $\sigma : [0, L] \rightarrow X$ such that $\sigma_{\varphi(n)}$ uniformly converges to σ as $n \rightarrow \infty$. Consider a subdivision $0 = t_0 \leq t_1 \leq \dots \leq t_K = L$. For all $n \geq 1$, by definition of $\sigma_{\varphi(n)}$,

$$\sum_{k=0}^{K-1} D_{\varphi(n)}(\sigma_{\varphi(n)}(t_k), \sigma_{\varphi(n)}(t_{k+1})) = D_{\varphi(n)}(x, y).$$

Letting $n \rightarrow \infty$ yields, by uniform convergence,

$$\sum_{k=0}^{K-1} D(\sigma(t_k), \sigma(t_{k+1})) = D(x, y).$$

Taking the supremum with respect to the subdivision, we get

$$D(\sigma) = D(x, y). \quad (2.1)$$

Besides, $\sigma(0) = x$ and $\sigma(L) = y$, thus σ is a D -geodesic from x to y . Consequently, $D \in \mathcal{D}_g^L$. \square

2.1.2 The highway method

The main result of this paragraph is Proposition 2.5, with states that a pseudometric $D \in \mathcal{D}_\mu$ may be approximated by considering the uniform space g then "activating" a large number of geodesics for D . Moreover, those geodesics may be chosen injective and pairwise disjoint. Some of the work, gathered in Lemma 2.3, may easily be adapted from Lemma 3.3 in [22].

Definition 2.2. Let $L > 0$ and $D \in \mathcal{D}_\mu^L$. Consider a sequence $(\sigma_k : [0, L] \rightarrow X)_{k \geq 1}$ of D -geodesics. We recursively define the sequence $(D[(\sigma_k)_{1 \leq k \leq K}])_{K \geq 0}$ of functions in \mathcal{F}_b as such : $D[\emptyset] = \mu$ and for all $K \geq 0$ and $x, y \in X$,

$$\begin{aligned} D[(\sigma_k)_{1 \leq k \leq K+1}](x, y) := & D[(\sigma_k)_{1 \leq k \leq K}](x, y) \wedge \min_{0 \leq s, t \leq L} \left(D[(\sigma_k)_{1 \leq k \leq K}](x, \sigma_{K+1}(s)) \right. \\ & \left. + D(\sigma_{K+1}(s), \sigma_{K+1}(t)) + D[(\sigma_k)_{1 \leq k \leq K}](\sigma_{K+1}(t), y) \right). \end{aligned} \quad (2.2)$$

Lemma 2.3. We adopt the same notations as in Definition 2.2.

(i) For all $K \geq 0$, $D[(\sigma_k)_{1 \leq k \leq K}] \in \mathcal{D}_\mu$.

(ii) For all $K \geq 0$,

$$D[(\sigma_k)_{1 \leq k \leq K}] \geq D[(\sigma_k)_{1 \leq k \leq K+1}] \geq D. \quad (2.3)$$

(iii) If the sequence $((\sigma_k(0), \sigma_k(L)))_{k \geq 1}$ is dense in X^2 for the usual topology, then

$$D = \lim_{K \rightarrow \infty} D[(\sigma_k)_{1 \leq k \leq K}]. \quad (2.4)$$

Definition 2.4. Let $L > 0$ and $D \in \mathcal{D}_\mu^L$. We say that a sequence $(\sigma_k : [0, L] \rightarrow X)_{k \geq 1}$ of D -geodesics is a *highway network* for D if they are injective, pairwise disjoint and (2.4) holds.

Proposition 2.5. Let $L > 0$ and $D \in \mathcal{D}_\mu^L$. Then D has a highway network. Moreover, any injective D -geodesic σ may be chosen as the first highway in the network.

The general idea is to consider a family of geodesics $(\hat{\sigma}_k)_{k \geq 1}$ whose endpoints form a dense subset of X , then get rid of the interesctions in order to create pairwise disjoint paths. The second part of the proposition will be a clear consequence of the proof. We rely on Lemma 2.6 to cut paths.

Lemma 2.6. Let $\gamma : [0, T] \rightarrow X$ be a continuous path and $X_0 \subseteq X$ be a compact set. There exists a countable family $([s_p, t_p])_{p \geq 1}$ of pairwise disjoint segments of $[0, T]$ such that

(i) For all $p \geq 1$, $\gamma([s_p, t_p]) \cap X_0 = \emptyset$.

(ii) For almost every $t \in [0, T] \setminus \left(\bigcup_{p \geq 1} [s_p, t_p] \right)^c$, $\gamma(t) \in X_0$.

Proof of Lemma 2.6. Since γ is continuous, $\gamma^{-1}(X \setminus X_0)$ is an open subset of $[0, T]$, thus a countable reunion of disjoint open intervals. Each of those can be covered almost everywhere by a countable reunion of disjoint segments. \square

Proof of Proposition 2.5. Let $((x_k, y_k))_{k \geq 1}$ be a dense sequence in X^2 . For all $k \geq 1$, consider a D -geodesic $\hat{\sigma}_k : [0, L] \rightarrow X$ from x_k to y_k . Up to removing loops and changing the parametrization, we may assume that they are injective. By Lemma 2.3,

$$D = \lim_{K \rightarrow \infty} D[(\hat{\sigma}_k)_{1 \leq k \leq K}]. \quad (2.5)$$

Let $k \geq 1$. Applying Lemma 2.6 to $\gamma = \hat{\sigma}_k$ and $X_0 = \bigcup_{i=1}^{k-1} \hat{\sigma}_i$, we know that there exists a family of pairwise disjoint segments $([s_{k,p}, t_{k,p}])_{p \geq 1}$ of $[0, L]$ such that

- (i) For all $p \geq 1$, $\hat{\sigma}_k([s_{k,p}, t_{k,p}]) \cap \left(\bigcup_{i=1}^{k-1} \hat{\sigma}_i \right) = \emptyset$.
- (ii) For almost every $t \in [0, L] \setminus \left(\bigcup_{p \geq 1} [s_{k,p}, t_{k,p}] \right)^c$, $\hat{\sigma}_k(t) \in \bigcup_{i=1}^{k-1} \hat{\sigma}_i$.

We denote by $\sigma_{k,p} : [0, L] \rightarrow X$ a 1-Lipschitz and injective reparametrization of $\sigma_k|_{[s_{k,p}, t_{k,p}]}$; note that it is a D -geodesic. Let $(\sigma_r)_{r \geq 1}$ be an enumeration of $(\sigma_{k,p})_{k,p \geq 1}$. It is sufficient to show

$$D = \lim_{R \rightarrow \infty} D[(\sigma_r)_{1 \leq r \leq R}]. \quad (2.6)$$

Let $\varepsilon > 0$. By (2.5) there exists $K > 0$ such that

$$\mathfrak{d}_\infty(D, D[(\hat{\sigma}_k)_{1 \leq k \leq K}]) \leq \varepsilon. \quad (2.7)$$

Besides, by definition of the $[s_{k,p}, t_{k,p}]$ and Lipschitz continuity, there exists $P > 0$ such that

$$\sum_{k=1}^K \sum_{p=P+1}^{\infty} \|\sigma_{k,p}\|_1 \leq \varepsilon. \quad (2.8)$$

Let $R > 0$ be an integer large enough so that $\{\sigma_{k,p}\}_{\substack{1 \leq k \leq K \\ 1 \leq p \leq P}} \subseteq \{\sigma_r\}_{1 \leq r \leq R}$, and $x, y \in X$. Any $D[(\hat{\sigma}_k)_{1 \leq k \leq K}]$ -geodesic between x and y is concatenation of a finite number of straight lines and subpaths of $\hat{\sigma}_k$, for $1 \leq k \leq K$. Thanks to (2.8), the former may be covered by subpaths of the $\sigma_{k,p}$, for $1 \leq k \leq K$ and $1 \leq p \leq P$, except along a $\|\cdot\|_1$ -length smaller than ε . Consequently, by triangle inequality,

$$D[(\sigma_r)_{1 \leq r \leq R}](x, y) \leq D[(\hat{\sigma}_k)_{1 \leq k \leq K}](x, y) + \|\mu\|_1 \varepsilon. \quad (2.9)$$

Applying (2.3), (2.7) and (2.9) and, we deduce that for large enough R , for all $x, y \in X$,

$$D(x, y) \leq D[(\sigma_r)_{1 \leq r \leq R}](x, y) \leq D(x, y) + (\|\mu\|_1 + 1)\varepsilon, \quad (2.10)$$

hence (2.6). \square

2.2 Integration along a Lipschitz path

Integrals along a Lipschitz path will appear naturally when we need to compute its length or cost (see e.g. (1.30)). We gather here the geometric measure theory tools we need to handle these objects, namely the *metric derivative* of a path, already defined by (1.19) and a special case of the so-called *area formula* (see Lemma 2.11). We also define the *gradient by paths* of a metric, which makes the link between the metric derivative of a path with respect to this metric, and its derivative in the usual sense (see Lemma 2.9). We rely on the monographs by Ambrosio and Tilli [1], and Krantz and Parks [18].

Lemma 2.7. *Let $\gamma : [0, T] \rightarrow X$ be a Lipschitz path.*

- (i) *For almost every $t \in [0, T]$, γ is differentiable at t , and for all $0 \leq t_1 \leq t_2 \leq T$,*

$$\gamma(t_2) - \gamma(t_1) = \int_{t_1}^{t_2} \gamma'(t) dt. \quad (2.11)$$

(ii) For all $D \in \mathcal{D}_\mu$, for almost every $t \in [0, T]$, the limit

$$|\dot{\gamma}|_D(t) := \lim_{h \rightarrow 0} \frac{D(\gamma(t), \gamma(t+h))}{|h|} \quad (2.12)$$

exists, and for all $0 \leq t_1 \leq t_2 \leq T$,

$$D(\gamma|_{[t_1, t_2]}) = \int_{t_1}^{t_2} |\dot{\gamma}|_D(t) dt. \quad (2.13)$$

Proof. The first part is a consequence of the fact that Lipschitz functions are absolutely continuous, thus satisfy the fundamental theorem of calculus (see e.g. Theorem 7.18 in [21]). The second part is an adaptation to pseudometrics of Theorem 4.1.6 in [1]. \square

Definition 2.8. Let $D \in \mathcal{D}_\mu$ and $z \in X$. The *gradient by paths* of D at z is defined as the function

$$(\text{P-grad } D)_z : \mathbb{R}^d \longrightarrow [0, \infty)$$

$$u \longmapsto \inf \left\{ \lim_{t \rightarrow 0} \frac{D(\gamma|_{[0, t]})}{t} \mid \begin{array}{l} \gamma : [0, T] \rightarrow X \text{ Lipschitz, } \gamma(0) = z, \\ \gamma \text{ is differentiable at } t = 0 \text{ and } \gamma'(0) = u \end{array} \right\} \quad (2.14)$$

Lemma 2.9. Let $D \in \mathcal{D}_\mu$, and $\gamma : [0, T] \rightarrow X$ be a Lipschitz path.

(i) For almost every $t \in [0, T]$,

$$|\dot{\gamma}|_D(t) = (\text{P-grad } D)_{\gamma(t)}(\gamma'(t)). \quad (2.15)$$

(ii) For all $0 \leq t_1 \leq t_2 \leq T$,

$$D(\gamma|_{[t_1, t_2]}) = \int_{t_1}^{t_2} (\text{P-grad } D)_{\gamma(t)}(\gamma'(t)) dt. \quad (2.16)$$

Proof. By (2.13) and Lebesgue's differentiation theorem (see e.g. Theorem 7.10 in [21]), for almost all $t \in (0, T)$,

$$|\dot{\gamma}|_D(t) = \lim_{h \rightarrow 0^+} \frac{D(\gamma(t), \gamma(t+h))}{h} = \lim_{h \rightarrow 0^+} \frac{D(\gamma|_{[t, t+h]})}{h}, \quad (2.17)$$

and γ is differentiable at t . Fix such t . By definition of the gradient by paths,

$$|\dot{\gamma}|_D(t) \geq (\text{P-grad } D)_{\gamma(t)}(\gamma'(t)). \quad (2.18)$$

Besides, let $\gamma_1 : [0, T_1] \rightarrow X$ be a Lipschitz path such that $\gamma_1(0) = \gamma(t)$ and $\gamma_1'(0) = \gamma'(t)$. By the triangle inequality,

$$\begin{aligned} |D(\gamma(t), \gamma(t+h)) - D(\gamma_1(0), \gamma_1(h))| &= |D(z, \gamma(t+h)) - D(z, \gamma_1(h))| \\ &\leq D(\gamma(t+h), \gamma_1(h)) \\ &\leq \mu(\gamma(t+h) - \gamma_1(h)) = o(h). \end{aligned}$$

Consequently, by (2.17),

$$\begin{aligned} |\dot{\gamma}|_D(t) &= \lim_{h \rightarrow 0^+} \frac{D(\gamma_1(0), \gamma_1(h))}{h} \\ &\leq \lim_{h \rightarrow 0^+} \frac{D(\gamma_1|_{[0, h]})}{h}. \end{aligned}$$

Taking the infimum over all paths γ_1 , we get

$$|\dot{\gamma}|_D(t) \leq (\text{P-grad } D)_{\gamma(t)}(\gamma'(t)). \quad (2.19)$$

Inequalities (2.18) and (2.19) give the first part of the lemma. The second part is a consequence of the first one and (2.13). \square

Definition 2.10. Given a Lipschitz, injective path γ and $z = \gamma(t) \in \gamma$, we say that z is a *regular point* of γ if the derivative $\gamma'(t)$ exists and is nonzero.

Lemma 2.11. *Let $\gamma : [0, T] \rightarrow X$ be a Lipschitz, injective path. Then*

(i) \mathcal{H}^1 -almost every point of γ is regular.

(ii) For all measurable function $\Phi : X \times \mathbb{R}^d \rightarrow \mathbb{R}^+$ such that for all $z \in X$, $\Phi(z, \cdot)$ is absolutely homogeneous,

$$\int_0^T \Phi(\gamma(t), \gamma'(t)) dt = \int_\gamma \Phi \left(z, \frac{\gamma'(\gamma^{-1}(z))}{\|\gamma'(\gamma^{-1}(z))\|_2} \right) \mathcal{H}^1(dz). \quad (2.20)$$

Proof. The first item is a consequence of the so-called area formula, in the version stated by Theorem 5.1.1 in [18], with $M = 1$, $N = d$, $f = \gamma$ and A the preimage by γ of the set of non-regular points of γ .

To prove the second one, note that

$$\begin{aligned} \int_0^T \Phi(\gamma(t), \gamma'(t)) dt &= \int_0^T \Phi(\gamma(t), \gamma'(t)) \mathbf{1}_{\gamma(t) \text{ is regular}} dt \\ &= \int_0^T \Phi \left(\gamma(t), \frac{\gamma'(t)}{\|\gamma'(t)\|_2} \right) \|\gamma'(t)\|_2 \mathbf{1}_{\gamma(t) \text{ is regular}} dt. \end{aligned}$$

Another version of the area formula, Corollary 5.1.13 in [18], applied for $M = 1$, $N = d$, $f = \gamma$ and $g(t) = \Phi \left(\gamma(t), \frac{\gamma'(t)}{\|\gamma'(t)\|_2} \right) \mathbf{1}_{\gamma(t) \text{ is regular}}$, gives

$$\begin{aligned} \int_0^T \Phi(\gamma(t), \gamma'(t)) dt &= \int_\gamma \Phi \left(z, \frac{\gamma'(\gamma^{-1}(z))}{\|\gamma'(\gamma^{-1}(z))\|_2} \right) \mathbf{1}_{z \text{ is regular}} \mathcal{H}^1(dz) \\ &= \int_\gamma \Phi \left(z, \frac{\gamma'(\gamma^{-1}(z))}{\|\gamma'(\gamma^{-1}(z))\|_2} \right) \mathcal{H}^1(dz). \end{aligned}$$

□

3 Elementary rate function

In this section we prove Theorem 1.1. Our general strategy follows a classic approach we first define $J_{\text{pp}}(x, \zeta)$ in the case where $x \in \mathbb{Z}^d$ with a classic subadditive argument (see Lemma 3.1), then extend to all $x \in \mathbb{Q}^d$ by homogeneity and to all $x \in \mathbb{R}^d$ by monotonicity (see Lemma 3.2). Equation (1.10) follows by stationarity. The characterization of the case $J_{\text{pp}}(x, \zeta) = 0$ is covered by Lemma 3.3.

Lemma 3.1. *For all $x \in \mathbb{Z}^d$ and $\zeta > a\|x\|_1$, the limit*

$$J_{\text{pp}}(x, \zeta) := \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(0, nx) \leq n\zeta) = \inf_{n \geq 1} -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(0, nx) \leq n\zeta) \quad (3.1)$$

exists, and it is finite. Moreover,

(i) For all $x \in \mathbb{Z}^d$, $\zeta > a\|x\|_1$ and $k \in \mathbb{N}^*$,

$$J_{\text{pp}}(kx, k\zeta) = kJ_{\text{pp}}(x, \zeta). \quad (3.2)$$

(ii) For all $x_1, x_2 \in \mathbb{Z}^d$, $\zeta_1 > a\|x_1\|_1$ and $\zeta_2 > a\|x_2\|_1$,

$$J_{\text{pp}}(x_1 + x_2, \zeta_1 + \zeta_2) \leq J_{\text{pp}}(x_1, \zeta_1) + J_{\text{pp}}(x_2, \zeta_2). \quad (3.3)$$

(iii) For all $x \in \mathbb{Z}^d$ and $\zeta > a\|x\|_1$,

$$J_{\text{pp}}(x, \zeta) = J_{\text{pp}}(|x|, \zeta). \quad (3.4)$$

(iv) For all $x_1, x_2 \in \mathbb{Z}^d$, $\zeta_1 > a\|x_1\|_1$ and $\zeta_2 > a\|x_2\|_1$, if $0 \leq x_1 \leq x_2$ and $\zeta_1 \geq \zeta_2$ then

$$J_{\text{pp}}(x_1, \zeta_1) \leq J_{\text{pp}}(x_2, \zeta_2). \quad (3.5)$$

For all $x \in \mathbb{Q}^d$ and $\zeta > a\|x\|_1$, we define

$$J_{\text{pp}}(x, \zeta) := \frac{1}{k} J_{\text{pp}}(kx, k\zeta), \quad (3.6)$$

where $k \in \mathbb{N}^*$ is any integer such that $kx \in \mathbb{Z}^d$; the choice does not matter thanks to (3.2). Moreover, for all $x \in \mathbb{Q}^d$, $\zeta > a\|x\|_1$ and $\lambda \in \mathbb{Q} \cap (0, \infty)$,

$$J_{\text{pp}}(\lambda x, \lambda \zeta) = \lambda J_{\text{pp}}(x, \zeta). \quad (3.7)$$

Lemma 3.2. *The function J_{pp} admits a continuous extension on \mathcal{X} that satisfies items (i), (ii) and (iii) in Theorem 1.1.*

Lemma 3.3. *For all $(x, \zeta) \in \mathcal{X}$, $J_{\text{pp}}(x, \zeta) > 0$ if and only if $\zeta < \mu(x)$.*

For all $u, v \in \mathbb{Z}^d$, $\zeta > 0$ and $\ell > 0$, we consider the event

$$\mathcal{E}(u, v, t, \ell) := \left\{ \text{There exists a discrete path } u \overset{\pi}{\rightsquigarrow} v \text{ with } \|\pi\| \leq \ell \text{ and } \tau(\pi) \leq t \right\}. \quad (3.8)$$

We first prove Theorem 1.1, assuming Lemmas 3.1, 3.2 and 3.3 are true. We will use several times the following straightforward fact: for all $n \in \mathbb{N}^*$, for all $u, v \in \llbracket 0, n \rrbracket^d$ and $t > a$,

$$\mathbb{P}\left(\mathbf{T}_{\llbracket 0, n \rrbracket^d}(u, v) \leq t\|u - v\|_1\right) \geq \nu([a, t])^{\|u - v\|_1}. \quad (3.9)$$

Proof of Theorem 1.1. We first show that for all distinct $x, y \in (0, 1)^d$, for small enough $\varepsilon > 0$, there exists $z \in \overline{\mathbb{B}}_1(y - x, \varepsilon\|y - x\|_1)$ such that

$$\overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}\left(\mathbf{T}_{\llbracket 0, n \rrbracket^d}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n\zeta\right) \leq J_{\text{pp}}(z, (1 - 2\varepsilon)\zeta) + 2\varepsilon - 2\varepsilon d \log \nu\left(\left[a, \frac{\zeta}{\|y - x\|_1}\right]\right) \quad (3.10)$$

and

$$\underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}\left(\mathbf{T}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n\zeta\right) \geq J_{\text{pp}}(z, (1 + 3\varepsilon)\zeta) + 2\varepsilon d \log \nu\left(\left[a, \frac{\zeta}{\|y - x\|_1}\right]\right). \quad (3.11)$$

Let $x, y \in (0, 1)^d$ be distinct,

$$\delta := d(\{x, y\}, \partial X) > 0 \text{ and } 0 < \varepsilon < \frac{\zeta - a\|y - x\|_1}{2\zeta + a\|y - x\|_1} \wedge \frac{\delta}{3}. \quad (3.12)$$

Let $z = z(x, y, \varepsilon) \in \mathbb{Q}^d$ be such that

$$\|y - x - z\|_1 \leq \varepsilon\|y - x\|_1. \quad (3.13)$$

Note that $(1 - 2\varepsilon)\zeta > a(1 + \varepsilon)\|y - x\|_1 \geq a\|z\|_1$. By definition of $J_{\text{pp}}(z, (1 - 2\varepsilon)\zeta)$ (see (3.1) and (3.6)), there exists $m \in \mathbb{N}^*$ such that $mz \in \mathbb{Z}^d$, and

$$-\frac{1}{m} \log \mathbb{P}(\mathbf{T}(0, mz) \leq m(1 - 2\varepsilon)\zeta) \leq J_{\text{pp}}(z, (1 - 2\varepsilon)\zeta) + \varepsilon.$$

By monotone convergence, there exists $\ell > 0$ such that

$$-\frac{1}{m} \log \mathbb{P}(\mathcal{E}(0, mz, m(1 - 2\varepsilon)\zeta, \ell)) \leq J_{\text{pp}}(z, (1 - 2\varepsilon)\zeta) + 2\varepsilon. \quad (3.14)$$

Let $n \in \mathbb{N}^*$ be such that

$$\frac{m\|y-x\|_1 + d}{n} < \varepsilon\|y-x\|_1 \text{ and } \frac{\ell}{n} < \varepsilon. \quad (3.15)$$

Let

$$K := \lfloor n/m \rfloor. \quad (3.16)$$

We have

$$\begin{aligned} \|\lfloor nx \rfloor + Kmz - \lfloor ny \rfloor\|_1 &\leq d + \|nx + Kmz - ny\|_1 \\ &\leq d + \|Kmx + Kmz - Kmy\|_1 + (n - Km)\|y-x\|_1 \\ &\leq d + Km\|x+z-y\|_1 + m\|y-x\|_1. \end{aligned}$$

Consequently, by (3.13) and (3.15),

$$\|\lfloor nx \rfloor + Kmz - \lfloor ny \rfloor\|_1 \leq 2\varepsilon n\|y-x\|_1. \quad (3.17)$$

Consider the event

$$\text{Fav} := \left(\bigcap_{k=0}^{K-1} \mathcal{E}(\lfloor nx \rfloor + kmz, \lfloor nx \rfloor + (k+1)mz, m(1-2\varepsilon)\zeta, \ell) \right) \cap \left\{ \mathbf{T}_{[0,n]^d}(\lfloor nx \rfloor + Kmz, \lfloor ny \rfloor) \leq 2\zeta\varepsilon n \right\}. \quad (3.18)$$

Since Fav is an intersection of decreasing events, by the FKG inequality, (3.9) and (3.17),

$$\mathbb{P}(\text{Fav}) \geq \left(\prod_{k=0}^{K-1} \mathbb{P}(\mathcal{E}(\lfloor nx \rfloor + kmz, \lfloor nx \rfloor + (k+1)mz, m(1-2\varepsilon)\zeta, \ell)) \right) \cdot \nu \left(\left[a, \frac{\zeta}{\|y-x\|_1} \right] \right)^{2\varepsilon n\|y-x\|_1}.$$

By stationarity of the model,

$$\mathbb{P}(\text{Fav}) \geq \mathbb{P}(\mathcal{E}(0, mz, m(1-2\varepsilon)\zeta, \ell))^K \cdot \nu \left(\left[a, \frac{\zeta}{\|y-x\|_1} \right] \right)^{2\varepsilon n\|y-x\|_1}.$$

Applying (3.14) gives

$$-\frac{1}{n} \log \mathbb{P}(\text{Fav}) \leq \frac{Km}{n} \cdot [J_{\text{pp}}(z, (1-2\varepsilon)\zeta) + 2\varepsilon] - 2\varepsilon\|y-x\|_1 \log \nu \left(\left[a, \frac{\zeta}{\|y-x\|_1} \right] \right),$$

thus

$$-\frac{1}{n} \log \mathbb{P}(\text{Fav}) \leq J_{\text{pp}}(z, (1-2\varepsilon)\zeta) + 2\varepsilon - 2\varepsilon d \log \nu \left(\left[a, \frac{\zeta}{\|y-x\|_1} \right] \right). \quad (3.19)$$

Assume that Fav occurs. By the inequalities $\frac{\ell}{n} \leq \varepsilon \leq \delta/3$ and (3.17), for all $k \in \llbracket 0, K-1 \rrbracket$,

$$\bar{\mathbf{B}}_1(\lfloor nx \rfloor + kmz, \ell) \subseteq [0, n]^d.$$

Consequently, for all $k \in \llbracket 0, K-1 \rrbracket$,

$$\mathbf{T}_{[0,n]^d}(\lfloor nx \rfloor + kmz, \lfloor nx \rfloor + (k+1)mz) \leq m(1-2\varepsilon)\zeta.$$

By triangle inequality,

$$\begin{aligned} \mathbf{T}_{[0,n]^d}(\lfloor nx \rfloor, \lfloor ny \rfloor) &\leq \left(\sum_{k=0}^{K-1} \mathbf{T}_{[0,n]^d}(\lfloor nx \rfloor + kmz, \lfloor nx \rfloor + (k+1)mz) \right) + \mathbf{T}_{[0,n]^d}(\lfloor nx \rfloor + Kmz, \lfloor ny \rfloor) \\ &\leq Km(1-2\varepsilon)\zeta + 2\varepsilon n\zeta \leq n\zeta, \end{aligned}$$

thus

$$\text{Fav} \subseteq \left\{ \mathbf{T}_{[0,n]^d}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n\zeta \right\}. \quad (3.20)$$

Combining (3.19) and (3.20) leads to

$$-\frac{1}{n} \log \mathbb{P} \left(\mathbf{T}_{[0, n]^d}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n\zeta \right) \leq J_{\text{pp}}(z, (1 - 2\varepsilon)\zeta) + 2\varepsilon - 2\varepsilon d \log \nu \left(\left[a, \frac{\zeta}{\|x - y\|_1} \right] \right).$$

Taking the superior limit as $n \rightarrow \infty$ gives (3.10).

We now turn to the proof of the lower bound (3.11). Let $m \in \mathbb{N}^*$ be such that $mz \in \mathbb{Z}^d$. Fix n and K as in (3.15) and (3.16). Consider the event

$$\text{Fav}^* := \{ \mathbf{T}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n\zeta \} \cap \left\{ \mathbf{T}_{[0, n]^d}(\lfloor nx \rfloor + Kmz, \lfloor ny \rfloor) \leq 2\zeta\varepsilon n \right\}. \quad (3.21)$$

By the FKG inequality and (3.9),

$$-\frac{1}{n} \log \mathbb{P}(\text{Fav}^*) \leq -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n\zeta) - 2\varepsilon d \log \nu \left(\left[a, \frac{\zeta}{\|x - y\|_1} \right] \right). \quad (3.22)$$

Besides, the triangle inequality gives

$$\text{Fav}^* \subseteq \{ \mathbf{T}(\lfloor nx \rfloor, \lfloor nx \rfloor + Kmz) \leq n(\zeta + 2\varepsilon) \} \subseteq \{ \mathbf{T}(\lfloor nx \rfloor, \lfloor nx \rfloor + Kmz) \leq Km(\zeta + 3\varepsilon) \}, \quad (3.23)$$

for large enough n . Consequently, by stationarity, for large enough n ,

$$\begin{aligned} & -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n\zeta) - 2\varepsilon d \log \nu \left(\left[a, \frac{\zeta}{\|x - y\|_1} \right] \right) \\ & \geq -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(\lfloor nx \rfloor, \lfloor nx \rfloor + Kmz) \leq Km(\zeta + 3\varepsilon)) \\ & = -\frac{Km}{n} \frac{1}{Km} \log \mathbb{P}(\mathbf{T}(0, Kmz) \leq Km(\zeta + 3\varepsilon)). \end{aligned}$$

Taking the inferior limit as $n \rightarrow \infty$ gives (3.11).

We now prove (1.10). If $x = y$, this is clear. If $x, y \in (0, 1)^d$ and are distinct, letting $\varepsilon \rightarrow 0$ in (3.10) and (3.11) gives, by continuity of J_{pp} on \mathcal{X} , the desired result (recall that z depends on ε). The remaining case is when $x, y \in [0, 1]^d$, are distinct and $\zeta > a\|x - y\|_1$. Let $0 < \varepsilon < \frac{1}{2} \left(1 - \frac{a\|x - y\|_1}{\zeta} \right)$. There exist distinct $\hat{x}, \hat{y} \in (0, 1)^d$ such that

$$(1 - 2\varepsilon)\zeta > a\|\hat{x} - \hat{y}\|_1,$$

and for large enough n ,

$$\| \lfloor nx \rfloor - \lfloor n\hat{x} \rfloor \|_1 \leq n\varepsilon\|x - y\|_1, \quad \| \lfloor ny \rfloor - \lfloor n\hat{y} \rfloor \|_1 \leq n\varepsilon\|x - y\|_1.$$

The triangle inequality gives for large enough n the inclusions

$$\begin{aligned} & \left\{ \mathbf{T}_{[0, n]^d}(\lfloor n\hat{x} \rfloor, \lfloor n\hat{y} \rfloor) \leq n(1 - 2\varepsilon)\zeta \right\} \cap \left\{ \mathbf{T}_{[0, n]^d}(\lfloor n\hat{x} \rfloor, \lfloor nx \rfloor) \leq n\zeta\varepsilon \right\} \\ & \cap \left\{ \mathbf{T}_{[0, n]^d}(\lfloor n\hat{y} \rfloor, \lfloor ny \rfloor) \leq n\zeta\varepsilon \right\} \subseteq \left\{ \mathbf{T}_{[0, n]^d}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n\zeta \right\} \end{aligned} \quad (3.24)$$

and

$$\begin{aligned} & \{ \mathbf{T}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n\zeta \} \cap \left\{ \mathbf{T}_{[0, n]^d}(\lfloor n\hat{x} \rfloor, \lfloor nx \rfloor) \leq n\zeta\varepsilon \right\} \\ & \cap \left\{ \mathbf{T}_{[0, n]^d}(\lfloor n\hat{y} \rfloor, \lfloor ny \rfloor) \leq n\zeta\varepsilon \right\} \subseteq \{ \mathbf{T}(\lfloor n\hat{x} \rfloor, \lfloor n\hat{y} \rfloor) \leq n(1 + 2\varepsilon)\zeta \}. \end{aligned} \quad (3.25)$$

By the FKG inequality, (3.9) and (1.10) for \hat{x} and \hat{y} , we have

$$J_{\text{pp}}(\hat{x} - \hat{y}, (1 - 2\varepsilon)\zeta) - 2\varepsilon \log \nu \left(\left[a, \frac{\zeta}{\|x - y\|_1} \right] \right) \geq \overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\mathbf{T}_{[0, n]^d}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n\zeta \right) \quad (3.26)$$

and

$$J_{\text{pp}}(\hat{x} - \hat{y}, (1 + 2\varepsilon)\zeta) + 2\varepsilon \log \nu \left(\left[a, \frac{\zeta}{\|x - y\|_1} \right] \right) \leq \underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(\lfloor nx \rfloor, \lfloor ny \rfloor) \leq n(1 + 2\varepsilon)\zeta). \quad (3.27)$$

By continuity of J_{pp} on \mathcal{X} , letting $\varepsilon \rightarrow 0$ gives (1.10) in full generality. \square

Proof of Lemma 3.1. By the FKG inequality and stationarity of the model, for all $x_1, x_2 \in \mathbb{Z}^d$ and $t_1, t_2 \geq 0$,

$$\mathbb{P}(\mathbf{T}(0, x_1) \leq t_1, \mathbf{T}(x_1, x_1 + x_2) \leq t_2) \geq \mathbb{P}(\mathbf{T}(0, x_1) \leq t_1) \cdot \mathbb{P}(\mathbf{T}(0, x_2) \leq t_2).$$

In particular, by triangle inequality,

$$\mathbb{P}(\mathbf{T}(0, x_1 + x_2) \leq t_1 + t_2) \geq \mathbb{P}(\mathbf{T}(0, x_1) \leq t_1) \cdot \mathbb{P}(\mathbf{T}(0, x_2) \leq t_2). \quad (3.28)$$

Let $x \in \mathbb{Z}^d$ and $\zeta > a\|x\|_1$. For all $n, m \in \mathbb{N}^*$, (3.28) with $x_1 = nx$, $x_2 = mx$, $t_1 = n\zeta$ and $t_2 = m\zeta$ implies

$$\mathbb{P}(\mathbf{T}(0, (n+m)x) \leq (n+m)\zeta) \geq \mathbb{P}(\mathbf{T}(0, nx) \leq n\zeta) \cdot \mathbb{P}(\mathbf{T}(0, mx) \leq m\zeta).$$

Fekete's lemma gives the existence of the limit in (3.1). The finiteness of the limit is a consequence of (3.9).

Let $x \in \mathbb{Z}^d$, $\zeta > a\|x\|_1$ and $k \in \mathbb{N}^*$. By (3.1), considering the extraction $n \mapsto kn$ yields (3.2).

Let $x_1, x_2 \in \mathbb{Z}^d$, $\zeta_1 > a\|x_1\|_1$ and $\zeta_2 > a\|x_2\|_1$. Plugging nx_i and $n\zeta_i$ into the inequality (3.28) yields

$$-\frac{1}{n} \log \mathbb{P}(\mathbf{T}(0, n(x_1 + x_2)) \leq n(\zeta_1 + \zeta_2)) \leq -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(0, nx_1) \leq n\zeta_1) - \frac{1}{n} \log \mathbb{P}(\mathbf{T}(0, nx_2) \leq n\zeta_2).$$

Letting $n \rightarrow \infty$ gives (3.3).

Equation (3.4) is a consequence of the invariance of the model with respect to the orthogonal symmetries of \mathbb{Z}^d .

Let $x = \sum_{i=1}^d x(i)e_i \in \mathbb{Z}^d$ and $\zeta > a\|x\|_1$. Assume that $x \geq 0$. Consider

$$y := -(x(1) + 1)e_1 + \sum_{i=2}^d x(i)e_i$$

and note that

$$2(x(1) + 1)x = y + (2x(1) + 1)(x + e_1).$$

By (3.2) and (3.3),

$$2(x(1) + 1)J_{\text{pp}}(x, \zeta) \leq J_{\text{pp}}(y, \zeta) + (2x(1) + 1)J_{\text{pp}}(x + e_1, \zeta).$$

Besides, (3.4) yields $J_{\text{pp}}(y, \zeta) = J_{\text{pp}}(x + e_1, \zeta)$, thus

$$2(x(1) + 1)J_{\text{pp}}(x, \zeta) \leq 2(x(1) + 1)J_{\text{pp}}(x + e_1, \zeta),$$

i.e.

$$J_{\text{pp}}(x, \zeta) \leq J_{\text{pp}}(x + e_1, \zeta).$$

The analogous inequality with any base vector e_i instead of e_1 holds. A straightforward induction argument gives (3.5). \square

Proof of Lemma 3.2. For all $(x, \zeta) \in \mathcal{X}$, we define

$$\hat{J}_{\text{pp}}(x, \zeta) := \inf \{ J_{\text{pp}}(x', \zeta') \mid (x', \zeta') \in \mathcal{X}, \quad x' \in \mathbb{Q}^d, \quad x' \geq |x|, \quad \zeta' \leq \zeta \}. \quad (3.29)$$

By (3.4) and (3.5), if $x \in \mathbb{Q}^d$ then $\hat{J}_{\text{pp}}(x, \zeta) = J_{\text{pp}}(x, \zeta)$. Item (i) is a consequence of (3.2) and (3.3). Items (ii) and (iii) hold by definition. The function \hat{J}_{pp} is convex on the open set \mathcal{X} therefore it is continuous. \square

Proof of Lemma 3.3. Let $(x, \zeta) \in \mathcal{X}$.

Direct implication: Assume that $\zeta \geq \mu(x)$. We prove that $J_{\text{pp}}(x, \zeta) = 0$. By lower semicontinuity of J_{pp} , it is sufficient to treat the case where $\zeta > \mu(x)$ and $x \in \mathbb{Q}^d$, which may be further reduced to $x \in \mathbb{Z}^d$. We conclude by convergence in probability of the rescaled passage time (see (1.4)).

Converse implication: Assume that $\zeta < \mu(x)$. Fix $\varepsilon := \frac{1}{2}(\mu(x) - \zeta)$. We first treat the case where the passage times τ_e are a.s. bounded by $b < \infty$. Theorem 6.12 in Boucheron-Lugosi-Massart (2013) [4] applied to the random variable $Z := \frac{1}{b}\mathbf{T}(0, \lfloor nx \rfloor)$, which depends only on a finite number of edge passage times and is a self-bounding function of these passage times, implies that for all $t > 0$,

$$\mathbb{P}(Z \leq \mathbb{E}[Z] - t) \leq \exp\left(-h\left(\frac{t}{\mathbb{E}[Z]}\right)\mathbb{E}[Z]\right), \quad (3.30)$$

where $h(u) = (1+u)\log(1+u) - u$. Consequently,

$$\begin{aligned} \mathbb{P}(\mathbf{T}(0, \lfloor nx \rfloor) \leq \mathbb{E}[\mathbf{T}(0, \lfloor nx \rfloor)] - t) &= \mathbb{P}\left(Z \leq \mathbb{E}[Z] - \frac{t}{b}\right) \\ &\leq \exp\left(-h\left(\frac{t}{\mathbb{E}[\mathbf{T}(0, \lfloor nx \rfloor)]}\right) \cdot \frac{\mathbb{E}[\mathbf{T}(0, \lfloor nx \rfloor)]}{b}\right). \end{aligned}$$

Besides, $\mathbb{E}[\mathbf{T}(0, \lfloor nx \rfloor)] \geq n(\mu(x) - \varepsilon)$ for large n , thus the choice $t := \varepsilon n$ gives, for large n ,

$$\begin{aligned} \mathbb{P}(\mathbf{T}(0, \lfloor nx \rfloor) \leq n\zeta) &\leq \mathbb{P}(\mathbf{T}(0, \lfloor nx \rfloor) \leq \mathbb{E}[\mathbf{T}(0, \lfloor nx \rfloor)] - \varepsilon n) \\ &\leq \exp\left(-h\left(\frac{\varepsilon n}{\mathbb{E}[\mathbf{T}(0, \lfloor nx \rfloor)]}\right) \cdot \frac{\mathbb{E}[\mathbf{T}(0, \lfloor nx \rfloor)]}{b}\right), \end{aligned}$$

hence

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(0, \lfloor nx \rfloor) \leq n\zeta) \geq h\left(\frac{\varepsilon}{\mu(x)}\right) \cdot \frac{\mu(x)}{b} > 0,$$

i.e. $J_{\text{pp}}(x, \zeta) > 0$.

We now turn to the general case. For all $b > a$ and $e \in \mathbb{E}^d$, we define $\tau_e^{(b)} := \tau_e \wedge b$ and denote by $\mu^{(b)}$ the associated time constant. Theorem 1.6 in Garet-Marchand-Procaccia-Théret (2017) [14],² states that

$$\lim_{b \rightarrow \infty} \mu^{(b)}(x) = \mu(x).$$

In particular there exists $b > a$ such that $\zeta < \mu^{(b)}(x)$. The straightforward inclusion

$$\{\mathbf{T}(0, \lfloor nx \rfloor) \leq \zeta\} \subseteq \{\mathbf{T}^{(b)}(0, \lfloor nx \rfloor) \leq \zeta\}$$

and the previous case concludes the proof. \square

4 Monotonous rate function

In this section we assume that (SubC) and (StrongShape) are satisfied and prove Theorem 1.7, which amounts to proving Propositions 4.1, 4.2, 4.3 and 4.4. Recall the definitions (1.26) and (1.27) of \bar{J}^- and \underline{J}^- .

Proposition 4.1. *Let $D \in \mathcal{D}_\mu$ and $(\sigma_k : [0, L] \rightarrow X)_{k \geq 1}$ be a highway network for D (see Definition 2.4). Then*

$$\bar{J}^-(D) \leq \sum_{k \geq 1} \int_0^L J_{\text{pp}}(\sigma'_k(t), |\sigma_k|_D(t)) dt. \quad (4.1)$$

Proposition 4.2. *Let $D \in \mathcal{D}_\mu$ and $(\gamma_k : [0, T_k] \rightarrow X)_{1 \leq k \leq K}$ be a finite family of 1-Lipschitz, injective and pairwise disjoint paths. Then*

$$\underline{J}^-(D) \geq \sum_{k=1}^K \int_0^{T_k} J_{\text{pp}}(\gamma'_k(t), |\gamma_k|_D(t)) dt. \quad (4.2)$$

²The result there is stated for $x \in \mathbb{Z}^d$, but the general case follows by standard arguments.

Every pseudometric $D \in \mathcal{D}_\mu$ has a highway network by Proposition 2.5, thus combining Propositions 4.1 and 4.2 gives $J^-(D) := \underline{J}^-(D) = \overline{J}^-(D)$. Moreover Equations (1.29), (1.30) hold.

Proposition 4.3. *For all $D \in \mathcal{D}_\mu$,*

$$J^-(D) = \int_X \max_{u \in \mathbb{S}_2} J_{\text{pp}}(u, (\text{P-grad } D)_z(u)) \mathcal{H}^1(dz). \quad (4.3)$$

Proposition 4.4. *Let $D_1, D_2 \in \mathcal{D}_\mu$ be distinct pseudometrics satisfying $D_1 \leq D_2$. If $J^-(D_2) < \infty$, then*

$$J^-(D_1) < J^-(D_2). \quad (4.4)$$

Assumption (StrongShape)'s usefulness stems from Lemma 4.5, a variant of a core lemma in the proof of the shape theorem (see Lemma 2.20 in [2]). We postpone its proof to Appendix A.

Lemma 4.5. *Under Assumption (StrongShape), there exists a constant $\kappa < \infty$ such that*

$$\lim_{n \rightarrow \infty} \min_{x \in \llbracket 0, n \rrbracket^d} \mathbb{P} \left(\begin{array}{l} \text{For all } y \in \llbracket 0, n \rrbracket^d, \text{ there exists a discrete path } x \overset{\pi}{\rightsquigarrow} y \text{ included} \\ \text{in } \llbracket 0, n \rrbracket^d, \text{ such that } \tau(\pi) \leq \kappa \|x - y\|_1 \text{ and } \|\pi\| \leq 2\|x - y\|_1 + 4 \end{array} \right) > 0. \quad (4.5)$$

We will say that $x \in \llbracket 0, n \rrbracket^d$ is a *hub* if the event in (4.5) occurs.

4.1 Upper bounding the monotonous rate function

In this section we prove Proposition 4.1. We fix a pseudometric $D \in \mathcal{D}_\mu$ and a highway network $(\sigma_k : [0, L] \rightarrow X)_{k \geq 1}$ for D . For all $K \in \mathbb{N}$, we define

$$D_K := D[\sigma_1, \dots, \sigma_K]. \quad (4.6)$$

We first show that Lemmas 4.6 and 4.7 imply Proposition 4.1, then prove said lemmas.

Lemma 4.6. *The metric induced by μ on X satisfies*

$$\overline{J}^-(\mu) = 0. \quad (4.7)$$

Lemma 4.7. *For all $K \in \mathbb{N}$,*

$$\overline{J}^-(D_{K+1}) \leq \overline{J}^-(D_K) + \int_0^L J_{\text{pp}}(\sigma'_{K+1}(t), |\dot{\sigma}_{K+1}|_D(t)) dt. \quad (4.8)$$

Proof of Proposition 4.1. By Lemma 4.6, $\overline{J}^-(D_0) = \overline{J}^-(\mu) = 0$. By induction, using (4.8), for all $K \geq 0$,

$$\overline{J}^-(D_K) \leq \sum_{k=1}^K \int_0^L J_{\text{pp}}(\sigma'_k(t), |\dot{\sigma}_k|_D(t)) dt. \quad (4.9)$$

Since $(D_K)_{K \geq 0}$ converges to D in \mathcal{F}_b and \overline{J}^- is lower semicontinuous, (4.1) holds. \square

Proof of Lemma 4.6. Let $\varepsilon > 0$. Consider a finite subset $\{x_1, \dots, x_K\}$ of X such that $X \subseteq \bigcup_{k=1}^K \overline{B}_1(x_k, \varepsilon)$. Recall the definition of hubs below Lemma 4.5. We define the event

$$\text{Fav}^\dagger := \left(\bigcap_{1 \leq k \leq K} \{\llbracket nx_k \rrbracket \text{ is a hub}\} \right) \cap \left(\bigcap_{1 \leq k_1 < k_2 \leq K} \left\{ \mathbf{T}_{[0, n]^d}(\llbracket nx_{k_1} \rrbracket, \llbracket nx_{k_2} \rrbracket) \leq n\mu(x_{k_1} - x_{k_2}) \right\} \right). \quad (4.10)$$

This event is an intersection of decreasing events, thus by the FKG inequality,

$$\mathbb{P}(\text{Fav}^\dagger) \geq \left(\prod_{1 \leq k \leq K} \mathbb{P}(\llbracket nx_k \rrbracket \text{ is a hub}) \right) \cdot \left(\prod_{1 \leq k_1 < k_2 \leq K} \mathbb{P}(\mathbf{T}_{[0, n]^d}(\llbracket nx_{k_1} \rrbracket, \llbracket nx_{k_2} \rrbracket) \leq n\mu(x_{k_1} - x_{k_2})) \right).$$

By Theorem 1.1 and Lemma 4.5,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{Fav}^\dagger) = 0. \quad (4.11)$$

Besides, we claim that for large enough n ,

$$\text{Fav}^\dagger \subseteq \text{LD}_n^-(\mu, 2(\|\mu\|_1 + 2\kappa)\varepsilon), \quad (4.12)$$

where $\|\mu\|_1$ is defined by (1.42). Indeed assume that Fav^\dagger occurs, and $n \geq \frac{d}{\varepsilon}$. Let $x, y \in X$. There exist $1 \leq k_1, k_2 \leq K$ such that

$$\|x - x_{k_1}\|_1 \leq \varepsilon \text{ and } \|y - x_{k_2}\|_1 \leq \varepsilon,$$

therefore

$$\| \lfloor nx \rfloor - \lfloor nx_{k_1} \rfloor \|_1 \leq 2n\varepsilon \text{ and } \| \lfloor ny \rfloor - \lfloor nx_{k_2} \rfloor \|_1 \leq 2n\varepsilon.$$

By triangle inequality,

$$\begin{aligned} \mathbf{T}_{[0, n]^d}(\lfloor nx \rfloor, \lfloor ny \rfloor) &\leq \mathbf{T}_{[0, n]^d}(\lfloor nx \rfloor, \lfloor nx_{k_1} \rfloor) + \mathbf{T}_{[0, n]^d}(\lfloor nx_{k_1} \rfloor, \lfloor nx_{k_2} \rfloor) + \mathbf{T}_{[0, n]^d}(\lfloor nx_{k_2} \rfloor, \lfloor ny \rfloor) \\ &\leq \kappa \| \lfloor nx \rfloor - \lfloor nx_{k_1} \rfloor \|_1 + n\mu(x_{k_1} - x_{k_2}) + \kappa \| \lfloor ny \rfloor - \lfloor nx_{k_2} \rfloor \|_1 \\ &\leq n\mu(x_{k_1} - x_{k_2}) + 4\kappa n\varepsilon. \end{aligned}$$

Besides, using the triangle inequality again gives

$$\begin{aligned} \mu(x_{k_1} - x_{k_2}) &\leq \mu(x_{k_1} - x) + \mu(x - y) + \mu(y - x_{k_2}) \\ &\leq \mu(x - y) + 2\|\mu\|_1\varepsilon. \end{aligned}$$

Consequently,

$$\widehat{\mathbf{T}}_n(x, y) \leq \mu(x - y) + 2(\|\mu\|_1 + 2\kappa)\varepsilon,$$

thus (4.12).

Equations (4.11) and (4.12) conclude the proof of the lemma. \square

Proof of Lemma 4.7. For all $K \in \mathbb{N}$. To simplify the notations we define $\sigma := \sigma_{K+1}$. Let $\varepsilon > 0$ and $P \in \mathbb{N}^*$. For all $0 \leq p \leq P$, define $t_p := \frac{pL}{P}$. Consider the event

$$\text{Fav}^\ddagger = \text{Fav}^\ddagger(n, \varepsilon, P) := \text{LD}_n^-(D_K, \varepsilon) \cap \left(\bigcap_{p=0}^{P-1} \left\{ \frac{1}{n} \mathbf{T}_{[0, n]^d}(\lfloor n\sigma(t_p) \rfloor, \lfloor n\sigma(t_{p+1}) \rfloor) \leq D(\sigma(t_p), \sigma(t_{p+1})) + \varepsilon \right\} \right). \quad (4.13)$$

We claim that

$$\overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{Fav}^\ddagger) \leq \overline{J}^-(D_K) + \int_0^L J_{\text{pp}}(\sigma'(t), |\dot{\sigma}|_D(t)) dt. \quad (4.14)$$

Indeed Fav^\ddagger is an intersection of decreasing events, thus the FKG inequality gives

$$\mathbb{P}(\text{Fav}^\ddagger) \geq \mathbb{P}(\text{LD}_n^-(D_K, \varepsilon)) \cdot \left(\prod_{p=0}^{P-1} \mathbb{P} \left(\frac{1}{n} \mathbf{T}_{[0, n]^d}(\lfloor n\sigma(t_p) \rfloor, \lfloor n\sigma(t_{p+1}) \rfloor) \leq D(\sigma(t_p), \sigma(t_{p+1})) + \varepsilon \right) \right),$$

therefore by Theorem 1.1,

$$\overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{Fav}^\ddagger) \leq \overline{J}^-(D_K) + \sum_{p=0}^{P-1} J_{\text{pp}}(\sigma(t_{p+1}) - \sigma(t_p), D(\sigma(t_p), \sigma(t_{p+1}))).$$

Since σ is a D -geodesic, $D(\sigma(t_p), \sigma(t_{p+1})) = D(\sigma|_{[t_p, t_{p+1}]})$ hence

$$\overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{Fav}^\ddagger) \leq \overline{J}^-(D_K) + \sum_{p=0}^{P-1} J_{\text{pp}}(\sigma(t_{p+1}) - \sigma(t_p), D(\sigma|_{[t_p, t_{p+1]}])).$$

Lemma 2.7 gives

$$\overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{Fav}^\dagger) \leq \overline{J}^-(D_K) + \sum_{p=0}^{P-1} J_{\text{pp}} \left(\int_{t_p}^{t_{p+1}} \sigma'(t) dt, \int_{t_p}^{t_{p+1}} |\dot{\sigma}|_D(t) dt \right). \quad (4.15)$$

The function J_{pp} is convex and homogeneous therefore Jensen's inequality in its multivariate version (see [19]) yields (4.14).

We now show that for all $n \in \mathbb{N}^*$,

$$\text{Fav}^\dagger \subseteq \text{LD}_n^-(D, (P+2)\varepsilon + \frac{4L\|\mu\|_1}{P}). \quad (4.16)$$

Let $n \in \mathbb{N}^*$. Assume that Fav^\dagger occurs. Let $x, y \in X$. By Definition 2.2, one of the following holds:

- (i) $D_{K+1}(x, y) = D_K(x, y)$.
- (ii) There exists $0 \leq s, t \leq L$ such that

$$D_{K+1}(x, y) = D_K(x, \sigma(s)) + D(\sigma(s), \sigma(t)) + D_K(\sigma(t), y).$$

In the case (i),

$$\widehat{\mathbf{T}}_n(x, y) \leq D_{K+1}(x, y) + \varepsilon \quad (4.17)$$

is straightforward. In the case (ii), there exist $p, q \in \llbracket 0, P \rrbracket$ such that $|s - t_p|, |t - t_q| \leq \frac{L}{P}$. Since σ is 1-Lipschitz and $D \leq D_K \leq \mu$, we have by triangle inequality,

$$D_{K+1}(x, y) \geq D_K(x, \sigma(t_p)) + D(\sigma(t_p), \sigma(t_q)) + D_K(\sigma(t_q), y) - \frac{4L\|\mu\|_1}{P}. \quad (4.18)$$

Up to exchanging x and y we may assume that $p \leq q$. By triangle inequality

$$\begin{aligned} \widehat{\mathbf{T}}_n(x, y) &\leq \widehat{\mathbf{T}}_n(x, \sigma(t_p)) + \sum_{i=p}^{q-1} \widehat{\mathbf{T}}_n(\sigma(t_i), \sigma(t_{i+1})) + \widehat{\mathbf{T}}_n(\sigma(t_q), y) \\ &\leq D_K(x, \sigma(t_p)) + \sum_{i=p}^{q-1} D(\sigma(t_i), \sigma(t_{i+1})) + D_K(\sigma(t_q), y) + (P+2)\varepsilon. \end{aligned}$$

Besides, σ is a D -geodesic therefore

$$\widehat{\mathbf{T}}_n(x, y) \leq D_K(x, \sigma(t_p)) + D(\sigma(t_p), \sigma(t_q)) + D_K(\sigma(t_q), y) + (P+2)\varepsilon. \quad (4.19)$$

Combining (4.18) and (4.19), we get

$$\widehat{\mathbf{T}}_n(x, y) \leq D_{K+1}(x, y) + (P+2)\varepsilon + \frac{4L\|\mu\|_1}{P}. \quad (4.20)$$

The inclusion (4.16) is a consequence of (4.17) and (4.20).

By (4.14) and (4.16),

$$\overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\text{LD}_n^-(D_{K+1}, (P+2)\varepsilon + \frac{4L\|\mu\|_1}{P}) \right) \leq \overline{J}^-(D_K) + \int_0^L J_{\text{pp}}(\sigma'(t), |\dot{\sigma}|_D(t)) dt. \quad (4.21)$$

Choosing $\varepsilon := P^{-2}$ and letting $P \rightarrow \infty$ yields (4.8). \square

4.2 Lower bounding the monotonous rate function

In this section we fix a pseudometric $D \in \mathcal{D}_\mu$ and prove Proposition 4.2. We first show that it is a consequence of Lemma 4.8 then prove the lemma.

Lemma 4.8. *For all finite families of injective, pairwise disjoint, 1-Lipschitz paths $(\gamma_k : [0, T_k] \rightarrow X)_{1 \leq k \leq K}$,*

$$\underline{J}^-(D) \geq \sum_{k=1}^K J_{\text{pp}}(\gamma_k(T_k) - \gamma_k(0), D(\gamma_k)). \quad (4.22)$$

Proof of Proposition 4.2. Let $(\gamma_k)_{1 \leq k \leq K}$ be a family of paths as in Proposition 4.2. Let $P \in \mathbb{N}^*$ and $0 < \delta < \frac{1}{P} \min_{1 \leq k \leq K} T_k$. For all $1 \leq p \leq P$ and $1 \leq k \leq K$, we define the paths

$$\gamma_{k,p} := \gamma_k|_{\left[\frac{(p-1)T_k}{P}, \frac{pT_k}{P}\right]} \text{ and } \gamma_{k,p}^{(\delta)} := \gamma_k|_{\left[\frac{(p-1)T_k}{P}, \frac{pT_k}{P} - \delta\right]}.$$

The paths $(\gamma_{k,p}^{(\delta)})_{\substack{1 \leq k \leq K \\ 1 \leq p \leq P}}$ are 1-Lipschitz, injective and pairwise disjoint therefore by Lemma 4.8,

$$\underline{J}^-(D) \geq \sum_{k=1}^K \sum_{p=1}^P J_{\text{pp}} \left[\gamma_k \left(\frac{pT_k}{P} - \delta \right) - \gamma_k \left(\frac{(p-1)T_k}{P} \right), D(\gamma_{k,p}^{(\delta)}) \right].$$

Letting $\delta \rightarrow 0$ and using the lower semicontinuity of J_{pp} , we get

$$\begin{aligned} \underline{J}^-(D) &\geq \sum_{k=1}^K \sum_{p=1}^P J_{\text{pp}} \left[\gamma_k \left(\frac{pT_k}{P} \right) - \gamma_k \left(\frac{(p-1)T_k}{P} \right), D(\gamma_{k,p}) \right] \\ &= \sum_{k=1}^K \int_0^{T_k} \frac{P}{T_k} J_{\text{pp}} \left[\gamma_k \left(\frac{T_k}{P} \left\lceil \frac{Pt}{T_k} \right\rceil \right) - \gamma_k \left(\frac{T_k}{P} \left(\left\lceil \frac{Pt}{T_k} \right\rceil - 1 \right) \right), D \left(\gamma_{k, \lceil \frac{Pt}{T_k} \rceil} \right) \right] dt. \end{aligned}$$

By homogeneity of J_{pp} ,

$$\underline{J}^-(D) \geq \sum_{k=1}^K \int_0^{T_k} J_{\text{pp}} \left[\frac{P}{T_k} \left[\gamma_k \left(\frac{T_k}{P} \left\lceil \frac{Pt}{T_k} \right\rceil \right) - \gamma_k \left(\frac{T_k}{P} \left(\left\lceil \frac{Pt}{T_k} \right\rceil - 1 \right) \right) \right], \frac{P}{T_k} D \left(\gamma_{k, \lceil \frac{Pt}{T_k} \rceil} \right) \right] dt. \quad (4.23)$$

By (2.11), (2.13) and Lebesgue's differentiation theorem (see e.g. Theorem 7.10 in [21]), for all $1 \leq k \leq K$ and almost every $t \in [0, T_k]$,

$$\lim_{P \rightarrow \infty} \frac{P}{T_k} \left[\gamma_k \left(\frac{T_k}{P} \left\lceil \frac{Pt}{T_k} \right\rceil \right) - \gamma_k \left(\frac{T_k}{P} \left(\left\lceil \frac{Pt}{T_k} \right\rceil - 1 \right) \right) \right] = \gamma'_k(t) \quad (4.24)$$

and

$$\lim_{P \rightarrow \infty} \frac{P}{T_k} D \left(\gamma_{k, \lceil \frac{Pt}{T_k} \rceil} \right) = |\gamma_k|_D(t). \quad (4.25)$$

For all t satisfying (4.24) and (4.25), by lower semicontinuity of J_{pp} ,

$$\liminf_{P \rightarrow \infty} J_{\text{pp}} \left[\frac{P}{T_k} \left[\gamma_k \left(\frac{T_k}{P} \left\lceil \frac{Pt}{T_k} \right\rceil \right) - \gamma_k \left(\frac{T_k}{P} \left(\left\lceil \frac{Pt}{T_k} \right\rceil - 1 \right) \right) \right], \frac{P}{T_k} D \left(\gamma_{k, \lceil \frac{Pt}{T_k} \rceil} \right) \right] \geq J_{\text{pp}}(\gamma'_k(t), |\gamma_k|_D(t)).$$

Consequently, taking the inferior limit as $P \rightarrow \infty$ in (4.23) and applying Fatou's lemma gives (4.2). \square

Proof of Lemma 4.8. If $\underline{J}^-(D) = \infty$, there is nothing to prove. Assume that $\underline{J}^-(D) < \infty$. Let $(\gamma_k)_{1 \leq k \leq K}$ be a family of paths as in the lemma. Fix $\delta > 0$ such that for all distinct $1 \leq k_1, k_2 \leq K$,

$$(\gamma_{k_1} + \overline{B}(0, 2\delta)) \cap (\gamma_{k_2} + \overline{B}(0, 2\delta)) = \emptyset. \quad (4.26)$$

For all $1 \leq k \leq K$, we will denote by $\bar{\gamma}_k$ the set $\gamma_k + \bar{\mathbf{B}}(0, 2\delta)$. Define

$$\eta := \frac{\delta \min_{u \in \mathbb{S}_1} \mu(u)}{2\|\mu\|_1} \leq \delta. \quad (4.27)$$

Note that

$$\min_{u \in \mathbb{S}_1} J_{\text{pp}}(\delta u, \|\mu\|_1 \eta) > 0. \quad (4.28)$$

Define

$$R := \left\lceil \frac{\underline{J}^-(D) + 1}{\min_{u \in \mathbb{S}_1} J_{\text{pp}}(\delta u, \|\mu\|_1 \eta)} \right\rceil. \quad (4.29)$$

Fix $P \in \mathbb{N}^*$ large enough so that $\frac{\max_{1 \leq k \leq K} T_k}{P} \leq \frac{\eta}{2}$, and $0 < \varepsilon \leq \frac{\eta \|\mu\|_1}{2}$. Let $n \geq 1$. For all $0 \leq p \leq P$ and $1 \leq k \leq K$, define $x_{k,p} := \left\lfloor n \gamma_k \left(\frac{p T_k}{P} \right) \right\rfloor$. Recall the definition of hub below Lemma 4.5. We consider the events

$$\mathcal{E}_1(n) := \{\text{For all } 1 \leq k \leq K \text{ and } 0 \leq p \leq P, x_{k,p} \text{ is a hub}\}, \quad (4.30)$$

and

$$\mathcal{E}_2(n) := \left\{ \begin{array}{l} \text{There exist pairwise disjoint discrete paths } z_1 \overset{\pi_1}{\rightsquigarrow} z'_1, \dots, z_R \overset{\pi_R}{\rightsquigarrow} z'_R \text{ in } \llbracket 0, n \rrbracket^d \\ \text{such that for all } r \in \llbracket 1, R \rrbracket, \|z_r - z'_r\|_1 \geq n\delta \text{ and } \tau(\pi_r) \leq n\|\mu\|_1 \eta \end{array} \right\}^c. \quad (4.31)$$

We define

$$\text{Fav}^{**} = \text{Fav}^{**}(n, \varepsilon) := \text{LD}_n^-(D, \varepsilon) \cap \mathcal{E}_1(n) \cap \mathcal{E}_2(n). \quad (4.32)$$

We claim that

$$\underline{J}^-(D) = \lim_{\varepsilon \rightarrow 0} \uparrow \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{Fav}^{**}(n, \varepsilon)). \quad (4.33)$$

Indeed $\mathcal{E}_1(n)$ is an intersection of a finite number of decreasing events whose probability is lower bounded by (4.5), thus the FKG inequality gives

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathcal{E}_1(n)) = 0.$$

Since $\text{LD}_n^-(D, \varepsilon)$ is also decreasing, using the FKG inequality again yields

$$\underline{J}^-(D) = \lim_{\varepsilon \rightarrow 0} \uparrow \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{LD}_n^-(D, \varepsilon) \cap \mathcal{E}_1(n)). \quad (4.34)$$

Besides, by the union bound and the BK inequality, for all $n \in \mathbb{N}^*$,

$$\mathbb{P}(\mathcal{E}_2(n)^c) \leq \sum_{\substack{(z_r)_{1 \leq r \leq R} \\ (z'_r)_{1 \leq r \leq R}}} \prod_{r=1}^R \mathbb{P}(\mathbf{T}(z_r, z'_r) \leq n\|\mu\|_1 \eta),$$

where the sum spans over families of points of $\llbracket 0, n \rrbracket^d$ such that for all $1 \leq r \leq R$, $\|z_r - z'_r\|_1 \geq n\delta$. In particular there exists a polynomial Pol such that for all $n \in \mathbb{N}^*$,

$$\mathbb{P}(\mathcal{E}_2(n)^c) \leq \text{Pol}(n) \sup_{\substack{z \in \mathbb{Z}^d \\ \|z\|_1 \geq n\delta}} \mathbb{P}(\mathbf{T}(0, z) \leq n\|\mu\|_1 \eta)^R.$$

Taking the log, multiplying by $-\frac{1}{n}$ and using (3.1), we get

$$\begin{aligned} -\frac{1}{n} \log \mathbb{P}(\mathcal{E}_2(n)^c) &\geq -\frac{1}{n} \log \text{Pol}(n) + R \inf_{\substack{z \in \mathbb{Z}^d \\ \|z\|_1 \geq n\delta}} -\frac{1}{n} \log \mathbb{P}(\mathbf{T}(0, z) \leq n\|\mu\|_1 \eta) \\ &\geq -\frac{1}{n} \log \text{Pol}(n) + R \min_{\|u\|_1 \geq 1} J_{\text{pp}}(\delta u, \|\mu\|_1 \eta) \end{aligned}$$

Applying Theorem 1.1(iii) gives

$$\begin{aligned} -\frac{1}{n} \log \mathbb{P}(\mathcal{E}_2(n)^c) &\geq -\frac{1}{n} \log \text{Pol}(n) + R \min_{u \in \mathbb{S}_1} J_{\text{pp}}(\delta u, \|\mu\|_1 \eta) \\ &\geq -\frac{1}{n} \log \text{Pol}(n) + \underline{J}^-(D) + 1, \end{aligned}$$

hence

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathcal{E}_2(n)^c) \geq \underline{J}^-(D) + 1. \quad (4.35)$$

Combining (4.34) and (4.35) gives (4.33).

Fix $1 \leq k \leq K$. We now show that for large enough n ,

$$\text{Fav}^{**} \subseteq \left\{ \frac{1}{n} \mathbf{T}_{n\bar{\gamma}_k}(\lfloor n\gamma_k(0) \rfloor, \lfloor n\gamma_k(T_k) \rfloor) \leq D(\gamma_k) + \varepsilon P + \frac{2\kappa T_k R}{P} \right\}. \quad (4.36)$$

Assume that Fav^{**} occurs, and $n \geq \frac{P(3d+4)}{T_k}$. For all $0 \leq p \leq P-1$, we denote by π_p a discrete $\mathbf{T}_{[0, n]^d}$ -geodesic from $x_{k,p}$ to $x_{k,p+1}$. We say that p is *good* if $\pi_p \subseteq n\bar{\gamma}_k$, and *bad* otherwise. If p is bad, we denote by $x_p \overset{\hat{\pi}_p}{\rightsquigarrow} y_p$ the largest subpath of π_p containing x_p and included in $n\bar{\gamma}_k$. For all bad integer p , we denote by $[p]$ the set of bad integers q such that x_p and x_q are connected by a discrete path included in

$$\mathcal{R} := \bigcup_{p \text{ bad}} \hat{\pi}_p.$$

Moreover, a bad integer p is called *maximal bad* if $p = \max[p]$. Let $(p(0), \dots, p(N))$ be the only subsequence of $\llbracket 0, P \rrbracket$ such that $p(0) = 0$, $p(N) = P$ and for all $i \in \llbracket 0, N-1 \rrbracket$:

- (i) If $p(i)$ is good then $p(i+1) = p(i) + 1$.
- (ii) If $p(i)$ is maximal bad then $p(i+1) = p(i) + 1$.
- (iii) If $p(i)$ is bad but not maximal bad then $p(i+1) = \max[p(i)]$.

Let $i \in \llbracket 0, r-1 \rrbracket$. In the case (i), since $\pi_p \subseteq n\bar{\gamma}_k$,

$$\frac{1}{n} \mathbf{T}_{n\bar{\gamma}_k}(x_{k,p(i)}, x_{k,p(i)+1}) = \frac{1}{n} \mathbf{T}_{[0, n]^d}(x_{k,p(i)}, x_{k,p(i)+1}). \quad (4.37)$$

Since $\text{LD}_n^-(D, \varepsilon)$ occurs,

$$\frac{1}{n} \mathbf{T}_{n\bar{\gamma}_k}(x_{k,p(i)}, x_{k,p(i)+1}) \leq D\left(\gamma_k\left(\frac{p(i)T_k}{P}\right), \gamma_k\left(\frac{p(i+1)T_k}{P}\right)\right) + \varepsilon. \quad (4.38)$$

In the case (ii), since $\mathcal{E}_1(n)$ occurs and $\|x_{k,p(i)} - x_{k,p(i)+1}\|_1 \leq \frac{nT_k}{P} + d$, there exists a discrete path $x_{p(i)} \overset{\pi'_{p(i)}}{\rightsquigarrow} x_{p(i)+1}$ included in $\llbracket 0, n \rrbracket^d$, such that

$$\tau(\pi'_{p(i)}) \leq \kappa\left(\frac{nT_k}{P} + d\right) \leq \frac{2nT_k\kappa}{P} \text{ and } \|\pi'_{p(i)}\| \leq 2\left(\frac{nT_k}{P} + d\right) + 4 \leq \frac{3nT_k}{P}.$$

Besides,

$$\begin{aligned} d(\pi'_{p(i)}, (n\bar{\gamma}_k)^c) &\geq d(x_p, (n\bar{\gamma}_k)^c) - \|\pi'_{p(i)}\| \\ &\geq d\left(n\gamma_k\left(\frac{pT_k}{P}\right), (n\bar{\gamma}_k)^c\right) - \left\|x_{k,p} - n\gamma\left(\frac{pT_k}{P}\right)\right\|_1 - \|\pi'_{p(i)}\| \\ &\geq 2n\delta - d - \frac{3nT_k}{P} \\ &\geq n\delta > 0. \end{aligned}$$

Consequently, $\pi'_{p(i)} \subseteq n\bar{\gamma}_k$, thus

$$\frac{1}{n} \mathbf{T}_{n\bar{\gamma}_k}(x_{k,p(i)}, x_{k,p(i+1)}) \leq \frac{2nT_k\kappa}{P}. \quad (4.39)$$

In the case (iii), there exists a self-avoiding path $x_{k,p(i)} \overset{\pi'_{p(i)}}{\rightsquigarrow} x_{k,p(i+1)}$ included in \mathcal{R} . It satisfies

$$\begin{aligned} \tau(\pi'_{p(i)}) &\leq \sum_{q \in [p(i)]} \tau(\hat{\pi}_q) \\ &\leq \sum_{q \in [p(i)]} \tau(\pi_q). \\ &= \sum_{q \in [p(i)]} \mathbf{T}_{[0,n]^d}(x_{k,q}, x_{k,q+1}). \end{aligned}$$

Besides, $\text{LD}_n^-(D, \varepsilon)$ occurs therefore

$$\frac{1}{n} \mathbf{T}_{n\bar{\gamma}_k}(x_{k,p(i)}, x_{k,p(i+1)}) \leq \left(\sum_{q \in [p(i)]} D\left(\gamma\left(\frac{qT_k}{P}\right), \gamma\left(\frac{(q+1)T_k}{P}\right)\right) \right) + \varepsilon \cdot \#[p(i)]. \quad (4.40)$$

By triangle inequality, (4.38), (4.39) and (4.40) yield

$$\begin{aligned} \frac{1}{n} \mathbf{T}_{n\bar{\gamma}_k}(x_{k,0}, x_{k,P}) &\leq \sum_{\substack{0 \leq p \leq P-1 \\ p \text{ good}}} \left(D\left(\gamma_k\left(\frac{pT_k}{P}\right), \gamma_k\left(\frac{(p+1)T_k}{P}\right)\right) + \varepsilon \right) + \frac{2T_k\kappa}{P} \#\{0 \leq p \leq P-1 \mid p \text{ is maximal bad}\} \\ &\quad + \sum_{\substack{0 \leq p \leq P-1 \\ p \text{ bad}}} \left(D\left(\gamma_k\left(\frac{pT_k}{P}\right), \gamma_{-kp}\left(\frac{(p+1)T_k}{P}\right)\right) + \varepsilon \right), \end{aligned}$$

thus

$$\frac{1}{n} \mathbf{T}_{n\bar{\gamma}_k}(x_{k,0}, x_{k,P}) = \sum_{p=0}^{P-1} D\left(\gamma_k\left(\frac{pT_k}{P}\right), \gamma_k\left(\frac{(p+1)T_k}{P}\right)\right) + \varepsilon P + \frac{2T_k\kappa}{P} \#\{0 \leq p \leq P-1 \mid p \text{ is maximal bad}\}. \quad (4.41)$$

Moreover, for all bad p , $\|x_p - y_p\| \geq n\delta$, and

$$\begin{aligned} \frac{1}{n} \tau(\hat{\pi}_p) &\leq D\left(\gamma_k\left(\frac{pT_k}{P}\right), \gamma\left(\frac{(p+1)T_k}{P}\right)\right) + \varepsilon \\ &\leq \frac{\|\mu\|_1 T_k}{P} + \varepsilon \leq \|\mu\|_1 \left(\frac{\eta}{2} + \frac{\eta}{2}\right) = \|\mu\|_1 \eta. \end{aligned}$$

Since the paths $\hat{\pi}_p$ for maximal bad integers p are pairwise disjoint and $\mathcal{E}_2(n)$ occurs, there are at most R maximal bad integers. Plugging this bound into (4.41) gives, for large enough n ,

$$\frac{1}{n} \mathbf{T}_{n\bar{\gamma}_k}(x_{k,0}, x_{k,P}) \leq D(\gamma_k) + \varepsilon P + \frac{2\kappa T_k R}{P}, \quad (4.42)$$

thus the inclusion (4.36).

By (4.36), for large enough n ,

$$\text{Fav}^{**}(n, \varepsilon) \subseteq \bigcap_{k=1}^K \left\{ \frac{1}{n} \mathbf{T}_{n\bar{\gamma}_k}([n\gamma_k(0)], [n\gamma_k(T_k)]) \leq D(\gamma_k) + \varepsilon P + \frac{2\kappa T_k R}{P} \right\}.$$

Note that the events the intersection are independent, thus by (4.33),

$$\underline{J}^-(D) \geq \sum_{k=1}^K \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\frac{1}{n} \mathbf{T}_{n\bar{\gamma}_k}(\lfloor n\gamma_k(0) \rfloor, \lfloor n\gamma_k(T_k) \rfloor) \leq D(\gamma_k) + \varepsilon P + \frac{2\kappa T_k R}{P} \right). \quad (4.43)$$

Theorem 1.1 gives

$$\underline{J}^-(D) \geq \sum_{k=1}^K J_{\text{pp}} \left(\gamma_k(T_k) - \gamma_k(0), D(\gamma_k) + \varepsilon P + \frac{2\kappa T_k R}{P} \right).$$

Since J_{pp} is lower semicontinuous, letting $\varepsilon \rightarrow 0$ then $P \rightarrow \infty$ leads to (4.22). \square

4.3 Intrinsic expression

In this section we prove Proposition 4.3. We fix $L > 0$, $D \in \mathcal{D}_\mu^L$ and a highway network $(\sigma_k : [0, L] \rightarrow X)_{k \geq 1}$ (see Definition 2.4). Our main argument is Lemma 4.9, which states that if $J^-(D) < \infty$, then $(\text{P-grad } D)_z(u) = \mu(u)$ except if z belongs to a highway, and u is tangent to the highway at z . We postpone its proof after the proof of Proposition 4.3.

Lemma 4.9. *Assume that $J^-(D) < \infty$. Then for \mathcal{H}^1 -almost all $z \in X \setminus \bigcup_{k \geq 1} \sigma_k$,*

$$(\text{P-grad } D)_z = \mu, \quad (4.44)$$

and for all $k \geq 1$, for \mathcal{H}^1 -almost all $z = \sigma_k(t) \in \sigma_k$, for all $u \in \mathbb{R}^d \setminus (\sigma'_k(t)\mathbb{R})$,

$$(\text{P-grad } D)_z(u) = \mu(u). \quad (4.45)$$

Proof of Proposition 4.3. For all $k \geq 1$, using (2.15) and Lemma 2.11, we get

$$\begin{aligned} \int_0^L J_{\text{pp}}(\sigma'_k(t), |\sigma_k|_D(t)) dt &= \int_0^L J_{\text{pp}}(\sigma'_k(t), (\text{P-grad } D)_{\sigma_k(t)}(\sigma'_k(t))) dt \\ &= \int_{\sigma_k} J_{\text{pp}} \left(\frac{\sigma'_k(\sigma_k^{-1}(z))}{\|\sigma'_k(\sigma_k^{-1}(z))\|_2}, (\text{P-grad } D)_z \left(\frac{\sigma'_k(\sigma_k^{-1}(z))}{\|\sigma'_k(\sigma_k^{-1}(z))\|_2} \right) \right) \mathcal{H}^1(dz). \end{aligned}$$

By (1.31),

$$J^-(D) = \sum_{k \geq 1} \int_{\sigma_k} J_{\text{pp}} \left(\frac{\sigma'_k(\sigma_k^{-1}(z))}{\|\sigma'_k(\sigma_k^{-1}(z))\|_2}, (\text{P-grad } D)_z \left(\frac{\sigma'_k(\sigma_k^{-1}(z))}{\|\sigma'_k(\sigma_k^{-1}(z))\|_2} \right) \right) \mathcal{H}^1(dz). \quad (4.46)$$

If $J^-(D) = \infty$, the result now follows from

$$J_{\text{pp}} \left(\frac{\sigma'_k(\sigma_k^{-1}(z))}{\|\sigma'_k(\sigma_k^{-1}(z))\|_2}, (\text{P-grad } D)_z \left(\frac{\sigma'_k(\sigma_k^{-1}(z))}{\|\sigma'_k(\sigma_k^{-1}(z))\|_2} \right) \right) \leq \max_{u \in \mathbb{S}_2} J_{\text{pp}}(u, (\text{P-grad } D)_z(u)), \quad (4.47)$$

for all $z \in \bigcup \sigma_k$. Otherwise, by Theorem 1.1(iv), for all $u \in \mathbb{S}_2$,

$$J_{\text{pp}}(u, \mu(u)) = 0.$$

In particular, by Lemma 4.9, (4.47) is an equality \mathcal{H}^1 -almost everywhere on the highway network, and the right-hand side is zero \mathcal{H}^1 -almost everywhere outside the highway network, giving the desired result. \square

Proof of Lemma 4.9. Let $0 < \varepsilon < 1$. Note that since J_{pp} is continuous on \mathcal{X} , Theorem 1.1(iv) implies

$$C(\varepsilon) := \min_{u \in \mathbb{S}_2} J_{\text{pp}}(u, (1 - \varepsilon)\mu(u)) > 0. \quad (4.48)$$

For all $k \geq 1$, we define X_k as the set of regular points of σ_k (see Definition 2.10), and

$$X_0 := X \setminus \bigcup_{k \geq 1} \sigma_k.$$

By Lemma 2.11,

$$\mathcal{H}^1 \left(X \setminus \left(\bigcup_{k \geq 0} X_k \right) \right) = 0. \quad (4.49)$$

To prove Lemma 4.9, it is sufficient to show that the subset

$$\begin{aligned} A_\varepsilon := & \{z \in X_0 \mid \exists u \in \mathbb{S}_2 \text{ s.t. } (\text{P-grad } D)_z(u) \leq (1 - 3\varepsilon)\mu(u)\} \\ & \cup \bigcup_{k \geq 1} \{\sigma_k(t) \in X_k \mid \exists u \in \mathbb{S}_2 \text{ not colinear to } \sigma'_k(t), \text{ s.t. } (\text{P-grad } D)_z(u) \leq (1 - 3\varepsilon)\mu(u)\} \end{aligned} \quad (4.50)$$

is \mathcal{H}^1 -negligible. Let $P \in \mathbb{N}^*$, $R > 0$ and $z \in A_\varepsilon$. By definition of $(\text{P-grad } D)_z$ and A_ε , there exists $u_z \in \mathbb{S}_2$ and a Lipschitz path $\gamma_z : [0, T] \rightarrow X$ such that $\gamma_z(0) = z$, $\gamma'_z(0) = u_z$,

$$\liminf_{t \rightarrow 0} \frac{D(\gamma_z|_{[0, t]})}{t} \leq (1 - 2\varepsilon)\mu(u_z),$$

and for small enough t ,

$$\gamma_z([0, t]) \cap \left(\bigcup_{p=1}^P \sigma_p \right) \subseteq \{z\} : \quad (4.51)$$

indeed, for all $1 \leq p \leq P$, either $z \notin \sigma_p$ and this is clear, or z is a regular point of σ_p , and u_z doesn't belong to the tangent line of σ_p at z . In particular, there exists $0 < r_z \leq \frac{R}{20}$ such that

$$D(\gamma|_{[0, r_z]}) \leq (1 - \varepsilon)\mu(\gamma_z(r_z) - \gamma_z(0)) \quad (4.52)$$

and for all $t \in (0, r_z]$,

$$\frac{t}{2} \leq \|\gamma_z(t) - \gamma_z(0)\|_2 \leq 2t. \quad (4.53)$$

Up to a reparametrization, γ_z may be assumed to be injective. Since A_ε is a bounded subset of \mathbb{R}^d , by Vitali's covering theorem (see e.g. Section 15.A in [16]), there exists a countable subset $\{z_i\}_{i \geq 1}$ of A_ε such that

$$A_\varepsilon \subseteq \bigcup_{i \geq 1} \bar{B}_2(z_i, 6r_{z_i}), \quad (4.54)$$

and the balls $\bar{B}_2(z_i, 2r_{z_i})$ are pairwise disjoint. To simplify the notations we will write r_i and γ_i instead of r_{z_i} and γ_{z_i} . By (4.54) and the definition of $\phi_R^1(A_\varepsilon)$ (see (1.43)),

$$\phi_R^1(A_\varepsilon) \leq 12 \sum_{i \geq 1} r_i. \quad (4.55)$$

We now upper bound the right-hand side of (4.55). First note that for all $i \geq 1$, by (4.53),

$$\gamma_i([0, r_i]) \subseteq \bar{B}(z_i, 2r_i).$$

In particular, for all $0 < \delta < 1$ the $\gamma_i([\delta r_i, r_i])$ are pairwise disjoint. Since they are also disjoint from the $(\sigma_p)_{1 \leq p \leq P}$ by (4.51), Proposition 4.2 gives

$$\sum_{i \geq 1} \int_{\delta r_i}^{r_i} J_{\text{PP}}(\gamma'_i(t), |\dot{\gamma}_i|_D(t)) dt + \sum_{p=1}^P \int_0^L J_{\text{PP}}(\sigma'_p(t), |\dot{\sigma}_p|_D(t)) dt \leq J^-(D).$$

By monotone convergence, letting $\delta \rightarrow 0$ gives

$$\sum_{i \geq 1} \int_0^{r_i} J_{\text{pp}}(\gamma'_i(t), |\dot{\gamma}_i|_D(t)) dt + \sum_{p=1}^P \int_0^L J_{\text{pp}}(\sigma'_p(t), |\dot{\sigma}_p|_D(t)) dt \leq J^-(D).$$

Hence, by Proposition 4.1,

$$\sum_{i \geq 1} \int_0^{r_i} J_{\text{pp}}(\gamma'_i(t), |\dot{\gamma}_i|_D(t)) dt \leq \sum_{p=P+1}^{\infty} \int_0^L J_{\text{pp}}(\sigma'_p(t), |\dot{\sigma}_p|_D(t)) dt. \quad (4.56)$$

Besides, for all $i \geq 1$, Jensen's inequality in its multivariate version (see [19]) and Lemma 2.7 gives

$$\int_0^{r_i} J_{\text{pp}}(\gamma'_i(t), |\dot{\gamma}_i|_D(t)) dt \geq J_{\text{pp}}\left(\gamma_i(r_i) - \gamma_i(0), D\left(\gamma|_{[0, r_i]}\right)\right).$$

By homogeneity of J_{pp} , (4.52) and (4.53),

$$\begin{aligned} \int_0^{r_i} J_{\text{pp}}(\gamma'_i(t), |\dot{\gamma}_i|_D(t)) dt &\geq \|\gamma_i(r_i) - \gamma_i(0)\|_2 J_{\text{pp}}\left(\frac{\gamma_i(r_i) - \gamma_i(0)}{\|\gamma_i(r_i) - \gamma_i(0)\|_2}, \frac{D\left(\gamma|_{[0, r_i]}\right)}{\|\gamma_i(r_i) - \gamma_i(0)\|_2}\right) \\ &\geq \frac{r_i}{2} \min_{u \in \mathbb{S}_2} J_{\text{pp}}(u, (1 - \varepsilon)\mu(u)) = \frac{r_i C(\varepsilon)}{2}. \end{aligned}$$

Combining this inequality with (4.56), we get

$$\sum_{i \geq 1} r_i \leq \frac{2}{C(\varepsilon)} \sum_{p=P+1}^{\infty} \int_0^L J_{\text{pp}}(\sigma'_p(t), |\dot{\sigma}_p|_D(t)) dt. \quad (4.57)$$

By (4.55) and (4.57),

$$\phi_R^1(A_\varepsilon) \leq \frac{24}{C(\varepsilon)} \sum_{p=P+1}^{\infty} \int_0^L J_{\text{pp}}(\sigma'_p(t), |\dot{\sigma}_p|_D(t)) dt.$$

Letting $R \rightarrow 0$ yields

$$\mathcal{H}^1(A_\varepsilon) \leq \frac{24}{C(\varepsilon)} \sum_{p=P+1}^{\infty} \int_0^L J_{\text{pp}}(\sigma'_p(t), |\dot{\sigma}_p|_D(t)) dt.$$

Since $J^-(D) < \infty$, the series on the right-hand side converges by (1.29), thus letting $P \rightarrow \infty$ gives $\mathcal{H}^1(A_\varepsilon) = 0$, which concludes the proof. \square

4.4 Strict monotonicity : proof of Proposition 4.4

Let $D_1, D_2 \in \mathcal{D}_\mu$ be distinct pseudometrics such that $D_1 \leq D_2$: in particular there exist $x, y \in X$ such that $D_1(x, y) < D_2(x, y)$. Also assume that $J^-(D_2) < \infty$. There exists a Lipschitz path $\gamma : [0, T] \rightarrow X$ such that $D_1(\gamma) < D_2(\gamma)$. In particular, by (2.16) and (2.20) there exists a Borel set $A \subset X$ such that $\mathcal{H}^1(A) > 0$, and for all $z \in A$, there exists $u \in \mathbb{S}_2$ such that

$$(\text{P-grad } D_1)_z(u) < (\text{P-grad } D_2)_z(u).$$

By Lemma 4.9 and Remark 1.2, for all $z \in A$,

$$\max_{u \in \mathbb{S}_2} J_{\text{pp}}(u, (\text{P-grad } D_1)_z(u)) < \max_{u \in \mathbb{S}_2} J_{\text{pp}}(u, (\text{P-grad } D_2)_z(u)).$$

Besides, the analogous weak inequality is true for all $z \in X$, thus Proposition 4.3 concludes. \square

5 Large deviation principle

The goal of this section is to prove Theorem 1.10. For all $b \in [1, \infty]$ and $e \in \mathbb{E}^d$, we define

$$\tau_e^{(b)} := \tau_e \wedge b. \quad (5.1)$$

We denote by $\widehat{\mathbf{T}}_n^{(b)}$, $\mu^{(b)}$, $J^{(b)}$ and $J^{-(b)}$ the analogues of $\widehat{\mathbf{T}}_n$, μ , J and J^- for the edge passage times $\left(\tau_e^{(b)}\right)_{e \in \mathbb{E}^d}$. For all $b < \infty$, we will introduce an auxiliary process $(\widetilde{\mathbf{T}}_n^{(b)})_{n \geq 1}$ with values in $\mathcal{D}_{b\|\cdot\|_1}$ (see (5.14)), such that:

(i) For all $\alpha > 0$, there exists $L > 0$ such that

$$\underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}\left(\widetilde{\mathbf{T}}_n^{(b)} \notin \mathcal{D}_{b\|\cdot\|_1}^L\right) \geq \alpha. \quad (5.2)$$

(ii) For all $n \geq 1$,

$$\mathfrak{d}_\infty\left(\widehat{\mathbf{T}}_n^{(b)}, \widetilde{\mathbf{T}}_n^{(b)}\right) \leq \frac{2bd}{n}. \quad (5.3)$$

Theorem 1.7 and (5.2) will imply that $(\widetilde{\mathbf{T}}_n^{(b)})_{n \geq 1}$ satisfies the LDP, with the rate function J . We will deduce the same for $(\widehat{\mathbf{T}}_n^{(b)})_{n \geq 1}$ by (5.3). We treat the general case by letting $b \rightarrow \infty$, under some good moment assumptions. To control the probability that $\widehat{\mathbf{T}}_n$ takes abnormally large values, we rely on Lemma 5.1, proven in Section B.

Lemma 5.1. *Assume (Moment). For all $\varepsilon > 0$,*

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \sup_{x, y \in X} \log \mathbb{P}\left(\widehat{\mathbf{T}}_n(x, y) \geq \mu(x - y) + \varepsilon\right) = +\infty. \quad (5.4)$$

5.1 Geodesic lengths

Lemma 5.2 essentially states that it is very atypical for the random metric $\mathbf{T}_{[0, n]^d}$ to have geodesics of length Ln , with large L . It will be the key argument in the proof of (5.2). Note that the bound (5.6) is uniform over all truncations of the passage times by $b \in [1, \infty]$. For all $b \in [1, \infty]$, $L > 0$ and $n \geq 1$, we define the event

$$\text{LongGeo}_n^{(b)}(L) := \left\{ \begin{array}{l} \text{There exists a discrete geodesic } \sigma \subseteq \llbracket 0, n \rrbracket^d \\ \text{for } \mathbf{T}_{[0, n]^d}^{(b)}, \text{ such that } \|\sigma\| \geq Ln \end{array} \right\}. \quad (5.5)$$

Lemma 5.2. *For all $\alpha > 0$, there exists $L > 0$ such that for all $b \in [1, \infty]$,*

$$\underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}\left(\text{LongGeo}_n^{(b)}(L)\right) \geq \alpha. \quad (5.6)$$

Proof. For all $L, C > 0$ and $n \geq 1$ we define the event

$$\mathcal{E}_n(L, C) := \left\{ \begin{array}{l} \text{There exists a self-avoiding discrete path } \pi \subseteq \llbracket 0, n \rrbracket^d \\ \text{such that } \|\pi\| \geq nL \text{ and } \tau^{(1)}(\pi) \leq nC \end{array} \right\}. \quad (5.7)$$

We claim that there exists $\beta > 0$ such that for large enough L ,

$$\underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathcal{E}_n(L, d\|\mu\|_1 + 1)) \geq \lfloor L \rfloor \beta, \quad (5.8)$$

By Proposition 5.8 in [17] (and a union bound on the possible starting point of a discrete path in $\llbracket 0, n \rrbracket^d$) there exists a constant $C_1 > 0$ such that

$$\beta := \underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathcal{E}_n(1, C_1)) > 0. \quad (5.9)$$

Let $L \geq \frac{d\|\mu\|_1+1}{C_1}$ be an integer. We have

$$\begin{aligned}\mathbb{P}(\mathcal{E}_n(L, d\|\mu\|_1 + 1)) &\leq \mathbb{P}(\mathcal{E}_n(L, LC_1)) \\ &\leq \mathbb{P}(\mathcal{E}_{nL}(1, C_1)).\end{aligned}$$

Consequently,

$$-\frac{1}{n} \log \mathbb{P}(\mathcal{E}_n(L, d\|\mu\|_1 + 1)) \geq L \cdot \left(-\frac{1}{nL} \log \mathbb{P}(\mathcal{E}_{nL}(1, C_1)) \right).$$

Letting $n \rightarrow \infty$ gives (5.8).

By Lemma 5.1,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \sup_{x, y \in \llbracket 0, n \rrbracket^d} \mathbb{P}(\mathbf{T}_{[0, n]^d}(x, y) \geq n(d\|\mu\|_1 + 1)) = \infty. \quad (5.10)$$

By union bound and (5.10),

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\exists x, y \in \llbracket 0, n \rrbracket^d, \mathbf{T}_{[0, n]^d}(x, y) \geq n(d\|\mu\|_1 + 1)) = \infty. \quad (5.11)$$

Let $L > 0$ be such that (5.8) holds and $b \in [1, \infty]$. We have the inclusion

$$\text{LongGeo}_n^{(b)}(L) \subseteq \left\{ \exists x, y \in \llbracket 0, n \rrbracket^d, \mathbf{T}_{[0, n]^d}(x, y) \geq n(d\|\mu\|_1 + 1) \right\} \cup \mathcal{E}_n(L, d\|\mu\|_1 + 1),$$

therefore by (5.8) and (5.11),

$$\underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{LongGeo}_n^{(b)}(L)) \geq [L]\beta,$$

which concludes the proof. \square

5.2 The continuous metric

In this section we fix $b \in [1, \infty)$ and define $\tilde{\mathbf{T}}_n^{(b)}$. We prove that it is a good approximation of $\widehat{\mathbf{T}}_n^{(b)}$ and it follows the LDP with the rate function $J^{(b)}$.

Definition 5.3. Let $n \geq 1$. We extend $\mathbf{T}_{[0, n]^d}^{(b)}$ to $[0, n]^d$ as follows. For all edges $(x^-, x^+), (y^-, y^+) \in \mathbb{E}(\llbracket 0, n \rrbracket^d)$ and all $x \in [x^-, x^+], y \in [y^-, y^+]$, we define

$$\mathbf{T}_{[0, n]^d}^{(b)}(x, y) := \min_{\substack{x' \in \{x^-, x^+\} \\ y' \in \{y^-, y^+\}}} \left(\|x - x'\|_1 \tau_{(x^-, x^+)} + \mathbf{T}_{[0, n]^d}^{(b)}(x', y') + \|y - y'\|_1 \tau_{(y^-, y^+)} \right). \quad (5.12)$$

We then define, for all $x, y \in [0, n]^d$,

$$\mathbf{T}_{[0, n]^d}^{(b)}(x, y) := (b\|x - y\|_1) \wedge \min_{x', y'} \left(b\|x - x'\|_1 + \mathbf{T}_{[0, n]^d}^{(b)}(x', y') + b\|y' - y\|_1 \right), \quad (5.13)$$

where the minimum is taken on all pairs of points, belonging to a pair of (maybe equal) edges in $\mathbb{E}(\llbracket 0, n \rrbracket^d)$. For all $x, y \in X$, we define

$$\tilde{\mathbf{T}}_n^{(b)}(x, y) := \frac{1}{n} \mathbf{T}_{[0, n]^d}^{(b)}(nx, ny). \quad (5.14)$$

Lemma 5.4. *The process $(\tilde{\mathbf{T}}_n^{(b)})_{n \geq 1}$ takes values in $\mathcal{D}_{b\|\cdot\|_1}$ and for all $x, y \in \frac{1}{n} \llbracket 0, n \rrbracket^d$,*

$$\tilde{\mathbf{T}}_n^{(b)}(x, y) = \widehat{\mathbf{T}}_n^{(b)}(x, y). \quad (5.15)$$

Moreover, for all $\alpha > 0$, there exists $L > 0$ that does not depend on b , such that

$$\underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\tilde{\mathbf{T}}_n^{(b)} \notin \mathcal{D}_{b\|\cdot\|_1}^L) \geq \alpha. \quad (5.16)$$

Proof. Let $n \geq 1$. It is clear that $\tilde{\mathbf{T}}_n^{(b)}$ is a pseudometric on X , is upper bounded by $b\|\cdot\|_1$ and satisfies (5.15). Let $x, y \in [0, n]^d$. It follows from the definition of $\mathbf{T}_{[0, n]^d}^{(b)}$ that there exists a $\mathbf{T}_{[0, n]^d}^{(b)}$ -geodesic σ from nx to ny that admits the decomposition

$$nx \overset{\sigma_1}{\rightsquigarrow} x' \overset{\sigma_2}{\rightsquigarrow} x'' \overset{\sigma_3}{\rightsquigarrow} y'' \overset{\sigma_4}{\rightsquigarrow} y' \overset{\sigma_5}{\rightsquigarrow} ny, \quad (5.17)$$

where:

- the points x' and y' belong to edges in $E\left(\llbracket 0, n \rrbracket^d\right)$, and the points x'', y'' belong to $\llbracket 0, n \rrbracket^d$,
- the paths $\sigma_1, \sigma_2, \sigma_4$ and σ_5 are straight lines,
- the path σ_3 is a discrete geodesic for $\mathbf{T}_{[0, n]^d}^{(b)}$ between x'' and y''

(some σ_i may be trivial). Moreover, σ_2 and σ_4 are subsets of edges in $E\left(\llbracket 0, n \rrbracket^d\right)$, thus

$$\|\sigma\|_1 \leq \|\sigma_3\| + 2dn + 2. \quad (5.18)$$

Fix $\alpha > 0$ and let $L > 0$ be the number provided by Lemma 5.6. Since $t \mapsto \frac{1}{t}\sigma(t)$ is a $\tilde{\mathbf{T}}_n^{(b)}$ -geodesic, we have

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}\left(\tilde{\mathbf{T}}_n^{(b)} \notin \mathcal{D}_{b\|\cdot\|_1}^{L+2d+2/n}\right) \geq \alpha,$$

thus

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}\left(\tilde{\mathbf{T}}_n^{(b)} \notin \mathcal{D}_{b\|\cdot\|_1}^{L+2d+1}\right) \geq \alpha,$$

which concludes the proof. \square

Lemma 5.5. *For all $n \geq 1$, almost surely*

$$\mathfrak{d}_\infty\left(\widehat{\mathbf{T}}_n^{(b)}, \tilde{\mathbf{T}}_n^{(b)}\right) \leq \frac{2bd}{n}. \quad (5.19)$$

In particular, replacing $\widehat{\mathbf{T}}_n^{(b)}$ by $\tilde{\mathbf{T}}_n^{(b)}$ in the definition of $\bar{J}^{(b)}$ (see (1.39)), $\underline{J}^{(b)}$ (see (1.40)) and $J^{-\cdot(b)}$ (see (1.26) and (1.27)) defines the same functions.

Proof. The pseudometrics $\widehat{\mathbf{T}}_n^{(b)}$ and $\tilde{\mathbf{T}}_n^{(b)}$ coincide on $\left(\frac{1}{n}\llbracket 0, n \rrbracket^d\right)^2$, thus the almost sure bound follows by triangle inequality. \square

Lemma 5.6. *The process $(\tilde{\mathbf{T}}_n^{(b)})_{n \geq 1}$ follows the LDP with the good rate function*

$$J^{(b)} : \mathcal{F}_b \longrightarrow [0, \infty]$$

$$D \longmapsto \begin{cases} \infty & \text{if } D \notin \mathcal{D}_{\mu^{(b)}}, \\ J^{-\cdot(b)}(D) & \text{if } D \in \mathcal{D}_{\mu^{(b)}}. \end{cases} \quad (5.20)$$

Proof. Let $D \in \mathcal{F}_b$. We claim that $(\tilde{\mathbf{T}}_n^{(b)})_{n \geq 1}$ follows the weak LDP with the rate function $J^{(b)}$, i.e.

$$\underline{J}^{(b)}(D) = \bar{J}^{(b)}(D) = J^{(b)}(D). \quad (5.21)$$

For all $\varepsilon > 0$, we define the events

$$\widetilde{\text{LD}}_n^{(b)}(D, \varepsilon) := \left\{ \mathfrak{d}_\infty\left(\tilde{\mathbf{T}}_n^{(b)}, D\right) \leq \varepsilon \right\} \quad (5.22)$$

and

$$\widetilde{\text{LD}}_n^{-\cdot(b)}(D, \varepsilon) := \left\{ \forall x, y \in X, \quad \tilde{\mathbf{T}}_n^{(b)}(x, y) \leq D(x, y) + \varepsilon \right\}, \quad (5.23)$$

i.e. the analogues of $\text{LD}_n(D, \varepsilon)$ and $\text{LD}_n^-(D, \varepsilon)$ with $\widehat{\mathbf{T}}_n$ replaced by $\widetilde{\mathbf{T}}_n^{(b)}$ (see (1.25) and (1.38)). We treat three cases differently.

Case 1: Assume that $D \notin \mathcal{D}_{b\|\cdot\|_1}$. Let $L > 0$. Since $\mathcal{D}_{b\|\cdot\|_1}^L$ is compact, there exists $\varepsilon > 0$ such that

$$\widetilde{\text{LD}}_n^{(b)}(D, \varepsilon) \subseteq \left\{ \widetilde{\mathbf{T}}_n^{(b)} \notin \mathcal{D}_{b\|\cdot\|_1}^L \right\}.$$

In particular,

$$\underline{J}^{(b)}(D) \geq \sup_{L>0} \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\widetilde{\mathbf{T}}_n^{(b)} \notin \mathcal{D}_{b\|\cdot\|_1}^L \right).$$

By (5.16), $\underline{J}^{(b)}(D) = \infty$, thus (5.21).

Case 2: Assume that $D \in \mathcal{D}_{b\|\cdot\|_1} \setminus \mathcal{D}_{\mu^{(b)}}$. Then there exists $x, y \in X$ such that $D(x, y) > \mu^{(b)}(x - y)$. There exists $\varepsilon > 0$ be such that

$$\widetilde{\text{LD}}_n(D, \varepsilon) \subseteq \left\{ \widetilde{\mathbf{T}}_n^{(b)}(x, y) \geq \mu^{(b)}(x - y) + \varepsilon \right\}.$$

In particular, by Lemma 5.1, $\underline{J}^{(b)}(D) = \infty$, thus (5.21).

Case 3: Assume that $D \in \mathcal{D}_{\mu^{(b)}}$. The inequality $\underline{J}^{(b)}(D) \geq J^{-, (b)}(D) = J^{(b)}(D)$ is clear. If $J^{-, (b)}(D) = \infty$, then (5.21) is proven. Otherwise by Lemma 5.4 there exists $L > 0$ such that

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\widetilde{\mathbf{T}}_n^{(b)} \notin \mathcal{D}_{b\|\cdot\|_1}^L \right) > J^{-, (b)}(D). \quad (5.24)$$

Fix $\varepsilon > 0$. Let $\delta \in (0, \varepsilon)$. Consider the compact sets

$$K_\varepsilon := \left\{ D' \in \mathcal{D}_{b\|\cdot\|_1}^L \mid D' \leq D, \text{ and there exists } x, y \in X \text{ such that } D'(x, y) \leq D(x, y) - \varepsilon \right\}$$

and

$$K_{\varepsilon, \delta} := \left\{ D' \in \mathcal{D}_{b\|\cdot\|_1}^L \mid D' \leq D + \delta, \text{ and there exists } x, y \in X \text{ such that } D'(x, y) \leq D(x, y) - \varepsilon \right\}$$

By Proposition 4.4, compactness and lower semicontinuity of $J^{-, (b)}$,

$$\min_{D' \in K_\varepsilon} J^{-, (b)}(D') > J^{-, (b)}(D). \quad (5.25)$$

Note that by (1.24),

$$\begin{aligned} \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\widetilde{\mathbf{T}}_n^{(b)} \in K_{\varepsilon, \delta} \right) &\geq \min_{D' \in K_{\varepsilon, \delta}} \underline{J}^{(b)}(D') \\ &\geq \min_{D' \in K_{\varepsilon, \delta}} J^{-, (b)}(D'). \end{aligned}$$

Using compactness and lower semicontinuity again yields

$$\lim_{\delta \rightarrow 0} \uparrow \lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\widetilde{\mathbf{T}}_n^{(b)} \in K_{\varepsilon, \delta} \right) \geq \min_{D' \in K_\varepsilon} J^{-, (b)}(D').$$

Consequently, by (5.25), for small enough $\delta \in (0, \varepsilon)$,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\widetilde{\mathbf{T}}_n^{(b)} \in K_{\varepsilon, \delta} \right) > J^{-, (b)}(D).$$

This implies that for small enough $\delta \in (0, \varepsilon)$,

$$\begin{aligned} \overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\widetilde{\text{LD}}_n^-(D, \delta) \right) &= \overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\widetilde{\text{LD}}_n^-(D, \delta) \cap \left\{ \widetilde{\mathbf{T}}_n^{(b)} \notin K_{\varepsilon, \delta} \right\} \right) \\ &\geq \overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\widetilde{\text{LD}}_n^-(D, \varepsilon) \right). \end{aligned}$$

Letting $\delta \rightarrow 0$ then $\varepsilon \rightarrow 0$ leads to $J^{-, (b)}(D) \geq J^{(b)}(D)$, which concludes the proof of (5.21).

Equation (5.21) and Lemma 1.6 imply that $(\tilde{\mathbf{T}}_n^{(b)})_{n \geq 1}$ follows the weak LDP, with the rate function $J^{(b)}$. Furthermore, Proposition 2.1 and (5.16) implies that $(\tilde{\mathbf{T}}_n^{(b)})_{n \geq 1}$ is exponentially tight, meaning that for all $\alpha > 0$, there exists a compact $K \subseteq \mathcal{F}_b$ such that

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\tilde{\mathbf{T}}_n^{(b)} \notin K) > \alpha. \quad (5.26)$$

Consequently, by Lemma 1.2.18 in [11], the process $(\tilde{\mathbf{T}}_n^{(b)})_{n \geq 1}$ follows the LDP with the good rate function $J^{(b)}$. \square

5.3 LDP for the classic passage time

We now prove that under the Assumptions (SubC) and (Moment), $(\hat{\mathbf{T}}_n^{(b)})_{n \geq 1}$ follows the LDP with the rate function J . For the truncated rescaled metric $\hat{\mathbf{T}}_n^{(b)}$, it is a consequence of Lemmas 5.5 and 5.6. For the untruncated one, we use a general large deviation theory result providing a LDP for a process whenever it is in some sense the limit of processes which all follow LDPs. Definition 5.7 and Theorem 5.8 are a reformulation of Definitions 4.2.10, 4.2.14 and Theorems 4.2.13, 4.2.16 in [11] for our framework.

Definition 5.7. Let $(\mathbb{X}, d_{\mathbb{X}})$ be a metric space. All the random variables mentioned in this definition are assumed to be defined on the probability space.

- (i) We say that two processes $(X_n)_{n \geq 1}$ and $(Y_n)_{n \geq 1}$ on \mathbb{X} are *exponentially equivalent* if for all $\varepsilon > 0$ and $n \geq 1$, the event $\{d_{\mathbb{X}}(X_n, Y_n) \geq \varepsilon\}$ is measurable, and with fixed $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(d_{\mathbb{X}}(X_n, Y_n) \geq \varepsilon) = \infty. \quad (5.27)$$

- (ii) We say that a family of processes $((X_n^{(b)})_{n \geq 1}, b \in [0, \infty))$ on \mathbb{X} is an *exponentially good approximation* of $(X_n)_{n \geq 1}$ as $b \rightarrow \infty$ if for all $\varepsilon > 0$, $b \geq 1$ and $n \geq 1$, the event $\{d_{\mathbb{X}}(X_n^{(b)}, X_n) \geq \varepsilon\}$ is measurable, and with fixed $\varepsilon > 0$,

$$\lim_{b \rightarrow \infty} \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(d_{\mathbb{X}}(X_n^{(b)}, X_n) \geq \varepsilon) = \infty. \quad (5.28)$$

Theorem 5.8. Let $(\mathbb{X}, d_{\mathbb{X}})$ be a metric space.

- (i) If the processes $(X_n)_{n \geq 1}$ and $(Y_n)_{n \geq 1}$ on \mathbb{X} are exponentially equivalent and $(X_n)_{n \geq 1}$ follows a LDP with a good rate function, then $(Y_n)_{n \geq 1}$ also follows a LDP, with the same rate function.
- (ii) Assume that the processes $(X_n^{(b)})_{n \geq 1}$ on \mathbb{X} are exponentially good approximations of $(X_n)_{n \geq 1}$ as $b \rightarrow \infty$, and for all $b \geq 1$, $(X_n^{(b)})_{n \geq 1}$ follows the LDP with a rate function $I^{(b)}$. Then $(X_n)_{n \geq 1}$ follows the weak LDP with the rate function

$$I : x \mapsto \lim_{\varepsilon \rightarrow 0} \uparrow \liminf_{b \rightarrow \infty} \inf_{\hat{x} \in B_{\mathbb{X}}(x, \varepsilon)} I^{(b)}(\hat{x}). \quad (5.29)$$

If furthermore I is a good rate function and satisfies, for all closed sets F ,

$$\inf_{x \in F} I(x) \leq \overline{\lim}_{b \rightarrow \infty} \inf_{x \in F} I^{(b)}(x), \quad (5.30)$$

then $(X_n)_{n \geq 1}$ follows the LDP with the rate function I .

Lemma 5.9. Assume (SubC) and (Moment). The processes $(\hat{\mathbf{T}}_n^{(b)})_{n \geq 1}$ are exponentially good approximations of $(\hat{\mathbf{T}}_n)_{n \geq 1}$ as $b \rightarrow \infty$.

Proof. Fix $\varepsilon > 0$, $\alpha > 0$. Let $L > 0$ be the number given by Lemma 5.2. For all $n \geq 1$ and $b \geq 1$, define

$$\text{LDGreedy}_n(b) := \left\{ \begin{array}{l} \text{There exists a self-avoiding discrete path } \pi \subseteq \llbracket 0, n \rrbracket^d, \\ \text{such that } \|\pi\| \leq Ln \text{ and } \tau(\pi) - \tau^{(b)}(\pi) \geq 2d\varepsilon n \end{array} \right\}. \quad (5.31)$$

Note that for all $n \geq 1$ and $b \geq 1$, we have

$$\left\{ \mathfrak{d}_\infty(\widehat{\mathbf{T}}_n^{(b)}, \widehat{\mathbf{T}}_n) \geq 2d\varepsilon \right\} \subseteq \text{LongGeo}_n^{(b)}(L) \cup \text{LDGreedy}_n(b). \quad (5.32)$$

By union bound and the estimate (5.6), it is thus sufficient to prove that for large enough b ,

$$\underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{LDGreedy}_n(b) \geq \alpha), \quad (5.33)$$

Lemma 4.2 in Dembo-Gandolfi-Kesten (2001) [12] provides an answer for a similar problem, in the framework of the so-called *greedy lattice animals*: they give an estimate for the upper-tail of the mass gathered by an animal of size n containing the origin, when the masses are scattered on the vertices of \mathbb{Z}^d . An animal there is defined as a finite connected subset of \mathbb{Z}^d . In particular, self-avoiding paths are animals. In order to adapt their result to our framework, let us introduce for all $v \in \mathbb{Z}^d$ and $z \in \{\pm e_i\}_{1 \leq i \leq d}$ the random variable

$$M_v(z) := (\tau_{(v, v+z)} - b)^+, \quad (5.34)$$

as well as

$$M_v := \sum_{z \in \{\pm e_i\}_{1 \leq i \leq d}} M_v(z). \quad (5.35)$$

For all z , the variables $(M_v(z))_{v \in \mathbb{Z}^d}$ are i.i.d. and for all discrete path π ,

$$\tau(\pi) - \tau^{(b)}(\pi) \leq \sum_{v \in \pi} M_v. \quad (5.36)$$

By Lemma 4.2 in [12] and (Moment), for large enough b , for all $z \in \{\pm e_i\}_{1 \leq i \leq d}$,

$$\underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\begin{array}{l} \text{There exists a self-avoiding discrete path } \pi \subseteq \mathbb{Z}^d, \\ \text{such that } 0 \in \pi, \|\pi\| \leq Ln \text{ and } \sum_{v \in \pi} M_v(z) \geq \varepsilon n \end{array} \right) \geq \alpha.$$

Consequently, the union bound and (5.36) give (5.33). \square

Proof of Theorem 1.10. Let $b < \infty$. By (5.19) the processes $(\widetilde{\mathbf{T}}_n^{(b)})_{n \geq 1}$ and $(\widehat{\mathbf{T}}_n^{(b)})_{n \geq 1}$ are exponentially equivalent. Consequently, by Theorem 5.8, $(\widehat{\mathbf{T}}_n^{(b)})_{n \geq 1}$ follows the LDP with the rate function $J^{(b)}$.

By Theorem 5.8(ii) and Lemma 5.9, $(\widehat{\mathbf{T}}_n)_{n \geq 1}$ follows the weak LDP with the rate function

$$\begin{aligned} J^{(\infty)} : \mathcal{F}_b &\longrightarrow [0, \infty] \\ D &\longmapsto \sup_{\delta > 0} \underline{\lim}_{b \rightarrow \infty} \inf_{\hat{D} \in \text{B}_\infty(D, \delta)} J^{(b)}(\hat{D}), \end{aligned} \quad (5.37)$$

where $\text{B}_\infty(D, \delta)$ denotes the open ball of center D and radius δ , for \mathfrak{d}_∞ . To prove the LDP with this rate function it is sufficient by the second part of Theorem 5.8(ii) to show that $J^{(\infty)}$ is a good rate function, and for all closed sets $F \subseteq \mathcal{F}_b$,

$$\inf_{D \in F} J^{(\infty)}(D) \leq \overline{\lim}_{b \rightarrow \infty} \inf_{D \in F} J^{(b)}(D). \quad (5.38)$$

By (5.16), using the same argument as in the proof of Lemma 5.6, Case 1, one show that for all $\alpha > 0$ there exists $L > 0$ such that for all $b \in [1, \infty]$,

$$\inf_{D \in \mathcal{F}_b \setminus \mathcal{D}_\mu^L} J^{(b)}(D) \geq \alpha. \quad (5.39)$$

Consequently, $J^{(\infty)}$ is a good rate function and (5.38) only needs to be proven for compact F . Let $F \subseteq \mathcal{F}_b$ be a compact set. For all $b \geq 1$, since $J^{(b)}$ is lower semicontinuous, it admits a minimizer D_b on F . By compactness there exists a sequence $(b_k)_{k \geq 1}$ that diverges to ∞ and $D \in F$ such that

$$\lim_{k \rightarrow \infty} D_{b_k} = D.$$

In particular, for all $\delta > 0$, for large enough k ,

$$\mathfrak{d}_\infty(D, D_{b_k}) < \delta,$$

thus

$$\begin{aligned} J^{(\infty)}(D) &\leq \varliminf_{k \rightarrow \infty} J^{(b_k)}(D_{b_k}) \\ &= \varliminf_{k \rightarrow \infty} \inf_{\hat{D} \in F} J^{(b_k)}(\hat{D}) \\ &\leq \overline{\lim}_{b \rightarrow \infty} \inf_{\hat{D} \in F} J^{(b)}(\hat{D}), \end{aligned}$$

thus (5.38).

We now prove that $J^{(\infty)} = J$. If $D \in \mathcal{F}_b \setminus \mathcal{D}_\mu$, then (5.39) implies $J^{(\infty)}(D) = \infty = J(D)$. Let $D \in \mathcal{D}_\mu^L$. To show the inequality $J^{(\infty)}(D) \geq J^-(D)$, notice that by an elementary inclusion, for all $\varepsilon > 0$,

$$\overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{LD}_n(D, \varepsilon)) \geq \overline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{LD}_n^-(D, \varepsilon)).$$

Moreover, since D is in the interior of $B_\infty(D, \varepsilon)$ and $\widehat{\mathbf{T}}_n$ follows the LDP with the rate function $J^{(\infty)}$, the left-hand side may be upper bounded by $J^{(\infty)}(D)$. Letting $\varepsilon \rightarrow 0$ gives $J^{(\infty)}(D) \geq J^-(D)$.

Let us prove the converse inequality. If $J^-(D) = \infty$, there is nothing to do. Assume the contrary. In particular there exists $L > 0$ such that

$$\inf_{\hat{D} \in \mathcal{F}_b \setminus \mathcal{D}_\mu^L} J^{(\infty)}(\hat{D}) > J^-(D). \quad (5.40)$$

For all $\varepsilon \geq 0$ we consider the closed set

$$F_\varepsilon := \left\{ \hat{D} \in \mathcal{F}_b \mid \forall x, y \in X, \hat{D}(x, y) \leq D(x, y) + \varepsilon \right\}, \quad (5.41)$$

and denote by K_ε the compact set $F_\varepsilon \cap \mathcal{D}_\mu^L$. Since $(\widehat{\mathbf{T}}_n)_{n \geq 1}$ follows the LDP with the rate function $J^{(\infty)}$, for all $\varepsilon > 0$,

$$\begin{aligned} \varliminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{LD}_n^-(D, \varepsilon)) &= \varliminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\widehat{\mathbf{T}}_n \in F_\varepsilon) \\ &\geq \inf_{\hat{D} \in F_\varepsilon} J^{(\infty)}(\hat{D}). \end{aligned}$$

By (5.40),

$$\varliminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\text{LD}_n^-(D, \varepsilon)) \geq \min_{\hat{D} \in K_\varepsilon} J^{(\infty)}(\hat{D}).$$

By compactness there exists a sequence $(\varepsilon_k)_{k \geq 1}$ decreasing to 0 and a converging sequence of metrics $(D_k)_{k \geq 1}$ such that for all $k \geq 1$, $D_k \in K_{\varepsilon_k}$ and realizes the minimum above. The limit of $(D_k)_{k \geq 1}$ belongs to $\bigcap_{\varepsilon > 0} K_\varepsilon = K_0$. Letting $k \rightarrow \infty$ gives, by lower semicontinuity of $J^{(\infty)}$,

$$J^-(D) \geq \min_{\hat{D} \in K_0} J^{(\infty)}(\hat{D}).$$

However, for all $\hat{D} \in K_0 \setminus \{D\}$, by Proposition 4.4,

$$J^{(\infty)}(\hat{D}) \geq J^-(\hat{D}) > J^-(D).$$

Consequently, $J^-(D) \geq J^{(\infty)}(D)$, which concludes the proof of $J = J^{(\infty)}$. \square

A Probability that a vertex is a hub

In this section we prove Lemma 4.5. Our arguments are adapted from Cox and Durrett's proof of the shape theorem (see Theorem 3.3 in [8]). We will use Lemma A.1 twice in order to dominate the passage time between two points by d independent sums of variables with distribution ν . The first step is to prove a similar result under a stronger moment condition, when the edge passage times may have short range dependence. Under the assumption $\mathbb{E}[\tau_e^{d+\xi}] < \infty$, (4.5) holds by standard estimates on i.i.d. sums (see Lemma A.2). To conclude under (StrongShape), we need an extra step consisting in bounding the passage time between two neighbors in $3\mathbb{Z}^d$ by the minimum of the passage times along the d paths provided by Lemma A.1. At the scale of those "long edges", Lemma A.2 can be applied.

Lemma A.1. *Fix $n \in \mathbb{N}^*$ and distinct vertices $x, y \in \llbracket 0, n \rrbracket^d$. There exist at least d pairwise disjoint except at their endpoints discrete paths π_1, \dots, π_d from x to y , with length $\|x - y\|_1$ or $\|x - y\|_1 + 2$.*

Proof. Write $x = (x_1, \dots, x_d)$ and $y = (y_1, \dots, y_d)$. Without loss of generality, we can assume the existence $i_0 \in \llbracket 0, d \rrbracket$, such that

- For all $i \in \llbracket 1, i_0 \rrbracket$, $x_i = y_i \neq n$.
- For all $i \in \llbracket i_0 + 1, d \rrbracket$, $x_i < y_i$.

Let $(x = z(0), \dots, z(r) = y)$ be any path from x to y with length $\|x - y\|_1$. For all $i \in \llbracket 1, i_0 \rrbracket$, we define π_i the path $(x, z(0) + e_i, z(1) + e_i, \dots, z(r) + e_i, y)$. For all $i \in \llbracket i_0 + 1, d \rrbracket$, we define π_i as the only path from x to y which is the concatenation of $d - i_0$ straight lines of respective directions $(e_i, e_{i+1}, \dots, e_d, e_{i_0+1}, \dots, e_{i-1})$. \square

Lemma A.2. *Let $(\tau'_e)_{e \in \mathbb{E}^d}$ be a family of identically distributed random variables. Assume that there exists $\xi > 0$ such that $\mathbb{E}[(\tau'_e)^{d+\xi}] < \infty$, and for all $e \in \mathbb{E}^d$, τ'_e is independent of the family passage times along edges having no common endpoint with e . Then there exists a constant $\kappa < \infty$ such that*

$$\lim_{n \rightarrow \infty} \min_{x \in \llbracket 0, n \rrbracket^d} \mathbb{P} \left(\begin{array}{l} \text{For all } y \in \llbracket 0, n \rrbracket^d, \text{ there exists a discrete path } x \overset{\pi}{\rightsquigarrow} y \text{ included} \\ \text{in } \llbracket 0, n \rrbracket^d, \text{ such that } \tau'(\pi) \leq \kappa \|x - y\|_1 \text{ and } \|\pi\| \leq \|x - y\|_1 + 2 \end{array} \right) > 0. \quad (\text{A.1})$$

Proof. Let $x, y \in \llbracket 0, n \rrbracket^d$ be distinct vertices and $\gamma_1, \dots, \gamma_d$ be the paths given by Lemma A.1, with removed endpoints. The γ_i have no common vertex, hence their passage times are independent. Fix $i \in \llbracket 1, d \rrbracket$. We denote by $\gamma_i^{(1)}$ and $\gamma_i^{(2)}$ the set of edges along γ_i , with odd indices and even indices respectively. By union bound and Markov's inequality,

$$\begin{aligned} & \mathbb{P}(\tau'(\gamma_i) - \mathbb{E}[\tau'(\gamma_i)] \geq \|x - y\|_1) \\ & \leq \mathbb{P} \left(\tau'(\gamma_i^{(1)}) - \mathbb{E}[\tau'(\gamma_i^{(1)})] \geq \frac{1}{2} \|x - y\|_1 \right) + \mathbb{P} \left(\tau'(\gamma_i^{(2)}) - \mathbb{E}[\tau'(\gamma_i^{(2)})] \geq \frac{1}{2} \|x - y\|_1 \right) \\ & \leq \frac{2^{d+\xi}}{\|x - y\|_1^{d+\xi}} \mathbb{E} \left[\left| \tau'(\gamma_i^{(1)}) - \mathbb{E}[\tau'(\gamma_i^{(1)})] \right|^{d+\xi} + \left| \tau'(\gamma_i^{(2)}) - \mathbb{E}[\tau'(\gamma_i^{(2)})] \right|^{d+\xi} \right]. \end{aligned} \quad (\text{A.2})$$

Since $\tau'(\gamma_i^{(1)})$ is sum of independent variables distributed as τ' , by Rosenthal's inequality (see Theorem 3 in [20]) there exists a constant C_2 , depending only on $d + \xi$, such that

$$\begin{aligned} \mathbb{E} \left[\left| \tau'(\gamma_i^{(1)}) - \mathbb{E}[\tau'(\gamma_i^{(1)})] \right|^{d+\xi} \right]^{1/(d+\xi)} & \leq C_2 \max \left(\left(\#\gamma_i^{(1)} \mathbb{E} \left[|\tau' - \mathbb{E}[\tau']|^{d+\xi} \right] \right)^{1/(d+\xi)}, \right. \\ & \left. \left(\#\gamma_i^{(1)} \mathbb{E} \left[|\tau' - \mathbb{E}[\tau']|^2 \right] \right)^{1/2} \right). \end{aligned} \quad (\text{A.3})$$

Consequently, there exists a constant C_3 , depending only on $d + \xi$ and the distribution of τ' , such that

$$\mathbb{E} \left[\left| \tau'(\gamma_i^{(1)}) - \mathbb{E}[\tau'(\gamma_i^{(1)})] \right|^{d+\xi} \right] \leq C_3 \|x - y\|_1^{(d+\xi)/2}.$$

The same goes for the paths $\gamma_i^{(2)}$. Applying this bound to (A.2) yields

$$\mathbb{P}(\tau'(\gamma_i) - \mathbb{E}[\tau'(\gamma_i)] \geq \|x - y\|_1) \leq \frac{C_3 2^{d+\xi+1}}{\|x - y\|_1^{(d+\xi)/2}}.$$

Consequently, plugging in the inequality $\|\gamma_i\| \leq \|x - y\|_1$ yields

$$\mathbb{P}(\tau'(\gamma_i) \geq (\mathbb{E}[\tau'_e] + 1)\|x - y\|_1) \leq \frac{C_3 2^{d+\xi+1}}{\|x - y\|_1^{(d+\xi)/2}}. \quad (\text{A.4})$$

Since the variables $\tau'(\gamma_i)$ for $i \in \llbracket 1, d \rrbracket$ are independent,

$$\mathbb{P}\left(\bigcap_{i=1}^d \{\tau'(\gamma_i) \geq (\mathbb{E}[\tau'_e] + 1)\|x - y\|_1\}\right) \leq \frac{C_3^d 2^{d(d+\xi+1)}}{\|x - y\|_1^{(d^2+d\xi)/2}}. \quad (\text{A.5})$$

Besides, by union bound and Markov's inequality,

$$\mathbb{P}\left(\max_{z \in \{\pm e_i\}_{1 \leq i \leq d}} (\tau_{(x, x+z)} \vee \tau_{(y, y+z)}) \geq \|x - y\|_1\right) \leq \frac{4d\mathbb{E}[(\tau'_e)^{d+\xi}]}{\|x - y\|_1^{d+\xi}}. \quad (\text{A.6})$$

The right-hand sides of (A.5) and (A.6) are summable over $y \in \mathbb{Z}^d \setminus \{x\}$, and only depends on $x - y$. Consequently, there exists an integer $r \geq 1$ and a constant $C_4 > 0$ such that for all $n \geq 1$ and $x \in \llbracket 0, n \rrbracket^d$,

$$\begin{aligned} & \sum_{\substack{y \in \llbracket 0, n \rrbracket^d \\ \|x - y\|_1 \geq r}} \left[\mathbb{P}\left(\bigcap_{i=1}^d \{\tau'(\gamma_i) \geq (\mathbb{E}[\tau'_e] + 1)\|x - y\|_1\}\right) \right. \\ & \left. + \mathbb{P}\left(\max_{z \in \{\pm e_i\}_{1 \leq i \leq d}} (\tau_{(x, x+z)} \vee \tau_{(y, y+z)}) \geq \|x - y\|_1\right) \right] \leq 1 - C_4. \end{aligned}$$

Using the union bound again, we get that for all $n \geq 1$ and $x \in \llbracket 0, n \rrbracket^d$,

$$\mathbb{P}\left(\begin{array}{l} \text{For all } y \in \llbracket 0, n \rrbracket^d \text{ such that } \|x - y\|_1 \geq r, \text{ there exists a discrete path } x \overset{\pi}{\rightsquigarrow} y \text{ included} \\ \text{in } \llbracket 0, n \rrbracket^d, \text{ such that } \tau'(\pi) \leq (\mathbb{E}[\tau'_e] + 3)\|x - y\|_1 \text{ and } \|\pi\| \leq \|x - y\|_1 + 2 \end{array}\right) \geq C_4.$$

In particular, (A.1) holds for $\kappa = \mathbb{E}[\tau'_e] + 3$. \square

Proof of Lemma 4.5. Assume (StrongShape). For all $x, y \in \llbracket 0, n \rrbracket^d$, we write $x \sim y$ if $x - y \in 3\mathbb{Z}^d$. We will denote by $[x]$ the equivalence class of x for \sim in $\llbracket 0, n \rrbracket^d$. If $x - y \in \{\pm 3e_i\}_{1 \leq i \leq d}$, we say that x and y are 3-neighbours and define $\tau'(x, y)$ as the minimum of the passage times along the d paths between x and y provided by Lemma A.1. Following the proof of Lemma 3.1 in [8], one shows that for all such x, y , $\mathbb{E}[\tau'(x, y)^{d+\xi}] < \infty$. Up to translation by $3v - \lfloor 3v \rfloor$ and rescaling by a factor 3, for all equivalence classes $[v]$ of \sim the family $(\tau'(x, y))$, spanning over pairs of 3-neighbours $(x, y) \in [v]^2$ satisfies the hypothesis of Lemma A.2. We say that a vertex $x \in \llbracket 0, n \rrbracket^d$ is a *pre-hub* if it satisfies the event in (A.1) for the lattice structure on $[x]$ induced by 3-neighbours, and the passage times τ' . If x is a pre-hub, then for all $y \in [x]$, there exists a discrete path $x \overset{\pi}{\rightsquigarrow} y$ included in $\llbracket 0, n \rrbracket^d$, such that

$$\tau(\pi) \leq \kappa \|x - y\|_1 \text{ and } \|\pi\| \leq \frac{5}{3}\|x - y\|_1 + \frac{10}{3} \leq 2\|x - y\|_1 + 4.$$

Given $x \in \llbracket 0, n \rrbracket^d$, consider the event

$$\mathcal{E}(n, x) := \left(\bigcap_{x' \in (x + [-2, 2]^d) \cap \llbracket 0, n \rrbracket^d} \{x' \text{ is a pre-hub}\} \right) \cap \left(\bigcap_{e \in \mathbb{E}((x + [-2, 2]^d) \cap \llbracket 0, n \rrbracket^d)} \{\tau_e \leq \kappa\} \right). \quad (\text{A.7})$$

It is an intersection of decreasing events, therefore by the FKG inequality and Lemma A.2,

$$\liminf_{n \rightarrow \infty} \min_{x \in \llbracket 0, n \rrbracket^d} \mathbb{P}(\mathcal{E}(n, x)) > 0. \quad (\text{A.8})$$

Let $x \in \llbracket 0, n \rrbracket^d$. We are left to show that on the event $\mathcal{E}(n, x)$, x is a hub. Assume that $\mathcal{E}(n, x)$ occurs and let $y \in \llbracket 0, n \rrbracket^d$. There exists $x' \in (x + [-2, 2]^d) \cap \llbracket 0, n \rrbracket^d$ such that $x' \sim y$ and $\|x - y\|_1 = \|x - x'\|_1 + \|x' - y\|_1$. There exists a path $x \overset{\pi_1}{\rightsquigarrow} x'$ with length $\|x - x'\|_1$, such that $\tau(\pi_1) \leq \kappa \|x - x'\|_1$. Moreover, since x' is a pre-hub, there exists a discrete path $x' \overset{\pi_2}{\rightsquigarrow} y$ included in $\llbracket 0, n \rrbracket^d$, such that $\tau(\pi_2) \leq \kappa \|x' - y\|_1$ and $\|\pi\| \leq 2\|x' - y\|_1 + 4$. The concatenation of π_1 and π_2 has the desired properties. \square

B Bound for upper-tail large deviations

In this section we prove Lemma 5.1, which states that under Assumption (Moment), the order of the upper-tail large deviation probability for the point-point time is lower than $\exp(-\alpha n)$, for all $\alpha > 0$. We essentially follow Kesten's proof ([17], Theorem 5.9) of the special case $x = 0, y = e_1$: for large n , we consider a large number of pairwise disjoint corridors, with fixed but large width m which are close to the segment $[nx, ny]$. On the large deviation event $\{\widehat{\mathbf{T}}_n(x, y) \geq \mu(x - y) + \varepsilon\}$, either the passage times across all corridors or the passage times from nx and ny to the entrances of each are abnormally large. The probability of each scenario is bounded by $\exp(-\alpha n)$, with arbitrary $\alpha > 0$.

Proof. We first fix distinct $x, y \in \mathring{X}$ and $\zeta > \mu(x - y)$ then prove the following weaker version of the result:

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\widehat{\mathbf{T}}_n(x, y) \geq \zeta) = +\infty. \quad (\text{B.1})$$

For all integers $n, m \geq 1$ and $v \in \mathbb{Z}^d$, we introduce the subset

$$\text{Corridor}(v, m, n) := \{z + s(y - x) \mid z \in \overline{\mathbf{B}}_2(v, m), s \in [0, n]\}. \quad (\text{B.2})$$

Let $\varepsilon > 0$ be such that $\zeta - \mu(x - y) \geq 4\varepsilon$. By adapting the argument at the bottom of p. 198 in [17], one shows the existence of $m \geq 1$ such that for all $n \geq m$,

$$\frac{\mathbb{E}[\mathbf{T}_{\text{Corridor}(0, m, n)}(0, \lfloor n(x - y) \rfloor)]}{n} \leq \mu(x - y) + \varepsilon.$$

In particular, by Talagrand's inequality (see e.g. [2], Theorem 3.13),

$$\alpha_m := \liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P}(\mathbf{T}_{\text{Corridor}(0, m, n)}(0, \lfloor n(x - y) \rfloor) \geq n(\mu(x - y) + 2\varepsilon)) > 0. \quad (\text{B.3})$$

Let us fix an integer r . For large enough n , there exist $v_1, \dots, v_r \in \llbracket 0, n \rrbracket^d$, such that:

- The sets $\text{Corridor}(v_i, m, n)$, for $i \in \llbracket 1, r \rrbracket$ are pairwise disjoint and included in $\llbracket 0, n \rrbracket^d$.
- For all $i \in \llbracket 1, r \rrbracket$,

$$\|\lfloor nx \rfloor - v_i\|_1 \leq \sqrt{n} \text{ and } \|\lfloor ny \rfloor - (v_i + \lfloor n(x - y) \rfloor)\|_1 \leq \sqrt{n}. \quad (\text{B.4})$$

By triangle inequality, we have the inclusion

$$\begin{aligned} \{\widehat{\mathbf{T}}_n(x, y) \geq \zeta\} &\subseteq \left(\bigcup_{i=1}^r \{\mathbf{T}_{\llbracket 0, n \rrbracket^d}(\lfloor nx \rfloor, v_i) > \varepsilon n\} \right) \\ &\cup \left(\bigcup_{i=1}^r \{\mathbf{T}_{\llbracket 0, n \rrbracket^d}(\lfloor ny \rfloor, (v_i + \lfloor n(x - y) \rfloor)) > \varepsilon n\} \right) \\ &\cup \left(\bigcap_{i=1}^r \{\mathbf{T}_{\llbracket 0, \text{Corridor}(v_i, m, n) \rrbracket^d}(v_i, v_i + \lfloor n(x - y) \rfloor) \geq (\mu(x - y) + 2\varepsilon)n\} \right). \end{aligned} \quad (\text{B.5})$$

Let $i \in \llbracket 1, r \rrbracket$. We claim that

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\mathbf{T}_{[0, n]^d}(\lfloor nx \rfloor, v_i) > \varepsilon n \right) = \infty. \quad (\text{B.6})$$

and

$$\lim_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\mathbf{T}_{[0, n]^d}(\lfloor ny \rfloor, (v_i + \lfloor n(x - y) \rfloor)) > \varepsilon n \right) = \infty. \quad (\text{B.7})$$

Indeed for all $\lambda > 0$, using Chernoff's inequality to bound the passage time along an oriented path from $\lfloor nx \rfloor$ and v_i and applying (B.4) provides

$$\mathbb{P} \left(\mathbf{T}_{[0, n]^d}(\lfloor nx \rfloor, v_i) > \varepsilon n \right) \leq \exp(-\lambda \varepsilon n) \mathbb{E}[\exp(\lambda \tau_e)]^{\sqrt{n}},$$

therefore

$$\underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\mathbf{T}_{[0, n]^d}(\lfloor nx \rfloor, v_i) > \varepsilon n \right) \geq \lambda \varepsilon.$$

By (Moment) this bound holds for all $\lambda > 0$, therefore we get (B.6). Equation (B.7) is proven similarly. Besides, by independence and stationarity,

$$\underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\bigcap_{i=1}^r \{ \mathbf{T}_{\text{Corridor}(v_i, m, n)}(v_i, v_i + \lfloor n(x - y) \rfloor) \geq (\mu(x - y) + 2\varepsilon)n \} \right) \geq \alpha_m r. \quad (\text{B.8})$$

Using the union bound on (B.5) and applying Equations (B.6), (B.7), (B.8) give

$$\underline{\lim}_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\widehat{\mathbf{T}}_n(x, y) \geq \zeta \right) \geq \alpha_m r.$$

Letting $r \rightarrow \infty$ yields (B.1).

We now turn to the proof of (5.4). Fix $\varepsilon > 0$ and $\alpha > 0$. We define

$$\lambda := \frac{\alpha + 1}{\varepsilon} \text{ and } \delta := \frac{\lambda \varepsilon - \alpha}{2 \log \mathbb{E}[\exp(\lambda \tau)]} \wedge \frac{1}{2}. \quad (\text{B.9})$$

Let $(x_i)_{1 \leq i \leq r}$ be a finite family of points in $\overset{\circ}{X}$ such that $X \subseteq \bigcup_{i=1}^r \overline{B}_1(x_i, \delta)$. Let $x, y \in X$. There exists $i, j \in \llbracket 1, r \rrbracket$ such that for all $n \geq 1/\delta$,

$$\max(\| \lfloor nx \rfloor - \lfloor nx_i \rfloor \|_1, \| \lfloor ny \rfloor - \lfloor nx_j \rfloor \|_1) \leq 2\delta n. \quad (\text{B.10})$$

By triangle inequality,

$$\begin{aligned} \left\{ \widehat{\mathbf{T}}_n(x, y) \geq \mu(x - y) + (3 + 2\|\mu\|_1)\varepsilon \right\} &\subseteq \left\{ \widehat{\mathbf{T}}_n(x, x_i) > \varepsilon \right\} \cup \left\{ \widehat{\mathbf{T}}_n(y, x_j) > \varepsilon \right\} \\ &\cup \left\{ \widehat{\mathbf{T}}_n(x_i, x_j) \geq \mu(x_i - x_j) + \varepsilon \right\}. \end{aligned} \quad (\text{B.11})$$

Using Chernoff's bound and applying (B.10) gives

$$\mathbb{P} \left(\mathbf{T}_{[0, n]^d}(nx, nx_i) > \varepsilon n \right) \leq \exp(-\varepsilon \lambda n) \mathbb{E}[\exp(\lambda \tau_e)]^{2\delta n},$$

thus, by definition of λ and δ ,

$$-\frac{1}{n} \log \mathbb{P} \left(\mathbf{T}_{[0, n]^d}(nx, nx_i) > \varepsilon n \right) \geq \alpha.$$

Similarly,

$$-\frac{1}{n} \log \mathbb{P} \left(\mathbf{T}_{[0, n]^d}(ny, nx_j) > \varepsilon n \right) \geq \alpha.$$

Consequently, applying the union bound to (B.11) and plugging in these two inequalities, we get

$$\begin{aligned}
& -\frac{1}{n} \log \mathbb{P} \left(\widehat{\mathbf{T}}_n(x, y) \geq \mu(x - y) + (3 + 2\|\mu\|_1)\varepsilon \right) \\
& \geq \alpha \wedge \min_{1 \leq i, j, \leq r} \left[\frac{1}{n} \log \mathbb{P} \left(\widehat{\mathbf{T}}_n(x_i, x_j) \geq \mu(x_i - x_j) + \varepsilon \right) \right] - \frac{\log 2}{n}.
\end{aligned} \tag{B.12}$$

Letting $n \rightarrow \infty$ and using (B.1) gives

$$\liminf_{n \rightarrow \infty} -\frac{1}{n} \log \mathbb{P} \left(\widehat{\mathbf{T}}_n(x, y) \geq \mu(x - y) + (3 + 2\|\mu\|_1)\varepsilon \right) \geq \alpha. \tag{B.13}$$

That bound holds for all $\alpha > 0$, hence (5.4). □

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