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Stabilization of aperiodic sampled-data switched affine systems to hybrid limit cycles

Carolina Albea^a, Mathias Serieye^b, Alexandre Seuret^a, Marc Jungers^c

^aUniversidad de Sevilla, Avd. de los Descubrimientos s/n, Sevilla, 41092, Spain.

^bUniversité de Toulouse, LAAS-CNRS, 7 avenue du Colonel Roche, Toulouse, 31031, France.

^cUniversité de Lorraine, CNRS, CRAN, F-54000, Nancy, France.

Abstract

This paper deals with the stabilization of aperiodic sampled-data switched affine systems to a predetermined hybrid limit cycle using a hybrid dynamical system representation and a control Lyapunov function approach. Some preliminaries on the hybrid dynamical system formalism provide the framework for modeling switched affine systems followed by definitions on hybrid limit cycles and related notions. The main result, based on simple Linear Matrix Inequalities (LMI), guarantees that the solutions to the closed-loop system converge to a hybrid limit cycle defined by the states, functioning modes with their corresponding dwell times. The theoretical results are evaluated on academic examples and demonstrate the potential and the originality of the method over the recent literature.

Keywords: Hybrid dynamical systems, Hybrid limit cycles, LMI, Control Lyapunov function.

1. Introduction

Hybrid systems represent a class of dynamical systems that can be described by a combination of continuous- and discrete-time dynamics. This allows us to portray a wide range of systems encountered on many occasions in different fields of application including power electronic, robotic or biology [11], [18], [12].

As a subclass, switched systems can be described as a family of regular systems and a rule orchestrating which one to follow at each time. This

rule is of great importance when considering switched affine systems, especially from the control point of view since it represents the only solution to control the system. Furthermore, the fact that the subsystems do not necessarily share a common equilibrium prevents, in general, from the asymptotic stabilization to a single equilibrium point. Numerous contributions have been motivated by the question of stabilizing switched affine systems to an operating point because of this characteristic [1], [10]. The drawback of many results in continuous-time is an arbitrarily high switching frequency to maintain the state trajectories in the vicinity of the desired equilibrium, generating eventually a Zeno behavior, i.e., an infinite number of switches in a bounded interval of the ordinary time (see [17, Def. 2.5]).

For the purpose of implementing the control input or due to physical constraints, it is then necessary to relax the control objectives. A common method is to design a sampled-data switching function, through either a continuous-time model [2],[20] or the discretized model [14],[27]. However, it is important to note that the resulting methods only ensure practical stability, meaning that the convergence is only guaranteed towards the surrounding of the desired operating point. Once a solution reaches that region, nothing can be said about the behavior of the state trajectories. To characterize in a thinner manner the steady state, an interesting solution is to consider the stabilization to a limit cycle.

Indeed, apart from equilibrium points, dynamical systems may show asymptotic behaviors such as limit cycles, that are defined as closed and isolated trajectories resulting from periodic solutions. Limit cycles are a specific case of limit sets and their study was initiated at first by H. Poincaré [24]. Through his researches, H. Poincaré provides a simple characterization of ω -limit sets for dynamical systems in dimension 2. Results that are not easy to generalize in higher dimensions nor in the hybrid system framework, remain relevant in this context. For instance, the reader may refer to the cases of piece-wise linear hybrid systems [16], switching systems [21], relay feedback systems [22], piece-wise affine systems [23] or, more generally, hybrid systems [30]. In most of the cases, the approach to analyze the stability of a limit cycle is to derive the Poincaré-Bendixon theorem. One of the main difficulties is, however, to determine the switching instants related to that limit cycle. A first step to circumvent this issue was to consider discrete-time switched affine systems such as in [13] and latter [28]. In both papers, the authors proposed a control law design based

on the resolution of a linear matrix inequality (LMI) problem originating from [7] on periodic systems. On the other hand, the two proposed control laws are different and so are the contributions. Reference [13] focuses on the convergence of the system's states to the periodic state-trajectories using a time-varying control law, while the authors from [28] came up with a time-independent control law allowing to extend the result to the case of uncertain switched affine systems, presenting then additional robustness compared to the first solution. In a second step, the authors of [13] derive their result to the case of continuous-time switched affine systems [15], where the steady state is forced to be periodic.

In this paper, the hybrid dynamical system framework is adopted to model switched affine systems following the paradigm given in [17]. Such a formulation has been already proposed in [33, 34] or in [2]. By using this formulation, this work focuses on the design of a closed-loop hybrid dynamical system, guaranteeing the Uniformly Globally Asymptotically Stability (UGAS) property of a given attractor by solving LMIs. This attractor is characterized by a hybrid limit cycle defined by the states, functioning modes and their corresponding dwell times, which is not necessarily assumed to be uniform for all modes. To do so, the definition of hybrid limit cycles and the conditions of their existence are adapted to the current formalism following the thread given in [28]. The remainder of the paper concerns to compare our result with previous ones and some illustrative examples with a short part on the selection of the cycle to minimize a given cost function aiming at evaluating the amplitude of the corresponding limit cycle of the plant and its distance to a desired functioning point.

Hence, the main contributions of this paper are:

- To provide UGAS property, which implies robustness of the system with respect to measurement errors, for instance [17, Chap.3].
- To design controllers that consider the aperiodic sampled-data switching, different to discrete-time or continuous-time approaches where the switching are periodic.
- To improve the transient time, which is an indirect result obtained of considering the real trajectory of the signals, i.e., comprising continuous-time and discrete-time signals.

Notations: Throughout the paper, \mathbb{N} (\mathbb{N}^*) denotes the set of (strictly positive) natural numbers, \mathbb{R} the real numbers, \mathbb{R}^n the n -dimensional Euclidean space and $\mathbb{R}^{n \times m}$ the set of all real $n \times m$ matrices. For any n in \mathbb{N} , matrix I_n denotes the identity matrix of $\mathbb{R}^{n \times n}$. When no confusion is possible, the subscript of this matrix that precises its dimension will be omitted. For any matrix M of $\mathbb{R}^{n \times n}$, the notation $M \succ 0$ ($M \prec 0$) means that M is symmetric positive (negative) definite and $\det(M)$ represents its determinant. The set of symmetric positive definite matrix of $\mathbb{R}^{n \times n}$ is denoted as \mathbb{S}_+^n . Notation $\text{diag}_{i=0}^N M_i$ stands for the block diagonal matrix composed of the elements M_i in the block position (i, i) . $\|\cdot\|$ denotes the Euclidean norm. For a symmetric positive definite matrix P and a vector x , we denote $\|x\|_P = \sqrt{x^\top P x}$, the weighted norm. For a symmetric matrix, $\lambda_m(\cdot)$ and $\lambda_M(\cdot)$ denote its minimal and maximal eigenvalues respectively.

2. Sampled-data switching affine model

2.1. System data

Consider the continuous-time switched affine system governed by

$$\begin{cases} \dot{x}(t) &= A_{\sigma(t)}x(t) + B_{\sigma(t)} \\ \sigma(t) &\in u(x(t_k)) \subset \mathbb{K}, \quad \forall t \in [t_k, t_{k+1}) \\ x_0 &\in \mathbb{R}^n, \end{cases} \quad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector and x_0 its initial condition, $\sigma(t)$ is a sampled-data switching signal which indicates the active mode in each time sampling interval $[t_k, t_{k+1})$ and u the state feedback control law to be designed. The sequence $\{t_k\}_{k \in \mathbb{N}}$ is a strictly increasing sequence of time instants for which it is assumed that there exist two positive scalars $T_m > 0$ and $T_M > 0$ such that the difference between two successive sampling instants verifies

$$t_{k+1} - t_k \in [T_m, T_M], \quad \forall k \in \mathbb{N}, \quad (2)$$

so that the sequence $\{t_k\}_{k \in \mathbb{N}}$ tends to infinity as k tends to infinity. Finally, system (1) is composed of K subsystems defined through matrices $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times 1}$ for all $i \in \mathbb{K} = \{1, 2, \dots, K\}$, which are assumed to be constant and known.

2.2. General hybrid dynamical model

Since switched affine systems, which may occur after given one or several dwell times defined a priori by the designer, represent a subclass of hybrid systems, it is reasonable to model (1) with the hybrid formalism given in [17]. The switches of the control law that may occur after given one or several dwell times defined *a priori* by the designer, correspond to the discrete-time behavior in hybrid systems while the continuous-time behavior is represented by the differential equation given in (1). More formally, the sampled-data model written in the framework of hybrid dynamical systems is given by

$$\mathcal{H}_0 : \begin{cases} \begin{bmatrix} \dot{x} \\ \dot{\sigma} \\ \dot{\tau} \\ \dot{T} \end{bmatrix} = f_0(x, \sigma, \tau, T) & (x, \sigma, \tau, T) \in \mathcal{C}_0 \\ \begin{bmatrix} x^+ \\ \sigma^+ \\ \tau^+ \\ T^+ \end{bmatrix} \in G_0(x, \sigma, \tau, T) & (x, \sigma, \tau, T) \in \mathcal{D}_0. \end{cases} \quad (3)$$

In this formulation, new state variables τ and T have been introduced and represent a timer that accounts for the time elapsed since the last jump and the length of the sampling interval, respectively. The state vector of the hybrid system denoted as (x, σ, τ, T) belongs to the set \mathbb{H}_0 given by

$$\mathbb{H}_0 = \mathbb{R}^n \times \mathbb{K} \times [0, T_M] \times [T_m, T_M]. \quad (4)$$

In equation (3), the (set-valued) maps f_0 and G_0 capture respectively the flow and jump maps, defined as follows:

$$f_0(x, \sigma, \tau, T) = \begin{bmatrix} A_\sigma x + B_\sigma \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad (x, \sigma, \tau, T) \in \mathbb{H}_0, \quad (5)$$

$$G_0(x, \sigma, \tau, T) = \begin{bmatrix} x \\ u(x, \sigma, \tau, T) \\ 0 \\ [T_m, T_M] \end{bmatrix} \quad (x, \sigma, \tau, T) \in \mathbb{H}_0,$$

where $u(x, \sigma, \tau, T)$ is the control law to be designed. Equation (3) also indicates the sets \mathcal{C}_0 and \mathcal{D}_0 , referred to as the "flow" and "jump" sets respectively. Depending in which set the state (x, σ, τ, T) is, it changes according to one of the dynamics. It is worth noting that, according to these maps, the switching signals $\sigma \in \mathbb{K}$ and $T \in [T_m, T_M]$ remain constant while (x, σ, τ, T) is in \mathcal{C}_0 and the state vector $x \in \mathbb{R}^n$ is constant when (x, σ, τ, T) is in \mathcal{D}_0 . To model the sampled-data switched affine system given in (1) using the hybrid formalism, the sets \mathcal{C}_0 and \mathcal{D}_0 must be designed such that the system jumps only when the variable τ is equal to $T \in [T_m, T_M]$. The flow and jump set are thus defined as follows

$$\begin{aligned} \mathcal{C}_0 &= \{(x, \sigma, \tau, T) \in \mathbb{H}_0 : \tau \leq T\} \\ \mathcal{D}_0 &= \{(x, \sigma, \tau, T) \in \mathbb{H}_0 : \tau = T\}. \end{aligned} \quad (6)$$

Other hybrid models can be derived for the sampled-data switched affine systems and the one given above is not unique. For instance, it is possible to remove variable T from \mathcal{H}_0 by modifying the timer. More precisely, one can introduce a timer τ' defined by $\dot{\tau}' = -1$ during flows and $\tau'^+ \in [T_m, T_M]$. The flow and jump sets would then be defined by conditions $\tau \geq 0$ and $\tau = 0$, respectively. The interest of introducing the variable T to denote the length of the sampling interval will appear in the next developments, more specifically in Section 4.

3. Hybrid limit cycles of switched affine systems

Through this section, the objective is to clarify the notion of *hybrid limit cycles* considered in this paper. Indeed, switched affine systems, as a particular class of hybrid and nonlinear systems, may exhibit a natural periodic behavior and various studies [3, 14, 27] have shown such a behavior when the control action is constrained by periodic updates for instance. It is then relevant to consider the convergence of the solutions to the switched affine systems to one or several time-varying steady states. Limit cycles generally refer to isolated and closed trajectories [31, 32] or to limit sets which are closed orbits [26]. In any case, limit cycles represent the stationary state of sustained oscillations, which do not depend on initial conditions but depend exclusively on the parameters of the system, i.e., they are intrinsic properties.

3.1. Definitions of hybrid limit cycles

Let us consider the following definitions borrowed from [17].

Definition 1. For any solution ϕ to hybrid system \mathcal{H}_0 , the subset $\text{dom } \phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ is a hybrid time domain if for each $(\mathcal{T}, J) \in \text{dom } \phi$,

$$\text{dom } \phi \cap ([0, \mathcal{T}] \times \{0, \dots, J\}) = \bigcup_{k=0}^J ([t_k, t_{k+1}] \times \{k\}).$$

Definition 2. A solution ϕ to system \mathcal{H}_0 is complete if $\text{dom } \phi$ is unbounded.

Definition 3 (Hybrid limit cycle). For a given complete solution ϕ^* to \mathcal{H}_0 , with a given state feedback $u^*(x, \sigma, \tau, T)$, the orbit $\phi^*(t, k)$, $(t, k) \in \text{dom } \phi^*$ is said to be a hybrid limit cycle under two conditions:

1. It is periodic, i.e., there exist a scalar $T^* > 0$ and an integer $N^* > 0$ such that

$$\phi^*(t+T^*, k+N^*) = \phi^*(t, k), \quad \forall (t, k) \in \text{dom } \phi^*.$$

2. It is isolated, i.e., for a given switching signal σ^* , there exists no other periodic orbit in its neighborhood.

According to the dynamics of hybrid system (3), the switching signal σ as well as the dwell time T are piece-wise constant functions. In the sequel, the switching sequence and the dwell time sequences we are referring to will stand for the sequence $\{\sigma(t_k, k)\}_{k=0}^{\infty}$ and $\{T(t_k, k)\}_{k=0}^{\infty}$. We will use notations ν and \mathbb{T} to denote the switching rules, i.e. $\{\nu(k)\}_{k=0}^{\infty} = \{\sigma(t_k, k)\}_{k=0}^{\infty}$, and the dwell time associated with each switching rule, i.e. $\{\mathbb{T}(k)\}_{k=0}^{\infty} = \{T(t_k, k)\}_{k=0}^{\infty}$, respectively. If a switching sequence is periodic, we will refer as a cycle in a set¹ $\mathfrak{C} \subset \mathbb{K}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ defined below.

Definition 4 (Set of cycles). The set of periodic sequences from \mathbb{N} to $\mathbb{K} \times \mathbb{R}$, denoted as $\mathfrak{C} \subset \mathbb{K}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$, are defined as

$$\mathfrak{C} = \left\{ \bar{\nu} = (\nu, \mathbb{T}) \in \mathbb{K}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}} : \exists N \in \mathbb{N}^*, \forall \ell \in \mathbb{N}, \quad \bar{\nu}(\ell+N) = \bar{\nu}(\ell) \right\}. \quad (7)$$

¹Notation $\mathbb{K}^{\mathbb{N}} \times \mathbb{R}^{\mathbb{N}}$ stands for the set of functions from \mathbb{N} to $\mathbb{K} \times \mathbb{R}$.

Finally, for a given cycle $\bar{\nu} \in \mathfrak{C}$, we define the minimal period of $\bar{\nu}$ as $N_{\bar{\nu}}$ and the minimal domain as $\mathbb{D}_{\bar{\nu}} = \{1, 2, \dots, N_{\bar{\nu}}\}$ and the following modulo notation is introduced

$$\lfloor \ell \rfloor_{\bar{\nu}} = ((\ell - 1) \bmod N_{\bar{\nu}}) + 1, \quad \forall \ell \in \mathbb{N}, \ell \geq 1.$$

That is, in particular, $\lfloor \ell \rfloor_{\bar{\nu}} = \ell$, for any $\ell \in \mathbb{D}_{\bar{\nu}}$ and $\lfloor N_{\bar{\nu}} + 1 \rfloor_{\bar{\nu}} = 1$.

3.2. Necessary and sufficient conditions of existence of periodic solutions

In the previous section, a quite general definition of *hybrid limit cycles* has been given. The interest is here to characterize the limit cycles of system (3), by introducing a periodic control law, which is defined using a given cycle $\bar{\nu} = (\nu, \mathbb{T})$ in \mathfrak{C} , i.e., the closed-loop system is given by \mathcal{H}_0 with the periodic control law $\sigma(t, k) = \nu(k)$ and $T(t, k) = \mathbb{T}(k)$, for all (t, k) in $\text{dom}(\phi)$.

Since the jumps occur eventually each time the timer τ reaches T , it allows us to take benefit of the discrete-time (linear) periodic system literature as for instance [6, 8]. The necessary and sufficient conditions to the existence of a periodic solution of ϕ^* were introduced in [28] and are adapted here to the hybrid system framework. Let us first introduce some notations that will be useful afterwards. With the periodic control law defined by $\bar{\nu} = (\nu, \mathbb{T})$, the relation between two successive switching instants is given, as in [26], by

$$x(t_{k+1}, k) = \Phi_{\bar{\nu}(k)} x(t_k, k) + \Gamma_{\bar{\nu}(k)}, \quad \forall k \in \mathbb{N}, \quad (8)$$

where

$$\Phi_{\bar{\nu}(k)} = e^{A_{\nu(k)} \mathbb{T}(k)} \quad \text{and} \quad \Gamma_{\bar{\nu}(k)} = \int_0^{\mathbb{T}(k)} e^{A_{\nu(k)} (\mathbb{T}(k) - s)} B_{\nu(k)} ds, \quad k \in \mathbb{N}. \quad (9)$$

With the periodic control law, it is clear that the state variables σ , τ and T are by construction periodic of period $N_{\bar{\nu}}$. To ensure that the orbit of \mathcal{H}_0 is periodic of period $N_{\bar{\nu}}$, the following equality must hold

$$x(t_k, k) = x(t_{k+N_{\bar{\nu}}}, k + N_{\bar{\nu}}).$$

The periodic orbit will then be denoted by ρ which is defined as follows

$$\begin{aligned} \rho(\tau, i) &= e^{A_{\nu(i)} \tau} \rho(0, i) + \int_0^\tau e^{A_{\nu(i)} (\tau - s)} B_{\nu(i)} ds, \quad \forall i \in \mathbb{D}_{\bar{\nu}}, \\ \rho(\mathbb{T}(i), i) &= \rho(0, \lfloor i + 1 \rfloor_{\bar{\nu}}). \end{aligned} \quad (10)$$

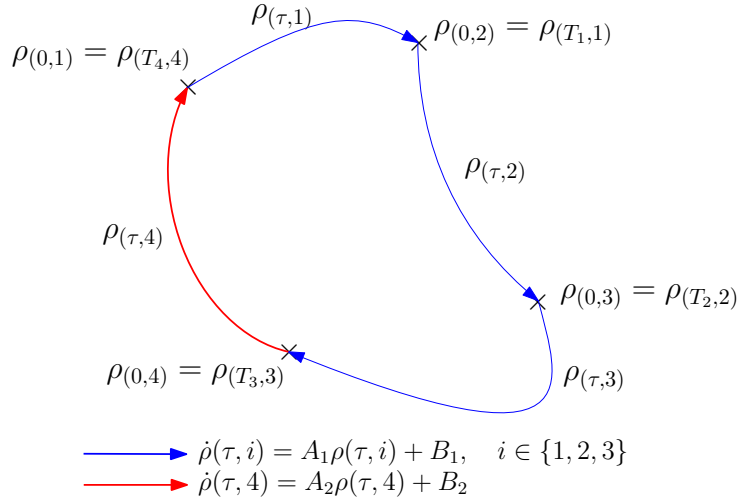


Figure 1: Illustration of the construction of a plant limit cycle for system (1) with $\mathbb{K} = \{1, 2\}$, $\nu = \{1, 1, 1, 2\}$, $\mathbb{T} = \{T_1, T_2, T_3, T_4\}$, where T_i , for all $i \in \mathbb{D}_{\bar{\nu}}$, are strictly positive scalars and $\mathbb{D}_{\bar{\nu}} = \{1, 2, 3, 4\}$. The blue (resp. red) arrows represent the trajectories of $\rho(\tau, i)$, for all $\tau \in [0, T_i]$ and $i \in \mathbb{D}_{\bar{\nu}}$ associated with mode 1 (resp. 2) such that $\nu(i) = 1$ (resp. $\nu(i) = 2$).

To better understand the previous calculations and the associated notations, Figure 1 is included to illustrate the following relevant notions:

- *plant limit cycle* $\{\rho(\tau, i)\}_{\tau \in [0, \mathbb{T}(i)], i \in \mathbb{D}_{\bar{\nu}}}$ and
- *hybrid limit cycle* $\{\rho(\tau, i), \tau, \nu(i), \mathbb{T}(i)\}_{\tau \in [0, \mathbb{T}(i)], i \in \mathbb{D}_{\bar{\nu}}}$.

Before stating the next lemma on the existence and uniqueness of periodic orbits of \mathcal{H}_0 , let us define the *monodromy matrix* associated with any cycle $\bar{\nu} = (\nu, \mathbb{T}) \in \mathfrak{C}$ of period $N_{\bar{\nu}}$, which corresponds to the transition matrix over the period of the cycle and which is given by

$$\bar{\Phi}_{\bar{\nu}} = \prod_{i=1}^{N_{\bar{\nu}}} \Phi_{\bar{\nu}(i)}, \quad (11)$$

where matrices $\Phi_{\bar{\nu}(i)}$ are given in (9).

Lemma 1. *For a given cycle $\bar{\nu} = (\nu, \mathbb{T}) \in \mathfrak{C}$, the hybrid system \mathcal{H}_0 admits a unique periodic solution $\phi^*(t, k)$ if and only if 1 is not an eigenvalue of $(\bar{\Phi}_{\bar{\nu}})^{N_{\bar{\nu}}}$.*

Moreover, if this assumption holds, the periodic solution is given by

$$\begin{bmatrix} x^*(t, k) \\ \sigma^*(t, k) \\ \tau^*(t, k) \\ T^*(t, k) \end{bmatrix} = \begin{bmatrix} \rho(\tau(t, k), \lfloor k - \delta \rfloor_{\bar{\nu}}) \\ \nu(k - \delta) \\ \tau(t, k) \\ \mathbb{T}(k - \delta) \end{bmatrix}, \quad \forall (t, k) \in \text{dom } \phi^*, \quad (12)$$

where $\delta \in \mathbb{D}_{\bar{\nu}}$ and ρ is defined in (10).

Remark 1. Before presenting the proof of this lemma, let us recall that a similar study of limit cycles has been already conducted in [13] in the context of discrete-time system and in [15] in the context of sampled-data systems. Nevertheless, these original contributions present the existence of periodic solutions (rather than limit cycles) as an assumption, while Lemma 1 provides a necessary and sufficient constructive condition for the existence of a periodic solution to \mathcal{H}_0 .

Proof. Noticing that (10) holds for any $(t, k) \in \text{dom } \phi$, in particular, at each jump, as x^* remains constant during jumps, this implies

$$\rho(0, \lfloor k + 1 \rfloor_{\bar{\nu}}) = \rho(\mathbb{T}(k), \lfloor k \rfloor_{\bar{\nu}}) = \Phi_{\bar{\nu}(k)} \rho(0, \lfloor k \rfloor_{\bar{\nu}}) + \Gamma_{\bar{\nu}(k)}, \quad \forall k \in \mathbb{N},$$

where $\Phi_{\bar{\nu}(i)}$ and $\Gamma_{\bar{\nu}(i)}$ are defined in (9) for all $i \in \mathbb{D}_{\bar{\nu}}$. Gathering them in the vector $\rho = [\rho^\top(0, 1), \dots, \rho^\top(0, N_{\bar{\nu}})]^\top$, simple calculations yield the following relation

$$(I_{nN_{\bar{\nu}}} - \mathbb{A}_{\bar{\nu}}) \rho = \mathbb{B}_{\bar{\nu}}, \quad (13)$$

where, if $N_{\bar{\nu}} = 1$, $\mathbb{A}_{\bar{\nu}} = \Phi_{\bar{\nu}(1)}$ and $\mathbb{B}_{\bar{\nu}} = \Gamma_{\bar{\nu}(1)}$ and, if $N_{\bar{\nu}} \geq 1$, they are given by

$$\mathbb{A}_{\bar{\nu}} = \left[\begin{array}{ccc|c} 0 & \dots & 0 & \Phi_{\bar{\nu}(N_{\bar{\nu}})} \\ \hline \Phi_{\bar{\nu}(1)} & & & 0 \\ & \ddots & & \vdots \\ & & \Phi_{\bar{\nu}(N_{\bar{\nu}}-1)} & 0 \end{array} \right], \quad \mathbb{B}_{\bar{\nu}} = \begin{bmatrix} \Gamma_{\bar{\nu}(N_{\bar{\nu}})} \\ \Gamma_{\bar{\nu}(1)} \\ \vdots \\ \Gamma_{\bar{\nu}(N_{\bar{\nu}}-1)} \end{bmatrix}. \quad (14)$$

The plant limit cycle $\rho(\tau(t, k), \lfloor k \rfloor_{\bar{\nu}})$, for all $(t, k) \in \text{dom } \phi^*$ is completely defined and is unique if there exists a solution to (13). Therefore, the proof is reduced to showing that the condition on the monodromy matrix $\bar{\Phi}_{\bar{\nu}}$ implies that (13) has a unique solution. First, it can be noticed that matrix $\mathbb{A}_{\bar{\nu}}$ has a close relation to matrix $\bar{\Phi}_{\nu}$, indeed, the following equality holds

$$\det(\lambda I_{nN_{\bar{\nu}}} - \mathbb{A}_{\bar{\nu}}) = \det(\lambda^{N_{\bar{\nu}}} I_n - \bar{\Phi}_{\bar{\nu}}). \quad (15)$$

Moreover, the spectrum of the cyclic augmented matrix $\mathbb{A}_{\bar{\nu}}$ is the set of all $N_{\bar{\nu}}$ -roots of the n eigenvalues of the monodromy matrix (see the argument of [6, page 322, Section 3.2]). Hence, the previous equation shows that the monodromy matrix elevated to the power $N_{\bar{\nu}}$ does not have the eigenvalue 1 if and only if the matrix $(I_{nN_{\bar{\nu}}} - \mathbb{A}_{\bar{\nu}})$ is non-singular and confirms the uniqueness of the periodic solution of period $N_{\bar{\nu}}$. ■

3.3. Necessary and sufficient conditions of existence of hybrid limit cycles

Note that the previous lemma only guarantees the existence and the uniqueness of a periodic solution to hybrid system \mathcal{H}_0 of period $N_{\bar{\nu}}$. However, the condition therein does not ensure that this solution is isolated, as required for being a limit cycle. The question is now to understand whether there exists other periodic solutions to the same system, but with a larger period $MN_{\bar{\nu}}$, for any integer $M \geq 1$. This is formulated in the following lemma.

Lemma 2. *For a given cycle $\bar{\nu} \in \mathcal{C}$, the hybrid system \mathcal{H}_0 admits a unique limit cycle if and only if 1 is not an eigenvalue of $(\bar{\Phi}_{\bar{\nu}})^M$, for any strictly positive integer M .*

Proof. The proof is available in [28, Lemma 1]. The underlying idea is to avoid the case where the monodromy matrix has an eigenvalue which is an M -root of 1, which would make the periodic trajectory not isolated. ■

To illustrate the difference between Lemmas 1 and 2, consider the following example, with $\mathbb{K} = \{1\}$, $A_1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, $B_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and the cycle of period $N_{\bar{\nu}} = 1$ and $\bar{\nu}_1 = (\nu_1(1) = 1, \mathbb{T}(1) = \pi/2)$. Simple computations yield

$$e^{A_1\tau} = \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix}, \quad \int_0^\tau e^{A_1(\tau-s)} B_1 ds = \begin{bmatrix} \sin(\tau) \\ 1 - \cos(\tau) \end{bmatrix},$$

and with $\tau = \mathbb{T}(1) = \pi/2$, we obtain $\bar{\Phi}_{\bar{\nu}_1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $\Gamma_{\bar{\nu}_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. In such a situation, the following statements hold:

- Consider cycle $\bar{\nu}_1 = (\{1\}, \{\frac{\pi}{2}\})$. Noting that the spectrum of $\bar{\Phi}_{\bar{\nu}_1} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ does not admit 1 as an eigenvalue, since the eigenvalues of $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are $\pm i$, which are 4-roots of 1. Therefore, Lemma 1 guarantees the existence of a unique periodic solution of period $\mathbb{T}(1)$, which is the solution

$$\rho(t, k) = \rho(\tau) = \begin{bmatrix} \cos(\tau) & -\sin(\tau) \\ \sin(\tau) & \cos(\tau) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} \sin(\tau) \\ 1 - \cos(\tau) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

that is constant during flows and jumps. However, this constant (and periodic) solution is not isolated as commented below.

- Consider the same example but with $\bar{\nu}_2 = (\{1, 1, 1, 1\}, \{\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\})$. The monodromy matrix associated with this cycle

$$\bar{\Phi}_{\bar{\nu}_2} = (\bar{\Phi}_{\bar{\nu}_1})^4 = I,$$

so that the conditions of Lemma 2 are not verified. Consequently, Lemma 2 states that the constant trajectory characterized above is not isolated. To see this issue, let us introduce, the solution given by

$$\rho(\tau, i) = \begin{bmatrix} -\varepsilon \sin(\tau + (i-1)\pi/2) \\ 1 + \varepsilon \cos(\tau + (i-1)\pi/2) \end{bmatrix},$$

for any scalar ε in \mathbb{R} . This solution is periodic of period $4T_s$. However, since ε is arbitrary, this periodic solution is not isolated.

Corollary 1. *If the monodromy matrix $\bar{\Phi}_\nu$ is Schur stable, then the hybrid system \mathcal{H}_0 admits a unique hybrid limit cycle.*

Proof. Recalling equation (15), if $\mathbb{A}_{\bar{\nu}}$ or, equivalently, $\bar{\Phi}_\nu$ is Schur stable, the magnitude of all their eigenvalues are strictly less than 1. Consequently, none of them can be an M -root of 1. Then, Lemma 2 ensures the existence and uniqueness of a limit cycle. ■

4. Problem formulation

Recall that the objective is to ensure the convergence and stability of any solution ϕ to ϕ^* of the hybrid system \mathcal{H}_0 . Therefore, it is more suited to rewrite \mathcal{H}_0 in a hybrid system that depends on the position of cycle (it will be defined by variable θ), instead of the functioning mode σ . We denote the hybrid state vector by $\xi = (x, \theta, \tau, T)$ and we define the hybrid system \mathcal{H} as follows

$$\mathcal{H} : \begin{cases} \dot{\xi} = f(\xi) & \xi \in \mathcal{C} \\ \xi^+ \in G(\xi) & \xi \in \mathcal{D}, \end{cases} \quad (16)$$

with function f and the set-valued map G now given by

$$f(\xi) = \begin{bmatrix} A_{\nu(\theta)}x + B_{\nu(\theta)} \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } G(\xi) = \left\{ \begin{bmatrix} x \\ q \\ 0 \\ \mathbb{T}(q) \end{bmatrix} \right\}_{q \in u(\xi)},$$

where $u(\xi)$ is a set valued map from $\mathbb{H} = \mathbb{R}^n \times \mathbb{D}_{\bar{\nu}} \times [0, T_M] \times \{\mathbb{T}(i)\}_{i \in \mathbb{D}_{\bar{\nu}}}$, where $T_M = \max_{i \in \mathbb{D}_{\bar{\nu}}} \mathbb{T}(i)$ and where only variable σ from \mathcal{H}_0 has been replaced by θ which lies in $\mathbb{D}_{\bar{\nu}}$. Hence the hybrid state vector ξ belongs to the set \mathbb{H} . Likewise, the flow set and the jump set are now

$$\mathcal{C} = \left\{ \xi \in \mathbb{H} : \xi = \begin{bmatrix} x \\ \theta \\ \tau \\ T \end{bmatrix}, \tau \leq T \right\} \text{ and } \mathcal{D} = \left\{ \xi \in \mathbb{H} : \xi = \begin{bmatrix} x \\ \theta \\ \tau \\ T \end{bmatrix}, \tau = T \right\}. \quad (17)$$

The well-posedness in the context of hybrid dynamical systems allows to apply several useful results [17, Chap. 6]. The selection of the control law $u(x, \theta, \tau, T)$ is critical to ensure this property. Following [17, Th. 6.8, p. 122], the sufficient conditions of well-posedness are summarized in the following proposition.

Proposition 1. [17, Th. 6.8, p. 122] *System $\mathcal{H}(f, G, \mathcal{C}, \mathcal{D})$ is well-posed if it verifies the basic hybrid conditions, stated below*

- *Sets \mathcal{C} and \mathcal{D} are closed subsets of $\mathbb{H} \subset \mathbb{R}^{n+3}$.*
- *$f : \mathbb{R}^{n+3} \rightrightarrows \mathbb{R}^{n+3}$ is outer semi-continuous and locally bounded relative to \mathcal{C} , $\mathcal{C} \subset \text{dom } F$, and $f(\xi)$ is convex for every $\xi \in \mathcal{C}$;*
- *$G : \mathbb{R}^{n+3} \rightrightarrows \mathbb{R}^{n+3}$ is outer semi-continuous and locally bounded relative to \mathcal{D} , $\mathcal{D} \subset \text{dom } G$.*

Now, we are in position to formulate the problem.

Problem 1. *Consider the hybrid system \mathcal{H} . Then the objectives are:*

- (i) *To design a control law that ensures the UGAS of the selected hybrid limit cycle designed by $\bar{\nu} = (\nu, \mathbb{T})$, where the dwell times are not necessarily uniform.*

- (ii) To select a hybrid limit cycle according to some system specifications: position of each $\rho(0, i)$, $i \in \mathbb{D}_{\bar{\nu}}$, distances to the operating point and amplitude of the trajectories.

5. Main result

5.1. Stabilization to a hybrid limit cycle

Stability properties of a hybrid limit cycle of \mathcal{H} associated with a given cycle $\bar{\nu} = (\nu, \mathbb{T}) \in \mathfrak{C}$ is analyzed here and later we will focus on the selection of a hybrid limit cycle.

Theorem 1. For a cycle $\bar{\nu} = (\nu, \mathbb{T}) \in \mathfrak{C}$, assume that there exist matrices P_i in \mathbb{S}_+^n , for $i \in \mathbb{D}_{\bar{\nu}}$, solution to the following inequalities

$$P_i \succ 0, \quad \Xi_i = \Phi_{\bar{\nu}(i)}^\top P_{[i+1]_{\bar{\nu}}} \Phi_{\bar{\nu}(i)} - P_i \prec 0, \quad \forall i \in \mathbb{D}_{\bar{\nu}}, \quad (18)$$

where matrices $\Phi_{\bar{\nu}(i)}$ are given in (9). Then, the following statements hold:

- (i) System \mathcal{H} admits a hybrid limit cycle (12), associated with cycle $\bar{\nu}$, i.e., ρ given in (10) is unique and isolated.
- (ii) System (16) with the control law

$$u(\xi) = \left\{ \underset{i \in \mathbb{D}_{\bar{\nu}}}{\operatorname{argmin}} (x - \rho(0, i))^\top P_i(0) (x - \rho(0, i)) \right\} \subset \mathbb{K}, \quad (19)$$

where the $\rho(\tau, i)$'s have been defined in (10), and where

$$P_i(\tau) = e^{-\tau A_{\nu(i)}^\top} P_i e^{-\tau A_{\nu(i)}}, \quad \forall (i, \tau) \in \mathbb{D}_{\bar{\nu}} \times [0, T_M],$$

satisfies the basic hybrid conditions in Proposition 1.

- (iii) The set

$$\mathcal{A} = \left\{ \xi \in \mathbb{H} : \xi = \begin{bmatrix} \rho(\tau, i) \\ i \\ \tau \\ \mathbb{T}(i) \end{bmatrix}, \tau \leq \mathbb{T}(i), \quad \forall i \in \mathbb{D}_{\bar{\nu}} \right\} \quad (20)$$

is an attractor, i.e., is UGAS for system (16) with the control law in (19).

(iv) Moreover, if the components $\{\rho(0, i)\}_{i \in \mathbb{D}_{\bar{\nu}}}$ are two by two different, then the state-dependent control law $u(\xi)$ converges in finite time to a periodic function, which is a shifted version of $\bar{\nu}$. Specifically, there exist $k_0 \in \mathbb{N}$ and an integer $\delta \in \mathbb{D}_{\bar{\nu}}$ such that

$$u(\xi(t_k, k)) = \{\nu(k + \delta)\}, \quad T(u(\xi(t_k, k))) = \mathbb{T}(k + \delta), \quad \forall k \geq k_0. \quad (21)$$

Proof. Each item shall be proven one by one.

Proof of (i): Consider $\bar{P} = \text{diag}_{i=1}^{N_{\bar{\nu}}} P_i$, where matrices P_i are solution to (18). Therefore, we can see from (14) that

$$\mathbb{A}_{\bar{\nu}}^\top \bar{P} \mathbb{A}_{\bar{\nu}} - \bar{P} = \text{diag}_{i=1}^{N_{\bar{\nu}}} \left(\Phi_{\bar{\nu}(i)}^\top P_{\lfloor i+1 \rfloor_{\bar{\nu}}} \Phi_{\bar{\nu}(i)} - P_i \right) \prec 0. \quad (22)$$

Hence, matrix $\mathbb{A}_{\bar{\nu}}$ is Schur stable as well as the associated monodromy matrix $\bar{\Phi}_{\bar{\nu}}$. The LMI condition given in (18) is a sufficient condition to the existence of a unique hybrid limit cycle as stated in Corollary 1.

Proof of (ii): The objective is here to show that \mathcal{H} satisfies the basic hybrid conditions in Proposition 1 with control law (19). The first and the second ones trivially hold since f is not a set valued map but a continuous function. The more challenging part relies on the last item of Proposition 1, which concerns the outer semi-continuity of G , more precisely of u in (19). Let us first recall the definition of outer semi-continuous functions in the context of this paper. The set-valued mapping

$$\Sigma : \quad \xi \mapsto \text{argmin}_{i \in \mathbb{D}_{\bar{\nu}}} (x - \rho(\tau, i))^\top \mathbb{P}_i(\tau) (x - \rho(\tau, i))$$

is outer semi-continuous (osc) at $\bar{\xi}$ if

$$\limsup_{\xi \rightarrow \bar{\xi}} \Sigma(\xi) \subset \Sigma(\bar{\xi}),$$

where the operator \limsup is defined for the set-valued mapping in [25, Section 5.B]).

Then, let us note that functions $\xi \mapsto (x - \rho(\tau, i))^\top \mathbb{P}_i(\tau) (x - \rho(\tau, i))$ are continuous and locally bounded, for all $i \in \mathbb{D}_{\bar{\nu}}$. Following the proof techniques in [25, Ex.5.22, p.162], the function Σ is outer semi-continuous, which is exactly the last item of Proposition 1. Hence, system \mathcal{H} with the control law (19) satisfies the basic hybrid conditions.

Proof of (iii): Consider the candidate Lyapunov function given by

$$V(\xi) = (x - \rho(\tau, \theta))^\top e^{-\varepsilon\tau} \mathbb{P}_\theta(\tau) (x - \rho(\tau, \theta)), \quad \forall \xi \in \mathbb{H}, \quad (23)$$

where ε is a sufficiently small positive scalar to be defined.

Let us first note that function $V(\xi)$ is quadratic in x for all $\theta, \tau, \mathbb{T}(i)$ in the compact set $\mathbb{D}_{\bar{\nu}} \times [0, T_M] \times \{\mathbb{T}(i)\}_{i \in \mathbb{D}_{\bar{\nu}}}$ and, thus, is locally Lipschitz, radially unbounded. To show this, let us first notice that, by assumption matrices P_i are symmetric positive definite and that matrices $e^{A_{\nu(i)}\tau}$ are non singular for all (i, τ) in $\mathbb{D}_{\bar{\nu}} \times [0, T_M]$. This implies that matrices $\mathbb{P}_i(\tau)$ are also symmetric positive definite for all (i, τ) in $\mathbb{D}_{\bar{\nu}} \times [0, T_M]$. Then, the continuity of matrices function considered over the compact set $\mathbb{D}_{\bar{\nu}} \times [0, T_M]$ ensures that there exists $\alpha_1 > 0$ such that

$$\alpha_1 I_n \prec \mathbb{P}_i(\tau), \quad \forall (i, \tau) \in \mathbb{D}_{\bar{\nu}} \times [0, T_M]. \quad (24)$$

Then, re-injecting this inequality in the definition of V , we get that

$$V(\xi) \geq \alpha_1 e^{-\varepsilon\tau} \|x - \rho(\theta, \tau)\|^2 \geq \alpha_1 e^{-\varepsilon T_M} |\xi|_{\mathcal{A}}^2, \quad \forall \xi \in \mathbb{H}, \quad (25)$$

which implies its strict positiveness for all $\xi \notin \mathcal{A}$. Moreover it can be easily verified that $V(\xi) = 0$ if $\xi \in \mathcal{A}$. The characterization of \mathcal{A} as a subset of \mathbb{H} such that $V(\xi) = 0$ and $\tau \leq \mathbb{T}(i)$ renders \mathcal{A} a closed set. In addition, since matrices P_i , and consequently $\mathbb{P}_i(\tau)$, are positive definite, \mathcal{A} is also bounded. Therefore, \mathcal{A} is a compact subset of \mathbb{H} .

Following the argument of [17, Th.3.18], one shall ensure that the derivative of $V(\xi)$ along the flows is strictly negative for all $\xi \in \mathcal{C} \setminus \mathcal{A}$ and that the difference of $V(\xi)$ across the jumps is strictly negative for all $\xi \in \mathcal{D} \setminus \mathcal{A}$. More formally, this means that there exists a sufficient small $\alpha > 0$ the next two inequalities need to hold

$$\langle \nabla V(\xi), f(\xi) \rangle \leq -\alpha |\xi|_{\mathcal{A}}^2 \quad \forall \xi \in \mathcal{C} \setminus \mathcal{A}, \quad (26)$$

$$\Delta V(\xi) = \max_{g \in \mathcal{G}(\xi) \cap (\mathcal{C} \cup \mathcal{D})} V(g) - V(\xi) \leq -\alpha |\xi|_{\mathcal{A}}^2 \quad \forall \xi \in \mathcal{D} \setminus \mathcal{A}, \quad (27)$$

First note that during flows $\dot{\rho}(\tau, \theta) = A_{\nu(\theta)}\rho(\tau, \theta) + B_{\nu(\theta)}$, which is direct from the definition given before ($x^*(t, k) = \rho(\tau, \lfloor k \rfloor_{\bar{\nu}}$). Hence, the next equations hold

$$\begin{aligned} \frac{d}{dt} [x - \rho(\tau, \theta)] &= A_{\nu(\theta)}x + B_{\nu(\theta)} - A_{\nu(\theta)}\rho(\tau, \theta) - B_{\nu(\theta)} \\ &= A_{\nu(\theta)}(x - \rho(\tau, \theta)), \\ \frac{d}{dt} (e^{-\varepsilon\tau} \mathbb{P}_\theta(\tau)) &= -e^{-\varepsilon\tau} (A_{\nu(\theta)}^\top \mathbb{P}_\theta(\tau) + \mathbb{P}_\theta(\tau) A_{\nu(\theta)} + \varepsilon \mathbb{P}_\theta(\tau)). \end{aligned}$$

The expression of $\frac{d}{dt}V(\xi) = \langle \nabla V(\xi), f(\xi) \rangle$ for any $\xi \in \mathcal{C} \setminus \mathcal{A}$ is given by

$$\begin{aligned} \langle \nabla V(\xi), f(\xi) \rangle &= (x - \rho(\tau, \theta))^\top \frac{d}{dt} \left(e^{-\varepsilon\tau} \mathbb{P}_\theta(\tau) \right) (x - \rho(\tau, \theta)) \\ &\quad + 2e^{-\varepsilon\tau} (x - \rho(\tau, \theta))^\top A_{\nu(\theta)}^\top \mathbb{P}_\theta(\tau) (x - \rho(\tau, \theta)) \\ &= -\varepsilon (x - \rho(\tau, \theta))^\top e^{-\varepsilon\tau} \mathbb{P}_\theta(\tau) (x - \rho(\tau, \theta)) = -\varepsilon V(\xi). \end{aligned} \quad (28)$$

Then, using inequality (25), we get

$$\langle \nabla V(\xi), f(\xi) \rangle \leq -\varepsilon \alpha_1 e^{-\varepsilon T_m} |\xi|_{\mathcal{A}}^2.$$

Since ε is a positive scalar, the previous inequality guarantees the satisfaction of the first Lyapunov condition (26).

To complete the proof, we still have to ensure (27), i.e., $\Delta V(\xi)$ is negative for all $\xi \in \mathcal{D} \setminus \mathcal{A}$. Let us prove that $V(x, q, 0, \mathbb{T}(q)) - V(x, \theta, \mathbb{T}(\theta), \mathbb{T}(\theta))$ is negative definite, $\forall \theta \in \mathbb{D}_{\bar{\nu}}$, $\forall q \in u(x, \theta, \mathbb{T}(\theta), \mathbb{T}(\theta))$. The control law selects the argument $i \in \mathbb{D}_{\bar{\nu}}$ that minimizes $(x - \rho(0, i))^\top \mathbb{P}_i(0) (x - \rho(0, i))$. Hence, we have

$$V(x, q, 0, \mathbb{T}(q)) \leq V(x, i, 0, \mathbb{T}(i)), \quad \forall i \in \mathbb{D}_{\bar{\nu}}, \forall q \in u(\xi).$$

In particular, selecting $i = \lfloor \theta + 1 \rfloor_{\bar{\nu}}$ yields

$$\begin{aligned} \Delta V(\xi) &\leq V(x, \lfloor \theta + 1 \rfloor_{\bar{\nu}}, 0, \mathbb{T}(\lfloor \theta + 1 \rfloor_{\bar{\nu}})) - V(x, \theta, \mathbb{T}(\theta), \mathbb{T}(\theta)) \\ &\leq (x - \rho(0, \lfloor \theta + 1 \rfloor_{\bar{\nu}}))^\top \mathbb{P}_{\lfloor \theta + 1 \rfloor_{\bar{\nu}}}(0) (x - \rho(0, \lfloor \theta + 1 \rfloor_{\bar{\nu}})) \\ &\quad - (x - \rho(\mathbb{T}(\theta), \theta))^\top e^{-\varepsilon \mathbb{T}(\theta)} \mathbb{P}_\theta(\mathbb{T}(\theta)) (x - \rho(\mathbb{T}(\theta), \theta)). \end{aligned}$$

Since equation (13) guarantees that $\rho(\mathbb{T}(\theta), \theta) = \rho(0, \lfloor \theta + 1 \rfloor_{\bar{\nu}})$, the following simplification can be made

$$\begin{aligned} \Delta V(\xi) &\leq (x - \rho(\mathbb{T}(\theta), \theta))^\top \left(\mathbb{P}_{\lfloor \theta + 1 \rfloor_{\bar{\nu}}}(0) - e^{-\varepsilon \mathbb{T}(\theta)} \mathbb{P}_\theta(\mathbb{T}(\theta)) \right) (x - \rho(\mathbb{T}(\theta), \theta)) \\ &\leq (x - \rho(\mathbb{T}(\theta), \theta))^\top \left(P_{\lfloor \theta + 1 \rfloor_{\bar{\nu}}} - e^{-\varepsilon \mathbb{T}(\theta)} \Phi_{\bar{\nu}(\theta)}^{-\top} P_\theta \Phi_{\bar{\nu}(\theta)}^{-1} \right) (x - \rho(\mathbb{T}(\theta), \theta)) \\ &\leq (x - \rho(\mathbb{T}(\theta), \theta))^\top \Phi_{\bar{\nu}(\theta)}^{-\top} \left(\Phi_{\bar{\nu}(\theta)}^\top P_{\lfloor \theta + 1 \rfloor_{\bar{\nu}}} \Phi_{\bar{\nu}(\theta)} - e^{-\varepsilon \mathbb{T}(\theta)} P_\theta \right) \Phi_{\bar{\nu}(\theta)}^{-1} (x - \rho(\mathbb{T}(\theta), \theta)). \end{aligned}$$

The previous inequality can be rewritten using the notation Ξ_i defined in the LMI conditions (18) of Theorem 1. Recalling that $e^{-\varepsilon T_m} \leq e^{-\varepsilon \mathbb{T}(\theta)}$, for all θ in $\mathbb{D}_{\bar{\nu}}$, it yields,

$$\Delta V(\xi) \leq (x - \rho(\mathbb{T}(\theta), \theta))^\top \Phi_{\bar{\nu}(\theta)}^{-\top} \left(\Xi_\theta + (1 - e^{-\varepsilon T_m}) P_\theta \right) \Phi_{\bar{\nu}(\theta)}^{-1} (x - \rho(\mathbb{T}(\theta), \theta)). \quad (29)$$

Since $1 - e^{-\varepsilon\mathbb{T}(\theta)}$ tends to zero as ε tends to zero, the satisfaction of LMI $\Xi_\theta \prec 0$ ensures that there exists sufficiently small $\varepsilon > 0$ and $\alpha_2 > 0$ such that

$$\Xi_\theta \prec -(1 - e^{-\varepsilon T_M})P_\theta - \alpha_2 \Phi_{\bar{\nu}(\theta)}^\top \Phi_{\bar{\nu}(\theta)}.$$

Re-injecting this expression upper bound into (29) leads to

$$\Delta V(\xi) \leq -\alpha_2 \|x - \rho(\mathbb{T}(\theta), \theta)\|^2 \leq -\alpha_2 |\xi|_{\mathcal{A}}^2, \quad (30)$$

which corresponds to the second Lyapunov condition in (27). Then proof is concluded by selecting $\alpha = \min(\alpha_1 \varepsilon e^{-\varepsilon T_m}, \alpha_2)$ to recover exactly conditions (26) and (27).

Proof of (iv): The proof of the last item is obtained by showing that there exists a sufficiently small and positive η such that if solutions to \mathcal{H} evolve in $\mathcal{S}_\eta = \{\xi(t_k, k) \in \mathbb{H}, V(\xi) \leq \eta^2\} \cap \mathcal{D}$, then the following condition is satisfied when the trajectory enters in \mathcal{D} :

$$\begin{aligned} & (x(t_{k+1}, k) - \rho(0, \lfloor \theta + 1 \rfloor_{\bar{\nu}}))^\top P_{\lfloor \theta + 1 \rfloor_{\bar{\nu}}} (x(t_{k+1}, k) - \rho(0, \lfloor \theta + 1 \rfloor_{\bar{\nu}})) \\ & < (x(t_{k+1}, k) - \rho(0, j))^\top P_j (x(t_{k+1}, k) - \rho(0, j)), \end{aligned} \quad (31)$$

for all $j \in \mathbb{D}_{\bar{\nu}}$, with $j \neq \lfloor \theta + 1 \rfloor_{\bar{\nu}}$, that is the solution to the next optimization problem (19) is $\lfloor \theta + 1 \rfloor_{\bar{\nu}}$. The convergence of the Lyapunov function to zero has been proven, hence, reaching the level set \mathcal{S}_η is always possible. The shift δ from equation (21) is determined at time k_0 (related to the time needed to reach \mathcal{S}_η) thanks to the solution θ of the optimization problem. One should notice that thanks to the equivalence of weighted norms, there exist constants $c_{i,j} > 0$, $\forall (i, j) \in \mathbb{D}_{\bar{\nu}}$, such that

$$\|x\|_{P_i} \leq c_{i,j} \|x\|_{P_j}. \quad (32)$$

For example, it is always possible to select $c_{i,j} = \sqrt{\lambda_M(P_i)/\lambda_m(P_j)}$. Then, thanks to item (ii) and $\xi \in \mathcal{S}_\varepsilon$, we have,

$$\|x(t_{k+1}, k) - \rho(0, \lfloor \theta + 1 \rfloor_{\bar{\nu}})\|_{P_{\lfloor \theta + 1 \rfloor_{\bar{\nu}}}} < \|x(t_k, k) - \rho(0, \theta)\|_{P_\theta} \leq \eta. \quad (33)$$

Now, from the fact that ξ is in \mathcal{S}_ε , together with equations (8) and (10), it follows

$$\begin{aligned} \|x(t_{k+1}, k) - \rho(0, \lfloor \theta + 1 \rfloor_{\bar{\nu}})\|_{P_\theta} &= \|\Phi_{\bar{\nu}(\theta)}(x(t_k, k) - \rho(0, \theta))\|_{P_\theta} \\ &\leq \|\Phi_{\bar{\nu}(\theta)}\|_{P_\theta} \|x(t_k, k) - \rho(0, \theta)\|_{P_\theta} \\ &\leq \|\Phi_{\bar{\nu}(\theta)}\|_{P_\theta} \eta, \end{aligned}$$

where $\|\Phi_{\bar{\nu}(\theta)}\|_{P_\theta}$ denotes the matrix norm induced by the weighted norm $\|\cdot\|_{P_\theta}$. Due to relations (32) and triangular inequalities, it yields

$$\begin{aligned} & \|\rho(0, \lfloor \theta + 1 \rfloor_{\bar{\nu}}) - \rho(0, j)\|_{P_\theta} - \|\Phi_{\bar{\nu}(\theta)}\|_{P_\theta} \varepsilon \\ & \leq \|\rho(0, \lfloor \theta + 1 \rfloor_{\bar{\nu}}) - \rho(0, j)\|_{P_\theta} - \|\Phi_{\nu(\theta)}(T_\theta)(x(t_k, k) - \rho(0, \theta))\|_{P_\theta} \\ & \leq \|\rho(0, \lfloor \theta + 1 \rfloor_{\bar{\nu}}) - \rho(0, j) + \Phi_{\bar{\nu}(\theta)}(x(t_k, k) - \rho(0, \theta))\|_{P_\theta} \\ & \leq \|x(t_{k+1}, k) - \rho(0, j)\|_{P_\theta} \\ & \leq c_{\theta, j} \|x(t_{k+1}, k) - \rho(0, j)\|_{P_j}, \quad \forall j \in \mathbb{D}_{\bar{\nu}}. \end{aligned}$$

Under the condition that all $\rho(0, i)$'s are different two by two, it is always possible to find a positive scalar ε such that the strict inequalities $0 < c_{\theta, j} \eta < \|\rho(0, \lfloor \theta + 1 \rfloor_{\bar{\nu}}) - \rho(0, j)\|_{P_\theta} - \|A_{\nu(\theta)}\|_{P_\theta} \eta$ hold for any $j \in \mathbb{D}_{\bar{\nu}}$, $j \neq \lfloor \theta + 1 \rfloor_{\bar{\nu}}$. Combining the two latter inequalities leads to

$$\eta < \|x(t_{k+1}, k) - \rho(0, j)\|_{P_j}, \quad \forall j \in \mathbb{D}_{\bar{\nu}} \setminus \{\lfloor \theta + 1 \rfloor_{\bar{\nu}}\}. \quad (34)$$

Then, inequalities (33) and (34) yield

$$\|x(t_{k+1}, k) - \rho(0, \lfloor \theta + 1 \rfloor_{\bar{\nu}})\|_{P_{\lfloor \theta + 1 \rfloor_{\bar{\nu}}}} \leq \eta < \|x(t_{k+1}, k) - \rho(0, j)\|_{P_j},$$

$$\forall j \in \mathbb{D}_{\bar{\nu}} \setminus \{\lfloor \theta + 1 \rfloor_{\bar{\nu}}\}.$$

This last inequality ensures that the argument that minimizes the quadratic terms is $\lfloor \theta + 1 \rfloor_{\bar{\nu}}$ and concludes the proof. ■

Theorem 1 provides the hybrid version of [28, Th.1], which was conducted only in the discrete-time framework. Whereas the LMI are the same as in [13] and [28], Theorem 1 does not only describe the stabilization to hybrid limit cycles composed of discrete limit cycles, i.e., the set of $\rho(0, i)$'s but the whole hybrid (continuous and discrete) trajectories that occur in between these discrete points. This makes the main differences with respect to [28, Th.1].

Note that the previous analysis can be simplified by consider the Lyapunov function (23) with $\varepsilon = 0$. However, this make that the Lyapunov function remains constant during flows. To guarantee the uniform asymptotic stability of the closed-loop system, one has to invoke a relaxed stability condition provided in [29, Th.1].

5.2. Optimal selection

At many occasions, the objective is to steer the state of the switched system as close as possible to a desired reference point $x_{\text{ref}} \in \mathbb{R}^n$ to meet

some practical specifications. [28, Prop.2] shows that the hybrid limit cycle ρ associated with a selected $\bar{\nu}$ is not affected by any coordinate transformation.

Theorem 1 presents a stabilization condition to a limit cycle that is characterized by an a priori given cycle ν . As different cycles lead to different limit cycles, an important question concerns the evaluation of the best or the most appropriate limit cycle to be selected according to a given cost function, which accounts for instance the distance of the plant limit cycle to a desired functioning point of interest, denoted here as x_{ref} in \mathbb{R}^n . This notion refers to the distance of a point x_{ref} to the union of $N_{\bar{\nu}}$ singletons $\{\rho(0, i)\}_{i \in \mathbb{D}_{\bar{\nu}}}$. This question was also raised in [15] and latter in [28]. Among the possible ways to build such a cost function, we consider here the following cost function, defined for a given sampling period T_i

$$J(\bar{\nu}, \rho_{\bar{\nu}}, x_{\text{ref}}) = J_1(\bar{\nu}, \rho_{\bar{\nu}}) + J_2(\bar{\nu}, \rho_{\bar{\nu}}, x_{\text{ref}}), \quad \forall (\bar{\nu}, \rho_{\bar{\nu}}, x_{\text{ref}}) \in \mathcal{C} \times (\mathbb{R}^n)^{N_{\bar{\nu}}} \times \mathbb{R}^n, \quad (35)$$

where $\rho_{\bar{\nu}}$ denotes here the values of the plant limit cycle issued from the unique solution to (13) for a given cycle $\bar{\nu}$. Moreover,

$$J_1(\bar{\nu}, \rho_{\bar{\nu}}) = \sup_{i \in \mathbb{D}_{\bar{\nu}}} \left(\sup_{j \in \mathbb{D}_{\bar{\nu}}} (\rho_{\bar{\nu}}(0, i) - \rho_{\bar{\nu}}(0, j))^\top H_1 (\rho_{\bar{\nu}}(0, i) - \rho_{\bar{\nu}}(0, j)) \right),$$

$$J_2(\bar{\nu}, \rho_{\bar{\nu}}, x_{\text{ref}}) = \left(x_{\text{ref}} - \frac{1}{N_{\bar{\nu}}} \sum_{i \in \mathbb{D}_{\bar{\nu}}} \rho_{\bar{\nu}}(0, i) \right)^\top H_2 \left(x_{\text{ref}} - \frac{1}{N_{\bar{\nu}}} \sum_{i \in \mathbb{D}_{\bar{\nu}}} \rho_{\bar{\nu}}(0, i) \right),$$

where H_1 and H_2 are two positive (semi-)definite matrices. The first term of the cost function aims at evaluating the amplitude of the oscillations within the limit cycle. The second one refers to the distance between the desired position x_{ref} to the average values of the plant limit cycle. The cost function defined in (35) is based on the knowledge of $\rho(0, i)$, $i \in \mathbb{D}_{\bar{\nu}}$. It might be possible to consider more complex cost functions, which consider the whole arcs $(\rho(\tau, i), i, \tau, T)$ in \mathcal{A} . Following the previous discussion, the following proposition is stated.

Proposition 2. *Consider a given bounded subset $\Omega \subset \mathcal{C}$, such that for any $\mu \in \Omega$, the monodromy matrix $\bar{\Phi}_\mu$ is Schur stable. For a given desired reference x_{ref} , the optimal cycle and plant limit cycle is the one that is solution to the following optimization problem*

$$\begin{aligned} \bar{\nu} \in & \operatorname{argmin}_{\mu \in \Omega} J(\mu, x_{\text{ref}}, \rho_\mu) \\ \text{s.t. } & \bar{\Phi}_\mu \text{ is Schur stable.} \end{aligned} \quad (36)$$

This optimization problem provides an optimal solution with respect to the distance of the $\rho_\mu(0, i)$ to the reference x_{ref} . However, other optimization metrics can be used. Note that the previous optimization problem does not involve the decision variables of the LMI condition.

6. Comparison with previous results

This section proposes a comparison between the present contribution and the few recent papers ([15], [13] and [28]) from the existing literature on limit cycles and switched affine systems.

- *Discrete vs. Sampled-data systems:* [13] and [28] deal with discrete-time switched affine systems arising from a periodic discretization. The analysis provided therein does not allow concluding on the inter-sampling behavior. On the contrary, Theorem 1 presents a contribution on aperiodic sampled-data control providing an extended analysis of [28], which also accounts for the inter-sampling period. This is made possible thanks to hybrid dynamical systems formulation, which takes into account both the continuous- and the discrete-time behaviors of the aperiodic sampled-data systems thanks to the associated Lyapunov function.
- *Interpretation of the LMI condition:* It is worth noting that the LMI condition in the theorem is the same as the one presented in [13] or [28], apart from a shift of index in the matrices P_i . However, as for the LMI condition for the stability analysis or stabilization of discrete-time and sampled-data systems, the same LMI condition can have different interpretations. In this paper, we show that the same LMI condition provided additional values to the contribution presented in [13] or [28]. More precisely, the LMI condition in this paper guarantees
 - the existence of a discrete-time limit cycle consisting of points $\{\rho(0, i)\}_{i \in \mathbb{D}_\nu}$, which was only an assumption in [13]. In addition, the authors of [13] only regarded the problem of periodic solutions of the switched affine systems, without studying the isolation property of this solution. As discussed in Section 3.3, the isolation property requires a dedicated analysis.

- The existence of a discrete-time hybrid limit cycle consisting of points $\{\rho(0, i), \nu(i), \mathbb{T}(i)\}_{i \in \mathbb{D}_{\bar{\nu}}}$. Indeed, in [28], we have demonstrated that the control law provided in [13] does not necessarily converge to a hybrid limit cycle, i.e., the state vector and the control law and its dwell time, but only to a plant limit cycle, i.e., only the projection of the hybrid limit cycle in x (see example 2 in [28]).
- The existence of a continuous-time hybrid limit cycle consisting of points $\{\rho(\tau, i), \nu(i), \tau, \mathbb{T}(i)\}_{i \in \mathbb{D}_{\bar{\nu}}, \tau \in [0, \mathbb{T}(i)]}$, including modes, timer and dwell time, in addition to the plant limit cycle.

These guarantees are obtained through the construction of our matrix \mathbb{A}_{ν} , which differs to the one used in [13]. This former is the sum of a block diagonal matrix composed of matrices $\Phi_{\bar{\nu}(i)}$ and a block circulant matrix composed of the identity matrix, i.e., the reverse process as \mathbb{A}_{ν} in (14). This small difference simplifies the proof of existence of a hybrid limit cycle since the LMI condition directly and elegantly ensures that matrix \mathbb{A}_{ν} is Schur stable.

- *Comparison of the control strategies of Theorem 1 and [13]:* First of all, from the intuitive point of view, the control law proposed in [13] aims at finding the mode that minimizes the increment of the Lyapunov function while our control law aims at minimizing the Lyapunov function itself. This small difference makes that the control law in [13] requires the exact knowledge of the system transition matrices $\Phi_{\bar{\nu}(i)}$, which prevents from extending the result to the uncertain case as demonstrated in [28]. This extension is not presented in this paper but will be treated in future work.
Second, the control law in [13] is time-dependent, while ours is a pure state-feedback, which makes it more robust with respect to the initial conditions. The authors of [13] have included an initialization process to find the best initial selection of the mode to be included in the control law. Our solution does not require such an initialization process.
- *Comparison of the control strategies of Theorem 1 and [15, Theorem 1]:* The authors of [15] propose a different solution to the stabilization problem of periodic sampled-data switched affine systems. The main differences are listed below:

- this paper addresses the case of periodic sampled-data switched affine system, while here we have introduced the possibility to consider several dwell times, that are not necessarily commensurate to sub-sampling time.
- The existence of a plant limit cycle is not studied therein but is presented as an assumption on the existence of a periodic function.
- The expression of the control law proposed in [15] is expressed using an integral of a quadratic expression of state variable over the inter-sampling period. Compared to our simple control law where the parameters are the solution of the LMI condition, the control law in [15] thus requires several manipulations and computations of integral to be evaluated. Even though these calculations are performed off-line, it is required to do them with caution.

7. Alternative stabilization conditions

The authors of [15] provide a stability condition in their Theorem 2 which is expressed in terms of a timer-dependent differential LMI condition, which is rewritten using our notation:

$$P(t)A_{\nu(i)} + A_{\nu(i)}^\top P(t) + \dot{P}(t) \prec 0, \quad \forall t \in \left[\sum_{j=1}^{i-1} \mathbb{T}(j), \sum_{j=1}^i \mathbb{T}(j) \right), \quad \forall i \in \mathbb{D}_{\bar{\nu}},$$

$$P(0) = P(\sum_{j \in \mathbb{D}_{\bar{\nu}}} \mathbb{T}(j)).$$

It is then relevant to understand the differences and the advantages between the condition (18) and the previous condition. To do so, the author of [9] studied equivalent formulation of LMI (18) in the case of periodic but also aperiodic sampling. In particular, a corollary of Theorem 1 in this paper, in which the sampling period is denoted as $\bar{T} = \sum_{j \in \mathbb{D}_{\bar{\nu}}} \mathbb{T}(j)$, is proposed here:

Proposition 3. *Condition (18) holds if and only if there exist timer-dependent matrix functions $S_i : [0, T_i] \rightarrow \mathbb{S}_+^n$ for all $i \in \mathbb{D}_{\bar{\nu}}$ and a sufficiently small $\varepsilon > 0$ such that*

$$\begin{aligned} A_{\nu(i)}^\top S_i(\tau) + S_i(\tau) A_{\nu(i)} + \dot{S}_i(\tau) &\prec 0, \quad \forall \tau \in [0, \mathbb{T}(i)], \quad \forall i \in \mathbb{D}_{\bar{\nu}} \\ S_{\lfloor i+1 \rfloor_{\bar{\nu}}}(0) - S_i(\mathbb{T}(i)) + \varepsilon I &\prec 0. \end{aligned} \quad (37)$$

Proof. The proof is direct by applying the equivalence between items (b) and (d) of Theorem 1 in [9] with

$$S_i \leftarrow S_{[i+1]_{\bar{\nu}}}, S_j \leftarrow S_i, A_i \leftarrow A_{\nu(i+1)} \frac{\mathbb{T}(i+1)}{\bar{T}}, \tau \leftarrow \tau \frac{\mathbb{T}(i+1)}{\bar{T}},$$

where the left-hand side refers to the notations used in [9] and the right-hand side to the notations of this paper. Then, this particular selection leads to

$$\begin{aligned} A_{\nu(i+1)}^\top S_{[i+1]_{\bar{\nu}}}(\tau) + S_{[i+1]_{\bar{\nu}}}(\tau) A_{\nu(i+1)} + \dot{S}_{[i+1]_{\bar{\nu}}}(\tau) &< 0, \quad \forall \tau \in [0, \mathbb{T}(i+1)], \\ S_{[i+1]_{\bar{\nu}}}(0) - S_i(\mathbb{T}(i)) + \varepsilon I &< 0, \end{aligned}$$

for all $i \in \mathbb{D}_{\bar{\nu}}$. Shifting back the first equation yields the results. ■

It can be easily understood that condition (37) is equivalent to the condition presented in Theorem 2 in [15], and consequently is equivalent to condition (18). To follow the developments presented in [15], we are in position to state the following result as a direct consequence of this proposition, which presents a discretization scheme to solve the differential LMI condition (37).

Corollary 2. *If there exist matrices $\{P_i\}_{i \in \mathbb{D}_{\bar{\nu}}}$ such that*

$$\begin{aligned} \Psi_{i1} &= A_{\nu(i)}^\top P_{[i+1]_{\bar{\nu}}} + P_{[i+1]_{\bar{\nu}}} A_{\nu(i)} + \frac{P_{[i+1]_{\bar{\nu}}} - P_i}{\mathbb{T}(i)} < 0, \\ \Psi_{i2} &= A_{\nu(i)}^\top P_i + P_i A_{\nu(i)} + \frac{P_{[i+1]_{\bar{\nu}}} - P_i}{\mathbb{T}(i)} < 0, \end{aligned} \quad (38)$$

for all $i \in \mathbb{D}_{\bar{\nu}}$, then inequalities (18) holds true.

The reverse implication is not true.

Proof. Consider the particular case with the matrix functions

$$S_i(\tau) = \frac{\mathbb{T}(i) - \tau}{\mathbb{T}(i)} (P_i - \varepsilon I) + \frac{\tau}{\mathbb{T}(i)} (P_{[i+1]_{\bar{\nu}}} + \varepsilon I), \quad \forall i \in \mathbb{D}_{\bar{\nu}},$$

where $\varepsilon > 0$ is a sufficiently small positive scalar such that $P_i \succ \varepsilon I$, for all $i \in \mathbb{D}_{\bar{\nu}}$. Then, conditions (37) with this particular case write

$$\begin{aligned} \frac{\mathbb{T}(i) - \tau}{\mathbb{T}(i)} (\Psi_{i2} - \varepsilon \Theta_2) + \frac{\tau}{\mathbb{T}(i)} (\Psi_{i1} - \varepsilon \Theta_1) &< 0, \\ (P_{[i+1]_{\bar{\nu}}} - \varepsilon I) - (P_{[i+1]_{\bar{\nu}}} + \varepsilon I) + \varepsilon I &= -\varepsilon I < 0, \end{aligned}$$

where $\Theta_\alpha = \frac{2}{\mathbb{T}(i)}I + (-1)^\alpha(A_{\nu(i)}^\top - A_{\nu(i)})$, where $\alpha = 1, 2$ are constant matrices. Hence, if conditions Ψ_{i1} and Ψ_{i2} are verified, then there exists a sufficiently small $\varepsilon > 0$ such that the previous inequality holds. ■

Remark 2. Corollary 2 shows that both inequalities $\Psi_{i1} \prec 0$ and $\Psi_{i2} \prec 0$ are sufficient to assess the stability of the closed-loop system. However, this discretization process is conservative, since the selection of functions S_i is very restrictive. As a consequence, conditions (38) are conservative compared to the necessary and sufficient condition provided in Theorem 1.

Remark 3. Corollary 2 presents a simple discretization process consisting of matrices S_i that depend linearly on τ . In [3, 15], the authors consider a more accurate discretization method where matrices S_i are piece-wise linear with respect to τ , which provide a less conservative condition. Nevertheless, since conditions (37) are equivalent to (18) in the case of constant and known matrices $(A_j, B_j)_{j \in \mathbb{K}}$, such conservative discretization is not needed.

8. Illustrative examples

In this section, different examples allow us to illustrate the results presented in this paper.

8.1. Example 1

This first example is borrowed from [2]. The continuous system of the form (1) is composed by the three following unstable modes:

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 0.5 \\ 0 & -1 \end{bmatrix}, & B_1 &= \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}, \\ A_2 &= \begin{bmatrix} 0.1 & 0 \\ -1 & -1 \end{bmatrix}, & B_2 &= \begin{bmatrix} -1 \\ -0.5 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, & B_3 &= \begin{bmatrix} 0 \\ 2 \end{bmatrix}. \end{aligned} \tag{39}$$

To compare the results obtained in [2] with ours, we consider the desired equilibrium point $x_{\text{ref}} = [0.1 \ 0.2]^\top$. The method employed to guarantee the stabilization of the system in the vicinity of x_{ref} in [2] is to define an attractive set where the state must converge to using a *periodic sampled-data*

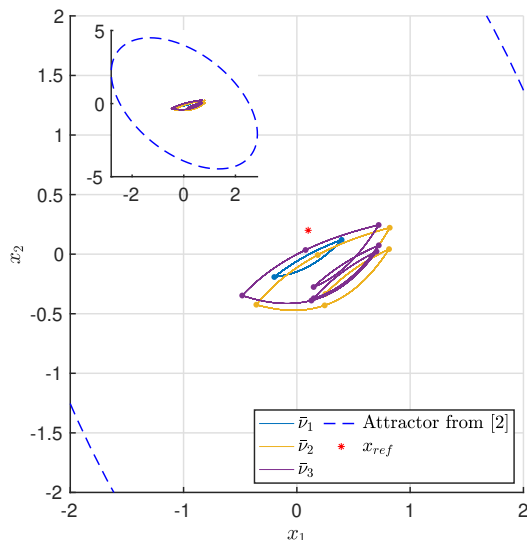


Figure 2: Example 1: Comparison between the attractive set from [2] and some limit cycles for $\mathbb{T}(i) = 0.6s$ for all $i \in \mathbb{D}_{\bar{\nu}}$. Note the window in the top-left corner giving a full view of the above mentioned attractor.

time-triggered control law. In order to highlight the differences between attractors, Figure 2 depicts some limit cycles of different lengths selected thanks to the minimization of the cost function J in (35) with $H_1 = 0$ and $H_2 = I_2$. Solving the optimization problem (36) with Ω given as follows

$$\Omega = \left\{ \bar{\mu} \in \mathfrak{C} : N_{\bar{\mu}} \leq 8, \mathbb{T}_i \in [1, 4] \cap \mathbb{N} \right\},$$

which means, in other words, that the length of cycles μ is less than or equal to 8, and the dwell times are limited to 1, 2, 3, 4, we have obtained the three limit cycles that deliver the lowest costs. The associated cycles are given by

$$\begin{aligned} \bar{\nu}_1 : \nu_1 &= \{2, 1\}, & \mathbb{T}_1 &= \{1, 1\}, \\ \bar{\nu}_2 : \nu_2 &= \{2, 1, 2, 1\}, & \mathbb{T}_2 &= \{1, 1, 2, 2\}, \\ \bar{\nu}_3 : \nu_3 &= \{2, 1, 2, 1, 2, 1\}, & \mathbb{T}_3 &= \{1, 1, 1, 1, 2, 2\}. \end{aligned}$$

Figure 3 shows the simulation of the closed-loop systems with three different switching control laws associated with cycles $\bar{\nu}_1$, $\bar{\nu}_2$ and $\bar{\nu}_3$, respectively. The plots in Figure 3 show the converge to the solutions to the desired hybrid limit cycles, after a short transient. In particular, the

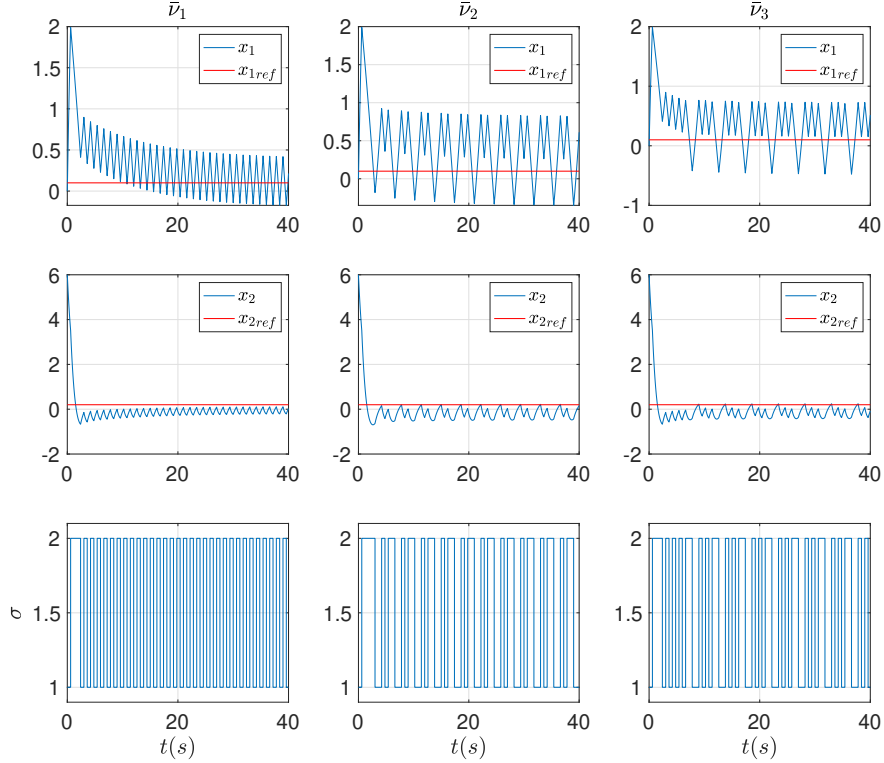


Figure 3: Example 1: Plots of the evolution of the plant state x and the switching control σ with respect to time for the cycles $\bar{\nu}_1$ (left), $\bar{\nu}_2$ (middle) and $\bar{\nu}_3$ (right).

switching signals σ follows a periodic behavior after the transient as stated in item (iv) of Theorem 1.

To complement Figure 3, the same simulations are presented in Figure 4, which shows trajectories of the plant state in the state space plan. Overall, Figures 3 and 4 show the impact of the *a priori* selection of the cycle $\bar{\nu}$ as the three solutions have very different behaviors.

Interestingly, one can see that these cycles do not contain the mode 3, therefore, the control law is restricted to a subset of \mathbb{K} . Figure 5 depicts the maximal absolute characteristic multiplier of the monodromy matrix for an arbitrarily selected cycle $\bar{\nu}_4 = (\nu_4 = \{3, 2, 1\}, \mathbb{T}_4 = \{2T_s, T_s, T_s\})$, for all $T_s < 10$. This shows that the stability of the monodromy matrix strongly depends on the selection of the sampling period T_s .

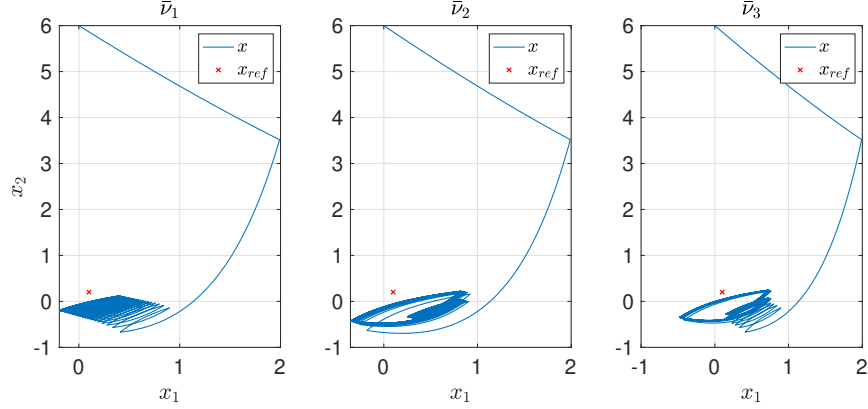


Figure 4: Example 1: Plots showing the evolution of the plant state x in the state space plan for the cycles $\bar{\nu}_1$ (left), $\bar{\nu}_2$ (middle) and $\bar{\nu}_3$ (right).

8.2. Example 2

This example is borrowed from [13] and [15] where the authors introduced several control laws to guarantee the Global Asymptotic Stability (GAS) of a predefined limit cycle. Consider system (1) with the following data

$$A_1 = \begin{bmatrix} -4 & -3 \\ -3 & 2.5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 4 & -1 \\ 1 & -2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ 8 \end{bmatrix}. \quad (40)$$

The objective is to make the state of the plant converges as close as possible to the reference $x_{\text{ref}} = [-9 \quad \star]^\top$, where \star means whatever value. Therefore, we selected $H_1 = H_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ in the cost function (35) to evaluate the candidate stable cycle. The solution to this optimization problem provides that the cycles

$$\begin{aligned} \bar{\nu}_1^* : \nu_1^* &= \{1, 1, 1, 1, 1, 1, 1, 1, 1, 2\}, \\ \mathbb{T}_1^* &= \{0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1, 0.1\}, \end{aligned} \quad (41)$$

$$\bar{\nu}_2^* : \nu_2^* = \{1, 1, 1, 2\}, \mathbb{T}_2^* = \{0.3, 0.3, 0.3, 0.1\}, \quad (42)$$

$$\bar{\nu}_3^* : \nu_3^* = \{1, 1, 2\}, \mathbb{T}_3^* = \{0.8, 0.1, 0.1\} \quad (43)$$

generate the closest limit cycle to x_{ref} , respectively. One can see in Figure 6, that the control law provided in this paper and from the control given in [13, Theorem 2] exhibit different trajectories of the closed-loop system.

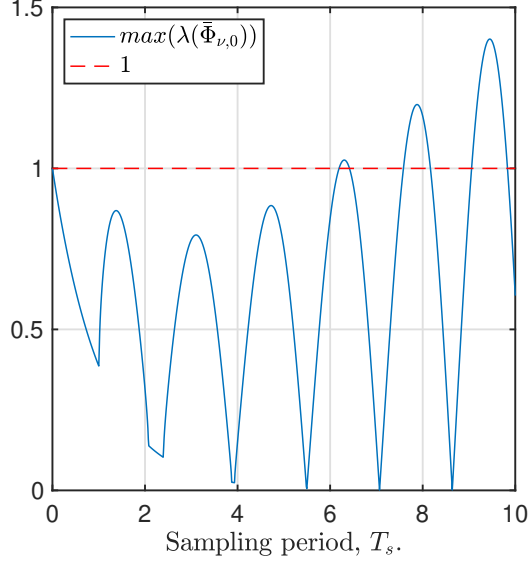


Figure 5: Example 1: Illustration of the maximal absolute characteristic multiplier of the monodromy matrix for the cycle $\bar{\nu}_4 = (\nu_4 = \{3 \ 2 \ 1\}, \mathbb{T}_4 = \{2T_s, T_s, T_s\})$ with respect to T_s .

Despite involving the solution to the same LMI condition, the control laws have different merits.

The main differences between the solution presented in this paper, which can be seen as the hybrid version of [28], has been already discussed in Section 4.3. The main difference refers to the time-varying nature of the control law, which is provided here:

$$u(x, \nu, k) = \operatorname{argmin}_{j \in \mathbb{K}} \begin{bmatrix} x - \rho(0, \lfloor k \rfloor_{\bar{\nu}}) \\ 1 \end{bmatrix}^\top \mathcal{L}_{k,j} \begin{bmatrix} x - \rho(0, \lfloor k \rfloor_{\bar{\nu}}) \\ 1 \end{bmatrix} \subset \mathbb{K}, \quad (44)$$

where $\mathcal{L}_{i,j} = \begin{bmatrix} A_j^\top P_{\lfloor i+1 \rfloor_{\bar{\nu}}} A_j - P_i & A_j^\top P_{\lfloor i+1 \rfloor_{\bar{\nu}}} b_{i,j} \\ * & b_{i,j}^\top P_{\lfloor i+1 \rfloor_{\bar{\nu}}} b_{i,j} \end{bmatrix}$ and with $b_{i,j} = A_j \rho(0, i) + B_j - \rho(0, \lfloor i+1 \rfloor_{\bar{\nu}})$ and with the clock-dependent Lyapunov function

$$V(x, \nu, k) = (x - \rho(0, \lfloor k \rfloor_{\bar{\nu}}))^\top P_{\lfloor k \rfloor_{\bar{\nu}}} (x - \rho(0, \lfloor k \rfloor_{\bar{\nu}})), \quad \forall x \in \mathbb{R}^n. \quad (45)$$

The underlying motivation behind this control law and the associated Lyapunov function is to find, for a given x , the best mode j in \mathbb{K} that minimizes the Lyapunov function at the next sampling instant. It can also be seen that, according to the definition of their Lyapunov function and

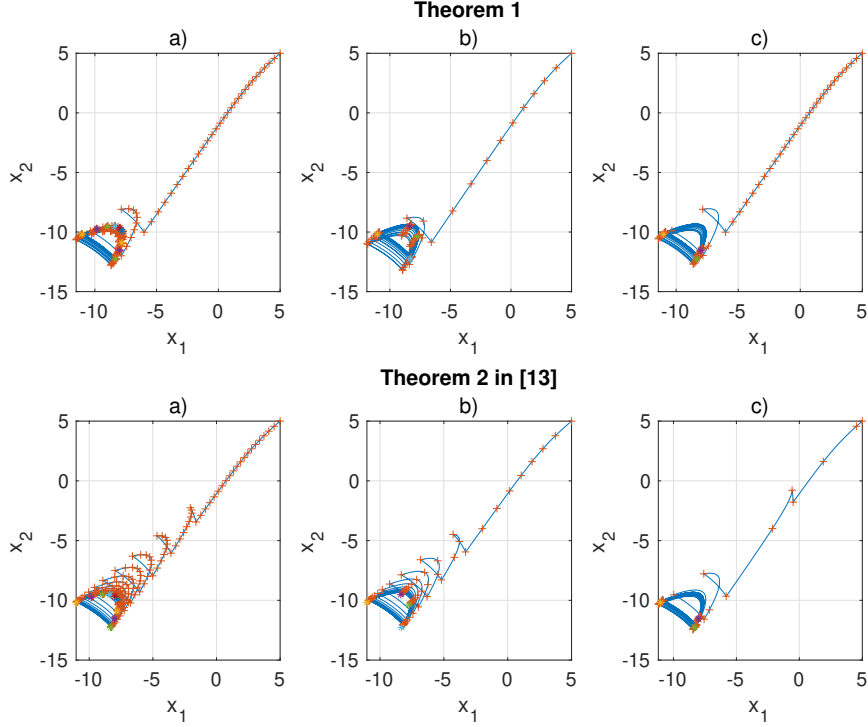


Figure 6: Example 2: Evolution of the states of the plant in the state space for cycles (41)–(43), using Theorem 1 and [13, Theorem 2]. Figures a), b) and c) are associated with the cycles (41), (42) and (43), respectively.

their control law, the position in the limit cycle is enforced to be $\rho(0, \lfloor k \rfloor_{\bar{\nu}})$, which is imposed by the time, while in our formulation, the control law only depends on x .

Another difference deals with the complexity of the control law. For a given cycle ν , control law (19) aims at selecting the best mode that minimizes the quadratic term in V , looking for the best position in the cycle. The control law (44) however finds the best mode that minimizes (45), at the next sampling instant $k + 1$. Therefore, depending on whether $N_{\bar{\nu}} > K$ or $N_{\bar{\nu}} < K$, control (19) or (44) can reduce the computational cost and the transient respectively.

The control law (19) for cycle $\bar{\nu}_1$ in (41) is parameterized with the fol-

lowing matrices:

$$\begin{aligned}
P_1 &= \begin{bmatrix} 0.4796 & 0.3865 \\ 0.3865 & 0.3662 \end{bmatrix}, P_2 = \begin{bmatrix} 0.4085 & 0.3482 \\ 0.3482 & 0.3535 \end{bmatrix}, P_3 = \begin{bmatrix} 0.3757 & 0.3358 \\ 0.3358 & 0.3591 \end{bmatrix}, \\
P_4 &= \begin{bmatrix} 0.3623 & 0.3371 \\ 0.3371 & 0.3749 \end{bmatrix}, P_5 = \begin{bmatrix} 0.3598 & 0.3463 \\ 0.3463 & 0.3975 \end{bmatrix}, P_6 = \begin{bmatrix} 0.3611 & 0.3589 \\ 0.3589 & 0.4238 \end{bmatrix}, \\
P_7 &= \begin{bmatrix} 0.3608 & 0.3717 \\ 0.3717 & 0.4521 \end{bmatrix}, P_8 = \begin{bmatrix} 0.3523 & 0.3804 \\ 0.3804 & 0.4799 \end{bmatrix}, P_9 = \begin{bmatrix} 0.3368 & 0.3866 \\ 0.3866 & 0.5086 \end{bmatrix}, \\
P_{10} &= \begin{bmatrix} 0.6212 & 0.4722 \\ 0.4722 & 0.4114 \end{bmatrix}.
\end{aligned}$$

Note that matrices P_i , for $i = 1, \dots, 9$ are very similar, which makes sense since they are all related to the use of the same mode 1. However, matrix P_{10} is clearly different from the others since it is the only one related to mode 2.

As the numerical values that are needed to compute the control law in [15] were not provided, a deepest comparison with the control law provided in this paper has not been included.

8.3. Example 3: DC-DC three cells-power converters

A more practical example is borrowed from [5] which also treats the asymptotic stability of a hybrid limit cycle. It is well known that power converters can be modeled as switched affine systems by considering current and voltage variables as the continuous states and the switching signal reflecting the positions of the power switches as the discrete-events. Hence, considering the DC-DC three-cells converter depicted in Figure 7, the state vector $x(t)$ gathers the capacitors voltages $v_1(t)$ and $v_2(t)$ and the load current $i(t)$. The control signals are the cell switches denoted as u_i and $\bar{u}_i = 1 - u_i$, such that $u_i = 1$ ($u_i = 0$) means that the upper switch is closed (opened) and the lower one is opened (closed). The different modes are derived from the combination of the cell switches as presented in Table 1 and they correspond to the system matrices defined as follows, for any u_1, u_2 and u_3 in $\{0, 1\}$

$$A(u_1, u_2, u_3) = \begin{bmatrix} 0 & 0 & \frac{u_2 - u_1}{C_1} \\ 0 & 0 & \frac{u_3 - u_2}{C_2} \\ \frac{u_1 - u_2}{L} & \frac{u_1 - u_3}{L} & -\frac{R}{L} \end{bmatrix}, \quad B(u_1, u_2, u_3) = \begin{bmatrix} 0 \\ 0 \\ \frac{V_{dc} u_3}{L} \end{bmatrix}. \quad (46)$$

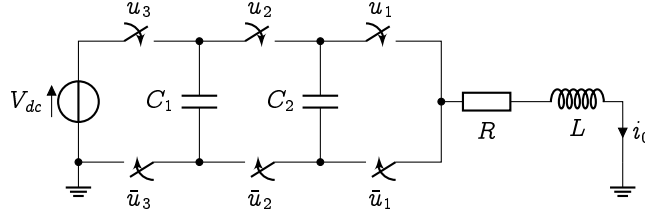


Figure 7: Example 3: Schematic of a three-cell converter.

First of all, note that there does not exist a linear combination of matrices A_i , that is Hurwitz stable, so that the stabilization conditions provided in [15] cannot be satisfied. It was also noticed in [19, Ex.3] that this system is not controllable if $R = 0$ because of the particular structure of matrices A_i 's making that there does not exist a cycle ν such that *the condition of Lemma 2 is verified for this system*. However, this is possible in the case where $R \neq 0$. In order to find all the candidate cycles, it is possible to follow the procedure from [28] to which we can add another step to the sieve to consider the technological constraint of the multi-cellular converter: the adjacency condition, i.e., only one switch can take place at each transition. In the consecutive papers [4, 5], the authors detail with a lot of specificity the different desired cyclic behaviors around the reference $x_{\text{ref}} = [V_{dc}/3 \ 2V_{dc}/3 \ I_{\text{ref}}]^T$ for the multilevel power converter considered while we only evaluate the cycles thanks to the cost function (35). However, the authors from [5] can only guarantee the local asymptotic stabilization of system (1), (46) to the following predefined limit cycle with the discrete signal sequence

$$\bar{\nu}_0 = (\nu_0 = \{1, 2, 1, 3, 1, 5\}, \mathbb{T}_0 = \{10^{-4}, 10^{-4}, 10^{-4}, 10^{-4}, 10^{-4}, 10^{-4}\}),$$

studied in [4]. On Figure 8, it is possible to observe the global stabilization to the limit cycle associated with ν_0 in the state space. The same simulation is illustrated on Figure 9 with a different view.

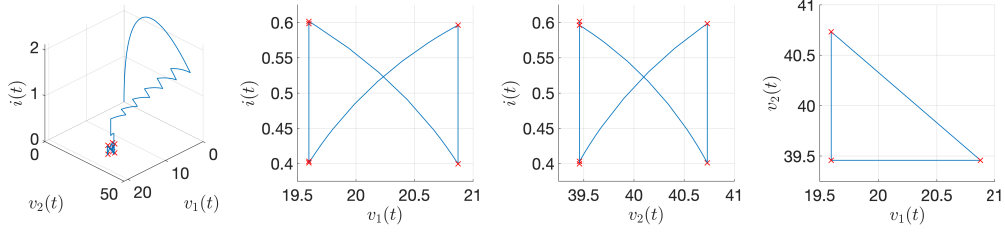


Figure 8: From left to right, the first figure depicts the global view of the evolution of the state x in a state space. The last three figures on the right-hand side are figures illustrating the plant limit cycle where the system converges. In each column, the blue line represents the evolution of the state x and the red crosses the point of the plant limit cycle $\rho(0, i)$, associated with the cycle \bar{v}_0 .

Table 1: Example 3: Converter parameters and switching states.

(a) DC-DC three-cells converter parameters

Parameters	V_{dc}	C_1, C_2	L	R	I_{ref}
Values	60V	40 μF	5mH	20 Ω	0.6A

(b) Values u_1, u_2 and u_3 for each mode i

i	1	2	3	4	5	6	7	8
u_1	0	1	0	1	0	1	0	1
u_2	0	0	1	1	0	0	1	1
u_3	0	0	0	0	1	1	1	1

The control law (19) is defined with the following matrices:

$$\begin{aligned}
 P_1 &= \begin{bmatrix} 0.9027 & 0.1167 & 0.1647 \\ 0.1167 & 0.6595 & 0.0634 \\ 0.1647 & 0.0634 & 0.1098 \end{bmatrix}, & P_2 &= \begin{bmatrix} 0.7236 & -0.0603 & -0.0970 \\ -0.0603 & 0.6930 & 0.0915 \\ -0.0970 & 0.0915 & 0.1201 \end{bmatrix}, \\
 P_3 &= \begin{bmatrix} 0.8429 & -0.1494 & -0.1512 \\ -0.1494 & 0.7595 & 0.1258 \\ -0.1512 & 0.1258 & 0.1163 \end{bmatrix}, & P_4 &= \begin{bmatrix} 0.7022 & -0.0790 & 0.0824 \\ -0.0790 & 0.7819 & -0.1420 \\ 0.0824 & -0.1420 & 0.1317 \end{bmatrix}, \\
 P_5 &= \begin{bmatrix} 0.7220 & -0.1253 & 0.0886 \\ -0.1253 & 0.8904 & -0.1732 \\ 0.0886 & -0.1732 & 0.1153 \end{bmatrix}, & P_6 &= \begin{bmatrix} 0.7846 & 0.0786 & 0.1241 \\ 0.0786 & 0.6472 & 0.0554 \\ 0.1241 & 0.0554 & 0.1191 \end{bmatrix}.
 \end{aligned}$$

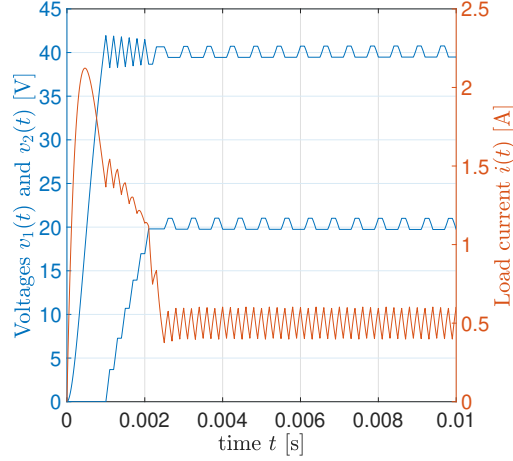


Figure 9: Example 3: Evolution of x function of the time.

9. Conclusion

Throughout this paper, the stabilization of continuous-time switched affine systems to hybrid limit cycles, using an aperiodic sampled-data control law has been provided. The hybrid limit cycles are not only defined by the state of the switched affine system (1), but also by the mode and their dwell time. Our solution extends the existing solution given in [15], which is shown to be a particular and conservative solution to the same problem. Although the main result seems to be similar to [13] and to [28], this extension requests the use of a new Lyapunov function to perform the stabilization analysis. In addition, the knowledge of the complete plant limit cycle, i.e., in continuous-time allows the designer to consider more complex cost functions to reduce the chattering effect for instance. On another side, compared to recent literature, the design of the global stabilizing switching control law (19) is based on simple LMI conditions, which are already known in the literature on periodic (linear) systems. Future work will consider robustness issues with respect to uncertainties affecting the system's matrices, i.e., in the situation, where matrices A_i and B_i are not assumed to be constant and known, and/or where the dwell-times can suffer from jitters' effect. Following the techniques presented in [28], this would require to extend the notion of hybrid limit cycles to a set of all the possible trajectories that are affected by the uncertainties.

Acknowledgments

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