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# THE SCHRÖDINGER EQUATION WITH FRACTIONAL LAPLACIAN ON HYPERBOLIC SPACES AND HOMOGENEOUS TREES

JEAN-PHILIPPE ANKER, GUENDALINA PALMIROTTA, AND YANNICK SIRE

ABSTRACT. We investigate dispersive and Strichartz estimates for the Schrödinger equation involving the fractional Laplacian in real hyperbolic spaces and their discrete analogues, homogeneous trees. Due to the Knapp phenomenon, the Strichartz estimates on Euclidean spaces for the fractional Laplacian exhibit loss of derivatives. A similar phenomenon appears on real hyperbolic spaces. However, such a loss disappears on homogeneous trees, due to the triviality of the estimates for small times.

## 1. INTRODUCTION

The aim of the present work is to derive dispersive and Strichartz estimates for Schrödinger equations associated to the *fractional Laplacian* on negatively curved manifolds like real hyperbolic spaces and their discrete counterparts, homogeneous trees. Specifically, consider the following Cauchy problem for the fractional Schrödinger equation:

$$\begin{cases} i\partial_t u + (-\Delta)^{\alpha/2} u = F(x, t) & (x, t) \in M \times \mathbb{R} \\ u|_{t=0} = u_0, \end{cases} \quad (1)$$

where  $M$  stands for either  $\mathbb{H}^n$  ( $n \geq 2$ ), the real hyperbolic space with its standard metric;  $\mathbb{T}_Q$  ( $Q \geq 2$ ), the homogeneous tree with  $Q + 1$  edges; or  $\mathbb{R}^n$  ( $n \geq 2$ ), the Euclidean space with the flat metric. The operator  $(-\Delta)^{\alpha/2}$  represents the Fourier multiplier of order  $\alpha \in (0, 2)$  associated to powers of the corresponding Laplacian on  $M$ . Here  $F(x, t)$  is a nonhomogeneous term defined on  $M \times \mathbb{R}$  and  $u_0$ , defined on  $M$ , stands for the initial data.

Positive powers of the Laplace-Beltrami operator, known as the fractional Laplacian, appear in several areas of mathematical physics such as relativistic theories (see e.g., [CMB90, DL83, FLS07, LY87, LY89]). Such operators are also reminiscent of stochastic processes with pure jumps since they are the infinitesimal generators of stable Levy processes (see the book by Bertoin [Ber96] for a detailed account). E.g. an attempt to reinterpret Feynman's path integral into the framework of Levy processes has been undertaken in [Las02].

Additionally, significant progress has been made in understanding such operators using techniques from partial differential equations. While a comprehensive review is beyond the scope of this work, we refer readers to the books and surveys [FRRO24, Caf12, CdPF<sup>+</sup>17, Váz14] for further insights.

We now outline our main results for  $M = \{\mathbb{H}^n, \mathbb{T}_Q\}$ , namely the dispersive estimates for Equation (1). As our focus is the harmonic analysis of the multipliers  $(-\Delta)^{\alpha/2}$  on rank 1 symmetric spaces of noncompact type, we state below the boundedness properties in  $L^p - L^q$  of the multiplier in question.

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**Theorem 1.1** (Dispersive estimates on hyperbolic spaces). *Consider  $M = \mathbb{H}^n$ .*

- Assume that  $n \geq 2$ ,  $1 < \alpha < 2$ ,  $0 \leq \sigma \leq \frac{n}{2}$ ,  $2 < q \leq \infty$  and let

$$m = \max \left\{ 2 \frac{n-\sigma}{\alpha}, \frac{n-2\sigma}{\alpha-1} \right\} = \begin{cases} 2 \frac{n-\sigma}{\alpha} & \text{if } \sigma \geq (1 - \frac{\alpha}{2})n, \\ \frac{n-2\sigma}{\alpha-1} & \text{if } \sigma \leq (1 - \frac{\alpha}{2})n. \end{cases}$$

Then the following dispersive estimates hold for  $t \in \mathbb{R}^*$  on  $\mathbb{H}^n$ :

$$\| (-\Delta)^{-\left(\frac{1}{2}-\frac{1}{q}\right)\sigma} e^{it(-\Delta)^{\alpha/2}} \|_{L^{q'} \rightarrow L^q} \lesssim |t|^{-\left(\frac{1}{2}-\frac{1}{q}\right)m}$$

for  $t$  small, say  $0 < |t| < 1$ , and

$$\| (-\Delta)^{-\sigma/2} e^{it(-\Delta)^{\alpha/2}} \|_{L^{q'} \rightarrow L^q} \lesssim |t|^{-\frac{3}{2}}$$

for  $t$  large, say  $|t| \geq 1$ .

- Assume that  $n \geq 3$ ,  $0 < \alpha < 1$ ,  $(1 - \frac{\alpha}{2})n \leq \sigma \leq n$  and  $2 < q \leq \infty$ . Then the following dispersive estimates hold for  $t \in \mathbb{R}^*$  on  $\mathbb{H}^n$ :

$$\| (-\Delta)^{-\left(\frac{1}{2}-\frac{1}{q}\right)\sigma} e^{it(-\Delta)^{\alpha/2}} \|_{L^{q'} \rightarrow L^q} \lesssim |t|^{-\left(\frac{1}{2}-\frac{1}{q}\right)2\frac{n-\sigma}{\alpha}}$$

for  $t$  small, say  $0 < |t| < 1$ , and

$$\| (-\Delta)^{-\sigma/2} e^{it(-\Delta)^{\alpha/2}} \|_{L^{q'} \rightarrow L^q} \lesssim |t|^{-\frac{3}{2}}$$

for  $t$  large, say  $|t| \geq 1$ .

In Theorem 1.1, in the case  $0 < \alpha < 1$ , we did not state the estimates whenever  $n = 2$  since the numerology gets more tedious. We refer the reader to the dedicated Section 4.2 for more details on this case.

Note that the  $L^p \rightarrow L^q$  boundedness of such multipliers, for fixed time  $t$ , was extensively investigated by Cowling, Giulini and Meda [GM90, CGM93, CGM95, CGM01, CGM02]. For specific  $\alpha$ , some results are known: For  $\alpha = 2$ , Equation (1) has been the subject of extensive study. Focusing specifically on symmetric spaces, key references include [AP09, IS09, APV11], while for higher-rank symmetric spaces, we refer to [AMP<sup>+</sup>23]. In addition, space-time linear estimates in the Euclidean space are the well-known Strichartz results of Ginibre-Velo [GV79, GV85] and Keel-Tao [KT98].

For  $\alpha \in (0, 2)$ , the Strichartz and dispersive estimates are due to Cho, Ozawa and Xia [COS11, CKS16]. Guo and Wang in [GW14] derived finer estimates for  $1 < \alpha < 2$ . This loss of derivatives is reminiscent of the Knapp phenomenon (see e.g., [GW14, Cor. 3.10.]). For  $\alpha = 1$ , Equation (1), often referred to as the *half-wave equation*, has been investigated in works such as [APV12, AP14, APV15].

In an influential paper [Gro87], Gromov investigates the so-called hyperbolic groups and provides a discrete analogue of the real hyperbolic space, namely the homogeneous trees introduced before. The next theorem provides dispersive estimates in such a setting.

**Theorem 1.2** (Dispersive estimate on homogeneous trees). *Consider  $M = \mathbb{T}_Q$ . Let  $0 < \alpha \leq 2$  and  $2 < q, \tilde{q} \leq \infty$ . Then the following dispersive estimate holds for  $t \in \mathbb{R}^*$ :*

$$\| e^{it(-\Delta)^{\alpha/2}} \|_{\ell^{q'}(\mathbb{T}_Q) \rightarrow \ell^{\tilde{q}}(\mathbb{T}_Q)} \lesssim (1 + |t|)^{-\frac{3}{2}}.$$

The previous statements follow from a fine analysis of the kernel of the propagator  $e^{it(-\Delta)^{\alpha/2}}$ . A standard argument then gives Strichartz estimates from the dispersive ones. As a quick inspection shows, the phase in the oscillatory integral of the linear solution changes convexity according to the powers  $\alpha$ . This requires to consider separately the

two regimes  $\alpha \in (0, 1)$  and  $\alpha \in (1, 2)$ . The kernel analysis is substantially more involved whenever  $\alpha$  is small. In particular, we observe degeneracies in this case which need to push the phase analysis an order more.

This phenomenon is dominant in the continuous setting of  $\mathbb{H}^n$  but disappears in the discrete one of  $\mathbb{T}_Q$  since in this case the *local* analysis of the kernel becomes trivial. We point out as well to the reference [Din17] for an investigation on closed manifolds and to [Din18] for one on asymptotically Euclidean manifolds.

More generally, it has already been observed in the Euclidean case that space-time estimates exhibit a loss of derivatives (see [GW14]) which is reminiscent of the Knapp phenomenon. However, it was observed by Guo and Wang that such a loss can be removed by assuming radial symmetry. More precisely, it was also shown in [GW14] by Guo and Wang that one can obtain optimal Strichartz estimates (i.e., without loss) if one restricts to  $\alpha \in (\frac{2}{2n-1}, 2)$ . In particular, the number  $\frac{2n}{2n-1}$  is larger than 1 and there is a gap between the Strichartz estimates for the wave operator  $\alpha = 1$  and the ones occurring for larger powers. The loss in question occurs at small scales in space and then can be removed in the case of homogeneous trees. It is worth mentioning that the range of admissible pairs for the Strichartz estimates on  $\mathbb{H}^n$  is larger than the one on Euclidean spaces. This was of course already observed for  $\alpha = 2$ . However, one still observes a loss of derivatives for general classes of data. A possible way to remove the loss would be to consider the case of data depending only on the geodesic distance to a given pole of  $\mathbb{H}^n$ .

The case of real hyperbolic space introduces several points departing substantially from the analysis in the Euclidean space.

- (1) First, we observe that the change of convexity in the phase is essential in dealing with the case of powers close to zero. This introduces a phase analysis much more technical than in the case close to 2.
- (2) Second, as already mentioned, the range of admissible exponents for the Strichartz estimates are broader than in the Euclidean case of Cho, Ozawa and Xia [COS11].
- (3) Interestingly enough, we notice that one can *remove* the loss of derivatives (the exponent  $\sigma$  in the previous theorems) provided  $\alpha < 1$  and paying the price of a much weaker dispersion.

The loss of derivatives introduces difficulties for the well-posedness theory for the nonlinear problem. Some partial results can be found in the work by Hong and the last author [HS15], as well as several subsequent contributions by many authors. As previously mentioned, the case of radial data is more favorable and a concentration-compactness/rigidity *à la Kenig-Merle*, for the energy-critical nonlinear Schrödinger (NLS) is performed in [GSWZ18].

In the present article, we refrain from developing nonlinear applications of our estimates. Our original motivation is to understand the structure of the kernel of the propagator on Riemannian symmetric spaces of rank one and noncompact type. We also exclude the case  $\alpha = 1$  in the continuous setting since it requires different techniques and is also contained in the literature (the half-wave theory, see [Tat01, APV11, MT11, MT12, AP14]).

Our paper is organized as follows: Section 2 is devoted to a summary of classical notations and preliminary tools of the harmonic analysis of symmetric spaces. Section 3 provides the refined kernel estimates, which are at the core of the proofs of the previous theorems 1.1 and 1.2. Section 4 gives the proofs of the dispersive and Strichartz estimates for the real hyperbolic spaces. Finally, Section 5 deals with the case of homogeneous trees.

*Notation.*

- Given two non-negative functions  $f$  and  $g$  defined on  $M$ , we write  $f \lesssim g$  if there exists a positive constant  $C$  such that  $f(x) \leq Cg(x)$  for all  $x \in M$ . The expression  $f \asymp g$  means that  $f(x) \lesssim g(x)$  and  $g(x) \lesssim f(x)$  for all  $x \in M$ .
- The Lebesgue spaces is denoted by  $(L^p := L^p(\mathbb{R}), \|\cdot\|_{L^p})$ ,  $p \geq 2$ , and the time-space Strichartz spaces  $L^p(\mathbb{R}; L^q(M))$ ,  $p, q \geq 2$ , of  $f$  on  $\mathbb{R} \times M$  is defined by

$$\|f\|_{L^p(\mathbb{R}; L^q(M))} := \left[ \int_{\mathbb{R}} \left( \int_M |f(t, x)|^q dx \right)^{\frac{p}{q}} dt \right]^{\frac{1}{p}}.$$

In the discrete setting, we will denote the Lebesgue space by its lower-case letter  $(\ell^p, \|\cdot\|_{\ell^p})$ .

## 2. HARMONIC ANALYSIS TOOLS ON HYPERBOLIC SPACES

This section is devoted to some preliminary results about harmonic analysis on hyperbolic spaces together with some definitions used throughout the paper.

**2.1. (Real) hyperbolic spaces.** We define  $M = \mathbb{H}^n$ , for  $n \geq 2$ , as the upper branch of a hyperboloid in  $\mathbb{R}^{n+1}$  with the metric induced by the Lorentzian metric in  $\mathbb{R}^{n+1}$  given by  $ds^2 - |dx|^2$ . More precisely, we take

$$\begin{aligned} \mathbb{H}^n &= \{(s, x) \in \mathbb{R} \times \mathbb{R}^n : s^2 - |x|^2 = 1\} \\ &= \{(s, x) \in \mathbb{R} \times \mathbb{R}^n : (s, x) = (\cosh r, (\sinh r)\omega) \text{ where } r \geq 0, \omega \in \mathbb{S}^{n-1}\}, \end{aligned}$$

with metric  $g_{\mathbb{H}^n} = dr^2 + \sinh^2 r d\omega^2$ , where  $d\omega^2$  is metric on the hypersphere  $\mathbb{S}^{n-1}$ . Notice that via a stereographic projection, one obtains the half-space model

$$\mathbb{H}^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_n > 0\},$$

with the metric  $g_{\mathbb{H}^n} = \frac{dx_1^2 + \dots + dx_n^2}{x_n^2}$ . By choosing coordinates  $x = (\tilde{x}, x_n)$ ,  $\tilde{x} = (x_1, \dots, x_{n-1})$ , we denote the volume element by

$$dV_{\mathbb{H}^n}(x) := (x_n)^{-n} d\tilde{x} dx_n.$$

The fact that we consider for simplicity the real case plays no role here except for simplicity of the presentation and all the formulas extend to *all* hyperbolic spaces, i.e., any rank one Riemannian symmetric spaces of noncompact type.

**2.2. Fractional Laplacian on hyperbolic spaces.** Under the parametrization of Sect. 2.1 the Laplace Beltrami operator is given by

$$\Delta_{\mathbb{H}^n} = x_n^2 \Delta - (n-2)x_n \partial_n. \quad (2)$$

Here  $\Delta$  denotes the Euclidean Laplacian in coordinates  $x_1, \dots, x_n \in \mathbb{R}^n$  and  $\partial_n = \frac{\partial}{\partial x_n}$  is the partial derivative with respect to  $x_n$ . Before defining its fractional representation, let us first recall in  $\mathbb{R}^n$  the Fourier transform, which is given by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-2\pi i x \cdot \xi} dx, \quad \xi \in \mathbb{R}^n.$$

Notice that the functions  $h_\xi(x) := e^{-2\pi i x \cdot \xi}$  are generalized (in the sense that they do not belong to  $L^2$ ) eigenfunctions of the Laplacian associated to the eigenvalue  $-4\pi^2 |\xi|^2$ . Moreover, the following inversion formula holds

$$f(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) e^{2\pi i x \cdot \xi} d\xi, \quad x \in \mathbb{R}^n.$$

Similarly, in  $\mathbb{H}^n$  we consider the generalized eigenfunctions of the Laplace Beltrami operator:

$$h_{\lambda,\theta}(x) = [x, (1, \theta)]^{i\lambda - \frac{n-1}{2}}, \quad x \in \mathbb{H}^n,$$

where  $\lambda \in \mathbb{R}$  and  $\theta \in \mathbb{S}^{n-1}$  and we denoted  $[\cdot, \cdot]$  the Lorentzian inner product, i.e.,  $[(s, x), (\tilde{s}, \tilde{x})] = s\tilde{s} - x\tilde{x}$ . Notice that

$$\Delta_{\mathbb{H}^n} h_{\lambda,\theta} = - \left( \lambda^2 + \frac{(n-1)^2}{4} \right) h_{\lambda,\theta}.$$

In analogy with the definition in  $\mathbb{R}^n$ , the Fourier transform can be defined as

$$\hat{f}(\lambda, \theta) = \int_{\mathbb{H}^n} f(x) h_{\lambda,\theta}(x) dx,$$

for  $\lambda \in \mathbb{R}$ ,  $\theta \in \mathbb{S}^{n-1}$ . Moreover, the following inversion formula holds:

$$f(x) = \int_{-\infty}^{\infty} \int_{\mathbb{S}^{n-1}} \bar{h}_{\lambda,\theta}(x) \hat{f}(\lambda, \theta) \frac{d\theta d\lambda}{|\mathbf{c}(\lambda)|^2},$$

where the Plancherel density

$$\frac{1}{|\mathbf{c}(\lambda)|^2} = \text{const.} \frac{|\Gamma(i\lambda + \frac{n-1}{2})|^2}{|\Gamma(i\lambda)|^2} \asymp \begin{cases} \lambda^2 & \text{if } |\lambda| \leq 1 \\ |\lambda|^{n-1} & \text{if } |\lambda| \geq 1 \end{cases}$$

involves the Harish-Chandra  $\mathbf{c}$ -function. It is easy to check by integration by parts for compactly supported functions, and consequently for every  $f \in L^2(\mathbb{H}^n)$ , that

$$\widehat{\Delta_{\mathbb{H}^n} f}(\lambda, \theta) = \int_{\mathbb{H}^n} f(y) \Delta_{\mathbb{H}^n} h_{\lambda,\theta}(y) dy = - \left( \lambda^2 + \frac{(n-1)^2}{4} \right) \hat{f}(\lambda, \theta).$$

Having in mind the theory of spherically symmetric multipliers, we define the *fractional Laplacian*  $(-\Delta_{\mathbb{H}^n})^{\alpha/2}$ , with  $0 < \alpha < 2$ , on the hyperbolic space  $\mathbb{H}^n$  by

$$\widehat{(-\Delta_{\mathbb{H}^n})^{\alpha/2} f}(\lambda, \theta) = \left( \lambda^2 + \frac{(n-1)^2}{4} \right)^{\alpha/2} \hat{f}(\lambda, \theta).$$

Hence we can write the Schrödinger solution as

$$\widehat{u}(\lambda, \theta; t) = \widehat{e^{it(-\Delta_{\mathbb{H}^n})^{\alpha/2}} u_0}(\lambda, \theta) = e^{it(\lambda^2 + \frac{(n-1)^2}{4})^{\alpha/2}} \widehat{u}_0(\lambda, \theta).$$

The (mild) solution to (1) on  $\mathbb{H}^n$  is given by

$$u(x, t) = \underbrace{e^{it(-\Delta_x)^{\alpha/2}} u_0(x)}_{\text{homogeneous}} - i \underbrace{\int_0^t e^{i(t-s)(-\Delta_x)^{\alpha/2}} F(x, s) ds}_{\text{inhomogeneous}}. \quad (3)$$

Our estimates and their proofs extend straightforwardly to all Riemannian symmetric spaces  $G/K$  of noncompact type of rank 1 (where  $G$  is a noncompact semisimple connected Lie group with finite centre and  $K$  is its maximal compact subgroup), i.e., to the four hyperbolic spaces:  $\mathbb{H}^n = \mathbb{H}^N(\mathbb{R})$ ,  $\mathbb{H}^N(\mathbb{C})$ ,  $\mathbb{H}^N(\mathbb{H})$ ,  $\mathbb{H}^2(\mathbb{O})$ , where  $\mathbb{H}$  (resp.  $\mathbb{O}$ ) denotes the field of quaternions (resp. octonions) in Table 1 and more generally to all Damek–Ricci spaces.

	$\mathbb{H}^n = \mathbb{H}^N(\mathbb{R})$	$\mathbb{H}^N(\mathbb{C})$	$\mathbb{H}^N(\mathbb{H})$	$\mathbb{H}^2(\mathbb{O})$
$G$	$\mathrm{SO}(n, 1)^\circ$	$\mathrm{SU}(n, 1)$	$\mathrm{Sp}(n, 1)$	$\mathrm{F}_{4(-20)}$
$K$	$\mathrm{SO}(n)$	$\mathrm{S}[\mathrm{U}(n) \times \mathrm{U}(1)]$	$\mathrm{Sp}(n)$	$\mathrm{Spin}(9)$
$n$	$N$	$2N$	$4N$	$16$
$\rho$	$\frac{N-1}{2}$	$N$	$2N+1$	$11$

TABLE 1. Symmetric spaces of rank 1. Here  $\mathrm{F}_{4(-20)}$  is an exponential non-compact Lie group,  $\mathrm{Spin}(9)$  is the spinor group of dimension 9,  $\mathrm{SO}(n, 1)^\circ$  is the connected component of the identity in the orthogonal Lorentz group  $\mathrm{SO}(n, 1)$ ,  $\mathrm{SO}(n)$  is the special orthogonal group,  $\mathrm{U}(n)$  is the unitary group, and  $\mathrm{SU}(n)$  is the special unitary group

**2.3. Asymptotic expansions of the spherical function.** We now state several well-known results about asymptotic expansions of the spherical function  $\varphi_\lambda$  which will be used throughout the proof of the kernel estimates in the next Section 3. To facilitate the reading, we collect those formulae into several lemmata.

We start the following integral formula due to Harish-Chandra, which holds for all  $r > 0$  (see for instance [GV88, Prop. 3.1.4], [Hel84, Ch. IV, Theorem 4.3] or [Koo84, p. 40]). We have

$$\begin{aligned} \varphi_\lambda(r) &= \int_K e^{(i\lambda - \rho)H(a_r k)} dk = \int_K e^{-(i\lambda + \rho)H(a_{-r} k)} dk \\ &= \frac{\Gamma(\frac{n}{2})}{\sqrt{\pi} \Gamma(\frac{n-1}{2})} \int_0^\pi (\sin \theta)^{n-2} (\cosh r \pm \sinh r \cos \theta)^{-\rho \mp i\lambda} d\theta, \end{aligned} \quad (4)$$

where  $H(a_r k_\theta) = \log(\cosh r + \sinh r \cos \theta) \in [-r, r]$ , with

$$a_r = \begin{pmatrix} \cosh r & 0 & 0 & \sinh r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & I_{n-2} & 0 \\ \sinh r & 0 & 0 & \cosh r \end{pmatrix} \quad \text{and} \quad k_\theta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & 0 & -\sin \theta \\ 0 & 0 & I_{n-2} & 0 \\ 0 & \sin \theta & 0 & \cos \theta \end{pmatrix}$$

in the hyperboloid model of  $\mathbb{H}^n$ .

The following result is a large scale asymptotic (see for instance [Koo84, Formula (2.17)]).

**Lemma 2.1** (Large scale asymptotic). *The following large scale converging expansion of the spherical functions holds: Let  $r_0 > 0$  be fixed. Then for every  $r > r_0$ , we have that*

$$\varphi_\lambda(r) = \mathbf{c}(\lambda) \Phi_\lambda(r) + \mathbf{c}(-\lambda) \Phi_{-\lambda}(r) \quad \forall \lambda \in \mathbb{C} \setminus i\mathbb{Z}, \quad (5)$$

holds, where

$$\begin{aligned} \Phi_\lambda(r) &= (2 \sinh r)^{i\lambda - \rho} {}_2F_1\left(\frac{\rho}{2} - i\frac{\lambda}{2}, -\frac{\rho-1}{2} - i\frac{\lambda}{2}; 1 - i\lambda; -\sinh^{-2} r\right) \\ &= (2 \sinh r)^{-\rho} e^{i\lambda r} \sum_{\ell=0}^{+\infty} \Gamma_\ell(\lambda) e^{-2\ell r}. \end{aligned} \quad (6)$$

The coefficients  $\Gamma_\ell(\lambda)$  in this expansion are inhomogeneous symbols of order 0 on  $\mathbb{R}$ . More precisely, there are constants  $\gamma \geq 0$  and  $C_j \geq 0$  ( $j \in \mathbb{N}$ ), such that

$$\left| \left(\frac{\partial}{\partial \lambda}\right)^j \Gamma_\ell(\lambda) \right| \leq C_j \ell^\gamma (1 + |\lambda|)^{-j} \quad \forall \ell \in \mathbb{N}^*, \forall \lambda \in \mathbb{R}. \quad (7)$$

**Remark 2.2.** Notice that we actually have  $\Gamma_0 \equiv 1$ , while the other  $\Gamma_\ell$  are inhomogeneous symbols of order  $-1$  (see for instance [APV15, Lem. 1]).

The following small scale asymptotic is due to Stanton and Tomas [ST78, Thm. 2.1].

**Lemma 2.3** (Small scale asymptotics). *The following small scale expansion holds: Let  $r_1 > 0$  be fixed. Then for every  $0 \leq r < r_1$ , we have*

$$\varphi_\lambda(r) \sim \sum_{m=0}^{+\infty} r^{2m} \tilde{b}_m(r) j_{m+\frac{n}{2}-1}(\lambda r), \quad (8)$$

where

$$\begin{aligned} j_\nu(z) &= \frac{\Gamma(\nu+1)}{\sqrt{\pi}\Gamma(\nu+\frac{1}{2})} \int_{-1}^{+1} (1-u^2)^{\nu-\frac{1}{2}} e^{izu} du = \sum_{m=0}^{+\infty} \frac{(-1)^m \Gamma(\nu+1)}{m! \Gamma(m+\nu+1)} \left(\frac{z}{2}\right)^{2m} \\ &= \Gamma(\nu+1) \left(\frac{z}{2}\right)^{-\nu} J_\nu(z), \end{aligned}$$

is a modified Bessel function and  $\tilde{b}_0(r) = \left(\frac{r}{\sinh r}\right)^{\frac{n-1}{2}}$  is the Jacobian of the exponential map, raised to the power  $-\frac{1}{2}$ . More precisely, for every  $M \in \mathbb{N}^*$ ,

$$\varphi_\lambda(r) = \sum_{0 \leq m < M} r^{2m} \tilde{b}_m(r) j_{m+\frac{n}{2}-1}(\lambda r) + r^{2M} \tilde{R}_M(\lambda, r),$$

where the coefficients  $\tilde{b}_m(r)$  are smooth even functions and

$$|\tilde{R}_M(\lambda, r)| \leq \tilde{C}_M (1+|\lambda r|)^{-\frac{n-1}{2}-M}.$$

**Remark 2.4.** *Such asymptotics are closely related to the dual Abel transform. Note that in [AP09, APV11, APV12, APV15, AP14], the inverse Abel transform was used instead of the asymptotic expansions we used in the present article.*

As noticed by Ionescu [Ion00, Prop. A2.(b)], by combining (8) with the classical asymptotics (see for instance [DLM, 10.17.3])

$$j_\nu(z) \sim e^{iz} \sum_{m=0}^{+\infty} \beta_{m,\nu} (iz)^{-m-\nu-\frac{1}{2}} + e^{-iz} \sum_{m=0}^{+\infty} \beta_{m,\nu} (-iz)^{-m-\nu-\frac{1}{2}},$$

one obtains, for every  $M \in \mathbb{N}^*$ , for every  $\Lambda > 0$ , for every  $\lambda \in \mathbb{R}^*$  and for every  $0 < r \leq 1$  such that  $|\lambda r| \geq \Lambda$ ,

$$\varphi_\lambda(r) = b_M(\lambda, r) e^{i\lambda r} + b_M(-\lambda, r) e^{-i\lambda r} + R_M(\lambda, r), \quad (9)$$

where

$$\left| \left(\frac{\partial}{\partial \lambda}\right)^j b_M(\lambda, r) \right| \leq C_M |\lambda r|^{-\frac{n-1}{2}} |\lambda|^{-j} \quad \forall 0 \leq j \leq M \quad (10)$$

and

$$|R_M(\lambda, r)| \leq C_M |\lambda r|^{-M-\frac{n-1}{2}}. \quad (11)$$

### 3. KERNEL ANALYSIS

This section is devoted to the results in the continuous setting. The analysis relies on fine kernel estimates and we observe the dichotomy between  $\alpha \in (0, 1)$  and  $\alpha \in (1, 2)$  which amounts to investigate the two different behaviours for the phase of the oscillatory integral. The integral expression in (3) involves the following propagator

$$e^{it(-\Delta)^{\alpha/2}} f(x) = f * k_t(x) = \int_{\mathbb{H}^n} f(y) k_t(d(x, y)) dy,$$

which is the radial convolution operator defined by the inverse spherical Fourier transform

$$k_t(r) = \text{const.} \int_{-\infty}^{+\infty} e^{it(\lambda^2 + \rho^2)^{\alpha/2}} \varphi_\lambda(r) \frac{d\lambda}{|c(\lambda)|^2} \quad \forall t \in \mathbb{R}^*, \forall r \geq 0. \quad (12)$$

Here  $\rho^2 = \left(\frac{n-1}{2}\right)^2$  is the bottom of the  $L^2$  spectrum of  $-\Delta$  on  $\mathbb{H}^n$ . Notice that, in comparison with the Hankel transform (i.e., the Fourier transform of radial functions in



$\mathbb{R}^n$ ), the modified Bessel functions are replaced in (12) by the spherical functions  $\varphi_\lambda(r)$  and the Plancherel density  $\lambda^{n-1}$  by  $|\mathbf{c}(\lambda)|^{-2}$ , where

$$\mathbf{c}(\lambda) = \frac{\Gamma(2\rho)}{\Gamma(\rho)} \frac{\Gamma(i\lambda)}{\Gamma(i\lambda+\rho)}.$$

**Remark 3.1.** Notice that  $|\varphi_\lambda(r)| \leq \varphi_0(r) \asymp (1+r)e^{-\rho r}$  for every  $\lambda \in \mathbb{R}$  and  $r \geq 0$ . Moreover, one should (often) factorize  $\mathbf{c}(\lambda)^{-1} = \lambda \left( i \frac{\Gamma(\rho)}{\Gamma(2\rho)} \frac{\Gamma(i\lambda+\rho)}{\Gamma(i\lambda+1)} \right)$  and use the fact that the parenthesis is an inhomogeneous symbol on  $\mathbb{R}$  of order  $\frac{n-3}{2}$ .

More generally we shall consider the operator  $(-\Delta_x)^{-\sigma/2} e^{it(-\Delta_x)^{\alpha/2}}$  for an additional smoothness  $\sigma \in \mathbb{C}$  with  $\operatorname{Re} \sigma \geq 0$ , and its kernel

$$k_t^\sigma(r) = \text{const.} \int_{-\infty}^{+\infty} (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} e^{it(\lambda^2 + \rho^2)^{\alpha/2}} \varphi_\lambda(r) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}. \quad (13)$$

This will lead us to analyze oscillatory integrals

$$\int_{-\infty}^{+\infty} a(\lambda) e^{it\psi(\lambda)} d\lambda$$

involving the phase

$$\psi(\lambda) = \psi_R(\lambda) = (\lambda^2 + \rho^2)^{\alpha/2} - R\lambda, \quad (14)$$

with  $t > 0$  and  $R = \frac{r}{t} \geq 0$ , and amplitudes  $a(\lambda)$  involving  $(\lambda^2 + \rho^2)^{-\sigma/2}$  and the  $\mathbf{c}$ -function. Without loss of generality, we may assume that  $t > 0$ . The function  $\psi$  will be the phase of the oscillatory integral associated to the propagator. As a consequence, the following technical lemmata are the basis of the kernel analysis (see Section 3.1).

**Lemma 3.2** (Phase for  $1 < \alpha < 2$ ). *Let  $1 < \alpha < 2$ . Then (14) has a single stationary point  $\lambda_1$ , which is nonnegative and comparable to*

$$\begin{cases} R & \text{if } R \leq 1, \\ R^{\frac{1}{\alpha-1}} & \text{if } R \geq 1. \end{cases}$$

Moreover, the following properties hold:

- (i)  $\psi''(\lambda)$ , which is positive and comparable to  $(|\lambda| + 1)^{-(2-\alpha)}$ , is an inhomogeneous symbol of order  $-(2-\alpha)$  on  $\mathbb{R}$ .
- (ii) Assume that  $R > 0$ . Then, for any fixed  $0 < \beta < 1$ ,  $|\psi'(\lambda)|$  is comparable to  $\frac{|\lambda + \lambda_1|}{(|\lambda + \lambda_1| + 1)^{2-\alpha}}$  when  $\lambda \in \mathbb{R} \setminus (\beta\lambda_1, \beta^{-1}\lambda_1)$ .

*Proof.* Let us compute the first two derivatives

$$\psi'(\lambda) = \alpha \lambda (\lambda^2 + \rho^2)^{-(1-\frac{\alpha}{2})} - R \quad (15)$$

and

$$\psi''(\lambda) = \alpha (\lambda^2 + \rho^2)^{-(2-\frac{\alpha}{2})} [(\alpha-1)\lambda^2 + \rho^2]. \quad (16)$$

(i) is an immediate consequence of (16). All claims about the stationary point  $\lambda_1$  follow from the equation  $\theta(\lambda_1) = R$  and from the behavior of the function

$$\theta(\lambda) = \alpha \lambda (\lambda^2 + \rho^2)^{-(1-\frac{\alpha}{2})} \quad (17)$$

(see Figure 1), which is odd, strictly increasing and comparable to

$$\begin{cases} \lambda & \text{on } [0, 1], \\ \lambda^{\alpha-1} & \text{on } [1, +\infty). \end{cases}$$

Actually  $\theta(\lambda) \leq \alpha \lambda^{\alpha-1}$  on  $[0, +\infty)$ , hence  $\lambda_1 \geq (R/\alpha)^{1/(\alpha-1)}$ . Finally, assume that  $R > 0$  and let us estimate

$$\psi'(\lambda) = \theta(\lambda) - \theta_1,$$

where  $\theta_1 = \theta(\lambda_1)$ . On the one hand,

$$|\psi'(\lambda)| \leq \theta(|\lambda|) + \theta_1.$$

On the other hand, notice that, for  $\beta \in (0, 1)$ ,

$$\frac{\theta(\beta\lambda)}{\theta(\lambda)} = \beta^{\alpha-1} \left(1 - \frac{(\beta^{-2}-1)\rho^2}{\lambda^2 + \beta^{-2}\rho^2}\right)^{1-\frac{\alpha}{2}} < \beta^{\alpha-1} \quad \forall \lambda > 0.$$

Hence

$$\theta(\lambda) - \theta_1 \geq \theta(\lambda) - \theta(\beta\lambda) \geq [1 - \beta^{\alpha-1}] \theta(\lambda)$$

if  $\lambda \geq \beta^{-1}\lambda_1$ , while

$$\theta_1 - \theta(\lambda) \geq \theta(\lambda_1) - \theta(\beta\lambda_1) \geq [1 - \beta^{\alpha-1}] \theta_1$$

if  $0 < \lambda \leq \beta\lambda_1$ . Moreover,

$$\theta_1 - \theta(\lambda) = \theta_1 + \theta(|\lambda|) \quad \forall \lambda \leq 0.$$

In conclusion,  $|\psi'(\lambda)|$  is comparable to

$$\theta(|\lambda|) + \theta_1 \asymp \frac{|\lambda| + \lambda_1}{(|\lambda| + \lambda_1 + 1)^{2-\alpha}}$$

when  $\lambda \notin (\beta\lambda_1, \beta^{-1}\lambda_1)$ . □

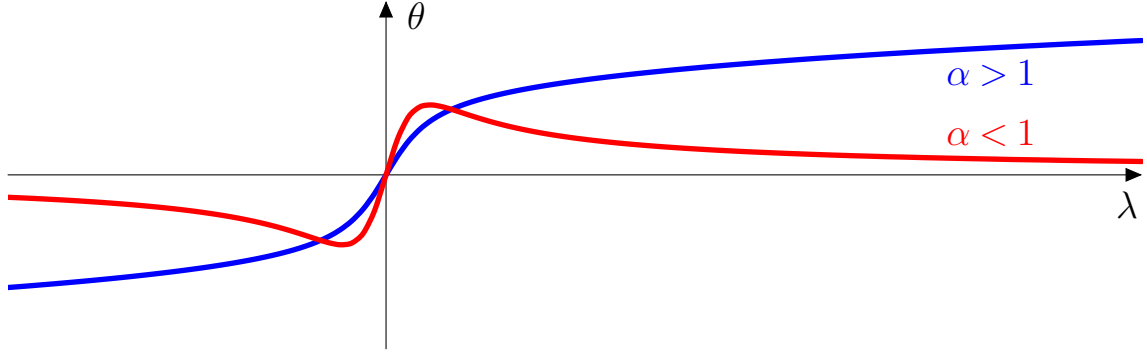


FIGURE 1. The function (17)

The function (17) behaves differently when  $0 < \alpha < 1$ . It is still odd and positive on  $(0, +\infty)$ . But now it increases between 0 and  $\lambda_0 := \frac{\rho}{\sqrt{1-\alpha}} > 0$ , where it reaches its maximum  $\theta_0 = \theta(\lambda_0) > 0$ , and decreases between  $\lambda_0$  and  $+\infty$ , where it tends to 0. Consequently the equation  $\theta(\lambda) = R$  may have 0, 1 or 2 solutions.

**Lemma 3.3** (Phase for  $0 < \alpha < 1$ ). *Let  $0 < \alpha < 1$ .*

(i) *If  $R > \theta_0$ , (14) has no stationary point. More precisely,*

$$|\psi'(\lambda)| \geq R - \theta_0 \quad \forall \lambda \in \mathbb{R}. \quad (18)$$

(ii) *If  $R = \theta_0$ , (14) has a single stationary point at  $\lambda_0$ , where  $\psi'$  and  $\psi''$  both vanish.*

(iii) *If  $0 < R < \theta_0$ , (14) has two stationary points:*

$$\begin{cases} \lambda_1 \in (0, \lambda_0), \text{ which is comparable to } R, \\ \lambda_2 \in (\lambda_0, +\infty), \text{ which is comparable to } R^{-\frac{1}{1-\alpha}}. \end{cases}$$

Moreover, for every  $0 < \beta < 1$ ,

$$|\psi'(\lambda)| \asymp (\min\{\lambda, \lambda_2\})^{-(1-\alpha)} \quad \forall \lambda \in [\lambda_0, +\infty) \setminus (\beta\lambda_2, \beta^{-1}\lambda_2). \quad (19)$$

(iv) *If  $R = 0$ , (14) has a single stationary point at the origin. More precisely,*

$$|\psi'(\lambda)| \asymp |\lambda| (|\lambda| + 1)^{-(2-\alpha)} \quad \forall \lambda \in \mathbb{R}.$$

(v) *Contrarily to  $\psi(\lambda)$  and  $\psi'(\lambda)$ ,  $\psi''(\lambda)$  and  $\psi'''(\lambda)$  don't depend on  $R$ . Moreover,*

- $\psi''(\lambda)$  is an even inhomogeneous symbol of order  $-(2-\alpha)$  on  $\mathbb{R}$ ,
- away from  $\lambda = \pm\lambda_0$ , where it vanishes,  $|\psi''(\lambda)|$  is comparable to  $(|\lambda| + 1)^{-(2-\alpha)}$ ,
- $\psi'''(\lambda)$  is an odd function on  $\mathbb{R}$ , which vanishes at  $\lambda = 0$  and  $\lambda = \pm\sqrt{3}\lambda_0$ .

**Remark 3.4.** *Actually, in (iii), we have*

$$\theta(\lambda) \leq \alpha(\lambda^2 + \rho^2)^{-(1-\alpha)/2} \text{ on } [0, +\infty), \text{ hence } \lambda_2 \leq \sqrt{\lambda_2^2 + \rho^2} \leq (R/\alpha)^{-1/(1-\alpha)}.$$

*Proof of Lemma 3.3.* Almost all claims are straightforward consequences of the expressions (15), (16) and of the above behavior of (17). The only exceptions are the last point, which follows from

$$\psi'''(\lambda) = \alpha(2-\alpha)\lambda[(1-\alpha)\lambda^2 - 3\rho^2](\lambda^2 + \rho^2)^{\frac{\alpha}{2}-3},$$

and (19), which is proved as (ii) in Lemma 3.2.  $\square$

Hence using the lemmata above, we estimate the kernel for the dichotomy between  $\alpha \in (0, 1)$  and  $\alpha \in (1, 2)$ .

**Theorem 3.5** (Kernel estimates). (i) *Assume that  $1 < \alpha < 2$  and  $0 \leq \sigma \leq \frac{n}{2}$ . Let  $\rho = \frac{n-1}{2}$ . Then the following estimates hold, for  $t \in \mathbb{R}^*$  and  $r \geq 0$ :*

- Large scale:  $|k_t^\sigma(r)| \lesssim \begin{cases} |t|^{-\frac{3}{2}}(1+r)e^{-\rho r} & \text{if } |t| \geq \max\{1, r\} \quad (\text{Subcases 1.1.1 and 2.1.1}), \\ |t|^{-\frac{1}{2}\frac{n-2\sigma}{\alpha-1}} r^{\frac{1}{2}\frac{n-2\sigma}{\alpha-1}-\frac{1}{2}} e^{-\rho r} & \text{if } r \geq \max\{1, |t|\} \quad (\text{Subcase 1.1.2}). \end{cases}$
- Small scale:  $|k_t^\sigma(r)| \lesssim \begin{cases} |t|^{-\frac{n-\sigma}{\alpha}} & \text{if } r^\alpha \leq |t| < 1 \quad (\text{Subcase 2.1.1}), \\ |t|^{-\frac{1}{2}\frac{n-2\sigma}{\alpha-1}} r^{\frac{1}{2}\frac{n-2\sigma}{\alpha-1}-\frac{n}{2}} & \text{if } |t| \leq r^\alpha < 1 \quad (\text{Subcase 2.1.2}). \end{cases}$

(ii) *Assume that  $0 < \alpha < 1$ ,  $\frac{n}{2} \leq \sigma \leq n$  and  $N > \frac{n+1}{2} - \sigma$ . Then the following estimates hold, for  $t \in \mathbb{R}^*$  and  $r \geq 0$ :*

- Large scale:
 
$$|k_t^\sigma(r)| \lesssim \begin{cases} |t|^{-\frac{3}{2}} + |t|^{-\frac{1}{2}\frac{2\sigma-n}{1-\alpha}} r^{\frac{1}{2}\frac{2\sigma-n}{1-\alpha}-\frac{n}{2}} & \text{if } 0 \leq r \leq 1 \leq |t| \quad (\text{Subcase 2.2.1}), \\ \left(\frac{r}{|t|}\right)^{\min\{\frac{3}{2}, \frac{1}{2}\frac{2\sigma-n}{1-\alpha}\}} r^{-\frac{1}{2}} e^{-\rho r} & \text{if } r \geq 1 \text{ and } \frac{r}{|t|} \leq \frac{1}{2}\theta_0 \quad (\text{Subcase 1.2.3}), \\ r^{-\frac{1}{3}} e^{-\rho r} & \text{if } r \geq 1 \text{ and } \frac{1}{2}\theta_0 < \frac{r}{|t|} < 2\theta_0 \quad (\text{Subcase 1.2.2}), \\ r^{-N} e^{-\rho r} & \text{if } r \geq 1 \text{ and } \frac{r}{|t|} \geq 2\theta_0 \quad (\text{Subcase 1.2.1}). \end{cases}$$
- Small scale:
 
$$|k_t^\sigma(r)| \lesssim \begin{cases} |t|^{-\frac{n-\sigma}{\alpha}} & \text{if } |t| \leq r^\alpha < 1 \quad (\text{Subcase 2.2.3}), \\ |t|^{-\frac{n-\sigma}{\alpha}} + |t|^{-\frac{1}{2}\frac{2\sigma-n}{1-\alpha}} r^{\frac{1}{2}\frac{2\sigma-n}{1-\alpha}-\frac{n}{2}} & \text{if } r^\alpha \leq |t| < 1 \quad (\text{Subcase 2.2.2}). \end{cases}$$

**Remark 3.6.** (i) *Assume that  $1 < \alpha < 2$  and  $0 \leq \sigma \leq \frac{n}{2}$ . Then the following inequalities are equivalent:*

$$\sigma \geq \left(1 - \frac{\alpha}{2}\right)n, \quad \frac{n-\sigma}{\alpha} \leq \frac{n}{2}, \quad \frac{1}{2}\frac{n-2\sigma}{\alpha-1} \leq \frac{n-\sigma}{\alpha}, \quad \frac{1}{2}\frac{n-2\sigma}{\alpha-1} \leq \frac{n}{2}.$$

Moreover, under these conditions, we have

$$|k_t^\sigma(r)| \lesssim |t|^{-\frac{1}{2}\frac{n-2\sigma}{\alpha-1}} r^{\frac{1}{2}\frac{n-2\sigma}{\alpha-1}-\frac{n}{2}} \leq |t|^{-\frac{1}{2}\frac{n-2\sigma}{\alpha-1}} |t|^{\frac{1}{2\alpha}\frac{n-2\sigma}{\alpha-1}-\frac{n}{2\alpha}} = |t|^{-\frac{n-\sigma}{\alpha}}$$

in the range  $0 < |t| \leq r^\alpha < 1$ .

(ii) *Assume that  $0 < \alpha < 1$  and  $\frac{n}{2} \leq \sigma \leq n$ . Then the following inequalities are equivalent:*

$$\sigma \geq \left(1 - \frac{\alpha}{2}\right)n, \quad \frac{n-\sigma}{\alpha} \leq \frac{n}{2}, \quad \frac{1}{2}\frac{2\sigma-n}{1-\alpha} \geq \frac{n-\sigma}{\alpha}, \quad \frac{1}{2}\frac{2\sigma-n}{1-\alpha} \geq \frac{n}{2}.$$

Moreover, in the range  $0 < r^\alpha < |t| < 1$ , we have

$$\max\left\{|t|^{-\frac{n-\sigma}{\alpha}}, |t|^{-\frac{1}{2}\frac{2\sigma-n}{1-\alpha}} r^{\frac{1}{2}\frac{2\sigma-n}{1-\alpha}-\frac{n}{2}}\right\} = \begin{cases} |t|^{-\frac{n-\sigma}{\alpha}} & \text{if } \sigma \geq \left(1 - \frac{\alpha}{2}\right)n, \\ |t|^{-\frac{1}{2}\frac{2\sigma-n}{1-\alpha}} r^{\frac{1}{2}\frac{2\sigma-n}{1-\alpha}-\frac{n}{2}} & \text{if } \sigma < \left(1 - \frac{\alpha}{2}\right)n. \end{cases}$$

**Remark 3.7.** *Assume that  $\sigma = \left(1 - \frac{\alpha}{2}\right)n$  and  $N > \frac{1+n(\alpha-1)}{2}$ . Then Theorem 3.5 boils down to*

$$|k_t^\sigma(r)| \lesssim \begin{cases} |t|^{-\frac{n}{2}}(1+r)^{\frac{n-1}{2}} e^{-\rho r} & \text{if } |t| \leq 1+r \\ |t|^{-\frac{3}{2}}(1+r) e^{-\rho r} & \text{if } |t| \geq 1+r \end{cases} \quad (20)$$

in the range  $1 < \alpha < 2$  and to

$$|k_t^\sigma(r)| \lesssim \begin{cases} |t|^{-\frac{n}{2}} & \text{if } 0 < |t| \leq 1 \text{ and } 0 \leq r \leq 1 \\ |t|^{-\min\{\frac{3}{2}, \frac{n}{2}\}} (1+r)^{\min\{1, \frac{n-1}{2}\}} e^{-\rho r} & \text{if } |t| \geq 1 \text{ and } 0 \leq \frac{r}{|t|} \leq \frac{1}{2}\theta_0 \\ r^{-\frac{1}{3}} e^{-\rho r} & \text{if } r \geq 1 \text{ and } \frac{1}{2}\theta_0 < \frac{r}{|t|} < 2\theta_0 \\ r^{-N} e^{-\rho r} & \text{if } r \geq 1 \text{ and } \frac{r}{|t|} \geq 2\theta_0 \end{cases} \quad (21)$$

in the range  $0 < \alpha < 1$ . In the limit case  $\alpha = 2$ , the kernel estimates (20) were obtained in [AP09] and [APV11].

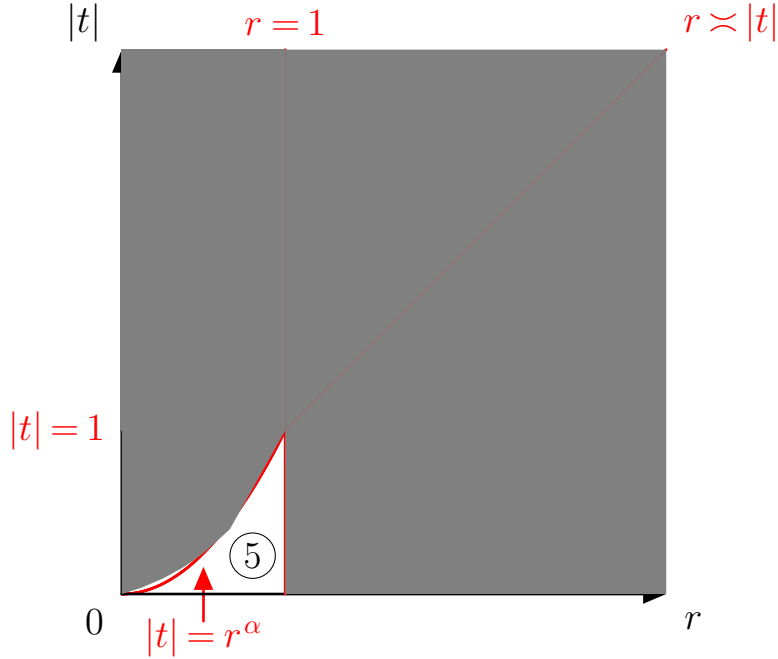


FIGURE 2. Different ranges for the kernel estimates in the case  $1 < \alpha < 2$ . The circled numbers correspond to the different cases in the proof of Theorem 3.5. More precisely, (1) corresponds to Subcase 1.1.1, (2) to Subcase 1.1.2, (3) – (4) to Subcase 2.1.1 and (5) to Subcase 2.1.2

**3.1. Proof of Theorem 3.5.** We first explain the global structure of the proof since this is quite technical. Due to the change of behaviour in the phase  $\psi$  in Equation (14), it is necessary to consider two regimes  $\alpha \in (0, 1)$  and  $\alpha \in (1, 2)$ . Of course, the case  $\alpha = 1$ , which is the half-wave has been treated extensively in the literature. As explained in the introduction, the change of “convexity” of  $\psi$  induces different losses which are a major difficulty for nonlinear applications. This also introduces several technical difficulties for the kernel analysis since for  $\alpha \in (0, 1)$  the phase has *two* stationary points (see Lemma 3.3) and one needs to go to the *third* order. Now for each of those ranges in  $\alpha$ , one needs to consider the different regimes in  $r$  and  $t$ , which is shown in the corresponding Figures 2 and 3. Because of similarities in the arguments, we prefer to split the proof into several parts according to:

- (1) First case of *large spatial scale*  $r \geq r_0 > 0$  with  $r_0$  fixed, will be treated in Section 3.1.1. Then we consider the subcases  $\alpha \in (0, 1)$  and  $\alpha \in (1, 2)$ .
- (2) Second case of *small spatial scale*  $0 \leq r \leq r_0$  can be found in Section 3.1.2. Then we consider again the subcases  $\alpha \in (0, 1)$  and  $\alpha \in (1, 2)$ .

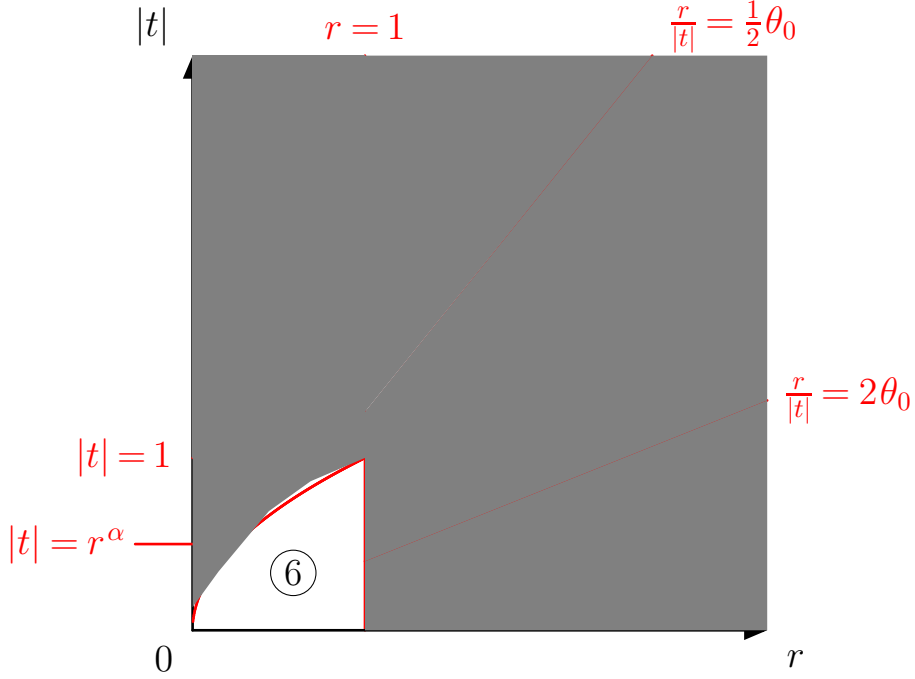


FIGURE 3. Different ranges for the kernel estimates in the case  $0 < \alpha < 1$ . The circled numbers correspond to the different cases in the proof of Theorem 3.5 (ii). More precisely,  $\textcircled{1} - \textcircled{2} - \textcircled{3}$  correspond to Subcase 1.2,  $\textcircled{4}$  to Subcase 2.2.1,  $\textcircled{5}$  to Subcase 2.2.2 and  $\textcircled{6}$  to Subcase 2.2.3

For each of the cases above, there is additional smallness to consider in the scaled variable  $\frac{r}{t}$ . We would like to emphasize that the range  $\alpha \in (0, 1)$  is the one presenting the most differences with the classical Laplacian estimates. This is because of the structure of the oscillatory integral term that such striking differences occur.

Through the proof, we will use the following version of the van der Corput Lemma (see [Ste93, Ch. VIII, Cor. p. 334]) when  $L = 2$  or  $3$ .

**Lemma 3.8.** *Let  $L \geq 2$  be an integer. Then there exists a constant  $C > 0$  such that*

$$\left| \int_I a(\lambda) e^{i\Psi(\lambda)} d\lambda \right| \leq C \{ \|a\|_\infty + \|a'\|_1 \} T^{-\frac{1}{L}},$$

for any interval  $I \subset \mathbb{R}$ , for any  $C^L$  function  $\Psi : I \rightarrow \mathbb{R}$  such that  $|\Psi^{(L)}| \geq T$  on  $I$ , and for any  $C^1$  function  $a : I \rightarrow \mathbb{C}$ .

3.1.1. *Case 1 - large spatial scale.* If  $r$  is bounded from below, let us say by  $1$ , we use the large scale expansion provided by Lemma 2.1. By substituting (5) and (6) in (13), we get

$$k_t^\sigma(r) = C (\sinh r)^{-\rho} \sum_{\ell=0}^{+\infty} e^{-2\ell r} k_{t,\ell}^\sigma(r),$$

where

$$k_{t,\ell}^\sigma(r) = \int_{-\infty}^{+\infty} a_\ell(\lambda) e^{it\psi_{\frac{r}{t}}(\lambda)} d\lambda, \quad (22)$$

with

$$a_\ell(\lambda) = \frac{\Gamma(i\lambda + \rho)}{\Gamma(i\lambda)} (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} \Gamma_\ell(-\lambda). \quad (23)$$

**Subcase 1.1.** Assume that  $1 < \alpha < 2$ .

*Subcase 1.1.1.* Consider first the case where  $\frac{r}{t}$  or equivalently  $\lambda_1$  remains bounded, let say  $\lambda_1 \leq 1$ , that is case (1) in Figure 2. Given an even bump function  $\chi_0 \in C_c^\infty(\mathbb{R})$  such that  $\chi_0 \equiv 1$  on  $[-2, 2]$ , let us split up

$$\int_{-\infty}^{+\infty} d\lambda = \int_{-\infty}^{+\infty} \chi_0(\lambda) d\lambda + \int_{-\infty}^{+\infty} [1 - \chi_0(\lambda)] d\lambda \quad (24)$$

in (22) and

$$k_{t,\ell}^\sigma(r) = k_{t,\ell}^{\sigma,0}(r) + k_{t,\ell}^{\sigma,\infty}(r) \quad (25)$$

accordingly. On the one hand, after an integration by parts based on

$$e^{it(\lambda^2 + \rho^2)^{\alpha/2}} = \frac{-i}{\alpha t \lambda} (\lambda^2 + \rho^2)^{1 - \frac{\alpha}{2}} \frac{\partial}{\partial \lambda} e^{it(\lambda^2 + \rho^2)^{\alpha/2}}, \quad (26)$$

we get

$$k_{t,\ell}^{\sigma,0}(r) = i \frac{r}{t} \int_{-\infty}^{+\infty} e^{it\psi_{\frac{r}{t}}(\lambda)} a_\ell^0(\lambda) d\lambda - \frac{1}{t} \int_{-\infty}^{+\infty} e^{it\psi_{\frac{r}{t}}(\lambda)} \frac{\partial}{\partial \lambda} a_\ell^0(\lambda) d\lambda,$$

where

$$a_\ell^0(\lambda) = \frac{1}{\alpha} \chi_0(\lambda) (\lambda^2 + \rho^2)^{1 - \frac{\alpha}{2} - \frac{\sigma}{2}} \frac{\Gamma(i\lambda + \rho)}{\Gamma(i\lambda + 1)} \Gamma_\ell(-\lambda)$$

is a smooth function with compact support. By using Lemma 3.8 with  $L = 2$ , together with Lemma 3.2 (i) and (7), we estimate

$$|k_{t,\ell}^{\sigma,0}(r)| \lesssim (1 + \ell)^\gamma t^{-\frac{3}{2}} r.$$

On the other hand, after  $N$  integrations by parts based on

$$e^{it\psi_{\frac{r}{t}}(\lambda)} = \frac{-i}{t\psi_{\frac{r}{t}}'(\lambda)} \frac{\partial}{\partial \lambda} e^{it\psi_{\frac{r}{t}}(\lambda)}, \quad (27)$$

we get

$$k_{t,\ell}^{\sigma,\infty}(r) = \left(\frac{i}{t}\right)^N \int_{-\infty}^{+\infty} e^{it\psi_{\frac{r}{t}}(\lambda)} a_\ell^\infty(\lambda) d\lambda,$$

where

$$a_\ell^\infty(\lambda) = \underbrace{\left\{ \frac{\partial}{\partial \lambda} \circ \frac{1}{\psi_{\frac{r}{t}}'(\lambda)} \circ \dots \circ \frac{\partial}{\partial \lambda} \circ \frac{1}{\psi_{\frac{r}{t}}'(\lambda)} \right\}}_{N \text{ times}} \{ [1 - \chi_0(\lambda)] a_\ell(\lambda) \}$$

is an inhomogeneous symbol of order  $\frac{n-1}{2} - \sigma - N\alpha$ , according to Lemma 3.2 (ii), Lemma 3.2 (i), (23) and (7). Hence

$$|k_{t,\ell}^{\sigma,\infty}(r)| \lesssim (1 + \ell)^\gamma t^{-N},$$

provided that  $N > \frac{n+1-2\sigma}{2\alpha}$ . By taking  $N = \max\{2, \lfloor \frac{n+1-2\sigma}{2\alpha} \rfloor + 1\}$  and by summing up over  $\ell$ , we conclude that

$$|k_t^\sigma(r)| \lesssim t^{-\frac{3}{2}} r e^{-\rho r} \quad (28)$$

when  $t \geq r \geq 1$ .

*Subcase 1.1.2.* Consider next the case where  $\lambda_1 \geq 1$ , that is case (2) in Figure 2. Given  $0 < \beta < 1$  and a bump function  $\chi_1 \in C_c^\infty(\mathbb{R})$  such that

$$\chi_1 \equiv \begin{cases} 1 & \text{on } [\beta, \frac{1}{\beta}], \\ 0 & \text{outside } (\frac{\beta}{2}, \frac{2}{\beta}), \end{cases}$$

let us now split up

$$a_\ell(\lambda) = \underbrace{\chi_1(\lambda_1^{-1}\lambda) a_\ell(\lambda)}_{= a_\ell^1(\lambda)} + \underbrace{[1 - \chi_1(\lambda_1^{-1}\lambda)] a_\ell(\lambda)}_{= a_\ell^\infty(\lambda)}$$

and

$$k_{t,\ell}^\sigma(r) = k_{t,\ell}^{\sigma,1}(r) + k_{t,\ell}^{\sigma,\infty}(r)$$

accordingly. On the one hand, by using Lemma 3.8 with  $L = 2$ , together with Lemma 3.2 (i), (23) and (7), we estimate

$$|k_{t,\ell}^{\sigma,1}(r)| \lesssim (1+\ell)^\gamma t^{-\frac{1}{2}} \lambda_1^{\frac{n+1-\alpha}{2}-\sigma} \asymp (1+\ell)^\gamma t^{-\frac{n-2\sigma}{2(\alpha-1)}} r^{\frac{n-2\sigma}{2(\alpha-1)}-\frac{1}{2}}.$$

On the other hand, after  $N$  integrations by parts based on (27),

$$k_{t,\ell}^{\sigma,\infty}(r) = \left(\frac{i}{t}\right)^N \int_{-\infty}^{+\infty} e^{it\psi_{\frac{r}{t}}(\lambda)} a_\ell^\infty(\lambda) \, d\lambda,$$

with

$$a_\ell^\infty(\lambda) = \underbrace{\left\{ \frac{\partial}{\partial \lambda} \circ \frac{1}{\psi_{\frac{r}{t}}(\lambda)} \circ \dots \circ \frac{\partial}{\partial \lambda} \circ \frac{1}{\psi_{\frac{r}{t}}(\lambda)} \right\}}_{N \text{ times}} \left\{ [1 - \chi_1(\lambda_1^{-1}\lambda)] a_\ell(\lambda) \right\}.$$

As

$$|a_\ell^\infty(\lambda)| \lesssim (1+\ell)^\gamma (|\lambda| + \lambda_1)^{-N(\alpha-1)} (1+|\lambda|)^{\frac{n-1}{2}-\sigma-N},$$

according to Lemma 3.2 (ii), Lemma 3.2 (i), (23) and (7), we have

$$\int_{|\lambda| \leq \lambda_1} |a_\ell^\infty(\lambda)| \, d\lambda \lesssim (1+\ell)^\gamma \lambda_1^{-N(\alpha-1)}$$

and

$$\int_{|\lambda| \geq \lambda_1} |a_\ell^\infty(\lambda)| \, d\lambda \lesssim (1+\ell)^\gamma \lambda_1^{\frac{n+1}{2}-\sigma-N\alpha},$$

provided that  $N > \frac{n+1}{2} - \sigma$ . Hence

$$|k_{t,\ell}^{\sigma,\infty}(r)| \lesssim (1+\ell)^\gamma t^{-N} \lambda_1^{-N(\alpha-1)} \asymp (1+\ell)^\gamma r^{-N}.$$

By summing up over  $\ell$ , we conclude that

$$|k_t^\sigma(r)| \lesssim t^{-\frac{n-2\sigma}{2(\alpha-1)}} r^{\frac{n-2\sigma}{2(\alpha-1)}-\frac{1}{2}} e^{-\rho r} \quad (29)$$

when  $t > 0$  and  $r \geq \max\{1, t\}$ .

**Subcase 1.2.** Assume that  $0 < \alpha < 1$ . The analysis of (22) depends again on the size of  $\frac{r}{t}$ .

*Subcase 1.2.1.* Firstly, if  $\frac{r}{t} \geq 2\theta_0$  (see Figure 3, case ①), the phase  $\psi_{\frac{r}{t}}$  has no stationary point and, after  $N$  integrations by parts based on (27), (22) becomes

$$k_{t,\ell}^\sigma(r) = \left(\frac{i}{r}\right)^N \int_{-\infty}^{+\infty} e^{it\psi_{\frac{r}{t}}(\lambda)} \underbrace{\left\{ \frac{\partial}{\partial \lambda} \circ \frac{r}{t\psi_{\frac{r}{t}}(\lambda)} \circ \dots \circ \frac{\partial}{\partial \lambda} \circ \frac{r}{t\psi_{\frac{r}{t}}(\lambda)} \right\}}_{N \text{ times}} a_\ell(\lambda) \, d\lambda,$$

where the amplitude is an inhomogeneous symbol of order  $\frac{n-1}{2} - \sigma - N$ , according this time to (18). Thus

$$|k_{t,\ell}^\sigma(r)| \lesssim (1+\ell)^\gamma r^{-N},$$

provided that  $N > \frac{n+1}{2} - \sigma$ , and

$$|k_t^\sigma(r)| \lesssim r^{-N} e^{-\rho r},$$

after summing up over  $\ell$ .

*Subcase 1.2.2.* Secondly, assume that  $\theta_0/2 \leq \frac{r}{t} \leq 2\theta_0$  (see Figure 3, case ②) and let  $0 < c_1 < 1 < c_2 < \sqrt{3}$  such that  $\theta(c_1\lambda_0) = \theta_0/2 = \theta(c_2\lambda_0)$ . Then all stationary points of the phase  $\psi_{\frac{r}{t}}$  are contained in  $[c_1\lambda_0, c_2\lambda_0]$ , according to Lemma 3.3. Let us split up (24) in (22) and (25) accordingly, where  $\chi_0 \in C_c^\infty(\mathbb{R})$  is a bump function such that  $\chi_0 = 1$  on a neighborhood of  $[c_1\lambda_0, c_2\lambda_0]$  and  $\text{supp } \chi_0 \subset (0, \sqrt{3}\lambda_0)$ . We estimate again

$$|k_{t,\ell}^{\sigma,0}(r)| \lesssim (1+\ell)^\gamma t^{-\frac{1}{3}},$$

by using Lemma 3.8, this time with  $L = 3$ , and

$$|k_{t,\ell}^{\sigma,\infty}(r)| \lesssim (1+\ell)^\gamma t^{-N},$$

by performing  $N$  integrations by parts based on (27). In conclusion,

$$|k_t^\sigma(r)| \lesssim r^{-\frac{1}{3}} e^{-\rho r},$$

as  $t$  and  $r$  are comparable under the present assumptions.

*Subcase 1.2.3.* Thirdly, in the remaining case  $0 < \frac{r}{t} < \theta_0/2$  (see Figure 3, case (3)), the phase  $\psi_{\frac{r}{t}}$  has two stationary points:  $\lambda_1 \in (0, c_1 \lambda_0)$  and  $\lambda_2 \in (c_2 \lambda_0, +\infty)$ . We shall isolate these two points by means of bump functions. Let  $\chi_0 \in C_c^\infty(\mathbb{R})$  and  $\chi_2 \in C_c^\infty(\mathbb{R})$  such that

$$\begin{cases} \chi_0 = 1 & \text{on } [-1, b_1 \lambda_0] \\ \text{supp } \chi_0 \subset [-2, b_2 \lambda_0] \end{cases} \quad \text{and} \quad \begin{cases} \chi_2 = 1 & \text{on } [b_3^{-1}, b_3] \\ \text{supp } \chi_2 \subset [b_4^{-1}, b_4] \end{cases} \quad (30)$$

where  $c_1 < b_1 < b_2 < 1 < b_3 < b_4 < c_2$ . Then  $\chi_0$  and  $\chi_2(\lambda_2^{-1} \cdot)$  are smooth bump functions around  $\lambda_1$  and  $\lambda_2$  respectively, whose supports are disjoint and don't contain  $\lambda_0$ . This follows indeed from the inequalities

$$b_2 \lambda_0 < \lambda_0 < b_4^{-1} c_2 \lambda_0 < b_4^{-1} \lambda_2.$$

Let us split up

$$\int_{-\infty}^{+\infty} d\lambda = \int_{-\infty}^{+\infty} \chi_0(\lambda) d\lambda + \int_{-\infty}^{+\infty} \chi_2(\lambda_2^{-1} \lambda) d\lambda + \int_{-\infty}^{+\infty} [1 - \chi_0(\lambda) - \chi_2(\lambda_2^{-1} \lambda)] d\lambda$$

in (22) and

$$k_{t,\ell}^\sigma(r) = k_{t,\ell}^{\sigma,0}(r) + k_{t,\ell}^{\sigma,2}(r) + k_{t,\ell}^{\sigma,\infty}(r) \quad (31)$$

accordingly. We estimate each term as we did in Subcase 1.1, using Lemma 3.3 instead of Lemma 3.2. This way, we obtain

$$\begin{cases} |k_{t,\ell}^{\sigma,0}(r)| \lesssim (1+\ell)^\gamma t^{-\frac{3}{2}} r, \\ |k_{t,\ell}^{\sigma,2}(r)| \lesssim (1+\ell)^\gamma t^{-\frac{2\sigma-n}{2(1-\alpha)}} r^{\frac{2\sigma-n}{2(1-\alpha)} - \frac{1}{2}}, \\ |k_{t,\ell}^{\sigma,\infty}(r)| \lesssim (1+\ell)^\gamma t^{-N}, \end{cases} \quad (32)$$

provided that  $\alpha N > \frac{n+1}{2} - \sigma$ , hence

$$|k_t^\sigma(r)| \lesssim \left(\frac{r}{t}\right)^{\min\left\{\frac{3}{2}, \frac{2\sigma-n}{2(1-\alpha)}\right\}} r^{-\frac{1}{2}} e^{-\rho r}. \quad (33)$$

**Remark 3.9.** *All results so far, which have been proved under the assumption  $r \geq 1$ , hold actually for  $r \geq r_0$  with  $r_0 > 0$  fixed.*

*3.1.2. Case 2 - small spatial scale.* If  $r$  is bounded above, let us say by 1, we use two expressions of the spherical functions  $\varphi_\lambda(r)$ , namely Harish-Chandra integral formula (4), with  $H(a_{\pm r} k) \in [-r, r]$  and Stanton-Tomas-Ionescu formula (9) (see also Lemma 2.3).

**Subcase 2.1.** Assume that  $1 < \alpha < 2$  and  $0 \leq r \leq 1$ ,  $t > 0$  (see Figure 2, cases (3) – (4) – (5)).

*Subcase 2.1.1.* Consider first the range  $r \leq t^{\frac{1}{\alpha}}$  (see Figure 2, cases (3) – (4)). By using (4), (13) becomes

$$k_t^\sigma(r) = \text{const.} \int_K e^{-\rho H(a_{-r} k)} \tilde{k}_t^\sigma(H(a_{-r} k)) dk, \quad (34)$$

where

$$\tilde{k}_t^\sigma(H) = \int_{-\infty}^{+\infty} |\mathbf{c}(\lambda)|^{-2} (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} e^{it(\lambda^2 + \rho^2)^{\alpha/2} - i\lambda H} d\lambda. \quad (35)$$



We estimate (35) when  $H \in [-r, r]$  by resuming the analysis in Subcase 1.1 and by dealing separately with the cases  $|H| \leq 1 \leq t$  and  $|H| \leq t^{\frac{1}{\alpha}} \leq 1$ . Let us elaborate.

• If  $|H| \leq 1 \leq t$  (see Figure 2, case (3)), the stationary point  $\lambda_1$  of the phase (14), with  $R = \frac{H}{t} \in [-1, 1]$ , remains bounded, according to Lemma 3.2, say  $|\lambda_1| \leq c$  for some constant  $c > 0$ . Let us split up

$$\int_{-\infty}^{+\infty} d\lambda = \int_{-\infty}^{+\infty} \chi\left(\frac{\lambda}{2c}\right) d\lambda + \int_{-\infty}^{+\infty} [1 - \chi\left(\frac{\lambda}{2c}\right)] d\lambda$$

in (35) and

$$\tilde{k}_t^\sigma(H) = \tilde{k}_t^{\sigma,0}(H) + \tilde{k}_t^{\sigma,\infty}(H) \quad (36)$$

accordingly, where  $\chi \in C_c^\infty(\mathbb{R})$  is an even bump function such that

$$0 \leq \chi \leq 1, \quad \chi = 1 \text{ on } [-1, 1], \quad \chi = 0 \text{ outside of } (-2, 2).$$

On the one hand, after an integration by parts based on (26), the first term in (36) becomes

$$\tilde{k}_t^{\sigma,0}(H) = C i \frac{H}{t} \int_{-\infty}^{+\infty} e^{it\psi_{\frac{H}{t}}(\lambda)} a_0(\lambda) d\lambda - \frac{C}{t} \int_{-\infty}^{+\infty} e^{it\psi_{\frac{H}{t}}(\lambda)} \frac{\partial}{\partial \lambda} a_0(\lambda) d\lambda,$$

where

$$a_0(\lambda) = \chi\left(\frac{\lambda}{2c}\right) \frac{\Gamma(i\lambda + \rho)}{\Gamma(i\lambda + 1)} \frac{\Gamma(-i\lambda + \rho)}{\Gamma(-i\lambda)} (\lambda^2 + \rho^2)^{1 - \frac{\alpha}{2} - \frac{\sigma}{2}}$$

is a smooth function with compact support. By using Lemma 3.8 with  $L = 2$ , we deduce that

$$|\tilde{k}_t^{\sigma,0}(H)| \lesssim t^{-\frac{3}{2}}. \quad (37)$$

On the other hand, after  $N$  integrations by parts based on

$$e^{it\psi_{\frac{H}{t}}(\lambda)} = \frac{-i}{t\psi'_{\frac{H}{t}}(\lambda)} \frac{\partial}{\partial \lambda} e^{it\psi_{\frac{H}{t}}(\lambda)}, \quad (38)$$

the second term in (36) becomes

$$\tilde{k}_t^{\sigma,\infty}(H) = C \left(\frac{i}{t}\right)^N \int_{-\infty}^{+\infty} e^{it\psi_{\frac{H}{t}}(\lambda)} a_\infty(\lambda) d\lambda,$$

where

$$a_\infty(\lambda) = \underbrace{\left\{ \frac{\partial}{\partial \lambda} \circ \frac{1}{\psi'_{\frac{H}{t}}(\lambda)} \circ \dots \circ \frac{\partial}{\partial \lambda} \circ \frac{1}{\psi'_{\frac{H}{t}}(\lambda)} \right\}}_{N \text{ times}} \left\{ [1 - \chi\left(\frac{\lambda}{2c}\right)] |\mathbf{c}(\lambda)|^{-2} (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} \right\}$$

is an inhomogeneous symbol of order  $n - 1 - \sigma - N\alpha$ , according to Lemma 3.2. Hence

$$|\tilde{k}_t^{\sigma,\infty}(H)| \lesssim t^{-N}, \quad (39)$$

provided that  $N > \frac{n-\sigma}{\alpha}$ . By taking  $N = \max\{2, \lfloor \frac{n-\sigma}{\alpha} \rfloor + 1\}$ , we obtain finally the bound  $O(t^{-\frac{3}{2}})$  for (35) and hence

$$|k_t^\sigma(r)| \lesssim t^{-\frac{3}{2}} e^{-\rho r} \quad (40)$$

when  $r \leq 1 \leq t$ .

• We proceed similarly in the case  $|H| \leq t^{\frac{1}{\alpha}} \leq 1$  (see Figure 2, case (4)), with the following few differences. The stationary point  $\lambda_1$  of the phase (14), with  $R = \frac{H}{t} \in [-t^{-\frac{\alpha-1}{\alpha}}, t^{-\frac{\alpha-1}{\alpha}}]$ , satisfies now  $|\lambda_1| \leq ct^{-1/\alpha}$ , for some constant  $c > 0$ . After splitting up

$$\int_{-\infty}^{+\infty} d\lambda = \int_{-\infty}^{+\infty} \chi\left(\frac{t^{1/\alpha}\lambda}{2c}\right) d\lambda + \int_{-\infty}^{+\infty} [1 - \chi\left(\frac{t^{1/\alpha}\lambda}{2c}\right)] d\lambda$$

in (35), the contribution of the first integral is estimated easily, while the contribution of the second integral is handled as above. Specifically, as  $|\mathbf{c}(\lambda)|^{-2} \lesssim (1 + \lambda)^{n-1}$ , we have

on the one hand

$$|\tilde{k}_t^{\sigma,0}(H)| \lesssim \int_{|\lambda| \leq 4ct^{-1/\alpha}} (1+\lambda)^{n-1-\sigma} d\lambda \lesssim t^{-\frac{n-\sigma}{\alpha}}. \quad (41)$$

On the other hand, after  $N$  integrations by parts based on (38), with  $N > \frac{n-\sigma}{\alpha}$ , we get

$$|\tilde{k}_t^{\sigma,\infty}(H)| \lesssim t^{-N} \int_{|\lambda| \geq 2ct^{-1/\alpha}} |\lambda|^{n-1-\sigma-N\alpha} d\lambda \lesssim t^{-\frac{n-\sigma}{\alpha}}. \quad (42)$$

In conclusion, (35) and hence (13) are  $O(t^{-\frac{n-\sigma}{\alpha}})$ , when  $r \leq t^{\frac{1}{\alpha}} \leq 1$ .

*Subcase 2.1.2.* Consider next the range  $0 < t^{\frac{1}{\alpha}} \leq r \leq 1$  (see Figure 2, case (5)). According to Lemma 3.2, there exists  $c > 0$  such that the stationary point of the phase (14), with  $R = \frac{r}{t} \geq 1$ , satisfies  $\lambda_1 > c(\frac{r}{t})^{1/(\alpha-1)}$ . Let us split up

$$\int_{-\infty}^{+\infty} d\lambda = \int_{-\infty}^{+\infty} \chi_0(r\lambda) d\lambda + \int_{-\infty}^{+\infty} \chi_1(\lambda_1^{-1}\lambda) d\lambda + \int_{-\infty}^{+\infty} \chi_\infty(\lambda) d\lambda \quad (43)$$

in (13) and

$$k_t^\sigma(r) = k_t^{\sigma,0}(r) + k_t^{\sigma,1}(r) + k_t^{\sigma,\infty}(r) \quad (44)$$

accordingly, where  $\chi_0, \chi_1 \in C_c^\infty(\mathbb{R})$  are even functions such that

$$\begin{cases} 0 \leq \chi_0 \leq 1, \chi_0 = 1 \text{ on } [-\frac{c}{8}, \frac{c}{8}], \text{ supp } \chi_0 \subset [-\frac{c}{4}, \frac{c}{4}], \\ 0 \leq \chi_1 \leq 1, \chi_1 = 1 \text{ on } [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2], \text{ supp } \chi_1 \subset [-4, -\frac{1}{4}] \cup [\frac{1}{4}, 4], \end{cases}$$

and  $\chi_\infty(\lambda) = 1 - \chi_0(r\lambda) - \chi_1(\lambda_1^{-1}\lambda)$ . Notice that the cutoff functions  $\chi_0(r \cdot)$  and  $\chi_1(\lambda_1^{-1} \cdot)$  have disjoint supports. By using  $|\mathbf{c}(\lambda)|^{-2} \lesssim (1+\lambda)^{n-1}$  and  $|\varphi_\lambda(r)| \leq 1$ , we estimate easily

$$|k_t^{\sigma,0}(r)| \lesssim \int_{|\lambda| \leq c_1/4r} (1+\lambda)^{n-\sigma-1} d\lambda \lesssim r^{-(n-\sigma)}.$$

Let us turn to the last two terms in (44). By using the small scale asymptotics (9) with  $\Lambda = \frac{c}{8}$ , we obtain the expressions

$$\begin{aligned} k_t^{\sigma,1}(r) &= 2C \overbrace{\int_{-\infty}^{+\infty} \chi_1(\lambda_1^{-1}\lambda) |\mathbf{c}(\lambda)|^{-2} (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} b_M(-\lambda, r) e^{it\psi_{\frac{r}{t}}(\lambda)} d\lambda}^I \\ &+ C \underbrace{\int_{-\infty}^{+\infty} \chi_1(\lambda_1^{-1}\lambda) |\mathbf{c}(\lambda)|^{-2} (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} R_M(\lambda, r) e^{it(\lambda^2 + \rho^2)^{\alpha/2}} d\lambda}_{II} \end{aligned} \quad (45)$$

and

$$\begin{aligned} k_t^{\sigma,\infty}(r) &= 2C \overbrace{\int_{-\infty}^{+\infty} \chi_\infty(\lambda) |\mathbf{c}(\lambda)|^{-2} (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} b_M(-\lambda, r) e^{it\psi_{\frac{r}{t}}(\lambda)} d\lambda}^{III} \\ &+ C \underbrace{\int_{-\infty}^{+\infty} \chi_\infty(\lambda) |\mathbf{c}(\lambda)|^{-2} (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} R_M(\lambda, r) e^{it(\lambda^2 + \rho^2)^{\alpha/2}} d\lambda}_{IV}. \end{aligned} \quad (46)$$

The main contribution arises from the first integral, which is estimated by using Lemma 3.8 with  $L = 2$ , together with Lemma 3.2 (i) and (10). This way we obtain

$$|I| \lesssim t^{-\frac{1}{2}} r^{-\frac{n-1}{2}} \lambda_1^{\frac{n-2\sigma}{2} - \frac{\alpha-1}{2}} \asymp t^{-\frac{n-2\sigma}{2(\alpha-1)}} r^{\frac{n-2\sigma}{2(\alpha-1)} - \frac{n}{2}}.$$

On the other hand, by using (11) with  $M > \frac{n+1}{2} + \sigma$ , we estimate

$$\begin{aligned} |III| &\lesssim r^{-\frac{n-1}{2}-M} \int_{\frac{1}{4}\lambda_1 \leq |\lambda| \leq 4\lambda_1} |\lambda|^{\frac{n-1}{2}-\sigma-M} d\lambda \\ &\asymp r^{-\frac{n-1}{2}-M} \lambda_1^{\frac{n-2\sigma}{2} - \frac{2M-1}{2}} \asymp \underbrace{\left(\frac{t}{r^\alpha}\right)^{\frac{2M-1}{2(\alpha-1)}}}_{\leq 1} t^{-\frac{n-2\sigma}{2(\alpha-1)}} r^{\frac{n-2\sigma}{2(\alpha-1)} - \frac{n}{2}} \end{aligned}$$

and  $|IV| \lesssim r^{-\frac{n-1}{2}-M} \int_{|\lambda| \geq c_1/8r} |\lambda|^{\frac{n-1}{2}-\sigma-M} d\lambda \asymp r^{-\frac{n-1}{2}-M} r^{-\frac{n+1}{2}+\sigma+M} = r^{-(n-\sigma)},$

Finally, the third integral in (46) is estimated by performing  $N$  integrations by parts based on (27), with  $N > \frac{n+1}{2} - \sigma$ , by using Lemma 3.2, together with (10), and by splitting up the integral

$$\int_{-\infty}^{+\infty} d\lambda = \int_{|\lambda| < \lambda_1} d\lambda + \int_{|\lambda| \geq \lambda_1} d\lambda.$$

This way we obtain

$$\begin{aligned} |III| &\lesssim t^{-N} r^{-\frac{n-1}{2}} \lambda_1^{-(\alpha-1)N} \underbrace{\int_{\frac{c_1}{8r} \leq |\lambda| < \lambda_1} |\lambda|^{\frac{n-1}{2}-\sigma-N} d\lambda}_{\lesssim r^{N+\sigma-\frac{n+1}{2}}} + t^{-N} r^{-\frac{n-1}{2}} \underbrace{\int_{|\lambda| \geq \lambda_1} |\lambda|^{\frac{n-1}{2}-\sigma-\alpha N} d\lambda}_{\lesssim \lambda_1^{\frac{n+1}{2}-\sigma-\alpha N}} \\ &\lesssim r^{-(n-\sigma)} + \underbrace{\left(\frac{t}{r^\alpha}\right)^{\frac{2N-1}{2(\alpha-1)}}}_{\leq 1} t^{-\frac{n-2\sigma}{2(\alpha-1)}} r^{\frac{n-2\sigma}{2(\alpha-1)} - \frac{n}{2}}. \end{aligned}$$

As

$$r^{-(n-\sigma)} = (r^\alpha)^{-\frac{1}{\alpha-1}(\frac{n}{2}-\sigma)} r^{+\frac{1}{\alpha-1}(\frac{n}{2}-\sigma)-\frac{n}{2}} \leq t^{-\frac{n-2\sigma}{2(\alpha-1)}} r^{\frac{n-2\sigma}{2(\alpha-1)} - \frac{n}{2}},$$

we conclude that

$$|k_t^\sigma(r)| \lesssim t^{-\frac{n-2\sigma}{2(\alpha-1)}} r^{\frac{n-2\sigma}{2(\alpha-1)} - \frac{n}{2}}.$$

**Subcase 2.2.** Assume that  $0 < \alpha < 1$  and  $0 \leq r \leq 1$ ,  $t > 0$  (see Figure 3, cases (4) – (5) – (6)).

*Subcase 2.2.1* (range (4) in Figure 3). Assume that  $0 \leq r < 1 \leq t$ , hence  $\frac{r}{t} < 1$ . Recall that (17) reaches its maximum  $\theta_0$  at  $\lambda_0 = \frac{\rho}{\sqrt{1-\alpha}} > 0$ . As in Subcase 1.2, let  $0 < c_1 < 1 < c_2 < \sqrt{3}$  such that  $\theta(c_1\lambda_0) = \theta_0/2 = \theta(c_2\lambda_0)$ . We may assume that  $\frac{r}{t} < \theta_0/2$  by reducing the range  $0 \leq r < r_0$ , according to Remark 3.9. Let us estimate  $k_t^\sigma(r)$  by considering again several cases.

- Assume first that  $r = 0$ . According to Lemma 3.3 (iv), the phase (14), with  $R = 0$ , has a single stationary point at the origin. Given an even bump function  $\chi_0 \in C_c^\infty(\mathbb{R})$  such that

$$\begin{cases} \chi_0 = 1 & \text{on } [-b_1\lambda_0, b_1\lambda_0] \\ \text{supp } \chi_0 \subset [-b_2\lambda_0, b_2\lambda_0] \end{cases}$$

where  $c_1 < b_1 < b_2 < 1$ , let us split up the integral as in (24) and the kernel

$$k_t^\sigma(0) = k_t^{\sigma,0}(0) + k_t^{\sigma,\infty}(0) \quad (47)$$

accordingly. On the one hand, after an integration by parts based on (26), we obtain

$$k_t^{\sigma,0}(0) = \frac{iC}{\alpha} t^{-1} \int_{-\infty}^{+\infty} e^{it(\lambda^2+\rho^2)^{\alpha/2}} \frac{\partial}{\partial \lambda} \left\{ \chi_0(\lambda) (\lambda^2+\rho^2)^{1-\frac{\alpha+\sigma}{2}} \lambda^{-1} |\mathbf{c}(\lambda)|^{-2} \right\} d\lambda$$

and deduce that

$$|k_t^{\sigma,0}(0)| \lesssim t^{-\frac{3}{2}}$$

by applying Lemma 3.8 with  $L = 2$ . On the other hand, after performing  $N$  integrations by parts based on (26), we obtain

$$k_t^{\sigma, \infty}(0) = C \left( \frac{i}{\alpha t} \right)^N \int_{-\infty}^{+\infty} e^{it(\lambda^2 + \rho^2)^{\alpha/2}} \times \underbrace{\left\{ \frac{\partial}{\partial \lambda} \circ \lambda^{-1} (\lambda^2 + \rho^2)^{1 - \frac{\alpha}{2}} \right\}^N \left\{ [1 - \chi_0(\lambda)] (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} |\mathbf{c}(\lambda)|^{-2} \right\}}_{O(|\lambda|^{n - \sigma - \alpha N - 1})} d\lambda,$$

which is  $O(t^{-N})$  if  $N > \frac{n - \sigma}{\alpha}$ . As a first conclusion, we obtain

$$|k_t^\sigma(0)| \lesssim t^{-\frac{3}{2}}$$

when  $r = 0$  and  $t \geq 1$ .

• Assume next that  $0 < r < \min\{1, r_0\}$ . According to Lemma 3.3 (iii), the phase (14), with  $R = \frac{r}{t} \in (0, \theta_0/2)$ , has two stationary points:  $\lambda_1 \in (0, c_1 \lambda_0)$ , which is comparable to  $\frac{r}{t}$ , and  $\lambda_2 \in (c_2 \lambda_0, +\infty)$ , which is comparable to  $(r/t)^{-1/(1-\alpha)}$ . Given another even bump function  $\chi_2 \in C_c^\infty(\mathbb{R})$  such that

$$\begin{cases} \chi_2 = 1 \text{ on } [-b_3, -b_3^{-1}] \cup [b_3^{-1}, b_3], \\ \text{supp } \chi_2 \subset [-b_4, -b_4^{-1}] \cup [b_4^{-1}, b_4], \end{cases}$$

where  $1 < b_3 < b_4 < c_2$ , let us split up

$$\begin{aligned} \int_{-\infty}^{+\infty} d\lambda &= \int_{-\infty}^{+\infty} \chi_0(\lambda) d\lambda + \int_{|\lambda| < \lambda_2} [1 - \chi_0(\lambda) - \chi_2(\lambda_2^{-1} \lambda)] d\lambda \\ &+ \int_{-\infty}^{+\infty} \chi_2(\lambda_2^{-1} \lambda) d\lambda + \int_{|\lambda| > \lambda_2} [1 - \chi_2(\lambda_2^{-1} \lambda)] d\lambda \end{aligned}$$

in (13) and

$$k_t^\sigma(r) = k_t^{\sigma, 0}(r) + k_t^{\sigma, 1}(r) + k_t^{\sigma, 2}(r) + k_t^{\sigma, \infty}(r) \quad (48)$$

accordingly. As far as the first term in (48) is concerned, we obtain

$$|k_t^{\sigma, 0}(r)| \lesssim t^{-\frac{3}{2}}$$

either by using the phase (14) with  $R = 0$  as above, or by using (4) and the phase (14) with  $R = \frac{H}{t}$  as in the proof of (37). We claim that the second term in (48) is  $O(t^{-N})$ , for every integer  $N > \frac{n - \sigma}{\alpha}$ . This is achieved by substituting (4) in

$$k_t^{\sigma, 1}(r) = C \int_{b_1 \lambda_0 < |\lambda| < b_3^{-1} \lambda_0} [1 - \chi_0(\lambda) - \chi_2(\lambda_2^{-1} \lambda)] \times e^{it(\lambda^2 + \rho^2)^{\alpha/2}} \varphi_\lambda(r) (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} |\mathbf{c}(\lambda)|^{-2} d\lambda$$

and by performing  $N$  integrations by parts based on (38) with  $H = H(a_{-r} k) \in [-r, r]$ , after observing that the stationary points of the phase (14), with  $R = \frac{H}{t}$ , remain outside  $\{\lambda \in \mathbb{R} \mid \lambda_1 < |\lambda| < \lambda_2\}$ . Let us turn to the third term in (48), which reads

$$k_t^{\sigma, 2}(r) = 2C \overbrace{\int_{b_4^{-1} \lambda_2 \leq |\lambda| \leq b_4 \lambda_2} \chi_2(\lambda_2^{-1} \lambda) e^{it\psi_r(\lambda)} b_M(-\lambda, r) (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} |\mathbf{c}(\lambda)|^{-2} d\lambda}^I + C \overbrace{\int_{b_4^{-1} \lambda_2 \leq |\lambda| \leq b_4 \lambda_2} \chi_2(\lambda_2^{-1} \lambda) e^{it(\lambda^2 + \rho^2)^{\alpha/2}} R_M(\lambda, r) (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} |\mathbf{c}(\lambda)|^{-2} d\lambda}^H$$

after substituting (9). We estimate the main term

$$|I| \lesssim r^{-\frac{n-1}{2}} t^{-\frac{1}{2}} \lambda_2^{\frac{n+1-\alpha}{2}-\sigma} \asymp t^{-\frac{1}{2}} \frac{2\sigma-n}{1-\alpha} r^{\frac{1}{2}} \frac{2\sigma-n-n}{1-\alpha}$$

by using Lemma 3.8 with  $L = 2$ , together with Lemma 3.3 (iii) and (10), and the remainder

$$|III| \lesssim r^{-\frac{n-1}{2}-M} \lambda_2^{\frac{n+1}{2}-\sigma-M} \asymp \underbrace{\left(\frac{r^\alpha}{t}\right)^{\frac{2M-1}{2(1-\alpha)}}}_{\leq 1} t^{-\frac{1}{2}} \frac{2\sigma-n}{1-\alpha} r^{\frac{1}{2}} \frac{2\sigma-n-n}{1-\alpha} \quad \forall M > \frac{n+1}{2} - \sigma$$

by using (11). Consider finally the last term in (48), which reads similarly

$$k_t^{\sigma, \infty}(r) = 2C \overbrace{\int_{|\lambda| > b_3 \lambda_2} [1 - \chi_2(\lambda_2^{-1} \lambda)] e^{it\psi_{\frac{r}{t}}(\lambda)} b_M(-\lambda, r) (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} |\mathbf{c}(\lambda)|^{-2} d\lambda}^{III} \\ + C \underbrace{\int_{|\lambda| > b_3 \lambda_2} [1 - \chi_2(\lambda_2^{-1} \lambda)] e^{it(\lambda^2 + \rho^2)^{\alpha/2}} R_M(\lambda, r) (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} |\mathbf{c}(\lambda)|^{-2} d\lambda}_{IV}.$$

We estimate

$$|III| \lesssim r^{-\frac{n-1}{2}} t^{-N} \lambda_2^{\frac{n+1}{2}-\sigma-\alpha N} \asymp \underbrace{\left(\frac{r^\alpha}{t}\right)^{\frac{2N-1}{2(1-\alpha)}}}_{\leq 1} t^{-\frac{1}{2}} \frac{2\sigma-n}{1-\alpha} r^{\frac{1}{2}} \frac{2\sigma-n-n}{1-\alpha} \quad \forall N > \frac{n+1}{2} - \sigma$$

by performing  $N$  integrations by parts based on (27) and by using Lemma 3.3 (iii) and (v), together with (10), and

$$|IV| \lesssim r^{-\frac{n-1}{2}-M} \lambda_2^{\frac{n+1}{2}-\sigma-M} \asymp \underbrace{\left(\frac{r^\alpha}{t}\right)^{\frac{2M-1}{2(1-\alpha)}}}_{\leq 1} t^{-\frac{1}{2}} \frac{2\sigma-n}{1-\alpha} r^{\frac{1}{2}} \frac{2\sigma-n-n}{1-\alpha} \quad \forall M > \frac{n+1}{2} - \sigma$$

by using (11). In conclusion,

$$|k_t^\sigma(r)| \lesssim t^{-\frac{3}{2}} + t^{-\frac{1}{2}} \frac{2\sigma-n}{1-\alpha} r^{\frac{1}{2}} \frac{2\sigma-n-n}{1-\alpha}$$

when  $r < \min\{1, r_0\}$  and  $t \geq 1$ .

**Remark 3.10.** We obtain in particular  $|k_t^\sigma(r)| \lesssim t^{-\frac{3}{2}} + t^{-\frac{n}{2}}$  when  $\sigma = (1 - \frac{\alpha}{2})n$ .

*Subcase 2.2.2* (range ⑤ in Figure 3). Assume that  $0 \leq r < 1$  and  $0 < t < 1$  satisfy  $r < t^{1/\alpha}$ , hence  $\frac{r}{t} < 1$ . We argue as in Subcase 2.2.1 with a few differences. By reducing  $0 \leq r < \min\{1, r_0\}$ , we may assume again that  $\frac{r}{t} < \theta_0/2$ .

- When  $r = 0$ , we split again  $k_t^\sigma(0)$  as in (47). On the one hand, we estimate trivially

$$|k_t^{\sigma, 0}(0)| \lesssim 1.$$

On the other hand, we estimate

$$|k_t^{\sigma, \infty}(0)| \lesssim t^{-\frac{n-\sigma}{\alpha}}$$

by splitting up

$$\int_{-\infty}^{+\infty} [1 - \chi(\lambda)] d\lambda = \int_{-\infty}^{+\infty} [1 - \chi(\lambda)] \chi\left(\frac{t^{1/\alpha}\lambda}{2}\right) d\lambda + \int_{-\infty}^{+\infty} [1 - \chi\left(\frac{t^{1/\alpha}\lambda}{2}\right)] d\lambda.$$

More precisely, the contribution of the first integral is trivially bounded by

$$\int_{1 \lesssim |\lambda| \lesssim t^{-1/\alpha}} |\lambda|^{n-\sigma-1} d\lambda \lesssim t^{-\frac{n-\sigma}{\alpha}}$$

while the contribution of the second integral is bounded, after  $N > \frac{n-\sigma}{\alpha}$  integrations by parts based on (26), by

$$\begin{aligned} & t^{-N} \int_{|\lambda| \gtrsim t^{-1/\alpha}} \overbrace{\left[ \left\{ \frac{\partial}{\partial \lambda} \circ \lambda^{-1} (\lambda^2 + \rho^2)^{1-\frac{\sigma}{2}} \right\}^N \left\{ \left[ 1 - \chi\left(\frac{t^{1/\alpha}\lambda}{2}\right) \right] (\lambda^2 + \rho^2)^{-\frac{\sigma}{2}} |\mathbf{c}(\lambda)|^{-2} \right\} \right]}^{\lesssim |\lambda|^{n-\sigma-\alpha N-1}} d\lambda \\ & \lesssim t^{-\frac{n-\sigma}{\alpha}}. \end{aligned}$$

As first conclusion, we obtain  $|k_t^\sigma(0)| \lesssim t^{-\frac{n-\sigma}{\alpha}}$  when  $r = 0$  and  $t \geq 1$ .

• When  $0 < r < \min\{1, r_0, t^{1/\alpha}\}$ , we split again  $k_t^\sigma(r)$  as in (44). This time, we estimate

$$|k_t^{\sigma,0}(r)| \lesssim 1$$

trivially and both  $k_t^{\sigma,2}(r)$ , *IV* by

$$t^{-\frac{1}{2}\frac{2\sigma-n}{1-\alpha}} r^{\frac{1}{2}\frac{2\sigma-n}{1-\alpha}-\frac{n}{2}},$$

as in Subcase 2.2.1. Finally, we split up

$$III = VII + VIII$$

according to

$$\begin{aligned} \int_{|\lambda| < \lambda_2} [1 - \chi_0(\lambda) - \chi_2(\lambda_2^{-1}\lambda)] d\lambda &= \int_{|\lambda| < \lambda_2} [1 - \chi_0(\lambda) - \chi_2(\lambda_2^{-1}\lambda)] \chi(t^{1/\alpha}\lambda) d\lambda \\ &+ \int_{|\lambda| < \lambda_2} [1 - \chi_0(\lambda) - \chi_2(\lambda_2^{-1}\lambda)] [1 - \chi(t^{1/\alpha}\lambda)] d\lambda. \end{aligned}$$

On the one hand, we estimate

$$|VII| \lesssim \int_{1 \lesssim |\lambda| \lesssim t^{-1/\alpha}} |\lambda|^{n-\sigma-1} d\lambda \lesssim t^{-\frac{n-\sigma}{\alpha}},$$

by using  $|\varphi_\lambda(r)| \leq 1$ , and

$$|VIII| \lesssim t^{-N} \int_{|\lambda| \gtrsim t^{-1/\alpha}} |\lambda|^{n-\sigma-\alpha N-1} d\lambda \lesssim t^{-\frac{n-\sigma}{\alpha}},$$

by substituting (4), by performing  $N > \frac{n-\sigma}{\alpha}$  integrations by parts based on (38) and by using (19).

In conclusion,

$$|k_t^\sigma(r)| \lesssim t^{-\frac{n-\sigma}{\alpha}} + t^{-\frac{1}{2}\frac{2\sigma-n}{1-\alpha}} r^{\frac{1}{2}\frac{2\sigma-n}{1-\alpha}-\frac{n}{2}}$$

when  $0 \leq r < \min\{1, r_0, t^{1/\alpha}\}$ .

**Remark 3.11.** We obtain in particular  $|k_t^\sigma(r)| \lesssim t^{-\frac{n}{2}}$  when  $\sigma = (1 - \frac{\alpha}{2})n$ .

*Subcase 2.2.3* (range ⑥ in Figure 3). Assume that  $0 < t \leq r^\alpha < 1$ . Notice that

$$\left(\frac{r}{t}\right)^{-\frac{1}{1-\alpha}} \leq t^{-1/\alpha}.$$

According to Lemma 3.3, there exists  $c > 0$  such that all critical of the phase (14), with  $R = \frac{r}{t} \in [0, \theta_0]$ , are contained in  $[0, cR^{-1/(1-\alpha)}] \subset [0, ct^{-1/\alpha}]$ . Given a smooth even bump function  $\chi$  on  $\mathbb{R}$  such that  $\chi = 1$  on  $[-1, 1]$  and  $\text{supp } \chi \subset [-2, 2]$ , let us split up

$$\int_{-\infty}^{+\infty} d\lambda = \int_{-\infty}^{+\infty} \chi\left(\frac{t^{1/\alpha}\lambda}{2c}\right) d\lambda + \int_{-\infty}^{+\infty} [1 - \chi\left(\frac{t^{1/\alpha}\lambda}{2c}\right)] d\lambda$$

and

$$k_t^\sigma(r) = k_t^{\sigma,0}(r) + k_t^{\sigma,\infty}(r)$$

accordingly. On the one hand, we estimate

$$|k_t^{\sigma,0}(r)| \lesssim \int_{|\lambda| \lesssim t^{-1/\alpha}} (|\lambda|+1)^{n-\sigma-1} d\lambda \lesssim t^{-\frac{n-\sigma}{\alpha}}$$

by using  $|\varphi_\lambda(r)| \leq 1$ . On the other hand, we estimate

$$|k_t^{\sigma,\infty}(r)| \lesssim t^{-N} \int_{|\lambda| \gtrsim t^{-1/\alpha}} |\lambda|^{n-\sigma-\alpha N-1} d\lambda \lesssim t^{-\frac{n-\sigma}{\alpha}},$$

by substituting (4), by performing  $N > \frac{n-\sigma}{\alpha}$  integrations by parts based on (38) and by using (19).

In conclusion,

$$|k_t^\sigma(r)| \lesssim t^{-\frac{n-\sigma}{\alpha}}$$

when  $0 < t \leq r^\alpha < 1$ . □

**Remark 3.12.** We obtain again  $|k_t^\sigma(r)| \lesssim t^{-\frac{n}{2}}$  when  $\sigma = (1 - \frac{\alpha}{2})n$ .

#### 4. DISPERSIVE AND STRICHARTZ ESTIMATES ON HYPERBOLIC SPACES

We now deduce from Theorem 3.5 dispersive and Strichartz estimates, following the standard strategy of Ginibre and Velo, combined with the Kunze–Stein phenomenon, as in [AP09, IS09, APV11, APV12, AP14, APV15]. More precisely, we will use the following version of the Kunze–Stein phenomenon (see [APV11, Thm. 4.2]).

**Lemma 4.1.** *Let  $2 \leq q < \infty$ . Then there exists a positive constant  $C$  such that, for every  $f \in L^{q'}(\mathbb{H}^n)$  and for every measurable radial function  $k$  on  $\mathbb{H}^n$ ,*

$$\|f * k\|_{L^q} \leq C \|f\|_{L^{q'}} \left[ \int_0^{+\infty} |k(r)|^{\frac{q}{2}} \varphi_0(r) (\sinh r)^{n-1} dr \right]^{\frac{2}{q}}.$$

**Remark 4.2.** Notice that

$$\varphi_0(r) (\sinh r)^{n-1} \asymp \frac{r^{n-1}}{(1+r)^{n-2}} e^{\frac{n-1}{2}r} \quad \forall r \geq 0. \quad (49)$$

The proof of the Strichartz estimates uses the standard  $TT^*$  argument, Young’s inequality extended to weak type spaces (see for instance [Gra14, Thm. 1.4.25]), the Christ–Kiselev Lemma [CK01] for the non-endpoint estimates and the Bourgain or Keel–Tao trick [KT98] for the endpoint estimates, as in [APV11, Sect. 6] or [AP14, Sect. 5] for instance. Therefore we generally omit proofs of the Strichartz estimates in this paper.

##### 4.1. Case $1 < \alpha < 2$ .

**Theorem 4.3** (Dispersive estimates). *Let  $1 < \alpha < 2$ ,  $0 \leq \sigma \leq \frac{n}{2}$ ,  $2 < q \leq \infty$  and set*

$$m = \max \left\{ 2 \frac{n-\sigma}{\alpha}, \frac{n-2\sigma}{\alpha-1} \right\} = \begin{cases} 2 \frac{n-\sigma}{\alpha} & \text{if } \sigma \geq (1 - \frac{\alpha}{2})n, \\ \frac{n-2\sigma}{\alpha-1} & \text{if } \sigma \leq (1 - \frac{\alpha}{2})n. \end{cases} \quad (50)$$

*Then the following dispersive estimates hold for  $t \in \mathbb{R}^*$ :*

$$\|(-\Delta)^{-\left(\frac{1}{2}-\frac{1}{q}\right)\sigma} e^{it(-\Delta)^{\alpha/2}}\|_{L^{q'} \rightarrow L^q} \lesssim |t|^{-\left(\frac{1}{2}-\frac{1}{q}\right)m}$$

*for  $t$  small, say  $0 < |t| < 1$ , and*

$$\|(-\Delta)^{-\sigma/2} e^{it(-\Delta)^{\alpha/2}}\|_{L^{q'} \rightarrow L^q} \lesssim |t|^{-\frac{3}{2}}$$

*for  $t$  large, say  $|t| \geq 1$ .*

**Remark 4.4.** *These estimates become in particular*

$$\|(-\Delta)^{-\sigma/2} e^{it(-\Delta)^{\alpha/2}}\|_{L^{q'} \rightarrow L^q} \lesssim \begin{cases} |t|^{-(\frac{1}{2}-\frac{1}{q})n} & \text{if } 0 < |t| < 1 \\ |t|^{-\frac{3}{2}} & \text{if } |t| \geq 1 \end{cases}$$

when  $\sigma = (\frac{1}{2} - \frac{1}{q})(2 - \alpha)n$ .

*Proof.* All estimates rely on the kernel estimates in Theorem 3.5 (i) extended straightforwardly to the vertical strip  $0 \leq \operatorname{Re} \sigma \leq \frac{n}{2}$  in  $\mathbb{C}$ . More precisely, the estimates for  $t$  small are obtained by interpolation for an analytic family of operators (see for instance [SW71, Ch. V, Thm. 4.1] or [Gra14, Thm. 1.3.7]) between

$$\|(-\Delta)^{-\sigma/2} e^{it(-\Delta)^{\alpha/2}}\|_{L^2 \rightarrow L^2} = 1 \quad \forall \sigma \in i\mathbb{R} \quad (51)$$

and

$$\|(-\Delta)^{-\sigma/2} e^{it(-\Delta)^{\alpha/2}}\|_{L^1 \rightarrow L^\infty} \lesssim \begin{cases} |t|^{-\frac{n-\operatorname{Re} \sigma}{\alpha}} & \text{if } \operatorname{Re} \sigma \geq (1 - \frac{\alpha}{2})n, \\ |t|^{-\frac{1}{2} \frac{n-2\operatorname{Re} \sigma}{\alpha-1}} & \text{if } \operatorname{Re} \sigma \leq (1 - \frac{\alpha}{2})n. \end{cases}$$

While the estimate for  $t$  large follows from

$$\left[ \int_0^{+\infty} |k_t^\sigma(r)|^{\frac{q}{2}} \varphi_0(r) (\sinh r)^{n-1} dr \right]^{\frac{2}{q}} \lesssim |t|^{-\frac{3}{2}}$$

and from Lemma 4.1. □

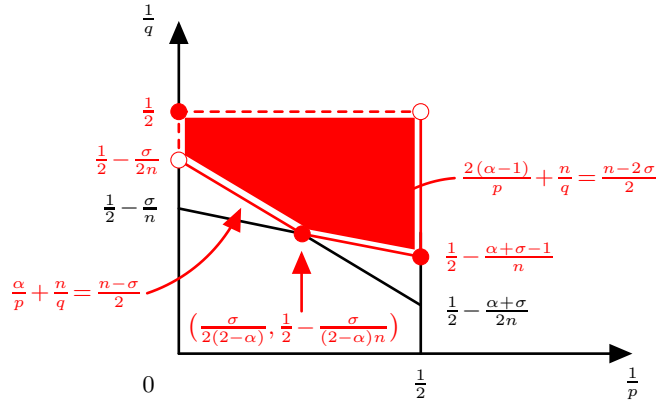


FIGURE 4. Admissible region  $\mathcal{R}_\alpha$  for fixed  $1 < \alpha < 2$  and  $0 \leq \sigma \leq \frac{n}{2}$

**Theorem 4.5** (Strichartz inequalities). *Assume that  $1 < \alpha < 2$  and let  $I = (-T, +T)$  be an open interval with  $T > 0$ . Then the following Strichartz inequalities hold for solutions of (1) on  $I \times \mathbb{H}^n$ :*

$$\|(-\Delta_x)^{-\sigma/4} u(t, x)\|_{L^p(I; L^q(\mathbb{H}^n))} \leq C \left\{ \|f\|_{L^2(\mathbb{H}^n)} + \|(-\Delta_x)^{\tilde{\sigma}/4} F(t, x)\|_{L^{\tilde{p}}(I; L^{\tilde{q}}(\mathbb{H}^n))} \right\}.$$

Here  $(\frac{1}{p}, \frac{1}{q}, \sigma)$  and  $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}}, \tilde{\sigma})$  belong to the admissible region (see Figure 4)

$\mathcal{R}_\alpha = \left\{ \left( \frac{1}{p}, \frac{1}{q}, \sigma \right) \in \left( 0, \frac{1}{2} \right) \times \left[ 0, \frac{1}{2} \right) \times \left[ 0, \frac{n}{2} \right] \mid \frac{\alpha}{p} + \frac{n}{q} \geq \frac{n-\sigma}{2} \text{ and } \frac{2(\alpha-1)}{p} + \frac{n}{q} \geq \frac{n-2\sigma}{2} \right\} \cup \left\{ \left( 0, \frac{1}{2}, 0 \right) \right\}$ , where  $m$  is defined in (50). Moreover  $C \geq 0$  depends on  $\alpha$ ,  $(p, q, \sigma)$  and  $(\tilde{p}, \tilde{q}, \tilde{\sigma})$  but not on  $T > 0$  and  $u$ .

**Remark 4.6.** *As already observed for the Schrödinger equation ( $\alpha = 2$ ) and for the wave equation ( $\alpha = 1$ ), the admissible region is much larger on hyperbolic spaces than on Euclidean spaces.*



4.2. **Case**  $0 < \alpha < 1$ . From the kernel estimates in Theorem 3.5 (ii), we deduce similarly the following inequalities.

**Theorem 4.7** (Dispersive estimates). (i) *Assume that  $n \geq 3$ ,  $0 < \alpha < 1$ ,  $(1 - \frac{\alpha}{2})n \leq \sigma \leq n$  and  $2 < q \leq \infty$ . Then the following dispersive estimates hold:*

$$\|(-\Delta)^{-\left(\frac{1}{2} - \frac{1}{q}\right)\sigma} e^{it(-\Delta)^{\alpha/2}}\|_{L^{q'} \rightarrow L^q} \lesssim |t|^{-\left(\frac{1}{2} - \frac{1}{q}\right)2\frac{n-\sigma}{\alpha}} \quad (52)$$

for  $t$  small, say  $0 < |t| < 1$ , and

$$\|(-\Delta)^{-\sigma/2} e^{it(-\Delta)^{\alpha/2}}\|_{L^{q'} \rightarrow L^q} \lesssim |t|^{-\frac{3}{2}} \quad (53)$$

for  $t$  large, say  $|t| \geq 1$ .

(ii) *In dimension  $n = 2$ , the small time estimate is the same, while the large time estimate reads*

$$\begin{aligned} \|(-\Delta)^{-\sigma/2} e^{it(-\Delta)^{\alpha/2}}\|_{L^{q'} \rightarrow L^q} &\lesssim |t|^{-\min\{\frac{3}{2}, \frac{\sigma-1}{1-\alpha}\}} \\ &= \begin{cases} |t|^{-\frac{\sigma-1}{1-\alpha}} & \text{if } 0 < \alpha \leq \frac{1}{3} \text{ and } 2 - \alpha \leq \sigma \leq 2, \\ |t|^{-\frac{\sigma-1}{1-\alpha}} & \text{if } \frac{1}{3} \leq \alpha < 1 \text{ and } 2 - \alpha \leq \sigma \leq \frac{5-3\alpha}{2}, \\ |t|^{-\frac{3}{2}} & \text{if } \frac{1}{3} \leq \alpha < 1 \text{ and } \frac{5-3\alpha}{2} \leq \sigma \leq 2. \end{cases} \end{aligned}$$

The estimate (53) is fine for  $q$  large but, as  $q$  tends to 2, the smoothness factor  $(-\Delta)^{-\sigma/2}$  doesn't vanish, as might be expected. Let us therefore refine (53) as follows.

**Corollary 4.8.** *Assume that  $n \geq 3$ ,  $0 < \alpha < 1$ ,  $(1 - \frac{\alpha}{2})n \leq \sigma \leq n$  and let  $2 < q \leq Q \leq \infty$ . Then the following dispersive estimate holds for  $t$  large, say  $|t| \geq 1$ :*

$$\|(-\Delta)^{-\frac{1/2-1/q}{1/2-1/Q}\frac{\sigma}{2}} e^{it(-\Delta)^{\alpha/2}}\|_{L^{q'} \rightarrow L^q} \lesssim |t|^{-\frac{1/2-1/q}{1/2-1/Q}\frac{3}{2}}. \quad (54)$$

*Proof.* This result is obtained by interpolation between the estimate (51) for  $q = 2$  and  $\sigma \in i\mathbb{R}$ , and the estimate (53) for  $q = Q$  and  $\sigma \in \mathbb{C}$  with  $(1 - \frac{\alpha}{2})n \leq \text{Re } \sigma \leq n$ .  $\square$

**Theorem 4.9** (Strichartz inequalities). *Assume that  $n \geq 3$ ,  $0 < \alpha < 1$  and let  $I = (-T, +T)$  be an open interval with  $T > 0$ . Then the following Strichartz inequalities hold for solutions of (1) on  $I \times \mathbb{H}^n$ :*

$$\|(-\Delta_x)^{-\frac{\sigma}{2}} u(t, x)\|_{L^p(I; L^q(\mathbb{H}^n))} \leq C \left\{ \|f\|_{L^2(\mathbb{H}^n)} + \|(-\Delta_x)^{\frac{\tilde{\sigma}}{2}} F(t, x)\|_{L^{\tilde{p}}(I; L^{\tilde{q}}(\mathbb{H}^n))} \right\}.$$

Here

$$\begin{cases} \sigma = \sigma(\beta, q) = \left(\frac{1}{2} - \frac{1}{q}\right)\left(1 - \frac{\beta}{2}\right)n \\ \tilde{\sigma} = \sigma(\tilde{\beta}, \tilde{q}) = \left(\frac{1}{2} - \frac{1}{\tilde{q}}\right)\left(1 - \frac{\tilde{\beta}}{2}\right)n \end{cases} \quad \text{with } \beta, \tilde{\beta} \in [0, \alpha]$$

and  $(\frac{1}{p}, \frac{1}{q}, \beta)$ ,  $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}}, \tilde{\beta})$  belong to the following admissible region (see Figures 5 and 6):

$$\mathcal{R}_\alpha = \left\{ \left(\frac{1}{p}, \frac{1}{q}, \beta\right) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \times [0, \alpha] \mid \left(\frac{1}{2} - \frac{1}{q}\right)\frac{\beta}{\alpha}\frac{n}{2} \leq \frac{1}{p} \leq \left(\frac{1}{2} - \frac{1}{q}\right)\frac{2-\beta}{2-\alpha}\frac{3}{2} \right\} \setminus \left\{ \left(\frac{1}{2}, 0, \frac{2}{n}\alpha\right) \right\}. \quad (55)$$

Moreover  $C \geq 0$  depends on  $n$ ,  $\alpha$ ,  $(p, q, \beta)$  and  $(\tilde{p}, \tilde{q}, \tilde{\beta})$  but not on  $T > 0$  and  $u$ .

**Remark 4.10.** *In this statement, the interval  $[0, \alpha]$  must be actually reduced to the smaller interval  $[0, \hat{\alpha}]$ , where*

$$\hat{\alpha} = \frac{6\alpha}{(2-\alpha)n+3\alpha} \begin{cases} = \alpha & \text{if } n = 3, \\ \in (0, \alpha) & \text{if } n > 3. \end{cases}$$

When  $n > 3$  and  $\hat{\alpha} < \beta \leq \alpha$ , we have indeed  $\frac{\beta}{\alpha}\frac{n}{2} > \frac{2-\beta}{2-\alpha}\frac{3}{2}$  and the admissibility condition boils down to the trivial endpoint  $p = \infty$ ,  $q = 2$ .

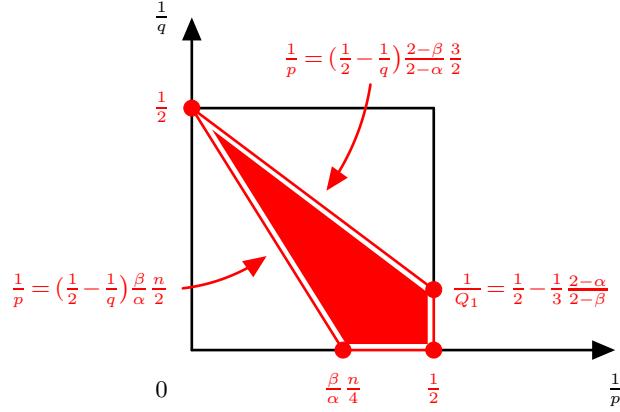


FIGURE 5. Admissible region  $\mathcal{R}_\alpha$  for  $n \geq 3$  and  $0 < \alpha < 1$ ,  $0 < \beta < \frac{2}{n}\alpha$  fixed.

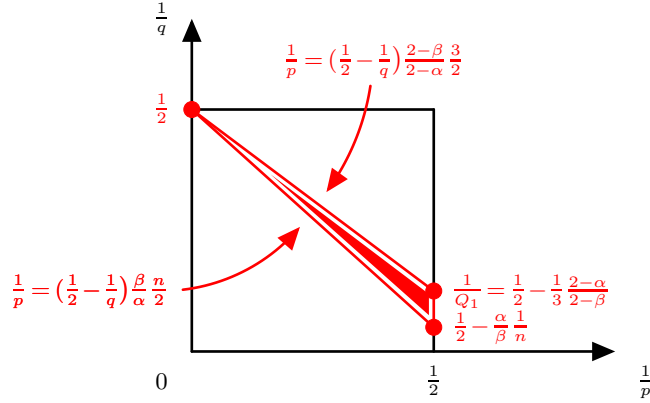


FIGURE 6. Admissible region  $\mathcal{R}_\alpha$  for  $n \geq 3$  and  $0 < \alpha < 1$ ,  $\frac{2}{n}\alpha < \beta < \hat{\alpha}$  fixed.

*Proof.* Referring to the proofs of [APV11, Thm. 6.3] and [AP14, Thm. 5.2], we shall be content to explain and comment the admissibility conditions

$$\begin{cases} \frac{1}{p} \geq \left(\frac{1}{2} - \frac{1}{q}\right) \frac{\beta}{\alpha} \frac{n}{2} \\ \frac{1}{p} \leq \left(\frac{1}{2} - \frac{1}{q}\right) \frac{2-\beta}{2-\alpha} \frac{3}{2} \\ \left(\frac{1}{p}, \frac{1}{q}, \beta\right) \neq \left(\frac{1}{2}, 0, \frac{2}{n}\alpha\right) \end{cases} \quad (56)$$

occurring in (55). Recall that the above-mentioned proofs consist mainly in estimating

$$\begin{aligned} & \left\| \int_{0 < |t-s| < 1} \| (-\Delta_x)^{-\sigma(\beta, q)} e^{i(t-s)(-\Delta_x)^{\alpha/2}} F(s, x) \|_{L_x^q} ds \right\|_{L_t^p} \\ & \lesssim \left\| \int_{-\infty}^{+\infty} \| F(s, x) \|_{L_x^{q'}} ds \right\|_{L_s^{p'}} \end{aligned} \quad (57)$$

and

$$\begin{aligned} & \left\| \int_{|t-s| \geq 1} \| (-\Delta_x)^{-\sigma(\beta, q)} e^{i(t-s)(-\Delta_x)^{\alpha/2}} F(s, x) \|_{L_x^q} ds \right\|_{L_t^p} \\ & \lesssim \left\| \int_{-\infty}^{+\infty} \| F(s, x) \|_{L_x^{q'}} ds \right\|_{L_s^{p'}}. \end{aligned} \quad (58)$$

On the one hand, we deduce (57) from the dispersive estimate (52), which yields

$$\| (-\Delta)^{-\sigma(\beta, q)} e^{i(t-s)(-\Delta)^{\alpha/2}} \|_{L^{q'} \rightarrow L^q} \lesssim |t-s|^{-\left(\frac{1}{2} - \frac{1}{q}\right) \frac{\beta}{\alpha} n},$$

and from Young's inequality (see for instance [Gra14, Thm. 1.4.25]) provided that  $(\frac{1}{2} - \frac{1}{q})\frac{\beta}{\alpha}n$  is  $< 1$  and  $\leq \frac{2}{p}$ . This way we obtain (57) under the assumptions

$$0 \leq \beta \leq \alpha, \quad 2 \leq p \leq \infty, \quad 2 < q \leq \infty, \quad \frac{1}{p} \geq (\frac{1}{2} - \frac{1}{q})\frac{\beta}{\alpha}\frac{n}{2}$$

and except for the endpoint

$$\left(\frac{1}{p}, \frac{1}{q}\right) = \begin{cases} \left(\frac{1}{2}, \frac{1}{2} - \frac{\alpha}{\beta}\frac{1}{n}\right) & \text{when } \beta > \frac{2}{n}\alpha, \\ \left(\frac{1}{2}, 0\right) & \text{when } \beta = \frac{2}{n}\alpha. \end{cases} \quad (59)$$

The first case is handled by the refined analysis in [KT98] while the second one is excluded.

On the other hand, we prove (58) under the assumptions

$$0 \leq \beta \leq \alpha, \quad 2 \leq p \leq \infty, \quad 2 < q \leq \infty, \quad \frac{1}{p} \leq (\frac{1}{2} - \frac{1}{q})\frac{2-\beta}{2-\alpha}\frac{3}{2}$$

by considering separately the ranges  $2 < q \leq Q_1$ ,  $Q_1 < q < Q_2$  and  $Q_2 \leq q \leq \infty$ , where

$$\frac{1}{Q_1} = \frac{1}{2} - \frac{1}{3}\frac{2-\alpha}{2-\beta} \in \left[\frac{1}{6}, \frac{1+\alpha}{6}\right] \quad \text{and} \quad \frac{1}{Q_2} = \frac{1}{2} - \frac{1}{2}\frac{2-\alpha}{2-\beta} \in \left[0, \frac{\alpha}{4}\right].$$

- Assume first that  $q \geq Q_2$ . Then

$$2\sigma(\beta, q) = \left(\frac{1}{2} - \frac{1}{q}\right)(2-\beta)n \geq \left(\frac{1}{2} - \frac{1}{Q_2}\right)(2-\beta)n = \left(1 - \frac{\alpha}{2}\right)n$$

and we deduce from (53) that (58) holds for every  $2 \leq p \leq \infty$ .

- Assume next that  $Q_1 < q < Q_2$ . Then

$$\frac{1/2 - 1/q}{1/2 - 1/Q_2} \frac{3}{2} > \frac{1/2 - 1/Q_1}{1/2 - 1/Q_2} \frac{3}{2} = 1$$

and we deduce from (54) with  $Q = Q_2$  that (58) holds for every  $2 \leq p \leq \infty$ .

- Assume finally that  $2 < q \leq Q_1$ . Then we obtain (58) under the assumption  $\frac{1}{p} \leq (\frac{1}{2} - \frac{1}{q})\frac{2-\beta}{2-\alpha}\frac{3}{2}$  by using (54) with  $Q = Q_2$  together with Young's inequality (see for instance [Gra14, Thm. 1.4.25]), except for the endpoint  $(p, q) = (2, Q_1)$ , where we use the refined analysis in [KT98]. Notice that this case is new compared to [AP09, APV12, AP14].

In conclusion, (57) and (58) hold under the conditions (56), which define a non-empty subset of  $[0, \frac{1}{2}] \times [0, \frac{1}{2}]$  provided that  $(\frac{1}{2} - \frac{1}{q})\frac{\beta}{\alpha}\frac{n}{2} \leq (\frac{1}{2} - \frac{1}{q})\frac{2-\beta}{2-\alpha}\frac{3}{2}$ , i.e.,  $\beta \leq \hat{\alpha}$ .  $\square$

**Remark 4.11.** *In dimension  $n = 2$ , Theorem 4.9 holds for the following admissible region:*

$$\begin{aligned} \mathcal{R}_\alpha &= \left\{ \left(\frac{1}{p}, \frac{1}{q}, \beta\right) \in \left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \times [0, \alpha] \mid \left(\frac{1}{2} - \frac{1}{q}\right)\frac{\beta}{\alpha} \leq \frac{1}{p} \leq \left(\frac{1}{2} - \frac{1}{q}\right) \min\left(\frac{3}{2}, \frac{1-\beta}{1-\alpha}\right)\frac{2-\beta}{2-\alpha} \right\} \\ &\quad \setminus \left\{ \left(\frac{1}{2}, 0, \alpha\right) \right\} \\ &= \begin{cases} \left\{ \left(\frac{1}{p}, \frac{1}{q}, \beta\right) \mid \left(\frac{1}{2} - \frac{1}{q}\right)\frac{\beta}{\alpha} \leq \frac{1}{p} \leq \left(\frac{1}{2} - \frac{1}{q}\right)\frac{3}{2}\frac{2-\beta}{2-\alpha} \right\} & \text{if } \alpha \in \left[\frac{1}{3}, 1\right) \text{ and } \beta \in \left[0, \frac{3\alpha-1}{2}\right], \\ \left\{ \left(\frac{1}{p}, \frac{1}{q}, \beta\right) \mid \left(\frac{1}{2} - \frac{1}{q}\right)\frac{\beta}{\alpha} \leq \frac{1}{p} \leq \left(\frac{1}{2} - \frac{1}{q}\right)\frac{1-\beta}{1-\alpha}\frac{2-\beta}{2-\alpha} \right\} \setminus \left\{ \left(\frac{1}{2}, 0, \alpha\right) \right\} & \text{if } \alpha \in \left(0, \frac{1}{3}\right] \text{ and } \beta \in [0, \alpha] \\ & \text{or if } \alpha \in \left[\frac{1}{3}, 1\right) \text{ and } \beta \in \left[\frac{3\alpha-1}{2}, \alpha\right]. \end{cases} \end{aligned}$$

For fixed  $\beta \in [0, \alpha)$ , the admissible set of couples  $(\frac{1}{p}, \frac{1}{q})$  looks like Figure 5, with

$$\begin{aligned} \frac{1}{Q_1} &= \frac{1}{2} - \frac{1}{2} \max\left\{\frac{2}{3}, \frac{1-\alpha}{1-\beta}\right\}\frac{2-\alpha}{2-\beta} \\ &= \begin{cases} \frac{1}{2} - \frac{1}{3}\frac{2-\alpha}{2-\beta} & \text{if } \alpha \in \left[\frac{1}{3}, 1\right) \text{ and } \beta \in \left[0, \frac{3\alpha-1}{2}\right], \\ \frac{1}{2} - \frac{1}{2}\frac{1-\alpha}{1-\beta}\frac{2-\alpha}{2-\beta} & \text{if } \alpha \in \left(0, \frac{1}{3}\right] \text{ and } \beta \in [0, \alpha] \text{ or if } \alpha \in \left[\frac{1}{3}, 1\right) \text{ and } \beta \in \left[\frac{3\alpha-1}{2}, \alpha\right], \end{cases} \end{aligned}$$

and, in the limit case  $\beta = \alpha$ , this set boils down to the diagonal

$$\left\{ \left(\frac{1}{p}, \frac{1}{q}\right) \in \left[0, \frac{1}{2}\right] \times \left(0, \frac{1}{2}\right] \mid \frac{1}{p} + \frac{1}{q} = \frac{1}{2} \right\}.$$

As a general observation, we would like to emphasize that the admissible range of exponents when the power of the Laplacian is below one, i.e., close to a very small diffusion, is smaller than the one for powers closer to the standard diffusion  $\alpha > 1$ . This is due to a combination of two effects: one is due to the necessary loss of derivatives  $\sigma$  which cannot be made arbitrarily small but also the behaviour of the kernel in this low diffusive case which is more similar to an Euclidean one. On the other hand, in the regime of higher diffusion, one observes a behaviour much more influenced by the negative curvature. From the point of view of nonlinear applications, this introduces substantial difficulties to prove well-posedness.

## 5. HOMOGENEOUS TREES

In this section, we consider the discrete analogs of hyperbolic spaces  $\mathbb{H}^n$  which are homogeneous trees and more precisely 0-hyperbolic space according to Gromov [Gro87]. Specifically, for  $Q \geq 2$ , a homogeneous tree of degree  $Q + 1$  is an infinite connected graph with no loops, in which every vertex is adjoint to  $Q + 1$  other vertices (see Figure 7). We denote by  $\mathbb{T}_Q$  the set of vertices in the homogeneous tree with  $Q + 1$  edges, equipped with the counting measure.

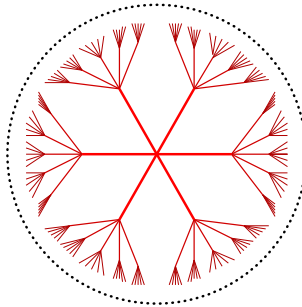


FIGURE 7. The homogeneous tree  $\mathbb{T}_5$

Recall that the combinatorial Laplacian on  $\mathbb{T}_Q$  is defined by

$$\Delta f(x) = \frac{1}{Q+1} \sum_{d(y,x)=1} f(y) - f(x)$$

and that its  $\ell^2$  spectrum is equal to  $[-\gamma_0 - 1, \gamma_0 - 1]$ , where  $\gamma_0 = \frac{2}{Q^{1/2} + Q^{-1/2}} \in (0, 1)$ . Here  $d(y, x)$  is the number of edges of the shortest path joining  $y$  and  $x$ . We refer to [Car73, FTM91, CMS98] for some basic tools of harmonic analysis on  $\mathbb{T}_Q$ .

Among earlier works about (1) on homogeneous trees, let us mention

- [Set98], which is devoted to the heat equation with continuous time associated with the Laplacian  $\Delta$  on  $\mathbb{T}_Q$ ,
- [Stó11], which is devoted to the heat equation with continuous time associated with the fractional Laplacian  $(-\Delta)^{\alpha/2}$  on  $\mathbb{T}_Q$ ,
- [MS99], which is devoted to the wave equation with continuous time associated with the shifted Laplacian  $\Delta + 1 - \gamma_0$  on  $\mathbb{T}_Q$ ,
- [Edd13a], which is devoted to the Schrödinger equation with continuous time associated with the Laplacian  $\Delta$  on  $\mathbb{T}_Q$ .

The equation (1) on  $\mathbb{T}_Q$  is solved and analyzed as the corresponding equation on  $\mathbb{H}^n$ . The main differences lie in the local (in time) analysis, which is trivial, and in the spectrum, which is compact. More precisely, we have again Duhamel's formula (3) where

$$e^{it(-\Delta)^{\alpha/2}} f = f * k_t$$

is the convolution operator defined by the radial kernel

$$k_t(r) = \text{const.} \int_0^{\tau/2} e^{it[1-\gamma(\lambda)]^{\alpha/2}} \varphi_\lambda(r) \frac{d\lambda}{|\mathbf{c}(\lambda)|^2}. \quad (60)$$

Here  $\tau = \frac{2\pi}{\log Q}$ ,  $\mathbf{c}(\lambda) = \frac{1}{Q^{1/2+Q^{-1/2}}} \frac{Q^{1/2+i\lambda} - Q^{-1/2-i\lambda}}{Q^{i\lambda} - Q^{-i\lambda}}$ ,  $\gamma(\lambda) = \frac{Q^{i\lambda} + Q^{-i\lambda}}{Q^{1/2+Q^{-1/2}}}$  and

$$\varphi_\lambda(r) = \mathbf{c}(\lambda) Q^{(-1/2+i\lambda)r} + \mathbf{c}(-\lambda) Q^{(-1/2-i\lambda)r} \quad (61)$$

is the spherical function of index  $\lambda \in \mathbb{C}$ . By substituting (61) in (60), we obtain

$$\begin{aligned} k_t(r) &= \text{const.} Q^{-r/2} \int_{\mathbb{R}/\tau\mathbb{Z}} \mathbf{c}(\lambda)^{-1} e^{it[1-\gamma(\lambda)]^{\alpha/2}} Q^{-i\lambda r} d\lambda \\ &= C Q^{-r/2} \int_{\mathbb{R}/2\pi\mathbb{Z}} \frac{\sin \lambda}{Q^{1/2} e^{i\lambda} - Q^{-1/2} e^{-i\lambda}} e^{it\psi(\lambda)} e^{-ir\lambda} d\lambda, \end{aligned} \quad (62)$$

where  $\psi(\lambda) = [1 - \gamma_0 \cos \lambda]^{\alpha/2}$ .

**Lemma 5.1** (Stationary phase analysis). *Let  $0 < \alpha < 2$  and assume that  $t > 0$ .*

(i) *The phase function  $\psi$  has two stationary points on the circle  $\mathbb{R}/2\pi\mathbb{Z}$ :*

$$\lambda_1 = 2\pi\mathbb{Z} \quad \text{and} \quad \lambda_2 = \pi + 2\pi\mathbb{Z}.$$

*Moreover,  $\psi''(\lambda_1) > 0$  and  $\psi''(\lambda_2) < 0$ .*

(ii) *There exists  $M > 0$  such that the phase function  $\psi_{t,r}(\lambda) := t\psi(\lambda) - r\lambda$  has*

$$\begin{cases} \text{no stationary point} & \text{if } \frac{r}{t} > M, \\ \text{one stationary point } \lambda_0 & \text{if } \frac{r}{t} = M, \\ \text{two stationary points } \lambda_1, \lambda_2 & \text{if } \frac{r}{t} < M. \end{cases}$$

(iii) *Moreover, we have the following additional information about the last case. For every  $\varepsilon > 0$ , there exist open subsets  $U, V$  in  $\mathbb{R}/2\pi\mathbb{Z}$  and a constant  $c > 0$  such that, whenever  $0 \leq \frac{r}{t} \leq M - \varepsilon$ ,*

- $\{\lambda_1, \lambda_2\} \subset U \subset \bar{U} \subset V$ ,
- $|\psi'_{t,r}| \geq ct$  outside  $U$  and  $|\psi''_{t,r}| \geq ct$  on  $V$ .

*Proof.* Let us compute the first derivatives

$$\begin{aligned} \psi'(\lambda) &= \frac{\alpha}{2} \gamma_0 [1 - \gamma_0 \cos \lambda]^{\alpha/2-1} \sin \lambda, \\ \psi''(\lambda) &= \frac{\alpha}{2} \gamma_0 [1 - \gamma_0 \cos \lambda]^{\alpha/2-2} \left[ -\frac{\alpha}{2} \gamma_0 \cos^2 \lambda + \cos \lambda - \left(1 - \frac{\alpha}{2}\right) \gamma_0 \right]. \end{aligned} \quad (63)$$

Consider the expression

$$\theta(\lambda) = -\frac{\alpha}{2} \gamma_0 \cos^2 \lambda + \cos \lambda - \left(1 - \frac{\alpha}{2}\right) \gamma_0,$$

which occurs in (63) and which is a  $2\pi$ -periodic even function on  $\mathbb{R}$ , with

$$\theta(0) = 1 - \gamma_0 > 0, \quad \theta\left(\frac{\pi}{2}\right) = -\left(1 - \frac{\alpha}{2}\right) \gamma_0 < 0, \quad \text{and} \quad \theta(\pi) = -1 - \gamma_0 < 0.$$

On  $[0, \pi]$ , the function  $\theta$  may increase before decreasing, as its derivatives

$$\theta'(\lambda) = [\alpha \gamma_0 \cos \lambda - 1] \sin \lambda$$

vanishes at  $\lambda = 0$ , at  $\lambda = \pi$  and at most once on  $(0, \frac{\pi}{2})$ . In particular,  $\theta$  and hence  $\psi''$

- vanishes at a single point  $\lambda_0$  in  $[0, \pi]$ , which belongs to  $(0, \frac{\pi}{2})$ ,
- is strictly positive on  $[0, \lambda_0)$ ,
- is strictly negative on  $(\lambda_0, \pi]$ .

Thus  $\psi'$ , which is a  $2\pi$ -periodic odd function on  $\mathbb{R}$ , increases (strictly) on  $[0, \lambda_0]$ , from  $\psi'(0) = 0$  to  $M = \psi'(\lambda_0) > 0$ , and decreases back on  $[\lambda_0, \pi]$ , from  $M$  to 0. Consequently, for every  $\mu \in [0, M)$ , the equation  $\psi'(\lambda) = \mu$  has exactly two solutions in  $(-\pi, \pi)$ :

$$\lambda_1(\mu) \in [0, \lambda_0) \quad \text{and} \quad \lambda_2(\mu) \in (\lambda_0, \pi].$$

Let  $\varepsilon \in (0, \frac{M}{3})$  and assume that  $\frac{r}{t} \leq M - 3\varepsilon$ . Then the phase function  $\psi_{t,r}$  has two stationary points in  $(-\pi, \pi]$ , namely  $\lambda_1 = \lambda_1(\frac{r}{t})$  and  $\lambda_2 = \lambda_2(\frac{r}{t})$ . Moreover,

- $U_1 = (-\lambda_1(\varepsilon), \lambda_1(M - 2\varepsilon)) \subset V_1 = (-\lambda_1(2\varepsilon), \lambda_1(M - \varepsilon))$  are neighborhoods of  $\lambda_1$  in  $(-\lambda_0, \lambda_0)$  such that  $|\psi'(\lambda) - \frac{r}{t}| \geq \varepsilon$  on  $[-\lambda_0, \lambda_0] \setminus U_1$  and  $\min_{\lambda \in \overline{V_1}} \psi''(\lambda) > 0$ ,
- $U_2 = (\lambda_2(M - 2\varepsilon), 2\pi - \lambda_2(\varepsilon)) \subset V_2 = (\lambda_2(M - \varepsilon), 2\pi - \lambda_2(2\varepsilon))$  are neighborhoods of  $\lambda_2$  in  $(\lambda_0, 2\pi - \lambda_0)$  such that  $|\psi'(\lambda) - \frac{r}{t}| \geq \varepsilon$  on  $[\lambda_0, 2\pi - \lambda_0] \setminus U_2$  and  $\min_{\lambda \in \overline{V_2}} |\psi''(\lambda)| > 0$ .

□

**Remark 5.2.** *In the limit case  $\alpha = 2$ , we have*

$$\psi(\lambda) = 1 - \gamma_0 \cos \lambda, \quad \psi'(\lambda) = \gamma_0 \sin \lambda, \quad \psi''(\lambda) = \gamma_0 \cos \lambda,$$

hence

$$\lambda_0 = \frac{\pi}{2}, \quad M = \gamma_0, \quad \lambda_1(\mu) = \arcsin \frac{\mu}{\gamma_0}, \quad \lambda_2(\mu) = \pi - \arcsin \frac{\mu}{\gamma_0}.$$

**Theorem 5.3** (Kernel estimate). *Assume that  $0 < \alpha \leq 2$ . Then the following kernel estimate holds:*

$$|k_t(r)| \lesssim Q^{-\frac{r}{2}} \quad \forall t \in \mathbb{R}^*, \forall r \in \mathbb{N}. \quad (64)$$

Moreover, there exists  $C > 0$  such that

$$|k_t(r)| \lesssim |t|^{-\frac{3}{2}} (1+r) Q^{-\frac{r}{2}} \quad (65)$$

if  $1+r \leq C|t|$ .

**Remark 5.4.**

- *In the limit case  $\alpha = 2$ , the slightly weaker estimate*

$$|k_t(r)| \lesssim \begin{cases} Q^{-r/2} & \text{if } 0 < |t| < 1 \\ |t|^{-3/2} (1+r)^2 Q^{-r/2} & \text{if } |t| \geq 1 \end{cases}$$

was obtained in [Edd13a, Prop. 3.1]. Note that a small error in [Edd13a, Prop. 3.1] was corrected in [Edd13b, Prop. 3.1].

- *The estimate (64) may be further improved, when  $|t|$  is large and  $\frac{r}{|t|}$  is bounded from below, but we will not need it.*

*Proof of Theorem 5.3.* Without loss of generality, we may assume that  $t > 0$ . The estimate (64) follows immediately from the expression (62), where the integrand is bounded. Let us improve (64) when  $t \geq 1$  and  $\frac{r}{t}$  is small, so that Lemma 5.1 (iii) applies. First, we perform an integration by parts based on

$$(\sin \lambda) e^{it\psi(\lambda)} = \frac{2}{\alpha\gamma_0} \frac{1}{t} [1 - \gamma_0 \cos \lambda]^{1-\alpha/2} \left(-i \frac{\partial}{\partial \lambda}\right) e^{it\psi(\lambda)}.$$

This way, (62) becomes

$$k_t(r) = C \frac{1}{t} Q^{-r/2} \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{i\psi_{t,r}(\lambda)} \{ra(\lambda) + ia'(\lambda)\} d\lambda, \quad (66)$$

where  $a(\lambda) = \frac{[1 - \gamma_0 \cos \lambda]^{1-\alpha/2}}{Q^{1/2} e^{i\lambda} - Q^{-1/2} e^{-i\lambda}}$  is bounded, as well as its derivatives. Next, we estimate (66) by stationary phase analysis based on Lemma 5.1 (iii). Specifically, given a smooth function  $\chi$  on  $\mathbb{R}/2\pi\mathbb{Z}$  such that  $\chi = 1$  on  $\overline{U}$  and  $\text{supp } \chi \subset V$ , we split up the integral in (66) as follows:

$$\int_{\mathbb{R}/2\pi\mathbb{Z}} d\lambda = \int_V \chi(\lambda) d\lambda + \int_{(\mathbb{R}/2\pi\mathbb{Z}) \setminus U} [1 - \chi(\lambda)] d\lambda.$$

On the one hand, the main estimate

$$\left| \int_V \chi(\lambda) e^{i\psi_{t,r}(\lambda)} \{ra(\lambda) + ia'(\lambda)\} d\lambda \right| \lesssim t^{-\frac{1}{2}}(1+r)$$

is obtained by applying Lemma 3.8 with  $L = 2$ . On the other hand, the remainder estimate

$$\left| \int_{(\mathbb{R}/2\pi\mathbb{Z}) \setminus U} [1 - \chi(\lambda)] e^{i\psi_{t,r}(\lambda)} \{ra(\lambda) + ia'(\lambda)\} d\lambda \right| \lesssim t^{-N}(1+r)$$

is obtained after  $N$  integrations by parts based on

$$e^{i\psi_{t,r}(\lambda)} = \frac{1}{\psi'_{t,r}(\lambda)} \left(-i \frac{\partial}{\partial \lambda}\right) e^{i\psi_{t,r}(\lambda)}.$$

This concludes the proof of (65).  $\square$

Let us turn to  $\ell^{q'}(\mathbb{T}_Q) \rightarrow \ell^{\tilde{q}}(\mathbb{T}_Q)$  mapping properties of the Schrödinger operator  $e^{it(-\Delta)^{\alpha/2}}$ . As in [Edd13a, Thm. 3.4 and Cor. 3.5], let us deduce the following result from Theorem 5.3.

**Corollary 5.5** (Dispersive estimate). *Let  $0 < \alpha \leq 2$  and  $2 < q, \tilde{q} \leq \infty$ . Then the following dispersive estimate holds for  $t \in \mathbb{R}^*$ :*

$$\|e^{it(-\Delta)^{\alpha/2}}\|_{\ell^{q'}(\mathbb{T}_Q) \rightarrow \ell^{\tilde{q}}(\mathbb{T}_Q)} \lesssim (1+|t|)^{-\frac{3}{2}}.$$

Similar, we can conclude from Theorem 5.3 and [Edd13a, Thm. 3.6] the following result for the inhomogeneous case.

**Corollary 5.6** (Strichartz estimates). *Let  $0 < \alpha \leq 2$  and  $I = (-T, +T)$  with  $T > 0$ . Then the following Strichartz estimates hold for solutions of (1) on  $I \times \mathbb{T}_Q$ :*

$$\|u(t, x)\|_{L^\infty(I, \ell^2(\mathbb{T}_Q))} + \|u(t, x)\|_{L^{\tilde{p}}(I, \ell^{\tilde{q}}(\mathbb{T}_Q))} \leq C \left\{ \|f\|_{\ell^2(\mathbb{T}_Q)} + \|F(t, x)\|_{L^{p'}(I, \ell^{q'}(\mathbb{T}_Q))} \right\}.$$

Here  $(\frac{1}{p}, \frac{1}{q})$  and  $(\frac{1}{\tilde{p}}, \frac{1}{\tilde{q}})$  belong to the square (see Figure 8)

$$\left[0, \frac{1}{2}\right] \times \left[0, \frac{1}{2}\right] \cup \left\{\left(0, \frac{1}{2}\right)\right\}$$

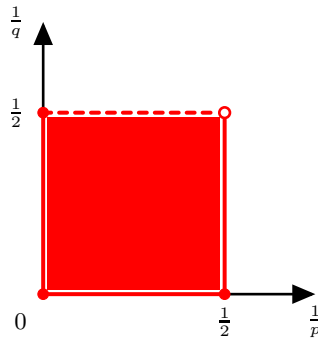


FIGURE 8. Admissible pairs for  $\mathbb{T}_Q$

and  $C \geq 0$  depends on  $\alpha$ ,  $(p, q)$ ,  $(\tilde{p}, \tilde{q})$  but not on  $T$  and  $u$ .

**Remark 5.7.** *Notice that, in the discrete setting and contrary to the continuous setting, the case  $\alpha = 1$  (half-wave equation) is similar to the general case  $0 < \alpha \leq 2$ .*

Let us mention that for the nonlinear Schrödinger (NLS) equation on homogenous trees

$$\begin{cases} i\partial_t u(x, t) + (-\Delta_x)^{\alpha/2} u(x, t) = \tilde{F}(u(x, t)) & (x, t) \in \mathbb{T}_Q \times \mathbb{R} \\ u(x, 0) = u_0 \end{cases} \quad (67)$$

where

$$\begin{cases} |\tilde{F}(u)| \lesssim |u|^\eta \\ |\tilde{F}(u) - \tilde{F}(v)| \lesssim \{|u|^{\eta-1} + |v|^{\eta-1}\}|u - v| \end{cases}$$

for some exponent  $\eta > 1$ , we get similar local and global well-posedness results as in [Edd13a, Thm. 4.1.] for  $\alpha = 2$ .

**Theorem 5.8.** *Let  $1 < \eta < \infty$ . Then the nonlinear Schrödinger equation (67) is*

- *locally well-posed for arbitrary initial data in  $\ell^2$ ,*
- *globally well-posed for small initial data in  $\ell^2$ ,*
- *globally well-posed for arbitrary initial data in  $\ell^2$  under the additional gauge invariant condition  $\text{Im}\{\tilde{F}(u)\bar{u}\} = 0$ .*

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