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# Time-delay Feedback Control of Fractional Chaotic Rössler Oscillator

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## Abstract:

In this research, we explore the utilization of the time-delayed feedback method to stabilize unstable steady states and aperiodic orbits within the chaotic fractional-order Rössler Oscillator. Employing the time-delay feedback control algorithm, we identify specific parameter ranges enabling the successful stabilization of unstable equilibria, considering variations in both feedback gain and time delay. Unlike previous research works where Caputo and Riemann-Liouville characterization of fractional derivatives are used, we are using Grünwald-Letnikov (GL) characterization because of its simplicity and ease of implementation and demonstrating the stability using the analytical and numerical analysis and plots of eigenvalues. Additionally, our analysis highlight the effectiveness of a sinusoidally modulated time delay in the control law, significantly expanding the stability region of steady states beyond the capabilities of the traditional time-delayed feedback scheme with a constant delay. Furthermore, the analysis of eigenvalues before and after applying the control strategy offers tangible insights into the system's stability dynamics.

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**Keywords:** Chaotic fractional-order Rössler Oscillator, Grünwald-Letnikov Characterization, Stabilization, Eigenvalues, Feedback gain, Time-delay.

## 1. INTRODUCTION

Fractional-order dynamical systems, such as Rössler, Chua, Lü, Chen, Duffing, and Lorenz, offer rich variability by incorporating memory and non-local interactions (Lovoie et al. (1976)). Fractional derivatives, characterized by methods like Caputo or Grünwald-Letnikov, enable control over past states' impact by adjusting the fractional derivative parameter ( $\alpha$ ) (Oliveira et al. (2014)). Varying  $\alpha$  allows analysis of chaotic and hyperchaotic behavior (Yousefpour et al. (2020)). The Caputo derivative and Grünwald-Letnikov are often approximated by finite differences of fractional order for their numerical application (Scherer et al. (2011)). The choice between Caputo and Grünwald-Letnikov (GL) characterizations for fractional derivatives in numerical applications varies. While Caputo is favored in some studies like (Gjurchinovski et al. (2010)) for its theoretical advantages, practical considerations often favor GL. (Betancur-Herrera and Muñoz-Galeano (2020)) demonstrated that GL significantly outperforms Caputo in terms of computational efficiency.

The idea in this work is to investigate a time-delayed feedback method, partially based on (Pyragas (2002)),

to regulate unstable equilibria and periodic orbits within the chaotic fractional-order Rössler system. Employing Grünwald-Letnikov characterization for fractional derivatives reduced time complexity and enhanced precision. The LevenbergMarquardt algorithm to analyze the characteristic equation post-control application is known to be efficient in solving nonlinear problems (Ranganathan (2004)), to ensures rapid convergence, adapting step sizes based on curvature, dampening noise-induced oscillations, and optimizing chaotic systems.

This paper is organized as follows. In section 2, a brief overview of Grünwald-Letnikov characterization for fractional derivatives is explained. In section 3, a short description of fractional Rössler Oscillator is explained. In section 4, time-delay control for constant time-delay and variable time delay is explained and necessary equations are demonstrated. In section 5, corresponding plots are shown and explained to demonstrate the chaos control and stabilization. In section 6, a summary of provided results are written and the open problems are also discussed.

## 2. FRACTIONAL DERIVATIVE CHARACTERIZATION

Fractional derivatives are mathematical operators that extend the concept of differentiation beyond integer orders. Unlike traditional derivatives, which indicate the rate of change of a function concerning a specific variable, fractional derivatives introduce the notion of fractional orders of differentiation, allowing for more in-depth and intricate analyses. In our study, the Grünwald-Letnikov (GL) characterization of fractional derivatives is utilized for comprehensive analysis.

The analytical formula for the Grünwald-Letnikov fractional derivative of a function  $f(t)$  at a point  $t_i$  with fractional order  $\alpha$  can be written as:

$$D_{GL}^\alpha f(t_i) = \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t^\alpha} \sum_{n=0}^{i-1} (-1)^n \binom{\alpha}{n} f(t_i - n\Delta t) \quad (1)$$

where  $\binom{\alpha}{n}$  is the generalized binomial coefficient Owolabi and Atangana (2019). It is further approximated by using a finite sum and a finite time step  $\Delta t = dt$ :

$$D_{GL}^\alpha f(t_i) \approx \frac{1}{dt^\alpha} \sum_{n=0}^{i-1} (-1)^n \binom{\alpha}{n} f(t_i - ndt) \quad (2)$$

The Grünwald-Letnikov characterization, an extension of the Euler method, incorporates fractional order binomial coefficients that are non-negative and defined recursively, enhancing its stability properties and computational efficiency. These coefficients serve as damping factors, ensuring the method's stability throughout computations (Scherer et al. (2011)). This approach, straightforward to implement numerically, proves accessible for analysis and computations.

## 3. FRACTIONAL CHAOTIC RÖSSLER OSCILLATOR

The Fractional Chaotic Rössler system is represented as follows:

$$\begin{aligned} D_{GL}^\alpha x &= -y - z \\ D_{GL}^\alpha y &= x + ay \\ D_{GL}^\alpha z &= b + z(x - c) \end{aligned} \quad (3)$$

Here,  $0 < \alpha < 1$  represents a real number, and  $D_0^\alpha$  denotes the fractional derivative, as per the Grünwald-Letnikov definition shown in equation (1). This dynamic system, characterized by the real parameters  $a, b, c$ , and  $\alpha$  serves as a fundamental model for investigating chaos phenomena, including bifurcations, control mechanisms, synchronization, and related studies (Čermák and Nechvátal (2018)). The traditional Rössler system, in its native state, displays chaotic and diverse dynamic behaviors. The introduction of fractional derivatives in the Rössler attractor results in even more intricate and complex dynamics. Exploring these dynamics provides profound insights into nonlinear systems and chaos theory.

Singular points are values of  $x, y$ , and  $z$  for which the time derivatives mentioned in either the classical or fractional Rössler oscillator are all equal to zero.

Hence, after setting the RHS of equation (3) to zero and checking the feasibility, we obtain the fixed points  $f_p^\pm$  as:

$$\left( x_{f_p^\pm}, y_{f_p^\pm}, z_{f_p^\pm} \right) = \left( \frac{c \pm \sqrt{c^2 - 4ab}}{2}, -\frac{-c \pm \sqrt{c^2 - 4ab}}{2a}, \frac{c \pm \sqrt{c^2 - 4ab}}{2a} \right).$$

The jacobian matrix (J) corresponding to equations (3) when they are equated to zero, can be expressed as:

$$J = \begin{bmatrix} 0 & -1 & -1 \\ 1 & a & 0 \\ z & 0 & x - c \end{bmatrix}$$

This matrix is used for the analysis of the eigenvalues before applying any control. For  $a = 0.5, b = 2, c = 4$ , and  $\alpha = 0.9$ , the eigenvalues for  $f_p^-$  are  $0.1802 + 0.9653i, 0.1802 - 0.9653i$ , and  $-3.5924$ . The eigenvalues for  $f_p^+$  are  $-0.0901 + 2.8976i, -0.0901 - 2.8976i$ , and  $0.4122$ .

## 4. TIME-DELAY CONTROL

We examine a nonlinear fractional-order dynamical system of  $n$  dimensions, employing a delayed feedback control method:

$$\begin{aligned} D_{GL}^{\alpha_1} x_1(t) &= f_1(\mathbf{x}(t)) + F_1(t) \\ D_{GL}^{\alpha_2} x_2(t) &= f_2(\mathbf{x}(t)) + F_2(t) \\ &\vdots \\ D_{GL}^{\alpha_n} x_n(t) &= f_n(\mathbf{x}(t)) + F_n(t) \end{aligned} \quad (4)$$

Here,

$$F_i(t) = \sum_{j=1}^n K_{ij} [x_j(t) - x_j(t - \tau)] \quad (5)$$

is the a delayed feedback component applied to the  $i^{th}$  component of the system, encompassing contributions from all system components (Gjurchinovski et al. (2010)). The feedback terms are characterized by gain factors  $K_{ij}$  and the system experiences a constant time delay denoted as  $\tau$ . Here,  $\mathbf{x} = (x_1, x_2, \dots, x_n)$  represents the state vector, while  $\mathbf{f} = (f_1, f_2, \dots, f_n)$  signifies the nonlinear vector field governing the dynamics of the unperturbed system. The notation  $D_{GL}^\alpha$  indicates the time fractional derivative according to the GL definition.

In this study, we have experimented three scenarios by applying the delayed feedback component with  $x$  component,  $y$  component and both  $x$  &  $y$  components of equation (3) and observed the effects. We did not consider to implement the control on the  $z$  component as dynamics of  $z$  is subject to the evolution of  $x$  and  $z$  comes to play a role only when  $x$  exceeds  $c$ .

By applying the feedback components on both  $x$  &  $y$  components of equation (3), we obtain:

$$\begin{aligned} D_{GL}^\alpha x(t) &= -y(t) - z(t) + K[x(t) - x(t - \tau)] \\ D_{GL}^\alpha y(t) &= x(t) + (a + K)y(t) - Ky(t - \tau) \\ D_{GL}^\alpha z(t) &= b + z(t)[x(t) - c] \end{aligned} \quad (6)$$

By applying the Laplace transform, the characteristic matrix ( $\Delta s$ ) can be expressed as:

$$\begin{bmatrix} (s^\alpha - K + Ke^{-\tau s}) & 1 & 1 \\ 1 & V & 0 \\ z & 0 & -(s^\alpha - (x - c)) \end{bmatrix} \quad (7)$$

Here,  $V = (-s^\alpha + a + K(1 - e^{-\tau s}))$ .

Similarly, the characteristic matrix ( $\Delta s$ ) when feedback components applied separately on  $x$  component can be expressed as:

$$\begin{bmatrix} (s^\alpha - K + Ke^{-\tau s}) & 1 & 1 \\ 1 & a - s^\alpha & 0 \\ -z & 0 & (s^\alpha - (x - c)) \end{bmatrix} \quad (8)$$

The characteristic matrix ( $\Delta s$ ) when feedback components applied separately on  $y$  component can be expressed as:

$$\begin{bmatrix} s^\alpha & 1 & 1 \\ -1 & (-s^\alpha - a - K(1 - e^{-\tau s})) & 0 \\ -z & 0 & (s^\alpha - (x - c)) \end{bmatrix} \quad (9)$$

*Theorem 1.* The equilibrium point  $f_P$  of the system described by equations (4) and (5) achieves local asymptotic stability if and only if all the roots  $s$  of the characteristic equation  $\det(\Delta s) = 0$  satisfy the following Matignon Criterion:  $|\arg(s)| \geq \frac{\alpha\pi}{2}$ . Here,  $s$  is the eigenvalue of the characteristic equation of the fractional order system. The matrix  $\Delta s$  is evaluated at the fixed point  $f_p^-$  which is shown as in (3).

The Matignon Stability Criterion provides a means to evaluate the stability of fractional-order systems without computing matrix eigenvalues, essential where traditional integer-order criteria are inapplicable (Matignon (1996)). Nonetheless, the stability criterion can also be verified through eigenvalue calculations following the analysis of the characteristic equation. In this context, the characteristic equation before applying the control algorithm and after applying the control algorithm are verified using the Matignon criterion. If all eigenvalues fall outside the closed angular sector defined by  $|\arg(\lambda_i)| \leq \frac{\alpha\pi}{2}$  (where  $\lambda_i$  is the  $i^{th}$  eigenvalue before applying the control) and  $|\arg(s)| \leq \frac{\alpha\pi}{2}$  (where  $s$  is the eigenvalue after applying the control), the stability is assured (Matignon (1996)).

*Theorem 2.* Let's consider an unstable equilibrium point  $f_p^+$  within the fractional-order system (4) when no control is applied ( $K_{ij} = 0$ ), and let  $J$  represent the Jacobian matrix at  $f_p^+$ . If  $J$  possesses an odd count of positive real eigenvalues, the Time-Delayed Feedback Control (TDFC) method is incapable of stabilizing the unstable equilibrium  $f_p^+$ , regardless of the values assigned to the control parameters  $K_{ij}$  and  $\tau$ .

After applying the control algorithm, modified eigenvalues are evaluated by setting the determinants of matrices to zero with the Levenberg-Marquardt algorithm. In the absence of control, equilibrium point  $f_p^+$  has one positive real eigenvalue, rendering it unstabilizable by the TDFC

method due to the Matignon stability criterion. This conclusion is supported by a numerical analysis showing no stability domain in the  $(K, \tau)$  parameter plane, confirmed by simulations. Conversely,  $f_p^-$  can be effectively controlled with TDFC, as evidenced by a stability domain delineated in Section 5. Here, black regions indicate stable combinations of control parameters  $K$  and  $\tau$ , ensuring successful control.

In Variable Delay Feedback Control (VDFC), a sinusoidally modulated signal  $F(t) = T_0 + \epsilon \sin(\omega t)$  is utilized, with  $T_0$ ,  $\epsilon$ , and  $\omega$  as user-defined constants and control parameters. VDFC adapts to changing conditions by varying the time delay in the feedback loop, enhancing stability and performance. The modulation introduces variability, enabling the system to counteract uncertainties and disturbances effectively. By dynamically adjusting the delay, undesired oscillations can be suppressed. The stability domain in the parameter plane  $(K, T_0)$  is determined by varying  $K$  and  $T_0$ , offering insights into control system behavior under different parameter combinations. On the other hand,  $F(t)$  is the feedback for both VDFC and TDFC. In VDFC,  $F(t) = K[y(t - T(t)) - y(t)]$ , where  $T(t) = T_0 + \epsilon \sin(\omega t)$ . If the feedback is applied to only  $y$ -component, the equation becomes,

$$D^\alpha y(t) = x(t) + ay(t) + K[y(t - T(t)) - y(t)] \quad (10)$$

$$D^\alpha y(t) = x(t) + ay(t) + K[y(t - T_0 + \epsilon \sin(\omega t)) - y(t)] \quad (11)$$

$D^\alpha x(t)$  and  $D^\alpha z(t)$  will remain identical.

## 5. RESULTS AND ANALYSIS

In this section, the plots of the fractional Rössler oscillator when feedback are applied to both  $x$  and  $y$  components are shown. Here the parameters are considered as:  $a = 0.5$ ,  $b = 2$ ,  $c = 4$ , and  $\alpha = 0.9$  for which the system is chaotic. In this context, delay ( $\tau$ ) is considered as **lag** as shown in the figures.

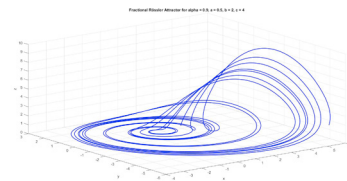


Fig. 1. Plot of chaotic oscillator for  $a = 0.5$ ,  $b = 2$ ,  $c = 4$  (without control action)

It is observed from fig. (1) that the phase plot is chaotic. This observation is confirmed by the Maximum Lyapunov Exponent shown later in this section.

It is observed from figures (2) and (3) that after applying the constant and variable time-delay feedback controls simultaneously, the phase plots become non-chaotic and these display periodicity. The results can be confirmed with the time responses of  $x$  and the Maximum Lyapunov Exponent shown later in this section.

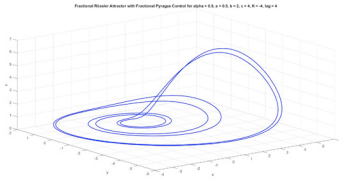


Fig. 2. Plot of oscillator for  $a = 0.5$ ,  $b = 2$ ,  $c = 4$ ,  $\alpha = 0.9$ ,  $K = -2$  and  $\tau = 4$

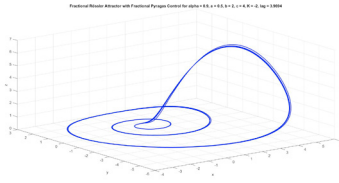


Fig. 3. Plot of oscillator for  $a = 0.5$ ,  $b = 2$ ,  $c = 4$ ,  $\alpha = 0.9$ ,  $K = -2$ ,  $\epsilon = 0.1$ ,  $\omega = 10$  and  $T_0 = 4$

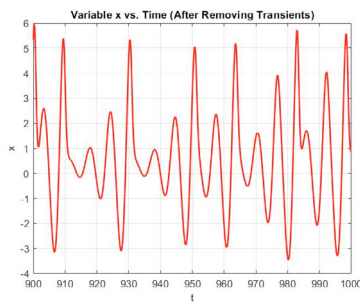


Fig. 4. Plot of chaotic time response for  $a = 0.5$ ,  $b = 2$ ,  $c = 4$ ,  $\alpha = 0.9$

From the figure (4), chaoticity is observed as there is no definite period and Maximum Lyapunov Exponent is positive.

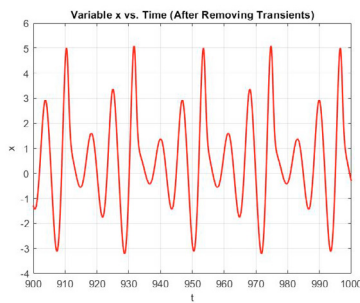


Fig. 5. Plot of periodic time response for  $a = 0.5$ ,  $b = 2$ ,  $c = 4$ ,  $\alpha = 0.9$ ,  $K = -2$  and  $\tau = 4$

From the fig. (5), it is observed that the phase plot is oscillating periodically and the control (TDFC) have suppressed the chaos. This similar scenario happens in case of the VDFC control too.

In this context, one bifurcation diagram (shown in fig. (6) computed for  $0.3 < \alpha < 1$  and  $-0.2 \leq a < 1$ . In this figure, until  $\alpha \approx 0.378$ , the fixed point  $f_p^-$  is stable, but after crossing that value, a Hopf bifurcation takes place, giving rise to a limit cycle. After  $\alpha > 0.73$ , other bifurcations lead to a chaotic attractor which is confirmed from fig.

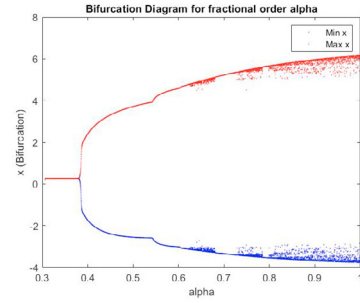


Fig. 6. Bifurcation Diagram of Fractional Rössler Oscillator for  $a = 0.5$ ,  $b = 2$ ,  $c = 4$ ,  $\alpha = 0.9$

(1). After analysing in detail the behaviour for alpha, we have observed that there is no significant alteration during  $0.6 \leq \alpha \leq 0.7$ .

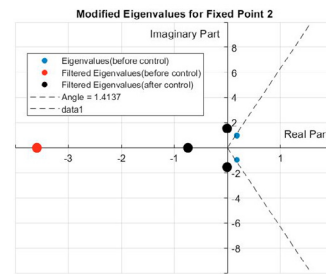


Fig. 7. Plot of the eigenvalues corresponding to the fixed point  $f_p^-$  before and after applying the TDFC (while  $K = -2$  and  $\tau = 4$ )

This fig. (7) shows that the chaos is taking place due to  $\alpha = 0.9$  and two complex conjugate eigenvalues (shown in blue) are not satisfying Matignon criterion. Thus, we need to use the time-delay feedback control to suppress the chaos and this is demonstrated by the black eigenvalues. The black eigenvalues are outside of the conic region and satisfying the Matignon criterion after applying the TDFC.

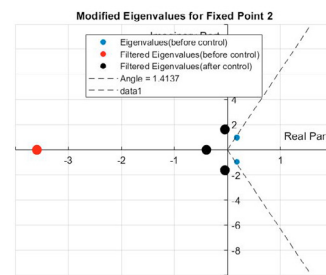


Fig. 8. Plot of the eigenvalues corresponding to the fixed point  $f_p^-$  before and after applying the VDFC (while  $K = -3$ ,  $\epsilon = 1$ ,  $\omega = 0.5$  and  $T_0 = 5$ )

The fig. (8) is generated before applying the VDFC and it shows that the chaos is taking place due to  $\alpha = 0.9$  and two complex conjugate eigenvalues (shown in blue) are not satisfying Matignon criterion. Thus, we have used the variable time-delay feedback control to suppress the chaos and this is demonstrated here. The black eigenvalues are outside of the conic region and satisfying the Matignon

criterion after applying the VDFC. Eigenvalues' locations pre and post-control provide confirmation of algorithm efficacy. Varying feedback gain and time delay parameters demonstrated stability regions.

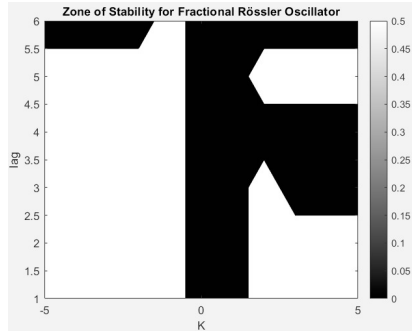


Fig. 9. The stability domain in the parameter plane ( $K$  vs  $\tau$ ) where the range of  $K$  and  $\tau$  are  $(-5 \rightarrow 5)$  and  $(1 \rightarrow 6)$

It is observed in fig. (9) that due to applying the feedback component to both  $x$  and  $y$  components, the region of stability is large. However, due to the constant time delay, the white stability region is separated by an unstable black region.

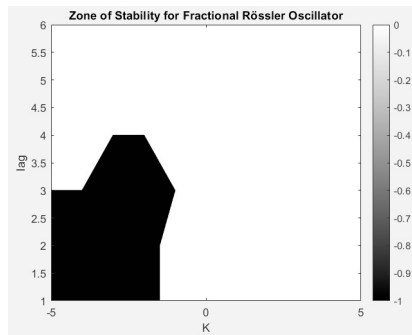


Fig. 10. The stability domain in the parameter plane ( $K$  vs  $\tau$ ) when feedback component is applied to the  $x$  component only and the range of  $K$  and  $\tau$  are  $(-5 \rightarrow 5)$  and  $(1 \rightarrow 6)$

It is observed in fig. (10) that due to applying the feedback component to only  $x$  component, the region of stability is quite small.

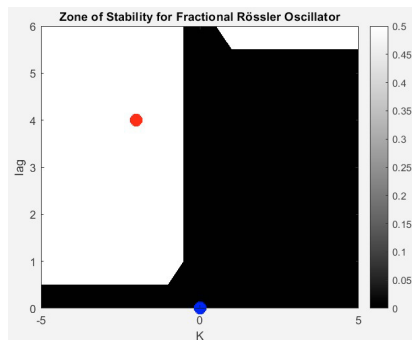


Fig. 11. The stability domain in the parameter plane ( $K$  vs  $\tau$ ) when feedback component is applied to the  $y$  component only and the range of  $K$  and  $\tau$  are  $(-5 \rightarrow 5)$  and  $(1 \rightarrow 6)$

It is observed in fig. (11) that due to applying the feedback component to only  $x$  component, the region of stability is also quite small.

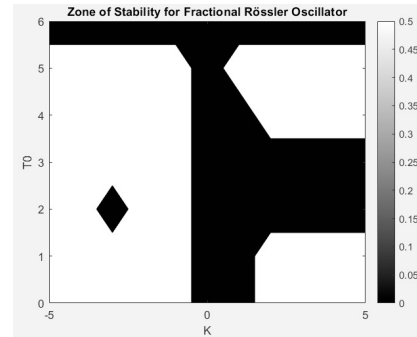


Fig. 12. The stability domain in the parameter plane ( $K$  vs  $T_0$ ) where the range of  $K$  and  $T_0$  are  $(-5 \rightarrow 5)$  and  $(0 \rightarrow 6)$

It is observed in fig. (12) that due to applying the feedback component to both  $x$  and  $y$  components, the region of stability is large. However, due to the variable time delay, the white stability region is increased as compared to the region of stability shown in fig. (9).

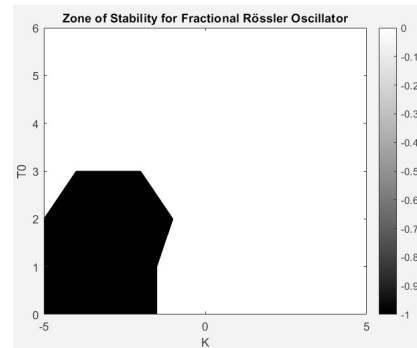


Fig. 13. The stability domain in the parameter plane ( $K$  vs  $T_0$ ) when feedback component is applied to the  $x$  component only and the range of  $K$  and  $T_0$  are  $(-5 \rightarrow 5)$  and  $(0 \rightarrow 6)$

It is observed in fig. (13) that due to applying the feedback component to  $x$  and component, the region of stability is smaller but it is still larger than in fig. (10).

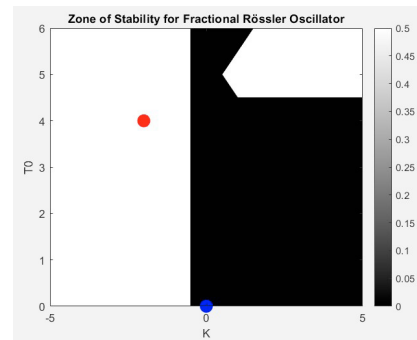


Fig. 14. The stability domain in the parameter plane ( $K$  vs  $T_0$ ) when feedback component is applied to the  $y$  component only and the range of  $K$  and  $T_0$  are  $(-5 \rightarrow 5)$  and  $(0 \rightarrow 6)$



It is observed in fig. (14) that due to applying the feedback component to  $x$  and component, the region of stability is quite smaller but it is still larger than in fig. (11).

In all the cases, the periodicity of the obtained orbits is verified with Fast Fourier Transform (FFT) and Maximum Lyapunov Exponent (MLE).. It turns out that after applying the time-delay control (both constant and variable), the stabilized orbits become periodic. For instance, after applying the variable time-delay control on  $y$ -component (while  $K = -2$ ,  $\epsilon = 1$ ,  $\omega = 0.5$  and  $T_0 = 5$ ),  $MLE = -0.56359$  which indicates stability. Similarly, after applying the constant time-delay control on  $y$ -component (while  $K = -2$  and  $\tau = 4$ ),  $MLE = -0.74028$  which indicates stability. However, without applying this control algorithm on  $y$ -component,  $MLE = 0.702$  which indicates chaoticity. The result is confirmed from the figures (11) and (14) where first red point (while  $K = -2$  and  $\tau = 4$ ) and second red point (while  $K = -2$ ,  $\epsilon = 1$ ,  $\omega = 0.5$  and  $T_0 = 5$ ) confirm the stability simulatenously as they are in the white stability region. Contrarily, the blue point in both the figures (11) and (14) in the black region, confirms instability because of the positive MLE. As the fractional order ( $\alpha$ ) decreases below 0.9, the complex-conjugate eigenvalues associated with equilibrium point gradually move outside the instability region defined by the Matignon criterion. Consequently, equilibrium stabilizes naturally, even without any control intervention. The critical threshold, marking the point where these eigenvalues cross the boundary of the conic surface separating distinct stability regions, can be precisely determined using criterion (1). In this specific scenario, the calculated value for  $\alpha$  is 0.8825.

Using Windows Home 11 OS, intel Core-i5 9300H processor and Matlab R2023b, we have found that Caputo-based method took 43 minutes to solve TDFC without calculating eigenvalues and stability zones, while GL achieved the same task in 1 second. The complexity of implementing Caputo algorithms mentioned by (Betancur-Herrera and Muñoz-Galeano (2020)) further tips the scales towards GL. This simplicity in implementation, especially evident in systems like the Fractional Rossler System, makes GL the preferred choice. Its efficiency not only reduces computational burden but also simplifies analysis and modeling, promoting precision in research efforts.

To summarize, in contrast to (Gjurchinovski et al. (2010)), this study focuses on GL characterization, and the calculation of fractional derivatives is performed according to equation (2). The phase phase and bifurcation diagrams are qualitatively similar, but the computational time is shorter and zone of stability has been altered. We have also considered variable frequency in order to investigate its impact on the stability performances and regions of stability. In VDFC, if  $T_0$ ,  $\omega$  or  $\epsilon$  is varied, we can observe that the delay is varied dynamically. For example, if  $\omega$  is increased from values 1  $\rightarrow$  5, the eigenvalues on the complex  $s$ -plane become more clustered and also the zone of stability (white region) decreases.

## 6. CONCLUSION AND FUTURE SCOPE

In conclusion, this study explored the control and stabilization of chaotic fractional-order Rössler systems using

Constant Time-Delayed Feedback Control (TDFC) and Variable Delay Feedback Control (VDFC) methods. By applying these techniques to the system's  $x$  and  $y$  components (and also only on  $x$  component), stability analysis was conducted using the Matignon Stability Criterion. For the TDFC method, a stability domain in the parameter plane ( $K, \tau$ ) was determined, indicating regions where chaotic behavior was effectively controlled. The VDFC method, incorporating sinusoidal modulation in time delay, demonstrated an expanded stability domain compared to TDFC, enhancing the system's controllability using a stability domain in the parameter plane ( $K, T_0$ ). This research provides valuable insights into controlling complex fractional-order systems, emphasizing the significance of fractional derivatives and modern control techniques in understanding and managing chaotic dynamics.

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