HYPERTRANSCENDENCE AND LINEAR DIFFERENCE EQUATIONS, THE EXPONENTIAL CASE

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Abstract. In this paper we study meromorphic solutions of linear shift difference equations with coefficients in $\mathbb{C}(x)$ involving the operator $\rho : y(x) \mapsto y(x+h)$, for some $h \in \mathbb{C}^*$. We prove that if f is a solution of an algebraic differential equation, then f belongs to a ring that is generated by periodic functions and exponentials. Our proof is based on the parametrized difference Galois theory initiated by Hardouin and Singer.

Contents

Introduction

Given a solution of a functional equation we might wonder if it is a solution of another kind of functional equation. If the two functional equations are too different we would expect that very few functions are solutions of both functional equations. The goal of this paper is to state a result in that direction. Before recalling the state or the art, let us state our main result and its framework.

Let ρ be the automorphism of $\mathbb{C}(x)$ defined by $\rho : y(x) \mapsto y(x+h)$ with $h \in \mathbb{C}^*$. Let $\mathcal{M}(\mathbb{C})$ be the field of meromorphic functions on $\mathbb{C}$. We say that $f \in \mathcal{M}(\mathbb{C})$ is differentially algebraic over $\mathbb{C}(x)$, if there exist $m \in \mathbb{N}$, $0 \neq P \in \mathbb{C}(x)[X_0, \dots, X_m]$ such that $P(f, \partial_x f, \dots, \partial_x^m f) = 0$. We say that f is differentially transcendental otherwise. Let $C_h := \{f \in \mathcal{M}(\mathbb{C}) | \rho(f) = f\}$ be the field of h-periodic functions. The goal of this paper is to prove:

Theorem 1. *Let $n \in \mathbb{N}^*$, and let $A \in \mathrm{GL}_n(\mathbb{C}(x))$. Let $\mathcal{Y} := (f_1, \dots, f_n)^\top \in (\mathcal{M}(\mathbb{C}))^n$ be a solution of $\rho(Y) = AY$. If every f_i is differentially algebraic*

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over $\mathbb{C}(x)$*, then there exist* $\ell \in \mathbb{N}^*$*,* $\lambda_1 \dots, \lambda_k \in \mathbb{C}$*, such that for all* $1 \leq i \leq n$*,* $f_i \in C_{\ell h}(x)[e^{\lambda_1 x}, \dots, e^{\lambda_k x}]$.

As a straightforward corollary, we find:

Corollary 2. *Let* $f \in \mathcal{M}(\mathbb{C})$ *such that*

$$a_0 f + \cdots + a_n \rho^n f = 0, \quad a_i \in \mathbb{C}(x), \quad a_0, a_n \neq 0.$$

If f *is differentially algebraic over* $\mathbb{C}(x)$*, then there exist* $\ell \in \mathbb{N}^*$*,* $\lambda_1 \dots, \lambda_k \in \mathbb{C}$*, such that* $f \in C_{\ell h}(x)[e^{\lambda_1 x}, \dots, e^{\lambda_k x}]$.

Proof of Corollary 2. Since ρ and ∂_x commute, for all $i \in \mathbb{N}$, $\rho^i(f)$ is differentially algebraic over $\mathbb{C}(x)$. Then, each entry of $\mathcal{Y} := (f, \rho(f) \dots, \rho^{n-1}(f))^\top \in \mathcal{M}(\mathbb{C})^n$ is differentially algebraic over $\mathbb{C}(x)$. Since

$$\rho(\mathcal{Y}) = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & 1 \\ -a_0/a_n & -a_1/a_n & \cdots & \cdots & -a_{n-1}/a_n \end{pmatrix} \mathcal{Y},$$

we conclude with Theorem 1. □

The first proof of differential transcendence of a function is due to Hölder, see [Höl87]. The author proved that the Gamma function is differentially transcendental. His approach mainly uses the functional equation $\rho(\Gamma) = x\Gamma$ (here $h = 1$).

More recently in [BG93], it is proved that if $f \in \mathcal{M}(\mathbb{C})$ is solution of a linear ρ-equation and a linear differential equation, then there exist $\ell \in \mathbb{N}^*$, $\lambda_1 \dots, \lambda_k \in \mathbb{C}$, such that $f \in C_{\ell h}(x)[e^{\lambda_1 x}, \dots, e^{\lambda_k x}]$. The strategy of the authors is to study the singularities of the solutions of a linear ρ-equation, linear differential equation, and prove that there is an incompatibility, unless the function is sufficiently simple. Similar results were known for q-difference equations ($\rho : y(x) \mapsto y(qx)$, $q \in \mathbb{C}^*$, $|q| \neq 1$) and Mahler equations ($\rho : y(x) \mapsto y(x^p)$, $p \in \mathbb{N}_{\geq 2}$), and were proved in an unified way in [SS19].

The question of showing that a solution of a linear ρ-equation is not solution of an algebraic differential equation is much more complicated. Contrary to the case of a linear differential equation, the number of unknowns grows too fast to hope to predict the potential equation in a brute-force search. Another approach that fails is the study of the singularities, since a solution of an algebraic differential equation may have a very large set of singularities.

Old results were known for affine equations of order one of the form $\rho(y) = ay + b$, when ρ is the shift operator considered above, the q-difference operator or a Mahler operator, see for instance [Höl87, Moo96, Mah30, Nis84, Ran92, Ish98, Har08, HS08, Ngu12].

For higher order equations we need the Galois theory of difference equations to be able to prove similar results. More precisely, given a linear ρ-system, we may associate a Galois group of matrices that encodes the algebraic (resp. algebraic and differential) relations among the solutions. The bigger this group is, and fewer relations there are. So if we are able to prove

that the Galois group is sufficiently large, we might expect that the solutions are differentially transcendental. The first attempt of this strategy could be found in [Har08] for a modern proof of Hölder's theorem of the differential transcendence of the Gamma function. This approach has been generalized into a complete theory in [HS08] and has been applied to prove that when the Galois group is big, the solutions are differentially transcendental, see [HS08, DHR18, DHR21, AS17, ADR21]. The general idea behind the latter papers is that when the solutions are differentially algebraic and the Galois group is big, then the functions are solutions of linear differential equations. We then use the results above mentioned of [SS19] to prove that they are too trivial and force the Galois group to be small, leading to a contradiction. The Galois theory of [HS08] has been applied to a very different context to prove in [DHRS18] that some generating series of walks in the quarter plane are differentially transcendental.

At this stage, we are able to prove that the solutions of linear ρ-equations are differentially transcendental or trivial only when the Galois group is big, or small (affine equations). It remains to treat the medium cases. This is the goal of [ADH21] where it is proved that when the Galois group is medium, then the ρ-equation is equivalent to a smaller one. Then an induction proof on the rank of the equation allows the authors to prove that either the solution is differentially transcendental, or it belongs to a small field. The results of [ADH21] are stated for meromorphic solutions at infinity and the shift operator, for the q-difference operator, and for the Mahler operator. For the shift case and meromorphic function on $\mathbb{C}$ which is the framework of this paper, the statement is much more complicated to set for two reasons:

- the exponential functions $e^{\lambda x}$, $\lambda \in \mathbb{C}$, are meromorphic solutions of linear ρ-equations and linear differential equations.
- the meromorphic functions that are ℓh-periodic for some $\ell \in \mathbb{N}^*$ are solutions of linear ρ-equations, and some of them are solution of differential equations.

Then, the set of functions that are both solutions of a linear ρ-equation and differential equation is bigger than the ground field $\mathbb{C}(x)$. To avoid this problem, the authors of [ADH21] consider the situation where f is solution of a linear ρ-equation in coefficients in $\mathbb{C}(x)$ and belong to a field of meromorphic function $F \subset \mathcal{M}(\mathbb{C})$, with $\rho(F) = F$, $\{f \in F | \rho(f) = f\} = \mathbb{C}$, and for all $\lambda_1 \ldots, \lambda_k \in \mathbb{C}$, $F \cap \mathbb{C}(x, e^{\lambda_1 x}, \ldots, e^{\lambda_k x}) = \mathbb{C}(x)$. Then it is proved that either $f \in \mathbb{C}(x)$, or f is differentially transcendental. Thus, Theorem 1 generalizes this result in this context since we avoid the assumptions on F. The strategy of the proof of Theorem 1 is in some sense similar to the one in the recent paper [dS22] that deals with another framework.

The paper is organized as follows. In Section 1 we give a reminder of the difference Galois theory and in Section 2, the parametrized difference Galois theory. In Section 3, we deal with difference Galois groups that are irreducible and connected, and in Section 4 we complete the proof of Theorem 1.

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1. Difference setting

The goal of this section is to give a short review of the main results of difference algebra and difference Galois theory that will be used in this paper. We refer to [Coh65] for more details on difference algebra and [vdPS97] for more details on difference Galois theory. In what follows, all fields are of characteristic zero, and all rings are unitary.

A difference ring is a ring (R,ρ) is a ring equipped with an automorphism. We define similarly the notion of difference fields, difference algebras, etc... When no confusion arises, we will denote by R the difference ring (R,ρ). The ring of constants is defined by $R^\rho := \{r \in R | \rho(r) = r\}$. If R is a field, R^ρ is also a field and will be called the field of constants.

Example 3. If we consider the notation of the introduction, $(\mathbb{C}(x),\rho)$ and $(\mathcal{M}(\mathbb{C}),\rho)$ are difference fields, and we have $\mathbb{C}(x)^\rho = \mathbb{C}$ and $\mathcal{M}(\mathbb{C})^\rho = C_h$.

A ρ-ideal $I \subset R$ is an ideal such that $\rho(I) \subset I$. We say that the difference ring is ρ-simple if the only ρ-ideals are $\{0\}$ and I.

Let $\mathbf{k}$ be a difference field. Let us assume that $\mathbf{C} := \mathbf{k}^\rho$ is algebraically closed. We consider the difference system

$$\rho(Y) = AY, \quad A \in \mathrm{GL}_n(\mathbf{k}). \tag{1.1}$$

A Picard-Vessiot ring extension for (1.1) over $\mathbf{k}$ is a difference ring extension $\mathcal{R}|\mathbf{k}$ such that

- There exists $U \in \mathrm{GL}_n(\mathcal{R})$, such that $\rho(U) = AU$, such matrix is called a fundamental matrix;
- $\mathcal{R} = \mathbf{k}[U, 1/\det(U)]$;
- $\mathcal{R}$ is a simple difference ring.

A Picard-Vessiot ring extension exists and is unique up to isomorphism of $\mathbf{k}$-ρ-algebras.

Given a Picard-Vessiot ring extension $\mathcal{R}|\mathbf{k}$, the Picard-Vessiot extension $\mathcal{Q}$ is the total ring of fractions of $\mathcal{R}$. We have $\mathcal{Q}^\rho = \mathcal{R}^\rho = \mathbf{k}^\rho = \mathbf{C}$. We define the difference Galois group as the group of ring automorphisms of $\mathcal{Q}$, leaving $\mathbf{k}$ invariant and commuting with ρ, that is

$$\mathrm{Gal}(\mathcal{Q}|\mathbf{k}) = \{\sigma \in \mathrm{Aut}(\mathcal{Q}|\mathbf{k}) | \sigma \circ \rho = \rho \circ \sigma\}.$$

For any fundamental matrix $U \in \mathrm{GL}_n(\mathcal{Q})$, an easy computation shows that $U^{-1}\sigma(U) \in \mathrm{GL}_n(\mathbf{C})$ for all $\sigma \in \mathrm{Gal}(\mathcal{Q}|\mathbf{k})$. By [vdPS97, Theorem 1.13], the faithful representation

$$\begin{array}{ccc} \mathrm{Gal}(\mathcal{Q}|\mathbf{k}) & \to & \mathrm{GL}_n(\mathbf{C}) \\ \sigma & \mapsto & U^{-1}\sigma(U) \end{array}$$

identifies $\mathrm{Gal}(\mathcal{Q}|\mathbf{k})$ with a linear algebraic subgroup $G \subset \mathrm{GL}_n(\mathbf{C})$. Choosing another fundamental matrix of solutions U leads to a conjugate representation.

Specific results for shift equations. In this paragraph, we consider the field $\mathbb{C}(x)$ equipped with the automorphism $\rho : y(x) \mapsto y(x+h)$, $h \in \mathbb{C}^*$, and give some specific results.

Remark 4. Note that $\mathcal{Y}(x)$ is a meromorphic solution of $\mathcal{Y}(x+h) = A(x)\mathcal{Y}(x)$ if and only if $\mathcal{Z}(x) := \mathcal{Y}(hx)$ is a meromorphic solution of $\mathcal{Z}(x+1) = A(hx)\mathcal{Z}(x)$. Furthermore, an entry of $\mathcal{Y}$ is differentially algebraic over $\mathbb{C}(x)$ if and only if the corresponding entry of $\mathcal{Z}$ is differentially algebraic over $\mathbb{C}(x)$. Hence, we may apply the specific results of [Pra86, HS08], originally stated for $h = 1$, for a general $h \in \mathbb{C}^*$ in this paper.

Fix $B \in \mathrm{GL}_n(\mathbb{C}(x))$; then there exists $V \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$ such that $\rho(V) = BV$ ([Pra86, Theorem 1]). Let $\overline{C_h}$ be the algebraic closure of C_h and consider $\overline{C_h} \otimes \mathcal{M}(\mathbb{C})$ equipped with the structure of a ρ-ring via $\rho(c \otimes f) = c \otimes \rho(f)$.

The following result will be used in the sequel.

Lemma 5. *The ring $\mathcal{R} = \overline{C_h}(x)[V, 1/\det(V)] \subset \overline{C_h} \otimes \mathcal{M}(\mathbb{C})$ is a Picard-Vessiot ring extension for $\rho(Y) = BY$ over $\overline{C_h}(x)$.*

Proof. We want to apply [DHR18, Lemma 2.3], in this context. Since there are no derivations yet, a differential closed field is just an algebraically closed field. So we want to use the above lemma with $F = C_h(x, V) \subset \mathcal{M}(\mathbb{C})$, $\widetilde{\mathbf{k}} = \overline{C_h}$, and $\mathbf{k} = C_h$. Note that $\mathbf{k} = C_h$ is not algebraically closed but this assumption is not used for the proof of the following statements. By [DHR18, Lemma 2.3], $\overline{C_h} \otimes C_h(x, V)$ is a simple difference ring and $(\overline{C_h} \otimes C_h(x, V))^\rho = \overline{C_h}$. By [HS08, Corollary 6.15], the total quotient ring of $\overline{C_h} \otimes C_h(x, V)$ has field of constants that is also $\overline{C_h}$. Let $\mathcal{Q}$ be the total quotient ring of $\mathcal{R}$. Then, since $\mathcal{R} \subset \overline{C_h} \otimes C_h(x, V)$, we have $\mathcal{R}^\rho = \mathcal{Q}^\rho = \overline{C_h}$. We conclude with [HS08, Propositions 6.17], see also [vdPS97, Corollary 1.24] that $\mathcal{R} = \overline{C_h}(x)[V, 1/\det(V)]$ is a Picard-Vessiot ring extension for $\rho(Y) = BY$ over $\overline{C_h}(x)$. □

The change of variables $Z := TY$, $T \in \mathrm{GL}_n(\mathbb{C}(x))$, transforms the system $\rho(Y) = AY$ with $A \in \mathrm{GL}_n(\mathbb{C}(x))$ into $\rho(Z) = BZ$ where $B := \rho(T)AT^{-1} \in \mathrm{GL}_n(\mathbb{C}(x))$. Let $A, B \in \mathrm{GL}_n(\mathbb{C}(x))$. We will say that $[A]$ and $[B]$ are equivalent over $\mathbb{C}(x)$ if there exists $T \in \mathrm{GL}_n(\mathbb{C}(x))$ such that $B = \rho(T)AT^{-1}$. The following theorem has been proved in [vdPS97, Propositions 1.20 and 1.21].

Theorem 6. *Let $G := \mathrm{Gal}(\mathcal{Q}|\mathbb{C}(x))$. There exists $T \in \mathrm{GL}_n(\mathbb{C}(x))$ such that $\rho(T)AT^{-1} \in G(\mathbb{C}(x))$.*

2. Differential setting

We refer to [HS08] for more details on what follows. A differential ring (R, δ) is a ring equipped with a derivation, that is an additive morphism satisfying the Leibnitz rule $\delta(fg) = f\delta(g) + \delta(f)g$. We define similarly the notion of differential fields, differential algebras, etc... The ring of δ-constants of R is defined by

$$R^\delta = \{r \in R \mid \delta(r) = 0\}.$$

If R is a field, R^δ is also a field and will be called the field of constants. Let (R, δ) be a differential ring extension for the differential field $(\mathbf{k}, \delta)$. We say that $f \in R$ is differentially algebraic over $\mathbf{k}$ if there exist $m \in \mathbb{N}$, and $0 \neq P \in \mathbf{k}[X_0, \ldots, X_m]$ such that $P(f, \delta f, \ldots, \delta^m f) = 0$. We say that f is differentially transcendental over $\mathbf{k}$ otherwise.

A (ρ,δ)-ring (R,ρ,δ) is a ring equipped with an automorphism ρ and a derivation δ that commutes with ρ. We define similarly the notion of (ρ,δ)-fields, etc...

Example 7. If we consider the notation of the introduction, $(\mathbb{C}(x),\rho,\partial_x)$ and $(\mathcal{M}(\mathbb{C}),\rho,\partial_x)$ are (ρ,∂_x)-fields, and we have $\mathbb{C}(x)^{\partial_x}=\mathcal{M}(\mathbb{C})^{\partial_x}=\mathbb{C}$.

We say that I is a (ρ,δ)-ideal of R if I is an ideal such that $\rho(I)\subset I$ and $\delta(I)\subset I$. We say that (R,ρ,δ) is (ρ,δ)-simple if its only (ρ,δ)-ideals are $\{0\}$ and R.

Let us denote by $R\{X_1,\dots,X_n\}_\delta$ the ring of differential polynomials in the indeterminates $\delta^i X_j$. We recall that a differential field (L,δ) is called differentially closed or δ-closed if, for every set of δ-polynomials $\mathcal{F}$, the system of δ-equations $\mathcal{F}=0$ has a solution in some δ-field extension of L if and only if it has a solution in L. There always exists a differential field extension that is differentially closed.

Assume that $\mathbf{k}$ is a (ρ,δ)-field with $\mathbf{C}=\mathbf{k}^\rho$ differentially closed and let $A\in \mathrm{GL}_n(\mathbf{k})$. A (ρ,δ)-Picard-Vessiot ring extension for $\rho(Y)=AY$ over $\mathbf{k}$ is a (ρ,δ)-ring extension $\mathcal{S}|\mathbf{k}$ such that

- There exists $U\in \mathrm{GL}_n(\mathcal{S})$;
- $\mathcal{S}=\mathbf{k}\{U,1/\det(U)\}_\delta$;
- $\mathcal{S}$ is a simple (ρ,δ)-ring.

A (ρ,δ)-Picard-Vessiot ring extensions exists and is unique up to isomorphisms of $\mathbf{k}$-(ρ,δ)-algebras. Given a (ρ,δ)-Picard-Vessiot ring extension $\mathcal{S}|\mathbf{k}$, the (ρ,δ)-Picard-Vessiot extension $\mathcal{Q}_S$ is the total ring of fractions of $\mathcal{S}$. We have $\mathcal{Q}_S^\rho=\mathcal{S}^\rho=\mathbf{k}^\rho=\mathbf{C}$. We define the (ρ,δ)-Galois group as the group of ring automorphisms of $\mathcal{Q}_S$, leaving $\mathbf{k}$ invariant and commuting with ρ and δ, that is

$$\mathrm{Gal}^\delta(\mathcal{Q}_S|\mathbf{k})=\{\sigma\in \mathrm{Aut}(\mathcal{Q}_S|\mathbf{k})|\sigma\circ\rho=\rho\circ\sigma,\quad \sigma\circ\delta=\delta\circ\sigma\}.$$

For any fundamental matrix $U\in \mathrm{GL}_n(\mathcal{Q}_S)$, an easy computation shows that $U^{-1}\sigma(U)\in \mathrm{GL}_n(\mathbf{C})$ for all $\sigma\in\mathrm{Gal}^\delta(\mathcal{Q}_S|\mathbf{k})$. By [HS08, Theorem 2.6], the faithful representation

$$\begin{array}{rcl}\mathrm{Gal}^\delta(\mathcal{Q}_S|\mathbf{k}) & \to & \mathrm{GL}_n(\mathbf{C})\\ \sigma & \mapsto & U^{-1}\sigma(U)\end{array}$$

identifies $\mathrm{Gal}^\delta(\mathcal{Q}_S|\mathbf{k})$ with a linear differential algebraic subgroup $H\subset \mathrm{GL}_n(\mathbf{C})$, that is a group of matrices whose entries satisfy a set of algebraic differential relations. Choosing another fundamental matrix of solutions U leads to a conjugate representation.

Specific results for shift equations. The field C_h may be equipped with a structure of differential field with the derivation $\delta:=\partial_x$.

Let $\widetilde{C_h}$ be a differentially closed field containing C_h. Consider $\widetilde{C_h}(x)$, that is equipped with a structure of (ρ,δ)-field with $\left(\widetilde{C_h}(x)\right)^\rho=\widetilde{C_h}$, $\rho(x)=x+h$ and $\delta(x)=1$. Note that ρ and δ commute. Consider $\widetilde{C_h}\otimes\mathcal{M}(\mathbb{C})$ equipped with the structure of a (ρ,δ)-ring via $\rho(c\otimes f)=c\otimes\rho(f)$ and $\delta(c\otimes f)=\delta(c)\otimes f+c\otimes\delta(f)$. Note that $(\widetilde{C_h}\otimes\mathcal{M}(\mathbb{C}))^\rho=\widetilde{C_h}$.

The following result that is the analogue of Lemma 5 and its proof is totally similar.

Lemma 8. *Let $B \in \mathrm{GL}_n(\mathbb{C}(x))$ and let $V \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$ such that $\rho(V) = BV$. The ring $\mathcal{S} = \widetilde{C_h}(x)\{V, 1/\det(V)\}_\delta$ is a (ρ, δ)-Picard-Vessiot ring extension for $\rho(Y) = BY$ over $\widetilde{C_h}(x)$.*

Let $B \in \mathrm{GL}_n(\mathbb{C}(x))$. Given $\ell \in \mathbb{N}^*$, we may iterate the difference system $\rho(Y) = BY$ by considering $\rho^\ell(Y) = B_{[\ell]}Y$, where $B_{[\ell]} = \rho^{\ell-1}(B) \times \cdots \times B$. Let $G_{[\ell]}$ be the difference Galois group of the system $\rho^\ell(Y) = B_{[\ell]}Y$ and G be the difference Galois group of $\rho(Y) = BY$.

The following lemma is a slight adaptation of [DHR21, Proposition 4.6] in the particular case where the parametric operator is the identity, see also [ADH21, Proposition 4.10].

Lemma 9. *Let $B \in \mathrm{GL}_n(\mathbb{C}(x))$. Let $\mathcal{Y} \in \mathcal{M}(\mathbb{C})^n$ be a nonzero vector solution of $\rho(Y) = BY$. Then, there exists $\ell \geq 1$, such that*

- *There exists a (ρ^ℓ, δ)-Picard-Vessiot extension for $\rho^\ell(Y) = B_{[\ell]}Y$ over $\widetilde{C_h}(x)$, that is a field, and with fundamental solution admitting $\mathcal{Y}$ as first column.*
- *$G_{[\ell]}$ is the connected component of the identity of G.*

Proof. As we can see in the proof of [DHR21, Proposition 4.6], there exists $\ell \geq 1$, such that the (ρ^ℓ, δ)-Picard-Vessiot extension for $\rho^\ell(Y) = B_{[\ell]}Y$ over $\widetilde{C_h}(x)$, is a field and $G_{[\ell]}$ is the connected component of the identity of G. By [Pra86, Theorem 1], let $V \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$ solution of $\rho^\ell(Y) = B_{[\ell]}Y$, with $\mathcal{Y}$ as first column. By Lemma 8 the ring $\widetilde{C_h}(x)\{V, 1/\det(V)\}_\delta$ is a (ρ^ℓ, δ)-Picard-Vessiot ring extension for $\rho^\ell(Y) = B_{[\ell]}Y$ over $\widetilde{C_h}(x)$. Since the (ρ^ℓ, δ)-Picard-Vessiot extension is a field, $\widetilde{C_h}(x)\{V, 1/\det(V)\}_\delta$ is an integral domain and its field of fraction is isomorphic to the (ρ^ℓ, δ)-Picard-Vessiot extension. □

3. Irreducible Galois group

Recall that we consider the system $\rho(Y) = AY$ with $A \in \mathrm{GL}_n(\mathbb{C}(x))$. By [Pra86, Theorem 1], there exists $U \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$ such that $\rho(U) = AU$. Let $\mathcal{Q}_\mathbb{C}$ be a Picard-Vessiot extension for $\rho(Y) = AY$ over $\mathbb{C}(x)$. In this section, following [ADH21, Section 5.2] we consider the case where $\mathrm{Gal}(\mathcal{Q}_\mathbb{C}|\mathbb{C}(x))$, the difference Galois group of $\rho(Y) = AY$ over $\mathbb{C}(x)$, seen as an algebraic subgroup of $\mathrm{GL}_n(\mathbb{C})$, is irreducible and connected. Recall that an algebraic subgroup $G \subset \mathrm{GL}_n(\mathbb{C})$ is said to be irreducible; if and only if for all $\mathbb{C}$-vector spaces $V \subset \mathbb{C}^n$, $G(V) \subset V$ implies that V is either $\{0\}$ or $\mathbb{C}^n$. The proof is very similar to [ADH21, Section 5.2], and only the points where this proof is really different to the one of [ADH21, Section 5.2] will be detailed.

Proposition 10. *Let us assume that $n \geq 2$. If $\mathrm{Gal}(\mathcal{Q}_\mathbb{C}|\mathbb{C}(x))$ is irreducible and connected, then every column of U contains at least one element that is differentially transcendental over $\widetilde{C_h}(x)$.*

As a key argument, we will use the following result due to [AS17, Lemma 5.1]. It says that the (ρ,δ)-Galois group must be as big as possible when the difference Galois group has an identity component that is semisimple.

Proposition 11. *Let $B \in \mathrm{GL}_n(\mathbb{C}(x))$. We let $\widetilde{G} \subset \mathrm{GL}_n(\widetilde{C_h})$ denote the difference Galois group of $\rho(Y)=BY$ over $\widetilde{C_h}(x)$, and $H \subset \mathrm{GL}_n(\widetilde{C_h})$ denote the (ρ,δ)-Galois group of $\rho(Y)=BY$ over $\widetilde{C_h}(x)$. If the identity component of $\widetilde{G}$ is semisimple, then $H=\widetilde{G}$.*

Proof of Proposition 10. Let us argue by contradiction assuming that there exists one column of the fundamental solution U whose coordinates are all differentially algebraic over $\widetilde{C_h}(x)$. By Lemma 5, $\widetilde{C_h}(x)[U,1/\det(U)]$ is a Picard-Vessiot ring extension for $\rho(Y)=AY$ over $\widetilde{C_h}(x)$ and by Lemma 8, $\widetilde{C_h}(x)\{V,1/\det(V)\}_\delta$ is a (ρ,δ)-Picard-Vessiot ring extension for $\rho(Y)=AY$ over $\widetilde{C_h}(x)$. The difference Galois group of $\rho(Y)=AY$ over $\widetilde{C_h}(x)$ is $\mathrm{Gal}(\mathcal{Q}_{\mathbb{C}}|\mathbb{C}(x))(\widetilde{C_h})$ and is therefore irreducible. Hence, the same reasoning as the proof of [ADH21, Proposition 5.4] shows that all entries of U are differentially algebraic over $\widetilde{C_h}(x)$. Then, $\det(U)$ is also differentially algebraic over $\widetilde{C_h}(x)$. We observe that the determinant $\det(U)$ is solution to the equation

$$\rho(y)=\det(A)y \tag{3.1}$$

and that the difference Galois group of this equation over $\mathbb{C}(x)$ is the group $\det(\mathrm{Gal}(\mathcal{Q}_{\mathbb{C}}|\mathbb{C}(x)))$. We claim that $\det(U)$ is differentially algebraic over $C_h(x)$. We find that $\frac{\partial_x\det(U)}{\det(U)}$ is solution of $\rho(y)-y=\frac{\partial_x\det(A)}{\det(A)}$. Let $\widehat{b}(x):=\frac{\partial_x\det(A)}{\det(A)}\in\mathbb{C}(x)$. By Remark 4, we may use [HS08, Proposition 3.1], to deduce that there exist a nonzero linear differential operator $\widetilde{L}$ with coefficients in $\widetilde{C_h}$, and $\widetilde{g}\in\widetilde{C_h}(x)$, such that $\widetilde{L}(\widehat{b})=\widetilde{g}(x+h)-\widetilde{g}(x)$. Since $\widetilde{L}$ has coefficients in $\widetilde{C_h}$, it follows that the poles of $\widetilde{L}(\widehat{b})$, seen as an element of $\widetilde{C_h}(x)$ are the poles of $\widehat{b}$, which are in $\mathbb{C}$, because $\widehat{b}(x)\in\mathbb{C}(x)$. With $0\neq\widetilde{L}(\widehat{b})=\widetilde{g}(x+h)-\widetilde{g}(x)$, it follows that the poles of $\widetilde{g}(x)\in\widetilde{C_h}(x)$ are also in $\mathbb{C}$. Taking the partial fraction decomposition yields that the equation $\widetilde{L}(\widehat{b})=\widetilde{g}(x+h)-\widetilde{g}(x)$ is equivalent to a polynomial equation with coefficients in $\mathbb{C}$. Since $\mathbb{C}$ is algebraically closed it has a solution in $\mathbb{C}$, proving the existence of a nonzero linear differential operator L with coefficients in $\mathbb{C}$, and $g'\in\mathbb{C}(x)$, such that $L(\widehat{b})=g'(x+h)-g'(x)$. Then, $L(\frac{\partial_x\det(U)}{\det(U)}(x))-g'(x)$ is ρ-invariant. This shows that $\det(U)(x)$ is differentially algebraic over $C_h(x)$. By Remark 4, we may use [HS08, Corollary 3.4], to deduce that there exist some nonzero elements $c\in\mathbb{C}$ and $g\in\mathbb{C}(x)$ such that $\det(A)=c\rho(g)/g$. Let $d\in\mathbb{C}$ such that $e^{dh}=c^{-1/n}$ and consider the matrix $V=Ue^{dx}\in\mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$. It is solution of $\rho(Y)=BY$ with $B=c^{-1/n}A$. We have $\det(B)=\rho(g)/g$. By Lemma 8, $\widetilde{C_h}(x)\{V,1/\det(V)\}_\delta$ is a (ρ,δ)-Picard-Vessiot ring extension for $\rho(Y)=BY$ over $\widetilde{C_h}(x)$. Let G be the difference Galois group of $\rho(Y)=BY$ over $\mathbb{C}(x)$, $\widetilde{G}$ the difference Galois group of $\rho(Y)=BY$ over $\widetilde{C_h}(x)$, and H the (ρ,δ)-Galois group over

$\widetilde{C_h}(x)$. Same reasons as in the proof of [ADH21, Proposition 5.4] shows that G is irreducible, the connected component G^0 of the identity of G is also irreducible, and there exists $\ell \geq 1$ such that the difference Galois group of $\rho^\ell(Y) = B_{[\ell]}Y$ over $\mathbb{C}(x)$ is G^0. Furthermore, $\det(B_{[\ell]}) = \rho^\ell(g)/g$ and therefore, $G^0 \subset \mathrm{SL}_n(\mathbb{C})$. With [ADH21, Lemma 4.2] we find that G^0 is primitive (we refer to the latter paper for the notion of primitive groups that will not be used again in the sequel). By [SU93, Proposition 2.3], we finally obtain that G^0 is semisimple. Then, $\widetilde{G}^0 = G^0(\widetilde{C_h})$, the identity component of $\widetilde{G}$, is semisimple too.

We infer from Proposition 11 that $H = \widetilde{G}$. We recall that all entries of U are differentially algebraic over $\widetilde{C_h}(x)$. Then the same holds for $V = Ue^{dx}$. The same reasoning as in the proof of [ADH21, Proposition 5.4] shows that G is a finite group. Since it is connected, we deduce that $G^0 = \{\mathrm{I}_n\}$, where we let I_n denote the identity matrix of size n. Since by assumption $n \geq 2$, this provides a contradiction with the fact that G^0 is irreducible. □

4. Proof of the main result

We are now ready to prove Theorem 1. Recall that $\mathcal{Y} := (f_1, \dots, f_n)^\top \in (\mathcal{M}(\mathbb{C}))^n$ is a solution of $\rho(Y) = AY$, with $A \in \mathrm{GL}_n(\mathbb{C}(x))$. Let $C_h^\infty = \cup_{\ell=1}^\infty C_{\ell h}$ and let $R_{h,\exp}$ be the ring of $C_h^\infty(x)$-linear combinations between $\{e^{\lambda_i x}, \lambda_i \in \mathbb{C}\}$. We have to prove that when all the entries of $\mathcal{Y}$ are differentially algebraic over $\mathbb{C}(x)$, they all belong to $R_{h,\exp}$. If $\mathcal{Y}$ is zero the result is clear. So let us assume that $\mathcal{Y}$ is nonzero. Let $\mathcal{Q}$ be the Picard-Vessiot extension for $\rho(Y) = AY$ over $\overline{C_h}(x)$ and let $\mathrm{Gal}(\mathcal{Q}|\overline{C_h}(x))$ be the difference Galois group. By Lemma 9, with $\delta = 0$, there exists $\ell \geq 1$, such that

- we have a Picard-Vessiot extension for $\rho^\ell(Y) = A_{[\ell]}Y$ over $(\overline{C_h}(x), \rho^\ell)$ that is a field;
- the fundamental solution U of the Picard-Vessiot extension over $(\overline{C_h}(x), \rho^\ell)$ admits $\mathcal{Y}$ as first column;
- the difference Galois group of $\rho^\ell(Y) = A_{[\ell]}Y$ over $\overline{C_h}(x)$ is the component of the identity of $\mathrm{Gal}(\mathcal{Q}|\overline{C_h}(x))$.

In virtue of [Pra86, Theorem 1], we may assume that $U \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$. Note that $(\mathbb{C}(x), \rho^\ell, \delta)$ is a (ρ^ℓ, δ)-field and $R_{\ell h,\exp} \subset R_{h,\exp}$ is a (ρ^ℓ, δ)-ring. So, without loss of generality, we may replace A by $A_{[\ell]}$, h by ℓh, and reduce to the case where the Picard-Vessiot extension $\mathcal{Q}$ is a field, the difference Galois group over $\overline{C_h}(x)$ is connected, and the Picard-Vessiot extension for $\rho(Y) = AY$ over $\overline{C_h}(x)$ admits a fundamental solution with $\mathcal{Y}$ as first column. Let $\mathcal{Q}_\mathbb{C}$ be the Picard-Vessiot extension for $\rho(Y) = AY$ over $\mathbb{C}(x)$ and let $\mathrm{Gal}(\mathcal{Q}_\mathbb{C}|\mathbb{C}(x))$ be the difference Galois group. Since $\mathrm{Gal}(\mathcal{Q}|\overline{C_h}(x)) = \mathrm{Gal}(\mathcal{Q}_\mathbb{C}|\mathbb{C}(x))(\overline{C_h})$, we find that $\mathrm{Gal}(\mathcal{Q}_\mathbb{C}|\mathbb{C}(x))$ is connected too.

Let us prove Theorem 1 by induction on n, the order of the equation. The global strategy of the proof will be quite similar to [ADH21], but the arguments will differ inside each step of the proof.

(1) Case $n = 1$ (initialization of the induction).
(2) Reduction to the affine case when $n \geq 2$.

(3) Case of affine equations.

Step 1: Case $n = 1$.

Let us prove Theorem 1 in the case $n = 1$.

Proposition 12. *Let $0 \neq a \in \mathbb{C}(x)$. If $f \in \mathcal{M}(\mathbb{C})$ is solution of $\rho(y) = ay$ and is differentially algebraic over $\mathbb{C}(x)$, then $f \in R_{h,\exp}$.*

Proof. If $f = 0$ the result is clear. Assume that $f \neq 0$. By Remark 4, we may apply [HS08, Corollary 3.4], to deduce that there exist nonzero elements $c \in \mathbb{C}$ and $d(x) \in \mathbb{C}(x)$ such that $f(x+h)/f(x) = a(x) = cd(x+h)/d(x)$. Let $c' \in \mathbb{C}$ such that $e^{c'h} = c$. We find that $c_1(x) := \frac{f(x)}{d(x)e^{c'x}}$ is ρ-invariant. We have proved that $c_1(x) \in C_h$ and therefore $f(x) = c_1(x)d(x)e^{c'x} \in R_{h,\exp}$ as expected. □

Step 2: Reduction to the affine case.

Let us now begin the induction step of the proof of Theorem 1. Let us fix $n \geq 2$ and assume that Theorem 1 holds for equations of order strictly less than n. We assume that each entry of $\mathcal{Y}$ is differentially algebraic over $\mathbb{C}(x)$ and prove that $\mathcal{Y}$ has entries in $R_{h,\exp}$. By Proposition 10, $\mathrm{Gal}(\mathcal{Q}_{\mathbb{C}}|\mathbb{C}(x))$ is reducible. By [ADH21, Lemma 4.4], there exists $T \in \mathrm{GL}_n(\mathbb{C}(x))$ such that $Z := T\mathcal{Y}$ is solution of a bloc system $\rho(Z) = BZ$ where $B = \begin{pmatrix} B_{1,1} & B_{1,2} \\ 0 & B_{2,2} \end{pmatrix}$, and $B_{i,i}$, $i = 1, 2$, has size $n_i < n$. Let us further assume that n_1 is minimal for this property. The goal of this step is to prove that when $n_1 > 1$ the results holds. More precisely, we want to prove the following:

Proposition 13. *If each entry of $\mathcal{Y}$ is differentially algebraic over $\mathbb{C}(x)$ and $n_1 > 1$, then $\mathcal{Y} \in (R_{h,\exp})^n$.*

Let G_1 be the difference Galois group of $\rho(Y) = B_{1,1}Y$ over $\mathbb{C}(x)$. The group G_1 is irreducible and connected, see [ADH21, Section 5.3].

Let $(g_1, \dots, g_n)^\top = T\mathcal{Y}$. The entries of $\mathcal{Y}$ are all differentially algebraic over $\mathbb{C}(x)$ if and only if the same holds for the entries of $(g_1, \dots, g_n)^\top$. Similarly, $\mathcal{Y} \in (R_{h,\exp})^n$ if and only if $(g_1, \dots, g_n)^\top \in (R_{h,\exp})^n$. So it suffices to show that if all the entries of $(g_1, \dots, g_n)^\top$ are differentially algebraic over $\mathbb{C}(x)$, then $(g_1, \dots, g_n)^\top \in (R_{h,\exp})^n$. Let $\mathcal{Y}_1$ be the vector $(g_1, \dots, g_{n_1})^\top$ and $\mathcal{Y}_2$ be the vector $(g_{n_1+1}, \dots, g_n)^\top$. We have $\rho(\mathcal{Y}_1) = B_{1,1}\mathcal{Y}_1 + B_{1,2}\mathcal{Y}_2$ and $\rho(\mathcal{Y}_2) = B_{2,2}\mathcal{Y}_2$. By induction hypothesis, when each entry of $\mathcal{Y}$ is differentially algebraic, we find that $\mathcal{Y}_2 \in (R_{h,\exp})^{n_2}$. Then, when each entry of $\mathcal{Y}$ is differentially algebraic over $\mathbb{C}(x)$, we have

$$\mathcal{Y} \in (R_{h,\exp})^n \iff \mathcal{Y}_1 \in (R_{h,\exp})^{n_1}. \tag{4.1}$$

Toward the proof of Proposition 13, we are going to prove that when each entry of $\mathcal{Y}$ is differentially algebraic over $\mathbb{C}(x)$, then the entries of $\mathcal{Y}_1$ belong to a certain ring. Since $\mathcal{Y}_2 \in (R_{h,\exp})^{n_2}$, there exist $\lambda_1, \dots, \lambda_k \in \mathbb{C}$ and $\ell \in \mathbb{N}^*$ such that the entries of $\mathcal{Y}_2$ belong to $C_{\ell h}(x)[e^{\lambda_1 x}, \dots, e^{\lambda_k x}]$. Up to replacing the ρ-equation by a ρ^ℓ-equation, we may reduce to the case where $\ell = 1$. Recall that $\widetilde{C_h}$ is a differentially closed field containing C_h. Let $\widetilde{K_0}$, be the (ρ, δ)-ring that is the total ring of fraction of the ring generated by the entries of $\mathcal{Y}_2$ over $\widetilde{C_h}(x)$.

Lemma 14. *If each entry of $\mathcal{Y}$ is differentially algebraic over $\mathbb{C}(x)$ and $n_1 > 1$, then $\mathcal{Y}_1 \in (\widetilde{K_0})^{n_1}$.*

Proof. Assume that each entry of $\mathcal{Y}$ is differentially algebraic over $\mathbb{C}(x)$. Let us prove the contrapositive: let us assume that $\mathcal{Y}_1 \notin (\widetilde{K_0})^{n_1}$ and let us prove that $n_1 = 1$. Recall that $U \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$ is such that $\rho(U) = AU$. Let $V = TU$; this is a fundamental solution of $\rho(Y) = BY$. By Lemma 8, $\mathcal{Q}_S$, the total ring of fractions of $\widetilde{C_h}(x)\{V, 1/\det(V)\}_\delta$, is a (ρ, δ)-Picard-Vessiot extension for $\rho(Y) = BY$ over $\widetilde{C_h}(x)$. Let $H = \mathrm{Gal}^\delta(\mathcal{Q}_S|\widetilde{C_h}(x))$ be the (ρ, δ)-Galois group of $\rho(Y) = BY$ over $\widetilde{C_h}(x)$. By the (ρ, δ)-Galois correspondence, see [HS08, Theorem 2.7], we deduce the existence of some $\sigma \in H$, such that $\sigma(g_i) = g_i$ for every i, $n_1 + 1 \leq i \leq n$ and

$$(\sigma(g_i))_{i\leq n}^\top \neq (g_i)_{i\leq n}^\top .$$

Hence $w := \mathcal{Y}_1 - \sigma(\mathcal{Y}_1)$ is a nonzero vector. Since the coordinates of $\mathcal{Y}_1$ are differentially algebraic over $\widetilde{C_h}(x)$ and σ belongs to the (ρ, δ)-Galois group H, the coordinates of $\sigma(\mathcal{Y}_1)$ are also differentially algebraic over $\widetilde{C_h}(x)$. Therefore the coordinates of w are differentially algebraic over $\widetilde{C_h}(x)$. Furthermore, $\mathcal{Y}_1$ and $\sigma(\mathcal{Y}_1)$ are both solution to the system

$$\rho(Y) = B_{1,1}Y + B_{1,2}\mathcal{Y}_2 .$$

It follows that

$$\rho(w) = B_{1,1}w .$$

Since we have

$$\widetilde{C_h} = \widetilde{C_h}(x)^\rho \subset (\mathcal{Q}_S)^\rho = \widetilde{C_h} ,$$

we find by an adaptation of Lemma 9, the existence of a positive integer s and a (ρ^s, δ)-Picard-Vessiot field extension $\mathcal{Q}_1$ for the system $\rho^s(Y) = (B_{1,1})_{[s]}Y$ over $\widetilde{C_h}(x)$ such that w is the first column of a fundamental matrix W. Furthermore, the difference Galois group of $\rho^s(Y) = (B_{1,1})_{[s]}Y$ over $\mathbb{C}(x)$ is equal to G_1 since the latter is connected. Let us build a fundamental solution of $\rho^s(Y) = (B_{1,1})_{[s]}Y$ with coefficients in $\mathcal{M}(\mathbb{C})$ and whose all coordinates of the first column are differentially algebraic over $\widetilde{C_h}(x)$. By [Pra86, Theorem 1], there exists $W_1 \in \mathrm{GL}_{n_1}(\mathcal{M}(\mathbb{C}))$ such that $\rho^s(W_1) = (B_{1,1})_{[s]}W_1$. By Lemma 8, the total ring of fractions of $\widetilde{C_h}(x)\{W_1, 1/\det(W_1)\}_\delta$ is a (ρ^s, δ)-Picard-Vessiot extension for $\rho^s(Y) = (B_{1,1})_{[s]}Y$ over $\widetilde{C_h}(x)$. Recall that the entries of w are differentially algebraic over $\widetilde{C_h}(x)$. Since the two (ρ^s, δ)-Picard-Vessiot ring extensions $\widetilde{C_h}(x)\{W, 1/\det(W)\}_\delta$ and $\widetilde{C_h}(x)\{W_1, 1/\det(W_1)\}_\delta$ are isomorphic, we deduce that without loss of generality, we may assume that the entries of the first column of W_1 are differentially algebraic over $\widetilde{C_h}(x)$ too. The group G_1 being connected and irreducible, Proposition 10 implies that $n_1 = 1$. This proves the result by contraposition. □

Let us descend from $\widetilde{K_0}$ to $\widetilde{C_h} \otimes R_{h,\exp}$.

Lemma 15. *If $\mathcal{Y}_1 \in (\widetilde{K_0})^{n_1}$, then $\mathcal{Y}_1 \in (\widetilde{C_h} \otimes R_{h,\exp})^{n_1}$.*

Proof. Recall that $\mathcal{Y}$ is solution of $\rho(Y) = AY$ and is the first column of the fundamental solution $U \in \mathrm{GL}_n(\mathcal{M}(\mathbb{C}))$. By Lemma 5, $\mathcal{R} = \widetilde{C_h}(x)[U, 1/\det(U)]$ is a Picard-Vessiot ring extension for $\rho(Y) = AY$ over $\widetilde{C_h}(x)$. Let us consider $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, such that the entries of $\mathcal{Y}_2$ belong to $C_h(x)[e^{\lambda_1 x}, \dots, e^{\lambda_k x}]$. Let $W = \mathrm{Diag}(e^{\lambda_1 x}, \dots, e^{\lambda_k x})$. By Lemma 5, $\mathcal{R}_1 = \widetilde{C_h}(x)[W, 1/\det(W)]$, is a Picard-Vessiot ring extension for $\rho(Y) = \mathrm{Diag}(e^{\lambda_1 h}, \dots, e^{\lambda_k h})Y$ over $\widetilde{C_h}(x)$. The following idea has been suggested by Charlotte Hardouin. Consider the ideal

$$I_1 = \{s \in \mathcal{R}_1 | s\mathcal{Y} \in \left(\mathcal{R}_1\right)^n\}.$$

Let us prove that it is a ρ-ideal. Let $s \in I_1$. We have

$$\rho(s)\mathcal{Y} = \rho(s)A^{-1}A\mathcal{Y} = A^{-1}\rho(s\mathcal{Y}).$$

With $s\mathcal{Y} \in \left(\mathcal{R}_1\right)^n$ we therefore deduce that we have $A^{-1}\rho(s\mathcal{Y}) \in \left(\mathcal{R}_1\right)^n$. Hence $\rho(s) \in I_1$. This shows that I_1 is a ρ-ideal. The latter is not reduced to (0) since it contains the common denominator of the entries of $\mathcal{Y}$. Since $\mathcal{R}_1$ is a simple ρ-ring, and $I_1 \neq (0)$, we deduce that $1 \in I_1$, proving that $\mathcal{Y}$ has its entries in $\widetilde{C_h}(x)[e^{\lambda_1 x}, \dots, e^{\lambda_k x}, e^{-(\lambda_1 + \dots + \lambda_k)x}]$. This completes the proof. □

Let us finish the proof of Proposition 13. Assume that the entries of $\mathcal{Y}$ are differentially algebraic over $\mathbb{C}(x)$ and $n_1 > 1$. We have seen that $\mathcal{Y}_2 = (g_{n_1+1}, \dots, g_n)^\top \in (R_{h,\exp})^{n_2}$ and by Lemma 14, and Lemma 15, $\mathcal{Y}_1 = (g_1, \dots, g_{n_1})^\top \in (\widetilde{C_h} \otimes R_{h,\exp})^{n_1}$. Since $T^{-1}(g_1, \dots, g_n)^\top = \mathcal{Y}$, it follows that $\mathcal{Y} \in (\widetilde{C_h} \otimes R_{h,\exp})^n$. The following lemma will terminate the proof of Proposition 13.

Lemma 16. *If $\mathcal{Y} \in (\widetilde{C_h} \otimes R_{h,\exp})^n$, then $\mathcal{Y} \in (R_{h,\exp})^n$.*

Proof. Let $0 \neq P \in C_h[x]$ of minimal degree such that we may write $P\mathcal{Y} = c_1\mathcal{Z}_1 + \dots + c_\kappa\mathcal{Z}_\kappa$ with $c_i \in \widetilde{C_h}$ and $\mathcal{Z}_i$ with entries that are $C_h[x]$-linear combination between the $\{e^{\lambda_i x}, \lambda_i \in \mathbb{C}\}$. Since $\mathcal{Y} \in (R_{h,\exp})^n$ if and only if $P\mathcal{Y} \in (R_{h,\exp})^n$ it suffices to show that $P\mathcal{Y} \in (R_{h,\exp})^n$. Let us assume that κ is minimal which implies that the $\mathcal{Z}_i$ are C_h linearly independent. Since functions solutions of linear difference equation form a ring it follows that every entries of $\mathcal{Z}_i$ are solution of linear difference equation in coefficients in $\mathbb{C}(x)$. Let us consider a difference equation such that every y_i are solutions, where y_i is the first entry of $\mathcal{Z}_i$. Let $\rho(Y) = \widetilde{A}Y$ be the corresponding system and let m be its size. Since $\rho(c_i) = c_i$ the first entry of $P\mathcal{Y}$ is solution of the same equation as the y_i. We use that fact that there are at most m linearly independent solutions, to deduce $\kappa \leq m$. By [Pra86, Theorem 1], let $W \in \mathrm{GL}_m(\mathcal{M}(\mathbb{C}))$ with $\rho(W) = \widetilde{A}W$ such that the κ first columns are the $(y_i, \rho(y_i), \dots, \rho^{m-1}(y_i))^\top$. Since U and W have coefficients in $\mathrm{GL}_m(\mathcal{M}(\mathbb{C}))$, there exists $C_0 \in \mathrm{GL}_m((\mathcal{M}(\mathbb{C}))^\rho) = \mathrm{GL}_n(C_h)$ such that $UC_0 = W$. This proves that the above c_i are in C_h, proving that $P\mathcal{Y} \in R_{h,\exp}$ as expected. □

Step 3: Affine case.

Let us finish the proof of Theorem 1. Let us recall that we assume that the entries of $\mathcal{Y}$ are differentially algebraic over $\mathbb{C}(x)$ and we have to prove

that $\mathcal{Y} \in (R_{h,\exp})^n$. By Proposition 13, it remains to consider the case where $n_1 = 1$. By (4.1) it suffices to show that $g_1 \in R_{h,\exp}$. Recall that g_1 is a differentially algebraic solution of the affine equation $\rho(g_1) = ag_1 + b$ where $a = B_{1,1} \in \mathbb{C}(x)$ and $b = B_{1,2}(g_2, \dots, g_n)^\top \in R_{h,\exp}$. Let $\widetilde{C_h}$ be a differentially closed field containing C_h. By [HS08, Proposition 3.8], the (ρ, δ)-Galois group of $\rho(y) = ay$ over $\widetilde{C_h}(x)$ is not all of $\mathrm{GL}_1(\widetilde{C_h})$. By [Pra86, Theorem 1], let $h_0 \in \mathcal{M}(\mathbb{C})$ be a nonzero solution of $\rho(y) = ay$. In virtue of Lemma 8, the total ring of fractions of $\widetilde{C_h}(x)\{h_0, 1/h_0\}_\delta$ is a (ρ, δ)-Picard-Vessiot extension for $\rho(Y) = aY$ over $\widetilde{C_h}(x)$. With [HS08, Proposition 6.26], h_0 is differentially algebraic over $\widetilde{C_h}(x)$. By Remark 4, we may use [HS08, Corollary 3.4], to find that there exist nonzero $c \in \mathbb{C}$ and $g \in \mathbb{C}(x)$ such that $a = c\rho(g)/g$. Let $c' \in \mathbb{C}$ such that $e^{c'h} = c$. We have $g_1 \in R_{h,\exp}$ if and only if $\hat{g} := \frac{g_1}{e^{c'x}g} \in R_{h,\exp}$. We have $\rho(\hat{g}) = \hat{g} + \frac{b}{\rho(e^{c'x}g)}$. The latter is solution of an equation of the form $\rho(y) = y + b'$ where $b' := \frac{b}{\rho(e^{c'x}g)} \in R_{h,\exp}$. So without loss of generality we may reduce to the case where $a = 1$. Then, g_1 is solution of $\rho(y) = y + b$ where $b \in R_{h,\exp}$. Let $\ell \in \mathbb{N}^*$ and $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, such that $b \in C_{\ell h}(x)[e^{\lambda_1 x}, \dots, e^{\lambda_k x}]$. We have $\rho^\ell(g_1) = g_1 + b_{[\ell]}$, with $b_{[\ell]} := b + \cdots + \rho^{\ell-1}(b) \in C_{\ell h}(x)[e^{\lambda_1 x}, \dots, e^{\lambda_k x}]$. Since $R_{\ell h,\exp} \subset R_{h,\exp}$ is (ρ^ℓ, δ)-ring, we may replace the ρ-equation by a ρ^ℓ-equation to reduce to the case $\ell = 1$. We deduce from the proof of Lemma 9 that there exists $\ell \geq 1$, such that $\widetilde{C_h}(x)\{e^{\lambda_1 x}, \dots, e^{\lambda_k x}, e^{-(\lambda_1+\cdots+\lambda_k)x}\}_\delta$ is a (ρ^ℓ, δ)-Picard-Vessiot ring extension for $\rho^\ell(Y) = \mathrm{Diag}(e^{\lambda_1 h}, \dots, e^{\lambda_k h})_{[\ell]}Y$ over $\widetilde{C_h}(x)$ that is an integral domain. Again, replacing the ρ-equation by a ρ^ℓ-equation we may reduce to the case where the (ρ, δ)-ring $\widetilde{C_h}(x)\{e^{\lambda_1 x}, \dots, e^{\lambda_k x}, e^{-(\lambda_1+\cdots+\lambda_k)x}\}_\delta$ in an integral domain and its field of fractions has $\widetilde{C_h}$ as field of constants.

By [HS08, Proposition 3.1], there exists a linear differential operator with coefficients in $\widetilde{C_h}$ and $g \in \mathrm{Frac}(\widetilde{C_h}(x)\{e^{\lambda_1 x}, \dots, e^{\lambda_k x}\}_\delta) = \widetilde{C_h}(x)(e^{\lambda_1 x}, \dots, e^{\lambda_k x})$, such that $L(b) = \rho(g) - g$. Consider $W = \mathrm{Diag}(e^{\lambda_1 x}, \dots, e^{\lambda_k x})$. The field $\widetilde{C_h}$ is algebraically closed. By a straightforward adaptation of Lemma 5, $\mathcal{R}_1 = \widetilde{C_h}(x)[e^{\lambda_1 x}, \dots, e^{\lambda_k x}, e^{-(\lambda_1+\cdots+\lambda_k)x}]$, is a Picard-Vessiot ring extension for $\rho(Y) = \mathrm{Diag}(e^{\lambda_1 h}, \dots, e^{\lambda_k h})Y$ over $\widetilde{C_h}(x)$. Consider the ideal

$$J_1 = \{s \in \mathcal{R}_1 | sg \in \mathcal{R}_1\}.$$

Let us prove that it is a ρ-ideal. Let $s \in J_1$. We have $\rho(sg) \in \mathcal{R}_1$. But $\rho(sg) = \rho(s)(g + L(b))$ and $\rho(s)L(b) \in \mathcal{R}_1$, proving that $\rho(s)g \in \mathcal{R}_1$. Hence $\rho(s) \in J_1$ showing that J_1 is a ρ-ideal. Since $\mathcal{R}_1$ is a simple ρ-ring, and by construction $J_1 \neq (0)$, we deduce that $1 \in J_1$ and therefore

$$g \in \widetilde{C_h}(x)[e^{\lambda_1 x}, \dots, e^{\lambda_k x}, e^{-(\lambda_1+\cdots+\lambda_k)x}].$$

Let us write $b = \sum b_i e^{\lambda_i x}$, $g = \sum \tilde{g}_i e^{\lambda_i x}$, $b_i, \tilde{g}_i \in \widetilde{C_h}(x)$, $\lambda_i \in \mathbb{C}$. Then, using the $\widetilde{C_h}(x)$-linear independence of the $e^{\lambda x}$, $\lambda \in \mathbb{C}$, we find equations of the form $L_i(b_i) = e^{h\lambda_i}\rho(\tilde{g}_i) - \tilde{g}_i$, where L_i is a linear differential operator with coefficients in $\widetilde{C_h}$. By Remark 4, we may use [HS08, Lemma 6.4], to deduce the existence of $\hat{g}_i \in \widetilde{C_h}(x)$ such that $b_i = e^{h\lambda_i}\rho(\hat{g}_i) - \hat{g}_i$. Then, there exists

$g' \in \mathrm{Vect}_{\widetilde{C_h}(x)}[e^{\lambda_1 x}, \dots, e^{\lambda_k x}]$, such that $b = \rho(g') - g'$. Equating the constant coefficients yields the existence of $g'' \in R_{h,\exp}$ such that $b = \rho(g'') - g''$. Recall that $b = \rho(g_1) - g_1$. We then find that $g_1 - g''$ is ρ-invariant. Hence, $g_1 - g'' \in C_h$ and therefore $g_1 \in R_{h,\exp}$. This completes the proof.

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