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# OPTIMAL TIME DECAY ESTIMATES FOR THE ELECTROMAGNETIC SCHRÖDINGER EQUATION

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**ABSTRACT.** The aim of this paper is to analyze the time decay of solutions to the Schrödinger equation with a magnetic potential and a positive scalar electric field, both of which are short-range, in asymptotically Euclidean spaces. We demonstrate that the decay rate of the solutions remains identical to that in the Euclidean case without an electromagnetic field. This analysis relies on specific differentiability properties of the spectral measure.

## 1. INTRODUCTION

Let  $d \geq 3$ . We consider the Schrödinger equation for a charged particle in an external electromagnetic field, given by

$$(1.1) \quad \begin{cases} i\partial_t u + P_{\mathbf{A},V} u = 0, & \text{on } \mathbb{R}_+ \times \mathbb{R}^d, \\ u|_{t=0} = f, & \text{on } \mathbb{R}^d, \end{cases}$$

where  $f \in L^2(\mathbb{R}^d)$ ,  $P_{\mathbf{A},V}$  is a general electromagnetic Laplacian operator on  $\mathbb{R}^d$ , close to the free electromagnetic Laplacian at infinity, and the magnetic potential vector  $\mathbf{A}(x) = (A_1(x), \dots, A_d(x))$  and the positive scalar electric field  $V$  are small at infinity. More precisely, the operator  $P_{\mathbf{A},V}$  is of the form

$$P_{\mathbf{A},V} = \frac{1}{w(x)} \sum_{1 \leq i, j \leq d} (D_i - A_i)[g^{ij}(x)(D_j - A_j)] + V(x), \quad D_j = -i\partial_{x_j},$$

where the positive function  $w(x)$  and the positive definite symmetric matrix  $(g^{ij}(x))$  have smooth coefficients.

**Assumption 1.1.** We assume that  $g^{ij}(x)$  and  $w(x)$  are long-range perturbations of  $\delta_{ij}$  (the Kronecker delta) and 1, respectively, in the sense that for some  $\rho_0 \in ]0, 1]$ , there exist constants  $C_\alpha > 0$ ,  $\alpha \in \mathbb{N}^d$ , such that for all  $x \in \mathbb{R}^d$ ,

$$(1.2) \quad |\partial^\alpha(g^{ij}(x) - \delta_{ij})| + |\partial^\alpha(w(x) - 1)| \leq C_\alpha \langle x \rangle^{-\rho_0 - |\alpha|}.$$

Here and throughout, we use the standard notation  $\langle x \rangle = (1 + |x|^2)^{\frac{1}{2}}$ .

Let us now state the assumptions on  $\mathbf{A}$  and  $V$ .

**Assumption 1.2.** The magnetic potential  $\mathbf{A}$  and the positive scalar electric field  $V$  are smooth, bounded, and short-range perturbations: there exists  $C_\alpha > 0$ ,  $\alpha \in \mathbb{N}^d$  such that for all  $x \in \mathbb{R}^d$ , we have

$$(1.3) \quad |\partial^\alpha \mathbf{A}(x)| \leq C_\alpha \langle x \rangle^{-1 - \rho_0 - |\alpha|},$$

and

$$(1.4) \quad V \geq 0, \quad |\partial^\alpha V(x)| \leq C_\alpha \langle x \rangle^{-2 - \rho_0 - |\alpha|}.$$

The goal of this paper is to extend the results of Bouclet and Burq [BB21] to the setting of an electromagnetic field. Specifically, we show that the temporal decay of solutions to the Schrödinger equation on asymptotically Euclidean backgrounds, in the presence of an electromagnetic field, remains identical to the optimal decay obtained in the case without such a field.

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This extension is achieved through an analysis of certain differentiability properties of the spectral measure, under the Assumptions 1.1, and 1.2, which will be detailed later.

For time decay estimates in the electromagnetic-free case on asymptotically Euclidean geometries, the reader can refer to the introduction of [BB21, Roy24] for an overview of results in this case, without magnetic influences.

The magnetic Schrödinger equation has attracted considerable attention from scientists in recent years, primarily due to the importance of dispersive estimates in the study of quantum mechanics and wave propagation in the presence of magnetic fields.

Here, we present some recent results on time decay for the free Schrödinger equation ( $w(x) = 1$  and  $g^{ij}(x) = \delta_{ij}$ ) with time-independent potentials. The dispersive decay for both linear and nonlinear magnetic Schrödinger equations is established in [DW23], with a decay rate of the solution in  $L^\infty$  being  $t^{-d/2}$ , where the magnetic field  $\mathbf{A}$  behaves at infinity like  $|x|^{-2-d}$ . Additionally, they assume that *zero is neither an eigenvalue nor a resonance* of the magnetic Schrödinger operator. This assumption is critical, as the spectral properties of the operator at zero influence the resolvent and, consequently, the decay behavior. Indeed, for a perturbation of the Laplacian by magnetic and electrostatic potentials, the behavior of the resolvent is significantly affected by the presence of eigenvalues or resonances at zero. This phenomenon is highlighted in the resolvent expansion described, for example, in the recent article [JK24], where  $\mathbf{A}$  and  $V$  behave at infinity like  $|x|^{-\beta}$  for  $\beta > 2$ . The expansion of the resolvent in the presence of these singularities provides important insights into the decay rates of solutions, especially when resonances or eigenvalues at zero are involved.

We recall that the well-posedness of solutions for the Schrödinger equations with time-dependent and time-independent magnetic fields was established in [AY14]. They prove that these equations uniquely generate unitary propagators under suitable conditions on the size and singularities of the time derivatives of the magnetic potentials. For the electromagnetic Schrödinger equation, under the most general conditions on the time-dependent potentials  $\mathbf{A}$  and  $V$ , the existence and uniqueness of the fundamental solution are established in [Yaj91]. This result addresses the fundamental question of well-posedness in the presence of general electromagnetic fields, which has been a central concern in mathematical quantum mechanics.

In a more general setting, it is natural to extend the long-time behavior results from the previous work to more general Schrödinger equations with electromagnetic fields. This generalization is crucial for applying dispersive estimates to a broader range of physical and mathematical models. In particular, our main result involves deriving dispersive decay in weighted  $L^2$ -spaces with an optimal decay rate of  $t^{-d/2}$ , with weaker assumptions on the electromagnetic fields compared to recent results like those in [DW23].

The main result of this paper is the following.

**Theorem 1.3.** *Let  $d \geq 3$  and  $\varphi \in C_0^\infty(\mathbb{R}_+)$ . Assume that Assumptions 1.1 and 1.2 hold. Then, for  $\nu > \lfloor \frac{d}{2} \rfloor + 2$ , we have*

$$\| \langle x \rangle^{-\nu} \varphi(P_{\mathbf{A},V}) e^{itP_{\mathbf{A},V}} \langle x \rangle^{-\nu} \|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \lesssim \langle t \rangle^{-\frac{d}{2}}.$$

The quadratic form associated with  $P_{\mathbf{A},V}$  is closed, and positive. Therefore, the spectrum of  $P_{\mathbf{A},V}$ ,  $\text{sp}(P_{\mathbf{A},V}) \subset [0, +\infty[$ , its resolvent set is contained in  $\mathbb{C} \setminus \mathbb{R}_+$ , and the resolvent at  $\lambda \pm i\epsilon$  is well-defined for  $\lambda, \epsilon \in \mathbb{R}$  and  $\epsilon \neq 0$ . Moreover, using tools from scattering theory (Mourre theory), see for example [JMP84, Agm75, KUR73], together with the fact that the self-adjoint operator  $P_{\mathbf{A},V}$  does not have eigenvalues embedded in  $\mathbb{R}_+$  (see Proposition 2.1), the following limits

$$(1.5) \quad (P_{\mathbf{A},V} - \lambda \pm i0)^{-1} = \lim_{\epsilon \rightarrow 0} (P_{\mathbf{A},V} - \lambda \pm i\epsilon)^{-1},$$

are well defined in the weighted  $L^2$  spaces. Next, we define the spectral measure associated with (1.5), given by the Stone formula (see, e.g. [RS75, Section VIII.3])

$$(1.6) \quad E'_{P_{\mathbf{A},V}}(\lambda) := \frac{1}{2i\pi} [(P_{\mathbf{A},V} - \lambda - i0)^{-1} - (P_{\mathbf{A},V} - \lambda + i0)^{-1}].$$

Since the self-adjoint operator  $P_{\mathbf{A},V}$  does not have eigenvalues embedded in  $\mathbb{R}_+$  and no resonance at 0 (see Propositions 2.1, 2.4), “*i.e.* the resolvent of the operator  $P_{\mathbf{A},V}$  is bounded at the edge point  $\lambda = 0$  of the continuous spectrum”, it follows that

$$\begin{aligned} \varphi(P_{\mathbf{A},V}) e^{itP_{\mathbf{A},V}} &= \frac{1}{2i\pi} \int_0^{+\infty} \varphi(\lambda) e^{it\lambda} [(P_{\mathbf{A},V} - \lambda - i0)^{-1} - (P_{\mathbf{A},V} - \lambda + i0)^{-1}] d\lambda \\ &= \frac{1}{2i\pi} \int_0^{+\infty} \varphi(\lambda) e^{it\lambda} E'_{P_{\mathbf{A},V}}(\lambda) d\lambda. \end{aligned}$$

Subsequently, to demonstrate Theorem 1.3, it suffices to estimate the latter integral in the weighted  $L^2$  space and therefore, estimating  $(P_{\mathbf{A},V} - \lambda \pm i0)^{-1}$  and the derivatives  $(P_{\mathbf{A},V} - \lambda \pm i0)^{-k}$ , for  $k \geq 1$ , as  $\lambda \rightarrow 0$ .

In addition, we derive uniform estimates on the resolvent, which play an important role in scattering theory (Theorem 1.4), and on the behavior of the spectral measure (Theorem 1.5), which is a key tool in the proof of Theorem 1.3.

**Theorem 1.4.** *Let  $d \geq 3$ ,  $\lambda_0 > 0$ ,  $k \in \mathbb{N}$  and  $\nu > k$ . Then there is a constant  $C > 0$  such that for all  $\lambda \in ]0, \lambda_0]$ , we have*

$$\left\| \langle x \rangle^{-\nu} (P_{\mathbf{A},V} - \lambda \pm i0)^{-k} \langle x \rangle^{-\nu} \right\| \leq C \lambda^{\min\{0, \frac{d}{2} - k\}},$$

if  $k \neq \frac{d}{2}$ , and

$$\left\| \langle x \rangle^{-\nu} (P_{\mathbf{A},V} - \lambda \pm i0)^{-\frac{d}{2}} \langle x \rangle^{-\nu} \right\| \leq C |\log \lambda|,$$

if  $d$  is even and  $k = \frac{d}{2}$ .

**Theorem 1.5.** *Let  $d \geq 3$ ,  $\lambda_0 > 0$ ,  $k \in \mathbb{N}$  and  $\nu > k$ . Then the function*

$$\lambda \rightarrow \langle x \rangle^{-\nu} E'_{P_{\mathbf{A},V}}(\lambda) \langle x \rangle^{-\nu}$$

is  $C^{k-1}$  on  $(0, \lambda_0]$  with values in (bounded) operators on  $L^2$ . If in addition  $\nu > \frac{d}{2}$ , then

$$\left\| \frac{d^j}{d\lambda^j} \langle x \rangle^{-\nu} E'_{P_{\mathbf{A},V}}(\lambda) \langle x \rangle^{-\nu} \right\| \leq C \lambda^{\frac{d}{2} - j - 1},$$

for all  $j \in \{1, \dots, k-1\}$ .

The proof of Theorem 1.4 follows a structure similar to that in [BB21] (see Section 5). Our approach is based on Mourre theory, which plays a key role in establishing resolvent estimates in weighted spaces. A central technical condition (Proposition 3.4) ensures the validity of an exact positive commutator estimate (Theorem 4.1). The main novelty of this paper lies in analyzing how the presence of the electromagnetic field perturbs the proof strategy developed in [BB21] (for example Proposition 3.4), while demonstrating that the principal ideas remain applicable in this extended framework.

We note that Theorem 1.3 deals with the contribution of low frequencies. Below, we get a non spectrally localized result, under the non-trapping condition in order to handle the high-frequency contribution. This non-trapping condition means that geodesics associated with the classical Hamiltonian  $\langle G(x)\xi, \xi \rangle_{\mathbb{R}^d}$  are not trapped.

**Corollary 1.6.** *Let  $d \geq 3$ . Under the non-trapping conditions. Then, for  $\nu > \lfloor \frac{d}{2} \rfloor + 2$ , we have*

$$\left\| \langle x \rangle^{-\nu} e^{itP_{\mathbf{A},V}} \langle x \rangle^{-\nu} \right\|_{L^2(\mathbb{R}^d) \rightarrow H^{\frac{d}{2}}(\mathbb{R}^d)} \lesssim \langle t \rangle^{-\frac{d}{2}}.$$

**Plan of the paper.** In Section 2, we provide some properties of the spectral analysis of the operator  $P_{\mathbf{A},V}$ . In Section 3, we prove some resolvent estimates using the decay of the coefficients at infinity (see (1.3) and (1.4)). In Section 4, we use the Mourre method to prove the main estimate for  $(P_{\mathbf{A},V} - \lambda \pm i0)^{-k}$ . In Section 5, we derive the local decay stated in Theorem 1.3, Theorem 1.5, and Corollary 1.6, and we prove the intermediate results needed to prove Theorem 1.4.

## 2. PRELIMINARY

This section focuses on the spectral analysis of the operator  $P_{\mathbf{A},V}$  in order to demonstrate that the limit in (1.5) exists and is finite.

**Proposition 2.1.** *Under the Assumptions 1.1 and 1.2, we have:*

$$\mathrm{sp}_{\mathrm{ess}}(P_{\mathbf{A},V}) = \mathrm{sp}_{\mathrm{ess}}(-\Delta) = [0, +\infty[ \quad \text{and} \quad \mathrm{sp}_{\mathrm{pp}}(P_{\mathbf{A},V}) \cap \mathrm{sp}_{\mathrm{ess}}(P_{\mathbf{A},V}) = \emptyset,$$

where  $\mathrm{sp}_{\mathrm{ess}}(P_{\mathbf{A},V})$  denotes the essential spectrum and  $\mathrm{sp}_{\mathrm{pp}}(P_{\mathbf{A},V})$  denotes the point spectrum.

*Proof.* Show that:  $\mathrm{sp}_{\mathrm{ess}}(P_{\mathbf{A},V}) = [0, +\infty[$ .

The trivial inclusion is

$$\mathrm{sp}_{\mathrm{ess}}(P_{\mathbf{A},V}) \subset [0, +\infty[,$$

because  $\mathrm{sp}_{\mathrm{ess}}(P_{\mathbf{A},V}) \subset \mathrm{sp}(P_{\mathbf{A},V}) \subset [0, +\infty[$ .

For the other inclusion, we write

$$w(x)P_{\mathbf{A},V} = -\mathrm{div}(g^{-1}\nabla) + i \mathrm{div}(g^{-1}\mathbf{A}) + 2i\mathbf{A} \cdot g^{-1}\nabla + g^{-1}\mathbf{A} \cdot \mathbf{A} + w(x)V(x).$$

To simplify the writing, we can take  $w = 1$ , and demonstrate the proposition. The case where  $w$  differs from 1 is obtained by the same reasoning. Then,

$$P_{\mathbf{A},V} = -\mathrm{div}(g^{-1}\nabla) + i \mathrm{div}(g^{-1}\mathbf{A}) + 2i\mathbf{A} \cdot g^{-1}\nabla + g^{-1}\mathbf{A} \cdot \mathbf{A} + V = -\Delta^g + 2i\mathbf{A} \cdot g^{-1}\nabla + W,$$

where  $W = i \mathrm{div}(g^{-1}\mathbf{A}) + g^{-1}\mathbf{A} \cdot \mathbf{A} + V$  and  $\Delta^g = \mathrm{div}(g^{-1}\nabla)$ .

We note that, under the Assumption 1.2 on  $\mathbf{A}$  and  $V$ , we get

$$\lim_{|x| \rightarrow +\infty} W(x) = 0.$$

Let us consider  $\chi \in C_0^\infty(\mathbb{R}^d)$  such that  $\|\chi\|_{L^2(\mathbb{R}^d)} = 1$ . For  $n \in \mathbb{N}$  and  $k \in \mathbb{R}$ , we consider

$$\chi_{n,k}(x) = e^{ik \cdot x} n^{-\frac{d}{2}} \chi(n^{-1}x - ne_1).$$

The sequence  $(\chi_{n,k})$  is  $L^2$ -normalized and converges to 0 weakly. We have

$$\|(P_{\mathbf{A},V} - k^2)\chi_{n,k}\| \leq \mathcal{O}(n^{-1}) + o(1) \xrightarrow{n \rightarrow \infty} 0.$$

By the Weyl criterion, we find that  $k^2 \in \mathrm{sp}_{\mathrm{ess}}(P_{\mathbf{A},V})$  for all  $k \in \mathbb{R}$ .

The second point of the proposition is a consequence of [KT06]. □

**Proposition 2.2.** *For all  $u \in \dot{H}^1(\mathbb{R}^d) := \{u \in L_{\mathrm{loc}}^2(\mathbb{R}^d) : \nabla u \in L^2(\mathbb{R}^d)\}$ , there exist  $c > 0$ , such that*

$$\langle P_{\mathbf{A},V}u, u \rangle_{L^2(\mathbb{R}^d)} \geq c \|\nabla|u|\|_{L^2(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d} V|u|^2 dx,$$

where  $\nabla|u|(x) = \mathrm{Re}\{\mathrm{Sgn}(f(x))\nabla f\}$  for almost every  $x \in \mathbb{R}^d$ .

**Remark 2.3.** Let  $d \geq 3$ . By the Hardy inequality on  $\mathbb{R}^d$ , we have that if  $u \in \dot{H}^1(\mathbb{R}^d)$ , then

$$\||x|^{-1}u\|_{L^2} \lesssim \|\nabla u\|_{L^2}.$$

In particular, by Assumption 1.2, for all  $1 \leq i \leq d$ ,  $(D_i - A_i)u \in L^2(\mathbb{R}^d)$ .

*Proof.* Let  $u \in \dot{H}^1(\mathbb{R}^d)$ . We introduce a family of cutoff functions  $\chi_k$ , defined by:

$$\chi_k(x) = \chi\left(\frac{x}{k}\right), \quad \forall k \in \mathbb{N}^*,$$

where  $\chi \in C_0^\infty(\mathbb{R}^n)$  satisfies  $0 \leq \chi \leq 1$ ,  $\chi = 1$  on the unit ball, and  $\text{supp}(\chi) \subset B(0, 2)$ . We have

$$\begin{aligned} & \langle P_{\mathbf{A}, V} u, \chi_k u \rangle_{L^2(\mathbb{R}^d)} \\ &= \frac{1}{w(x)} \int_{\mathbb{R}^d} P_{\mathbf{A}, V} u \chi_k \bar{u}(x) dx \\ &= \frac{1}{w(x)} \sum_{1 \leq i, j \leq d} \int_{\mathbb{R}^d} (D_i - A_i) [g^{ij}(x) (D_j - A_j)] u(x) \chi_k \bar{u}(x) dx + \int_{\mathbb{R}^d} V \chi_k |u|^2 dx \\ &= \frac{1}{w(x)} \sum_{1 \leq i, j \leq d} \int_{\mathbb{R}^d} g^{ij}(x) (D_j - A_j) u(x) \overline{(D_i - A_i) \chi_k u(x)} dx + \int_{\mathbb{R}^d} V \chi_k |u|^2 dx \\ &= \frac{1}{w(x)} \sum_{1 \leq i, j \leq d} \int_{\mathbb{R}^d} g^{ij}(x) (D_j - A_j) u(x) [\chi_k \overline{(D_i - A_i) u} + \overline{u(x) D_i \chi_k}] dx + \int_{\mathbb{R}^d} V \chi_k |u|^2 dx. \end{aligned}$$

Consequently,

(2.1)

$$\begin{aligned} \langle P_{\mathbf{A}, V} u, \chi_k u \rangle_{L^2(\mathbb{R}^d)} &= \frac{1}{w(x)} \sum_{1 \leq i, j \leq n} \left[ \underbrace{\int_{\mathbb{R}^n} \chi_k g^{ij}(x) (D_j - A_j) u(x) \overline{(D_i - A_i) u(x)} dx}_{(I)} \right. \\ &\quad \left. + \underbrace{\int_{\mathbb{R}^n} g^{ij}(x) (D_j - A_j) u(x) \overline{u(x) D_i \chi_k} dx}_{(II)} \right] + \int_{\mathbb{R}^d} V \chi_k |u|^2 dx. \end{aligned}$$

Estimate (II): We have

$$\begin{aligned} & \left| \int_{\mathbb{R}^n} g^{ij}(x) (D_j - A_j) u(x) \bar{u}(x) D_i \chi_k dx \right| \\ & \leq \int_{\mathbb{R}^n} |g^{ij}(x) (D_j - A_j) u(x) \bar{u}(x) D_i \chi_k| dx \\ & \leq \int_{k \leq |x| \leq 2k} |g^{ij}(x) (D_j - A_j) u(x) \bar{u}(x) D_i \chi_k| dx \\ & \leq \frac{1}{k} \int_{k \leq |x| \leq 2k} \left| g^{ij}(x) (D_j - A_j) u(x) \bar{u}(x) D_i \chi\left(\frac{x}{k}\right) \right| dx \\ & \leq \frac{C}{k} \int_{k \leq |x| \leq 2k} \left| (D_j - A_j) u(x) \bar{u}(x) D_i \chi\left(\frac{x}{k}\right) \right| dx \\ & \leq \frac{C}{k} \|(D_j - A_j) u\|_{L^2(\mathbb{R}^d)} \left[ \int_{k \leq |x| \leq 2k} \left| \bar{u}(x) D_i \chi\left(\frac{x}{k}\right) \right|^2 dx \right]^{1/2} \\ & \leq \frac{C}{k} \|(D_j - A_j) u\|_{L^2(\mathbb{R}^d)} \left[ \int_{k \leq |x| \leq 2k} \langle x \rangle^2 \left| \langle x \rangle^{-1} u(x) D_i \chi\left(\frac{x}{k}\right) \right|^2 dx \right]^{1/2} \\ & \leq C \|(D_j - A_j) u\|_{L^2(\mathbb{R}^d)} \left[ \int_{k \leq |x| \leq 2k} \left| \langle x \rangle^{-1} u(x) D_i \chi\left(\frac{x}{k}\right) \right|^2 dx \right]^{1/2}. \end{aligned}$$

Then, as  $k \rightarrow +\infty$ , by the dominated convergence theorem and with Remark 2.3, we obtain that:  $(II) \rightarrow 0$ . Since,  $\chi_k \xrightarrow{k \rightarrow +\infty} 1$ , then

$$(I) \xrightarrow{k \rightarrow +\infty} \int_{\mathbb{R}^n} g^{ij}(x)(D_j - A_j)u(x)\overline{(D_i - A_i)u(x)}dx.$$

We deduce that:

$$\langle P_{\mathbf{A},V}u, \chi_k u \rangle_{L^2(\mathbb{R}^d)} \xrightarrow{k \rightarrow +\infty} \frac{1}{w(x)} \int_{\mathbb{R}^n} g^{ij}(x)(D_j - A_j)u(x)\overline{(D_i - A_i)u(x)}dx + \int_{\mathbb{R}^d} V|u|^2 dx.$$

Therefore,

$$\langle P_{\mathbf{A},V}u, u \rangle_{L^2(\mathbb{R}^d)} = \frac{1}{w(x)} \sum_{1 \leq i, j \leq d} \int_{\mathbb{R}^d} g^{ij}(x)(D_i - A_i(x))u(x)\overline{(D_j - A_j(x))u(x)} dx + \int_{\mathbb{R}^d} V|u|^2 dx.$$

From (1.2), there exists  $C > 0$ , such that

$$\begin{aligned} \langle P_{\mathbf{A},V}u, u \rangle_{L^2(\mathbb{R}^d)} &\geq C \sum_{1 \leq i, j \leq d} \int_{\mathbb{R}^d} \delta_{ij}(D_i - A_i(x))u(x)\overline{(D_j - A_j(x))u(x)} dx + \int_{\mathbb{R}^d} V|u|^2 dx \\ &\geq C \|(\nabla - A(x))u\|_{L^2(\mathbb{R}^d)}^2 + \int_{\mathbb{R}^d} V|u|^2 dx. \end{aligned}$$

By the diamagnetic inequality (see, e.g., [LL01, Thm. 7.21] or [23, Thm. 2.1.1])

$$|(\nabla - A(x))u|^2 \geq |\nabla |u||^2,$$

we complete the proof.  $\square$

**Proposition 2.4.** *Zero is a regular point of  $P_{\mathbf{A},V}$ . Namely, zero is neither an eigenvalue nor a resonance of  $P_{\mathbf{A},V}$ , i.e., there is no non-trivial solution  $u \in \dot{H}^1(\mathbb{R}^d)$  to  $P_{\mathbf{A},V}u = 0$ .*

*Proof.* Let  $u \in \dot{H}^1(\mathbb{R}^d)$  such that  $P_{\mathbf{A},V}u = 0$ , and show that  $u = 0$ . From Proposition 2.2, since  $V \geq 0$ , we obtain  $\nabla |u| = 0$  almost everywhere  $x \in \mathbb{R}^d$ , and thus  $u$  is constant. Since  $u \in L^2_{\text{loc}}(\mathbb{R}^d)$ , then  $u = 0$  almost everywhere  $x \in \mathbb{R}^d$ , hence  $u = 0$ .  $\square$

### 3. RESOLVENT ESTIMATE

This section focuses on the resolvent estimates of the operator  $P_{\mathbf{A},V}$ .

**Proposition 3.1.** *For all  $u \in H^2(\mathbb{R}^d)$ , we have*

$$(3.1) \quad |e^{-tP_{\mathbf{A},V}}u| \leq e^{-tP_{0,V}}|u|.$$

*Proof.* Let  $u \in H^2(\mathbb{R}^d)$ . Let  $\varepsilon > 0$  and define

$$u_\varepsilon = (|u|^2 + \varepsilon^2)^{\frac{1}{2}}.$$

A simple calculation shows that

$$P_{0,0}u_\varepsilon \leq \text{Re} \left[ \left( \frac{\bar{u}}{u_\varepsilon} \right) P_{\mathbf{A},0}u \right].$$

Since  $u_\varepsilon \rightarrow |u|$  and  $\frac{\bar{u}}{u_\varepsilon} \rightarrow \text{sgn}(\bar{u})$ , we obtain

$$(3.2) \quad P_{0,V}|u| \leq \text{Re}[\text{sgn}(\bar{u})P_{\mathbf{A},V}u],$$

where  $\text{sgn}(u) = \frac{u}{|u|}$  for  $u \neq 0$ , and 0 if  $u = 0$ . For more details, see [Kat72, Section 5].

Now, multiplying by  $\lambda > 0$  and adding  $|u|$  to (3.2), we obtain

$$(\lambda P_{0,V} + \text{Id})|u| \leq \text{Re}[\text{sgn}(\bar{u})(\lambda P_{\mathbf{A},V} + \text{Id})u] \leq |(\lambda P_{\mathbf{A},V} + \text{Id})u|,$$

for all  $u \in H^2(\mathbb{R}^d)$ . This inequality implies that

$$|(\lambda P_{\mathbf{A},V} + \text{Id})^{-1}u| \leq (\lambda P_{0,V} + \text{Id})^{-1}|u|,$$

and then for all  $n \in \mathbb{N}^*$ ,

$$|(\lambda P_{\mathbf{A},V} + \text{Id})^{-n} u| \leq (\lambda P_{0,V} + \text{Id})^{-n} |u|.$$

Then, we obtain the estimate (3.1) using the fact that:

$$e^{-tP_{\mathbf{A},V}} = \lim_{n \rightarrow \infty} \left( \frac{t}{n} P_{\mathbf{A},V} + \text{Id} \right)^{-n}.$$

For more details, see [Sim79, Theorem 1].  $\square$

**Proposition 3.2.** *For all  $s \in [0, \frac{d}{4}]$ ,  $\sigma > 2s$  and  $k > s$ , there is  $C > 0$  such that for all  $\lambda > 0$ ,*

$$\| (P_{\mathbf{A},V}/\lambda + 1)^{-k} \langle x \rangle^{-\sigma} \|_{L^2 \rightarrow L^2} \leq C \lambda^s.$$

*Proof.* By Hölder's inequality, we can consider  $\langle x \rangle^{-\sigma}$  as a bounded operator from  $L^2 \rightarrow L^p$ . To complete the proof of Proposition 3.2, it suffices to show that

$$\| (P_{\mathbf{A},V}/\lambda + 1)^{-k} \|_{L^p \rightarrow L^2} \lesssim \lambda^s.$$

From the Nash inequality [Nas58, Cou96], for all  $t > 0$  and  $p \in [1, 2]$ , there exists  $C > 0$  such that:

$$\| e^{-tP_{0,V}} \|_{L^p \rightarrow L^2} \leq C t^{-\frac{d}{2}(\frac{1}{p} - \frac{1}{2})}.$$

Then, from Proposition 3.1, we obtain that

$$(3.3) \quad \| e^{-t(P_{\mathbf{A},V}/\lambda)} \|_{L^p \rightarrow L^2} \leq C \lambda^s t^{-s},$$

where  $p = \frac{2d}{4s+d} \in [1, 2]$ . We use the identity

$$(P_{\mathbf{A},V}/\lambda + 1)^{-k} = \frac{1}{\Gamma(k)} \int_0^{+\infty} e^{-t(P_{\mathbf{A},V}/\lambda + 1)} t^{k-1} dt,$$

and (3.3), we get

$$\| (P_{\mathbf{A},V}/\lambda + 1)^{-k} \|_{L^p \rightarrow L^2} \leq C \lambda^s \int_0^{+\infty} e^{-t} t^{k-1-s} dt.$$

Since  $e^{-t} t^{k-1-s}$  is integrable, the conclusion follows.  $\square$

**Lemma 3.3.** *For all  $\sigma > 0$  and  $s \in [0, \min(\frac{d}{4}, 1)] \cap [0, \frac{\sigma}{2}]$ , there exist  $C > 0$ , such that for all  $\lambda > 0$ ,  $\zeta \in \mathbb{C}$  with  $|\text{Im}(\zeta)| > 0$ , we have*

$$(3.4) \quad \left\| (P_{\mathbf{A},V}/\lambda - \zeta)^{-1} \langle x \rangle^{-\sigma} \right\| \leq C \frac{\langle \zeta \rangle}{|\text{Im}(\zeta)|} \lambda^s.$$

*Proof.* We have

$$\begin{aligned} \left\| (P_{\mathbf{A},V}/\lambda - \zeta)^{-1} \langle x \rangle^{-\sigma} \right\| &= \left\| (P_{\mathbf{A},V}/\lambda - \zeta)^{-1} (P_{\mathbf{A},V}/\lambda + 1) (P_{\mathbf{A},V}/\lambda + 1)^{-1} \langle x \rangle^{-\sigma} \right\| \\ &\leq \left\| (P_{\mathbf{A},V}/\lambda - \zeta)^{-1} (P_{\mathbf{A},V}/\lambda + 1) \right\| \left\| (P_{\mathbf{A},V}/\lambda + 1)^{-1} \langle x \rangle^{-\sigma} \right\|. \end{aligned}$$

Using the fact that

$$\left\| (P_{\mathbf{A},V}/\lambda - \zeta)^{-1} (P_{\mathbf{A},V}/\lambda + 1) \right\| \leq C \frac{\langle \zeta \rangle}{|\text{Im}(\zeta)|},$$

and applying Proposition 3.2, we conclude the proof of (3.4).  $\square$

**Proposition 3.4.** *Let  $d \geq 3$ . For all  $\sigma, \epsilon > 0$ , there exist  $\lambda_0 > 0$ , and  $f \in C_0^\infty((0, \infty), [0, 1])$  such that  $f = 1$  near 1 and*

$$\left\| \left\langle \lambda^{\frac{1}{2}} x \right\rangle^{-\sigma} f(\lambda^{-1} P_{\mathbf{A},V}) \right\| \leq \epsilon,$$

for  $0 < \lambda \leq \lambda_0$ .



*Proof.* We start by decomposing  $\langle \lambda^{\frac{1}{2}}x \rangle^{-\sigma} f(\lambda^{-1}P_{\mathbf{A},V})$  as follows:

$$(3.5) \quad \langle \lambda^{\frac{1}{2}}x \rangle^{-\sigma} f(\lambda^{-1}P_{\mathbf{A},V}) = \langle \lambda^{\frac{1}{2}}x \rangle^{-\sigma} f(-\Delta/\lambda) + \langle \lambda^{\frac{1}{2}}x \rangle^{-\sigma} [f(P_{\mathbf{A},V}/\lambda) - f(-\Delta/\lambda)].$$

Using [BB21, Lemma 3.4], we can select  $f$  such that

$$\left\| \langle \lambda^{\frac{1}{2}}x \rangle^{-\sigma} f(\Delta/\lambda) \right\| \leq \frac{\epsilon}{2}.$$

To finish the proof, it suffices to show that, if  $\lambda$  is small enough,

$$\left\| \langle \lambda^{\frac{1}{2}}x \rangle^{-\sigma} [f(P_{\mathbf{A},V}/\lambda) - f(-\Delta/\lambda)] \right\| \leq \frac{\epsilon}{2}.$$

By the Helffer-Sjöstrand formula (see [DS99, Section 8]), we have

$$(3.6) \quad f(P_{\mathbf{A},V}/\lambda) - f(-\Delta/\lambda) = \frac{1}{\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{\zeta}}(\zeta) (P_{\mathbf{A},V}/\lambda - \zeta)^{-1} \left[ \frac{P_{\mathbf{A},V}}{\lambda} - \frac{-\Delta}{\lambda} \right] \left( -\frac{\Delta}{\lambda} - \zeta \right)^{-1} d\lambda(\zeta).$$

To estimate the norm of the integrand, we write

$$\begin{aligned} P_{\mathbf{A},V} + \Delta &= \frac{1}{w(x)} \sum_{1 \leq i, j \leq d} (D_i - A_i) [g^{ij}(x) (D_j - A_j)] + V(x) + \Delta \\ &= \frac{1}{w(x)} \left[ \sum_{1 \leq i, j \leq d} D_i [(g^{ij}(x) - \delta^{ij}(x)) D_j] - \operatorname{div}(g^{-1}A) + 2iA \cdot g^{-1}\nabla + g^{-1}A \cdot A \right] \\ &\quad + \frac{w(x) - 1}{w(x)} \Delta + V(x). \end{aligned}$$

We remark that we can write

$$P_{\mathbf{A},V} + \Delta = \sum_{|\alpha| \leq 2} a_{\alpha}(x) \partial^{\alpha},$$

with  $a_{\alpha} \in S^{-\rho_0 + |\alpha| - 2}(\mathbb{R}^d)$ . Thus,

$$\begin{aligned} \left\| (P_{\mathbf{A},V}/\lambda - \zeta)^{-1} \left[ \frac{P_{\mathbf{A},V}}{\lambda} - \frac{-\Delta}{\lambda} \right] \left( -\frac{\Delta}{\lambda} - \zeta \right)^{-1} \right\| &= \left\| (P_{\mathbf{A},V}/\lambda - \zeta)^{-1} \sum_{|\alpha| \leq 2} a_{\alpha}(x) \frac{\partial^{\alpha}}{\lambda} \left( -\frac{\Delta}{\lambda} - \zeta \right)^{-1} \right\| \\ &\leq \sum_{|\alpha| \leq 2} \left\| (P_{\mathbf{A},V}/\lambda - \zeta)^{-1} a_{\alpha}(x) \left( -\frac{\Delta}{\lambda} - \zeta \right)^{-1} \frac{\partial^{\alpha}}{\lambda} \right\|. \end{aligned}$$

Using Proposition 3.2, Lemma 3.3 and the spectral theorem, we deduce that:

- For  $|\alpha| = 2$ , for  $s$  near  $\frac{\rho_0}{2}$ , we have

$$\begin{aligned} \left\| (P_{\mathbf{A},V}/\lambda - \zeta)^{-1} a_{\alpha}(x) \left( -\frac{\Delta}{\lambda} - \zeta \right)^{-1} \frac{\partial^{\alpha}}{\lambda} \right\| &\leq \left\| (P_{\mathbf{A},V}/\lambda - \zeta)^{-1} \langle x \rangle^{-\rho_0} \right\| \left\| \left( -\frac{\Delta}{\lambda} - \zeta \right)^{-1} \frac{\partial^{\alpha}}{\lambda} \right\| \\ &\leq C \frac{\langle \zeta \rangle^2}{|\operatorname{Im}(\zeta)|^2} \lambda^s \left\| \left( -\frac{\Delta}{\lambda} + 1 \right)^{-1} \left( -\frac{\Delta}{\lambda} \right) \right\| \\ &\leq C \frac{\langle \zeta \rangle^2}{|\operatorname{Im}(\zeta)|^2} \lambda^s. \end{aligned}$$

- For  $|\alpha| = 1$ , for  $\tilde{s}$  near  $\frac{\rho_0}{4} + \frac{1}{2}$ , we have

$$\begin{aligned} \left\| (P_{\mathbf{A},V/\lambda} - \zeta)^{-1} a_\alpha(x) \left( -\frac{\Delta}{\lambda} - \zeta \right)^{-1} \frac{\partial^\alpha}{\lambda} \right\| &\leq \left\| (P_{\mathbf{A},V/\lambda} - \zeta)^{-1} \langle x \rangle^{-\rho_0-1} \right\| \left\| \left( -\frac{\Delta}{\lambda} - \zeta \right)^{-1} \frac{\partial^\alpha}{\lambda} \right\| \\ &\leq C \frac{\langle \zeta \rangle^2}{|\operatorname{Im}(\zeta)|^2} \lambda^{\tilde{s}-\frac{1}{2}} \left\| \left( -\frac{\Delta}{\lambda} + 1 \right)^{-1} \frac{(-\Delta)^{1/2}}{\lambda^{1/2}} \right\| \\ &\leq C \frac{\langle \zeta \rangle^2}{|\operatorname{Im}(\zeta)|^2} \lambda^{\tilde{s}-\frac{1}{2}}. \end{aligned}$$

- For  $|\alpha| = 0$ , for  $s_1$  near  $\frac{\rho_0}{4} + \frac{1}{2}$  and  $s_2$  near  $\frac{1}{2}$ , we have

$$\begin{aligned} \left\| (P_{\mathbf{A},V/\lambda} - \zeta)^{-1} a_\alpha(x) \left( -\frac{\Delta}{\lambda} - \zeta \right)^{-1} \frac{\partial^\alpha}{\lambda} \right\| &\leq \left\| (P_{\mathbf{A},V/\lambda} - \zeta)^{-1} \langle x \rangle^{-\rho_0-1} \right\| \left\| \langle x \rangle^{-1} \left( -\frac{\Delta}{\lambda} - \zeta \right)^{-1} \frac{\partial^\alpha}{\lambda} \right\| \\ &\leq C \frac{\langle \zeta \rangle^2}{|\operatorname{Im}(\zeta)|^2} \lambda^{s_1-1} \left\| \langle x \rangle^{-1} \left( -\frac{\Delta}{\lambda} + 1 \right)^{-1} \right\| \\ &\leq C \frac{\langle \zeta \rangle^2}{|\operatorname{Im}(\zeta)|^2} \lambda^{s_1+s_2-1}. \end{aligned}$$

From the previous estimates, we deduce that the integral in (3.6) is bounded by  $\frac{\epsilon}{2}$  for sufficiently small  $\lambda$ , since  $\frac{\partial \tilde{f}}{\partial \zeta}$  vanishes to infinite order on  $\operatorname{Im}(\zeta) = 0$  and is compactly supported.

Thus, combining (3.5) and (3.6), we conclude the proof.  $\square$

#### 4. MOURRE ESTIMATES

The main ingredient to prove such resolvent estimates near the real axis is the Mourre commutators method. We use a modified version of the usual conjugate operator of Schrödinger-type operator, namely the generator of dilations, which is defined by

$$(4.1) \quad T = -\frac{x \cdot i\nabla + i\nabla \cdot x}{2}.$$

For  $t \in \mathbb{R}$ ,  $u \in L^2$  and  $x \in \mathbb{R}^d$ , this operator satisfies the relation:

$$(4.2) \quad e^{itT} u(x) = e^{\frac{td}{2}} u(e^t x).$$

To localize the operator in space, we introduce a cut-off function  $\chi \in C_0^\infty(\mathbb{R}^d, [0, 1])$ , which is equal to 1 on a neighborhood of 0. Using this, we define the modified conjugate operator as

$$(4.3) \quad T^\lambda = -\frac{(1 - \chi(\lambda^{\frac{1}{2}} x))x \cdot i\nabla + i\nabla \cdot x(1 - \chi(\lambda^{\frac{1}{2}} x))}{2}.$$

To account for the self-adjointness of  $P_{\mathbf{A},V}$  in  $L^2(w(x)dx)$ , we introduce the operator

$$T_w^\lambda := w^{-\frac{1}{2}} T^\lambda w^{\frac{1}{2}}.$$

Using the abstract Mourre commutators method and applying Proposition 3.4, we have the following theorem.

**Theorem 4.1.** *Let  $d \geq 3$  and  $\lambda_0 > 0$ . Then, for each  $k \in \mathbb{N}$ , there exists a constant  $C > 0$  such that for all  $\lambda \in (0, \lambda_0]$*

$$(4.4) \quad \|(T_w^\lambda - i)^{-k} (P_{\mathbf{A},V/\lambda} - 1 \pm i0)^{-k} (T_w^\lambda + i)^{-k}\| \leq C.$$

For the proof of Theorem 4.1, we will use pseudodifferential calculus. To this end, we need some definitions.

**Definition 4.2.** A function  $q(x, \xi)$  belongs to the class  $S^{m,n}(\mathbb{R}^d \times \mathbb{R}^d)$  if, for all multi-indices  $\alpha, \beta \in \mathbb{N}^d$ , it satisfies the inequality:

$$|\partial_x^\alpha \partial_\xi^\beta q(x, \xi)| \leq C_{\alpha,\beta} \langle \xi \rangle^{m-|\beta|} \langle x \rangle^{n-|\alpha|}.$$

For a symbol  $q(x, \xi) \in S^{m,n}$ , the associated pseudodifferential operator  $\text{Op}(q)$  is defined by the quantization formula:

$$(\text{Op}(q)u)(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} e^{i(x-y)\cdot\xi} q(x, \xi) u(y) dy d\xi,$$

where  $u$  is a test function (or a distribution) on  $\mathbb{R}^d$ .

*Proof.* Using tools from scattering theory (Mourre theory, see for example [JMP84]), to demonstrate Theorem 4.4, it suffices to show the following positive commutator estimate

$$(4.5) \quad f(P_{\mathbf{A},V}/\lambda) [P_{\mathbf{A},V}/\lambda, iT_w^\lambda] f(P_{\mathbf{A},V}/\lambda) \geq f^2(P_{\mathbf{A},V}/\lambda),$$

where  $f \in C_0^\infty(]0, +\infty[)$  such that  $f = 1$  near 1.

To prove (4.5), we outline some proof ideas inspired by Section 2 of [BB21]. First, observe that:

$$\begin{aligned} P_{\mathbf{A},V} - (-\Delta) &= \frac{1}{w(x)} \sum_{1 \leq i,j \leq d} \left[ [(D_i - A_i)g^{ij}(x)] D_j + (-D_i g^{ij}(x) + A_i(x)g^{ij}(x)) A_j(x) + \right. \\ &\quad \left. + (g^{ij}(x) - w(x)\delta_{ij}) D_i D_j - g^{ij}(x) D_j A_i(x) \right] + V(x). \end{aligned}$$

Then, we can write

$$(4.6) \quad P_{\mathbf{A},V}/\lambda + \Delta/\lambda = e^{i\tau T} p_\lambda(x, D) e^{-i\tau T},$$

where  $\tau = \ln(\lambda^{\frac{1}{2}})$  (corresponding to a rescaling by  $\lambda^{\frac{1}{2}}$ , see (4.6)), and the symbol  $p_\lambda(x, \xi)$  is given by:

$$\begin{aligned} p_\lambda(x, \xi) &= \frac{\lambda^{-\frac{1}{2}}}{w(\lambda^{\frac{1}{2}}x)} \sum_{1 \leq i,j \leq d} [(D_i - A_i(\lambda^{\frac{1}{2}}x))g^{ij}(\lambda^{\frac{1}{2}}x)\xi_j] \\ &\quad + \frac{1}{w(\lambda^{\frac{1}{2}}x)} \sum_{1 \leq i,j \leq d} (g^{ij}(\lambda^{\frac{1}{2}}x) - w(\lambda^{\frac{1}{2}}x)\delta_{ij})\xi_i\xi_j \\ &\quad + \frac{\lambda^{-1}}{w(\lambda^{\frac{1}{2}}x)} \sum_{1 \leq i,j \leq d} (-D_i g^{ij}(\lambda^{\frac{1}{2}}x) + A_i(\lambda^{\frac{1}{2}}x)g^{ij}(\lambda^{\frac{1}{2}}x))A_j(\lambda^{\frac{1}{2}}x) \\ &\quad - \frac{\lambda^{-1}}{w(\lambda^{\frac{1}{2}}x)} \sum_{1 \leq i,j \leq d} g^{ij}(\lambda^{\frac{1}{2}}x)D_j A_i(\lambda^{\frac{1}{2}}x) + \lambda^{-1}V(\lambda^{\frac{1}{2}}x). \end{aligned}$$

Now, examining the components of  $p_\lambda(x, \xi)$ , we deduce the following symbolic properties:

- $(D_i - A_i(\lambda^{\frac{1}{2}}x))g^{ij}(\lambda^{\frac{1}{2}}x)\xi_j \in \lambda^{\frac{\rho_0+1}{2}} S^{1,-1-\rho_0}$ .
- $(g^{ij}(\lambda^{\frac{1}{2}}x) - w(\lambda^{\frac{1}{2}}x)\delta_{ij})\xi_i\xi_j \in \lambda^{\frac{\rho_0}{2}} S^{2,-\rho_0}$ .
- $(-D_i g^{ij}(\lambda^{\frac{1}{2}}x) + A_i(\lambda^{\frac{1}{2}}x)g^{ij}(\lambda^{\frac{1}{2}}x))A_j(\lambda^{\frac{1}{2}}x) \in \lambda^{1+\rho_0} S^{0,-2-2\rho_0}$ .
- $g^{ij}(\lambda^{\frac{1}{2}}x)D_j A_i(\lambda^{\frac{1}{2}}x)$  and  $V(\lambda^{\frac{1}{2}}x) \in \lambda^{1+\frac{\rho_0}{2}} S^{0,-2-\rho_0}$ .

From the previous writing, we deduce that:

$$(4.7) \quad p_\lambda(x, D) \in \lambda^{\frac{\rho_0}{2}} \text{Op}(S^{2,-\rho_0}).$$

On the other hand, we have

$$T_w^\lambda = (1 - \chi(\lambda^{\frac{1}{2}}x))T + \frac{i}{2}(1 - \chi)(\lambda^{\frac{1}{2}}x) \frac{x \cdot \nabla w(x)}{w(x)} + \frac{i}{2}(\lambda^{\frac{1}{2}}x) \cdot \nabla \chi(\lambda^{\frac{1}{2}}x) = e^{i\tau T} t_\lambda(x, D) e^{-i\tau T}$$

where

$$t_\lambda(x, \xi) = (1 - \chi)(x) \left( x \cdot \xi + \frac{d}{2i} \right) + \frac{i}{2} (1 - \chi)(x) \frac{\lambda^{-\frac{1}{2}} x \cdot \nabla w(\lambda^{-\frac{1}{2}} x)}{w(\lambda^{-\frac{1}{2}} x)} + \frac{i}{2} x \cdot \nabla \chi(x).$$

One can rewrite its symbolic form as

$$t_\lambda(x, \xi) = (1 - \chi)(x) \left( x \cdot \xi + \frac{d}{2i} \right) + (1 - \chi)(x) \lambda^{\frac{\rho_0}{2}} S^{0, -\rho_0} + C_0^\infty(\mathbb{R}^d \setminus \{0\}).$$

Using the symbolic form of  $T_w^\lambda$  and  $P_{\mathbf{A}, V}$ , we have

$$\begin{aligned} [P_{\mathbf{A}, V}/\lambda, iT_w^\lambda] &= [P_{\mathbf{A}, V}/\lambda + \Delta/\lambda, iT_w^\lambda] + [-\Delta/\lambda, iT_w^\lambda] \\ &= e^{i\tau T} \left( i [p_\lambda(x, D), t_\lambda(x, D)] \right) e^{-i\tau T} + e^{i\tau T} \left( i [|\xi|^2, t_\lambda(x, D)] \right) e^{-i\tau T} \\ &= e^{i\tau T} \left( i [p_\lambda(x, D), t_\lambda(x, D)] \right) e^{-i\tau T} + 2(1 - \chi)(\lambda^{\frac{1}{2}} x) (-\Delta/\lambda) \\ &\quad + e^{i\tau T} \left( i \left[ |\xi|^2, (1 - \chi)(x) \lambda^{\frac{\rho_0}{2}} S^{0, -\rho_0} + C_0^\infty(\mathbb{R}^d \setminus \{0\}) \right] \right) e^{-i\tau T} \end{aligned}$$

By the pseudo-differential calculus, we can compute

$$\begin{aligned} &[p_\lambda(x, D), t_\lambda(x, D)] \\ &= \left[ \lambda^{\frac{\rho_0}{2}} \text{Op}(S^{2, -\rho_0}), (1 - \chi)(x) \left( x \cdot \xi + \frac{d}{2i} \right) + (1 - \chi)(x) \lambda^{\frac{\rho_0}{2}} \text{Op}(S^{0, -\rho_0}) + C_0^\infty(\mathbb{R}^d \setminus \{0\}) \right] \\ &= \left[ \lambda^{\frac{\rho_0}{2}} \text{Op}(S^{2, -\rho_0}), (1 - \chi)(x) \text{Op}(S^{1, 1}) + (1 - \chi)(x) \lambda^{\frac{\rho_0}{2}} \text{Op}(S^{0, -\rho_0}) + C_0^\infty(\mathbb{R}^d \setminus \{0\}) \right] \\ &= \lambda^{\frac{\rho_0}{2}} \text{Op}(S^{2, -\rho_0}) \Phi_1(x), \end{aligned}$$

where  $\Phi_1$  equal 0 near 0. For an arbitrary  $N$ , we also have

$$\begin{aligned} &\left[ |\xi|^2, (1 - \chi)(x) \lambda^{\frac{\rho_0}{2}} S^{0, -\rho_0} + C_0^\infty(\mathbb{R}^d \setminus \{0\}) \right] \\ &= \left[ \text{Op}(S^{2, 0}), (1 - \chi)(x) \lambda^{\frac{\rho_0}{2}} \text{Op}(S^{0, -\rho_0}) + C_0^\infty(\mathbb{R}^d \setminus \{0\}) \right] \\ &= \lambda^{\frac{\rho_0}{2}} \text{Op}(S^{2, -\rho_0}) \Phi_2(x) + \text{Op}(S^{2, -N}) \Phi_3(x), \end{aligned}$$

where  $\Phi_2$  and  $\Phi_3$  are equal to 0 near 0. Thus, using (4.6) and (4.7), we conclude:

$$\begin{aligned} &[P_{\mathbf{A}, V}/\lambda, iT_w^\lambda] \\ &= 2(1 - \chi)(\lambda^{\frac{1}{2}} x) (-\Delta/\lambda) + \lambda^{\frac{\rho_0}{2}} e^{i\tau T} \text{Op}(S^{2, -\rho_0}) e^{-i\tau T} \Phi(\lambda^{\frac{1}{2}} x) \\ &\quad + \left\langle \lambda^{\frac{1}{2}} x \right\rangle^{-N} e^{i\tau T} \text{Op}(S^{2, 0}) e^{-i\tau T} \Phi_3(\lambda^{\frac{1}{2}} x) \\ &= 2(1 - \chi)(\lambda^{\frac{1}{2}} x) (-\Delta/\lambda - P_{\mathbf{A}, V}/\lambda) + 2(1 - \chi)(\lambda^{\frac{1}{2}} x) (P_{\mathbf{A}, V}/\lambda) + \lambda^{\frac{\rho_0}{2}} e^{i\tau T} \text{Op}(S^{2, -\rho_0}) e^{-i\tau T} \Phi(\lambda^{\frac{1}{2}} x) \\ &\quad + \left\langle \lambda^{\frac{1}{2}} x \right\rangle^{-N} e^{i\tau T} \text{Op}(S^{2, 0}) e^{-i\tau T} \Phi_3(\lambda^{\frac{1}{2}} x) \\ &= 2(1 - \chi)(\lambda^{\frac{1}{2}} x) (P_{\mathbf{A}, V}/\lambda) + \lambda^{\frac{\rho_0}{2}} e^{i\tau T} \text{Op}(S^{2, -\rho_0}) e^{-i\tau T} \Phi(\lambda^{\frac{1}{2}} x) \\ &\quad + \left\langle \lambda^{\frac{1}{2}} x \right\rangle^{-N} e^{i\tau T} \text{Op}(S^{2, 0}) e^{-i\tau T} \Phi_3(\lambda^{\frac{1}{2}} x), \end{aligned}$$

where  $\Phi$  equal 0 near 0. For  $N = \rho_0$ , we deduce that

$$[P_{\mathbf{A}, V}/\lambda, iT_w^\lambda] - 2P_{\mathbf{A}, V}/\lambda = -2\chi(\lambda^{\frac{1}{2}} x) (P_{\mathbf{A}, V}/\lambda) + \left\langle \lambda^{\frac{1}{2}} x \right\rangle^{-\rho_0} e^{i\tau T} \text{Op}(S^{2, 0}) e^{-i\tau T} \tilde{\Phi}(\lambda^{\frac{1}{2}} x),$$

where where  $\tilde{\Phi}$  equal 0 near 0.

Take  $f_1$  satisfying Proposition 3.4, and compose  $[P_{\mathbf{A},V/\lambda}, iT_w^\lambda] - 2P_{\mathbf{A},V/\lambda}$  on the right and on the left with  $f_1(P_{\mathbf{A},V/\lambda})$ , we obtain

$$\begin{aligned}
& \left\| f_1(P_{\mathbf{A},V/\lambda}) ([P_{\mathbf{A},V/\lambda}, iT_w^\lambda] - 2P_{\mathbf{A},V/\lambda}) f_1(P_{\mathbf{A},V/\lambda}) \right\| \\
& \leq \left\| f_1(P_{\mathbf{A},V/\lambda}) \chi(\lambda^{\frac{1}{2}}x) (P_{\mathbf{A},V/\lambda}) f_1(P_{\mathbf{A},V/\lambda}) \right\| \\
& + \left\| f_1(P_{\mathbf{A},V/\lambda}) \left\langle \lambda^{\frac{1}{2}}x \right\rangle^{-\rho_0} e^{i\tau T} \text{Op}(S^{2,0}) e^{-i\tau T} \tilde{\Phi}(\lambda^{\frac{1}{2}}x) f_1(P_{\mathbf{A},V/\lambda}) \right\| \\
& \leq \left\| f_1(P_{\mathbf{A},V/\lambda}) \chi(\lambda^{\frac{1}{2}}x) (P_{\mathbf{A},V/\lambda}) (P_{\mathbf{A},V/\lambda} + i)^{-1} \right\| \left\| (P_{\mathbf{A},V/\lambda} + i) f_1(P_{\mathbf{A},V/\lambda}) \right\| \\
& + \left\| f_1(P_{\mathbf{A},V/\lambda}) \left\langle \lambda^{\frac{1}{2}}x \right\rangle^{-\rho_0} e^{i\tau T} \text{Op}(S^{2,0}) e^{-i\tau T} \tilde{\Phi}(\lambda^{\frac{1}{2}}x) (P_{\mathbf{A},V/\lambda} + i)^{-1} \right\| \left\| (P_{\mathbf{A},V/\lambda} + i) f_1(P_{\mathbf{A},V/\lambda}) \right\| \\
& \leq \left\| f_1(P_{\mathbf{A},V/\lambda}) \left\langle \lambda^{\frac{1}{2}}x \right\rangle^{-\rho_0} \right\| \left\| \chi(\lambda^{\frac{1}{2}}x) \left\langle \lambda^{\frac{1}{2}}x \right\rangle^{\rho_0} (P_{\mathbf{A},V/\lambda}) (P_{\mathbf{A},V/\lambda} + i)^{-1} \right\| \left\| (P_{\mathbf{A},V/\lambda} + i) f_1(P_{\mathbf{A},V/\lambda}) \right\| \\
& + \left\| f_1(P_{\mathbf{A},V/\lambda}) \left\langle \lambda^{\frac{1}{2}}x \right\rangle^{-\rho_0} \right\| \left\| e^{i\tau T} \text{Op}(S^{2,0}) e^{-i\tau T} \tilde{\Phi}(\lambda^{\frac{1}{2}}x) (P_{\mathbf{A},V/\lambda} + i)^{-1} \right\| \left\| (P_{\mathbf{A},V/\lambda} + i) f_1(P_{\mathbf{A},V/\lambda}) \right\|.
\end{aligned}$$

Using the spectral theorem and the support of  $\chi$ , we have the following estimates:

$$\left\| (P_{\mathbf{A},V/\lambda} + i) f_1(P_{\mathbf{A},V/\lambda}) \right\| \leq C, \quad \left\| \chi(\lambda^{\frac{1}{2}}x) \left\langle \lambda^{\frac{1}{2}}x \right\rangle^{\rho_0} (P_{\mathbf{A},V/\lambda}) (P_{\mathbf{A},V/\lambda} + i)^{-1} \right\| \leq C.$$

Using standard techniques for constructing a parametrix of  $(P_{\mathbf{A},V/\lambda} + i)^{-1}$  (see [BB21, Proposition 2.6]), there exists a constant  $C$ , independent of  $\lambda$  such that:

$$\left\| e^{i\tau T} \text{Op}(S^{2,0}) e^{-i\tau T} \tilde{\Phi}(\lambda^{\frac{1}{2}}x) (P_{\mathbf{A},V/\lambda} + i)^{-1} \right\| \leq C.$$

Therefore, we obtain:

$$\left\| f_1(P_{\mathbf{A},V/\lambda}) ([P_{\mathbf{A},V/\lambda}, iT_w^\lambda] - 2P_{\mathbf{A},V/\lambda}) f_1(P_{\mathbf{A},V/\lambda}) \right\| \lesssim \left\| f_1(P_{\mathbf{A},V/\lambda}) \left\langle \lambda^{\frac{1}{2}}x \right\rangle^{-\rho_0} \right\|.$$

By Proposition 3.2, the right hand side is arbitrarily small. Hence, we simply have:

$$f_1(P_{\mathbf{A},V/\lambda}) ([P_{\mathbf{A},V/\lambda}, iT_w^\lambda] - 2P_{\mathbf{A},V/\lambda}) f_1(P_{\mathbf{A},V/\lambda}) \geq -\frac{1}{2}.$$

Moreover, when the support of  $f_1$  is small enough around 1, we have

$$2f_1(P_{\mathbf{A},V/\lambda}) (P_{\mathbf{A},V/\lambda}) f_1(P_{\mathbf{A},V/\lambda}) \geq \frac{3}{2} f_1^2(P_{\mathbf{A},V/\lambda}).$$

Thus, after composition by  $f(P_{\mathbf{A},V/\lambda})$  with  $f$  equal 1 near 1, (4.5) holds.  $\square$

## 5. MAIN RESULT: TIME DECAY ESTIMATES

In this section, we prove Theorems 1.3, 1.4,1.5, and Corollary 1.6. To do so, we will use the following proposition.

**Proposition 5.1.** *Let  $d \geq 3$ ,  $k \in \mathbb{N}$  and  $f \in C_0^\infty(\mathbb{R})$ . Let  $\nu \geq k$  and  $s \in [0, \frac{d}{4}]$  be such that  $\nu > 2s$ . Then*

$$(5.1) \quad \left\| (T_w^\lambda + i)^k f(P_{\mathbf{A},V/\lambda}) \langle x \rangle^{-\nu} \right\| \leq C \lambda^s,$$

as long as  $\lambda > 0$  belongs to a bounded set.

*Proof.* The proof is similar to that of Proposition 4.1 from [BB21], without a magnetic field and an electric potential because, by (4.6),  $P_{\mathbf{A},V}$  has the same symbolic structure as (2.8) of [BB21].  $\square$

By combining (4.4) with Proposition 5.2, we can directly derive bounds on the spectrally localized resolvent.

**Theorem 5.2.** *Let  $d \geq 3$ ,  $f \in C_0^\infty(\mathbb{R})$ ,  $k \in \mathbb{N}$  and  $\lambda_0 > 0$ . If  $\nu \geq k$  and  $s \in [0, \frac{d}{4}]$  are such that  $\nu > 2s$ , then*

$$\left\| \langle x \rangle^{-\nu} f(P_{\mathbf{A},V}/\lambda) (P_{\mathbf{A},V} - \lambda \pm i0)^{-k} \langle x \rangle^{-\nu} \right\| \leq C \lambda^{2s-k}$$

for all  $\lambda \in (0, \lambda_0]$ . In particular, if  $\nu > \frac{d}{2}$  and  $\nu \geq k$ , then

$$\left\| \langle x \rangle^{-\nu} f(P_{\mathbf{A},V}/\lambda) (P_{\mathbf{A},V} - \lambda \pm i0)^{-k} \langle x \rangle^{-\nu} \right\| \leq C \lambda^{\frac{d}{2}-k}.$$

This theorem allows us to obtain a bound on the resolvent that is still localized, though less strongly. Since the proof is analogous to Proposition 4.4 in [BB21], it is not provided here.

**Proposition 5.3.** *Let  $\lambda_0 > 0$ , there exists  $F \in C_0^\infty(\mathbb{R})$  (and actually for all  $F$ ) equal to 1 near  $(0, \lambda_0]$  such that for  $k \geq 1$  and  $\nu > k$ , one has*

$$\left\| \langle x \rangle^{-\nu} F(P_{\mathbf{A},V}) (P_{\mathbf{A},V} - \lambda \pm i0)^{-k} \langle x \rangle^{-\nu} \right\| \leq C \lambda^{2s-k},$$

for all  $\lambda \in (0, \lambda_0]$  and  $k \neq \frac{d}{2}$ .

If  $d$  is even and  $k = \frac{d}{2}$ , we have

$$\left\| \langle x \rangle^{-\nu} F(P_{\mathbf{A},V}) (P_{\mathbf{A},V} - \lambda \pm i0)^{-\frac{d}{2}} \langle x \rangle^{-\nu} \right\| \leq C |\log \lambda|.$$

*Proof of Theorem 1.4.* Select  $F$  as described in Proposition 5.3. The result follows because  $(1 - F(P_{\mathbf{A},V})) (P_{\mathbf{A},V} - \lambda \pm i0)^{-k}$  is uniformly bounded with respect to  $\lambda$  on  $L^2$ , as ensured by the spectral theorem.  $\square$

*Proof of Theorem 1.5.* From (1.6), we have

$$\begin{aligned} \frac{d^j}{d\lambda^j} \langle x \rangle^{-\nu} E'_{P_{\mathbf{A},V}}(\lambda) \langle x \rangle^{-\nu} &= \frac{1}{2i\pi} \frac{d^j}{d\lambda^j} \langle x \rangle^{-\nu} [(P_{\mathbf{A},V} - \lambda - i0)^{-1} - (P_{\mathbf{A},V} - \lambda + i0)^{-1}] \langle x \rangle^{-\nu} \\ &= \frac{1}{2i\pi} \langle x \rangle^{-\nu} \frac{d^j}{d\lambda^j} [(P_{\mathbf{A},V} - \lambda - i0)^{-1} - (P_{\mathbf{A},V} - \lambda + i0)^{-1}] \langle x \rangle^{-\nu} \\ &= \frac{j!}{2i\pi} \langle x \rangle^{-\nu} [(P_{\mathbf{A},V} - \lambda - i0)^{-1-j} - (P_{\mathbf{A},V} - \lambda + i0)^{-1-j}] \langle x \rangle^{-\nu}. \end{aligned}$$

By Theorem 5.2, we complete the proof.  $\square$

*Proof of Theorem 1.3.* Let  $m = \lfloor \frac{d}{2} \rfloor$ . For  $\nu > \lfloor \frac{d}{2} \rfloor + 2$ , we have

$$\begin{aligned} (it)^m \langle x \rangle^{-\nu} \varphi(P_{\mathbf{A},V}) e^{itP_{\mathbf{A},V}} \langle x \rangle^{-\nu} &= \frac{1}{2i\pi} \int_0^{+\infty} \varphi(\lambda) \langle x \rangle^{-\nu} \partial_\lambda^m (e^{it\lambda}) E'_{P_{\mathbf{A},V}}(\lambda) \langle x \rangle^{-\nu} d\lambda \\ &= \frac{(-1)^m}{2i\pi} \int_0^{+\infty} e^{it\lambda} \partial_\lambda^m [\langle x \rangle^{-\nu} \varphi(\lambda) E'_{P_{\mathbf{A},V}}(\lambda) \langle x \rangle^{-\nu}] d\lambda \\ &= \frac{(-1)^m}{2i\pi} \int_0^{t^{-1}} e^{it\lambda} \partial_\lambda^m [\langle x \rangle^{-\nu} \varphi(\lambda) E'_{P_{\mathbf{A},V}}(\lambda) \langle x \rangle^{-\nu}] d\lambda + \\ &\quad + \frac{(-1)^m}{2i\pi} \int_{t^{-1}}^{+\infty} e^{it\lambda} \partial_\lambda^m [\langle x \rangle^{-\nu} \varphi(\lambda) E'_{P_{\mathbf{A},V}}(\lambda) \langle x \rangle^{-\nu}] d\lambda. \end{aligned}$$

On the one hand, by Theorem 1.5, we have

$$\left\| \int_0^{t^{-1}} e^{it\lambda} \partial_\lambda^m [\langle x \rangle^{-\nu} \varphi(\lambda) E'_{P_{\mathbf{A},V}}(\lambda) \langle x \rangle^{-\nu}] d\lambda \right\|_{L^2} \lesssim \int_0^{t^{-1}} \lambda^{\frac{d}{2}-1-m} d\lambda \lesssim t^{-\frac{d}{2}+m}$$

On the other hand, by Theorem 1.5, we also have:

$$\begin{aligned}
& t \left\| \int_{t^{-1}}^{+\infty} e^{it\lambda} \partial_\lambda^m \left[ \langle x \rangle^{-\nu} \varphi(\lambda) E'_{P_{\mathbf{A},V}}(\lambda) \langle x \rangle^{-\nu} \right] d\lambda \right\|_{L^2} \\
&= \left\| \int_{t^{-1}}^{+\infty} \partial_\lambda e^{it\lambda} \partial_\lambda^m \left[ \langle x \rangle^{-\nu} \varphi(\lambda) E'_{P_{\mathbf{A},V}}(\lambda) \langle x \rangle^{-\nu} \right] d\lambda \right\|_{L^2} \\
&= \left\| \partial_\lambda^m \left[ \langle x \rangle^{-\nu} \varphi(t^{-1}) E'_{P_{\mathbf{A},V}}(t^{-1}) \langle x \rangle^{-\nu} \right] \right\|_{L^2} + \left\| \int_{t^{-1}}^{+\infty} e^{it\lambda} \partial_\lambda^{m+1} \left[ \langle x \rangle^{-\nu} \varphi(\lambda) E'_{P_{\mathbf{A},V}}(\lambda) \langle x \rangle^{-\nu} \right] d\lambda \right\|_{L^2} \\
&\lesssim t^{-\frac{d}{2}+m+1}.
\end{aligned}$$

Combining these estimates, we conclude the desired time decay rate for the evolution operator.  $\square$

*Proof of Corollary 1.6.* For  $\varphi \in C_0^\infty(\mathbb{R}_+)$ , we have

$$\begin{aligned}
& \left\| \langle x \rangle^{-\nu} e^{itP_{\mathbf{A},V}} \langle x \rangle^{-\nu} \right\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \\
&= \left\| \langle x \rangle^{-\nu} \varphi(P_{\mathbf{A},V}) e^{itP_{\mathbf{A},V}} \langle x \rangle^{-\nu} \right\|_{L^2 \rightarrow L^2} + \left\| \langle x \rangle^{-\nu} (1 - \varphi(P_{\mathbf{A},V})) e^{itP_{\mathbf{A},V}} \langle x \rangle^{-\nu} \right\|_{L^2 \rightarrow L^2}.
\end{aligned}$$

The first norm follows from Theorem 1.3. For the second norm, if the non-trapping condition holds, we have, by [JMP84, Rob92],

$$\left\| \langle x \rangle^{-\nu} (1 - \varphi(P_{\mathbf{A},V})) e^{itP_{\mathbf{A},V}} \langle x \rangle^{-\nu} \right\|_{L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)} \lesssim \langle t \rangle^{-\nu'}, \quad \text{for } \nu' < \nu.$$

In particular, for  $\nu' = \frac{d}{2}$ , we obtain the desired estimate for the  $L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$  operator norm. From the dyadic partition (see [BB21, Section 6]), this can be replaced by the  $L^2(\mathbb{R}^d) \rightarrow H^{\frac{d}{2}}(\mathbb{R}^d)$  norm.  $\square$

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