



QUANTITATIVE INDEX, NOVIKOV CONJECTURE AND COARSE DECOMPOSABILITY

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QUANTITATIVE INDEX, NOVIKOV CONJECTURE AND COARSE DECOMPOSABILITY

HERVÉ OYONO-OYONO AND GUOLIANG YU

ABSTRACT. We define for families of finite metric spaces quantitative assembly map estimates that take into account propagation phenomena for pseudo-differential calculus. We relate these estimates to the Novikov conjecture and we show that they fit nicely under coarse decompositions. As an application, we provide a geometric proof of Novikov conjecture for groups with finite decomposition complexity.

Keywords: quantitative operator K -theory, spaces with finite asymptotic dimension, group with finite complexity decomposition, Novikov conjecture, Coarse Baum-Connes conjecture.

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INTRODUCTION

Since the pioneering work of Atiyah and Singer on the index theory of elliptic differential operators, operator algebras and their K -theory have become a central tool in the study of topological invariants in differential geometry. The most spectacular example is probably the Novikov conjecture on the homotopy invariance of higher signatures which can be stated as follows. Let Γ be a finitely generated group and let M be a compact oriented manifold with fundamental group Γ . Let B_Γ be the classifying space for Γ and let $f_\Gamma : M \rightarrow B_\Gamma$ be the classifying map. We denote by $\mathbb{L}(M) \in H^*(M, \mathbb{Q})$ the Pontryagin-Hirzebruch class of M . Recall that we have a natural isomorphism $H^*(\Gamma, \mathbb{Q}) \cong H^*(B_\Gamma, \mathbb{Q})$, so any x in $H^*(\Gamma, \mathbb{Q})$ can be pulled back to an element $f_\Gamma^*(x)$ in $H^*(M, \mathbb{Q})$. The rational numbers

$$\langle f_\Gamma^*(x), \mathbb{L}(M) \cap [M] \rangle$$

are called higher signatures of M (here $[M]$ stands for the fundamental class for M).

Conjecture (Novikov). *The higher signatures are oriented homotopy invariant.*

By introducing in his work on the Novikov conjecture the bivariant K -theory [11], Kasparov provided a far reaching generalization to the concept of index with a wide range of important applications to geometry. He proved the Novikov conjecture for discrete subgroups of Lie groups. His approach has been extended successfully to the case of groups acting on an euclidian building [12], of "bolic" groups (including Gromov hyperbolic groups) [13] and had culminated in the proof of Novikov conjecture for groups which are amenable at infinity [9, 25].

Following the development by Roe of a higher index theory for elliptic differential operators on non-compact complete riemannian manifolds [18], the 90's have seen the emergence of a more geometric approach. It has generated a great deal of interest as it has highlighted geometric obstructions for the Baum-Connes conjecture leading to the only known counterexamples [8]. It has also striking applications to secondary invariants [21, 22]. The foundational principle is the fact that

the higher indices have their receptacles in the K -theory of a canonical C^* -algebra, called the Roe algebra. The Roe algebra only depends on the large scale geometry of the underlying space. It can be viewed as an algebra of order zero pseudo-differential calculus and inherits from the geometry a propagation structure. For a finitely generated group (viewed as a metric space using any word length) the computability of the Roe algebra using higher indices implies (under some very mild condition) the Novikov conjecture. A prominent achievement of this is the proof of the Novikov conjecture for finitely generated groups with finite asymptotic dimension [24] by proving that the K -theory of a localized version of the Roe algebra can be computed using a "cut-and-pasting" techniques. This was done by introducing for these algebras a quantitative version of the K -theory which takes only into account propagation up to a given order. At each order there is a Mayer-Vietoris type decomposition which allowed to compute the quantitative K -theory and then, the K -theory is obtained by letting the order goes to infinity. Quantitative K -theory has been generalized in [15] to the setting of filtered C^* -algebras by the two authors. Thereafter, they introduced in [16, Section 3.1] geometric controlled assembly maps that compute locally the K -theory of the Roe algebra. For a finite metric space X and a C^* -algebra A , these maps have source the K -homology with coefficients in A of the Rips complex of X and range the quantitative K -theory of $A \otimes \mathcal{K}(\ell^2(X))$. Here $\mathcal{K}(\ell^2(X))$ stands for the algebra of matrices with entries in X and the filtration is provided by the metric. Quantitative assembly maps (QAM) estimates for metric spaces where stated for purpose of computing the K -theory of Roe algebras from the geometric controlled assembly maps by uniform approximations [16, Section 4.2]. It was then shown that if Γ is a finitely generated group such that the QAM-estimates are uniformly satisfied by the family of its finite subsets, then Γ satisfies the Novikov conjecture. It is worth pointing out that unlike the proof of the Novikov conjecture for groups amenable at infinity, this QAM-estimates only involve finite dimension analysis and finite linear algebra.

The aim of this paper is to establish a stability result for uniform QAM-estimates under geometric decomposition that we shall explain in more details now. Geometric decomposability in the context of metric spaces has been brought in by by Guentner, Tessera and Yu in their study of topological rigidity for manifolds [5]. For a given positive number R , a family of discrete metric spaces \mathcal{X} is said to be R -decomposable with respect to another family \mathcal{Y} if for any metric space X in \mathcal{X} , there exists a decomposition $X = X^{(1)} \cup X^{(2)}$ such that $X^{(i)}$ is for $i = 1, 2$ the R -disjoint union (i.e at mutual distance greater than R) of metric spaces in \mathcal{Y} . We are now in position to state the main result of this

paper (Theorem 2.13). Let \mathcal{X} be a family of finite metric spaces. Assume that for any $R > 0$, there exists a family of finite metric spaces \mathcal{Y} such that

- \mathcal{X} is R -decomposable with respect to \mathcal{Y} ;
- the family $\tilde{\mathcal{Y}}$ of all subsets of all metric spaces in \mathcal{Y} satisfies uniformly the QAM-estimates.

Then the family \mathcal{X} satisfies (under some uniform boundedness condition) uniformly the QAM-estimates. A special case of interest is the class of families with finite decomposition complexity [5]. A class \mathcal{C} of families of proper discrete metric spaces is said to be closed under coarse decomposition if any family \mathcal{X} of discrete proper metric spaces which is R -decomposable with respect to a family \mathcal{Y} in \mathcal{C} for any positive number R lies indeed in \mathcal{C} . Let us define the class \mathcal{C}_{fdc} as the smallest class of families that contains uniformly bounded families of discrete metric spaces and which is closed under coarse decomposability. As uniformly bounded families of discrete metric spaces satisfy obviously uniformly the QAM-estimates, we obtain as a consequence of our main result that any family \mathcal{X} in \mathcal{C}_{fdc} with uniformly bounded geometry satisfies uniformly the QAM-estimates (Theorem 2.20). If X is a discrete proper metric space, we say that X has finite decomposition complexity if the single family $\{X\}$ is in \mathcal{C}_{fdc} [5]. Examples of such metric spaces are provided by metric spaces with finite asymptotic dimension. As a consequence of Theorem 2.20, we get that if Γ is a finitely generated group with finite decomposition complexity which admits a classifying space with finite homotopy type, then Γ satisfies the Novikov conjecture. This result was already known as group with finite decomposition complexity are amenable [6] but as already emphasized, our proof does not involve infinite dimension analysis.

Let us now explain the general idea of the proof of the main result of this paper. In [17] was stated a controlled Mayer-Vietoris exact sequence in quantitative K -theory. It was indeed motivated by K -theory computability of Roe type algebras under the geometric decompositions described above and thus result in six-term controlled exact sequences on the right hand-side of the quantitative assembly maps. These decompositions also give rise to Mayer-Vietoris exact sequences in K -homology on the left-hand side of the quantitative assembly maps. Our proof then amounts to prove the compatibility of these two sequences under the quantitative assembly maps. The major obstacle is that the two boundary maps do not have the same nature. In K -homology, the boundary maps is constructed using homotopy theory whereas in a quantitative K -theory, we use an approximating lifting property for almost unitaries. The first step to circumvent this obstacle is to show that as it is the case for usual assembly maps, local quantitative coarse assembly map factorize through quantitative index maps. These quantitative index maps are indeed defined for any compact metric space Z

and is valued for a fixed non degenerate standard Z -module \mathcal{H}_Z in the quantitative K -theory of the algebra $\mathcal{K}(\mathcal{H}_Z)$ of compact operators on \mathcal{H}_Z (or in $A \otimes \mathcal{K}(\mathcal{H}_Z)$ for the version with coefficients). The second step is to prove that for finite simplicial complexes and for small propagation (depending only on the dimension), the quantitative index maps are (up to some rescaling) isomorphisms. This is done once again by using controlled Mayer-Vietoris exact sequence in quantitative K -theory. Checking the compatibility of the boundary maps is a quite tedious chore but it is worth noticing that it can be straightforwardly generalized to the equivariant case (in particular with respect to groupoid actions). The proof of our main result then amounts to show that for any finite metric space X in \mathcal{X} , then the map induced in quantitative K -theory by some canonical morphism $\mathcal{K}(\mathcal{H}_{P_d(X)}) \rightarrow \mathcal{K}(\ell^2(X))$ is an isomorphism up to uniformly controlled rescaling if the same holds for any metric spaces in $\hat{\mathcal{Y}}$. Geometric decompositions give also rise to controlled Mayer-Vietoris exact sequence to compute the quantitative $\mathcal{K}(\mathcal{H}_{P_d(X)})$. The two controlled exact sequences being now intertwined by natural morphisms, this is achieved by using a controlled version of the five lemma.

By analogy with metric spaces, the notion of geometric decomposition can be also defined for groupoids [7, 14]. An interesting perspective would be to extend the main result of the paper to prove the permanence property of the Baum-Connes conjecture for étale groupoids under geometric decomposition [14, Theorem 6.8] without the assumption of the existence of a γ -element in the sense of [20, Proposition 5.20 and Remark 5.21]. To reach this goal, we need to develop the analogous of the theory of standard module in the framework of cocompact proper action of groupoids (see [1] for the case of ample groupoids).

Outline of the paper.

In Section 1, we review from [15, 16] the main features of quantitative K -theory for filtered C^* -algebras. We draw attention of the reader on the fact that the definition of quantitative K -theory that we give in this paper is slightly different from the original one given in [11][Section 3.1]. We have changed the range of the control ε and allowed larger homotopies in the equivalence relations between almost projections and almost unitaries. The purpose for this is to define later on in this section the rescaling operator which allows to reduce the control ε in quantitative K -theory.

The main theorem is stated in Section 2. We first recall from [16] the construction of the geometric controlled assembly map which can be viewed as a geometric version of the controlled Baum-Connes assembly maps (see [15]). Next we give the formulation of the quantitative assembly maps estimates for a proper discrete metric space. We recall the link with the coarse Baum-Connes conjecture and with the Novikov conjecture. We then state the main result of this paper. We

apply this result to metric spaces with finite decomposition complexity introduced in [6] and we obtain as a consequence the Novikov conjecture for finitely generated groups with finite decomposition complexity (under the standard assumption on classifying spaces).

Section 3 is devoted to the definition and to the study of the quantitative index map. After some reminders about non-degenerated standard modules and propagation for their operators, we introduce for a compact metric space X and a non-degenerated standard X -modules \mathcal{H}_X the fundamental class in quantitative K -theory. This is a family of rank 1 projections in $C(X, \mathcal{K}(\mathcal{H}_X))$ with arbitrary small propagation and which are moreover compatible with respect to the structure maps in quantitative K -theory. This leads to the definition of quantitative indices. We then study some naturality properties for this map and we state the fundamental theorem for quantitative indices : if X is a finite simplicial complex, then for small propagation depending only on the dimension, quantitative indices are (up to some controlled rescaling) isomorphisms. This theorem will be proved in Section 4. We end the section by proving that the geometric quantitative assembly maps factorizes through quantitative index maps.

In Section 4, we introduce the central tool of this paper : the controlled six terms exact sequence in quantitative K -theory associated to a Mayer-Vietoris pair decomposition for a filtered C^* -algebra [17]. We show that a classical Mayer-Vietoris decomposition for a compact metric space gives rise to a Mayer-Vietoris pair for the right hand side of the quantitative index map. We prove by an arduous computation that the Mayer-Vietoris boundary maps in K -homology and in quantitative K -theory are intertwined by the quantitative index maps. As a consequence, we prove by using a controlled version of the five Lemma that for small propagation, the quantitative index is an isomorphism up to some rescaling.

In Section 5, we collect results from Section 3 and 4 to show that the proof of the main theorem reduces to replace in the QAM-estimates the geometric assembly maps by the morphisms induced in quantitative K -theory by some canonical map $\mathcal{K}(\mathcal{H}_{P_d(X)}) \rightarrow \mathcal{K}(\ell^2(X))$. Noticing that geometric decompositions give rise to compatible Mayer-Vietoris pair decompositions both for $\mathcal{K}(\mathcal{H}_{P_d(X)})$ and $\mathcal{K}(\ell^2(X))$, this is done by using once again a five Lemma type argument for the associated controlled six terms exact sequences in quantitative K -theory.

1. OVERVIEW ON QUANTITATIVE K -THEORY

In this section, we recall the basic concepts of quantitative K -theory for filtered C^* -algebras and collect the main results of [15] concerning quantitative K -theory that we shall use throughout this paper. The structure of filtered C^* -algebras allows us to talk about the propagation scale of elements in the C^* -algebras. Roughly speaking, quantitative

K -theory is the abelian groups of K -theory elements at each given scale.

Definition 1.1. *A filtered C^* -algebra A is a C^* -algebra equipped with a family $(A_r)_{r>0}$ of closed linear subspaces indexed by positive numbers such that:*

- $A_r \subseteq A_{r'}$ if $r \leq r'$;
- A_r is stable under involution;
- $A_r \cdot A_{r'} \subseteq A_{r+r'}$;
- the subalgebra $\bigcup_{r>0} A_r$ is dense in A .

If A is unital, we also require that the identity 1 is an element of A_r for every positive number r . The elements of A_r are said to have **propagation** r .

Let A and A' be respectively C^* -algebras filtered by $(A_r)_{r>0}$ and $(A'_r)_{r>0}$. A homomorphism of C^* -algebras $\phi : A \rightarrow A'$ is a **filtered homomorphism** (or a **homomorphism of filtered C^* -algebras**) if $\phi(A_r) \subseteq A'_r$ for any positive number r .

If A is not unital, let us denote by A^+ its unitarization, i.e.,

$$A^+ = \{(x, \lambda); x \in A, \lambda \in \mathbb{C}\}$$

with the product

$$(x, \lambda)(x', \lambda') = (xx' + \lambda x' + \lambda' x, \lambda \lambda')$$

for all (x, λ) and (x', λ') in A^+ . Then A^+ is filtered with

$$A_r^+ = \{(x, \lambda); x \in A_r, \lambda \in \mathbb{C}\}.$$

We also define $\rho_A : A^+ \rightarrow \mathbb{C}; (x, \lambda) \mapsto \lambda$.

1.1. Definition of quantitative K -theory. Let A be a unital filtered C^* -algebra. For any positive numbers r and ε , we call

- an element u in A an ε - r -unitary if u belongs to A_r , $\|u^* \cdot u - 1\| < \varepsilon$ and $\|u \cdot u^* - 1\| < \varepsilon$. The set of ε - r -unitaries on A will be denoted by $U^{\varepsilon, r}(A)$.
- an element p in A an ε - r -projection if p belongs to A_r , $p = p^*$ and $\|p^2 - p\| < \varepsilon$. The set of ε - r -projections on A will be denoted by $P^{\varepsilon, r}(A)$.

Notice that an ε - r -unitary is invertible, and that if p is an ε - r -projection in A , then it has a spectral gap around $1/2$ and then gives rise by functional calculus to a projection $\kappa_0(p)$ in A such that $\|p - \kappa_0(p)\| < 2\varepsilon$.

Recall from [16, Lemma 1.7] the following result that will be used quite extensively throughout the paper.

Lemma 1.2. *Let A be a C^* -algebra filtered by $(A_r)_{r>0}$.*

- (i) If p is an ε - r -projection in A and q is a self-adjoint element of A_r such that $\|p - q\| < \frac{\varepsilon - \|p^2 - p\|}{4}$, then q is an ε - r -projection. In particular, if p is an ε - r -projection in A and if q is a self-adjoint element in A_r such that $\|p - q\| < \varepsilon$, then q is a 5ε - r -projection in A and p and q are connected by a homotopy of 5ε - r -projections.
- (ii) If A is unital, u is an ε - r -unitary and v is an element of A_r such that $\|u - v\| < \frac{\varepsilon - \|u^*u - 1\|}{3}$, then v is an ε - r -unitary. In particular, if u is an ε - r -unitary and v is an element of A_r such that $\|u - v\| < \varepsilon$, then v is a 4ε - r -unitary in A and u and v are connected by a homotopy of 4ε - r -unitaries.
- (iii) If p is a projection in A and q is a self-adjoint element of A_r such that $\|p - q\| < \frac{\varepsilon}{4}$, then q is an ε - r -projection.
- (iv) If A is unital, u is a unitary in A and v is an element of A_r such that $\|u - v\| < \frac{\varepsilon}{3}$, then v is an ε - r -unitary.

We get as a corollary:

Corollary 1.3. *Let u be an ε - r -unitary in a unital filtered C^* -algebra A , then $\text{diag}(u, u^*)$ and I_2 are homotopic as 3ε - $2r$ -unitaries in $M_2(A)$.*

For any integer n , we set $U_n^{\varepsilon, r}(A) = U^{\varepsilon, r}(M_n(A))$ and $P_n^{\varepsilon, r}(A) = P^{\varepsilon, r}(M_n(A))$. For any unital filtered C^* -algebra A , any positive numbers ε and r and any positive integer n , we consider the inclusions

$$P_n^{\varepsilon, r}(A) \hookrightarrow P_{n+1}^{\varepsilon, r}(A); p \mapsto \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$U_n^{\varepsilon, r}(A) \hookrightarrow U_{n+1}^{\varepsilon, r}(A); u \mapsto \begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}.$$

This allows us to define

$$U_\infty^{\varepsilon, r}(A) = \bigcup_{n \in \mathbb{N}} U_n^{\varepsilon, r}(A)$$

and

$$P_\infty^{\varepsilon, r}(A) = \bigcup_{n \in \mathbb{N}} P_n^{\varepsilon, r}(A).$$

For a unital filtered C^* -algebra A , $\varepsilon \in (0, 1/100)$ and $r > 0$, we define the following equivalence relations on $P_\infty^{\varepsilon, r}(A) \times \mathbb{N}$ and on $U_\infty^{\varepsilon, r}(A)$:

- if p and q are elements of $P_\infty^{\varepsilon, r}(A)$, l and l' are positive integers, $(p, l) \sim (q, l')$ if there exists a positive integer k and an element h of $P_\infty^{25\varepsilon, r}(A[0, 1])$ such that $h(0) = \text{diag}(p, I_{k+l'})$ and $h(1) = \text{diag}(q, I_{k+l})$.
- if u and v are elements of $U_\infty^{\varepsilon, r}(A)$, $u \sim v$ if there exists an element h of $U_\infty^{25\varepsilon, 2r}(A[0, 1])$ such that $h(0) = u$ and $h(1) = v$.

If p is an element of $P_\infty^{\varepsilon,r}(A)$ and l is an integer, we denote by $[p, l]_{\varepsilon,r}$ the equivalence class of (p, l) modulo \sim and if u is an element of $U_\infty^{\varepsilon,r}(A)$ we denote by $[u]_{\varepsilon,r}$ its equivalence class modulo \sim .

Definition 1.4. *Let r and ε be positive numbers with $\varepsilon < 1/100$. We define:*

- (i) $K_0^{\varepsilon,r}(A) = P_\infty^{\varepsilon,r}(A) \times \mathbb{N} / \sim$ for A unital and
 $K_0^{\varepsilon,r}(A) = \{[p, l]_{\varepsilon,r} \in K_0(A^+) \text{ such that } \text{rank } \kappa_0(\rho_A(p)) = l\}$
for A non unital ($\kappa_0(\rho_A(p))$ being the spectral projection associated to $\rho_A(p)$);
- (ii) $K_1^{\varepsilon,r}(A) = U_\infty^{\varepsilon,r}(A^+) / \sim$, with $A = A^+$ if A is already unital.

Then $K_0^{\varepsilon,r}(A)$ turns to be an abelian group [15, Lemma 1.15] where

$$[p, l]_{\varepsilon,r} + [p', l']_{\varepsilon,r} = [\text{diag}(p, p'), l + l']_{\varepsilon,r}$$

for any $[p, l]_{\varepsilon,r}$ and $[p', l']_{\varepsilon,r}$ in $K_0^{\varepsilon,r}(A)$. According to [15, Lemma 1.15], $K_1^{\varepsilon,r}(A)$ is equipped with a structure of abelian group such that

$$[u]_{\varepsilon,r} + [u']_{\varepsilon,r} = [\text{diag}(u, v)]_{\varepsilon,r},$$

for any $[u]_{\varepsilon,r}$ and $[u']_{\varepsilon,r}$ in $K_1^{\varepsilon,r}(A)$.

We draw attention of the readers on the fact that this definition of controlled K -theory differs slightly from the original one given in [15][Section 3.1]. We have increased control of homotopies for purpose of rescaling (see Section 1.2). As quantitative objects (see Section 1.3), the two objects are related by a controlled isomorphism. Furthermore, controlled morphisms and isomorphisms and controlled exact sequences for both objects are in natural correspondance with each other.

Recall from [15, Corollaries 1.19 and 1.21] that for any positive numbers r and ε with $\varepsilon < 1/100$, then

$$K_0^{\varepsilon,r}(\mathbb{C}) \rightarrow \mathbb{Z}; [p, l]_{\varepsilon,r} \mapsto \text{rank } \kappa_0(p) - l$$

is an isomorphism and $K_1^{\varepsilon,r}(\mathbb{C}) = \{0\}$.

We have for any filtered C^* -algebra A and any positive numbers r, r' , ε and ε' with $\varepsilon \leq \varepsilon' < 1/100$ and $r \leq r'$ natural group homomorphisms

- $\iota_0^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \rightarrow K_0(A); [p, l]_{\varepsilon,r} \mapsto [\kappa_0(p)] - [I_l]$ (where $\kappa_0(p)$ is the spectral projection associated to p);
- $\iota_1^{\varepsilon,r} : K_1^{\varepsilon,r}(A) \rightarrow K_1(A); [u]_{\varepsilon,r} \mapsto [u]$;
- $\iota_*^{\varepsilon,r} = \iota_0^{\varepsilon,r} \oplus \iota_1^{\varepsilon,r}$;
- $\iota_0^{\varepsilon,\varepsilon',r,r'} : K_0^{\varepsilon,r}(A) \rightarrow K_0^{\varepsilon',r'}(A); [p, l]_{\varepsilon,r} \mapsto [p, l]_{\varepsilon',r'}$;
- $\iota_1^{\varepsilon,\varepsilon',r,r'} : K_1^{\varepsilon,r}(A) \rightarrow K_1^{\varepsilon',r'}(A); [u]_{\varepsilon,r} \mapsto [u]_{\varepsilon',r'}$;
- $\iota_*^{\varepsilon,\varepsilon',r,r'} = \iota_0^{\varepsilon,\varepsilon',r,r'} \oplus \iota_1^{\varepsilon,\varepsilon',r,r'}$

If some of the indices r, r' or $\varepsilon, \varepsilon'$ are equal, we shall not repeat it in $\iota_*^{\varepsilon,\varepsilon',r,r'}$. Furthermore, when the range (resp. source, source and range) is obvious, we shall write $\iota_*^{\varepsilon,r,-}$ (resp. $\iota_*^{-,\varepsilon',r'}$, $\iota_*^{-,-}$) instead of $\iota_*^{\varepsilon,\varepsilon',r,r'}$.

Notice that with this definition, the family of morphisms

$$(\iota_*^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \rightarrow K_*(A))_{\varepsilon \in (0,1/100), r > 0}$$

gives rise to an isomorphism

$$\lim_{r > 0} K_*^{\varepsilon,r}(A) \xrightarrow{\simeq} K_*(A), \quad (1)$$

where the inductive limit on the left hand side is taken with respect to the inductive system

$$\iota_*^{\varepsilon,r,r'} : K_*^{\varepsilon,r}(A) \rightarrow K_*^{\varepsilon,r'}(A)$$

for $r' \geq r > 0$.

If $\phi : A \rightarrow B$ is a homomorphism of filtered C^* -algebras, then since ϕ preserves ε - r -projections and ε - r -unitaries, it obviously induces for any positive number r and any $\varepsilon \in (0, 1/100)$ a group homomorphism

$$\phi_*^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \longrightarrow K_*^{\varepsilon,r}(B).$$

Moreover the quantitative K -theory is homotopy invariant with respect to homotopies that preserves propagation [15, Lemma 1.26]. There is also a quantitative version of Morita equivalence [15, Proposition 1.28].

Proposition 1.5. *If A is a filtered algebra and \mathcal{H} is a separable Hilbert space, then the homomorphism*

$$A \rightarrow \mathcal{K}(\mathcal{H}) \otimes A; a \mapsto \begin{pmatrix} a & & \\ & 0 & \\ & & \ddots \end{pmatrix}$$

induces a $(\mathbb{Z}_2$ -graded) group isomorphism (the Morita equivalence)

$$\mathcal{M}_A^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \rightarrow K_*^{\varepsilon,r}(\mathcal{K}(\mathcal{H}) \otimes A)$$

for any positive number r and any $\varepsilon \in (0, 1/100)$.

1.2. Rescaling the control. In this subsection, we describe a rescaling process that allows to fix the control in quantitative K -theory.

Recall that if q is an ε - r projection, then $\kappa_0(q)$ stands for the spectral projection corresponding to q' . If u is an ε - r unitary, we set $\kappa_1(u) = (u^*u)^{-1/2}u$ (then $\kappa_1(u)$ is a unitary).

Lemma 1.6. *There exists a non increasing function*

$$(0, 1/8) \rightarrow [1, +\infty); \varepsilon \mapsto l_\varepsilon$$

such that for any unital filtered C^ -algebra A , any ε in $(0, 1/8)$ and any $r > 0$, then the following holds*

- (i) *for any ε - r -projection q in A , there exists an $\varepsilon/2$ - $l_\varepsilon r$ -projection q' such that $\|\kappa_0(q) - q'\| < \varepsilon/8$.*
- (ii) *for any ε -unitary u in A , there exists an $\varepsilon/2$ - $l_\varepsilon r$ -unitary u' such that $\|\kappa_1(u) - u'\| < \varepsilon/6$.*

Moreover, if J is an ideal in A and if $q - 1$ (resp. $u - 1$) lies in J , then we can choose q' (resp. u') such that $q' - 1$ (resp. $u' - 1$) lies also in J .

Proof. Let us first prove the even case. Let us fix α in $(0, 1 - 1/\sqrt{2})$. Then the spectrum of q is included in $[-\alpha, \alpha] \cup [1 - \alpha, 1 + \alpha]$ for any ε - r -projection q with ε in $(0, 1/8)$. Let us fix $\kappa_0 : \mathbb{R} \rightarrow \mathbb{R}$ a continuous function such $\kappa_0(t) = 0$ for t in $[-\alpha, \alpha]$ and $\kappa_0(t) = 1$ for t in $[1 - \alpha, 1 + \alpha]$ and let $(Q_N)_{N \in \mathbb{N}}$ be a polynomial sequence such that $(Q_N)_{N \in \mathbb{N}}$ converge uniformly to κ_0 on $[-\alpha, \alpha] \cup [1 - \alpha, 1 + \alpha]$. For ε in $(0, 1/8)$, let N_ε be the smallest integer such that $\|Q_{N_\varepsilon}(t) - \kappa_0(t)\| < \varepsilon/8$ for all t in $[-\alpha, \alpha] \cup [1 - \alpha, 1 + \alpha]$. Set then $l_\varepsilon = \max\{d^o Q_k; k = 0, \dots, N_\varepsilon\}$. Then $q' = Q_{N_\varepsilon}(q)$ belongs to $A_{l_\varepsilon r}$ and $\|\kappa_0(q) - q'\| < \varepsilon/8$. In particular, according to the third point of Lemma 1.2, we see that q' is an $\varepsilon/2 - l_\varepsilon r$ -projection.

In the same way for the odd case, using the power serie of $t \mapsto (1 + t)^{-1/2}$, we see that there exists:

- a non increasing function $(0, 1/4) \rightarrow \mathbb{N}; \varepsilon \mapsto l_\varepsilon$
- for any ε in $(0, 1/4)$ a polynomial function Q_ε

such that

- The degree of Q_ε is l_ε ;
- $Q_\varepsilon(1) = 1$
- $|Q_\varepsilon(t) - t^{-1/2}| < \varepsilon/8$ for all t in $[3/4, 5/4]$.

If we set $u' = Q_\varepsilon(u^*u)u$, then we have that $\|\kappa_1(u) - u'\| < \varepsilon/6$ and in view of the fourth point of Lemma 1.2, we see that u' is an $\varepsilon/2 - (2l_\varepsilon + 1)r$ -unitary. \square

If q is an ε - r -projection in A , with ε in $(0, 1/200)$, we deduce from the first point of Lemma 1.2 that two elements that satisfy the condition of the first item of the lemma are homotopic as $\frac{5}{4}\varepsilon$ - r -projections and hence there exists a well defined homomorphism

$$L_{A,0}^{\varepsilon,r} : K_0^{\varepsilon,r}(A) \longrightarrow K_0^{\varepsilon/2, l_\varepsilon r}(A),$$

natural with respect to homomorphisms of filtered C^* -algebras, such that if A is unital, then $L_{A,0}^{\varepsilon,r}([q, k]_{\varepsilon,r}) = [q', k]_{\varepsilon/2, l_\varepsilon r}$ for any ε - r -projection q in some $M_n(A)$, any integer k and any $\varepsilon/2 - l_\varepsilon r$ -projection q' in $M_n(A)$ such that $\|\kappa_0(q) - q'\| < \varepsilon/8$.

In the same way, there exists a well defined natural homomorphism

$$L_{A,1}^{\varepsilon,r} : K_1^{\varepsilon,r}(A) \longrightarrow K_1^{\varepsilon/2, l_\varepsilon r}(A)$$

such that if A is unital, then $L_{A,1}^{\varepsilon,r}([u]_{\varepsilon,r}) = [u']_{\varepsilon/2, l_\varepsilon r}$ for any ε - r -unitary u in some $M_n(A)$ and any $\varepsilon/2 - l_\varepsilon r$ -unitary u' in $M_n(A)$ satisfying $\|\kappa_1(u) - u'\| < \varepsilon/8$.

For any ε in $(0, 1/200)$ and $r > 0$, let us set $l_{rs,\varepsilon} = l_\varepsilon$ and

$$L_{A,*}^{\varepsilon,r} = L_{A,0}^{\varepsilon,r} \oplus L_{A,1}^{\varepsilon,r} : K_*^{\varepsilon,r}(A) \longrightarrow K_*^{\varepsilon/2, l_{rs,\varepsilon} r}(A).$$

We clearly have that

$$L_{A,*}^{\varepsilon',r'} \circ \iota_*^{\varepsilon,r,-} = \iota_*^{-,\varepsilon'/2,l_{rs,\varepsilon'}r'} \circ L_{A,*}^{\varepsilon,r}$$

for any $\varepsilon, \varepsilon', r$ and r' such that $0 < \varepsilon \leq \varepsilon' < 1/200$, $0 < r \leq r'$ and $l_{rs,\varepsilon}r \leq l_{rs,\varepsilon'}r'$.

Lemma 1.7. *For any unital filtered C^* -algebra A , any ε in $(0, 1/200)$ and any $r > 0$, then*

- (i) $\iota_*^{\varepsilon/2,\varepsilon,l_\varepsilon r} \circ L_{A,*}^{\varepsilon,r} = \iota_*^{\varepsilon,r,l_\varepsilon r};$
- (ii) $L_{A,*}^{\varepsilon,r} \circ \iota_*^{\varepsilon/2,\varepsilon,r} = \iota_*^{\varepsilon/2,l_\varepsilon r}.$

Proof. We can assume without loss of generality that A is unital. Let us prove the first point in the even case. Let q be an ε - r -projection in some $M_n(A)$ and let q' be an $\varepsilon/2$ - $l_\varepsilon r$ -projection in $M_n(A)$ such that $\|\kappa_0(q) - q'\| < \varepsilon/8$ as in Lemma 1.6. Then

$$\begin{aligned} \|q - q'\| &\leq \|q - \kappa_0(q)\| + \|\kappa_0(q) - q'\| \\ &< \frac{17\varepsilon}{8}. \end{aligned}$$

In view of the first point of Lemma 1.2, we obtain that q and q' are homotopic as 25ε - $l_\varepsilon r$ -projections and we deduce the result for the first point in the even case. The odd case is quite similar.

For the second point, let q be an $\varepsilon/2$ - r -projection in some $M_n(A)$ and let us consider q as a ε - r -projection. Pick then an $\varepsilon/2$ - $l_\varepsilon r$ -projection q' in $M_n(A)$ such that $\|\kappa_0(q) - q'\| < \varepsilon/8$. Then

$$\begin{aligned} \|q - q'\| &\leq \|q - \kappa_0(q)\| + \|\kappa_0(q) - q'\| \\ &< \frac{9\varepsilon}{8} \end{aligned}$$

and hence q and q' are homotopic as $\frac{25\varepsilon}{2}$ - $l_\varepsilon r$ -projections and we deduce the result of the second point in the even case, the odd case being similar. \square

1.3. Quantitative objects. In order to study the properties of quantitative K -theory, we introduce the following concept of quantitative object.

Definition 1.8. *A quantitative object is a family $\mathcal{O} = (O^{\varepsilon,r})_{0 < \varepsilon < 1/100, r > 0}$ of abelian groups, together with a family of group homomorphisms*

$$\iota_{\mathcal{O}}^{\varepsilon,\varepsilon',r,r'} : O^{\varepsilon,r} \rightarrow O^{\varepsilon',r'}$$

for $0 < \varepsilon \leq \varepsilon' < 1/100$ and $0 < r \leq r'$ such that

- $\iota_{\mathcal{O}}^{\varepsilon,\varepsilon,r,r} = \text{Id}_{O^{\varepsilon,r}}$ for any $0 < \varepsilon < 1/100$ and $r > 0$;
- $\iota_{\mathcal{O}}^{\varepsilon',\varepsilon'',r',r''} \circ \iota_{\mathcal{O}}^{\varepsilon,\varepsilon',r,r'} = \iota_{\mathcal{O}}^{\varepsilon,\varepsilon'',r,r''}$ for any $0 < \varepsilon \leq \varepsilon' \leq \varepsilon'' < 1/100$ and $0 < r \leq r' \leq r''$;

As for structure maps attached to quantitative K -theory, when some of the indices r, r' or $\varepsilon, \varepsilon'$ are equal, we shall not repeat it in $\iota_{\mathcal{O}}^{\varepsilon,\varepsilon',r,r'}$.

Example 1.9. Of course, our prominent example will be quantitative K -theory $\mathcal{K}_*(A) = (K_*^{\varepsilon,r}(A))_{0 < \varepsilon < 1/100, r > 0}$ of a filtered C^* -algebras A with structure maps $\iota_*^{\varepsilon,\varepsilon',r,r'} : K_*^{\varepsilon,r}(A) \longrightarrow K_*^{\varepsilon',r'}(A)$.

In order to define the relevant notion of morphisms between quantitative objects, we need to introduce control pairs.

Definition 1.10. A control pair is a pair (λ, h) , where

- $\lambda \geq 1$;
- $h : (0, \frac{1}{100\lambda}) \rightarrow [1, +\infty)$; $\varepsilon \mapsto h_\varepsilon$ is a non-increasing map.

The set of control pairs is equipped with a partial order: $(\lambda, h) \leq (\lambda', h')$ if $\lambda \leq \lambda'$ and $h_\varepsilon \leq h'_\varepsilon$ for all $\varepsilon \in (0, \frac{1}{100\lambda})$. If (λ, h) and (λ', h') are two control pairs, define

$$h * h' : \left(0, \frac{1}{100\lambda\lambda'}\right) \rightarrow (0, +\infty); \varepsilon \mapsto h_{\lambda'\varepsilon} h'_\varepsilon.$$

Then $(\lambda\lambda', h * h')$ is again a control pair.

Definition 1.11. Let (λ, h) be a control pair and let $\mathcal{O} = (O^{\varepsilon,r})_{0 < \varepsilon < 1/100, r > 0}$ and $\mathcal{O}' = (O'^{\varepsilon,r})_{0 < \varepsilon < 1/100, r > 0}$ be quantitative objects. A (λ, h) -controlled morphism

$$\mathcal{F} : \mathcal{O} \rightarrow \mathcal{O}'$$

is a family $\mathcal{F} = (F^{\varepsilon,r})_{0 < \varepsilon < \frac{1}{100\lambda}, r > 0}$ of groups homomorphisms

$$F^{\varepsilon,r} : \mathcal{O}^{\varepsilon,r} \rightarrow \mathcal{O}'^{\lambda\varepsilon, h_\varepsilon r}$$

such that for any positive numbers $\varepsilon, \varepsilon', r$ and r' with $0 < \varepsilon \leq \varepsilon' < \frac{1}{100\lambda}$, $r \leq r'$ and $h_\varepsilon r \leq h_{\varepsilon'} r'$, we have

$$F^{\varepsilon',r'} \circ \iota_{\mathcal{O}}^{\varepsilon,\varepsilon',r,r'} = \iota_{\mathcal{O}'}^{\lambda\varepsilon, \lambda\varepsilon', h_\varepsilon r, h_{\varepsilon'} r'} \circ F^{\varepsilon,r}.$$

When it is not necessary to specify the control pair, we just say that \mathcal{F} is a controlled morphism. If $\mathcal{O} = (O^{\varepsilon,r})_{0 < \varepsilon < 1/100, r > 0}$ is a quantitative object, let us define the identity $(1, 1)$ -controlled morphism $\mathcal{Id}_{\mathcal{O}} = (Id_{O^{\varepsilon,r}})_{0 < \varepsilon < 1/100, r > 0} : \mathcal{O} \rightarrow \mathcal{O}$. Observe that if A and B are filtered C^* -algebras and if $\mathcal{F} : \mathcal{K}_*(A) \rightarrow \mathcal{K}_*(B)$ is a (λ, h) -controlled morphism, then \mathcal{F} induces a morphism $F : K_*(A) \rightarrow K_*(B)$ uniquely defined by $\iota_*^{\varepsilon,r} \circ F^{\varepsilon,r} = F \circ \iota_*^{\varepsilon,r}$.

In some situation, as for instance controlled boundary maps of controlled Mayer-Vietoris pair (see Section 4.2), we deal with family $F^{\varepsilon,r} : \mathcal{O}^{\varepsilon,r} \rightarrow \mathcal{O}'^{\lambda\varepsilon, h_\varepsilon r}$ of group morphisms defined indeed only up to a certain order.

Definition 1.12. Let (λ, h) be a control pair, let $\mathcal{O} = (O^{\varepsilon,s})_{0 < \varepsilon < 1/100, s > 0}$ and $\mathcal{O}' = (O'^{\varepsilon,s})_{0 < \varepsilon < 1/100, s > 0}$ be quantitative objects and let r be a positive number. A (λ, h) -controlled morphism of order r

$$\mathcal{F} : \mathcal{O} \rightarrow \mathcal{O}'$$

is a family $\mathcal{F} = (F^{\varepsilon,s})_{0 < \varepsilon < \frac{1}{100\lambda}, 0 < s < \frac{r}{h_\varepsilon}}$ of groups homomorphisms

$$F^{\varepsilon,s} : \mathcal{O}^{\varepsilon,s} \rightarrow \mathcal{O}'^{\lambda\varepsilon, h_\varepsilon s}$$

such that for any positive numbers $\varepsilon, \varepsilon', s$ and s' with $0 < \varepsilon \leq \varepsilon' < \frac{1}{100\lambda}$, $s \leq s'$ and $h_\varepsilon s \leq h_{\varepsilon'} s' < r$, we have

$$F^{\varepsilon',s'} \circ \iota_{\mathcal{O}}^{\varepsilon,\varepsilon',s,s'} = \iota_{\mathcal{O}'}^{\lambda\varepsilon,\lambda\varepsilon',h_\varepsilon s,h_{\varepsilon'} s'} \circ F^{\varepsilon,s}.$$

Let $\mathcal{O} = (O^{\varepsilon,r})_{0 < \varepsilon < 1/100, r > 0}$, $\mathcal{O}' = (O'^{\varepsilon,r})_{0 < \varepsilon < 1/100, r > 0}$ and $\mathcal{O}'' = (O''^{\varepsilon,r})_{0 < \varepsilon < 1/100, r > 0}$ be quantitative objects, let

$$\mathcal{F} = (F^{\varepsilon,r})_{0 < \varepsilon < \frac{1}{100\alpha}, r > 0} : \mathcal{O} \rightarrow \mathcal{O}'$$

be a (α, k) -controlled morphism, and let

$$\mathcal{G} = (G^{\varepsilon,r})_{0 < \varepsilon < \frac{1}{100\alpha'}, r > 0} : \mathcal{O}' \rightarrow \mathcal{O}''$$

be a (α', k') -controlled morphism. Then $\mathcal{G} \circ \mathcal{F} : \mathcal{O} \rightarrow \mathcal{O}''$ is the $(\alpha'\alpha, k' * k)$ -controlled morphism defined by the family

$$(G^{\alpha\varepsilon, k_\varepsilon r} \circ F^{\varepsilon,r} : O^{\varepsilon,r} \rightarrow O''^{\alpha'\alpha\varepsilon, k_\varepsilon k'_{\alpha\varepsilon} r})_{0 < \varepsilon < \frac{1}{100\alpha\alpha'}, r > 0}. \quad (2)$$

Notice that if $\mathcal{F} : \mathcal{O} \rightarrow \mathcal{O}'$ and $\mathcal{G} : \mathcal{O}' \rightarrow \mathcal{O}''$ are respectively a (α, k) -controlled morphism and (α', k') -controlled morphism of order r , then equation (2) defines a $(\alpha'\alpha, k' * k)$ -controlled morphism $\mathcal{G} \circ \mathcal{F} : \mathcal{O} \rightarrow \mathcal{O}''$ of order r .

Notation 1.13. Let $\mathcal{O} = (O^{\varepsilon,r})_{0 < \varepsilon < 1/100, r > 0}$ and $\mathcal{O}' = (O'^{\varepsilon,r})_{0 < \varepsilon < 1/100, r > 0}$ be quantitative objects, let $\mathcal{F} = (F^{\varepsilon,r})_{0 < \varepsilon < \frac{1}{100\alpha}, r > 0} : \mathcal{O} \rightarrow \mathcal{O}'$ (resp. $\mathcal{G} = (G^{\varepsilon,r})_{\frac{1}{100\alpha'}, r > 0} : \mathcal{O} \rightarrow \mathcal{O}'$) be an (α, k) -controlled morphism (resp. an (α', k') -controlled morphism) and let (λ, h) be a control pair. We write $\mathcal{F} \stackrel{(\lambda, h)}{\sim} \mathcal{G}$ if

- $(\alpha, k) \leq (\lambda, h)$ and $(\alpha', k') \leq (\lambda, h)$.
- for every ε in $(0, \frac{1}{100\lambda})$ and $r > 0$, we have

$$\iota_{\mathcal{O}'}^{\alpha\varepsilon, \lambda\varepsilon, k_\varepsilon r, h_\varepsilon r} \circ F^{\varepsilon,r} = \iota_{\mathcal{O}'}^{\alpha'\varepsilon, \lambda\varepsilon, k'_\varepsilon r, h_\varepsilon r} \circ G^{\varepsilon,r}.$$

Recall from [16] the definition of a controlled isomorphisms.

Definition 1.14. Let (λ, h) be a control pair, and let $\mathcal{F} : \mathcal{O} \rightarrow \mathcal{O}'$ be a (α, k) -controlled morphism with $(\alpha, k) \leq (\lambda, h)$. \mathcal{F} is called (λ, h) -invertible or a (λ, h) -isomorphism if there exists a controlled morphism $\mathcal{G} : \mathcal{O}' \rightarrow \mathcal{O}$ such that $\mathcal{G} \circ \mathcal{F} \stackrel{(\lambda, h)}{\sim} \text{Id}_{\mathcal{O}}$ and $\mathcal{F} \circ \mathcal{G} \stackrel{(\lambda, h)}{\sim} \text{Id}_{\mathcal{O}'}$. The controlled morphism \mathcal{G} is called a (λ, h) -inverse for \mathcal{F} .

In particular, if A and B are filtered C^* -algebras and if $\mathcal{F} : \mathcal{K}_*(A) \rightarrow \mathcal{K}_*(B)$ is a (λ, h) -isomorphism, then the induced morphism $F : K_*(A) \rightarrow K_*(B)$ is an isomorphism and its inverse is induced by a controlled morphism (indeed induced by any (λ, h) -inverse for \mathcal{F}).

In order to define the quantitative index map, we need to consider the special case of controlled morphisms with constant source (see Section 3.4).

Definition 1.15. Let Z be an abelian group and let $\mathcal{O} = (O^{\varepsilon,s})_{0 < \varepsilon < 1/100, s > 0}$ be a quantitative object. A projective morphism $\mathcal{F} : Z \rightarrow \mathcal{O}$ is a family of group homomorphisms

$$\mathcal{F} = (F^{\varepsilon,s} : Z \rightarrow \mathcal{O}^{\varepsilon,s})_{0 < \varepsilon < 1/100, s > 0}$$

such that for every positive numbers $\varepsilon, \varepsilon', s$ and s' with $\varepsilon \leq \varepsilon' < 1/100$ and $s \leq s'$, then $F^{\varepsilon',s'} = \iota_{\mathcal{O}}^{\varepsilon,\varepsilon',s,s'} \circ F^{\varepsilon,s}$.

Remark 1.16. Let

$$\mathcal{F} = (F^{\varepsilon,s} : Z \rightarrow \mathcal{O}^{\varepsilon,s})_{0 < \varepsilon < 1/100, s > 0}$$

be a projective morphism.

- (i) let us fix ε_0 in $(0, 1/100)$ and $s_0 > 0$. Then \mathcal{F} is completely determined by the knowledge of $(F^{\varepsilon,s})_{0 < \varepsilon < \varepsilon_0, 0 < s \leq s_0}$.
- (ii) for any fixed ε in $(0, 1/100)$, the projective morphism F induces a morphism

$$\begin{aligned} Z &\longrightarrow \varprojlim_{s > 0} O^{\varepsilon,s} \\ x &\mapsto (F^{\varepsilon,s}(x))_{s > 0}, \end{aligned}$$

where the right hand side is the projective limit with respect to the inductive system $(\iota_{\mathcal{O}}^{\varepsilon,s,s'})_{0 < s \leq s'}$.

If $\Lambda : Z' \rightarrow Z$ is a homomorphism of abelian group, then

$$\mathcal{F} \circ \Lambda \stackrel{\text{def}}{=} (F^{\varepsilon,s} \circ \Lambda)_{0 < \varepsilon < 1/100, s > 0} : Z' \longrightarrow \mathcal{O}$$

is a projective morphism and if $\mathcal{G} : \mathcal{O} \rightarrow \mathcal{O}'$ is a (α, k) -controlled morphism, then

$$\mathcal{G} \circ \mathcal{F} \stackrel{\text{def}}{=} \left(G^{\frac{\varepsilon}{\alpha}, \frac{s}{k\varepsilon/\alpha}} \circ F^{\frac{\varepsilon}{\alpha}, \frac{s}{k\varepsilon/\alpha}} \right)_{0 < \varepsilon < 1/100, s > 0} : Z \longrightarrow \mathcal{O}'$$

is a projective morphism. In order to study the behaviour of quantitative objects for small propagation, we introduce the following notion of h -isomorphism.

Definition 1.17. Let $(0, 1/200) \rightarrow [1, +\infty); \varepsilon \mapsto h_\varepsilon$ be a non decreasing function. A projective morphism

$$\mathcal{F} = (F^{\varepsilon,s})_{0 < \varepsilon < 1/100, s > 0} : Z \rightarrow \mathcal{O}$$

is called a h -isomorphism if for any ε in $(0, 1/200)$,

- $F^{\varepsilon,s} : Z \rightarrow \mathcal{O}^{\varepsilon,r}$ is one-to-one for any positive number r with $r < 1/h_\varepsilon$;
- For any positive number r with $r < 1/h_\varepsilon^2$ and any y in $\mathcal{O}^{\varepsilon,r}$, there exists x in Z such that $F^{\varepsilon,h_\varepsilon r}(x) = \iota_{\mathcal{O}}^{\varepsilon,r,h_\varepsilon r}(y)$.

In other words, this means that $F^{\varepsilon,r}$ is one-to-one onto $\iota_{\mathcal{O}}^{\varepsilon,r/h_\varepsilon,r}(O^{\varepsilon,r/h_\varepsilon,r})$ for any positive number r with $r < 1/h_\varepsilon$.

The restriction to $(0, 1/200)$ of the range of the control is for controlled rescaling reasons that will become clear later on when stating the properties of the quantitative index map. The meaning of this definition is that up to a rescaling, Z and $O^{\varepsilon,r}$ are isomorphic for small propagation. In view of the second point of Remark 1.16, this implies in particular that for any ε in $(0, 1/200)$, the projective morphism \mathcal{F} induces an isomorphism

$$Z \xrightarrow{\simeq} \varprojlim_{s>0} O^{\varepsilon,s}.$$

Let us recall now the definition of the controlled exact sequence for quantitative objects. This controlled exact sequence is an important tool in computing quantitative K -theory for filtered C^* -algebras.

Definition 1.18. Let (λ, h) be a control pair,

- Let $\mathcal{O} = (O_{\varepsilon,s})_{0<\varepsilon<\frac{1}{100}, s>0}$, $\mathcal{O}' = (O'_{\varepsilon,s})_{0<\varepsilon<\frac{1}{100}, s>0}$ and $\mathcal{O}'' = (O''_{\varepsilon,s})_{0<\varepsilon<\frac{1}{100}, s>0}$ be quantitative objects, let

$$\mathcal{F} = (F^{\varepsilon,s})_{0<\varepsilon<\frac{1}{100\alpha}, s>0} : \mathcal{O} \rightarrow \mathcal{O}'$$

be a (α, k) -controlled morphism and let

$$\mathcal{G} = (G^{\varepsilon,s})_{0<\varepsilon<\frac{1}{100\alpha'}, s>0} : \mathcal{O}' \rightarrow \mathcal{O}''$$

be a (α', k') -controlled morphism with $(\alpha', k') \leq (\lambda, h)$. Then the composition

$$\mathcal{O} \xrightarrow{\mathcal{F}} \mathcal{O}' \xrightarrow{\mathcal{G}} \mathcal{O}''$$

is said to be (λ, h) -exact at \mathcal{O}' if $\mathcal{G} \circ \mathcal{F} = 0$ and if for any $0 < \varepsilon < \frac{1}{100\lambda\alpha}$, any $s > 0$ and any y in $O'^{\varepsilon,s}$ such that $G^{\varepsilon,s}(y) = 0$ in $O''_{\alpha'\varepsilon, k'_\varepsilon r}$, there exists an element x in $O^{\lambda\varepsilon, h_\varepsilon s}$ such that

$$F^{\lambda\varepsilon, h_\varepsilon s}(x) = \iota_{\mathcal{O}'}^{\varepsilon, \alpha\lambda\varepsilon, r, k_{\lambda\varepsilon} h_\varepsilon s}(y)$$

in $O^{\alpha\lambda\varepsilon, k_{\lambda\varepsilon} h_\varepsilon s}$.

- A sequence of controlled morphisms

$$\cdots \mathcal{O}_{k-1} \xrightarrow{\mathcal{F}_{k-1}} \mathcal{O}_k \xrightarrow{\mathcal{F}_k} \mathcal{O}_{k+1} \xrightarrow{\mathcal{F}_{k+1}} \mathcal{O}_{k+2} \cdots$$

is called (λ, h) -exact if for every k , the composition

$$\mathcal{O}_{k-1} \xrightarrow{\mathcal{F}_{k-1}} \mathcal{O}_k \xrightarrow{\mathcal{F}_k} \mathcal{O}_{k+1}$$

is (λ, h) -exact at \mathcal{O}_k .

- a controlled morphism $\mathcal{F} : \mathcal{O} \rightarrow \mathcal{O}'$ is called (λ, \mathbf{h}) -injective if

$$0 \rightarrow \mathcal{O} \xrightarrow{\mathcal{F}} \mathcal{O}'$$

is (λ, h) -exact;

- a controlled morphism $\mathcal{F} : \mathcal{O} \rightarrow \mathcal{O}'$ is called (λ, h) -**surjective** if

$$\mathcal{O} \xrightarrow{\mathcal{F}} \mathcal{O}' \rightarrow 0$$

is (λ, h) -exact.

It is then obvious that if a controlled morphism \mathcal{F} is a (λ, h) -isomorphism, then there exists a controlled pairs (λ', h') depending only on (λ, h) such that \mathcal{F} is (λ', h') -injective and (λ', h') -surjective. Conversely, if \mathcal{F} is (λ', h') -injective and (λ'', h'') -surjective, then there exists a controlled pair (λ, h) depending only on (λ', h') and (λ'', h'') such that \mathcal{F} is a (λ, h) -isomorphism.

The notion of (λ, h) -exactness of a composition and of a sequence can be extended to the setting of controlled morphism of order r .

Definition 1.19. Let (λ, h) be a control pair,

Let $\mathcal{O} = (O_{\varepsilon, s})_{0 < \varepsilon < \frac{1}{100}, s > 0}$, $\mathcal{O}' = (O'_{\varepsilon, s})_{0 < \varepsilon < \frac{1}{100}, s > 0}$ and $\mathcal{O}'' = (O''_{\varepsilon, s})_{0 < \varepsilon < \frac{1}{100}, s > 0}$ be quantitative objects, let

$$\mathcal{F} = (F^{\varepsilon, s})_{0 < \varepsilon < \frac{1}{100\alpha}, 0 < s < \frac{r_1}{h\varepsilon}} : \mathcal{O} \rightarrow \mathcal{O}'$$

be a (α, k) -controlled morphism of order r_1 and let

$$\mathcal{G} = (G^{\varepsilon, s})_{0 < \varepsilon < \frac{1}{100\alpha'}, 0 < s < \frac{r_2}{h\varepsilon}} : \mathcal{O}' \rightarrow \mathcal{O}''$$

be a (α', k') -controlled morphism order r_2 with $(\alpha', k') \leq (\lambda, h)$. For r in $(0, \min\{r_1, r_2\})$, the composition

$$\mathcal{O} \xrightarrow{\mathcal{F}} \mathcal{O}' \xrightarrow{\mathcal{G}} \mathcal{O}''$$

is said to be (λ, h) -exact at \mathcal{O}' and at order r if

- for any $0 < \varepsilon < \frac{1}{100\lambda\alpha}$, any $0 < s < \frac{r}{k\varepsilon h\alpha\varepsilon}$ and any x in $\mathcal{O}_{\varepsilon, s}$, then $G^{\alpha\varepsilon, k\alpha\varepsilon s} \circ F^{\varepsilon, s}(x) = 0$ in $\mathcal{O}''_{\alpha\alpha'\varepsilon, k\varepsilon k'\alpha\varepsilon s}$.
- for any $0 < \varepsilon < \frac{1}{100\lambda\alpha}$, any $0 < s < \frac{r}{k\lambda\varepsilon h\varepsilon}$ and any y in $\mathcal{O}'^{\varepsilon, s}$ such that $G^{\varepsilon, s}(y) = 0$ in $\mathcal{O}''_{\alpha'\varepsilon, k'\varepsilon r}$, there exists an element x in $\mathcal{O}^{\lambda\varepsilon, h\varepsilon s}$ such that

$$F^{\lambda\varepsilon, h\varepsilon s}(x) = \iota_{\mathcal{O}'}^{\varepsilon, \alpha\lambda\varepsilon, r, k\lambda\varepsilon h\varepsilon s}(y)$$

in $\mathcal{O}^{\alpha\lambda\varepsilon, k\lambda\varepsilon h\varepsilon s}$.

1.4. Six-term controlled exact sequence. Examples of controlled exact sequences in quantitative K -theory are provided by controlled six term exact sequences associated to a completely filtered extensions of C^* -algebras [15, Section 3].

Definition 1.20. Let A be a C^* -algebra filtered by $(A_r)_{r>0}$, let J be an ideal of A and set $J_r = J \cap A_r$. The extension of C^* -algebras

$$0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$$

is called a *completely filtered extension* of C^* -algebras if the bijective continuous linear map

$$A_r/J_r \longrightarrow (A_r + J)/J$$

induced by the inclusion $A_r \hookrightarrow A$ is a complete isometry i.e for any integer n , any positive number r and any x in $M_n(A_r)$, then

$$\inf_{y \in M_n(J_r)} \|x + y\| = \inf_{y \in M_n(J)} \|x + y\|.$$

Notice that in this case, the ideal J is filtered by $(A_r \cap J)_{r>0}$ and A/J is filtered by $(A_r + J)_{r>0}$. A particular case of completely filtered extension of C^* -algebra is the case of a filtered and semi-split extension of C^* -algebras [15, Lemma 3.3] (or a semi-split extension of filtered algebras) i.e extension

$$0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0,$$

such that

- A is filtered by $(A_r)_{r>0}$;
- there exists a completely positive and completely contractive (if A is not unital) cross-section

$$s : A/J \rightarrow A$$

such that

$$s(A_r + J) \subseteq A_r$$

for any number $r > 0$.

For any extension of C^* -algebras

$$0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$$

we denote by $\partial_{J,A,*} : K_*(A/J) \rightarrow K_*(J)$ the associated (odd degree) boundary map.

Proposition 1.21. *There exists a control pair $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$ such that for any completely filtered extension of C^* -algebras*

$$0 \longrightarrow J \longrightarrow A \xrightarrow{q} A/J \longrightarrow 0,$$

there exists an $(\alpha_{\mathcal{D}}, k_{\mathcal{D}})$ -controlled morphism of odd degree

$$\mathcal{D}_{J,A,*} = (\partial_{J,A,*}^{\varepsilon,r})_{0 < \varepsilon < \frac{1}{100\alpha_{\mathcal{D}}}, r > 0} : K_*(A/J) \rightarrow K_*(J)$$

which induces $\partial_{J,A,} : K_*(A/J) \rightarrow K_*(J)$ in K -theory.*

Moreover the controlled boundary map enjoys the expected naturality properties with respect to completely filtered extensions. We have in the setting of controlled exactness the analogue of the six terms exact sequence in K -theory.

Theorem 1.22. *There exists a control pair (λ, h) such that for any completely filtered extension of C^* -algebras*

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{q} A/J \longrightarrow 0,$$

the following six-term sequence is (λ, h) -exact

$$\begin{array}{ccccc} \mathcal{K}_0(J) & \xrightarrow{j_*} & \mathcal{K}_0(A) & \xrightarrow{q_*} & \mathcal{K}_0(A/J) \\ \mathcal{D}_{J,A,*} \uparrow & & & & \mathcal{D}_{J,A,*} \downarrow \\ \mathcal{K}_1(A/J) & \xleftarrow{q_*} & \mathcal{K}_1(A) & \xleftarrow{j_*} & \mathcal{K}_1(J) \end{array}$$

Moreover, by construction (see [15, Section 3.2]), the controlled boundary map is compatible with the rescaling defined in Section 1.2.

Proposition 1.23. *For any completely filtered extension of C^* -algebras*

$$0 \longrightarrow J \xrightarrow{j} A \xrightarrow{q} A/J \longrightarrow 0,$$

if we set $(\alpha, k) = (\alpha_{\mathcal{D}}, k_{\mathcal{D}})$ and $l = l_{rs}$ (with notation of Section 1.2, then we have

$$\begin{aligned} \partial_{J,A,*}^{\varepsilon/2, l_{\varepsilon} r} \circ L_{A/J,*}^{\varepsilon, r} &= \iota_*^{-, \alpha\varepsilon/2, k_{\varepsilon}/2 l_{\varepsilon} r} \circ L_{J,*}^{\alpha\varepsilon, k_{\varepsilon} r} \circ \partial_{J,A,*}^{\varepsilon, r} \\ &= L_{J,*}^{\alpha\varepsilon, \frac{k_{\varepsilon}/2 l_{\varepsilon} r}{l_{\alpha\varepsilon}}} \circ \partial_{J,A,*}^{\varepsilon, \frac{k_{\varepsilon}/2 l_{\varepsilon} r}{k_{\varepsilon} l_{\alpha\varepsilon}}} \circ \iota_*^{\varepsilon, r, -} \end{aligned}$$

for any ε in $(0, \frac{1}{200\alpha})$ and any $r > 0$.

1.5. Tensorisation in KK -theory and controlled morphisms.

Let A be a C^* -algebra and let B be a filtered C^* -algebra filtered by $(B_r)_{r>0}$. Let us define $A \otimes B_r$ as the closure of the algebraic tensor product of A and B_r in the spatial tensor product $A \otimes B$. Then the C^* -algebra $A \otimes B$ is filtered by $(A \otimes B_r)_{r>0}$. If $f : A_1 \rightarrow A_2$ is a homomorphism of C^* -algebras, let us set

$$f_B : A_1 \otimes B \rightarrow A_2 \otimes B; a \otimes b \mapsto f(a) \otimes b.$$

Recall from [11] that for any C^* -algebras A_1, A_2 and B , G. Kasparov defined a tensorization map

$$\tau_B : KK_*(A_1, A_2) \rightarrow KK_*(A_1 \otimes B, A_2 \otimes B).$$

If B is a filtered C^* -algebra, then for any z in $KK_*(A_1, A_2)$ the morphism

$$K_*(A_1 \otimes B) \longrightarrow K_*(A_2 \otimes B); x \mapsto x \otimes_{A_1 \otimes B} \tau_B(z)$$

is induced by a control morphism [15, Theorem 4.4].

Theorem 1.24. *There exists a control pair $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$ with $k_{\mathcal{T}} > 1$ such that*

- *for any filtered C^* -algebra B ;*
- *for any C^* -algebras A_1 and A_2 ;*
- *for any element z in $KK_*(A_1, A_2)$,*

There exists an $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$ -controlled morphism

$$\mathcal{T}_B(z) = (\tau_B^{\varepsilon, r}(z))_{\varepsilon \in (0, \frac{1}{100\alpha_{\mathcal{T}}}), r > 0} : \mathcal{K}_*(A_1 \otimes B) \rightarrow \mathcal{K}_*(A_2 \otimes B)$$

with same degree as z that satisfies the following:

- (i) $\mathcal{T}_B(z) : \mathcal{K}_*(A_1 \otimes B) \rightarrow \mathcal{K}_*(A_2 \otimes B)$ induces the right multiplication by $\tau_B(z)$ in K -theory;
- (ii) For any elements z and z' in $KK_*(A_1, A_2)$. Then

$$\mathcal{T}_B(z + z') = \mathcal{T}_B(z) + \mathcal{T}_B(z').$$

- (iii) Let A'_1 be a filtered C^* -algebra and let $f : A_1 \rightarrow A'_1$ be a homomorphism of C^* -algebras. Then we have $\mathcal{T}_B(f^*(z)) = \mathcal{T}_B(z) \circ f_{B,*}$ for all z in $KK_*(A'_1, A_2)$.
- (iv) Let A'_2 be a C^* -algebra and let $g : A'_2 \rightarrow A_2$ be a homomorphism of filtered C^* -algebras. Then we have $\mathcal{T}_B(g_*(z)) = g_{B,*} \circ \mathcal{T}_B(z)$ for any z in $KK_*(A_1, A'_2)$.
- (v) $\mathcal{T}_B([Id_{A_1}]) \stackrel{(\alpha_{\mathcal{T}}, k_{\mathcal{T}})}{\sim} Id_{\mathcal{K}_*(A_1 \otimes B)}$.
- (vi) For any C^* -algebra D and any element z in $KK_*(A_1, A_2)$, we have $\mathcal{T}_B(\tau_D(z)) = \mathcal{T}_{B \otimes D}(z)$.
- (vii) For any semi-split extension of C^* -algebras $0 \rightarrow J \rightarrow A \rightarrow A/J \rightarrow 0$ with corresponding element $[\partial_{J,A}]$ of $KK_1(A/J, J)$ that implements the boundary map, we have

$$\mathcal{T}_B([\partial_{J,A}]) = \mathcal{D}_{J \otimes B, A \otimes B}.$$

Moreover, \mathcal{T}_B is compatible with Kasparov products [15, Theorem 4.5].

Theorem 1.25. *There exists a control pair (λ, h) such that the following holds :*

let A_1, A_2 and A_3 be separable C^ -algebras and let B be a filtered C^* -algebra. Then for any z in $KK_*(A_1, A_2)$ and any z' in $KK_*(A_2, A_3)$, we have*

$$\mathcal{T}_B(z \otimes_{A_2} z') \stackrel{(\lambda, h)}{\sim} \mathcal{T}_B(z') \circ \mathcal{T}_B(z).$$

We also have the following naturality result with respect to the filtered C^* -algebras.

Proposition 1.26. *Let A_1 and A_2 be separable C^* -algebras, let B_1 and B_2 be filtered C^* -algebras and let $f : B_1 \rightarrow B_2$ be a morphism of filtered C^* -algebras. Let us consider $f_{A_1} : A_1 \otimes B_1 \rightarrow A_1 \otimes B_2$ and $f_{A_2} : A_2 \otimes B_1 \rightarrow A_2 \otimes B_2$ the morphisms of filtered C^* -algebra induced by f on tensor product. Then*

$$f_{A_2,*} \circ \mathcal{T}_{B_1}(z) = \mathcal{T}_{B_2}(z) \circ f_{A_1,*}$$

for any z in $KK_(A_1, A_2)$.*

Remark 1.27. We can extend Proposition 1.26 to the slightly more general following situation. Let $f : B_1 \rightarrow B_2$ be a homomorphism of C^* -algebras and assume that there exist a positive number θ with $\theta \geq 1$ and a non negative number β such that $f(B_{1,r}) \subseteq B_{2,\theta r+\beta}$ for all positive number r . Then f induces for every positive numbers ε and r with $\varepsilon < 1/100$ and for any C^* -algebra A a morphism

$$f_{A,*}^{\varepsilon,r} : K_*^{\varepsilon,r}(A \otimes B_1) \longrightarrow K_*^{\varepsilon,\theta r+\beta}(A \otimes B_2).$$

With notations of Proposition 1.26, we have with $(\alpha, k) = (\alpha_{\mathcal{T}}, k_{\mathcal{T}})$

$$\iota_*^{-,\alpha\varepsilon,k\varepsilon(\theta r+\beta)} \circ f_{A_2,*}^{\alpha\varepsilon,k\varepsilon r} \circ \tau_{B_1}^{\varepsilon,r}(z) = \tau_{B_2}^{\varepsilon,\theta r+\beta}(z) \circ f_{A_1,*}^{\varepsilon,r}$$

for all positive numbers ε and r with $\varepsilon < \frac{1}{100\alpha}$.

By construction of \mathcal{T}_B (see [15, Section 3.2]) and in view of Proposition 1.23, this quantitative version of Kasparov transformation is compatible with the rescaling defined in Section 1.2.

Proposition 1.28. *Let us set $(\alpha, k) = (\alpha_{\mathcal{T}}, k_{\mathcal{T}})$ and $l = l_{r_s}$ (with notation of Section 1.2). Then for any filtered C^* -algebra B , for any C^* -algebras A_1 and A_2 and any element z in $KK_*(A_1, A_2)$, we have*

$$\begin{aligned} \tau_B^{\varepsilon/2, l\varepsilon r}(z) \circ L_{A_1 \otimes B,*}^{\varepsilon,r} &= \iota_*^{-,\alpha\varepsilon/2, k\varepsilon/2 l\varepsilon r} \circ L_{A_2 \otimes B,*}^{\alpha\varepsilon, k\varepsilon r} \circ \tau_B^{\varepsilon,r}(z) \\ &= L_{A_2 \otimes B,*}^{\alpha\varepsilon, \frac{k\varepsilon/2 l\varepsilon r}{l_{\alpha\varepsilon}}} \circ \tau_B^{\varepsilon, \frac{k\varepsilon/2 l\varepsilon r}{k\varepsilon l_{\alpha\varepsilon}}}(z) \circ \iota_*^{\varepsilon,r,-} \end{aligned}$$

for any ε in $(0, \frac{1}{200\alpha})$ and any $r > 0$.

2. QUANTITATIVE ASSEMBLY MAPS ESTIMATES AND COARSE DECOMPOSITION

In [16], the authors developed a geometric version of the controlled Baum-Connes assembly maps (see [15]) and of the related quantitative statements. It was proved that if the family of finite subsets of a proper discrete metric space Σ satisfies uniformly these geometric quantitative statements, then Σ satisfies the coarse Baum-Connes conjecture. Due to the descent principle [18], in the particular case of a finitely generated group with finite type classifying space, this implies the Novikov conjecture. In this section, we recall the setting for these geometric statement and we state the main result of this paper: uniform geometric quantitative statements for families of finite metric spaces with uniformly bounded geometry are stable under coarse decompositions. We apply this result to metric spaces with finite geometric complexity introduced in [6]. We prove that the family of finite subsets of such a metric space with bounded geometry satisfies uniformly the geometric quantitative statements.

2.1. Geometric controlled assembly maps. Let Σ be a proper discrete metric space. For any C^* -algebra A , set $A_\Sigma = A \otimes \mathcal{K}(\ell^2(\Sigma))$. Then the distance d on Σ induces a filtration on A_Σ in the following way: let r be a positive number and $T = (T_{\sigma,\sigma'})_{(\sigma,\sigma') \in \Sigma^2}$ be an element in $A_{\ell^2(\Sigma)}$, with $T_{\sigma,\sigma'}$ in A for any σ and σ' in Σ^2 . Then T has propagation r if $T_{\sigma,\sigma'} = 0$ for σ and σ' in Σ such that $d(\sigma, \sigma') > r$. We define the Rips complex of degree d of Σ as the set $P_d(\Sigma)$ of probability measure on Σ with support of diameter less than d . Then $P_d(\Sigma)$ is a locally finite simplicial complex and is locally compact when endowed with the simplicial topology. Let us define then

$$K_*(P_d(\Sigma), A) \stackrel{\text{def}}{=} \lim_{Z \subseteq P_d(\Sigma)} KK_*(C(Z), A),$$

where Z runs through compact subsets of $P_d(\Sigma)$. In [16] was constructed for any C^* -algebra A local quantitative coarse assembly maps

$$\nu_{\Sigma, A, *}^{\varepsilon, r, d} : K_*(P_d(\Sigma), A) \rightarrow K_*^{\varepsilon, r}(A_\Sigma),$$

with $d > 0$, $\varepsilon \in (0, 1/100)$ and $r \geq r_{d, \varepsilon}$, where the map

$$(0, +\infty) \times (0, 1/100) \rightarrow (0, +\infty); (\varepsilon, d) \mapsto r_{d, \varepsilon}$$

is non increasing in ε , non decreasing in d and satisfies $r_{d, \varepsilon} > 2d$ for all positive numbers ε and d with $\varepsilon < 1/100$. Let us recall now the definition of these quantitative assembly maps:

Notice first that any x in $P_d(\Sigma)$ can be written as a finite convex combination

$$x = \sum_{\sigma \in \Sigma} \lambda_\sigma(x) \delta_\sigma$$

where

- δ_σ is the Dirac probability measure at σ in Σ .
- for every σ in Σ , the coordinate function

$$\lambda_\sigma = \lambda_{\Sigma, \sigma} : P_d(\Sigma) \rightarrow [0, 1]$$

is continuous and its support is the closure of the set of probability measures on Σ for which σ is in the support.

Let X be a compact subset of $P_d(\Sigma)$. Let us define

$$P_X : C(X) \otimes \ell^2(\Sigma) \rightarrow C(X) \otimes \ell^2(\Sigma)$$

by

$$(P_X \cdot h)(x, \sigma) = \lambda_\sigma^{1/2}(x) \sum_{\sigma' \in \Sigma} h(x, \sigma') \lambda_{\sigma'}^{1/2}(x), \quad (3)$$

for any h in $C(X) \otimes \ell^2(\Sigma)$. Since $\sum_{\sigma \in \Sigma} \lambda_\sigma = 1$, it is straightforward to check that P_X is a projection in $C(X) \otimes \mathcal{K}(\ell^2(\Sigma)) = C(X)_\Sigma$ with propagation less than $2d$ and hence gives rise in particular to a class $[P_X]$ in $K_0^{\varepsilon, r}(C(X)_\Sigma)$ for any ε in $(0, 1/100)$ and any positive number r with $r \geq 2d$. Let us set $r_{d, \varepsilon} = 2dk_{\varepsilon/\alpha}$, for any positive number ε and r with $\varepsilon < 1/100$, where $(\alpha, k) = (\alpha_{\mathcal{T}}, k_{\mathcal{T}})$ is the control pair of Theorem

1.24. For any C^* -algebra A , any $\varepsilon \in (0, 1/100)$ and any $r \geq r_{d,\varepsilon}$, the map

$$\begin{aligned} KK_*(C(X), A) &\longrightarrow K_*^{\varepsilon,r}(A_\Sigma) \\ x &\mapsto \left(\tau_{\mathcal{K}(\ell^2(\Sigma))}^{\varepsilon/\alpha, r/k_{\varepsilon/\alpha}}(x) \right) ([P_X, 0]_{\varepsilon/\alpha, r/k_{\varepsilon/\alpha}}) \end{aligned} \quad (4)$$

is compatible with inductive limit over compact subsets of $P_d(\Sigma)$ and hence gives rise to a local coarse assembly map

$$\nu_{\Sigma, A, *}^{\varepsilon, r, d} : K_*(P_d(\Sigma), A) \longrightarrow K_*(A_\Sigma).$$

The map $\nu_{\Sigma, \bullet, *}^{\varepsilon, r, d}$ is natural for C^* -algebras and induces in K -theory the index map, i.e the maps $\iota_*^{\varepsilon, r} \circ \nu_{\Sigma, A, *}^{\varepsilon, r, d}$ is up to Morita equivalence given for any compact subset Z of $P_d(\Sigma)$ by the morphism in the inductive limit $KK_*(C(Z), A) \rightarrow K_*(A)$ induced by the map $Z \rightarrow \{pt\}$. Moreover, the maps $\nu_{\Sigma, A, *}^{\bullet, \bullet, \bullet}$ are compatible with structure morphisms and with inclusion of Rips complexes:

- $\iota_*^{\varepsilon, r, d} \circ \nu_{\Sigma, A, *}^{\varepsilon, r, d} = \nu_{\Sigma, A, *}^{\varepsilon', r', d}$ for any positive numbers $\varepsilon, \varepsilon', r, r'$ and d such that $\varepsilon \leq \varepsilon' < 1/100$, $r_{d,\varepsilon} \leq r$, $r_{d,\varepsilon'} \leq r'$ and $r \leq r'$;
- $\nu_{\Sigma, A, *}^{\varepsilon, r, d} \circ q_{d, d', *} = \nu_{\Sigma, A, *}^{\varepsilon, r, d}$ for any positive numbers ε, r, d and d' such that $\varepsilon < 1/100$, $d \leq d'$ and $r_{d', \varepsilon} \leq r$, where

$$q_{d, d', *} = q_{d, d'}^\Sigma : K_*(P_d(\Sigma), A) \rightarrow K_*(P_{d'}(\Sigma), A)$$

is the homomorphism induced by the restriction from $P_{d'}(\Sigma)$ to $P_d(\Sigma)$.

The construction of the map $\nu_{\Sigma, A, *}^{\varepsilon, r, d}$ has to be compare with the construction of the coarse assembly map in [19, Section 2.3].

Lemma 2.1. *Let Σ be a proper discrete metric space and let Σ' be a subset of Σ . Let us consider :*

- $j_{\Sigma, \Sigma'}^\# : \mathcal{K}(\ell^2(\Sigma')) \hookrightarrow \mathcal{K}(\ell^2(\Sigma))$ the natural inclusion;
- $j_{P_d(\Sigma), P_d(\Sigma'), *} : K_*(P_d(\Sigma'), A) \rightarrow K_*(P_d(\Sigma), A)$ the homomorphism induced by the restriction from $P_d(\Sigma)$ to $P_d(\Sigma')$.

Then we have the equality

$$\nu_{\Sigma, A, *}^{\varepsilon, r, d} \circ j_{P_d(\Sigma), P_d(\Sigma'), *} = j_{\Sigma, \Sigma', A, *}^\# \circ \nu_{\Sigma', A, *}^{\varepsilon, r, d}$$

for any positive numbers ε, r, d such that $\varepsilon < 1/100$, and $r_{d,\varepsilon} \leq r$, where

$$j_{\Sigma, \Sigma', A}^\# : A_{\Sigma'} \longrightarrow A_\Sigma$$

is the filtered morphism induced by $j_{\Sigma, \Sigma'}^\#$.

Proof. Let X be a compact subset of $P_d(\Sigma') \subseteq P_d(\Sigma)$ and let P_X in $C(X) \otimes \mathcal{K}(\ell^2(\Sigma))$ and P'_X in $C(X) \otimes \mathcal{K}(\ell^2(\Sigma'))$ be the projections of equation (3) respectively obtained from Σ and Σ' . It is straightforward to check that

$$j_{\Sigma, \Sigma', C(X)}^\#(P'_X) = P_X. \quad (5)$$

Now the result is a consequence of the naturality in the filtered C^* -algebra of \mathcal{T}_\bullet stated in Proposition 1.26. \square

Let us set $l = l_{rs}$ (with notation of Section 1.2 and let X be a compact subset of $P_d(\Sigma)$. It is clear that $L_{C(X) \otimes \mathcal{K}(\ell^2(\Sigma))}^{\varepsilon, r}([P_X, 0]_{\varepsilon, r}) = [P_X, 0]_{\varepsilon/2, l_{\varepsilon} r}$ for $r > 2d$ and hence according to Proposition 1.28, the local quantitative assembly maps are compatible with the rescaling map.

Proposition 2.2. *Let Σ be a proper discrete metric space and let A be a C^* -algebra A . For any positive numbers ε , r and d with $\varepsilon < 1/200$ and $r_{d, \varepsilon} \leq r$, let us set $k'_\varepsilon = \frac{l_{\varepsilon/\alpha} k_{\frac{\varepsilon}{2\alpha}}}{k_{\varepsilon/\alpha}}$. Then, we have*

$$\nu_{\Sigma, A, *}^{\varepsilon/2, k'_\varepsilon r, d} = \iota_{*, \varepsilon/2, k'_\varepsilon r, d}^- \circ L_{A_{\Sigma, *}}^{\varepsilon, r} \circ \nu_{\Sigma, A, *}^{\varepsilon, r, d}.$$

2.2. Quantitative Assembly Maps estimates. Let us consider for d, d', r, r' and ε positive numbers with $d \leq d'$, $\varepsilon < 1/100$, $r_{d, \varepsilon} \leq r'$ and $r \leq r'$ the following statements:

- $QI_{\Sigma, A, *}(d, d', r, \varepsilon)$: for any element x in $K_*(P_d(\Sigma), A)$, then $\nu_{\Sigma, A, *}^{\varepsilon, r, d}(x) = 0$ in $K_*^{\varepsilon, r}(A \otimes \mathcal{K}(\ell^2(\Sigma)))$ implies that $q_{d, d', *}(x) = 0$ in $K_*(P_{d'}(\Sigma), A)$.
- $QS_{\Sigma, A, *}(d, r, r', \varepsilon)$: for every y in $K_*^{\varepsilon, r}(A_\Sigma)$, there exists an element x in $K_*(P_d(\Sigma), A)$ such that

$$\nu_{\Sigma, A, *}^{\varepsilon, r', d}(x) = \iota_{*, \varepsilon, r', d}^-(y).$$

Definition 2.3.

- Let Σ be a proper discrete metric space. We say that Σ satisfies the Quantitative Assembly Map estimates (QAM-estimates) if
 - for every positive numbers ε , r and d with $\varepsilon < \frac{1}{200}$ and $r \geq r_{d, \varepsilon}$, there exists a positive number d' with $d' \geq d$ such that $QI_{\Sigma, A, *}(d, d', r, \varepsilon)$ holds for any C^* -algebras A ;
 - for every positive numbers ε , r with $\varepsilon < \frac{1}{200}$ there exists positive numbers d and r' with $r_{d, \varepsilon} \leq r'$ and $r \leq r'$ and such that $QS_{\Sigma, A, *}(d, r, r', \varepsilon)$ holds for any C^* -algebras A ;
- Let \mathcal{Y} be a family of proper discrete metric spaces. We say that the family \mathcal{Y} satisfies uniformly the QAM-estimates if
 - for every positive numbers ε , r and d with $\varepsilon < \frac{1}{200}$ and $r \geq r_{d, \varepsilon}$, there exists a positive number d' with $d' \geq d$ such that $QI_{\Sigma, A, *}(d, d', r, \varepsilon)$ holds for any C^* -algebras A and any proper discrete metric space Σ in \mathcal{Y} ;
 - for every positive numbers ε , r with $\varepsilon < \frac{1}{200}$ there exists positive numbers d and r' with $r_{d, \varepsilon} \leq r'$ and $r \leq r'$ and such that $QS_{\Sigma, A, *}(d, r, r', \varepsilon)$ holds for any C^* -algebras A and any proper discrete metric space Σ in \mathcal{Y} .

Remark 2.4. According to Lemma 1.7 and Proposition 2.2, it is enough to check the QAM-estimates for a fixed ε in $(0, 1/200)$:

Assume that there exists ε in $(0, 1/200)$ such that

- for every positive numbers r and d with $r \geq r_{d,\varepsilon}$, there exists a positive number d' with $d' \geq d$ such that $QI_{\Sigma,A,*}(d, d', r, \varepsilon)$ holds for any C^* -algebras A ;
- for every positive number r , there exists positive numbers d and r' with $r_{d,\varepsilon} \leq r'$ and $r \leq r'$ and such that $QS_{\Sigma,A,*}(d, r, r', \varepsilon)$ holds for any C^* -algebras A .

Then Σ satisfies the QAM-estimates.

We have a similar statement in the uniform case.

The next two results, proved in [16, Theorem 4.10 and 4.11], provide numerous examples of discrete proper metric space that satisfy the QAM-estimates.

Theorem 2.5. *Let Γ be a finitely generated group provided with any word metric. Assume that Γ satisfies the Baum-Connes conjecture. Then Γ as a metric space satisfies the QAM-estimates.*

Before stating the second result, we recall the definition of coarse embeddability.

Definition 2.6. *A proper discrete metric space (Σ, d) coarsely embeds into a Hilbert space \mathcal{H} if there exist two functions $\rho_{\pm} : (0, +\infty) \rightarrow (0, +\infty)$ and a map $f : \Sigma \rightarrow \mathcal{H}$ such that*

- $\lim_{x \rightarrow +\infty} \rho_{\pm}(x) = +\infty$.
- $\rho_{-}(d(\sigma, \sigma')) \leq \|f(\sigma) - f(\sigma')\| \leq \rho_{+}(d(\sigma, \sigma'))$ for all σ and σ' in Σ .

Recall that a discrete metric space has bounded geometry if every ball of a given radius has uniformly bounded cardinal.

Theorem 2.7. *Let Σ be a discrete metric space with bounded geometry that coarsely embeds into a Hilbert space, then Σ satisfy the QAM-estimates.*

The QAM-estimates are also related to the Novikov conjecture in the following way [16, Theorem 5.1].

Theorem 2.8. *Let (Σ, d) be a proper discrete metric space and assume that the family of finite subsets of Σ satisfies uniformly the QAM-estimates. Then Σ satisfies the coarse Baum-Connes conjecture.*

In view of the descent principle [18], we have the following consequence.

Corollary 2.9. *Let Γ be a finitely generated group which admits a finite type classifying space and let us equip Γ with any word metric. Assume that the family of finite subsets of Γ satisfies uniformly the QAM-estimates. Then Γ satisfies the Novikov conjecture.*

Notice that Theorems 2.5, 2.7 and 2.8 were originally stated with a rescaling which is now taken into account by enlarging homotopies in the definition of quantitative K -theory.

Next we show that for a proper discrete metric space Σ we can reduce QAM-estimates for Σ to uniform QAM-estimates for the family of finite subsets of Σ .

Proposition 2.10. *Let Σ be a discrete proper metric space and assume that the family of finite subsets of Σ satisfies uniformly the QAM-estimates. Then Σ satisfies the QAM-estimates.*

Proof. Let us fix σ_0 in Σ and set

$$\Sigma_n = \{\sigma \in \Sigma \text{ such that } d(\sigma, \sigma_0) < n\}.$$

According to [15, Proposition 1.28], the sequence of inclusions

$$j_{\Sigma_n, \Sigma}^{\#} : \mathcal{K}(\ell^2(\Sigma_n)) \hookrightarrow \mathcal{K}(\ell^2(\Sigma))$$

induces an isomorphism

$$\lim_n K^{\varepsilon, r}(A_{\Sigma_n}) \xrightarrow{\simeq} K^{\varepsilon, r}(A_{\Sigma}).$$

In the same way the sequence of restrictions of $P_d(\Sigma)$ to $P_d(\Sigma_n)$ induces an isomorphism

$$\lim_n K_*(P_d(\Sigma_n), A) \xrightarrow{\simeq} K_*(P_d(\Sigma), A)$$

and according to Lemma 2.1, we see that if $QI_{\Sigma_n, A, *}(d, d', r, \varepsilon)$ holds for every n , then $QI_{\Sigma, A, *}(d, d', r, \varepsilon)$ is also satisfied. We can proceed similarly for the QS -statement \square

2.3. The main theorem. In this subsection, we state the main Theorem of this paper. This theorem will be proved in Section 5. Let us first recall from [6] the definition of coarse decomposability for discrete proper metric spaces.

Definition 2.11. *Let \mathcal{X} and \mathcal{Y} be families of discrete metric spaces and let R be a positive number. The family \mathcal{X} is R -decomposable with respect to \mathcal{Y} if for any X in \mathcal{X} , there exists $X^{(1)}$ and $X^{(2)}$ two subsets of X such that*

- $X = X^{(1)} \cup X^{(2)}$;
- for $i = 1, 2$, then $X^{(i)}$ is the R -disjoint union of metric spaces in \mathcal{Y} , i.e there exist $(X_j^{(i)})_{j \in J_i}$ a family of metric space in \mathcal{Y} such that $X^{(i)} = \bigsqcup_{j \in J_i} X_j^{(i)}$ and $d(X_j^{(i)}, X_{j'}^{(i)}) > R$ if $j \neq j'$ (in this situation, we shall write $X^{(i)} =_R \bigsqcup_{j \in J_i} X_j^{(i)}$).

We also need to generalize the concept of bounded geometry to families of discrete proper metric spaces.

Definition 2.12. Let \mathcal{X} be a family of discrete proper metric spaces. Then \mathcal{X} has uniformly bounded geometry if for every positive number r , there exists an integer n_r such that for any metric space in \mathcal{X} , balls of radius r have cardinal bounded by n_r .

If \mathcal{X} is a family of finite metric spaces, we denote by $\widetilde{\mathcal{X}}$ the family of all subsets of all metric spaces in \mathcal{X} .

Theorem 2.13. Let \mathcal{X} be a family of finite metric spaces and assume that for every positive number R , there exists a family \mathcal{Y} of finite metric spaces with uniformly bounded geometry such that

- \mathcal{X} is R -decomposable with respect to \mathcal{Y} ;
- the family $\widetilde{\mathcal{Y}}$ satisfies uniformly the QAM-estimates;

Then the family \mathcal{X} satisfies uniformly the QAM-estimates.

This theorem will be proved in Section 5.

2.4. Application to metric space with finite decomposition complexity. Spaces with finite geometric decomposition were introduced in [5, 6]. As an application of Theorem 2.13, we show that family of finite subsets of a space with finite decomposition complexity satisfies uniformly the QAM-estimates. Recall first the concept of coarse decomposition for a class of proper discrete metric spaces.

Definition 2.14. Let \mathcal{C} be a class of family of proper discrete metric spaces. We say that \mathcal{C} is closed under coarse decomposition if the following is satisfied : let \mathcal{X} be a family of discrete proper metric spaces such that for every positive number R , there exists a family \mathcal{Y} in \mathcal{C} such that \mathcal{X} is R -decomposable with respect to \mathcal{Y} . Then \mathcal{X} belongs to \mathcal{C} .

Example 2.15. The class of families of discrete metric spaces with uniformly bounded geometry is closed under coarse decomposition.

Let us define as in [5, 6] the class \mathcal{C}_{fdc} as the smallest class of family that contains uniformly bounded families of discrete metric spaces and which is closed under coarse decomposability. Typical examples of family of metric spaces in \mathcal{C}_{fdc} are provided by spaces with finite asymptotic dimension. Recall that for a metric space X and a positive number r , a cover $(U_i)_{i \in \mathbb{N}}$ has r -multiplicity n if any ball of radius r in X intersects at most n elements in $(U_i)_{i \in \mathbb{N}}$.

Definition 2.16. Let Σ be a proper discrete metric set. Then Σ has **finite asymptotic dimension** if there exists an integer m such that for any positive number r , there exists a uniformly bounded cover $(U_i)_{i \in \mathbb{N}}$ with finite r -multiplicity $m + 1$. The smallest integer that satisfies this condition is called the **asymptotic dimension** of Σ .

We have the following characterization of finite asymptotic dimension [4].

Proposition 2.17. *Let Σ be a proper discrete metric space and let m be an integer. Then the following assertions are equivalent:*

- (i) Σ has asymptotic dimension m ;
- (ii) *For every positive number r there exist $m+1$ subsets $X^{(1)}, \dots, X^{(m+1)}$ of Σ such that*
 - $\Sigma = X^{(1)} \cup \dots \cup X^{(m+1)}$;
 - *for $i = 1, \dots, m+1$, then $X^{(i)}$ is the r -disjoint union of a uniformly bounded family $(X_k^{(i)})_{k \in \mathbb{N}}$ of subsets of $X^{(i)}$, i.e. $X^{(i)} = \cup_{k \in \mathbb{N}} X_k^{(i)}$, $d_i(X_k^{(i)}, X_l^{(i)}) \geq r$ if $k \neq l$ and there exists a positive number C such $\text{diam } X_k^{(i)} \leq C$ for all integer k .*

Example 2.18. If T is a tree, then T has asymptotic dimension equal to 1.

Let Σ be a proper metric space with asymptotic dimension m , then there exists a sequence of positive numbers $(R_k)_{k \in \mathbb{N}}$ and for any integer k a cover $(U_i^{(k)})_{i \in \mathbb{N}}$ of Σ such that ([4, Page 27] and [24])

- $R_{k+1} > 4R_k$ for every integer k ;
- $U_i^{(k)}$ has diameter less than R_k for every integers i and k ;
- for any integer k , the R_k -multiplicity of $(U_i^{(k+1)})_{i \in \mathbb{N}}$ is $m+1$.

The sequence $(R_k)_{k \in \mathbb{N}}$ is called **the m-growth** of Σ .

Lemma 2.19. *Let m be an integer, let $(R_k)_{k \in \mathbb{N}}$ be a sequence of positive numbers such that $R_{k+1} > 4R_k$ for every integer k . Let $(\Sigma_i)_{i \in \mathbb{N}}$ be a family of proper metric spaces with asymptotic dimension m and m -growth $(R_k)_{k \in \mathbb{N}}$. Then the family $(\Sigma_i)_{i \in \mathbb{N}}$ belongs to \mathcal{C}_{fdc} .*

Proof. Let us equip $\Sigma = \bigsqcup_{i \in \mathbb{N}} \Sigma_i$ with a distance d_Σ such that the inclusion $\Sigma_i \hookrightarrow \Sigma$ is isometric for any integer i and $d_\Sigma(\Sigma_i, \Sigma_j) \geq i+j$ for all integers i and j with $i \neq j$. Then Σ has asymptotic dimension m and hence according to [3], the metric space Σ embeds uniformly in a product of trees $\prod_{j=1}^n T_j$. Let d be the metric on $X = \prod_{j=1}^n T_j$ and d_i the distance on Σ_i when i runs through integers. Then there exists two non-decreasing functions $\rho_\pm : [0, +\infty) \rightarrow [0, +\infty)$ and for every integer i a map $f_i : \Sigma_i \rightarrow \prod_{j=1}^n T_j$ such that

- $\lim_{r \rightarrow +\infty} \rho_\pm(r) = +\infty$;
- $\rho_-(d_i(x, y)) \leq d(f_i(x), f_i(y)) \leq \rho_+(d_i(x, y))$ for all integer i and all x and y in Σ_i .

If $n = 1$, then X is a tree and then the result holds in view Example 2.18. A straightforward induction shows that if Σ embeds uniformly in a product of n trees, then the family $(\Sigma_i)_{i \in \mathbb{N}}$ belongs to \mathcal{C}_{fdc} . \square

Theorem 2.20. *Let \mathcal{X} be a family of finite metric space in \mathcal{C}_{fdc} with uniformly bounded geometry, then \mathcal{X} satisfies uniformly the QAM-estimates.*

Proof. Let \mathcal{C}'_{fdc} be the class of families of finite metric spaces with uniformly bounded geometry that belongs to \mathcal{C}_{fdc} . Then the class \mathcal{C}'_{fdc} is indeed the smallest class of families of finite metric spaces such that

- \mathcal{C}'_{fdc} contains all families of uniformly bounded finite metric spaces with uniformly bounded geometry;
- \mathcal{C}'_{fdc} is closed under coarse decomposition.

Its also straightforward to check that $\widetilde{\mathcal{X}}$ belongs to \mathcal{C}'_{fdc} if \mathcal{X} does. Notice first that if \mathcal{X} is a family of uniformly bounded metric spaces with uniformly bounded geometry then \mathcal{X} satisfies uniformly the QAM-estimates:

- the QI -statement is a consequence on one hand of that there exists a positive number d such that $P_d(X)$ is contractible for every X in \mathcal{X} and on the other hand of that $\nu_{X,A,*}^{\bullet,\bullet,\bullet}$ induces in K -theory the map induced by $P_d(X) \rightarrow \{pt\}$.
- since there exists a positive number r such that $\mathcal{K}(\ell^2(X))_r = \mathcal{K}(\ell^2(X))$ for all X in \mathcal{X} , the QS -statement is a consequence of the isomorphism of equation (1).

If a family of finite metric space \mathcal{X} is R -decomposable with respect to a family of finite metric space \mathcal{Y} , then $\widetilde{\mathcal{X}}$ is clearly also R -decomposable with respect to the family of finite metric space $\widetilde{\mathcal{Y}}$. Then the class of family \mathcal{X} of finite metric spaces with uniformly bounded geometry such that $\widetilde{\mathcal{X}}$ satisfies the QAM-estimates is according to Theorem 2.13 and in view of Example 2.15 closed under coarse decomposition and contains family of metric spaces with uniformly bounded diameter and uniformly bounded geometry. Hence it contains \mathcal{C}'_{fdc} . \square

As a consequence of this theorem, we can deduce the Novikov conjecture for group with finite asymptotic dimension which admits a classifying space of finite type (see [24] for the original proof). The argument of the proof can be extended to the case of groups with finite decomposition complexity. This class of groups was introduced in [5] in their study of topological rigidity for manifolds.

Definition 2.21. *A proper discrete metric space Σ has finite decomposition complexity if the single family $\{\Sigma\}$ is in \mathcal{C}_{fdc} .*

Since the family of finite subsets of a discrete metric space with finite decomposition complexity is in \mathcal{C}_{fdc} , we deduce from Theorem 2.20 the following.

Theorem 2.22. *Let Σ be a metric space with bounded geometry and finite decomposition complexity, then the family of finite subsets of Σ satisfies uniformly the QAM-estimates.*

As a consequence of Corollary 2.9, we deduce

Corollary 2.23. *Let Γ be a finitely generated group with finite decomposition complexity and which admits a classifying space of finite type, then Γ satisfies Novikov conjecture.*

Notice that the Novikov conjecture was indeed already known in this situation since the property (A) holds for finitely generated groups with finite decomposition complexity [6][Theorem 4.3] and hence under the extra assumptions of the corollary, the Novikov conjecture is satisfied [25]. But as mentioned before, our proof does not involve infinite dimension analysis.

3. K -HOMOLOGY AND SMALL PROPAGATION

In this section, we define a quantitative version of the index map for K -homology of a compact metric space X . This quantitative index map is valued in the quantitative K -theory for the compact operators algebra of any non degenerate X -standard module and induces in K -theory the usual index map. As we shall see, when X is a finite simplicial complex, then the quantitative index map is a h -isomorphism in sense of Definition 1.17

3.1. X -modules. Recall that if X is a compact space, then an X -module is a pair (\mathcal{H}_X, ρ_X) , where \mathcal{H}_X is a separable Hilbert space and $\rho_X : C(X) \rightarrow \mathcal{L}(\mathcal{H}_X)$ is a representation of $C(X)$ on \mathcal{H}_X . An X -module (\mathcal{H}_X, ρ_X) is called

- non degenerate if $\rho_X(1) = Id_{\mathcal{H}_X}$;
- standard if no non-zero function acts as a compact operator on \mathcal{H}_X (in particular ρ_X is one-to-one).

As usual, if (\mathcal{H}_X, ρ_X) is an X -module, then for f in $C(X)$ and ξ in \mathcal{H}_X we denote for short $\rho_X(f) \cdot \xi$ by $f \cdot \xi$.

Example 3.1. Let μ be a borelian measure on X such that $\text{supp } \mu = X$. Then $L^2(X, \mu)$ is a non degenerate X -module for the representation given by the pointwise multiplication and if \mathcal{H} is a separable Hilbert space, then $L^2(X, \mu) \otimes \mathcal{H}$ is a non degenerate standard X -module. Indeed, if X has no isolated point, then the representation on $L^2(X, \mu)$ is standard.

Let us first recall the definition of support for vectors and operators on an X -module.

- Let X be a compact space, let (\mathcal{H}_X, ρ_X) be an X -module and let ξ be a vector of \mathcal{H}_X . Then the support of ξ , denoted by $\text{supp } \xi$, is the complementary in X of the open subset of elements x for which there exists f in $C(X)$ with $f(x) \neq 0$ and $f \cdot \xi = 0$.
- Let X and Y be compact spaces, let (\mathcal{H}_X, ρ_X) and (\mathcal{H}_Y, ρ_Y) be respectively a X -module and a Y -module and let $T : \mathcal{H}_X \rightarrow$

\mathcal{H}_Y be a bounded operator. Then the support of T , denoted by $\text{supp } T$, is the complementary in $X \times Y$ of the open subset of pairs (x, y) for which there exist f in $C(Y)$ and g in $C(X)$ with $f(y) \neq 0$, $g(x) \neq 0$ and $f \cdot T \cdot g = 0$;

Let us now define for a compact space X the restriction of an X -module to a closed subspace Y . Let (\mathcal{H}_X, ρ_X) be a X -module. Define then \mathcal{H}_X^Y as the set of elements in \mathcal{H}_X with support in Y . For an element ξ in \mathcal{H}_X , then ξ belongs to \mathcal{H}_X^Y if and only if $f \cdot \xi = 0$ for all f in $C_0(X \setminus Y)$. In particular, \mathcal{H}_X^Y is a closed linear subspace of \mathcal{H}_X and hence is an Hilbert space. Let f be a continuous function in $C(Y)$ and let \tilde{f} in $C(X)$ be a continuous function such that $\tilde{f}|_Y = f$. It is straightforward to check that if ξ lies in \mathcal{H}_X^Y , then $\rho_X(\tilde{f}) \cdot \xi$ does not depend on the choice of the lift \tilde{f} for f . Hence, if we set $\rho_X^Y(f) \cdot \xi = \rho_X(\tilde{f}) \cdot \xi$, then $(\mathcal{H}_X^Y, \rho_X^Y)$ is a Y -module that we shall call the restriction of (\mathcal{H}_X, ρ_X) to Y . Moreover, if (\mathcal{H}_X, ρ_X) is non degenerate then $(\mathcal{H}_X^Y, \rho_X^Y)$ is also non degenerate.

3.2. Coarse geometry and non degenerate standard modules.

Definition 3.2. Let X be a compact metric space, let (\mathcal{H}_X, ρ_X) and $(\mathcal{H}'_X, \rho'_X)$ be two X -modules, let $T : \mathcal{H}_X \rightarrow \mathcal{H}'_X$ be a bounded operator and let r be a positive number. The operator T is said to have propagation (less than) r if

$$\text{supp } T \subseteq \{(x, x') \in X \times X \text{ such that } d(x, x') \leq r\}.$$

If X is a compact metric space and if (\mathcal{H}_X, ρ_X) is an X -module, the algebra of compact operator $\mathcal{K}(\mathcal{H}_X)$ is then filtered by $(\mathcal{K}(\mathcal{H}_X)_r)_{r>0}$, where $\mathcal{K}(\mathcal{H}_X)_r$ is for any positive number r the set of compact operators on \mathcal{H}_X with propagation less than r . If A is a C^* -algebra, let $A \otimes \mathcal{K}(\mathcal{H}_X)_r$ be the closure of the algebraic tensor product of A and $\mathcal{K}(\mathcal{H}_X)_r$ in the C^* -tensor product $A \otimes \mathcal{K}(\mathcal{H}_X)$, then $A_{\mathcal{H}_X} \stackrel{\text{def}}{=} A \otimes \mathcal{K}(\mathcal{H}_X)$ is filtered by $(A \otimes \mathcal{K}(\mathcal{H}_X)_r)_{r>0}$.

Recall from [10] that if \mathcal{H}_X and \mathcal{H}'_X are two non degenerate standard X -modules, then for every positive number δ , there exists an isometry $V : \mathcal{H}_X \rightarrow \mathcal{H}'_X$ with propagation less than δ . The isometry V gives rise by conjugation for any positive number ε and r with $\varepsilon < 1/100$ to a homomorphism

$$\text{Ad}_{V,A} : A_{\mathcal{H}_X} \rightarrow A_{\mathcal{H}'_X}$$

such that $\text{Ad}_{V,A}(A_{\mathcal{H}_X,r}) \subseteq A_{\mathcal{H}'_X,r+2\delta}$ and hence to a group morphism

$$\text{Ad}_{V,A,*}^{\varepsilon,r} : K_*^{\varepsilon,r}(A_{\mathcal{H}_X}) \rightarrow K_*^{\varepsilon,r+2\delta}(A_{\mathcal{H}'_X}).$$

Lemma 3.3. With notations above, for any isometries $V_1 : \mathcal{H}_X \rightarrow \mathcal{H}'_X$ and $V_2 : \mathcal{H}_X \rightarrow \mathcal{H}'_X$ with propagation less than δ , then

$$\text{Ad}_{V_1,A,*}^{\varepsilon,r} = \text{Ad}_{V_2,A,*}^{\varepsilon,r}.$$

Proof. Let us set for t in $[0, 1]$

$$R_t = \begin{pmatrix} \cos \frac{\pi t}{2} & -\sin \frac{\pi t}{2} \\ \sin \frac{\pi t}{2} & \cos \frac{\pi t}{2} \end{pmatrix}$$

and define then the isometry

$$W_t : \mathcal{H}_X \longrightarrow \mathcal{H}_{X'} \oplus \mathcal{H}_{X'}; \xi \mapsto R_t \operatorname{diag}(V_1, V_2) R_{-t} \cdot (\xi \oplus 0).$$

Then

$$\operatorname{Ad}_{W_t, A} : \mathcal{A}_{\mathcal{H}_X} \longrightarrow M_2(\mathcal{A}_{\mathcal{H}_{X'}})$$

is a homomorphism that increase propagation by 2δ and $(\operatorname{Ad}_{W_t, A})_{t \in [0, 1]}$ is a homomorphism homotopy between $\operatorname{diag}(\operatorname{Ad}_{V_0, A}, 0)$ and $\operatorname{diag}(\operatorname{Ad}_{V_1, A}, 0)$. \square

As consequence, we get

Corollary 3.4. *Let X be a compact metric space, let \mathcal{H}_X and \mathcal{H}'_X be two non degenerate standard X -modules. For any positive number r , choose $V_r : \mathcal{H}_X \rightarrow \mathcal{H}'_X$ a partial isometry with propagation less than $r/2$. Then*

- (i) $\Lambda_{\mathcal{H}_X, \mathcal{H}'_X, *} \stackrel{\text{def}}{=} (\operatorname{Ad}_{V_r, A, *})_{0 < \varepsilon < 1/4, r > 0} : K_*(\mathcal{A}_{\mathcal{H}_X}) \rightarrow K_*(\mathcal{A}_{\mathcal{H}'_X})$ is a $(1, 2)$ -controlled morphism that not depend on the choice of the family $(V_r)_{r > 0}$.
- (ii) $\Lambda_{\mathcal{H}_X, \mathcal{H}'_X, *}$ is $(1, 4)$ -invertible.

Let X and Y be two compact metric spaces, let \mathcal{H}_X be a standard non degenerate X -module, let \mathcal{H}_Y be a standard non degenerate Y -module and let $f : X \rightarrow Y$ be a C -Lipschitz map. Consider $\mathcal{H}_X \oplus \mathcal{H}_Y$ as a Y -module by using the homomorphism

$$C(Y) \rightarrow C(X) \oplus C(Y); h \mapsto (h \circ f) \oplus h.$$

Then $\mathcal{H}_X \oplus \mathcal{H}_Y$ is a standard non degenerate Y -module. For δ a positive number, fix an isometry $V : \mathcal{H}_X \oplus \mathcal{H}_Y \rightarrow \mathcal{H}_Y$ with propagation less than δ . Define then the partial isometry

$$V_f : \mathcal{H}_X \rightarrow \mathcal{H}_Y; \xi \mapsto V(\xi \oplus 0).$$

The support of V_f lies in

$$\{(x, y) \in X \times Y \text{ such that } d(y, f(x)) < \delta\}.$$

For any C^* -algebra A , let $\operatorname{Ad}_{V_f, A} : \mathcal{A}_{\mathcal{H}_X} \rightarrow \mathcal{A}_{\mathcal{H}_Y}$ the element obtained by conjugation. According to Lemma 3.3, we see that

$$\operatorname{Ad}_{V_f, A, *} : K_*^{\varepsilon, r}(\mathcal{A}_{\mathcal{H}_X}) \rightarrow K_*^{\varepsilon, Cr+2\delta}(\mathcal{A}_{\mathcal{H}_Y})$$

does not depend on the partial isometry $V : \mathcal{H}_X \oplus \mathcal{H}_Y \rightarrow \mathcal{H}_Y$. In particular, for $\delta = Cr/2$ we obtain then a family of morphisms

$$f_{A, *}^{\sharp, \varepsilon, r} : K_*^{\varepsilon, r}(\mathcal{A}_{\mathcal{H}_X}) \rightarrow K_*^{\varepsilon, 2Cr}(\mathcal{A}_{\mathcal{H}_Y})$$

compatible with the structure map and hence

$$f_{A,*}^\# \stackrel{\text{def}}{=} (f_{A,*}^{\#, \varepsilon, r})_{0 < \varepsilon < 1/100} : \mathcal{K}_*(A_{\mathcal{H}_X}) \rightarrow \mathcal{K}_*(A_{\mathcal{H}_Y}) \quad (6)$$

is a $(1, 2C)$ -controlled morphism.

Remark 3.5.

- (i) In view of the proof of Lemma 3.3 and with the above notations, for any positive number r and any partial isometry $V_r : \mathcal{H}_X \rightarrow \mathcal{H}_Y$ with support in

$$\{(x, y) \in X \times Y \text{ such that } d(y, f(x)) < Cr/2\},$$

then we have $Ad_{V_r, A, *}^{\varepsilon, r} = f_{A,*}^{\#, \varepsilon, r}$.

- (ii) Let X be a compact metric space and let Y be a closed subset of X . Let (\mathcal{H}_X, ρ_X) be a non degenerate standard module such that $(\mathcal{H}_X^Y, \rho_X^Y)$ (the restriction of (\mathcal{H}_X, ρ_X) to Y) is standard. Let $j_{X,Y} : Y \hookrightarrow X$ be the inclusion. On the other hand, let $j_{X,Y}^\# : \mathcal{K}(\mathcal{H}_X^Y) \hookrightarrow \mathcal{K}(\mathcal{H}_X)$ be the inclusion of $\mathcal{K}(\mathcal{H}_X^Y)$ as a corner in $\mathcal{K}(\mathcal{H}_X)$ via the orthogonal decomposition $\mathcal{H}_X = \mathcal{H}_X^Y \oplus^\perp \mathcal{H}'$. Then the induced morphism $j_{X,Y,A}^\# : A_{\mathcal{H}_X^Y} \hookrightarrow A_{\mathcal{H}_X}$ is for any C^* -algebra A a morphism of filtered C^* -algebras. The morphism induced in quantitative K -theory by $j_{X,Y,A}^\#$ coincide up to a rescaling of the propagation by 2 with the morphism constructed in equation (6) from the isometric inclusion $j_{X,Y}$.

Let X, Y and Z be compact metric spaces, let $\mathcal{H}_X, \mathcal{H}_Y$ and \mathcal{H}_Z be respectively standard non degenerate standard X -module, Y -module and Z -module and let $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ be respectively C -Lipschitz and C' -Lipschitz maps. Then we can easily check that

$$(g \circ f)_{A,*}^\# \stackrel{(1, 4CC')}{\sim} g_{A,*}^\# \circ f_{A,*}^\# \quad (7)$$

for any C^* -algebra A . Let X and Y be two compact metric spaces and let C be a positive number.

Definition 3.6. A family of maps $(g_t)_{t \in [0,1]}$ with $g_t : X \rightarrow Y$ continuous for all t in $[0, 1]$ is a C -Lipschitz homotopy if

- $[0, 1] \times X \rightarrow Y; (t, x) \mapsto g_t(x)$ is continuous;
- g_t is C -Lipschitz for all t in $[0, 1]$.

Proposition 3.7. Let X and Y be two compact metric spaces, let C be a positive number and let $(g_t)_{t \in [0,1]}$ be a C -Lipschitz homotopy. Then for any C^* -algebra A , any non degenerate standard X -module \mathcal{H}_X and any non degenerate standard Y -module \mathcal{H}_Y , we have

$$g_{0,A,*}^\# = g_{1,A,*}^\#.$$

Proof. Let δ be a positive number with $\delta < r/2$ and pick

$$t_0 = 1 < t_1 < \dots < t_{n-1} < t_n = 1$$

such that $d(g_{t_i}(x), g_{t_{i-1}}(x)) < \delta/4$ for all $i = 1, \dots, n$ and all x in X . For every $i = 1, \dots, n$, choose $V_i : \mathcal{H}_X \rightarrow \mathcal{H}_Y$ an isometry with support in

$$\{(x, y) \in X \times Y \text{ such that } d(g_{t_i}(x), y) < \delta/4\}.$$

Let us set for t in $[0, 1]$

$$R_t = \begin{pmatrix} \cos \frac{\pi t}{2} & -\sin \frac{\pi t}{2} \\ \sin \frac{\pi t}{2} & \cos \frac{\pi t}{2} \end{pmatrix}$$

and define then the isometry

$$W_t : \mathcal{H}_X \longrightarrow \mathcal{H}_Y \oplus \mathcal{H}_Y; \xi \mapsto R_{\frac{t-t_i}{t_{i+1}-t_i}} \text{diag}(V_i, V_{i+1}) R_{\frac{t-t_i}{t_{i+1}-t_i}}^* \cdot (\xi \oplus 0)$$

for $t_i \leq t \leq t_{i+1}$ and $1 \leq i \leq n-1$. Then W_t has for all t in $[0, 1]$ support in

$$\{(x, y) \in X \times Y \text{ such that } d(g_t(x), y) < \delta/2\}.$$

Hence W_t induces by conjugation a homomorphism

$$\text{Ad}_{W_t, A} : A_{\mathcal{H}_X} \longrightarrow M_2(A_{\mathcal{H}_Y})$$

such that

$$\text{Ad}_{W_t, A}(\mathcal{A}_{\mathcal{H}_X, r}) \subseteq M_2(A_{\mathcal{H}_Y, Cr+2\delta}).$$

Moreover $(\text{Ad}_{W_t, A})_{t \in [0, 1]}$ is a homotopy of homomorphisms between $\text{diag}(\text{Ad}_{V_0, A}, 0)$ and $\text{diag}(\text{Ad}_{V_1, A}, 0)$ which respectively implement $g_{0, A, *}^{\sharp, \varepsilon, r}$ and $g_{1, A, *}^{\sharp, \varepsilon, r}$. \square

3.3. Fundamental class. Let us now define the fundamental class in $K_0^{\varepsilon, r}(C(X) \otimes \mathcal{H}_X)$ for a compact metric space X and for any ε in $(0, 1/100)$ and any positive number r . This class plays the role of the class in K -theory of a rank one projection. Let us first consider μ a borelian measure on X such that $\text{supp } \mu = X$. Let r be a positive number and let us choose a continuous function $\phi : X \times X \rightarrow \mathbb{R}$ supported in

$$\{(x, y) \in X \times X \text{ such that } d(x, y) < r/2\}$$

and such that for any x in X , then

$$\int \phi(x, y) d\mu(y) = 1.$$

Consider the operator $E_\phi : C(X) \otimes L^2(X, \mu) \longrightarrow C(X) \otimes L^2(X, \mu)$ defined by

$$E_\phi \cdot \xi(x, y) = \phi^{1/2}(x, y) \int_X \phi^{1/2}(x, z) \xi(x, z) d\mu(z).$$

Then E_ϕ is a projection in $C(X) \otimes \mathcal{K}(L^2(X, \mu))$ with propagation less than r . Indeed, $E_\phi(x)$ is supported in $B(x, r/2) \times B(x, r/2)$ for every x in X . If ϕ and ϕ' are two continuous functions supported in

$$\{(x, y) \in X \times X; \text{ such that } d(x, y) < r/2\}$$

and such that

$$\int \phi(x, y) d\mu(y) = \int \phi'(x, y) d\mu(y) = 1$$

for any x in X , then $(E_{(1-t)\phi+t\phi'})_{t \in [0,1]}$ is a homotopy of projections with propagation less than r between E_ϕ and $E_{\phi'}$.

Let \mathcal{H}_X be a non degenerate standard X -module, let \mathcal{H} be a separable Hilbert space and let us fix e_0 a rank one projection on \mathcal{H} . For any positive numbers r , let $V : L^2(X, \mu) \rightarrow \mathcal{H}_X$ be an isometry with propagation less than $r/4$. Let E_ϕ the projection constructed as above with $\phi : X \times X \rightarrow \mathbb{R}$ supported in

$$\{(x, y) \in X \times X; \text{ such that } d(x, y) < r/4\}.$$

According to Lemma 3.3, for any positive number ε with $\varepsilon < 1/100$, the element

$$[E_X]_{\varepsilon, r} \stackrel{\text{def}}{=} Ad_{V, *}^{\varepsilon, r/2}([E_\phi \otimes e_0, 0]_{\varepsilon, r/2}) \quad (8)$$

in $K_0^{\varepsilon, r}(C(X) \otimes \mathcal{H}_X)$ does not depend on the choices of V and of ϕ . It does not depend neither on e_0 . Moreover, for any positive number $\varepsilon, \varepsilon', r$ and r' with $\varepsilon \leq \varepsilon' < 1/100$ and $r < r'$, then we have $\iota_{*, \varepsilon', r, r'}^{\varepsilon, \varepsilon'}([E_X]_{\varepsilon, r}) = [E_X]_{\varepsilon', r'}$. For any continuous map $f : X \rightarrow Y$ where X and Y are compact sets, any filtered C^* -algebra A and any positive numbers ε and r with $\varepsilon < 1/4$, Let

$$f_A^{*, \varepsilon, r} : K_*^{\varepsilon, r}(C(Y) \otimes A) \rightarrow K_*^{\varepsilon, r}(C(X) \otimes A)$$

be the morphism induced by $C(Y) \rightarrow C(X) : h \mapsto h \circ f$.

Remark 3.8. Let X be a compact metric space, let Y be closed subset of A and let (\mathcal{H}_X, ρ_X) be a non degenerate standard X -module with restriction to \mathcal{H}_X^Y non degenerate. Then, with notations of point (ii) of Remark 3.5, we have

$$j_{X, Y, C(Y), *}^{\sharp, \varepsilon, r}[E_Y]_{\varepsilon, r} = j_{X, Y, \mathcal{H}(\mathcal{H}_X)}^*[E_X]_{\varepsilon, r}.$$

This identity can indeed be extended to C -Lipschitz maps.

Lemma 3.9. *Let X and Y be two compact metric spaces, let \mathcal{H}_X and \mathcal{H}_Y be respectively non degenerate X -module and Y -modules. Let $f : X \rightarrow Y$ be a C -Lipschitz map for some positive numbers C . Then for any positive numbers ε and r with ε in $(0, 1/100)$, we have*

$$f_{C(X), *}^{\sharp, \varepsilon, r}[E_X]_{\varepsilon, r} = f_{\mathcal{H}(\mathcal{H}_Y)}^*[E_Y]_{\varepsilon, 2Cr}.$$

Proof. Let μ_X and μ_Y be respectively borelian measures on X and Y such that $\text{supp } \mu_X = X$ and $\text{supp } \mu_Y = Y$. Let us choose $\phi_0 : X \times X \rightarrow [0, 1]$ and $\phi_1 : Y \times Y \rightarrow [0, 1]$ continuous functions such that

- ϕ_0 is supported in $\{(x, x') \in X \times X \text{ such that } d(x, x') < \frac{r}{2C}\}$;
- ϕ_1 is supported in $\{(y, y') \in Y \times Y \text{ such that } d(y, y') < r/2\}$;
- $\int_X \phi_0(x, x') d\mu_X(x') = 1$ for any x in X ;
- $\int_Y \phi_1(y, y') d\mu_Y(y') = 1$ for any y in Y .

Define for any x in X

$$\phi_0(x, \bullet) : X \rightarrow [0, 1]; x' \mapsto \phi_0(x, x')$$

and for any y in Y

$$\phi_1(y, \bullet) : Y \rightarrow [0, 1]; y' \mapsto \phi_1(y, y').$$

For any x in X then $\phi_0^{1/2}(x, \bullet)$ is a unit vector in $L^2(X, \mu_X)$ and for any y in Y then $\phi_1^{1/2}(y, \bullet)$ is a unit vector in $L^2(Y, \mu_Y)$.

Let us consider for any t in $[0, 1]$ the $C(X)$ -linear operator

$$E_t : C(X) \otimes (L^2(X, \mu_X) \oplus L^2(Y, \mu_Y)) \longrightarrow C(X) \otimes (L^2(X, \mu_X) \oplus L^2(Y, \mu_Y))$$

which is pointwise at x in X the rank one projection associated to the unit vector

$$t^{1/2} \phi_0^{1/2}(x, \bullet) \oplus (1-t)^{1/2} \phi_1^{1/2}(f(x), \bullet)$$

in $L^2(X, \mu_X) \oplus L^2(Y, \mu_Y)$. If we view $L^2(X, \mu_X) \otimes L^2(Y, \mu_Y)$ as a Y -module acted upon for any h in $C(Y)$ by multiplication $(h \circ f) \oplus h$, then $E_t(x)$ is supported in $\{y \in Y \text{ such that } d(f(x), y) < r/2\}^2$ for every x in X . Hence E_t is for all t in $[0, 1]$ a projection of $C(X) \otimes \mathcal{K}(L^2(X, \mu_X) \otimes L^2(Y, \mu_Y))$ with propagation less than r . Moreover,

- E_0 is the image of $E_{\phi_0} \in C(X) \otimes \mathcal{K}(L^2(X, \mu_X))$ under conjugation by the (isometrical) inclusion $L^2(X, \mu_X) \hookrightarrow L^2(X, \mu_X) \oplus L^2(Y, \mu_Y)$;
- the pull-back of E_1 by f (which belongs to $C(Y) \otimes (\mathcal{K}(L^2(X, \mu_X) \oplus L^2(Y, \mu_Y)))$) is the image of $E_{\phi_1} \in C(Y) \otimes \mathcal{K}(L^2(Y, \mu_Y))$ under conjugation by the inclusion $L^2(Y, \mu_Y) \hookrightarrow L^2(X, \mu_X) \oplus L^2(Y, \mu_Y)$

Using the definition of $f_{C(X),*}^{\sharp, \varepsilon, r}[E_X]_{\varepsilon, r}$, the result is now a consequence of Corollary 3.4. \square

3.4. Quantitative index map. First, let us fix some notations. Let X be a compact space, let (ρ_X, \mathcal{H}_X) be an X -module and let A be a C^* -algebra. Recall that we have set $\widetilde{A_{\mathcal{H}_X}} \stackrel{\text{def}}{=} A \otimes \mathcal{K}(\mathcal{H}_X)$. If A is unital, we can identify the unitarisation $\widetilde{A_{\mathcal{H}_X}}$ of $A_{\mathcal{H}_X}$ with the algebra $A \otimes \mathcal{K}(\mathcal{H}_X) + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X}$ (viewed as adjointable operators on the right Hilbert A -module $A \otimes \mathcal{H}_X$). If A is non unital, with unitarisation \widetilde{A} , then we identify $\widetilde{A_{\mathcal{H}_X}}$ with the subalgebra

$$A \otimes \mathcal{K}(\mathcal{H}_X) + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X}$$

of

$$\widetilde{A} \otimes \mathcal{K}(\mathcal{H}_X) + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X}.$$

If B is a C^* -algebra and $f : A \rightarrow B$ is a $*$ -homomorphism, then $f_{\mathcal{H}_X} : A_{\mathcal{H}_X} \rightarrow B_{\mathcal{H}_X}$ stands for the $*$ -homomorphism induced by f .

If z is an element in $KK_*(A, B)$ then $\tau_{\mathcal{H}_X}(x)$ stands for the element $\tau_{\mathcal{K}(\mathcal{H}_X)}(z)$ of $KK_*(A_{\mathcal{H}_X}, B_{\mathcal{H}_X}) = KK_*(A \otimes \mathcal{K}(\mathcal{H}_X), B \otimes \mathcal{K}(\mathcal{H}_X))$ and in the same way

$$\mathcal{T}_{\mathcal{H}_X}(x) = \left(\tau_{\mathcal{H}_X}^{\varepsilon, r}(x) \right)_{\varepsilon \in (0, \frac{1}{100\alpha}), r > 0}$$

stands for

$$\mathcal{T}_{\mathcal{K}(\mathcal{H}_X)}(x) = \left(\tau_{\mathcal{H}_X}^{\varepsilon, r}(x) \right)_{\varepsilon \in (0, \frac{1}{100\alpha}), r > 0},$$

with $(\alpha, k) = (\alpha_{\mathcal{T}}, k_{\mathcal{T}})$.

We are now in a position to define the quantitative index map. For X a compact space and A a C^* -algebra, we set $K_*(X, A) \stackrel{\text{def}}{=} KK_*(C(X), A)$.

Definition 3.10. *Let X be a compact metric space, let (\mathcal{H}_X, ρ_X) be a non degenerate standard X -module and let A be a C^* -algebra. For any positive number ε and r with $\varepsilon < 1/100$, the ε - r -index is the map*

$$\begin{aligned} \text{Ind}_{X, A, *}^{\varepsilon, r} : K_*(X, A) &\longrightarrow K_*^{\varepsilon, r}(A_{\mathcal{H}_X}) \\ x &\longmapsto \tau_{\mathcal{H}_X}^{\varepsilon/\alpha, r/k_{\varepsilon/\alpha}}(x)([E_X]_{\varepsilon/\alpha, r/k_{\varepsilon/\alpha}}) \end{aligned}$$

with $(\alpha, k) = (\alpha_{\mathcal{T}}, k_{\mathcal{T}})$.

It is straightforward to check that

$$\iota_*^{\varepsilon, \varepsilon', r, r'} \circ \text{Ind}_{X, A, *}^{\varepsilon, r} = \text{Ind}_{X, A, *}^{\varepsilon', r'}$$

for any positive number $\varepsilon, \varepsilon', r$ and r' with $\varepsilon < \varepsilon' < 1/100$ and $r < r'$ and hence

$$\text{Ind}_{X, A, *} \stackrel{\text{def}}{=} (\text{Ind}_{X, A, *}^{\varepsilon, r})_{0 < \varepsilon < 1/100, r > 0} : K_*(X, A) \longrightarrow \mathcal{K}_*(A_{\mathcal{H}_X})$$

is a projective morphism in the sense of Definition 1.15. Furthermore, the class in $K_0(C(X)_{\mathcal{H}_X})$ of the projection E_{ϕ} constructed above is up to Morita equivalence the class of the trivial projection 1 in $K_0(C(X))$ and hence we get that $\iota_*^{\varepsilon, r} \circ \text{Ind}_{X, A, *}^{\varepsilon, r}$ is up to Morita equivalence the usual index map.

Remark 3.11. Let \mathcal{H}_X and \mathcal{H}'_X be two non degenerate standard X modules and let us denote respectively by $\text{Ind}_{X, A}^{\mathcal{H}_X}$ and $\text{Ind}_{X, A}^{\mathcal{H}'_X}$ the quantitative index map relatively to \mathcal{H}_X and \mathcal{H}'_X , then according to Corollary 3.4 and in view of remark 1.27 regarding to the functoriality of \mathcal{T}_{\bullet} , we obtain that

$$\text{Ind}_{X, A, *}^{\mathcal{H}'_X} = \Lambda_{X, \mathcal{H}_X, \mathcal{H}'_X, *} \circ \text{Ind}_{X, A, *}^{\mathcal{H}_X}.$$

By naturality of $\mathcal{T}_{\mathcal{H}_X}$, we have the following functoriality for the controlled index:

Proposition 3.12. *Let X be a compact metric space, let (\mathcal{H}_X, ρ_X) be a non degenerate standard X -module and let $f : A \rightarrow B$ be C^* -algebra homomorphism. Then*

$$\text{Ind}_{X, B, *} \circ f_* = f_{\mathcal{H}_X, *} \circ \text{Ind}_{X, A, *},$$

where $f_{\mathcal{H}_X} : A_{\mathcal{H}_X} \rightarrow B_{\mathcal{H}_X}$ is induced by f .

The quantitative index map is compatible with the rescaling map defined in Section 1.2.

Proposition 3.13. *Let A be a C^* -algebra, let X be a compact metric space and let (\mathcal{H}_X, ρ_X) be a non degenerate standard X -module. Then*

$$L_{A_{\mathcal{H}_X},*}^{\varepsilon,r} \circ \text{Ind}_{X,A,*}^{\varepsilon,r} = \text{Ind}_{X,A,*}^{\varepsilon/2, l_{\varepsilon} r}$$

for any ε in $(0, 1/200)$ (with $l = l_{rs}$ as in Section 1.2).

Proof. Let E_X as in Equation (8). Since E_X is a projection, we have that

$$L_{C(X)\mathcal{H}_X,*}^{\varepsilon,r}([E_X, 0]_{\varepsilon,r}) = [E_X, 0]_{\varepsilon/2, l_{\varepsilon} r}$$

for any ε in $(0, 1/200)$ and any $r > 0$. The result is then a consequence of Proposition 1.28. \square

Moreover, we have the following compatibility result for the quantitative index with respect to Kasparov product.

Proposition 3.14. *Let X be a compact metric space, let (\mathcal{H}_X, ρ_X) be a non degenerate standard X -module, let A and B be C^* -algebras and let z be an element in $KK_*(A, B)$. Then*

$$\text{Ind}_{X,B,*} \circ (\bullet \otimes z) = \mathcal{T}_{\mathcal{H}_X}(z) \circ \text{Ind}_{X,A,*},$$

where $\bullet \otimes z : K_*(X, A) \rightarrow K_*(X, B)$ is the right Kasparov product by z .

Proof. For propagation small enough, this is a consequence of Theorem 1.25. The general case follows then from the first point of Remark 1.16. \square

As a consequence of Lemma 3.9, of point (iii) of Theorem 1.24 and of Remark 1.27, we see that the quantitative index is also functorial in spaces.

Lemma 3.15. *Let X and Y be two compact metric spaces, let \mathcal{H}_X and \mathcal{H}_Y be respectively non degenerate X -module and Y -module. Let $f : X \rightarrow Y$ be a C -Lipschitz map for some positive number C and denote by $f_* : K_*(X, A) \rightarrow K_*(Y, A)$ the induced morphism. Then we have*

$$\text{Ind}_{Y,A,*} \circ f_* = f_{A,*}^{\sharp} \circ \text{Ind}_{X,A,*}.$$

Remark 3.16. In the special case of inclusion, let X be a compact metric space, let Y be a closed subset of A and let (\mathcal{H}_X, ρ_X) be a non degenerate standard X -module with standard restriction to \mathcal{H}_X^Y . Then with notations point ii) of Remark 3.5 and according to Remark 3.8, we have

$$\text{Ind}_{X,A,*} \circ j_{X,Y,*} = j_{X,Y,A,*}^{\sharp} \circ \text{Ind}_{Y,A,*}.$$

We are now in a position to state the main theorem of this section: for a finite simplicial complex, then the quantitative index is a h -isomorphism in the sense of Definition 1.17 with h depending only on the dimension of the complex.

Let X be a finite simplicial complex provided with the simplicial topology. We equip X with the normalized spherical metric (see [23, Section 7.2]).

- Let Δ be a (geometrical) simplex of X of dimension n with vertices x_0, \dots, x_n . Let x be an element of Δ with barycentric coordinates (t_0, \dots, t_n) with respect to x_0, \dots, x_n and set

$$f_\Delta(x) = \frac{1}{(t_0^2 + \dots + t_n^2)^{1/2}}(t_0, \dots, t_n).$$

Then $f_\Delta(x)$ belongs to the unit n -sphere. We define d on Δ by

$$\cos \frac{\pi}{2} d(x, y) = \langle f_\Delta(x), f_\Delta(y) \rangle$$

for any x and y in Δ .

- If x and y belong to the same connected component of X , we set $d(x, y) = \inf\{d(x_0, x_1) + \dots + d(x_{n-1}, x_n)\}$, where the infimum is taken over sequence $x = x_0, x_1, \dots, x_{n-1}, x_n = y$ such that x_i and x_{i+1} lie for $i = 0, \dots, n-1$ in a common simplex;
- If x and y lie in distinct connected components, we set $d(x, y) = +\infty$.

Theorem 3.17. *There exists a non increasing function*

$$(0, 1/200) \rightarrow [1, +\infty); \varepsilon \mapsto h_\varepsilon$$

such that for any finite simplicial complex X of dimension less than m , any non degenerate standard X -module \mathcal{H}_X and any C^ -algebra A , then*

$$\text{Ind}_{X,A,*} : K_*(X, A) \longrightarrow \mathcal{K}_*(A_{\mathcal{H}_X})$$

is a $h^{(m^2)}$ -isomorphism (in sense of definition 1.17).

This theorem will be proved in Section 4.5.

Corollary 3.18. *Let X be a finite simplicial complex. Then for any non degenerate standard X -module \mathcal{H}_X , any C^* -algebra A and any fixed ε in $(0, 1/200)$, the map*

$$\begin{aligned} K_*(X, A) &\longrightarrow \varprojlim_{r>0} K_*(A_{\mathcal{H}_X}) \\ x &\mapsto (\text{Ind}_{X,A}^{\varepsilon,r})_{r>0} \end{aligned}$$

is an isomorphism.

We end this section by showing that the quantitative assembly maps for finite metric spaces can indeed be recovered from the quantitative index. Let Σ be a finite metric space. For any integer d , we equipped $P_d(\Sigma)$ the Rips complex of degree d of Σ with the normalized spherical metric. Let $\mathcal{H}_{\Sigma,d}$ be a non degenerate standard $P_d(\Sigma)$ -module. Consider then the isometry

$$V_{\mathcal{H}_{\Sigma,d}} : \mathcal{H}_{\Sigma,d} \longrightarrow \ell^2(\Sigma, \mathcal{H}_{\Sigma,d}) \quad (9)$$

defined by

$$V_{\mathcal{H}_{\Sigma,d}} \cdot \xi(\sigma) = \lambda_\sigma^{1/2} \cdot \xi$$

for all ξ in $\mathcal{H}_{\Sigma,d}$ and σ in Σ . We equip $\ell^2(\Sigma, \mathcal{H}_{\Sigma,d})$ with the Σ -module structure given by pointwise multiplication by $C(\Sigma)$.

Lemma 3.19. *If T is an element of $A_{\mathcal{H}_{\Sigma,d}}$ with propagation r , then*

$$Ad_{V_{\mathcal{H}_{\Sigma,d},A}} \cdot T \stackrel{\text{def}}{=} (Id_A \otimes V_{\mathcal{H}_{\Sigma,d}}) \cdot T \cdot (Id_A \otimes V_{\mathcal{H}_{\Sigma,d}}^*)$$

has propagation $2d(r+1)$.

Proof. A pair $(\sigma, \sigma') \in \Sigma^2$ is in the support of $Ad_{V_{\mathcal{H}_{\Sigma,d},A}}$ if and only if there exists (x, x') in the support of T such that σ is in the support of x and σ' is in the support of x' . The result is then a consequence of [23, Corollary 7.2.6]. \square

Hence, in view of Proposition 1.5, for every positive numbers ε and r with $\varepsilon < 1/100$, then $Ad_{V_{\mathcal{H}_{\Sigma,d},A}}$ induces a morphism

$$Ad_{V_{\mathcal{H}_{\Sigma,d},A,*}}^{\varepsilon,r} : K_*^{\varepsilon,r}(A_{\mathcal{H}_{\Sigma,d}}) \longrightarrow K_*^{\varepsilon,2d(r+1)}(A_{\ell^2(\Sigma, \mathcal{H}_d)}) \xrightarrow{\cong} K_*^{\varepsilon,2d(r+1)}(A_\Sigma)$$

(recall that $A_\Sigma = A \otimes \mathcal{K}(\ell^2(\Sigma))$) such that

$$\iota_*^{\varepsilon,2d(r+1),2d(r'+1)} \circ Ad_{V_{\mathcal{H}_{\Sigma,d},A,*}}^{\varepsilon,r} = Ad_{V_{\mathcal{H}_{\Sigma,d},A,*}}^{\varepsilon,r'} \circ \iota_*^{\varepsilon,r,r'}. \quad (10)$$

for every positive number r' with $r' \geq r$

Proposition 3.20. *Let Σ be a finite metric space, let A be a C^* -algebra and let ε , d and r be positive numbers with $\varepsilon < 1/100$ and $r \geq k_{\mathcal{T},\varepsilon/\alpha_{\mathcal{T}}} - 1$. Then for any non degenerate standard $P_d(\Sigma)$ -module $\mathcal{H}_{\Sigma,d}$, we have up to Morita equivalence*

$$\nu_{\Sigma,A,*}^{\varepsilon,2d(r+1),d} = Ad_{V_{\mathcal{H}_{\Sigma,d},A,*}}^{\varepsilon,r} \circ Ind_{P_d(\Sigma),A,*}^{\varepsilon,r}.$$

Proof. Let us first assume that $\mathcal{H}_{\Sigma,d} = L^2(P_d(\Sigma), \mu) \otimes \mathcal{H}$, where \mathcal{H} is a separable Hilbert space and μ is a borelian measure with $\text{supp } \mu = X$. For sake of simplicity, let us forget the factor \mathcal{H} . Notice that since $r_{d,\varepsilon} = 2dk_{\mathcal{T},\varepsilon/\alpha_{\mathcal{T}}}$, we deduce that $2d(r+1) \geq r_{d,\varepsilon}$. Then according to Remark 1.27, to equation (10) and to the definitions of $\nu_{\Sigma,A,*}^{\varepsilon,2d(r+1),d}$ and $Ind_{P_d(\Sigma),A,*}^{\varepsilon,r}(P_d(\Sigma), A)$, this amounts to show with notations of Sections 2.1 and 3.3 that for any positive number r , for any continuous map

$$\phi : P_d(\Sigma) \times P_d(\Sigma) \rightarrow \mathbb{R}$$

supported in

$$\{(x, y) \in P_d(\Sigma) \times P_d(\Sigma) \text{ such that } d(x, y) < r/2\}$$

and such that $\int_{P_d(\Sigma)} \phi(x, y) d\mu(y) = 1$ and for any continuous function $\psi : P_d(\Sigma) \rightarrow [0, 1]$ such that $\int_{P_d(\Sigma)} \psi(x) d\mu(x) = 1$, then

$$Ad_{V_{\mathcal{H}_{\Sigma,d}, C(P_d(\Sigma)), *}}^{\varepsilon, r}[E_\phi, 0]_{\varepsilon, r} = [e_\psi \otimes P_{P_d(\Sigma)}]_{\varepsilon, 2d(r+1)},$$

where $e_\psi : L^2(P_d(\Sigma), \mu) \rightarrow L^2(P_d(\Sigma), \mu)$ is the orthogonal projection corresponding to the unit vector $\psi^{1/2} \in L^2(P_d(\Sigma), \mu)$. Let us view $\ell^2(\Sigma, L^2(P_d(\Sigma), \mu))$ as functions $P_d(\Sigma) \times \Sigma \rightarrow \mathbb{C}$. Define then

$$\varphi_0(x, \bullet, \bullet) : P_d(\Sigma) \times \Sigma \rightarrow \mathbb{R}; (y, \sigma) \mapsto \phi(x, y) \lambda_\sigma(y).$$

and for any t in $[0, 1]$

$$\varphi_t(x, \bullet, \bullet) : P_d(\Sigma) \times \Sigma \rightarrow \mathbb{R}; (y, \sigma) \mapsto (1 - t)\varphi_0(x, y, \sigma) + t\lambda_\sigma(x)\psi(y).$$

Then $Ad_{V_{\mathcal{H}_{\Sigma,d}, C(X)}}(E_\phi)$ is pointwise at $x \in P_d(\Sigma)$ the rank one projection associated to the unit vector $\varphi_0^{1/2}(x, \bullet, \bullet)$ in $\ell^2(\Sigma, L^2(P_d(\Sigma), \mu))$. For every t in $[0, 1]$ and x in $P_d(\Sigma)$, the projection associated to the unit vector $\varphi_t^{1/2}(x, \bullet, \bullet)$ in $\ell^2(\Sigma, L^2(P_d(\Sigma), \mu))$ has propagation less than $2d(r+1)$. Hence we get in this way a homotopy of projection of propagation less than $2d(r+1)$ in $C(P_d(\Sigma)) \otimes \mathcal{K}(L^2(P_d(\Sigma), \mu) \otimes \ell^2(\Sigma))$ between $Ad_{V_{\mathcal{H}_{\Sigma,d}, C(X)}}(E_\phi)$ and $e_\psi \otimes P_{P_d(\Sigma)}$.

In the general case, we use Remark 3.11 and the argument of the proof of Lemma 3.3 to reduce to the previous case. \square

4. MAYER-VIETORIS SIX TERM EXACT SEQUENCES IN K -HOMOLOGY AND IN QUANTITATIVE K -THEORY

4.1. Mayer-Vietoris exact sequence in K -homology. Let (X, d) be a compact metric space, let X_1 and X_2 be closed subsets of X such that $X = \overset{\circ}{X}_1 \cup \overset{\circ}{X}_2$ and let A be a C^* -algebra. Then we have the following six-term exact sequence [2, Theorem 21.5.1]:

$$\begin{array}{ccccc} K_0(X_1 \cap X_2, A) & \xrightarrow{(j_{X_1, X_2, *}, j_{X_1, X_2, *})} & K_0(X_1, A) \oplus K_0(X_2, A) & \xrightarrow{j_{X_1, *}, -j_{X_2, *}} & K_0(X, A) \\ \partial_{X_1, X_2, *} \uparrow & & & & \downarrow \partial_{X_1, X_2, *} \\ K_1(X, A) & \xleftarrow{j_{X_1, *}, -j_{X_2, *}} & K_1(X_1, A) \oplus K_1(X_2, A) & \xleftarrow{(j_{X_1, X_2, *}, j_{X_2, X_1, *})} & K_1(X_1 \cap X_2, A) \end{array} \quad (11)$$

where

- j_{X_1} , j_{X_2} , j_{X_1, X_2} and j_{X_2, X_1} are respectively induced by the restrictions to $X_1 \subseteq X$, $X_2 \subseteq X$, $X_1 \cap X_2 \subseteq X_1$ and $X_1 \cap X_2 \subseteq X_2$;
- the boundary maps in K -homology $\partial_{X_1, X_2, *}$ are given by left multiplication by an element $[\partial_{X_1, X_2}]$ in $KK_1(C(X_1 \cap X_2), C(X))$ canonically associated to the pair (X_1, X_2) and that we shall describe from [2, Section 21.2] next.

Let $C(X_1, X_2)$ be the algebra of continuous functions

$$f : ([0, 1/2] \times X_1) \cup ([1/2, 1] \times X_2) \rightarrow \mathbb{C}$$

(where $([0, 1/2] \times X_1) \cup ([1/2, 1] \times X_2) \subseteq [0, 1] \times X$ is equipped with the induced topology) such that

- $f(0, x) = 0$ for all x in X_1 ;
- $f(1, x) = 0$ for all x in X_2 .

Let $SC(X_1 \cap X_2)$ be the suspension algebra of $C(X_1 \cap X_2)$, i.e the algebra of continuous functions $f : [0, 1] \times (X_1 \cap X_2) \rightarrow \mathbb{C}$ such that $f(0, x) = f(1, x) = 0$ for all x in $X_1 \cap X_2$. By restriction to

$$[0, 1] \times (X_1 \cap X_2) \subseteq ([0, 1/2] \times X_1) \cup ([1/2, 1] \times X_2),$$

we get a surjective $*$ -homomorphism

$$\Psi_{X_1, X_2} : C(X_1, X_2) \rightarrow SC(X_1 \cap X_2)$$

with contractible kernel (the direct sum of two cone algebras) and hence Ψ_{X_1, X_2} induces an invertible element $[\Psi_{X_1, X_2}]$ in $KK_*(C(X_1, X_2), SC(X_1 \cap X_2))$. The restriction to

$$\{1/2\} \times X \subseteq ([0, 1/2] \times X_1) \cup ([1/2, 1] \times X_2)$$

gives rise to a $*$ -homomorphism

$$\Psi'_{X_1, X_2} : C(X_1, X_2) \rightarrow C(X).$$

Let $[\partial]$ be the element of $KK_1(\mathbb{C}, C(0, 1))$ that corresponds to the boundary element of the extension

$$0 \longrightarrow C(0, 1) \longrightarrow C(0, 1) \xrightarrow{ev_1} \mathbb{C} \longrightarrow 0,$$

(ev_1 being the evaluation at 1). Then we have (see [2, Section 21.2])

$$[\partial_{X_1, X_2}] = \tau_{C(X_1 \cap X_2)}([\partial]) \otimes \Psi'_{X_1, X_2, *} [\Psi_{X_1, X_2, *}]^{-1}. \quad (12)$$

4.2. Controlled Mayer-Vietoris pair. We refer to [17, Section 2] for the definition of Complete Intersection Approximation (CIA) property and of controlled Mayer-Vietoris pairs.

Let us introduce some further notations. Let (X, d) be a compact metric space.

- For a closed subset Y of X and δ a positive number, we set

$$Y_\delta = \{x \in Y \text{ such that } d(x, Y) \leq \delta\}.$$

- If (\mathcal{H}_X, ρ_X) is a X -module and Y is a closed subset of X , we set with notations of Section 3.1 $\mathcal{H}_Y = \mathcal{H}_X^Y$;
- If (\mathcal{H}_X, ρ_X) is a X -module, r is a positive number, and X_1 and X_2 are two closed subsets of X , we set

$$\mathcal{K}(\mathcal{H}_{X_1}, \mathcal{H}_{X_2})_r = \mathcal{K}(\mathcal{H}_{X_1}, \mathcal{H}_{X_2}) \cap \mathcal{K}(\mathcal{H}_X)_r.$$

- with above notations, if A is a C^* -algebra, we set $A_{\mathcal{H}_{X_1}, \mathcal{H}_{X_2}, r} = A \otimes \mathcal{K}(\mathcal{H}_{X_1}, \mathcal{H}_{X_2})_r$.

Let X_1 and X_2 be two closed subsets of X such $X = \mathring{X}_1 \cup \mathring{X}_2$. Then we have $\mathcal{H}_X = \mathcal{H}_{X_1} + \mathcal{H}_{X_2}$. Let us fix δ a positive number and let $X_1^{(\delta)}$ and $X_2^{(\delta)}$ be two closed subsets of X such that $X_{1,\delta} \subseteq X_1^{(\delta)}$ and $X_{2,\delta} \subseteq X_2^{(\delta)}$. Then for any positive number r with $r < \delta$

$$\mathcal{K}(\mathcal{H}_X)_r = \mathcal{K}(\mathcal{H}_{X_1}, \mathcal{H}_{X_1^{(\delta)}})_r + \mathcal{K}(\mathcal{H}_{X_2}, \mathcal{H}_{X_2^{(\delta)}})_r,$$

and similarly for any C^* -algebra A ,

$$A_{\mathcal{H}_X, r} = A_{\mathcal{H}_{X_1}, \mathcal{H}_{X_1^{(\delta)}}} + A_{\mathcal{H}_{X_2}, \mathcal{H}_{X_2^{(\delta)}}}.$$

Moreover it is straightforward to check that if $5r < \delta$, then

$$\left(A_{\mathcal{H}_{X_1}, \mathcal{H}_{X_1^{(\delta)}}}, A_{\mathcal{H}_{X_2}, \mathcal{H}_{X_2^{(\delta)}}}, A_{\mathcal{H}_{X_1^{(\delta)}}}, A_{\mathcal{H}_{X_2^{(\delta)}}} \right)$$

is a controlled Mayer-Vietoris pair of order r with coercitivity 1. According to [17, Section 3], then for some universal control pairs $(\alpha, k) = (\alpha_{MV}, k_{MV})$ and (λ, k) , there exists a (α, k) -controlled morphism of order r and of odd degree

$$\mathcal{D}_{X_1, X_2, A, *}^{(\delta)} = (\partial_{X_1, X_2, A, *}^{\varepsilon, s, (\delta)})_{0 < \varepsilon < \frac{1}{100\alpha_D}, 0 < s < \frac{r}{k\varepsilon}} : \mathcal{K}_*(A_{\mathcal{H}_X}) \longrightarrow \mathcal{K}_*(A_{\mathcal{H}_{X_1^{(\delta)}} \cap X_2^{(\delta)}}) \quad (13)$$

such that the following diagram is (λ, h) -exact at order r

$$\begin{array}{ccccc} \mathcal{K}_0(A_{\mathcal{H}_{X_1^{(\delta)}} \cap X_2^{(\delta)}}) & \xrightarrow{(j_{X_1, X_2, A, *}^{\sharp, (\delta)}, j_{X_2, X_1, A, *}^{\sharp, (\delta)})} & \mathcal{K}_0(A_{\mathcal{H}_{X_1^{(\delta)}}}) \oplus \mathcal{K}_0(A_{\mathcal{H}_{X_2^{(\delta)}}}) & \xrightarrow{j_{X_1, A, *}^{\sharp, (\delta)} - j_{X_2, A, *}^{\sharp, (\delta)}} & \mathcal{K}_0(A_{\mathcal{H}_X}) \\ \uparrow \mathcal{D}_{X_1, X_2, A, *}^{(\delta)} & & & & \downarrow \mathcal{D}_{X_1, X_2, A, *}^{(\delta)} \\ \mathcal{K}_1(A_{\mathcal{H}_X}) & \xleftarrow{j_{X_1, A, *}^{\sharp, (\delta)} - j_{X_2, A, *}^{\sharp, (\delta)}} & \mathcal{K}_1(A_{\mathcal{H}_{X_1^{(\delta)}}}) \oplus \mathcal{K}_1(A_{\mathcal{H}_{X_2^{(\delta)}}}) & \xleftarrow{(j_{X_1, X_2, A, *}^{\sharp, (\delta)}, j_{X_2, X_1, A, *}^{\sharp, (\delta)})} & \mathcal{K}_1(A_{\mathcal{H}_{X_1^{(\delta)}} \cap X_2^{(\delta)}}) \end{array}.$$

where

- $j_{X_1, A}^{\sharp, (\delta)} : A_{\mathcal{H}_{X_1^{(\delta)}}} \hookrightarrow A_{\mathcal{H}_X}$;
- $j_{X_2, A}^{\sharp, (\delta)} : A_{\mathcal{H}_{X_2^{(\delta)}}} \hookrightarrow A_{\mathcal{H}_X}$;
- $j_{X_1, X_2, A}^{\sharp, (\delta)} : A_{\mathcal{H}_{X_1^{(\delta)}} \cap X_2^{(\delta)}} \hookrightarrow A_{\mathcal{H}_{X_1^{(\delta)}}}$;
- $j_{X_2, X_1, A}^{\sharp, (\delta)} : A_{\mathcal{H}_{X_1^{(\delta)}} \cap X_2^{(\delta)}} \hookrightarrow A_{\mathcal{H}_{X_2^{(\delta)}}}$;

are the obvious inclusions. By construction, the Mayer-Vietoris controlled boundary map is compatible with the rescaling defined in Section 1.2.

Lemma 4.1. *With above notations and with $(\alpha, k) = (\alpha_{MV}, k_{MV})$ and $l = l_{rs}$ as in Section 1.2, we have*

$$\begin{aligned} \partial_{X_1, X_2, A, *}^{\varepsilon/2, l\varepsilon s, (\delta)} \circ L_{A_{\mathcal{H}_X}, *}^{\varepsilon, s} &= \iota_*^{-, \alpha\varepsilon/2, k\varepsilon/2 l\varepsilon s} \circ L_{A_{\mathcal{H}_{X_1^{(\delta)}} \cap X_2^{(\delta)}}, *}^{\alpha\varepsilon, k\varepsilon s} \circ \partial_{X_1, X_2, A, *}^{\varepsilon, s, (\delta)} \\ &= L_{A_{\mathcal{H}_{X_1^{(\delta)}} \cap X_2^{(\delta)}}, *}^{\alpha\varepsilon, \frac{k\varepsilon/2 l\varepsilon s}{l\alpha\varepsilon}} \circ \partial_{X_1, X_2, A, *}^{\varepsilon, \frac{k\varepsilon/2 l\varepsilon s}{k\varepsilon l\alpha\varepsilon}, (\delta)} \circ \iota_*^{\varepsilon, s, -} \end{aligned}$$

for any ε in $(0, \frac{1}{200\alpha})$ and any s in $(0, \frac{r}{l_\varepsilon k_\varepsilon/2})$.

The fact that the quantitative index intertwines with $\mathcal{D}_{X_1, X_2, A, *}^{(\delta)}$ and $\partial_{X_1^{(\delta)}, X_2^{(\delta)}, *}$ is the key point for the proof of Theorem 3.17 and will be proved in the next two subsections. For that purpose, we need to introduce a coarse Mayer-Vietoris pair for

$$A_{\mathcal{H}_X, \rho_X} \stackrel{\text{def}}{=} A \otimes \mathcal{K}(\mathcal{H}_X) + A \otimes \rho_X(C(X))$$

where A is a C^* -algebra and (\mathcal{H}_X, ρ_X) is a non degenerate standard X -module. Let us consider the following C^* -subalgebras of $A_{\rho_X, \mathcal{H}_X}$:

$$A_{\mathcal{H}_X, \rho_X}^{(1)} = A \otimes \left(\mathcal{K}(\mathcal{H}_{X_1^{(\delta)}}) + \rho_X(C_0(\dot{X}_1)) \right)$$

and

$$A_{\mathcal{H}_X, \rho_X}^{(2)} = A \otimes \left(\mathcal{K}(\mathcal{H}_{X_2^{(\delta)}}) + \rho_X(C_0(\dot{X}_2)) \right).$$

Notice that if z is an element of $A_{\mathcal{H}_X, \rho_X}$ with propagation r with $r < \delta$ and if f is in $A \otimes \rho_X(C_0(\dot{X}_i))$ for $i = 1, 2$, then $f \cdot z$ is in $A_{\mathcal{H}_X, \rho_X}^{(i)}$.

For any C^* -algebra A and any r in $(0, \delta/5)$, then

$$\left(A_{\mathcal{H}_{X_1}, \mathcal{H}_{X_1^{(\delta)}, r}} + A \otimes \rho_X(C_0(\dot{X}_1)), A_{\mathcal{H}_{X_2}, \mathcal{H}_{X_2^{(\delta)}, r}} + A \otimes \rho_X(C_0(\dot{X}_2)), A_{\mathcal{H}_X, \rho_X}^{(1)}, A_{\mathcal{H}_X, \rho_X}^{(2)} \right)$$

is a controlled Mayer-Vietoris pair of order r with coercitivity 1 for $A_{\mathcal{H}_X, \rho_X}$ and hence, as before there exists for $(\alpha, k) = (\alpha_{MV}, k_{MV})$ a (α, k) -controlled morphism of order r and of odd degree

$$\mathcal{D}_{X_1, X_2, A, *}^{(\delta)} = (\partial_{X_1, X_2, A, *}^{\varepsilon, s, (\delta)})_{0 < \varepsilon < \frac{1}{100\alpha}, 0 < s < r} : \mathcal{K}_*(A_{\mathcal{H}_X, \rho_X}) \longrightarrow \mathcal{K}_*(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}) \quad (14)$$

such that the following diagram is (λ, h) -exact at order r

$$\begin{array}{ccccc} \mathcal{K}_0(A_{\mathcal{H}_X, \rho_X}^{(1,2)}) & \xrightarrow{(j'_{X_1, X_2, A, *}^\sharp, j'_{X_2, X_1, A, *}^\sharp)} & \mathcal{K}_0(A_{\mathcal{H}_X, \rho_X}^{(1)}) \oplus \mathcal{K}_0(A_{\mathcal{H}_X, \rho_X}^{(2)}) & \xrightarrow{j'_{X_1, A, *}^\sharp - j'_{X_2, A, *}^\sharp} & \mathcal{K}_0(A_{\mathcal{H}_X, \rho_X}) \\ \uparrow \mathcal{D}'_{X_1, X_2, A, *} & & & & \downarrow \mathcal{D}'_{X_1, X_2, A, *} \\ \mathcal{K}_1(A_{\mathcal{H}_X, \rho_X}) & \xleftarrow{j'_{X_1, A, *}^\sharp - j'_{X_2, A, *}^\sharp} & \mathcal{K}_1(A_{\mathcal{H}_X, \rho_X}^{(1)}) \oplus \mathcal{K}_1(A_{\mathcal{H}_X, \rho_X}^{(2)}) & \xleftarrow{(j'_{X_1, X_2, A, *}^\sharp, j'_{X_2, X_1, A, *}^\sharp)} & \mathcal{K}_1(A_{\mathcal{H}_X, \rho_X}^{(1,2)}) \end{array}$$

where

- $A_{\mathcal{H}_X, \rho_X}^{(1,2)} \stackrel{\text{def}}{=} A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}} + A \otimes \rho_X(C_0(\dot{X}_1 \cap \dot{X}_2));$
- $j'_{X_1, A}^\sharp : A_{\mathcal{H}_X, \rho_X}^{(1)} \hookrightarrow A_{\mathcal{H}_X, \rho_X};$
- $j'_{X_2, A}^\sharp : A_{\mathcal{H}_X, \rho_X}^{(2)} \hookrightarrow A_{\mathcal{H}_X, \rho_X};$
- $j'_{X_1, X_2, A}^\sharp : A_{\mathcal{H}_X, \rho_X}^{(1,2)} \hookrightarrow A_{\mathcal{H}_X, \rho_X}^{(1)};$
- $j'_{X_2, X_1, A}^\sharp : A_{\mathcal{H}_X, \rho_X}^{(1,2)} \hookrightarrow A_{\mathcal{H}_X, \rho_X}^{(2)};$

are the obvious inclusions.

In what follows we will respectively identify the unitarisation of $A_{\mathcal{H}_X, \rho_X}^{(1)}$, $A_{\mathcal{H}_X, \rho_X}^{(2)}$ and $A_{\mathcal{H}_X, \rho_X}^{(1,2)}$ with the subalgebras $A_{\mathcal{H}_X, \rho_X}^{(1)} + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X}$, $A_{\mathcal{H}_X, \rho_X}^{(2)} + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X}$ and $A_{\mathcal{H}_X, \rho_X}^{(1,2)} + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X}$ of $\widetilde{A_{\mathcal{H}_X}}$ if A is unital and of

$\tilde{A}\mathcal{H}_X + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X}$ viewed as adjointable operator in the right Hilbert \tilde{A} -module $\tilde{A} \otimes \mathcal{H}_X$ if not.

4.3. Compatibility of boundaries. Next proposition, together with the naturality of the quantitative index of Proposition 3.12, shows that the quantitative index map intertwines the Mayer-Vietoris six terms exact sequence in K -homology of diagram (11) of Section 4.1 with the above six term controlled Mayer-Vietoris exact sequence.

Proposition 4.2. *For any compact metric space X , any positive number δ , any closed subsets X_1 and X_2 of X , any closed subsets $X_1^{(\delta)}$ and $X_2^{(\delta)}$ of X and any standard X -module (\mathcal{H}_X, ρ_X) such that*

- $X = \overset{\circ}{X}_1 \cup \overset{\circ}{X}_2$;
- $X_{i,\delta} \subseteq X_i^{(\delta)}$ for $i = 1, 2$;
- the restriction of (\mathcal{H}_X, ρ_X) to

$$\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}} = \{\xi \in \mathcal{H}_X \text{ such that } \text{supp } \xi \subseteq X_1^{(\delta)} \cap X_2^{(\delta)}\}$$

is standard,

then the following diagram is commutative

$$\begin{array}{ccc} K_*(X, A) & \xrightarrow{\partial_{X_1^{(\delta)}, X_2^{(\delta)}, *}} & K_{*+1}(X_1^{(\delta)} \cap X_2^{(\delta)}, A) \\ \text{\scriptsize $Ind_{X,A,*}$} \downarrow & & \downarrow \text{\scriptsize $Ind_{X_1^{(\delta)} \cap X_2^{(\delta)}, A,*}$} \\ \mathcal{K}_*(A_{\mathcal{H}_X}) & \xrightarrow{\mathcal{D}_{X_1, X_2, A, *}^{(\delta)}} & \mathcal{K}_{*+1}(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}) \end{array}$$

where on the top row, $\partial_{X_1^{(\delta)}, X_2^{(\delta)}, *}$ is the left multiplication by the element $[\partial_{X_1^{(\delta)}, X_2^{(\delta)}}]$ of $KK_1(C(X_1^{(\delta)} \cap X_2^{(\delta)}), C(X))$ described in Section 4.1.

Remark 4.3. The above diagram a priori makes sense only up to order $\delta/5$. But indeed $\mathcal{D}_{X_1, X_2, A, *}^{(\delta)} \circ Ind_{X, A, *}$ can be extended to any positive number in the following way: let ε and r be positive numbers with $\varepsilon < 1/100$, pick a positive number s with $k_{\varepsilon/\alpha} s < \delta/5$ (with $(\alpha, k) = (\alpha_{MV}, k_{MV})$) and $s \leq r$. Then

$$\partial_{X_1, X_2, A, *}^{(\delta), \varepsilon, r} \circ Ind_{X, A, *}^{\varepsilon/\alpha, r/k_{\varepsilon/\alpha}} \stackrel{\text{def}}{=} \iota_*^{\varepsilon, s, r} \circ \partial_{X_1, X_2, A, *}^{(\delta), \varepsilon/\alpha, s/k_{\varepsilon/\alpha}} \circ Ind_{X, A, *}^{\varepsilon/\alpha, s/k_{\varepsilon/\alpha}}$$

does not depends on the choice of s and hence define a projective morphism of odd degree

$$\begin{aligned} \mathcal{D}_{X_1 \cap X_2, A, *}^{(\delta)} \circ Ind_{X, A, *} &= \\ (\partial_{X_1, X_2, A, *}^{(\delta), \varepsilon, r} \circ Ind_{X, A, *}^{\varepsilon/\alpha, r/k_{\varepsilon/\alpha}})_{\varepsilon \in (0, 1/100), r > 0} : K_*(X, A) &\longrightarrow \mathcal{K}_{*+1}(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}). \end{aligned}$$

Proposition 4.2 will be proved in next subsection. The end of this subsection is devoted to a carefull study of the projective morphism

$$Ind_{X_1^{(\delta)} \cap X_2^{(\delta)}, A, *} \circ \partial_{X_1^{(\delta)}, X_2^{(\delta)}, *} : K_*(X, A) \longrightarrow \mathcal{K}_{*+1}(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}).$$

Let $f : X \rightarrow [0, 1]$ be a continuous function such that $f(x) = 1$ if x is in $X \setminus \overset{\circ}{X}_1$ and $f(x) = 0$ if x is in $X \setminus \overset{\circ}{X}_2$. Let $(p_r)_{r>0}$ be a family of projections in $C(X_1 \cap X_2)_{\mathcal{H}_{X_1 \cap X_2}}$ such that

- p_r has propagation less than r for every positive number r .
- $\iota_*^{\varepsilon, r, r'}([p_r, 0]_{\varepsilon, r} = [p_{r'}, 0]_{\varepsilon, r'})$ in $K_0^{\varepsilon, r'}(C(X_1 \cap X_2)_{\mathcal{H}_{X_1 \cap X_2}})$ for every positive numbers ε , r and r' with $\varepsilon < 1/100$ and $r \leq r'$.

Consider then for every positive number r

$$U_{f,r} : X \longrightarrow \mathcal{K}(\mathcal{H}_{X_1 \cap X_2}) + \mathbb{C} \otimes Id_{\mathcal{H}_{X_1 \cap X_2}} \quad (15)$$

defined by

- $U_{f,r}(x) = Id_{\mathcal{H}_{X_1 \cap X_2}}$ if $x \notin X_1 \cap X_2$;
- $U_{f,r}(x) = e^{2i\pi f(x)} p_r(x) + Id_{\mathcal{H}_{X_1 \cap X_2}} - p_r(x)$ if x is in $X_1 \cap X_2$.

Then $U_{f,r}$ is a unitary in $\widetilde{C(X)_{\mathcal{H}_{X_1 \cap X_2}}}$ with propagation less than r and $\iota_*^{\varepsilon, r, r'}([U_{f,r}]_{\varepsilon, r}) = [U_{f,r'}]_{\varepsilon, r'}$ in $K_1^{\varepsilon, r'}(C(X)_{\mathcal{H}_{X_1 \cap X_2}})$ for every positive numbers ε , r and r' with $\varepsilon < 1/100$ and $r \leq r'$. For every positive number ε and r , with $\varepsilon < 1/100$, define then for $(\alpha, k) = (\alpha_{MV}, k_{MV})$

$$\begin{aligned} \Lambda^{\varepsilon, r} : K_1(X, A) &\longrightarrow K_0^{\varepsilon, r}(A_{\mathcal{H}_{X_1 \cap X_2}}) \\ x &\mapsto \tau_{\mathcal{H}_{X_1 \cap X_2}}^{\varepsilon/\alpha, r/k_{\varepsilon/\alpha}}(x)([U_{f,r/k_{\varepsilon/\alpha}}]_{\varepsilon/\alpha, r/k_{\varepsilon/\alpha}}) \end{aligned}$$

and

$$\begin{aligned} \Lambda'^{\varepsilon, r} : K_1(X, A) &\longrightarrow K_0^{\varepsilon, r}(A_{\mathcal{H}_{X_1 \cap X_2}}) \\ x &\mapsto \tau_{\mathcal{H}_{X_1 \cap X_2}}^{\varepsilon/\alpha, r/k_{\varepsilon/\alpha}}([\partial_{X_1, X_2}] \otimes x)([p_{r/k_{\varepsilon/\alpha}}, 0]_{\varepsilon/\alpha, r/k_{\varepsilon/\alpha}}) \end{aligned}$$

Then

$$\Lambda = (\Lambda^{\varepsilon, r})_{0 < \varepsilon, 1/100, r > 0} : K_1(X, A) \longrightarrow \mathcal{K}_0(A_{\mathcal{H}_{X_1 \cap X_2}})$$

and

$$\Lambda' = (\Lambda'^{\varepsilon, r})_{0 < \varepsilon, 1/100, r > 0} : K_1(X, A) \longrightarrow \mathcal{K}_0(A_{\mathcal{H}_{X_1 \cap X_2}})$$

are projective morphisms in sense of Definition 1.15.

Lemma 4.4. *With above notations, we have $\Lambda = \Lambda'$.*

Proof. Let

$$\Psi_{X_1, X_2} : C(X_1, X_2) \rightarrow SC(X_1 \cap X_2)$$

and

$$\Psi'_{X_1, X_2} : C(X_1, X_2) \rightarrow C(X)$$

being as in equation (12). Recall that

- the element $[\Psi_{X_1, X_2}]$ induced by Ψ_{X_1, X_2} is invertible as an element in $KK_*(C(X_1, X_2), SC(X_1 \cap X_2))$;
- $[\partial_{X_1, X_2}] = \tau_{C(X_1 \cap X_2)}([\partial]) \otimes \Psi'_{X_1, X_2, *}[\Psi_{X_1, X_2, *}]^{-1}$;

(recall that $[\partial \in KK_1(\mathbb{C}, C(0, 1))$ corresponds to the boundary map of the Bott extension $0 \rightarrow C(0, 1) \rightarrow C(0, 1] \xrightarrow{ev_1} \mathbb{C} \rightarrow 0$). Then, according to Theorems 1.24 and 1.25 we get that there exists a control pair depending only on $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$ such that

$$\mathcal{T}_{\mathcal{H}_{X_1 \cap X_2}}([\partial_{X_1, X_2}]) \stackrel{(\lambda, h)}{\sim} \Psi'_{X_1, X_2, \mathcal{H}_{X_1 \cap X_2}, *} \circ \mathcal{T}_{\mathcal{H}_{X_1 \cap X_2}}([\Psi_{X_1, X_2, *}^{-1}]^{-1}) \circ \mathcal{T}_{\mathcal{H}_{X_1 \cap X_2}}(\tau_{C(X_1 \cap X_2)}([\partial])),$$

where $\Psi'_{X_1, X_2, \mathcal{H}_{X_1 \cap X_2}} : C(X_1, X_2)_{\mathcal{H}_{X_1 \cap X_2}} \rightarrow C(X)_{\mathcal{H}_{X_1 \cap X_2}}$ is the $*$ -homomorphism induced by Ψ'_{X_1, X_2} . For any x in $K_*(X, A)$, we define

$$\mathcal{F}(x) = \mathcal{T}_{\mathcal{H}_{X_1 \cap X_2}}(x) \circ \Psi'_{X_1, X_2, \mathcal{H}_{X_1 \cap X_2}, *} \circ \mathcal{T}_{\mathcal{H}_{X_1 \cap X_2}}([\Psi_{X_1, X_2, *}^{-1}]^{-1}) \circ \mathcal{T}_{\mathcal{H}_{X_1 \cap X_2}}((\tau_{C(X_1 \cap X_2)}([\partial]))).$$

Then

$$\mathcal{F}(x) = (F^{\varepsilon, r})_{0 < \varepsilon < \frac{1}{100\alpha_{\mathcal{F}}}, r > 0} : K_*(C(X)_{\mathcal{H}_{X_1 \cap X_2}}) \rightarrow K(A_{\mathcal{H}_{X_1 \cap X_2}})$$

is a $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$ -controlled morphism for some control pair $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$ depending only on $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$. According to Theorems 1.24 and 1.25, there exists a control pair (λ, h) , depending only on $(\alpha_{\mathcal{F}}, k_{\mathcal{F}})$ such that

$$\mathcal{F}(x) \stackrel{(\lambda, h)}{\sim} \mathcal{T}_{\mathcal{H}_{X_1 \cap X_2}}(x \otimes [\partial_{X_1, X_2}]) \quad (16)$$

for all x in $K_*(X, A)$. Let $f : X \rightarrow [0, 1]$ be a continuous function as above such that $f(x) = 1$ if x is in $X \setminus \mathring{X}_1$ and $f(x) = 0$ if x is in $X \setminus \mathring{X}_2$ and defined then $F : X \times [0, 1] \rightarrow [0, 1]$ by

- $F(x, t) = 2f(x)t$ if $0 \leq t \leq 1/2$;
- $F(x, t) = 2(f(x) + t - f(x)t) - 1$ if $1/2 \leq t \leq 1$

Consider then for $(p_r)_{r>0}$ a family of projections in $C(X_1 \cap X_2)_{\mathcal{H}_{X_1 \cap X_2}}$ as above

$$\begin{aligned} W_r : (X_1 \times [0, 1/2]) \cup (X_2 \times [1/2, 1]) &\longrightarrow K(\mathcal{H}_{X_1 \cap X_2}) + \mathbb{C}Id_{\mathcal{H}_{X_1 \cap X_2}} \\ (x, t) &\mapsto e^{2i\pi F(x, t)} p_r(x) + Id_{\mathcal{H}_{X_1 \cap X_2}} - p_r(x). \end{aligned}$$

Then W_r is a unitary in $C(\widehat{X_1, X_2})_{\mathcal{H}_{X_1 \cap X_2}}$ with propagation less than r . We clearly have

$$\Psi_{X_1, X_2, \mathcal{H}_{X_1 \cap X_2}, *}^{\varepsilon, r}([W]_{\varepsilon, r}) = [U_{f, r}]_{\varepsilon, r}.$$

By using the homotopy of ε - r -unitaries

$$\begin{aligned} [0, 1] \times (X_1 \cap X_2) \times [0, 1] &\longrightarrow C(X_1 \cap X_2)_{\mathcal{H}_{X_1 \cap X_2}} + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_{X_1 \cap X_2}} \\ (s, x, t) &\mapsto e^{2i\pi(sF(x, t) + (1-s)t)} p_r(x) + 1 \otimes Id_{\mathcal{H}_{X_1 \cap X_2}} - p_r(x), \end{aligned}$$

we see that $\Psi_{X_1, X_2, \mathcal{H}_{X_1 \cap X_2}, *}^{\varepsilon, r}([W]_{\varepsilon, r})$ is equal to the class in $K_1^{\varepsilon, r}(SC(X_1 \cap X_2)_{\mathcal{H}_{X_1 \cap X_2}})$ represented by the ε - r -unitary defined by

$$\begin{aligned} [0, 1] &\longrightarrow C(X_1 \cap X_2)_{\mathcal{H}_{X_1 \cap X_2}} + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_{X_1 \cap X_2}} \\ t &\mapsto e^{2i\pi t} p_r(x) + 1 \otimes Id_{\mathcal{H}_{X_1 \cap X_2}} - p_r(x). \end{aligned}$$

Hence, according to [15, Proposition 3.9] and to point (vii) of Theorem 1.24 and up to possibly rescaling the control pair $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$ we get that

$$\Psi_{X_1, X_2, \mathcal{H}_{X_1 \cap X_2}, *}^{\alpha_{\mathcal{T}}, k_{\mathcal{T}}, \varepsilon, r}([W]_{\alpha_{\mathcal{T}}, k_{\mathcal{T}}, \varepsilon, r}) = \tau_{\mathcal{H}_{X_1 \cap X_2}}^{\varepsilon, r}(\tau_{C(X_1 \cap X_2)}([\partial])([p_r, 0]_{\varepsilon, r})).$$

In view of Theorem 1.25, there exists a control pair (α, k) , depending only on $(\alpha_{\mathcal{T}}, k_{\mathcal{T}})$ such that $\mathcal{T}_{\mathcal{H}_{X_1, X_2}}([\Psi_{X_1, X_2}]^{-1})$ is an (α, k) -inverse for $\Psi_{X_1, X_2, \mathcal{H}_{X_1 \cap X_2}, *}$. We deduce from equation (16) that there exists $\alpha \geq \lambda$ such that

$$\tau_{\mathcal{H}_{X_1 \cap X_2}}^{\varepsilon, r}(x)([U_{f, r}]_{\varepsilon, r}) = \tau_{\mathcal{H}_{X_1 \cap X_2}}^{\varepsilon, r}([\partial_{X_1, X_2}] \otimes x)([p_r, 0]_{\varepsilon, r}) \quad (17)$$

for ε in $(0, \frac{1}{100\lambda})$ and $r > 0$. The result is then a consequence of the first point of Remark 1.16. \square

Let (X, d) be a compact metric space, let (\mathcal{H}_X, ρ_X) be a standard X -modules and let X_1 and X_2 two closed subsets of X such that $X = \overset{\circ}{X}_1 \cup \overset{\circ}{X}_2$. Let δ and r be positive numbers with $5r < \delta$ and let $p : X \rightarrow \mathcal{K}(\mathcal{H}_X) + Id_{\mathcal{H}_X}$ be a continuous map such that $p(x)$ is a projection with support in $B(x, r/2) \times B(x, r/2)$ for every x in X . For any continuous function $f : X \rightarrow [0, 1]$ such that $f(x) = 0$ if $x \notin X_2$ and $f(x) = 1$ if $x \notin X_1$, then

$$e^{2i\pi f(x)p(x)} = e^{2i\pi f(x)}p(x) + Id_{\mathcal{H}_X} - p(x)$$

is for any x in X a unitary with propagation r in the unitarisation of $C(X)_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}$. Two continuous functions from X to $[0, 1]$ vanishing on $X \setminus \overset{\circ}{X}_1$ and identically equal to 1 on $X \setminus X_2$ give rise to homotopic unitaries with propagation r , i.e. connected by a unitary of the unitarisation of $C([0, 1] \times X)_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}$ with propagation r . Moreover two continuous maps $p_0, p_1 : X \rightarrow \mathcal{K}(\mathcal{H}_X) + Id_{\mathcal{H}_X}$ such that $p_0(x)$ and $p_1(x)$ are projection with support in $B(x, r/2) \times B(x, r/2)$ and which are connected by a homotopy

$$[0, 1] \times X \longrightarrow \mathcal{K}(\mathcal{H}_X) + Id_{\mathcal{H}_X}; (t, x) \mapsto p_t(x)$$

such that $p_t(x)$ is a projection with support in $B(x, r/2) \times B(x, r/2)$ for any t in $[0, 1]$ and any x in X gives rise to homotopic unitaries. Apply this to the class $[E_X]_{\varepsilon, r}$, we obtain a class $[U_{X_1, X_2}^{(\delta)}]_{\varepsilon, r}$ in $K_1^{\varepsilon, r}(C(X)_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}})$ which satisfies

$$\iota_*^{-, \varepsilon', r'}([U_{X_1, X_2}^{(\delta)}]_{\varepsilon, r}) = [U_{X_1, X_2}^{(\delta)}]_{\varepsilon', r'}$$

for every positive numbers $\varepsilon, \varepsilon', r$ and r' with $\varepsilon \leq \varepsilon' < 1/100$ and $r' \leq r < \delta/5$.

For any positive number ε and r with $\varepsilon < 1/100$, pick any positive number s with $s \leq r$ and $s < k_{\varepsilon/\alpha}\delta/5$ for $(\alpha, k) = (\alpha_{\mathcal{T}}, k_{\mathcal{T}})$. Then

$$\begin{aligned} \partial \text{Ind}_{X_1, X_2, A, *}^{(\delta), \varepsilon, r} : K_1(X, A) &\longrightarrow K_0^{\varepsilon, r}(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}) \\ x &\mapsto \iota_*^{\varepsilon, s, r} \tau_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}^{\varepsilon/\alpha, s/k_{\varepsilon/\alpha}}(x)([U_{X_1, X_2}^{(\delta)}]_{\varepsilon/\alpha, s/k_{\varepsilon/\alpha}}) \end{aligned}$$

does not depend on the choice of s and

$$\partial \text{Ind}_{X_1, X_2, A, *}^{(\delta)} \stackrel{\text{def}}{=} (\partial \text{Ind}_{X_1, X_2, A, *}^{(\delta), \varepsilon, r})_{0 < \varepsilon < 1/100, 0 < r} : K_1(X, A) \longrightarrow \mathcal{K}_0(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}})$$

is a projective morphism.

Lemma 4.5. *We above notations, we have*

$$\partial \text{Ind}_{X_1, X_2, A, *}^{(\delta)} = \text{Ind}_{X_1^{(\delta)} \cap X_2^{(\delta)}, A, *} \circ \partial_{X_1^{(\delta)} \cap X_2^{(\delta)}, A, *}.$$

Proof. Let us set $Y = X_1^{(\delta)} \cap X_2^{(\delta)}$. Let (\mathcal{H}_X, ρ_X) be a non degenerate standard X -module with standard restriction to Y . Let μ be a measure on X , let μ' be the restriction of μ to Y and assume that $\text{supp } \mu = X$ and $\text{supp } \mu' = Y$. Fix a Hilbert space \mathcal{H} and let choose isometries $V_1 : \mathcal{H}_Y \rightarrow L^2(Y, \mu') \otimes \mathcal{H}$ and $V_2 : \mathcal{H}_X \rightarrow L^2(X, \mu) \otimes \mathcal{H}$ with propagation less than $r/2$. By considering the square

$$\begin{array}{ccc} L^2(Y, \mu') \otimes \mathcal{H} & \longrightarrow & L^2(X, \mu) \otimes \mathcal{H} \\ V_1 \downarrow & & \downarrow V_2 \\ \mathcal{H}_Y & \longrightarrow & \mathcal{H}_X \end{array},$$

where the horizontal maps are inclusions and according to the first point of Remark 3.5, we can indeed assume without loss of generality that $\mathcal{H}_X = L^2(X, \mu) \otimes \mathcal{H}$ for \mathcal{H} a separable Hilbert space and thus $\mathcal{H}_Y = L^2(Y, \mu') \otimes \mathcal{H}$. As in the definition of $[E_X]_{\varepsilon, r}$ and $[E_Y]_{\varepsilon, r}$, let us choose $\phi_{0, r} : X \times X \rightarrow [0, 1]$ and $\phi_{1, r} : Y \times Y \rightarrow [0, 1]$ continuous functions such that

- $\phi_{0, r}$ is supported in $\{(x, x') \in X \times X \text{ such that } d(x, x') < r/2\}$;
- $\phi_{1, r}$ is supported in $\{(y, y') \in Y \times Y \text{ such that } d(y, y') < r/2\}$;
- $\int_X \phi_{0, r}(x, x') d\mu(x') = 1$ for any x in X ;
- $\int_Y \phi_{1, r}(y, y') d\mu'(y') = 1$ for any y in Y .

Then $y \mapsto \phi_1(y, \bullet)$ can be viewed as a continuous function from Y valued in $L^1(Y, \mu') \subseteq L^1(Y, \mu)$. The idea is to compare $[U_{X_1, X_2}^{(\delta)}]_{\varepsilon, r}$ with the class of the unitary of Equation (15) arising from $E_{\phi_{0, r}}$. Define then for any t in $[0, 1]$

$$\phi_{t, r} : Y \rightarrow L^1(X, \mu) : x \mapsto (1 - t)\phi_0(y, \bullet) + t\phi_1(y, \bullet).$$

For all y in Y and for all t , the support of $\phi_{t, r}(y)$ is included in

$$\{x \in X \text{ such that } d(y, x) < r\}.$$

Consider then the operator

$$E_{t,r} : C(Y) \otimes L^2(X, \mu) \longrightarrow C(Y) \otimes L^2(X, \mu)$$

defined by

$$E_{t,r} \cdot \xi(y, x) = \phi_{t,r}^{1/2}(y, x) \int_X \phi_{t,r}^{1/2}(y, x') \xi(y, x') d\mu_X(x'),$$

with ξ in $C(Y) \otimes L^2(X, \mu)$ viewed as a function in $Y \times X$. Then $E_{t,r}(y)$ is for all t in $[0, 1]$ a projection of $C(Y)_{L^2(X, \mu)} = C(Y) \otimes \mathcal{K}(L^2(X, \mu))$ with propagation less than r . If $f : X \rightarrow [0, 1]$ is a continuous function such that $f(x) = 0$ if $x \notin X_2$ and $f(x) = 1$ if $x \notin X_1$, define for $r < \delta/5$

$$U_{t,r}(x) : X \longrightarrow \mathcal{K}(L^2(Y, \mu')) + \mathbb{C} \cdot Id_{L^2(Y, \mu')}$$

by

- $U_{t,r}(x) = Id_{L^2(Y, \mu')}$ if $x \notin Y$;
- $U_{t,r}(x) = e^{2i\pi f(x)} E_{t,r}(x) + Id_{L^2(Y, \mu')} - E_{t,r}(x)$ if x is in Y .

We obtain in this way a homotopy of unitaries with propagation r in the unitarisation of $C(X)_{L^2(Y, \mu')}$. Since $[U_{0,r}(x)]_{\varepsilon,r} = [U_{X_1, X_2}^{(\delta)}]_{\varepsilon,r}$, the result is now a consequence of Lemma 4.4 applied to the family $(E_{1,r})_{r>0}$. \square

4.4. Proof of Proposition 4.2. This proof will require some preliminary work. Let consider a semi-split extension

$$0 \rightarrow J \rightarrow A \rightarrow C(X) \rightarrow 0$$

where J is a closed two sided ideal in a unital C^* -algebra A . Let $(E_r)_{r>0}$ be a family of projections in $C(X) \otimes \mathcal{K}(\mathcal{H}_X)$ such that for every positive number r and for every x in X , then $E_r(x)$ has support in $B(x, r/2) \times B(x, r/2)$ and $[E_X]_{\varepsilon,r} = [E_r, 0]_{\varepsilon,r}$ for every positive numbers ε and r with $\varepsilon < 1/100$. Let $f : X \rightarrow [0, 1]$ be a continuous function such that $\text{supp } f \subseteq \overset{\circ}{X}_2$ and $\text{supp}(1 - f) \subseteq \overset{\circ}{X}_1$

Let r_0 be a positive number such that $|f(x) - f(x')| < \frac{\varepsilon}{16\pi}$ if $d(x, x') < r_0$. We can assume without loss of generality that $E_r(x)$ has support in $B(x, r_0/2) \times B(x, r_0/2)$ for every positive number r and for every x in X . Let then z_r in $A_{\mathcal{H}_X}$ be a lift for E_r with propagation r with respect to the semi-split filtered extension

$$0 \rightarrow J_{\mathcal{H}_X} \rightarrow A_{\mathcal{H}_X} \rightarrow C(X)_{\mathcal{H}_X} \rightarrow 0$$

such that $\|z_r\| \leq 1$. Let $g : [0, 1] \rightarrow [0, 1]$ be defined by

- $g(t) = 0$ if $0 \leq t \leq 1/4$;
- $g(t) = 2(t - 1/4)$ if $1/4 \leq t \leq 3/4$;
- $g(t) = 1$ if $3/4 \leq t \leq 1$.

Let us fix a positive number δ and consider, with the notations of Section 4.2 the following elements

- $u : [0, 1] \rightarrow A_{\mathcal{H}_X, \rho_X}$,
- $v : [0, 1] \rightarrow C(X)_{\mathcal{H}_X, \rho_X}^{(1)} + \mathbb{C} \cdot 1 \otimes \mathcal{H}_X$,

$$\bullet w : [0, 1] \rightarrow C(X)_{\mathcal{H}_X, \rho_X}^{(2)} + \mathbb{C} \cdot 1 \otimes \mathcal{H}_X$$

defined for θ in $[0, 1]$ by

$$\begin{aligned} \bullet u(\theta) &= (1 \otimes \rho_X(\exp(2i\pi\theta g \circ f))) \cdot \exp(2i\pi(1 - \theta)z_r) \\ \bullet v(\theta) &= (1 \otimes \rho_X(\exp(2i\pi\theta(g \circ f - 1)))) \cdot E_r + 1 \otimes Id_{\mathcal{H}_X} - E_r; \\ \bullet w(\theta) &= E_r + (1 \otimes \rho_X(\exp(2i\pi\theta g \circ f))) \cdot (1 \otimes Id_{\mathcal{H}_X} - E_r). \end{aligned}$$

Since

$$\|(1 \otimes \rho_X(\exp(2i\pi\theta g \circ f) - 1)) \cdot E_r - ((\exp(2i\pi\theta g \circ f) - 1) \otimes Id_{\mathcal{H}_X}) \cdot E_r\| < \varepsilon/4 \quad (18)$$

and

$$\|E_r \cdot (1 \otimes \rho_X(\exp(2i\pi\theta g \circ f) - 1)) - E_r \cdot ((\exp(2i\pi\theta g \circ f) - 1) \otimes Id_{\mathcal{H}_X})\| < \varepsilon/4$$

we deduce that

$$\|(1 \otimes \rho_X(\exp(2i\pi\theta g \circ f) - 1)) \cdot E_r - E_r \cdot (1 \otimes \rho_X(\exp(2i\pi\theta g \circ f) - 1))\| < \varepsilon/2 \quad (19)$$

Hence, v and w are ε - r -unitaries and moreover, $v - 1 \otimes Id_{\mathcal{H}_X}$ lies in $C([0, 1] \times X)_{\mathcal{H}_X, \rho_X}^{(1)}$ that $w - 1 \otimes Id_{\mathcal{H}_X}$ lies in $C([0, 1] \times X)_{\mathcal{H}_X, \rho_X}^{(2)}$.

For next proposition, we need to recall from [17, Section 2] the notion of ε - r - N -invertible for positive number ε , r and N with $\varepsilon < 1/100$: an element x of a unital C^* -algebra B filtered by $(B_r)_{r>0}$ is ε - r - N -invertible if

- x is in B_r ;
- $\|x\| < N$;
- there exists an element y in B_r with $\|y\| < N$ and such that $\|xy - 1\| < \varepsilon$ and $\|yx - 1\| < \varepsilon$.

Such an element y is called an ε - r - N -inverse for x .

Proposition 4.6. *For some universal control pair (α, k) and positive number N and with notations as above, there exist P_1 and P_2 respectively $\alpha\varepsilon$ - $k_\varepsilon r$ - N -invertible elements in $M_2(A[0, 1]_{\mathcal{H}_X, \rho_X}^{(1)} + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X})$ and $M_2(A[0, 1]_{\mathcal{H}_X, \rho_X}^{(2)} + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X})$ (with $A[0, 1] = C([0, 1], A)$) connected to I_2 as $\alpha\varepsilon$ - $k_\varepsilon r$ - N -invertible for any positive numbers ε and r with $\varepsilon < \frac{1}{100\alpha}$ and $r < \frac{\delta}{5k_\varepsilon}$ and such that*

- (i) $\|P_1 P_2 - \text{diag}(u, u^*)\| < \alpha\varepsilon$;
- (ii) $\|\dot{P}_1 - \text{diag}(v, v^*)\| < \alpha\varepsilon$, where \dot{P}_1 is the image of P_1 in

$$M_2(C(X)_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X})$$

relatively to the extension

$$0 \rightarrow A_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] \rightarrow B_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] \rightarrow C(X)_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] \rightarrow 0;$$

- (iii) $\|\dot{P}_2 - \text{diag}(w, w^*)\| < \alpha\varepsilon$, where \dot{P}_2 is the image of P_2 in

$$M_2(C(X)_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X})$$

relatively to the extension

$$0 \rightarrow A_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] \rightarrow B_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] \rightarrow C(X)_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] \rightarrow 0.$$

- (iv) $P_1(1)$ is an $\alpha\varepsilon$ - $k_\varepsilon r$ - N -invertible of $M_2(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}) + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X}$ connected to $I_2 \otimes Id_{\mathcal{H}_X}$ as an $\alpha\varepsilon$ - $k_\varepsilon r$ - N -invertible of $M_2(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}) + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X}$.

Moreover,

- $P_1(0)$ and $P_2(0)$ lie respectively in $M_2(A_{\mathcal{H}_{X_1^{(\delta)}}}) + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X}$ and $M_2(A_{\mathcal{H}_{X_2^{(\delta)}}}) + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X}$
- $P_1(1) - I_2 \otimes Id_{\mathcal{H}_X}$ and $P_1^{-1}(1) - I_2 \otimes Id_{\mathcal{H}_X}$ have coefficient in $A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}$.

Proof. Let us fix a continuous function $\phi : [0, 1] \rightarrow [0, 1]$ such that $\phi(1) = 1$ and $\phi(t) = 0$ if t is in $[0, 3/4]$ and let us then fix $(\phi_n)_{n \in \mathbb{N}}$ a sequence of real polynomial functions converging uniformly to ϕ on $[0, 1]$ and such that $|\phi_n(t)| \leq 2$, $\phi_n(0) = 0$, $\phi_n(1) = 1$ and $d^o \phi_n \geq n$ for every integer n and every t in $[0, 1]$. For ε a positive number let l_ε be the smallest integer such that

- for all integer k with $k > l_\varepsilon$ and all t in $[0, 1]$ then $|\phi_k(t) - \phi(t)| < \varepsilon$;
- $\sum_{k > l_\varepsilon} \frac{(2\pi)^k}{k!} < \varepsilon/3$.

If a and b are elements in a C^* -algebra and θ in $[0, 1]$, let us set

$$\psi_\varepsilon^{(1)}(\theta, a, b) = \phi(a) \sum_{k=1}^{l_\varepsilon} \frac{(2i\pi(1-\theta)b)^k}{k!} + (1-\phi(a)) \cdot (\exp(-2i\pi\theta g(a)) - 1) \phi_{l_\varepsilon}(b)$$

and

$$\psi_\varepsilon^{(2)}(\theta, a, b) = \exp(2i\pi\theta g(a)) \cdot \left(\sum_{k=0}^{l_\varepsilon} \frac{(2i\pi(1-\theta)b)^k}{k!} \right).$$

Notice that the degree in b of $\psi_\varepsilon^{(1)}(\theta, a, b)$ is less than

$$h_\varepsilon = \sup\{d^o \phi_k; k = 0, \dots, l_\varepsilon\}.$$

We define for any θ in $[0, 1]$

$$\begin{aligned} u'(\theta) &= \psi_\varepsilon^{(2)}(\theta, 1 \otimes \rho_X(f), z_r); \\ y(\theta) &= 1 \otimes Id_{\mathcal{H}_X} + \psi_\varepsilon^{(1)}(\theta, 1 \otimes \rho_X(1-f), z_r) + \psi_\varepsilon^{(1)}(\theta, 1 \otimes \rho_X(f), 1 \otimes Id_{\mathcal{H}_X} - z_r)^*; \\ u_1(\theta) &= 1 \otimes Id_{\mathcal{H}_X} + \psi_\varepsilon^{(1)}(\theta, 1 \otimes \rho_X(1-f), z_r) + (1 \otimes \rho_X(1-g \circ f)) \cdot (u'(\theta) - y(\theta)) \\ u_2(\theta) &= \psi_\varepsilon^{(1)}(\theta, 1 \otimes \rho_X(f), 1 - z_r)^* + (1 \otimes \rho_X(g \circ f)) \cdot (u'(\theta) - y(\theta)) \end{aligned}$$

Then for every θ in $[0, 1]$ and $r < \frac{\delta}{5h_\varepsilon}$,

- $u'(\theta)$ and $y(\theta)$ are elements of $A_{\mathcal{H}_X, \rho_X}$ with propagation less than $h_\varepsilon r$;
- $u_1(\theta) - 1 \otimes Id_{\mathcal{H}_X}$ is an element in $A_{\mathcal{H}_X, \rho_X}^{(1)}$ with propagation less than $h_\varepsilon r$;
- $u_2(\theta)$ is an element in $A_{\mathcal{H}_X, \rho_X}^{(2)}$ with propagation less than $h_\varepsilon r$;
- $u'(\theta) = u_1(\theta) + u_2(\theta)$;
- $\|u_1(\theta)\| \leq 15$ and $\|u_2(\theta)\| \leq 15$.

Moreover,

$$\|u(\theta) - u'(\theta)\| < \varepsilon/3$$

and hence $u'(\theta)$ is an ε - $h_\varepsilon r$ -unitary in $A_{\mathcal{H}_X, \rho_X}$.

Let us set then $X(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ and $Y(y) = \begin{pmatrix} 1 & 0 \\ y & 1 \end{pmatrix}$ and define then for θ in $[0, 1]$

$$P_1(\theta) = X(u_2(\theta))X(u_1(\theta))Y(-u_1^*(\theta))X(u_1(\theta)) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} X(-u_2(\theta))$$

and

$$P_2(\theta) = X(u_2(\theta)) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} X(-u_1(\theta))Y(-u_2^*(\theta))X(u_1(\theta))X(u_2(\theta)) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then for some constant N and some control pair (α, k) independent of X, X_1, X_2, A and B , for all θ in $[0, 1]$ and for all positive number r with $r < \frac{\delta}{5k_\varepsilon}$,

- $P_1(\theta)$ and $P_2(\theta)$ are invertible elements in $M_2(A_{\mathcal{H}_X, \rho_X})$;
- $P_1(\theta) - I_2$ and $P_1(\theta)^{-1} - I_2$ have coefficients in $A_{\mathcal{H}_X, \rho_X}^{(1)}$ and propagation less than $k_\varepsilon r$;
- $P_2(\theta) - I_2$ and $P_2(\theta)^{-1} - I_2$ have coefficients in $A_{\mathcal{H}_X, \rho_X}^{(2)}$ and propagation less than $k_\varepsilon r$;
- $P_1(\theta), P_2(\theta), P_1(\theta)^{-1}$ and $P_2(\theta)^{-1}$ are bounded in norm by N ;
- $\|P_1(\theta)P_2(\theta) - \text{diag}(u(\theta), u^*(\theta))\| < \alpha\varepsilon$;
- $P_1(0)$ and $P_2(0)$ lie respectively in $M_2(A_{\mathcal{H}_{X_1^{(\delta)}}}) + \mathbb{C} \cdot 1 \otimes \mathcal{H}_X$ and $M_2(A_{\mathcal{H}_{X_2^{(\delta)}}}) + \mathbb{C} \cdot 1 \otimes \mathcal{H}_X$;
- $P_1(1) - I_2 \otimes \mathcal{H}_X$ and $P_1^{-1}(1) - I_2 \otimes \mathcal{H}_X$ have coefficients in $A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}$;
- $P_1(1)$ is connected to $I_2 \otimes Id_{\mathcal{H}_X}$ as an $\alpha\varepsilon - k_\varepsilon r - N$ -invertible of $M_2(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}) + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X}$.

Hence if we set for $i = 1, 2$

$$\begin{aligned} P_i : [0, 1] &\longrightarrow M_2(A_{\mathcal{H}_X, \rho_X}) \\ \theta &\longmapsto P_i(\theta), \end{aligned}$$

then P_i is an $\varepsilon - k_\varepsilon r - N$ -invertible element in

$$M_2(A_{\mathcal{H}_X, \rho_X}^{(i)}[0, 1]) + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X}$$

connected to $I_2 \otimes Id_{\mathcal{H}_X}$ as an ε - $k_\varepsilon r$ - N -invertible.

Let us consider the filtered semi-split extension

$$0 \rightarrow A_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] \rightarrow B_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] \rightarrow C(X)_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] \rightarrow 0$$

and let us give an approximation of the image of P_1 in the C^* -algebra $M_2(C(X)_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1]) + \mathbb{C} \cdot I_2 \otimes \mathcal{H}_X$. In this computation, we can replace in the definition of $P_1(\theta)$.

- $u'(\theta)$ by

$$(1 \otimes \rho_X(\exp(2i\pi\theta g \circ f))) \cdot \exp(2i\pi(1 - \theta)E_r);$$

- $\psi_\varepsilon^{(1)}(\theta, 1 \otimes \rho_X(1 - f), z_r)$ by

$$\begin{aligned} (1 \otimes \rho_X(\phi \circ (1 - f)))(e^{2i\pi(1-\theta)} - 1) \\ + 1 \otimes \rho_X((1 - \phi \circ (1 - f))(\exp(-2i\pi\theta g \circ (1 - f)) - 1))) \cdot E_r \\ = (1 \otimes \rho_X(\exp(-2i\pi\theta g \circ (1 - f)) - 1)) \cdot E_r \\ = (1 \otimes \rho_X(\exp(2i\pi\theta(g \circ f - 1)) - 1)) \cdot E_r \end{aligned}$$

- $\psi_\varepsilon^{(1)}(\theta, 1 \otimes \rho_X(f), 1 \otimes Id_{\mathcal{H}_X} - z_r)$ by

$$(1 \otimes \rho_X(\exp(-2i\pi\theta g \circ f) - 1) \cdot (1 \otimes Id_{\mathcal{H}_X} - E_r));$$

- $y(\theta)$ by

$$(1 \otimes \rho_X(\exp(2i\pi\theta(g \circ f - 1)))) \cdot E_r + (1 \otimes \rho_X(\exp(2i\pi\theta g \circ f)) \cdot (1 \otimes Id_{\mathcal{H}_X} - E_r)) = u'(\theta);$$

- $u_1(\theta)$ by

$$1 \otimes Id_{\mathcal{H}_X} - E_r + (1 \otimes \rho_X(\exp(2i\pi\theta(g \circ f - 1))) \cdot E_r);$$

- $u_2(\theta)$ by $(1 \otimes \rho_X(\exp(2i\pi\theta g \circ f))) \cdot (1 \otimes Id_{\mathcal{H}_X} - E_r)$.

In these computations, we have used the identities $(1 - g)\phi = 0$ and $g(t) + g(1 - t) = 1$ for all t in $[0, 1]$ and in the approximation of $y(\theta)$ and of $u_2(\theta)$, the fact that according to (19), then E_r and $\rho_X(\exp(2i\pi\theta g \circ f - 1))$ almost commute. Using again this almost commutation, we see that the image of P_1 in $M_2(C(X)_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1]) + \mathbb{C} \cdot I_2 \otimes \mathcal{H}_X$ is closed to the unitary

$$\begin{aligned} [0, 1] &\rightarrow M_2(C(X)_{\mathcal{H}_X, \rho_X}^{(1)}) + \mathbb{C} \cdot I_2 \otimes \mathcal{H}_X \\ \theta &\mapsto \text{diag}(v_\theta, v_\theta^*) \end{aligned} \quad (20)$$

In the same way, if we consider the filtered semi-split extension

$$0 \rightarrow A_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] \rightarrow B_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] \rightarrow C(X)_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] \rightarrow 0,$$

then the image of P_2 in $M_2(C(X)_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1]) + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X}$ is closed to the unitary

$$\begin{aligned} [0, 1] &\rightarrow M_2(C(X)_{\mathcal{H}_X, \rho_X}^{(2)}) + \mathbb{C} \cdot I_2 \otimes \mathcal{H}_X \\ \theta &\mapsto \text{diag}(w_\theta, w_\theta^*) \end{aligned} \quad (21)$$

with for all θ in $[0, 1]$.

□

As a corollary we get with notations of Proposition 4.6

Corollary 4.7. *For some control pair (λ, h) depending only on the control pair (α, k) and on the positive number N of Proposition 4.6, then for any positive numbers ε and r with $\varepsilon < 1/100\lambda$ and $r < \frac{\delta}{5h\varepsilon}$, there exist W_1 and W_2 respectively $\lambda\varepsilon$ - $h_\varepsilon r$ -unitary elements in $M_2(A[0, 1]_{\mathcal{H}_X, \rho_X}^{(1)} + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X})$ and $M_2(A[0, 1]_{\mathcal{H}_X, \rho_X}^{(2)} + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X})$ connected to $I_2 \otimes Id_{\mathcal{H}_X}$ as $\lambda\varepsilon$ - $h_\varepsilon r$ -unitaries such that*

- (i) $\|W_1 W_2 - \text{diag}(u, u^*)\| < \lambda\varepsilon$;
- (ii) $\|\dot{W}_1 - \text{diag}(v, v^*)\| < \lambda\varepsilon$, where \dot{W}_1 is the image of W_1 in

$$M_2(C(X)_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X})$$

relatively to the extension

$$0 \rightarrow A_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] \rightarrow B_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] \rightarrow C(X)_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] \rightarrow 0;$$

- (iii) $\|\dot{W}_2 - \text{diag}(w, w^*)\| < \lambda\varepsilon$, where \dot{W}_2 is the image of W_2 in

$$M_2(C(X)_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X})$$

relatively to the extension

$$0 \rightarrow A_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] \rightarrow B_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] \rightarrow C(X)_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] \rightarrow 0.$$

- (iv) $W_1(0)$ and $W_2(0)$ lie respectively in $M_2(A_{\mathcal{H}_{X_1^{(\delta)}}} + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X})$ and $M_2(A_{\mathcal{H}_{X_2^{(\delta)}}} + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X})$
- (v) $W_1(1)$ is an $\lambda\varepsilon$ - $h_\varepsilon r$ -unitary of $M_2(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}} + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X})$ connected to $I_2 \otimes Id_{\mathcal{H}_X}$ as a $\lambda\varepsilon$ - $l_\varepsilon r$ -unitary which belongs to $M_2(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}} + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X})$.

Proof. In view of [17, Lemma 2.4], there exists a control pair (λ, h) and a positive number N' depending only on N and on the controlled pair (α, k) of Proposition 4.6 for which the following is satisfied :

For any positive number ε with $\varepsilon < \frac{1}{100\lambda}$, there exists a polynomial function Q_ε of degree less than h_ε with $Q_\varepsilon(1) = 1$ such that $Q_\varepsilon(P_1^* P_1)$ and $Q_\varepsilon(P_2^* P_2)$ respectively admit positive $\lambda\varepsilon$ - $h_\varepsilon r$ - N' -inverse in $M_2(A_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X})$ and $M_2(A_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X})$ and such that if we set $W_1 = P_1 Q_\varepsilon(P_1^* P_1)$ and $W_2 = Q_\varepsilon(P_2^* P_2) P_2$, then

- W_1 and W_2 are $\lambda\varepsilon$ - $h_\varepsilon r$ -unitaries in $M_2(A_{\mathcal{H}_X, \rho_X}[0, 1] + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X})$;
- $W_1 - I_2 \otimes Id_{\mathcal{H}_X}$ has coefficients in $A_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1]$;
- $W_2 - I_2 \otimes Id_{\mathcal{H}_X}$ has coefficients in $A_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1]$;
- $\|P_1 - |P_1| W_1\| < \lambda\varepsilon$ and $\|P_2 - W_2 |P_2^*|\| < \lambda\varepsilon$;
- W_1 and W_2 are respectively homotopic to $I_2 \otimes Id_{\mathcal{H}_X}$ as $\lambda\varepsilon$ - $h_\varepsilon r$ -unitaries in $M_2(A_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1] + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X})$ and $M_2(A_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1] + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X})$;

- $W_1(0)$ and $W_2(0)$ lie respectively in $M_2(A_{\mathcal{H}_{X_1^{(\delta)}}}) + \mathbb{C} \cdot 1 \otimes \mathcal{H}_X$ and $M_2(A_{\mathcal{H}_{X_2^{(\delta)}}}) + \mathbb{C} \cdot 1 \otimes \mathcal{H}_X$.

Proceeding as in the proof of [17, Proposition 2.11] and possibly rescaling λ , we see that $\|W_1 W_2 - \text{diag}(u, u^*)\| < \lambda \varepsilon$. Moreover, since \dot{P}_1 is closed to a unitary, then $|\dot{P}_1|$ is closed to $I_2 \otimes \mathcal{H}_X$ in $M_2(C(X)^{(1)}_{\mathcal{H}_X, \rho_X} [0, 1]) + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X}$ and therefore, up to rescaling λ , we get that $\|\dot{W}_1 - \text{diag}(v, v^*)\| < \lambda \varepsilon$ and similarly that $\|\dot{W}_2 - \text{diag}(w, w^*)\| < \lambda \varepsilon$.

Since $Q_\varepsilon(1) = 1$ and since $P_1(1) - I_2 \otimes Id_{\mathcal{H}_X}$ has coefficients in $A_{\mathcal{H}_{X_1^{(\delta)}} \cap \mathcal{H}_{X_2^{(\delta)}}}$, then the same holds for $W_1(1) - I_2 \otimes Id_{\mathcal{H}_X}$ and $W_1(1)$ is connected to $I_2 \otimes \mathcal{H}_X$ as a $\lambda \varepsilon$ - $l_\varepsilon r$ -unitary of $M_2(A_{\mathcal{H}_{X_1^{(\delta)}} \cap \mathcal{H}_{X_2^{(\delta)}}}) + \mathbb{C} \cdot I_2 \otimes \mathcal{H}_X$. \square

End of proof of Proposition 4.2. With notations of Lemma 4.5, this amounts to proving that

$$\mathcal{D}_{X_1, X_2, A, *}^{(\delta)} \circ Ind_{X, A, *} = \partial Ind_{X_1, X_2, A, *}^{(\delta)}.$$

Let us give the proof for $K_1(X, A)$, the even case being deduced from controlled Bott periodicity. Then any element of $K_1(X, A)$ corresponds to a semi-split extension

$$0 \rightarrow A \rightarrow B \rightarrow C(X) \rightarrow 0,$$

where A is closed two-sided ideal in B . We can assume without loss of generality that B is unital.

Recall that

- $f : X \rightarrow [0, 1]$ is a continuous function such that $\text{supp } f \subseteq \mathring{X}_2$ and $\text{supp } 1 - f \subseteq \mathring{X}_1$;
- r_0 is a positive number with $r_0 \leq r$ such that $|f(x) - f(x')| < \frac{\varepsilon}{16\pi}$ if $d(x, x') < r_0$;
- the function $g : [0, 1] \rightarrow [0, 1]$ is defined by $g(t) = 0$ if $0 \leq t \leq 1/4$, $g(t) = 2(t - 1/4)$ if $1/4 \leq t \leq 3/4$ and $g(t) = 1$ if $3/4 \leq t \leq 1$;
- $E_r(x)$ is a projection in $C(X) \otimes \mathcal{K}(\mathcal{H}_X)$ with support in $B(x, r') \times B(x, r')$ with $r' = \frac{1}{2} \min\{r_0, r\}$ and such that $[E_X]_{\varepsilon, r} = [E_r, 0]_{\varepsilon, r}$ for every positive numbers ε and r with $\varepsilon < 1/100$;
- $u(\theta) = (1 \otimes \rho_X(\exp(2i\pi\theta g \circ f))) \cdot \exp(2i\pi(1 - \theta)z_r)$ for all θ in $[0, 1]$ with z_r a lift with propagation r of E_r in $A_{\mathcal{H}_X}$.

In order to have a more readable end of proof of the proposition, we divide it into three steps.

Step 1: Let (λ, h) be a control pair and let W_1 and W_2 be respectively $\lambda \varepsilon$ - $h_\varepsilon r$ -unitary elements in $M_2(A[0, 1]_{\mathcal{H}_X, \rho_X}^{(1)}) + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X}$ and $M_2(A[0, 1]_{\mathcal{H}_X, \rho_X}^{(2)}) + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X}$ as in Corollary 4.7. According to

points (ii) and (iii) of Corollary 4.7, up to enlarging λ and since for $i = 1, 2$ the extension

$$0 \rightarrow A_{\mathcal{H}_X, \rho_X}^{(i)}[0, 1] \rightarrow B_{\mathcal{H}_X, \rho_X}^{(i)}[0, 1] \rightarrow C(X)_{\mathcal{H}_X, \rho_X}^{(i)}[0, 1] \rightarrow 0$$

is semi-split filtered, there exists an element y_1 in $M_2(A_{\mathcal{H}_X, \rho_X}^{(1)}[0, 1]) + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X}$ with propagation less than $2h_\varepsilon$ such that

$$\|W_1^* \text{diag}(I_2 \otimes Id_{\mathcal{H}_X}, 0)W_1 - y_1 - \text{diag}(I_2 \otimes Id_{\mathcal{H}_X}, 0)\| < \lambda\varepsilon$$

and an element y_2 in $M_2(A_{\mathcal{H}_X, \rho_X}^{(2)}[0, 1]) + \mathbb{C} \cdot I_2 \otimes Id_{\mathcal{H}_X}$ with propagation less than $2h_\varepsilon$ such that

$$\|W_2 \text{diag}(I_2 \otimes Id_{\mathcal{H}_X}, 0)W_2^* - y_2 - \text{diag}(I_2 \otimes Id_{\mathcal{H}_X}, 0)\| < \lambda\varepsilon.$$

Since W_2 is close to $W_1^* \cdot \text{diag}(u, u^*)$, we can assume up to rescaling λ that

$$\|y_1 - y_2\| < \lambda\varepsilon.$$

Using the CIA property and up to rescaling the control pair (λ, h) , we deduce that there exists q a $\lambda\varepsilon$ - $h_\varepsilon r$ -projection in $M_2(A_{\mathcal{H}_X, \rho_X}[0, 1])$ such that

- $q - \text{diag}(1 \otimes Id_{\mathcal{H}_X}, 0)$ has coefficients in

$$A_{\mathcal{H}_X, \rho_X}^{(1,2)}[0, 1] = A[0, 1] \otimes \mathcal{H}(\mathcal{H}_{X_1^{(\delta)}} \cap \mathcal{H}_{X_2^{(\delta)}}) + A[0, 1] \otimes \rho_X(C_0(\mathring{X}_1 \cap \mathring{X}_2));$$

- $\|y_1 + \text{diag}(1 \otimes Id_{\mathcal{H}_X}, 0) - q\| < \lambda\varepsilon$ and $\|y_2 + \text{diag}(1 \otimes Id_{\mathcal{H}_X}, 0) - q\| < \lambda\varepsilon$.

Step 2: Let

$$\mathcal{D}_{A_{\mathcal{H}_X}, B_{\mathcal{H}_X}, *} = (\partial_{A_{\mathcal{H}_X}, B_{\mathcal{H}_X}, *}^{\varepsilon, r})_{0 < \varepsilon < \frac{1}{100\alpha_D}, r > 0} : \mathcal{K}_*(C(X)_{\mathcal{H}_X}) \longrightarrow \mathcal{K}_{*+1}(A_{\mathcal{H}_X})$$

be the controlled boundary map associated to the semi-split filtered extension

$$0 \longrightarrow A_{\mathcal{H}_X} \longrightarrow B_{\mathcal{H}_X} \longrightarrow C(X)_{\mathcal{H}_X} \longrightarrow 0$$

and

$$\mathcal{D}_{A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}, B_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}, *} = (\partial_{A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}, B_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}, *}^{\varepsilon, r})_{0 < \varepsilon < \frac{1}{100\alpha_D}, r > 0} : \mathcal{K}_*(C(X)_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}) \longrightarrow \mathcal{K}_{*+1}(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}})$$

be the controlled boundary associated to the semi-split filtered extension

$$0 \longrightarrow A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}} \longrightarrow B_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}} \longrightarrow C(X)_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}} \longrightarrow 0.$$

Since $W_1(0)$ and $W_2(0)$ lie indeed respectively in $M_2(A_{\mathcal{H}_{X_1^{(\delta)}}}) + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X}$ and $M_2(A_{\mathcal{H}_{X_2^{(\delta)}}}) + \mathbb{C} \cdot 1 \otimes Id_{\mathcal{H}_X}$ and since $W_1(0)W_2(0)$ is close to $\text{diag}(\exp(2i\pi z_r), \exp(-2i\pi z_r))$, we see that, up to rescaling (λ, h) and possibly (α_D, k_D) , the class $[q(0), 1]_{\lambda\varepsilon, h_\varepsilon r}$ coincides with the image of

$$\iota_*^{-, \lambda\varepsilon, h_\varepsilon r} \circ \partial_{X_1, X_2, *}^{(\delta), \alpha_D \varepsilon, k_D, \varepsilon r} \circ \partial_{A_{\mathcal{H}_X}, B_{\mathcal{H}_X}, *}^{\varepsilon, r}([E_X]_{\varepsilon, r})$$

under the morphism

$$\mathcal{K}_*(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}) \rightarrow \mathcal{K}_*(A_{\mathcal{H}_X, \rho_X}^{(1,2)}) \quad (22)$$

induced by the inclusion

$$\mathcal{K}(\mathcal{H}_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}) \hookrightarrow \mathcal{K}(\mathcal{H}_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}) + \rho_X(C_0(\mathring{X}_1 \cap \mathring{X}_2)).$$

On the other hand,

- according to points (ii) and (v) of Corollary 4.7;
- since

$$v(1) = (1 \otimes \rho_X(\exp(2i\pi \circ f))) \cdot E_r + 1 \otimes Id_{\mathcal{H}_X} - E_r$$

is according of Equation (18) close and hence homotopic to

$$(\exp(2i\pi \circ f) \otimes Id_{H_X}) \cdot E_r + 1 \otimes Id_{\mathcal{H}_X} - E_r$$

as an ε - r -unitary in $C(X)_{X_1^{(\delta)} \cap X_2^{(\delta)}}$,

we see that up to rescaling (λ, h) , then $[q(1), 1]_{\lambda\varepsilon, h\varepsilon r}$ coincides with the image of

$$\iota_*^{-, \lambda\varepsilon, h\varepsilon r} \circ \partial_{A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}, B_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}, *}^{\varepsilon, r}([U_{X_1, X_2}^{(\delta)}]_{\varepsilon, r})$$

under the morphism of equation (22).

Step 3: As we have seen in the previous step,

$$\iota_*^{-, \lambda\varepsilon, h\varepsilon r} \circ \partial_{A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}, B_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}, *}^{\varepsilon, r}([U_{X_1, X_2}^{(\delta)}]_{\varepsilon, r})$$

and

$$\iota_*^{-, \lambda\varepsilon, h\varepsilon r} \circ \partial_{X_1, X_2, *}^{(\delta), \alpha_{\mathcal{D}\varepsilon}, k_{\mathcal{D}, \varepsilon} r} \circ \partial_{A_{\mathcal{H}_X}, B_{\mathcal{H}_X}, *}^{\varepsilon, r}([E_X]_{\varepsilon, r})$$

have same images under the morphism of equation (22). Since the extension

$$0 \longrightarrow A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}} \longrightarrow A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}}^{(1,2)} \longrightarrow A \otimes \rho_X(C_0(V_1 \cap V_2)) \longrightarrow 0$$

is filtered and split, then according to [15, Corollary 4.9], there exists a control pair (λ', h') such that the morphism of equation (22) is (λ', h') -injective as defined in Definition 1.18. Hence we get the result for $K_1(X, A)$ at least for small ε . For ε in $(0, 1/100)$, the result is a consequence of the first point of Remark 1.16.

Let us prove now the result in the even case. Let us consider the following diagram

$$\begin{array}{ccccccc} \mathcal{K}_0(X, A) & \xrightarrow{\otimes \tau_A([\partial])} & \mathcal{K}_1(X, SA) & \xrightarrow{\otimes [\partial]_{X_1^{(\delta)}, X_2^{(\delta)}}} & \mathcal{K}_0(X_1^{(\delta)} \cap X_2^{(\delta)}, SA) & \xrightarrow{\otimes \tau_A([\partial]^{-1})} & \mathcal{K}_1(X_1^{(\delta)} \cap X_2^{(\delta)}, A) \\ \downarrow \text{Ind}_{X, A, *} & & \downarrow \text{Ind}_{X, SA, *} & & \downarrow \text{Ind}_{X_1^{(\delta)}, X_2^{(\delta)}, SA, *} & & \downarrow \text{Ind}_{X_1^{(\delta)}, X_2^{(\delta)}, A, *} \\ \mathcal{K}_0(A(\mathcal{H}_X)) & \xrightarrow{\mathcal{T}_{HX}(\tau_A([\partial]))} & \mathcal{K}_1(SA(\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}})) & \xrightarrow{\mathcal{D}_{X_1, X_2, SA, *}^{(\delta)}} & \mathcal{K}_0(SA(\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}})) & \xrightarrow{\mathcal{T}_{\mathcal{H}_X}(\tau_A([\partial]^{-1}))} & \mathcal{K}_1(A(\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}})) \end{array},$$

where $\otimes \bullet$ stands for the left Kasparov product. According to Proposition 3.14 and to the even case, the three squares are commutative. The composition on the top row is

$$\otimes[\partial_{X_1^{(\delta)}, X_2^{(\delta)}}] : K_0(X, A) \longrightarrow K_1(X_1^{(\delta)} \cap X_2^{(\delta)}, A).$$

The result for ε small enough is then a consequence of the definition of

$$\mathcal{D}_{X_1, X_2, A}^{(\delta)} : \mathcal{K}_0(SA(\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}})) \longrightarrow \mathcal{K}_1(A(\mathcal{H}_{X_1^{(\delta)} \cap X_1^{(\delta)}}))$$

(see [17, Section 3.2]). For ε in $(0, 1/100)$, the result is a consequence of the first point of Remark 1.16. \square

4.5. Proof of Theorem 3.17. Let X be a compact metric space, let A be a C^* -algebra, let \mathcal{H}_X be a non degenerate standard X -module and let $(0, 1/200) \rightarrow [1, +\infty); \varepsilon \mapsto h_\varepsilon$ be a non increasing function. Recall from definition 1.17 that

$$Ind_{X, A, *} : K_*(X, A) \longrightarrow \mathcal{K}(A_{\mathcal{H}_X})$$

is a h -isomorphism if for any ε in $(0, 1/200)$.

- $Ind_{X, A, *}^{\varepsilon, r}$ is one-to-one for any r in $(0, 1/h_\varepsilon)$;
- for any r in $(0, 1/h_\varepsilon^2)$ and any y in $K_*^{\varepsilon, r}(A_{\mathcal{H}_X})$, there exists x in $K_*(X, A)$ such that $Ind_{X, A, *}^{\varepsilon, h_\varepsilon r}(x) = \iota_*^{\varepsilon, r, h_\varepsilon r}(y)$.

The proof of Theorem 3.17 will be by induction. We first prove a Mayer-Vietoris type stability result for h -isomorphism of $Ind_{X, A, *}$.

Lemma 4.8. *There exists a non increasing function*

$$(0, 1/200) \rightarrow [1, +\infty); \varepsilon \mapsto k_\varepsilon$$

such that the following is satisfied. Assume we are given

- X a compact metric space and X_1 and X_2 two closed subsets such that $X = \overset{\circ}{X}_1 \cup \overset{\circ}{X}_2$;
- non increasing functions $(0, 1/200) \rightarrow [1, +\infty); \varepsilon \mapsto h_\varepsilon$ and $(0, 1/200) \rightarrow [1, +\infty); \varepsilon \mapsto h'_\varepsilon$;
- $\delta > \frac{5}{\inf\{h_\varepsilon h'_\varepsilon; \varepsilon \in (0, 1/200)\}}$ and for $i = 1, 2$ a closed subset $X_i^{(\delta)}$ with $\{x \in X \text{ such that } d(x, X_i) \leq \delta\} \subseteq X_i^{(\delta)}$

for which

$$Ind_{X_1^{(\delta)} \cap X_2^{(\delta)}, A, *} : K_*(X_1^{(\delta)} \cap X_2^{(\delta)}, A) \rightarrow \mathcal{K}(A_{\mathcal{H}_{1,2}})$$

is for any non degenerate standard $X_1^{(\delta)} \cap X_2^{(\delta)}$ -module $\mathcal{H}_{1,2}$ a h -isomorphism and

$$Ind_{X_i^{(\delta)}, A, *} : K_*(X_i^{(\delta)}, A) \rightarrow \mathcal{K}(A_{\mathcal{H}_i})$$

is for any non degenerate standard $X_i^{(\delta)}$ -module \mathcal{H}_i a h' -isomorphism for $i = 1, 2$.

Then,

$$\text{Ind}_{X,A,*} : K_*(X, A) \rightarrow \mathcal{K}(A_{\mathcal{H}_X})$$

is a $kh'h'$ -isomorphism for any non degenerate standard X -module \mathcal{H}_X .

Proof. The lemma is proved by using a 5-lemma argument. Let (α, k) be a control pair such that the controlled Mayer-Vietoris sequence of Section 4.2 is (α, k) -exact. We can assume without loss of generality that $\alpha = 2^{n_0}$ for some integer n_0 . We can also assume that the restrictions of \mathcal{H}_X to $X_1^{(\delta)}$, $X_2^{(\delta)}$ and $X_1^{(\delta)} \cap X_2^{(\delta)}$ are also standard and then use the notations of Section 4.2. Let us first prove the injectivity part. Let us prove indeed that $\text{Ind}_{X,A,*}^{\varepsilon,r}$ is one-to-one for any ε in $(0, \frac{1}{200\alpha})$ and any r in $(0, \frac{1}{k_\varepsilon h_{\alpha\varepsilon}^2 h'_{\alpha\varepsilon}})$. The general case will follow using a rescaling argument. Let x be an element of $K_*(X, A)$ such that $\text{Ind}_{X,A,*}^{\varepsilon,r}(x) = 0$ in $K_*^{\varepsilon,r}(A_{\mathcal{H}_X})$. In view of Proposition 4.2, we have

$$\text{Ind}_{X_1^{(\delta)} \cap X_2^{(\delta)}, A,*}^{\alpha\varepsilon, k_\varepsilon r} \circ \partial_{X_1^{(\delta)}, X_2^{(\delta)}, *} (x) = 0$$

with $\alpha\varepsilon$ in $(0, 1/200)$ and $k_\varepsilon r < \frac{1}{h_{\alpha\varepsilon}^2 h'_{\alpha\varepsilon}} < \frac{1}{h_{\alpha\varepsilon}}$. Since by assumption, $\text{Ind}_{X_1^{(\delta)} \cap X_2^{(\delta)}, A,*}^{\alpha\varepsilon, k_\varepsilon r}$ is one-to-one, we deduce that $\partial_{X_1^{(\delta)}, X_2^{(\delta)}, *} (x) = 0$ in $K_*(X_1^{(\delta)} \cap X_2^{(\delta)}, A)$. Using the Mayer-Vietoris exact sequence in K -homology for the pair $(X_1^{(\delta)}, X_2^{(\delta)})$, we deduce with notations of Section 4.1 that there exist x'_1 in $K_*(X_1^{(\delta)}, A)$ and x'_2 in $K_*(X_2^{(\delta)}, A)$ such that

$$j_{X_1^{(\delta)}, *} (x'_1) - j_{X_2^{(\delta)}, *} (x'_2) = x.$$

Let us set then $y'_1 = \text{Ind}_{X_1^{(\delta)}, A,*}^{\varepsilon,r}(x'_1)$ and $y'_2 = \text{Ind}_{X_2^{(\delta)}, A,*}^{\varepsilon,r}(x'_2)$. By Lemma 3.15 and with notations of Section 4.2, we deduce that

$$j_{X_1, A,*}^{\sharp, (\delta), \varepsilon, r}(y'_1) - j_{X_2, A,*}^{\sharp, (\delta), \varepsilon, r}(y'_2) = 0$$

and hence, by controlled exactness of the quantitative Mayer-Vietoris sequence, we deduce that there exists y'' in $K_*^{\alpha\varepsilon, k_\varepsilon r}(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}})$ such that

$$j_{X_1, X_2, A,*}^{\sharp, (\delta), \alpha\varepsilon, k_\varepsilon r}(y'') = \iota_*^{-, \alpha\varepsilon, k_\varepsilon r}(y'_1)$$

and

$$j_{X_2, X_1, A,*}^{\sharp, (\delta), \alpha\varepsilon, k_\varepsilon r}(y'') = \iota_*^{-, \alpha\varepsilon, k_\varepsilon r}(y'_2).$$

Since $\alpha\varepsilon \in (0, 1/200)$ and $k_\varepsilon r < 1/h_{\alpha\varepsilon}^2$, then by assumption on $\text{Ind}_{X_1^{(\delta)} \cap X_2^{(\delta)}, A,*}$, there exists x'' in $K_*(X_1^{(\delta)} \cap X_2^{(\delta)}, A)$ such that

$$\text{Ind}_{X_1^{(\delta)} \cap X_2^{(\delta)}, A,*}^{\alpha\varepsilon, h_{\alpha\varepsilon} k_\varepsilon r}(x'') = \iota_*^{-, \alpha\varepsilon, h_{\alpha\varepsilon} k_\varepsilon r}(y'').$$

Since $k_\varepsilon h_{\alpha\varepsilon} r < 1/h'_{\alpha\varepsilon}$, then $\text{Ind}_{X_i^{(\delta)}, A,*}^{\alpha\varepsilon, h_{\alpha\varepsilon} k_\varepsilon r}$ is one-to-one for $i = 1, 2$ and hence, we deduce from Lemma 3.15 that $x'_1 = j_{X_1^{(\delta)}, X_2^{(\delta)}, *} (x'')$ and $x'_2 = j_{X_2^{(\delta)}, X_1^{(\delta)}, *} (x'')$. We deduce from this that $x = 0$ and hence $\text{Ind}_{X,A,*}^{\varepsilon,r}$ is one-to-one if ε is in $(0, \frac{1}{200\alpha})$ and r is in $(0, \frac{1}{k_\varepsilon h_{\alpha\varepsilon}^2 h'_{\alpha\varepsilon}})$. Since we have

chosen $\alpha = 2^{n_0}$, we see using Proposition 3.13 that $Ind_{X,A,*}^{\varepsilon,r}$ is one-to-one if ε is in $(0, \frac{1}{200})$ and r is in $(0, \frac{1}{k'_\varepsilon h_\varepsilon h'_\varepsilon})$ with $k'_\varepsilon = l_\varepsilon \cdots l_{\varepsilon/2^{n_0-1}} k_{\varepsilon/2^{n_0}}$ and $l = l_{rs}$ (with notations of Section 1.2).

Let us prove now the surjectivity statement. We can assume without loss of generality that (α, k) is greater than the control pair of [17, Corollary 3.6]. Let us set $l'_\varepsilon = l_\varepsilon \cdots l_{\varepsilon/2^{n_0-1}}$ with l_ε as above and $k'_\varepsilon = l'_\varepsilon k_\varepsilon k_{\varepsilon/\alpha}$. Let us prove first that for any ε in $(0, \frac{1}{200\alpha})$, any r in $(0, \frac{1}{k'_\varepsilon h_{\alpha\varepsilon} h'^2_{\alpha\varepsilon}})$ and any y in $K_*^{\varepsilon,r}(A_{\mathcal{H}_X})$, there exists x_y in $K_*(X, A)$ such that

$$Ind_{X,A,*}^{\alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} h'_{\alpha\varepsilon} r}(x_y) = \iota_*^{-, \alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} h'_{\alpha\varepsilon} r}(y).$$

With notations as above, let us set

$$y' = \iota_*^{-, \alpha\varepsilon, k_\varepsilon r} \circ \partial_{X_1, X_2, A, *}^{(\delta), \varepsilon, r}(y).$$

Since $\alpha\varepsilon$ is in $(0, 1/200)$ and $k_\varepsilon r < \frac{1}{l'_\varepsilon k_{\varepsilon/\alpha} h_{\alpha\varepsilon} h'^2_{\alpha\varepsilon}} \leq \frac{1}{h_{\alpha\varepsilon}^2}$, there exists x' in $K_{*+1}(X_1^{(\delta)} \cap X_2^{(\delta)}, A)$ such that

$$Ind_{X,A,*}^{\alpha\varepsilon, h_{\alpha\varepsilon} k_\varepsilon r}(x') = \iota_*^{-, \alpha\varepsilon, h_{\alpha\varepsilon} k_\varepsilon r}(y').$$

Let us set then

$$x'_1 = j_{X_1^{(\delta)}, X_2^{(\delta)}, *} (x') \in K_{*+1}(X_1^{(\delta)}, A)$$

and

$$x'_2 = j_{X_2^{(\delta)}, X_1^{(\delta)}, *} (x') \in K_{*+1}(X_2^{(\delta)}, A).$$

According to Lemma 3.15, we have

$$Ind_{X_i^{(\delta)}, A, *}^{\alpha\varepsilon, h_{\alpha\varepsilon} k_\varepsilon r}(x'_i) = 0$$

for $i = 1, 2$ and hence, since $\alpha\varepsilon \in (0, 1/200)$ and

$$h_{\alpha\varepsilon} k_\varepsilon r < \frac{1}{h'_{\alpha\varepsilon}},$$

we deduce by injectivity of $Ind_{X_i^{(\delta)}, A, *}^{\alpha\varepsilon, h_{\alpha\varepsilon} k_\varepsilon r}$ that $x'_i = 0$ for $i = 1, 2$. Hence, by exactness of the Mayer-Vietoris sequence in K -homology, there exists an element x in $K_*(X, A)$ such that

$$\partial_{X_1^{(\delta)}, X_2^{(\delta)}, *} (x) = x'.$$

Using Proposition 4.2, we see that

$$\iota_*^{-, \alpha\varepsilon, k_\varepsilon h_{\alpha\varepsilon} r} \circ \partial_{X_1, X_2}^{(\delta), \varepsilon, r}(Ind_{X,A,*}^{\varepsilon,r}(x) - y) = 0$$

in $K_{*+1}^{\alpha\varepsilon, k_\varepsilon h_{\alpha\varepsilon} r}(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}})$. Let us set

$$y_\alpha = L_{A_{\mathcal{H}_X}, *}^{\varepsilon/2^{n_0-1}, l_{\varepsilon/2^{n_0-2}} \cdots l_\varepsilon r} \circ \cdots \circ L_{A_{\mathcal{H}_X}, *}^{\varepsilon/2, l_\varepsilon r} \circ L_{A_{\mathcal{H}_X}, *}^{\varepsilon, r}(y).$$

Then y_α lies in $K_*^{\varepsilon/\alpha, l'_\varepsilon r}(A_{\mathcal{H}_X})$ and in view of Lemma 4.1 and of Proposition 3.13, we see that

$$\iota_*^{-, \varepsilon, k_\varepsilon/\alpha h_{\alpha\varepsilon} l'_\varepsilon r} \circ \partial_{X_1, X_2}^{(\delta), \varepsilon/\alpha, l'_\varepsilon r}(Ind_{X, A, *}^{\varepsilon/\alpha, l'_\varepsilon r}(x) - y_\alpha) = 0$$

in $K_{*+1}^{\varepsilon, k_\varepsilon/\alpha h_{\alpha\varepsilon} l'_\varepsilon r}(A_{\mathcal{H}_{X_1^{(\delta)} \cap X_2^{(\delta)}}})$.

Then according to [17, Corollary 3.6], there exists y_i in $K_*^{\alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} r}(A_{\mathcal{H}_{X_i^{(\delta)}}})$ such that

$$\begin{aligned} j_{X_1, A, *}^{\sharp, (\delta), \alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} r}(y_1) - j_{X_2, A, *}^{\sharp, (\delta), \alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} r}(y_2) &= Ind_{X, A, *}^{\alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} r}(x) - \iota_*^{-, \alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} r}(y_\alpha) \\ &= Ind_{X, A, *}^{\alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} r}(x) - \iota_*^{-, \alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} r}(y), \end{aligned}$$

where the second equality holds in view of the first point of Lemma 1.7 (recall that $k'_\varepsilon = k_\varepsilon k_{\varepsilon/\alpha} l'_\varepsilon$). Since $\alpha\varepsilon$ is in $(0, 1/200)$ and $k'_\varepsilon h_{\alpha\varepsilon} r < \frac{1}{h_{\alpha\varepsilon}^2}$, we see by surjectivity assumption for $Ind_{X_i^{(\delta)}, A, *}^{\alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} r}$ for $i = 1, 2$ that there exists x_i in $K_*(X_i^{(\delta)}, A)$ such that

$$Ind_{X_i^{(\delta)}, A, *}^{\alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} h'_{\alpha\varepsilon} r}(x_i) = \iota_*^{-, \alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} h'_{\alpha\varepsilon} r}(y_i).$$

Eventually we set

$$x_y = x - j_{X_1^{(\delta)}, *}^{(\delta)}(x_1) + j_{X_2^{(\delta)}, X_2^{(\delta)}, *}^{(\delta)}(x_2)$$

in $K_*(X, A)$. Using Lemma 3.15, we obtain that

$$\iota_*^{-, \alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} h'_{\alpha\varepsilon} r}(y) = Ind_{X, A, *}^{\alpha\varepsilon, k'_\varepsilon h_{\alpha\varepsilon} h'_{\alpha\varepsilon} r}(x_y).$$

Now, for ε in $(0, 1/200)$, r in $(0, \frac{1}{l'_\varepsilon k'_\varepsilon/\alpha h_\varepsilon h_\varepsilon^2})$ and y in $K_*^{\varepsilon, r}(A_{\mathcal{H}_X})$, we can apply the preceding result to

$$L_{A_{\mathcal{H}_X}, *}^{\varepsilon/2^{n_0-1}, l_{\varepsilon/2^{n_0-2}} \dots l_{\varepsilon/2} l_\varepsilon r} \circ \dots \circ L_{A_{\mathcal{H}_X}, *}^{\varepsilon/2, l_\varepsilon r} \circ L_{A_{\mathcal{H}_X}, *}^{\varepsilon, r}(y).$$

We deduce from Proposition 3.13 that there exist x in $K_*(X, A)$ such that

$$Ind_{X, A, *}^{\varepsilon, l'_\varepsilon k'_\varepsilon/\alpha h_\varepsilon h'_\varepsilon r}(x) = y.$$

□

Let us first prove Theorem 3.17 for a finite simplicial complex of dimension 0.

Lemma 4.9. *Let Z_0 be a finite metric set such that for some $\delta > 0$, then $d(x, x') \geq \delta$ for every distinct points x and x' in Z_0 and let A be a C^* -algebra. Then $Ind_{Z_0, A, *}^{\varepsilon, r}$ is an isomorphism for any ε in $(0, 1/200)$ and any r in $(0, \delta)$.*

Proof. We have an isomorphism

$$K_*(Z_0, A) \xrightarrow{\cong} K_*(A)^{Z_0} \quad (23)$$

induced by the family of evaluation at points in Z_0 . Any non degenerate standard Z_0 -module $(\rho_{Z_0}, \mathcal{H}_{Z_0})$ is unitary equivalent to $(\rho, \ell^2(Z_0) \otimes \mathcal{H})$ where \mathcal{H} is a separable Hilbert space and ρ is the pointwise multiplication. Hence we can assume without loss of generality that $(\mathcal{H}_{Z_0}, \rho_{Z_0}) = (\ell^2(Z_0) \otimes \mathcal{H}, \rho)$ and then we have a Morita equivalence $K_*^{\varepsilon, r}(A_{\mathcal{H}_{Z_0}}) \cong K_*^{\varepsilon, r}(A \otimes \mathcal{K}(\ell^2(Z_0)))$. Any operator in $A \otimes \mathcal{K}(\ell^2(Z_0))$ with propagation less than δ is actually diagonal. Hence if r is a positive number with $r < \delta$, then any ε - r projection of $A \otimes \mathcal{K}(\ell^2(Z_0))$ is given by a family $(q_z)_{z \in Z_0}$ of self-adjoint elements in A such that $\|q_z^2 - q_z\| < \varepsilon$. Since ε lies in $(0, 1/200)$, then the spectra of q_z lies in $(-\infty, 1/5) \cup (4/5, +\infty)$. Let us fix a continuous function $\kappa_0 : \mathbb{R} \rightarrow [0, 1]$ such that

- $\kappa_0(t) = 0$ if $t \leq 1/5$
- $\kappa_0(t) = 1$ if $t \geq 4/5$.

If $P_n(A)$ stands for the set of projections in $M_n(A)$, it is then straightforward to see that

$$P_n^{\varepsilon, r}(A \otimes \mathcal{K}(\ell^2(Z_0))) \longrightarrow P_n(A)^{Z_0}; \text{diag}(q_z)_{z \in Z_0} \mapsto (\kappa(q_z))_{z \in Z_0}$$

induces an isomorphism

$$K_0^{\varepsilon, r}(A \otimes \mathcal{K}(\ell^2(Z_0))) \xrightarrow{\cong} K_0(A)^{Z_0}.$$

Similarly, in the odd case, the map

$$U^{\varepsilon, r}(A \otimes \mathcal{K}(\ell^2(Z_0))) \longrightarrow GL(A)^{Z_0}; \text{diag}(u_z)_{z \in Z_0} \mapsto (u_z)_{z \in Z_0}$$

induces an isomorphism

$$K_1^{\varepsilon, r}(A \otimes \mathcal{K}(\ell^2(Z_0))) \xrightarrow{\cong} K_1(A)^{Z_0}.$$

Under these identifications and that of equation (23), then

$$\text{Ind}_{Z_0, A, *}: K_*(Z_0, A) \longrightarrow K_*(A \otimes \mathcal{K}(\mathcal{H}_{Z_0}))$$

is the identity map for any positive number ε and r with $\varepsilon < 1/200$ and $r < \delta$. \square

As a first application of Lemma 4.8, we have the following estimates for boundaries of simplices. Let $\partial\Delta_n$ be the boundary of the standard simplex of dimension n equipped with the normalized spherical metric.

Lemma 4.10. *There exists a non increasing function*

$$(0, 1/200) \rightarrow [1, +\infty); \varepsilon \mapsto h_\varepsilon,$$

such that for any integer n , for any C^ -algebra A and for any non degenerate standard $\partial\Delta_n$ -module $\mathcal{H}_{\partial\Delta_n}$, then*

$$\text{Ind}_{\partial\Delta_n, A, *}: K_*(\partial\Delta_n, A) \longrightarrow K_*(A_{\mathcal{H}_{\partial\Delta_n}})$$

is a h^n -isomorphism.

Proof. Let us view $\partial\Delta_n$ as embedded in \mathbb{S}_n and let us consider for α in $(0, 1)$

$$\partial\Delta_{n,\alpha}^+ = \left\{ \cos \theta \cdot v + \sin \theta \cdot e_{n+1}; \theta \in \left[\frac{\alpha\pi}{2}, \pi/2 \right], v \in \mathbb{S}_{n-1} \subseteq \mathbb{S}_n \right\} \cap \partial\Delta_n$$

and

$$\partial\Delta_{n,\alpha}^- = \left\{ \cos \theta \cdot v + \sin \theta \cdot e_{n+1}; \theta \in \left[0, \frac{\alpha\pi}{2} \right], v \in \mathbb{S}_{n-1} \subseteq \mathbb{S}_n \right\} \cap \partial\Delta_n.$$

Then for any β in $(0, \alpha)$, we have

$$\{v \in \partial\Delta_n \text{ such that } d(v, \partial\Delta_{n,\alpha}^+) < \beta\} \subseteq \partial\Delta_{n,\alpha-\beta}^+$$

and for any β in $(0, 1 - \alpha)$, we have

$$\{v \in \partial\Delta_n \text{ such that } d(v, \partial\Delta_{n,\alpha}^-) < \beta\} \subseteq \partial\Delta_{n,\alpha+\beta}^-.$$

For δ in $(0, 1/6)$, we set $X_1 = \partial\Delta_{n,1/3}^+$, $X_1^\delta = \partial\Delta_{n,1/3-\delta}^+$, $X_2 = \partial\Delta_{n,2/3}^-$ and $X_2^\delta = \partial\Delta_{n,2/3+\delta}^-$.

We have thus that

- X_1^δ can be contracted to $\{e_{n+1}\}$ by a 1-Lipschitz homotopy;
- X_2^δ is homotopic to Δ_{n-1} by using a $3/2$ -Lipschitz homotopy which in turn can be contracted to its center by a 1-Lipschitz homotopy;
- $X_1^\delta \cap X_2^\delta$ is homotopic to

$$\left\{ \frac{1}{\sqrt{2}}(v + e_{n+1}); v \in \mathbb{S}_n \right\} \cap \partial\Delta_n \quad (24)$$

by a 3-Lipschitz homotopy. The metric subspace of equation (24) is a rescaling of $\partial\Delta_{n-1}$ with a factor $\frac{1}{\sqrt{2}}$. Using equation (7) of Section 3.2, Proposition 3.7 and Lemma 3.15 and in view of Lemma 4.9, we see that $\text{Ind}_{X_i^{(\delta)}, A, *}$ is a C -isomorphism for some constant C that does not depend on n and that we can choose such that $C > 30$. Pick then δ in $(5/C, 1/6)$, the result is then by induction a consequence of Lemma 4.8. \square

We are now in a position to achieve our goal.

Proof of Theorem 3.17 : Let X be a connected finite simplicial complex of dimension n . Let us prove the result by induction for $k = 0, \dots, n$ and any simplicial subcomplex of X of dimension k . According to Lemma 23, the result clearly holds for subcomplexes of dimension 0. Let us fix k in $1, \dots, n$ and let Z be simplicial subcomplex of X of dimension k . Let Z'_0 be the set of centers of k -simplexes in Z . For any σ in Z'_0 let Δ_σ be the k -dimension simplex that contains σ and define for any λ in $(0, 1)$ the set

$$\Delta_{\sigma,\lambda} = \{x \in \Delta_\sigma; \exists x' \in \Delta_\sigma / d(x, x') = (1-\lambda)d(x', \sigma) \text{ and } d(x, \sigma) = \lambda d(x', \sigma)\}.$$

In other word, an element x of Δ_σ belongs to $\Delta_{\sigma,\lambda}$ if there exists x' in Δ_σ such that x is in the geodesic arc between σ and x' at distance $\lambda d(x', \sigma)$ from σ and $(1 - \lambda)d(x', \sigma)$ from x' . Let us set then

$$Z_1 = \bigsqcup_{\sigma \in Z'_0} \Delta_{\sigma, 2/3},$$

$$Z_2 = \overline{Z \setminus \bigsqcup_{\sigma \in Z'_0} \Delta_{\sigma, 1/3}},$$

and for δ in $(0, \frac{2}{3\pi^2(k+1)})$,

$$Z_1^{(\delta)} = \bigsqcup_{\sigma \in Z'_0} \Delta_{\sigma, 2/3 + \delta\pi\sqrt{\frac{k+1}{4}}},$$

and

$$Z_2^{(\delta)} = \overline{Z \setminus \bigsqcup_{\sigma \in Z'_0} \Delta_{\sigma, 1/3 - \delta\pi\sqrt{\frac{k+1}{4}}}.$$

A lengthy but straightforward computation shows that for $(0, \frac{2}{3\pi^2(k+1)})$ and $i = 1, 2$, we have

$$\{x \in Z \text{ such that } d(x, Z_i) < \delta\} \subseteq Z_i^{(\delta)}.$$

Using equation (7) of Section 3.2, Proposition 3.7 and Lemma 3.15, we see that

- $\Delta_{\sigma, 2/3 + \delta\pi\sqrt{\frac{k+1}{4}}}$ being homotopic to $\{\sigma\}$ by a C -Lipschitz homotopy with C which depends neither on σ , k nor δ , we deduce that $Z_1^{(\delta)}$ is homotopic to Z'_0 by a C -Lipschitz homotopy. Noticing that the distance between two points in Z'_0 is greater than $\frac{4}{\pi\sqrt{k+1}}$, we get according to Lemma 4.9 that $Ind_{Z_1^{(\delta)}, A, *}$ is a $C'\sqrt{k+1}$ -isomorphism for $C' > 1$ that depends neither on k nor δ .
- since

$$Z_1^{(\delta)} \cap Z_2^{(\delta)} = \bigsqcup_{\sigma \in Z'_0} \Delta_{\sigma, 2/3 + \delta\pi\sqrt{\frac{n+1}{4}}} \cap \overline{\Delta_\sigma \setminus \Delta_{\sigma, 1/3 - \delta\pi\sqrt{\frac{n+1}{4}}}}$$

with

$$\Delta_{\sigma, 2/3 + \delta\pi\sqrt{\frac{n+1}{4}}} \cap \overline{\Delta_\sigma \setminus \Delta_{\sigma, 1/3 - \delta\pi\sqrt{\frac{n+1}{4}}}}$$

homotopic to the boundary of Δ_σ by a C -Lipschitz homotopy with C which depends neither on k nor δ , then according to Lemma 4.10, there exists a non increasing function

$$(0, 1/200) \rightarrow [1, +\infty); \varepsilon \mapsto h_\varepsilon$$

that we can choose satisfying $h^n > 10\pi^2(n+1)$ for all positive integer n and such that $Ind_{Z_1^{(\delta)} \cap Z_2^{(\delta)}, A, *}$ is a h^k -isomorphism.

Since $Z_2^{(\delta)}$ is homotopic to the $k - 1$ -skeleton of Z by a C -Lipschitz homotopy with C independent of k , and in view once again of equation (7) of Section 3.2, Proposition 3.7 and Lemma 3.15, we can then apply Lemma 4.8 to $Z = Z_1 \cup Z_2$, h , $h' = C'\sqrt{k+1}$ and Z_1^δ and Z_2^δ for δ in $(\frac{1}{2\pi^2(n+1)}, \frac{2}{3\pi^2(n+1)})$. The result follows then by induction.

5. QUANTITATIVE ASSEMBLY MAP ESTIMATES AND COARSE DECOMPOSABILITY

In this section, we give the proof of Theorem 2.13.

5.1. Preliminary.

Definition 5.1. *Let X and Y be two discrete proper metric spaces such that $Y \subseteq X$ and let s be a positive number. The metric space Y is called a s -net for X if for any element x in X there exists an element y in Y such that $d(x, y) < s$.*

Lemma 5.2. *Let X be a proper metric space, let s be a positive number, let Y be a s -net for X and let \mathcal{H} be a separable Hilbert space. Then for any positive number d and for any rank 1 projection e on \mathcal{H} , there exists a continuous map*

$$\phi_d : P_d(X) \longrightarrow P_{d+2s}(Y)$$

and a unitary map

$$V_d : \ell^2(X, \mathcal{H}) \rightarrow \ell^2(Y, \mathcal{H})$$

such that

- (i) for any z in $P_d(X)$, then $\text{supp } \phi_d(z)$ has support contained in

$$\{y \in Y \text{ such that } d(y, \text{supp } z) < s\}$$

(where $\text{supp } z$ is the support of z as a probability measure on X);

- (ii) V_d has support contained in

$$\{(x, y) \in X \times Y \text{ such that } d(x, y) < s\};$$

- (iii) for any compact subset Z of $P_d(X)$ and for any positive number r with $r > 2d+4s$, then $Ad_{V_d, C(Z)} \cdot (P_Z \otimes e)$ and $\phi_{d, \ell^2(X, \mathcal{H})}^*(P_{\phi_d(Z)} \otimes e)$ are homotopic as ε -($r+2s$)-projections in $C(Z)_{\ell^2(Y, \mathcal{H})}$, where

- $\phi_{d, \ell^2(X, \mathcal{H})}^* : C(\phi_d(Z))_{\ell^2(X, \mathcal{H})} \longrightarrow C(Z)_{\ell^2(X, \mathcal{H})}$ is the map induced by ϕ_d ;
- P_Z and $P_{\phi_d(Z)}$ are respectively the projections in $C(Z)_Y = C(Z) \otimes \mathcal{K}(\ell^2(Y))$ and $C(\phi_d(Z))_X = C(\phi_d(Z)) \otimes \mathcal{K}(\ell^2(X))$ constructed in Section 2.1.

Proof. Let $(\lambda_{X,x})_{x \in X}$ and $(\lambda_{Y,y})_{y \in Y}$ be respectively the coordinate systems for the Rips complexes of X and Y . For each x in X , pick \tilde{x} in Y such that $d(x, \tilde{x}) < s$ and set $X_y = \{x \in X, \tilde{x} = y\}$ for any y in Y . Define then $\phi_d : P_d(X) \rightarrow P_{d+2s}(Y)$ by

$$\lambda_{Y,y}(\phi_d(z)) = \sum_{x \in X_y} \lambda_{X,x}(z)$$

for any z in $P_d(X)$ and any y in Y . Then $\phi_d(z)$ is clearly a probability measure supported in

$$\{y \in Y \text{ such that } d(y, \text{supp } z) < s\}.$$

Let η be a vector in \mathcal{H} with $\|\eta\| = 1$ such that e is the projection on $\mathbb{C} \cdot \eta$ and let us choose for any y in Y a unitary

$$W_y : \ell^2(X_y) \otimes \mathcal{H} \rightarrow \mathcal{H}$$

such that

$$W_y \cdot \left(\frac{1}{\sqrt{\#X_y}} \chi_{X_y} \otimes \eta \right) = \eta,$$

where χ_{X_y} is the characteristic function of X_y viewed as an element in $\ell^2(X_y)$. Define then

$$V_d = \bigoplus_{y \in Y} W_y : \ell^2(X) \otimes \mathcal{H} \cong \bigoplus_{y \in Y} \ell^2(X_y) \otimes \mathcal{H} \rightarrow \ell^2(Y) \otimes \mathcal{H}.$$

The support of V_d is obviously in $\{(x, y) \in X \times Y \text{ such that } d(x, y) < s\}$. Now for any z in Z , then $\phi_{d, \mathcal{H}(\ell^2(X) \otimes \mathcal{H})}^* \cdot (P_{\phi_d(Z)} \otimes e)(z)$ is the projection associated to

$$\left(\left(\sum_{x \in X_y} \lambda_{X,x}(z) \right)^{1/2} \otimes \eta \right)_{y \in Y}.$$

Define for all x in X

$$\lambda'_x = \frac{1}{\#X_{\tilde{x}}} \sum_{x' \in X_{\tilde{x}}} \lambda_{X,x'}.$$

Consider then for any t in $[0, 1]$ and any x in Z the projection $q_t(z)$ on

$$(((1-t)\lambda_{X,x}(z) + t\lambda'_x(z))^{1/2} \otimes \eta)_{x \in X}.$$

If $r > 2d + 4s$, then $(q_t)_{t \in [0,1]}$ is a projection in $C(Z) \otimes \mathcal{K}(\ell^2(X) \otimes \mathcal{H})$ with propagation less than r such that $q_0 = P_Z \otimes e$ and

$$Ad_{V_d, C(Z)} \cdot q_1 = \phi_{d, \mathcal{H}(\ell^2(X) \otimes \mathcal{H})}^* \cdot (P_{\phi_d(Z)} \otimes e).$$

□

Remark 5.3. Let A be a C^* -algebra and let d, s, X and Y be as in the lemma and recall that $A_X = A \otimes \mathcal{K}(\ell^2(X))$ and $A_Y = A \otimes \mathcal{K}(\ell^2(Y))$.

(i) According to Lemma 3.3, then the compositions

$$\mathcal{K}(\ell^2(X) \otimes \mathcal{H}) \xrightarrow{Ad_{V_d}} \mathcal{K}(\ell^2(Y) \otimes \mathcal{H}) \hookrightarrow \mathcal{K}(\ell^2(X) \otimes \mathcal{H})$$

and

$$\mathcal{K}(\ell^2(Y) \otimes \mathcal{H}) \hookrightarrow \mathcal{K}(\ell^2(X) \otimes \mathcal{H}) \xrightarrow{Ad_{V_d}} \mathcal{K}(\ell^2(Y) \otimes \mathcal{H})$$

respectively induce for $r > 2d+4s$ and up to Morita equivalence

$$\iota_*^{\varepsilon, r, r+2s} : K_*^{\varepsilon, r}(A_X) \longrightarrow K_*^{\varepsilon, r+2s}(A_X)$$

and

$$\iota_*^{\varepsilon, r, r+2s} : K_*^{\varepsilon, r}(A_Y) \longrightarrow K_*^{\varepsilon, r+2s}(A_Y).$$

(ii) The compositions

$$P_d(X) \xrightarrow{\phi_d} P_{d+2s}(Y) \hookrightarrow P_{d+2s}(X)$$

and

$$P_d(Y) \hookrightarrow P_d(X) \xrightarrow{\phi_d} P_{d+2s}(Y)$$

are respectively homotopic to the inclusions $P_d(X) \hookrightarrow P_{d+2s}(X)$ and $P_d(Y) \hookrightarrow P_{d+2s}(Y)$ and hence induce respectively the morphisms

$$q_{d, d+2s, *}^X : K_*(P_d(X), A) \longrightarrow K_*(P_{d+2s}(X), A)$$

and

$$q_{d, d+2s, *}^Y : K_*(P_d(Y), A) \longrightarrow K_*(P_{d+2s}(Y), A).$$

(iii)

$$q_{d+2s, d'+2s, *}^Y \circ \phi_{d, *} = \phi_{d', *} \circ q_{d, d', *}^X$$

for any positive number d' with $d' \geq d$;

(iv)

$$\iota_*^{\varepsilon, r+2s, r'+2s} \circ Ad_{V_d, *}^{\varepsilon, r} = Ad_{V_{d'}, *}^{\varepsilon, r'} \circ \iota_*^{\varepsilon, r, r'}$$

for any positive number r' with $r' \geq r$, $r \geq 2d+4s$ and $r' \geq 2d'+4s$.

Lemma 5.4. *Let \mathcal{X} and \mathcal{Y} be families of proper discrete metric spaces. Assume that there exists a positive number s such that*

- *for any X in \mathcal{X} , there exists a set Y in \mathcal{Y} contained in X as a s -net;*
- *for any Y in \mathcal{Y} , there exists X in \mathcal{X} that contains Y as a s -net.*

Then the following assertions are equivalent:

- (i) *the family \mathcal{X} satisfies the QAM-estimates;*
- (ii) *the family \mathcal{Y} satisfies the QAM-estimates.*

Proof. Let X be an element of \mathcal{X} and let Y be a s -net for X that belongs to \mathcal{Y} , let

$$j_{X,Y}^\sharp : \mathcal{K}(\ell^2(Y)) \rightarrow \mathcal{K}(\ell^2(X))$$

be the inclusion and for any C^* -algebra A , let

$$j_{P_d(X), P_d(Y), *}: K_*(P_d(Y), A) \longrightarrow K_*(P_d(X), A)$$

be the morphism induced by the inclusion $P_d(Y) \hookrightarrow P_d(X)$. Then according to Lemma 2.1, we have

$$\nu_{A,X,*}^{\varepsilon,r,d} \circ j_{P_d(X), P_d(Y), *} = j_{X,Y,A,*}^{\sharp,\varepsilon,r} \circ \nu_{A,Y,*}^{\varepsilon,r,d}, \quad (25)$$

where

$$j_{X,Y,A,*}^\sharp : \mathcal{K}_*(A_Y) \longrightarrow \mathcal{K}_*(A_X)$$

is the morphism induced by $j_{X,Y}^\sharp$. According to Remark 1.27 and Lemma 5.2, we also have (up to Morita equivalence)

$$Ad_{V_d,A,*}^{\varepsilon,r} \circ \nu_{A,X,*}^{\varepsilon,r,d} = \nu_{A,Y,*}^{\varepsilon,r+2s,d+2s} \circ \phi_{d,*} \quad (26)$$

for every number r such that $r \geq r_{d+2s,\varepsilon}$, where

$$\phi_{d,*} : K_*(P_d(X), A) \longrightarrow K_*(P_{d+2s}(X), A)$$

is the morphism induces by ϕ_d as in Lemma 5.2. In view of Remark 5.3 and using Equations (25) and (26), then for any positive numbers ε, d, d', r and r' with $\varepsilon < \frac{1}{200}$, then

- $QI_{X,A,*}(d, d', r, \varepsilon)$ implies $QI_{Y,A,*}(d, d' + 2s, r, \varepsilon)$ if $r \geq r_{d,\varepsilon}$ and $d' \geq d$;
- $QI_{Y,A,*}(d + 2s, d', r + 2s, \varepsilon)$ implies $QI_{X,A,*}(d, d' + 2s, r, \varepsilon)$ if $r \geq r_{d+2s,\varepsilon}$ and $d' \geq d + 2s$;
- $QS_{X,A,*}(d, r, r', \varepsilon)$ implies $QS_{Y,A,*}(d + 2s, r, r' + 2s, \varepsilon)$ if $r' \geq r$ and $r' > r_{d+2s,\varepsilon} - 2s$;
- $QS_{Y,A,*}(d, r + 2s, r', \varepsilon)$ implies $QS_{X,A,*}(d, r, r', \varepsilon)$ if $r' \geq r + 2s$ and $r' > r_{d,\varepsilon}$.

□

5.2. Notations. For convenience of the readers, we collect in this section some notations that we shall use throughout the proof of Theorem 2.13.

- If Z is compact metric space, then for any subset Y of Z and any positive number s , we set $Y_s = \{z \in Z \text{ such that } d(Y, z) \leq s\}$;
- If X_1 and X_2 are two subsets of a finite metric space X and A is a C^* -algebra, then $A_{X_1, X_2, r} = (A \otimes \mathcal{K}(\ell^2(X_1), \ell^2(X_2))) \cap A_{X, r}$ (recall that $A_X = A \otimes \mathcal{K}(\ell^2(X))$);
- if (\mathcal{H}, ρ) is a Z -module,
 - we set $A_{\mathcal{H}} = A \otimes \mathcal{K}(\mathcal{H})$. Then $A_{\mathcal{H}}$ is filtered by

$$(A_{\mathcal{H}, r})_{r>0} = (A \otimes \mathcal{K}(\mathcal{H})_r)_{r>0};$$

- if \mathcal{H}' is a closed linear subspace of \mathcal{H} , we view $A_{\mathcal{H}'}$ as a corner sub- C^* -algebra of $A_{\mathcal{H}}$ and $Id_{\mathcal{H}'}$ as the projection onto \mathcal{H}' ;
- If \mathcal{H}' and \mathcal{H}'' are two closed linear subsets of \mathcal{H} , we view

$$A_{\mathcal{H}', \mathcal{H}''} \stackrel{\text{def}}{=} A \otimes \mathcal{K}(\mathcal{H}', \mathcal{H}'')$$

as a closed subset of $A_{\mathcal{H}}$ and we set $A_{\mathcal{H}', \mathcal{H}'', r} = A_{\mathcal{H}', \mathcal{H}''} \cap A_{\mathcal{H}, r}$;

- let X be a finite metric space, let d be a positive number and let (\mathcal{H}_d, ρ_d) be a $P_d(X)$ -module.

– $V_{\mathcal{H}_d} : \mathcal{H}_d \rightarrow \ell^2(X, \mathcal{H}_d)$ is the isometry defined by

$$(V_{\mathcal{H}_d} \cdot \xi)(x) = \rho(\lambda_x^{1/2}) \cdot \xi$$

for any ξ in \mathcal{H}_d , where $(\lambda_x)_{x \in X}$ is the coordinates system in $P_d(X)$.

Let X be a finite metric space and let $X^{(1)}$ and $X^{(2)}$ be two subsets of X such that $X^{(1)} \cup X^{(2)}$. Let δ_0 be a non negative number and let δ_1 and r be positive numbers such that $\delta_1 > 5r$. With notations as above and for any C^* -algebra A ,

$$\left(A_{X_{\delta_0}^{(1)}, X_{\delta_0+\delta_1}^{(1)}, r}, A_{X_{\delta_0}^{(2)}, X_{\delta_0+\delta_1}^{(2)}, r}, A_{X_{\delta_0+\delta_1}^{(1)}}, A_{X_{\delta_0+\delta_1}^{(2)}} \right) \quad (27)$$

is a r -controlled Mayer-Vietoris pair for A_X . We denote respectively by

$$j_{X,A}^{(1),\#} : A_{X_{\delta_0+\delta_1}^{(1)}} \longrightarrow A_X,$$

$$j_{X,A}^{(2),\#} : A_{X_{\delta_0+\delta_1}^{(2)}} \longrightarrow A_X,$$

$$j_{X,A}^{(1,2),\#} : A_{X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}} \longrightarrow A_{X_{\delta_0+\delta_1}^{(1)}}$$

and

$$j_{X,A}^{(2,1),\#} : A_{X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}} \longrightarrow A_{X_{\delta_0+\delta_1}^{(2)}}$$

the morphisms of filtered C^* -algebras induced by the inclusion $\ell^2(X_{\delta_0+\delta_1}^{(1)}) \hookrightarrow \ell^2(X)$, $\ell^2(X_{\delta_0+\delta_1}^{(2)}) \hookrightarrow \ell^2(X)$, $\ell^2(X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}) \hookrightarrow \ell^2(X_{\delta_0+\delta_1}^{(1)})$ and $\ell^2(X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}) \hookrightarrow \ell^2(X_{\delta_0+\delta_1}^{(2)})$. Let

$$\mathcal{D}_{X_{\delta_0+\delta_1}^{(1)}, X_{\delta_0+\delta_1}^{(2)}, A, *} = (\partial_{X_{\delta_0+\delta_1}^{(1)}, X_{\delta_0+\delta_1}^{(2)}, A, *}^{\varepsilon, s})_{0 < \varepsilon < \frac{1}{4\alpha}, 0 < s < \frac{r}{k\varepsilon}} : \mathcal{K}_*(A_X) \longrightarrow \mathcal{K}_*(A_{X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}})$$

be the associated controlled Mayer-Vietoris boundary map (of order r) with $(\alpha, k) = (\alpha_{MV}, k_{MV})$.

Let d be a positive number and let (\mathcal{H}_d, ρ_d) be a non degenerate $P_d(X)$ -module. For $i = 1, 2$, and $\delta > 0$, let $\mathcal{H}_{d,\delta}^{(i)}$ be the set of elements

of \mathcal{H}_d with support in $P_d(X_\delta^{(i)})$. Assume now that $\delta_0 > 2d$ and $\delta_1 > (2d+1)(5r+1)$. It is straightforward to check that

$$P_d(X) = \widehat{P_d(X_{\delta_0}^{(1)})} \cup \widehat{P_d(X_{\delta_0}^{(2)})}. \quad (28)$$

Let us describe the r -controlled Mayer-Vietoris for $A_{\mathcal{H}_d}$ associated to the decomposition $P_d(X) = P_d(X_{\delta_0}^{(1)}) \cup P_d(X_{\delta_0}^{(2)})$. Since $P_d(Y)_s \subseteq P_d(Y_{2d(s+1)})$ for any subset Y of X and any positive number s ,

$$\left(A_{\mathcal{H}_{d,\delta_0}^{(1)}, \mathcal{H}_{d,\delta_0+\delta_1}^{(1)}, r}, A_{\mathcal{H}_{d,\delta_0}^{(2)}, \mathcal{H}_{d,\delta_0+\delta_1}^{(1)}, r}, A_{\mathcal{H}_{d,\delta_0+\delta_1}^{(1)}}, A_{\mathcal{H}_{d,\delta_0+\delta_1}^{(2)}} \right) \quad (29)$$

is a r -controlled Mayer-Vietoris pair for $A_{\mathcal{H}_d}$. We denote respectively by

$$\begin{aligned} j_{P_d(X), A}^{(1), \sharp} : A_{\mathcal{H}_{d,\delta_0+\delta_1}^{(1)}} &\longrightarrow A_{\mathcal{H}_d}, \\ j_{P_d(X), A}^{(2), \sharp} : A_{\mathcal{H}_{d,\delta_0+\delta_1}^{(2)}} &\longrightarrow A_{\mathcal{H}_d}, \\ j_{P_d(X), A}^{(1,2), \sharp} : A_{\mathcal{H}_{d,\delta_0+\delta_1}^{(1)} \cap \mathcal{H}_{d,\delta_0+\delta_1}^{(2)}} &\longrightarrow A_{\mathcal{H}_{d,\delta_0+\delta_1}^{(1)}} \end{aligned}$$

and

$$j_{P_d(X), A}^{(2,1), \sharp} : A_{\mathcal{H}_{d,\delta_0+\delta_1}^{(1)} \cap \mathcal{H}_{d,\delta_0+\delta_1}^{(2)}} \longrightarrow A_{\mathcal{H}_{d,\delta_0+\delta_1}^{(2)}}$$

the morphisms of filtered C^* -algebras induced by the inclusions $\mathcal{H}_{d,\delta_0+\delta_1}^{(1)} \hookrightarrow \mathcal{H}_d$, $\mathcal{H}_{d,\delta_0+\delta_1}^{(2)} \hookrightarrow \mathcal{H}_d$, $\mathcal{H}_{d,\delta_0+\delta_1}^{(1)} \cap \mathcal{H}_{d,\delta_0+\delta_1}^{(2)} \hookrightarrow \mathcal{H}_{d,\delta_0+\delta_1}^{(1)}$ and $\mathcal{H}_{d,\delta_0+\delta_1}^{(1)} \cap \mathcal{H}_{d,\delta_0+\delta_1}^{(2)} \hookrightarrow \mathcal{H}_{d,\delta_0+\delta_1}^{(2)}$.

On the same way, for K -homology, we denote respectively by

$$\begin{aligned} j_{P_d(X), * }^{(1)} : K_*(P_d(X_{\delta_0+\delta_1}^{(1)}), A) &\longrightarrow K_*(P_d(X), A), \\ j_{P_d(X), * }^{(2)} : K_*(P_d(X_{\delta_0+\delta_1}^{(2)}), A) &\longrightarrow K_*(P_d(X), A), \\ j_{P_d(X), * }^{(1,2)} : K_*(P_d(X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}), A) &\longrightarrow K_*(P_d(X_{\delta_0+\delta_1}^{(1)}), A) \end{aligned}$$

and

$$j_{P_d(X), * }^{(2,1)} : K_*(P_d(X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}), A) \longrightarrow K_*(P_d(X_{\delta_0+\delta_1}^{(2)}), A),$$

the morphisms induced by the inclusions $P_d(X_{\delta_0+\delta_1}^{(1)}) \hookrightarrow P_d(X)$, $P_d(X_{\delta_0+\delta_1}^{(2)}) \hookrightarrow P_d(X)$, $P_d(X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}) \hookrightarrow P_d(X_{\delta_0+\delta_1}^{(1)})$ and $P_d(X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}) \hookrightarrow P_d(X_{\delta_0+\delta_1}^{(2)})$.

Let

$$\mathcal{D}_{X,d,\delta_0,\delta_1,A,*} = (\partial_{X,d,\delta_0,\delta_1,A,*}^{\varepsilon,r})_{0 < \varepsilon < \frac{1}{100\alpha}, 0 < s < \frac{r}{k\varepsilon}} : \mathcal{K}_*(A_{\mathcal{H}_d}) \longrightarrow \mathcal{K}_*(A_{\mathcal{H}_{d,\delta_0+\delta_1}^{(1)} \cap \mathcal{H}_{d,\delta_0+\delta_1}^{(2)}})$$

be the associated controlled Mayer-Vietoris boundary map at order r with $(\alpha, k) = (\alpha_{MV}, k_{MV})$.

Assume now that there exists a positive number R such for $i = 1, 2$ then

$$X^{(i)} =_R \bigsqcup_{j=1}^{n_i} X_j^{(i)}$$

and let us fix a positive number δ such that $2\delta < R$. Let s and s' be positive numbers with $2s < R - 2\delta$ and $s \leq s'$. Let us define for $i = 1, 2$

$$\iota'_{X,A,*}^{(i),\varepsilon,s,s'} : \bigoplus_{j=1}^{n_i} K_*^{\varepsilon,s}(A_{X_{j,\delta}^{(i)}}) \longrightarrow \bigoplus_{j=1}^{n_i} K_*^{\varepsilon,s'}(A_{X_{j,\delta}^{(i)}})$$

and

$$\iota'_{X,A,*}^{(1,2),\varepsilon,s,s'} : \bigoplus_{j=1,\dots,n_1,l=1,\dots,n_2} K_*^{\varepsilon,s}(A_{X_{j,\delta}^{(1)} \cap X_{l,\delta}^{(2)}}) \longrightarrow \bigoplus_{j=1,\dots,n_1,l=1,\dots,n_2} K_*^{\varepsilon,s'}(A_{X_{j,\delta}^{(1)} \cap X_{l,\delta}^{(2)}})$$

as the morphisms obtained by increasing the control from s to s' . As usual, we should write $\iota'_{X,A,*}^{(i),-,\varepsilon,s'}$ (resp. $\iota'_{X,A,*}^{(i),\varepsilon,s,-}$) when the source (resp. the range) is obvious and similarly for $\iota'_{X,A,*}^{(1,2),\varepsilon,s,s'}$. Let

$$j'_{X,A,*}^{(1,2),\sharp,\varepsilon,s'} : \bigoplus_{j=1,\dots,n_1,l=1,\dots,n_2} K_*^{\varepsilon,s'}(A_{X_{j,\delta}^{(1)} \cap X_{l,\delta}^{(2)}}) \longrightarrow \bigoplus_{j=1}^{n_1} K_*^{\varepsilon,s'}(A_{X_{\delta,j}^{(1)}})$$

be the morphism induced by the inclusion

$$\bigoplus_{j=1,\dots,n_1,l=1,\dots,n_2} A_{X_{j,\delta}^{(1)} \cap X_{l,\delta}^{(2)}} \hookrightarrow \bigoplus_{j=1}^{n_1} A_{X_{j,\delta}^{(1)}}$$

and let

$$j'_{X,A,*}^{(2,1),\sharp,\varepsilon,s'} : \bigoplus_{j=1,\dots,n_1,l=1,\dots,n_2} K_*^{\varepsilon,s'}(A_{X_{j,\delta}^{(1)} \cap X_{l,\delta}^{(2)}}) \longrightarrow \bigoplus_{l=1}^{n_2} K_*^{\varepsilon,s'}(A_{X_{l,\delta}^{(2)}})$$

be the morphism induced by the inclusion

$$\bigoplus_{j=1,\dots,n_1,l=1,\dots,n_2} A_{X_{j,\delta}^{(1)} \cap X_{l,\delta}^{(2)}} \hookrightarrow \bigoplus_{l=1}^{n_2} A_{X_{l,\delta}^{(2)}}$$

Let us set also for any positive number d

$$P'_d(X_\delta^{(i)}) = \bigsqcup_{j=1}^{n_i} P_d(X_{j,\delta}^{(i)}) \subseteq P_d(X_\delta^{(i)}) \text{ for } i = 1, 2;$$

$$P'_d(X_\delta^{(1)} \cap X_\delta^{(2)}) = P'_d(X_\delta^{(1)}) \cap P'_d(X_\delta^{(2)}) = \bigsqcup_{j=1,\dots,n_1;l=1,\dots,n_2} P_d(X_{j,\delta}^{(1)} \cap X_{l,\delta}^{(2)});$$

$$P_{d,\delta}(X) = P'_d(X_\delta^{(1)}) \cup P'_d(X_\delta^{(2)}).$$

For $i = 1, 2$ and ε, s positive numbers with $\varepsilon < 1/100$ and $s \geq r_{\varepsilon, d}$, let us define

$$\nu_{X_\delta^{(i)}, A, *}^{\varepsilon, s, d} : K_*(P'_d(X_\delta^{(i)}), A) \longrightarrow \bigoplus_{j=1}^{n_i} K_*^{\varepsilon, s}(A_{X_{j, \delta}^{(i)}})$$

as the composition

$$K_*(P'_d(X_\delta^{(i)}), A) \xrightarrow{\cong} \bigoplus_{j=1}^{n_i} K_*(P_d(X_{j, \delta}^{(i)}), A) \xrightarrow{\bigoplus_{j=1}^{n_i} \nu_{X_{j, \delta}^{(i)}, A, *}^{\varepsilon, s, d}} \bigoplus_{j=1}^{n_i} K_*^{\varepsilon, s}(A_{X_{j, \delta}^{(i)}})$$

where the first map is induced by the isomorphism

$$C(P'_d(X_\delta^{(i)})) \xrightarrow{\cong} \bigoplus_{j=1}^{n_i} C(P_d(X_{j, \delta}^{(i)})).$$

Define similarly

$$\nu_{X_\delta^{(1)} \cap X_\delta^{(2)}, A, *}^{\varepsilon, s, d} : K_*(P'_d(X_\delta^{(1)} \cap X_\delta^{(2)}), A) \longrightarrow \bigoplus_{j=1, \dots, n_1; l=1, \dots, n_2} K_*^{\varepsilon, s}(A_{X_{j, \delta}^{(1)} \cap X_{l, \delta}^{(2)}}).$$

Assume that $d < R - 2\delta$, in which case we have $P_d^\delta(X) = P_d(X)$, $P'_d(X_\delta^{(i)}) = P_d(X_\delta^{(i)})$ for $i = 1, 2$ and $P'_d(X_\delta^{(1)} \cap X_\delta^{(2)}) = P_d(X_\delta^{(1)} \cap X_\delta^{(2)})$. Let d' be a positive number with $d \leq d'$. For any C^* algebra A , let us define

$$\begin{aligned} q'_{d, d', \delta, *} &: K_*(P_d(X), A) \longrightarrow K_*(P_{d', \delta}(X), A), \\ q'^{(1)}_{d, d', \delta, *} &: K_*(P_d(X_\delta^{(1)}), A) \longrightarrow K_*(P'_{d'}(X_\delta^{(1)}), A), \\ q'^{(2)}_{d, d', \delta, *} &: K_*(P_d(X_\delta^{(2)}), A) \longrightarrow K_*(P'_{d'}(X_\delta^{(2)}), A) \end{aligned}$$

and

$$q'^{(1,2)}_{d, d', \delta, *} : K_*(P_d(X_\delta^{(1)} \cap X_\delta^{(2)}), A) \longrightarrow K_*(P'_{d'}(X_\delta^{(1)} \cap X_\delta^{(2)}), A)$$

as the morphism induced by the inclusion $P_d(X) \hookrightarrow P_{d', \delta}(X)$.

On the other hand, let $(\mathcal{H}_{d', \delta}, \rho_{d', \delta})$ be a non degenerate $P_{d', \delta}(X)$ -module, let \mathcal{H}_d be the set of elements in $\mathcal{H}_{d', \delta}$ with support in $P_d(X)$ and let ρ_d be the restriction of $\rho_{d', \delta}$ to \mathcal{H}_d . For $i = 1, 2$, let $\mathcal{H}_{d', \delta}^{(i)}$ be the set of elements of $\mathcal{H}_{d', \delta}$ with support in $P'_{d'}(X_\delta^{(i)})$.

Let then

$$q'^{\sharp}_{d, d', \delta} : \mathcal{K}(\mathcal{H}_d) \longrightarrow \mathcal{K}(\mathcal{H}_{d', \delta})$$

be the natural inclusion and let us denote respectively by

$$\begin{aligned} q'^{(1), \sharp}_{d, d', \delta} &: \mathcal{K}(\mathcal{H}_{d, \delta}^{(1)}) \longrightarrow \mathcal{K}(\mathcal{H}_{d', \delta}^{(1)}) \\ q'^{(2), \sharp}_{d, d', \delta} &: \mathcal{K}(\mathcal{H}_{d, \delta}^{(2)}) \longrightarrow \mathcal{K}(\mathcal{H}_{d', \delta}^{(2)}) \end{aligned}$$

and

$$q'^{(1,2), \sharp}_{d, d', \delta} : \mathcal{K}(\mathcal{H}_{d, \delta}^{(1)} \cap \mathcal{H}_{d, \delta}^{(2)}) \longrightarrow \mathcal{K}(\mathcal{H}_{d', \delta}^{(1)} \cap \mathcal{H}_{d', \delta}^{(2)})$$

the restriction of $q'^{\sharp}_{d, d', \delta}$ to $\mathcal{H}_{d, \delta}^{(1)}$, $\mathcal{H}_{d, \delta}^{(2)}$ and $\mathcal{H}_{d, \delta}^{(1)} \cap \mathcal{H}_{d, \delta}^{(2)}$.

Since the inclusion $P_d(X) \hookrightarrow P_{d'}(X)$ is contractive, we see that $q_{d,d',\delta}^\sharp$ is a morphism of filtered C^* -algebras. Notice that if moreover (\mathcal{H}_d, ρ_d) and $(\mathcal{H}_{d',\delta}, \rho_{d',\delta})$ are standard modules and up to a propagation rescaling, then $q_{d,d',\delta}^\sharp$ induces in quantitative K -theory the morphism corresponding to the inclusion $P_d(X) \hookrightarrow P_{d',\delta}(X)$ constructed in Section 3.2 (see Remark 3.5).

As for $P_d(X)$, we denote respectively by

$$j_{P_{d',\delta}(X),A}^{(1),\sharp} : A_{\mathcal{H}_{d',\delta}^{(1)}} \longrightarrow A_{\mathcal{H}_{d',\delta}},$$

$$j_{P_{d',\delta}(X),A}^{(2),\sharp} : A_{\mathcal{H}_{d',\delta}^{(2)}} \longrightarrow A_{\mathcal{H}_{d',\delta}},$$

$$j_{P_{d',\delta}(X),A}^{(1,2),\sharp} : A_{\mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)}} \longrightarrow A_{\mathcal{H}_{d',\delta}^{(1)}}$$

and

$$j_{P_{d',\delta}(X),A}^{(2,1),\sharp} : A_{\mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)}} \longrightarrow A_{\mathcal{H}_{d',\delta}^{(2)}}$$

the morphisms of filtered C^* -algebras induced by the inclusions $\mathcal{H}_{d',\delta}^{(1)} \hookrightarrow \mathcal{H}_{d',\delta}^{(2)}$, $\mathcal{H}_{d',\delta}^{(2)} \hookrightarrow \mathcal{H}_{d',\delta}^{(1)}$, $\mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)} \hookrightarrow \mathcal{H}_{d',\delta}^{(1)}$ and $\mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)} \hookrightarrow \mathcal{H}_{d',\delta}^{(2)}$.

In the same way, for K -homology, we denote for $i = 1, 2$ by

$$j_{P_{d',\delta}(X),*}^{(i)} : K_*(P_{d'}'(X_\delta^{(i)}), A) \longrightarrow K_*(P_{d',\delta}(X), A),$$

the morphisms induced by the inclusion $P_{d'}'(X_\delta^{(i)}) \hookrightarrow P_{d',\delta}(X)$.

Eventually let

$$V_{\mathcal{H}_{d',\delta}} : \mathcal{H}_{d',\delta} \longrightarrow \ell^2(X, \mathcal{H}_{d',\delta})$$

be the isometry defined by

$$(V_{\mathcal{H}_{d',\delta}} \cdot \xi)(x) = \rho(\lambda_x^{1/2}) \cdot \xi$$

for any ξ in $\mathcal{H}_{d',\delta}$ and any x in X . Let

$$V_{\mathcal{H}_{d',\delta}^{(1)}} : \mathcal{H}_{d',\delta}^{(1)} \longrightarrow \ell^2(X_\delta^{(1)}, \mathcal{H}_{d',\delta}^{(1)}),$$

$$V_{\mathcal{H}_{d',\delta}^{(2)}} : \mathcal{H}_{d',\delta}^{(2)} \longrightarrow \ell^2(X_\delta^{(2)}, \mathcal{H}_{d',\delta}^{(2)})$$

and

$$V_{\mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)}} : \mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)} \longrightarrow \ell^2(X_\delta^{(1)} \cap X_\delta^{(2)}, \mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)})$$

be respectively the restriction of $V_{\mathcal{H}_{d',\delta}}$ to $\mathcal{H}_{d',\delta}^{(1)}$, $\mathcal{H}_{d',\delta}^{(2)}$ and $\mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)}$.

5.3. The QI -statement. In this subsection, we set $(\alpha, k) = (\alpha_{MV}, k_{MV})$.

Lemma 5.5. *There exists a control pair (λ, h) with $(\lambda, h) \geq (\alpha, k)$ such that the following holds :*

Let X be a finite metric space and let $X^{(1)}$ and $X^{(2)}$ be two subsets of X such that $X = X^{(1)} \cup X^{(2)}$, then

- *for any C^* -algebra A ;*
- *for any positive numbers $\varepsilon, s, d, \delta_0, \delta_1$ and r with $\delta_0 > 2d$, $\delta_1 > (2d + 1)(5r + 1)$, $\varepsilon < \frac{1}{100\lambda}$, $s \geq r_{d, \alpha\varepsilon}$ and $s < \frac{r}{k_\varepsilon}$,*

we have

$$\nu_{X_{\delta_0+\delta_1}^{(1)} \cap X_{d, \delta_0+\delta_1}^{(2)}, A, *}^{\alpha\varepsilon, k_\varepsilon s, d} \circ \partial_{P_d(X_{\delta_0+\delta_1}^{(1)}), P_d(X_{\delta_0+\delta_1}^{(2)}, *)} = \partial_{X_{\delta_0+\delta_1}^{(1)}, X_{\delta_0+\delta_1}^{(2)}, A, *}^{\varepsilon, s} \circ \nu_{X, A, *}^{\varepsilon, s, d}.$$

Proof. Let us choose a non degenerate standard $P_d(X)$ -module (\mathcal{H}_d, ρ_d) such that the restriction to $P_d(X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)})$ is also standard. Then $Ad_{V_{\mathcal{H}_d, A}}$ clearly maps the controlled r -controlled Mayer-Vietoris pair of equation (29) to the one of equation (27) and hence intertwines up to Morita equivalence the associated controlled boundary maps. The lemma is then a consequence of Propositions 3.20 and 4.2. \square

Lemma 5.6. *There exists a control pair (λ, h) with $(\lambda, h) \geq (\alpha, k)$ such that*

- *for any finite metric space X ;*
- *for any C^* -algebra A ;*
- *for any subsets $X^{(1)}$ and $X^{(2)}$ of X and any $R > 0$ such that*
 - $X = X^{(1)} \cup X^{(2)}$;
 - $X^{(i)} =_R \bigsqcup_{k=1}^{n_i} X_k^{(i)}$ for $i = 1, 2$.
- *for any positive numbers $d, d', \delta_0, \delta_1$ and s with $2(\delta_0 + \delta_1) < R$, $d < 2s$, $d \leq d'$, $\delta_0 > 2d$ and $\delta_1 > (2d + 1)(5s + 1)$, $s < R/2 - \delta_0 - \delta_1$,*

the following is satisfied:

- *let $(\mathcal{H}_{d', \delta_0+\delta_1}, \rho_{d', \delta_0+\delta_1})$ be a non degenerate $P_{d', \delta_0+\delta_1}(X)$ -module and let (\mathcal{H}_d, ρ_d) be its restriction to the set of elements in $\mathcal{H}_{d', \delta_0+\delta_1}$ with support in $P_d(X)$;*
- *let $\varepsilon_0, \varepsilon$ and s_0 be positive numbers such that $\varepsilon_0 \leq \varepsilon < \frac{1}{100\lambda}$, $h_{\varepsilon_0} s_0 < s$ and $2d(h_{\varepsilon} s_0 + 1) < s$;*
- *let y be an element in $K_1^{\varepsilon_0, s_0}(A_{\mathcal{H}_d})$ such that*
 - $\iota_*^{\varepsilon, s} \circ Ad_{V_{\mathcal{H}_d, A}, *}^{\varepsilon_0, s_0}(y) = 0$ in $K_1^{\varepsilon, s}(A_X)$ (up to Morita equivalence);
 - $q_{d, d', \delta_0+\delta_1, A, *}^{(1, 2), \#, \alpha\varepsilon_0, k_{\varepsilon_0} s_0} \circ \partial_{X, d, \delta_0, \delta_1, A, *}^{\varepsilon_0, s_0}(y) = 0$ in $K_0^{\alpha\varepsilon_0, k_{\varepsilon_0} s_0}(A_{\mathcal{H}_{d', \delta_0+\delta_1}^{(1)} \cap \mathcal{H}_{d', \delta_0+\delta_1}^{(2)}})$,

Then there exists

- $y^{(1)}$ in $K_1^{\lambda\varepsilon_0, h_{\varepsilon_0} s_0}(A_{\mathcal{H}_{d', \delta_0+\delta_1}^{(1)}})$;

- $y^{(2)}$ in $K_1^{\lambda\varepsilon_0, h\varepsilon_0 s_0}(A_{\mathcal{H}_{d', \delta_0+\delta_1}^{(2)}})$;
- z in $\bigoplus_{j=1, \dots, n_1; l=1, \dots, n_2} K_1^{\lambda\varepsilon, 2(d'+1)(h\varepsilon s+1)}(A_{X_{j, \delta_0+\delta_1}^{(1)} \cap X_{l, \delta_0+\delta_1}^{(2)}})$

such that

- (i) $j_{P_{d', \delta_0+\delta_1}(X), A, *}^{(1), \sharp, \lambda\varepsilon_0, h\varepsilon_0 s_0}(y^{(1)}) - j_{P_{d', \delta_0+\delta_1}(X), A, *}^{(2), \sharp, \lambda\varepsilon_0, h\varepsilon_0 s_0}(y^{(2)}) = \iota_*^{-, \lambda\varepsilon_0, h\varepsilon_0 s_0} \circ q_{d, d', \delta_0+\delta_1, A, *}^{\sharp, \varepsilon_0, s_0}(y)$
in $K_1^{\lambda\varepsilon_0, h\varepsilon_0 s_0}(A_{\mathcal{H}_{d', \delta_0+\delta_1}})$;
- (ii) $\iota'_{X, A, *}^{(1), -, \lambda\varepsilon, 2(d'+1)(h\varepsilon s+1)} \circ \text{Ad}_{V_{\mathcal{H}_{d', \delta_0+\delta_1}^{(1)}}}^{\lambda\varepsilon_0, h\varepsilon_0 s_0}(y^{(1)}) = j_{X, A, *}^{\sharp, (1, 2), \lambda\varepsilon, 2(d'+1)(h\varepsilon s+1)}(z)$
in $K_1^{\lambda\varepsilon', h\varepsilon' r}(A_{X_{\delta_0+\delta_1}^{(1)}})$;
- (iii) $\iota'_{X, A, *}^{(2), -, \lambda\varepsilon, 2(d'+1)(h\varepsilon s+1)} \circ \text{Ad}_{V_{\mathcal{H}_{d', \delta_0+\delta_1}^{(2)}}}^{\lambda\varepsilon_0, h\varepsilon_0 s_0}(y^{(2)}) = j_{X, A, *}^{\sharp, (2, 1), \lambda\varepsilon, 2(d'+1)(h\varepsilon s+1)}(z)$
in $K_1^{\lambda\varepsilon, 2(d'+1)(h\varepsilon s+1)}(A_{X_{\delta_0+\delta_1}^{(2)}})$ where the equalities in points (ii) and (iii) are indeed up to Morita equivalence.

Proof. Let u be a ε_0 - s_0 -unitary in some $M_n(\widetilde{A_{\mathcal{H}_d}})$ such that $y = [u]_{\varepsilon_0, s_0}$ in $K_1^{\varepsilon_0, s_0}(A_{\mathcal{H}_d})$. In view of [17, Lemma 1.8], we can assume without loss of generality that $u - I_n \otimes \text{Id}_{\mathcal{H}_d}$ is in $M_n(A_{\mathcal{H}_d})$. As in the construction of the controlled Mayer-Vietoris boundary [17, Section 3.2] and possibly taking a larger n , let us pick two $\alpha\varepsilon_0$ - $k\varepsilon_0$ - s_0 -unitaries v_1 and v_2 in $\widetilde{M_{2n}(A_{\mathcal{H}_d})}$ and an $\alpha\varepsilon_0$ - $k\varepsilon_0$ - s_0 -projection p in $\widetilde{M_{2n}(A_{\mathcal{H}_d})}$ such that

- $v_i - I_{2n} \otimes \text{Id}_{\mathcal{H}_d}$ lies in $M_{2n}(A_{\mathcal{H}_{d, \delta_0+\delta_1}^{(i)}})$ for $i = 1, 2$;
- $p - \text{diag}(I_n \otimes \text{Id}_{\mathcal{H}_d}, 0)$ lies in $M_{2n}(A_{\mathcal{H}_{d, \delta_0+\delta_1}^{(1)} \cap \mathcal{H}_{d, \delta_0+\delta_1}^{(2)}})$
- $\|\text{diag}(u, u^*) - v_1 v_2\| < \alpha\varepsilon_0$;
- $\|p - v_1^* \text{diag}(I_n \otimes \text{Id}_{\mathcal{H}_d}, 0) v_1\| < \alpha\varepsilon_0$;
- $\|p - v_2 \text{diag}(I_n \otimes \text{Id}_{\mathcal{H}_d}, 0) v_2^*\| < \alpha\varepsilon_0$;
- $[p, n]_{\alpha\varepsilon_0, k\varepsilon_0 s_0} = \partial_{X, d, \delta_0, \delta_1, A, *}^{\varepsilon_0, s_0}(y)$ in $K_0^{\alpha\varepsilon_0, k\varepsilon_0 s_0}(A_{\mathcal{H}_d^{X(1), \delta_0+\delta_1} \cap \mathcal{H}_d^{X(2), \delta_0+\delta_1}})$.

Let us set

$$\begin{aligned} u' &= u - I_n \otimes \text{Id}_{\mathcal{H}_d} + I_n \otimes \text{Id}_{\mathcal{H}_{d', \delta_0+\delta_1}} \\ v'_1 &= v_1 - I_{2n} \otimes \text{Id}_{\mathcal{H}_d} + I_{2n} \otimes \text{Id}_{\mathcal{H}_{d', \delta_0+\delta_1}} \\ v'_2 &= v_2 - I_{2n} \otimes \text{Id}_{\mathcal{H}_d} + I_{2n} \otimes \text{Id}_{\mathcal{H}_{d', \delta_0+\delta_1}} \\ p' &= p - \text{diag}(I_n \otimes \text{Id}_{\mathcal{H}_d}, 0) + \text{diag}(I_n \otimes \text{Id}_{\mathcal{H}_{d', \delta_0+\delta_1}}, 0), \end{aligned}$$

(recall that $\text{Id}_{\mathcal{H}_d}$ is viewed as the projection on $\mathcal{H}_d \subseteq \mathcal{H}_{d', \delta_0+\delta_1}$). Apply then [17, Lemma 3.5] to $\widetilde{A_{\mathcal{H}_{d', \delta_0+\delta_1}}}$, $\widetilde{A_{\mathcal{H}_{d', \delta_0+\delta_1}^{(1)}}}$, $\widetilde{A_{\mathcal{H}_{d', \delta_0+\delta_1}^{(2)}}}$, u' , u'^* , v'_1 , v'_2 and p' , we see that for some control pair (λ, h) depending only on $(\alpha, k) = (\alpha_{MV}, k_{MV})$ and such that $(\lambda, h) \geq (\alpha, k)$ and possibly taking a larger n , there exist two $\lambda\varepsilon_0$ - $h\varepsilon_0$ - s_0 -unitaries w_1 and w_2 in $M_n(\widetilde{A_{\mathcal{H}_{d', \delta_0+\delta_1}}})$ such that

- $w_i - I_{2n} \otimes Id_{\mathcal{H}_{d', \delta_0 + \delta_1}}$ lies in $A_{\mathcal{H}_{d', \delta_0 + \delta_1}}^{(i)}$ for $i = 1, 2$;
- $\|\text{diag}(u', I_n \otimes Id_{\mathcal{H}_{d', \delta_0 + \delta_1}}) - w_1 w_2\| < \lambda \varepsilon_0$.

Let us set then

- $y^{(1)} = [w_1]_{\lambda \varepsilon_0, h_{\varepsilon_0} s_0}$ in $K_1^{\lambda \varepsilon_0, h_{\varepsilon_0} s_0}(A_{\mathcal{H}_{d', \delta_0 + \delta_1}}^{(1)})$;
- $y^{(2)} = [w_2^*]_{\lambda \varepsilon_0, h_{\varepsilon_0} s_0}$ in $K_1^{\lambda \varepsilon_0, h_{\varepsilon_0} s_0}(A_{\mathcal{H}_{d', \delta_0 + \delta_1}}^{(2)})$.

Point (i) is then a consequence of the equality

$$[u']_{\varepsilon_0, s_0} = q_{d, d', \delta_0 + \delta_1, *}^{\varepsilon_0, s_0}([u]_{\varepsilon_0, s_0}).$$

Notice that

$$[Ad_{V_{\mathcal{H}_{d', \delta_0 + \delta_1}, M_n(A)}}(u') + I_n \otimes Id_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})} - I_n \otimes V_{\mathcal{H}_{d', \delta_0 + \delta_1}} V_{\mathcal{H}_{d', \delta_0 + \delta_1}}^*]_{\varepsilon_0, 2d(s_0 + 1)}$$

is the image of

$$[Ad_{V_{\mathcal{H}_d, M_n(A)}}(u) + I_n \otimes Id_{\ell^2(X, \mathcal{H}_d)} - I_n \otimes V_{\mathcal{H}_d} V_{\mathcal{H}_d}^*]_{\varepsilon_0, 2d(s_0 + 1)}$$

under the morphism

$$K_1^{\varepsilon_0, 2d(s_0 + 1)}(A_{\ell^2(X, \mathcal{H}_d)}) \longrightarrow K_1^{\varepsilon, 2d(s_0 + 1)}(A_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})})$$

induced by the inclusion $A_{\ell^2(X, \mathcal{H}_d)} \hookrightarrow A_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})}$ and hence

$$[Ad_{V_{\mathcal{H}_{d', \delta_0 + \delta_1}, M_n(A)}}(u') + I_n \otimes Id_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})} - I_n \otimes V_{\mathcal{H}_{d', \delta_0 + \delta_1}} V_{\mathcal{H}_{d', \delta_0 + \delta_1}}^*]_{\varepsilon, s} = 0$$

in $K_1^{\varepsilon, s}(A_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})})$.

Hence, in view of [17, Proposition 2.12], and up to rescaling (λ, h) and to enlarging n , we see that there exists two $\lambda \varepsilon$ - $h_{\varepsilon} s$ -unitaries V_1 and V_2 in $M_n(\overline{A_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})}})$ such that

- $V_i - I_{2n} \otimes Id_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})}$ is in

$$\bigoplus_{j=1}^{n_i} M_n(A_{\ell^2(X_{j, \delta_0 + \delta_1}^{(i)}, \mathcal{H}_{d', \delta_0 + \delta_1})})$$

for $i = 1, 2$;

- V_i is homotopic to $I_n \otimes Id_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})}$ as a $\lambda \varepsilon$ - $h_{\varepsilon} s$ -unitary in

$$\bigoplus_{j=1}^{n_i} M_n(A_{\ell^2(X_{j, \delta_0 + \delta_1}^{(i)}, \mathcal{H}_{d', \delta_0 + \delta_1})}) + \mathbb{C} \cdot I_n \otimes Id_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})};$$

- $\|Ad_{V_{\mathcal{H}_{d', \delta_0 + \delta_1}, M_n(A)}}(u') + I_n \otimes Id_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})} - I_n \otimes V_{\mathcal{H}_{d', \delta_0 + \delta_1}} V_{\mathcal{H}_{d', \delta_0 + \delta_1}}^* - V_1 V_2\| < \lambda \varepsilon$.

Let us set for $i = 1, 2$

$$W_i = Ad_{V_{\mathcal{H}_{d', \delta_0 + \delta_1}, M_n(A)}}(w_i) + I_n \otimes Id_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})} - I_n \otimes V_{\mathcal{H}_{d', \delta_0 + \delta_1}} V_{\mathcal{H}_{d', \delta_0 + \delta_1}}^*.$$

For $i = 1, 2$, then W_i is a $\lambda \varepsilon_0$ - $2d'(h_{\varepsilon_0} s_0 + 1)$ -unitary in

$$M_n(A_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})}) + \mathbb{C} \cdot I_n \otimes Id_{\ell^2(X, \mathcal{H}_{d', \delta_0 + \delta_1})}$$

and moreover, since the propagation of w_i is less than $h_{\varepsilon_0} s_0$ then $W_i - I_n \otimes Id_{\ell^2(X, \mathcal{H}_{d'}, \delta_0 + \delta_1)}$ lies in

$$\bigoplus_{j=1}^{n_i} M_n(A_{\ell^2(X_{j, \delta_0 + \delta_1}^{(i)}, \mathcal{H}_{d'}, \delta_0 + \delta_1)}).$$

Furthermore, we have

$$\|V_1 V_2 - W_1 W_2\| < 2\lambda\varepsilon$$

and hence

$$\|V_2 W_2^* - V_1^* W_1\| < 12\lambda\varepsilon.$$

Using the *CIA*-property for the pair of algebras

$$\left(\bigoplus_{j=1}^{n_1} A_{\ell^2(X_{j, \delta_0 + \delta_1}^{(1)}, \mathcal{H}_{d'}, \delta_0 + \delta_1)}, \bigoplus_{j=1}^{n_1} A_{\ell^2(X_{j, \delta_0 + \delta_1}^{(2)}, \mathcal{H}_{d'}, \delta_0 + \delta_1)} \right),$$

we deduce that there exists a $12\lambda\varepsilon \cdot 2(d' + 1)(h_{\varepsilon} s + 1)$ -unitary U in the unitarization of

$$\bigoplus_{j=1, \dots, n_1; l=1, \dots, n_2} M_n(A_{\ell^2(X_{j, \delta_0 + \delta_1}^{(1)} \cap X_{l, \delta_0 + \delta_1}^{(2)}, \mathcal{H}_{d'}, \delta_0 + \delta_1)})$$

such that

$$\|U - V_1^* W_1\| < 12\lambda\varepsilon$$

and

$$\|U - V_2 W_2^*\| < 12\lambda\varepsilon.$$

In particular, in view of point (ii) of Lemma 1.2, we deduce that

- U and W_1 are homotopic as $48\lambda\varepsilon \cdot 2(d' + 1)(h_{\varepsilon} s + 1)$ -unitaries in

$$\bigoplus_{k=1}^{n_i} M_n(A_{\ell^2(X_{k, \delta_0 + \delta_1}^{(1)}, \mathcal{H}_{d'}, \delta_0 + \delta_1)}) + \mathbb{C} \cdot I_n \otimes Id_{\ell^2(X, \mathcal{H}_{d'}, \delta_0 + \delta_1)};$$

- U and W_2^* are homotopic as $48\lambda\varepsilon \cdot 2(d' + 1)(h_{\varepsilon} s + 1)$ -unitaries in

$$\bigoplus_{k=1}^{n_i} M_n(A_{\ell^2(X_{k, \delta_0 + \delta_1}^{(2)}, \mathcal{H}_{d'}, \delta_0 + \delta_1)}) + \mathbb{C} \cdot I_n \otimes Id_{\ell^2(X, \mathcal{H}_{d'}, \delta_0 + \delta_1)};$$

Let us replace 48λ by λ and let then set $z = [U]_{\lambda\varepsilon, 2(d' + 1)(h_{\varepsilon} s + 1)}$ (up to Morita equivalence) in $K_1^{\lambda\varepsilon, 2(d' + 1)(h_{\varepsilon} s + 1)}(A_{\ell^2(X_{\delta_0 + \delta_1}^{(1)} \cap X_{\delta_0 + \delta_1}^{(2)})}$). We have

$$\begin{aligned} j_{X, A, *}^{\sharp, (1, 2), \lambda\varepsilon, 2(d' + 1)(h_{\varepsilon} s + 1)}(z) &= [j_{X, M_n(A)}^{\sharp, (1, 2)}(U)]_{\lambda\varepsilon, 2(d' + 1)(h_{\varepsilon} s + 1)} \\ &= [W_1]_{\lambda\varepsilon, 2(d' + 1)(h_{\varepsilon} s + 1)} \\ &= \iota_{*}^{(1) -, \lambda\varepsilon, 2(d' + 1)(h_{\varepsilon} s + 1)} \circ Ad_{V_{\mathcal{H}_{d'}, \delta_0 + \delta_1}^{\lambda\varepsilon_0, h_{\varepsilon_0} s_0}, A, *} (y^{(1)}) \end{aligned}$$

and similarly

$$j'_{X,A,*}^{\sharp,(2,1),\lambda\varepsilon,h\varepsilon s}(z) = \iota_{*}^{(2),-,\lambda\varepsilon,2(d'+1)(h\varepsilon s+1)} \circ \text{Ad}_{V_{\mathcal{H}_{d',\delta_0+\delta_1}^{(2)},A,*}}^{\lambda\varepsilon_0,h\varepsilon_0 s_0}(y^{(2)}).$$

and hence we deduce points (ii) and (iii). \square

End of the proof of the QI statement. Let \mathcal{X} be a family of finite metric spaces satisfying the assumptions of Theorem 2.13. Let ε , s and d be positive numbers with $\varepsilon < \frac{1}{200}$ and $s \geq r_{d,\varepsilon}$. Let us show that there exists a positive number d' such that for any X in \mathcal{X} and any C^* -algebra A , then $QI_{X,A,*}(d, d', s, \varepsilon)$ holds. Using controlled Bott Periodicity, we can reduce to the odd case. As before, we set $(\alpha, k) = (\alpha_{MV}, k_{MV})$ and for sake of simplicity, we shall also denote by (α, k) the control pairs of Lemmas 5.5 and 5.6.

Let us fix four positive numbers r , δ_0 , δ_1 and R such that $k_\varepsilon s < r$, $\delta_0 > 2d$, $\delta_1 > (2d+1)(5r+1)$ and $R > 2(\delta_0 + \delta_1) + 2r$. Let \mathcal{Y} be a family of finite metric spaces with uniformly bounded geometry such that

- \mathcal{X} is R -decomposable with respect to \mathcal{Y} ;
- $\widetilde{\mathcal{Y}}$ satisfies uniformly the QAM-estimates.

For any positive number δ , let $\widetilde{\mathcal{X}}_\delta$ be the set of all subsets of metric spaces in \mathcal{X} that contains a metric space in \mathcal{Y} as a δ -net. According to Lemma 5.4, the family $\widetilde{\mathcal{X}}_\delta$ satisfies uniformly the QAM-estimates.

Step 1: Assume that $\varepsilon < \frac{1}{200\alpha}$ (the general case will be obtained by rescaling using Proposition 2.2). Let X be a metric space in \mathcal{X} and let $X^{(1)}$ and $X^{(2)}$ be two subsets of X such that

- $X = X^{(1)} \cup X^{(2)}$;
- $X^{(i)} =_R \bigsqcup_{j=1}^{n_i} X_j^{(i)}$ with $X_j^{(i)}$ in \mathcal{Y} for $j = 1, \dots, n_i$ and $i = 1, 2$.

Recall from equation (28) that

$$P_d(X) = \overbrace{P_d(X_{\delta_0+\delta_1}^{(1)})}^{\circ} \cup \overbrace{P_d(X_{\delta_0+\delta_1}^{(2)})}^{\circ}.$$

Let d' be a positive number with $d' \geq d$ such that $QI_{X',A,*}(d, d', k_\varepsilon s, \varepsilon)$ holds for all metric spaces in $\widetilde{\mathcal{X}}_{\delta_0+\delta_1}$. Notice that

$$K_*(P_d(X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}), A) \cong \bigoplus_{j=1, \dots, n_1, l=1, \dots, n_2} K_*(P_d(X_{j,\delta_0+\delta_1}^{(1)} \cap X_{l,\delta_0+\delta_1}^{(2)}), A)$$

and

$$K_*^{\alpha\varepsilon, k_\varepsilon s}(A_{X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}}) \cong \bigoplus_{j=1, \dots, n_1, l=1, \dots, n_2} K_*^{\alpha\varepsilon, k_\varepsilon s}(A_{X_{j,\delta_0+\delta_1}^{(1)} \cap X_{l,\delta_0+\delta_1}^{(2)}})$$

and that under these two identifications, then

$$\nu_{X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}, A, *}^{\alpha\varepsilon, k_\varepsilon s, d} = \bigoplus_{j=1, \dots, n_1, l=1, \dots, n_2} \nu_{X_{j, \delta_0+\delta_1}^{(1)} \cap X_{l, \delta_0+\delta_1}^{(2)}, A, *}^{\alpha\varepsilon, k_\varepsilon s, d}. \quad (30)$$

Let x be a element of $K_1(P_d(X), A)$ such that $\nu_{X, A, *}^{\varepsilon, s, d}(x) = 0$ in $K_1^{\varepsilon, s}(A_X)$.

Since $QI_{X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}, A, *}^{\alpha\varepsilon, k_\varepsilon s, d}(d, d', k_\varepsilon s, \alpha\varepsilon)$ holds, we deduce from Lemma 5.5 and from equation (30) that

$$q'_{d, d', \delta_0+\delta_1, *}^{(1,2)} \circ \partial_{P_d(X_{\delta_0+\delta_1}^{(1)}), P_d(X_{\delta_0+\delta_1}^{(2)}), *} (x) = 0. \quad (31)$$

Step 2: Let $n_{d'}$ be an integer such that for any metric space in \mathcal{Y} , balls of radius d' have cardinal bounded by $n_{d'}$. In view of Theorem 3.17, let $(0, 1/200) \rightarrow [1, +\infty); \varepsilon \mapsto h_\varepsilon$ be a non increasing function such that for any finite simplicial complexe Z with dimension less than $n_{d'}$ and for any C^* -algebra A , then $Ind_{Z, A, *}$ is a h -isomorphism. Let us fix a non degenerate standard module $P_{d', \delta_0+\delta_1}(X)$ -module $(\mathcal{H}_{d', \delta_0+\delta_1}, \rho_{d', \delta_0+\delta_1})$ such that the restrictions to elements of $\mathcal{H}_{d', \delta_0+\delta_1}$ with support in $P_d(X)$, $P'_{d'}(X_{\delta_0+\delta_1}^{(1)})$, $P'_{d'}(X_{\delta_0+\delta_1}^{(2)})$ and $P'_{d'}(X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)})$ are standard. Let (\mathcal{H}_d, ρ_d) be the restriction of $(\mathcal{H}_{d', \delta_0+\delta_1}, \rho_{d', \delta_0+\delta_1})$ to element of $\mathcal{H}_{d', \delta_0+\delta_1}$ with support in $P_d(X)$. Let us pick two positive numbers ε_0 and s_0 with $\varepsilon_0 < \varepsilon$, $s_0 < \frac{1}{k_{\varepsilon_0} h_{\alpha\varepsilon_0}^2}$, $k_{\varepsilon_0} s_0 < s$ and $2d(k_{\varepsilon_0} s_0 + 1) < s$ (recall that $s \geq r_{d, \varepsilon} > 2d$). Let us set then

$$y = Ind_{P_d(X), A, *}^{\varepsilon_0, s_0}(x)$$

in $K_1^{\varepsilon_0, s_0}(A_{\mathcal{H}_d})$. According to Proposition 3.20, we see that

$$\iota_*^{-, \varepsilon, s} \circ Ad_{V_{\mathcal{H}_d}, A, *}^{\varepsilon_0, s_0}(y) = 0.$$

On the other hand,

$$\begin{aligned} & q'_{d, d', \delta_0+\delta_1, A, *}^{(1,2), \sharp, \alpha\varepsilon_0, k_{\varepsilon_0} s_0} \circ \partial_{X, d, \delta_0, \delta_1, A, *}^{\varepsilon_0, s_0}(y) \\ &= q'_{d, d', \delta_0+\delta_1, A, *}^{(1,2), \sharp, \alpha\varepsilon_0, k_{\varepsilon_0} s_0} \circ \partial_{X, d, \delta_0, \delta_1, A, *}^{\varepsilon_0, s_0} \circ Ind_{X, A, *}^{\varepsilon_0, s_0}(x) \\ &= q'_{d, d', \delta_0+\delta_1, A, *}^{(1,2), \sharp, \alpha\varepsilon_0, k_{\varepsilon_0} s_0} \circ Ind_{P_d(X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}), A, *}^{\alpha\varepsilon_0, k_{\varepsilon_0} s_0} \circ \partial_{P_d(X_{\delta_0+\delta_1}^{(1)}), P_d(X_{\delta_0+\delta_1}^{(2)}), *} (x) \\ &= Ind_{P_{d', \delta_0+\delta_1} X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}, A, *}^{\alpha\varepsilon_0, k_{\varepsilon_0} s_0} \circ q'_{d, d', *}^{(1,2)} \circ \partial_{P_d(X_{\delta_0+\delta_1}^{(1)}), P_d(X_{\delta_0+\delta_1}^{(2)}), *} (x) \\ &= 0, \end{aligned}$$

where

- the second equality is a consequence of Proposition 4.2;
- the third equality is a consequence of Remark 3.16;
- the last equality is a consequence of equation (31).

Step 3: Let $y^{(1)}$ in $K_1^{\alpha\varepsilon_0, k\varepsilon_0 s_0}(A_{\mathcal{H}_{d', \delta_0 + \delta_1}^{(1)}})$, $y^{(2)}$ in $K_1^{\alpha\varepsilon_0, k\varepsilon_0 s_0}(A_{\mathcal{H}_{d', \delta_0 + \delta_1}^{(2)}})$ and z in $\bigoplus_{j=1, \dots, n_1; l=1, \dots, n_2} K_1^{\alpha\varepsilon, 2(d'+1)(k_\varepsilon s+1)}(A_{X_{j, \delta_0 + \delta_1}^{(1)} \cap X_{l, \delta_0 + \delta_1}^{(2)}})$ be as in Lemma 5.6 with respect to y . Pick then positive numbers d'' and s' with $s' \geq r_{d'', \alpha\varepsilon}$ and $s' \geq 2(d'+1)(k_\varepsilon s+1)$ and such that

$$QS_{X', A, *}(d'', 2(d'+1)(k_\varepsilon s+1), s', \alpha\varepsilon)$$

holds for any metric space X' in $\widetilde{\mathcal{X}_{\delta_0 + \delta_1}}$.

Then there exists an element x_z in

$$K_1(P'_{d''}(X_{\delta_0 + \delta_1}^{(1)} \cap X_{\delta_0 + \delta_1}^{(2)}), A) \cong \bigoplus_{j=1, \dots, n_1; l=1, \dots, n_2} K_1(P_{d''}(X_{j, \delta_0 + \delta_1}^{(1)} \cap X_{l, \delta_0 + \delta_1}^{(2)}), A)$$

such that under above identifications

$$\nu'^{\alpha\varepsilon, s', d''}_{X_{\delta_0 + \delta_1}^{(1)} \cap X_{\delta_0 + \delta_1}^{(2)}, A, *} (x_z) = z', \quad (32)$$

where z' is the image of z under the map

$$\bigoplus_{j=1, \dots, n_1; l=1, \dots, n_2} K_1^{\alpha\varepsilon, 2(d'+1)(k_\varepsilon s+1)}(A_{X_{j, \delta_0 + \delta_1}^{(1)} \cap X_{l, \delta_0 + \delta_1}^{(2)}}) \longrightarrow \bigoplus_{j=1, \dots, n_1; l=1, \dots, n_2} K_1^{\alpha\varepsilon, s'}(A_{X_{j, \delta_0 + \delta_1}^{(1)} \cap X_{l, \delta_0 + \delta_1}^{(2)}})$$

(increasing control).

Since $k\varepsilon_0 s_0 < 1/h_{\alpha\varepsilon_0}^2$ and $\text{Ind}_{P'_{d''}(X_{\delta_0 + \delta_1}^{(i)}, A, *)}$ is a h -isomorphism, there exists an element $x^{(i)}$ in $K_1(P'_{d''}(X_{\delta_0 + \delta_1}^{(i)}), A)$ for $i = 1, 2$ such that

$$\text{Ind}_{P'_{d''}(X_{\delta_0 + \delta_1}^{(i)}, A, *)}^{\alpha\varepsilon_0, h_{\alpha\varepsilon_0} k\varepsilon_0 s_0} (x^{(i)}) = \iota_*^{-, \alpha\varepsilon_0, h_{\alpha\varepsilon_0} k\varepsilon_0 s_0} (y^{(i)}). \quad (33)$$

For $i = 1, 2$, let $x'^{(i)}$ in $K_1(P'_{d''}(X_{\delta_0 + \delta_1}^{(i)}), A)$ be the image of $x^{(i)}$ under the morphism induced by the inclusion $P'_{d''}(X_{\delta_0 + \delta_1}^{(i)}) \hookrightarrow P'_{d''}(X_{\delta_0 + \delta_1}^{(i)})$. Then

$$\begin{aligned} \nu'^{\alpha\varepsilon, s', d''}_{X_{\delta_0 + \delta_1}^{(i)}, A, *} (x'^{(i)}) &= \iota_{X, A, *}^{(i), -, \alpha\varepsilon, s'} \circ \text{Ad}_{V_{\mathcal{H}_{d', \delta_0 + \delta_1}^{(i)}}}^{\alpha\varepsilon_0, k\varepsilon_0 s_0}, A, * (y^{(i)}) \\ &= j_{P_{d'', \delta_0 + \delta_1}(X), A, *}^{(1, 2), \sharp, \alpha\varepsilon, s'} \circ \iota_*^{-, -} (z) \\ &= j_{P_{d'', \delta_0 + \delta_1}(X), A, *}^{(1, 2), \sharp, \alpha\varepsilon, s'} \circ \nu'^{\alpha\varepsilon, s', d''}_{X_{\delta_0 + \delta_1}^{(1)} \cap X_{\delta_0 + \delta_1}^{(2)}, A, *} (x_z). \end{aligned}$$

Step 4: Let d''' be a positive number with $d''' \geq d''$ and such that $QI_{X', A, *}(d'', d''', s', \alpha\varepsilon)$ holds for every metric space X' in $\widetilde{\mathcal{X}_{\delta_0 + \delta_1}}$. Since

$$P'_{d''}(X_{\delta_0 + \delta_1}^{(i)}) = \bigsqcup_{j=1, \dots, n_i} P_{d''}(X_{j, \delta_0 + \delta_1}^{(i)})$$

and according to Lemma 2.1 and to equation (32), we deduce that in $K_1(P'_{d'''}(X_{\delta_0+\delta_1}^{(i)}), A)$, the image of $x'^{(i)}$ under the morphism induced by the inclusion $P'_{d''}(X_{\delta_0+\delta_1}^{(i)}) \hookrightarrow P'_{d'''}(X_{\delta_0+\delta_1}^{(i)})$ and the image of x_z under the morphism induced by the inclusion $P'_{d''}(X_{\delta_0+\delta_1}^{(1)} \cap X_{\delta_0+\delta_1}^{(2)}) \hookrightarrow P'_{d'''}(X_{\delta_0+\delta_1}^{(i)})$ coincides for $i = 1, 2$. As a consequence of equation (33) and of the injectivity of $Ind_{P'_{d''}(X_{\delta_0+\delta_1}^{(i)}), A, *}^{\alpha\varepsilon_0, h_{\alpha\varepsilon_0} k_{\varepsilon_0} s_0}$ and since

$$j_{P'_{d', \delta_0+\delta_1}(X), A, *}^{(1), \sharp, \alpha\varepsilon_0, k_{\varepsilon_0} s_0}(y^{(1)}) - j_{P'_{d', \delta_0+\delta_1}(X), A, *}^{(2), \sharp, \alpha\varepsilon_0, k_{\varepsilon_0} s_0}(y^{(2)}) = \iota_*^{-, \alpha\varepsilon_0, k_{\varepsilon_0} s_0} \circ q'_{d, d', \delta_0+\delta_1, A, *}^{\sharp, \varepsilon_0, s_0}(y)$$

in $K_1^{\alpha\varepsilon_0, k_{\varepsilon_0} s_0}(A_{\mathcal{H}_{d', \delta_0+\delta_1}})$, we deduce from Remark 3.16 that

$$j_{P'_{d', \delta_0+\delta_1}(X), A, *}^{(1)}(x^{(1)}) - j_{P'_{d', \delta_0+\delta_1}(X), A, *}^{(2)}(x^{(2)}) = q'_{d, d', \delta_0+\delta_1, A, *}(x)$$

and hence

$$j_{P'_{d'', \delta_0+\delta_1}(X), A, *}^{(1)}(x'^{(1)}) - j_{P'_{d'', \delta_0+\delta_1}(X), A, *}^{(2)}(x'^{(2)}) = q'_{d, d'', \delta_0+\delta_1, A, *}(x)$$

and therefore $q_{d, d'', *}(x) = 0$. Hence the condition $QI_{X, A, *}(d, d''', s, \varepsilon)$ holds.

5.4. The QS-statement. Let \mathcal{X} be a family of finite metric spaces and let \mathcal{Y} be a family of finite metric spaces with uniformly bounded geometry such that for any positive number R ,

- \mathcal{X} is R -decomposable with respect to \mathcal{Y} ;
- $\tilde{\mathcal{Y}}$ satisfies uniformly the QAM-estimates.

As for the QI -statement, by using controlled Bott Periodicity we can reduce it to the odd case and we set $(\alpha, k) = (\alpha_{MV}, k_{MV})$.

Let ε and s be positive numbers with $\varepsilon < \frac{1}{200\alpha}$ (the general case will follow by rescaling). Let X be a metric space in \mathcal{X} and let z be an element in $K_1^{\varepsilon, s}(A_X)$. Let us fix δ and R two positive numbers such that $10k_\varepsilon s < 10r < 2\delta < R$. Let $X = X^{(1)} \cup X^{(2)}$ be a R -decomposition of X in \mathcal{Y} with $X^{(i)} =_R \bigsqcup_{j=1}^{n_i} X_j^{(i)}$ for $i = 1, 2$. Then $X_\delta^{(i)} =_{R-2\delta} \bigsqcup_{j=1}^{n_i} X_{j, \delta}^{(i)}$ for $i = 1, 2$ and as mentioned in Section 5.2 (see equation (27)), for any C^* -algebra A ,

$$(A_{X^{(1)}, X_\delta^{(1)}, r}, A_{X^{(2)}, X_\delta^{(2)}, r}, A_{X_\delta^{(1)}}, A_{X_\delta^{(2)}})$$

is a r -controlled Mayer-Vietoris pair for A_X . Recall from Lemma 5.4 that the family $\tilde{\mathcal{X}}_\delta$ satisfies uniformly the QAM-estimates.

Step 1: Let s' and d be positive numbers with $s' \geq r_{d, \alpha\varepsilon}$, $s' \geq k_\varepsilon s$ and such that the condition $QS_{X', A, *}(k_\varepsilon s, s', d, \alpha\varepsilon)$ holds for every metric space X' in $\tilde{\mathcal{X}}_\delta$. Since

$$K_*^{\varepsilon, s}(A_{X_\delta^{(1)} \cap X_\delta^{(2)}}) \cong \bigoplus_{k=1, \dots, n_1; l=1, \dots, n_2} K_*^{\varepsilon, s}(A_{X_{k, \delta}^{(1)} \cap X_{l, \delta}^{(2)}})$$

and under the identification

$$K_0(P'_d(X_\delta^{(1)} \cap X_\delta^{(2)}), A) \cong \bigoplus_{j=1, \dots, n_1; l=1, \dots, n_2} K_0(P_d(X_{j,\delta}^{(1)} \cap X_{l,\delta}^{(2)}), A),$$

there exists an element x' in $K_0(P'_d(X_\delta^{(1)} \cap X_\delta^{(2)}), A)$ such that

$$\nu_{X_\delta^{(1)} \cap X_\delta^{(2)}, A, *}^{\alpha\varepsilon, s', d}(x') = \iota_{*}^{(1,2), -, \alpha\varepsilon, s'} \circ \partial_{X_\delta^{(1)}, X_\delta^{(2)}, A, *}^{\varepsilon, s}(z).$$

Since

$$j_{X, A, *}^{(1,2), \#} \circ \mathcal{D}_{X_\delta^{(1)}, X_\delta^{(2)}, A, *} = 0$$

and

$$j_{X, A, *}^{(2,1), \#} \circ \mathcal{D}_{X_\delta^{(1)}, X_\delta^{(2)}, A, *} = 0,$$

using Lemma 2.1 we deduce that

$$\nu_{X_\delta^{(1)}, A, *}^{\alpha\varepsilon, s', d} \circ j_{P_{d,\delta}(X), *}^{(1,2)}(x') = 0 \quad (34)$$

and

$$\nu_{X_\delta^{(2)}, A, *}^{\alpha\varepsilon, s', d} \circ j_{P_{d,\delta}(X), *}^{(2,1)}(x') = 0, \quad (35)$$

where

$$j_{P_{d,\delta}(X), *}^{(1,2)} : K_*(P'_d(X_\delta^{(1)} \cap X_\delta^{(2)}), A) \longrightarrow K_*(P'_d(X_\delta^{(1)}), A)$$

and

$$j_{P_{d,\delta}(X), *}^{(2,1)} : K_*(P'_d(X_\delta^{(1)} \cap X_\delta^{(2)}), A) \longrightarrow K_*(P'_d(X_\delta^{(2)}), A)$$

are respectively the morphism induced by the inclusions

$$P'_d(X_\delta^{(1)} \cap X_\delta^{(2)}) \hookrightarrow P'_d(X_\delta^{(1)})$$

and

$$P'_d(X_\delta^{(1)} \cap X_\delta^{(2)}) \hookrightarrow P'_d(X_\delta^{(2)}).$$

Step 2: Let d' be a positive number with $d' \geq d$ and such that $QI_{X', A, *}(d, d', s', \alpha\varepsilon)$ holds for every metric space X' in $\widetilde{\mathcal{X}}_\delta$. We deduce from equation (34) that

$$\begin{aligned} j_{P_{d',\delta}(X), *}^{(1,2)} \circ q_{d,d',\delta,*}^{\prime(1,2)}(x') &= q_{d,d',\delta,*}^{\prime(1)} \circ j_{P_{d,\delta}(X), *}^{(1,2)}(x') \\ &= 0, \end{aligned}$$

and similarly from equation (35) that

$$j_{P_{d',\delta}(X), *}^{(2,1)} \circ q_{d,d',\delta,*}^{\prime(2,1)}(x') = 0.$$

Let (λ, h) be the control pair of [17, Lemma 3.8]. For sake of simplicity, we also denote by h the non increasing function of Step 2 of the proof of the QI -statement in Section 5.3. Let us pick ε_0 and s_0 two positive numbers such that $\varepsilon_0 < \frac{1}{200\lambda}$ and $s_0 < \frac{1}{h_{\lambda\varepsilon_0}^2 h_{\varepsilon_0}}$. Let us fix $(\mathcal{H}_{d',\delta}, \rho_{d',\delta})$ a

non degenerate standard $P_{d',\delta}(X)$ -module with standard restriction to $\mathcal{H}_{d',\delta}^{(1)}$, $\mathcal{H}_{d',\delta}^{(2)}$ and $\mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)}$. Let us set then

$$y' = \text{Ind}_{P_{d'}(X_\delta^{(1)} \cap X_\delta^{(2)}), A, *}^{\varepsilon_0, s_0} \circ q'_{d, d', \delta, *}^{(1,2)}(x')$$

in $K_0^{\varepsilon_0, s_0}(A_{\mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)}})$. From Remark 3.16, we deduce that

$$j_{P_{d',\delta}(X), A, *}^{(1,2), \sharp, \varepsilon_0, s_0}(y') = 0$$

in $K_0^{\varepsilon_0, s_0}(A_{\mathcal{H}_{d',\delta}^{(1)}})$ and

$$j_{P_{d',\delta}(X), A, *}^{(2,1), \sharp, \varepsilon_0, s_0}(y') = 0$$

in $K_0^{\varepsilon_0, s_0}(A_{\mathcal{H}_{d',\delta}^{(2)}})$. Let then q be an ε_0 - s_0 -projection in some $M_n(\widetilde{A_{\mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)}}})$ and let l be an integer with $l \leq n$ such that $y' = [q, l]_{\varepsilon_0, s_0}$ in $K_0^{\varepsilon_0, s_0}(A_{\mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)}})$.

We can assume without loss of generality that

- $q - \text{diag}(I_l \otimes Id_{\mathcal{H}_{d',\delta}}, 0)$ is in $M_n(A_{\mathcal{H}_{d',\delta}^{(1)} \cap \mathcal{H}_{d',\delta}^{(2)}})$ (see [17, Lemma 1.7]);
- q is homotopic to $\text{diag}(I_l \otimes Id_{\mathcal{H}_{d',\delta}}, 0)$ as an ε_0 - s_0 -projection in $M_n(\widetilde{A_{\mathcal{H}_{d',\delta}^{(i)}}})$ for $i = 1, 2$.

Applying [17, Lemma 3.8], we see that for an integer N with $N \geq n$, there exist w_1 and w_2 two $\lambda_{\varepsilon_0} h_{\varepsilon_0} s_0$ -unitaries in $M_N(\widetilde{A_{\mathcal{H}_{d',\delta}}})$, u an $\lambda_{\varepsilon_0} h_{\varepsilon_0} s_0$ -unitary in $M_n(\widetilde{A_{\mathcal{H}_{d',\delta}}})$, v an $\lambda_{\varepsilon_0} h_{\varepsilon_0} s_0$ -unitary in $M_{N-n}(\widetilde{A_{\mathcal{H}_{d',\delta}}})$ such that

- $w_i - I_N \otimes Id_{\mathcal{H}_{d',\delta}}$ is in $M_N(A_{\mathcal{H}_{d',\delta}^{(i)}})$ for $i = 1, 2$;
-

$$\|w_1^* \text{diag}(I_n \otimes Id_{\mathcal{H}_{d',\delta}}, 0) w_1 - \text{diag}(q, 0)\| < \lambda_{\varepsilon_0}$$

and

$$\|w_2 \text{diag}(I_n \otimes Id_{\mathcal{H}_{d',\delta}}, 0) w_2^* - \text{diag}(q, 0)\| < \lambda_{\varepsilon_0}.$$

- $\|\text{diag}(u, v) - w_1 w_2\| < \lambda_{\varepsilon_0}$.

Step 3: Let U be an ε - s -unitary in some $M_m(\widetilde{A_{\ell^2(X, \mathcal{H}_{d',\delta})}})$ such that $z = -[U]_{\varepsilon, s}$ in $K_1^{\varepsilon, s}(A_X)$ (up to Morita equivalence). According to [17, Lemma 1.8], we can assume without loss of generality that $U - I_m \otimes Id_{\ell^2(X, \mathcal{H}_{d',\delta})}$ is in $M_m(A_{\ell^2(X, \mathcal{H}_{d',\delta})})$. Up to stabilization and by construction of the Mayer-Vietoris controlled boundary map, there exist W_1 and W_2 two $\alpha\varepsilon$ - $k_\varepsilon s$ -unitaries in $M_{2m}(\widetilde{A_{\ell^2(X, \mathcal{H}_{d',\delta})}})$ and Q an $\alpha\varepsilon$ - $k_\varepsilon s$ -projection in $M_{2m}(\widetilde{A_{\ell^2(X, \mathcal{H}_{d',\delta})}})$ such that

- $W_i - I_{2m} \otimes Id_{\ell^2(X, \mathcal{H}_{d',\delta})}$ is in $M_{2m}(A_{\ell^2(X_\delta^{(i)}, \mathcal{H}_{d',\delta})})$ for $i = 1, 2$;
- $Q - \text{diag}(I_m \otimes Id_{\ell^2(X, \mathcal{H}_{d',\delta})}, 0)$ is in $M_{2m}(A_{\ell^2(X_\delta^{(1)} \cap X_\delta^{(2)}, \mathcal{H}_{d',\delta})})$

- $\|\text{diag}(U, U^*) - W_1 W_2\| < \alpha\varepsilon$;
- $\|W_1^* \text{diag}(I_m \otimes Id_{\ell^2(X, \mathcal{H}_{d', \delta})}, 0) W_1 - Q\| < \alpha\varepsilon$

and

$$\|W_2 \text{diag}(I_n \otimes Id_{\mathcal{H}_{d', \delta}}, 0) W_2^* - Q\| < \alpha\varepsilon;$$

- up to Morita equivalence, then

$$\partial_{X_\delta^{(1)}, X_\delta^{(2)}, A, *}^{\varepsilon, s}(z) = -[Q, m]_{\alpha\varepsilon, k_\varepsilon s}$$

$$\text{in } K_0^{\alpha\varepsilon, k_\varepsilon s}(A_{X_\delta^{(1)} \cap X_\delta^{(2)}}).$$

Let s'' be a positive number such that $s'' \geq s'$ and $s'' > 2d'(h_{\varepsilon_0} s_0 + 1)$. Applying [17, Lemma 3.5] to

$$\text{diag}(Ad_{V_{\mathcal{H}_{d', \delta}, A}} u + I_n \otimes Id_{\mathcal{H}_{d', \delta}} - I_n \otimes V_{\mathcal{H}_{d', \delta}} V_{\mathcal{H}_{d', \delta}}^*, U),$$

$$\text{diag}(Ad_{V_{\mathcal{H}_{d', \delta}, A}} v + I_{N-n} \otimes Id_{\mathcal{H}_{d', \delta}} - I_{N-n} \otimes V_{\mathcal{H}_{d', \delta}} V_{\mathcal{H}_{d', \delta}}^*, U^*)$$

and to the matrices obtained from

$$\text{diag}(Ad_{V_{\mathcal{H}_{d', \delta}, A}} w_1 + I_N \otimes Id_{\mathcal{H}_{d', \delta}} - I_n \otimes V_{\mathcal{H}_{d', \delta}} V_{\mathcal{H}_{d', \delta}}^*, W_1),$$

$$\text{diag}(Ad_{V_{\mathcal{H}_{d', \delta}, A}} w_2 + I_N \otimes Id_{\mathcal{H}_{d', \delta}} - I_n \otimes V_{\mathcal{H}_{d', \delta}} V_{\mathcal{H}_{d', \delta}}^*, W_2)$$

and

$$\text{diag}(Ad_{V_{\mathcal{H}_{d', \delta}, A}} \text{diag}(q, 0) + I_N \otimes Id_{\mathcal{H}_{d', \delta}} - I_n \otimes V_{\mathcal{H}_{d', \delta}} V_{\mathcal{H}_{d', \delta}}^*, Q)$$

by swapping the order of the coordinates $n+1, \dots, N$ and $N+1, \dots, N+m$, we see that for some control pair (λ', h') that depends only on (α, k) and (λ, h) with $(\lambda', h') \geq (\lambda, h)$, there exists $U^{(1)}$ and $U^{(2)}$ two $\lambda'\varepsilon - h'_\varepsilon s''$ -unitary in some $M_{n'}(\widetilde{A_X})$ such that

- $U^{(i)} - I_{n'} \otimes Id_{\ell^2(X)}$ is in $\bigoplus_{k=1}^{n_i} M_{n'}(A_{\ell^2(X_{k, \delta}^{(i)})})$ for $i = 1, 2$;
- up to Morita equivalence, we have

$$[U^{(1)}]_{\lambda'\varepsilon, h'_\varepsilon s''} + [U^{(2)}]_{\lambda'\varepsilon, h'_\varepsilon s''} =$$

$$[Ad_{V_{\mathcal{H}_{d', \delta}, A}} u + I_n \otimes Id_{\mathcal{H}_{d', \delta}} - I_n \otimes V_{\mathcal{H}_{d', \delta}} V_{\mathcal{H}_{d', \delta}}^*]_{\lambda'\varepsilon, h'_\varepsilon s''} - \iota_*^{\lambda'\varepsilon, h'_\varepsilon s''}(y)$$

$$\text{in } K_1^{\lambda'\varepsilon, h'_\varepsilon s''}(A_{\ell^2(X)}).$$

Step 4: Assume from now on that $\varepsilon < \frac{1}{200\lambda'}$, the general case being deduced by rescaling using Proposition 2.2. Let s''' and d'' be positive numbers with $d'' \geq d'$, $s''' > h'_\varepsilon s''$, $s''' > r_{d'', \lambda'\varepsilon}$ and $s''' > 2d''(h_{\varepsilon_0} h_{\lambda\varepsilon_0} s_0 + 1)$ such that the condition $QS_{X', A, *}(d'', h'_\varepsilon s'', s''', \lambda'\varepsilon)$ holds for any metric space X' in $\widetilde{\mathcal{X}}_\delta$ and any C^* -algebra A . Then for $i = 1, 2$, there exists $x'^{(i)}$ in $K_1(P'_{d''}(X_\delta^{(i)}), A) \cong \bigoplus_{j=1}^{n_i} K_1(P_{d''}(X_{j, \delta}^{(i)}), A)$ such that up to Morita equivalence, then

$$\nu'_{X_\delta^{(i)}, A, *}^{\lambda'\varepsilon, s''', d''}(x'^{(i)}) = [U^{(i)}]_{\lambda'\varepsilon, s''', d''}$$

in $K_1^{\lambda'\varepsilon, s'''}(\bigoplus_{j=1}^{n_i} A_{X_{j,\delta}^{(i)}})$. Since $\text{Ind}_{P_{d',\delta}(X), A, *}$ is a h -isomorphism, there exists a unique element x' in $K_1(P_{d',\delta}(X), A)$ such that

$$\text{Ind}_{P_{d',\delta}(X), A, *}^{\lambda\varepsilon_0, h_{\lambda\varepsilon_0} h_{\varepsilon_0} s_0}(x') = [u]_{\lambda\varepsilon_0, h_{\lambda\varepsilon_0} h_{\varepsilon_0} s_0}$$

in $K_1^{\lambda\varepsilon_0, h_{\lambda\varepsilon_0} h_{\varepsilon_0} s_0}(A_{\mathcal{H}_{d',\delta}})$. Let x in $K_1(P_{d''}(X), A)$ be the image of x' under the morphism induced by the inclusion $P_{d',\delta}(X) \hookrightarrow P_{d''}(X)$. By choosing a non degenerate standard module for $P_{d''}(X)$ that restricts to $(\mathcal{H}_{d',\delta}, \rho_{d',\delta})$ on $P_{d',\delta}(X)$, it is straightforward to check using Lemma 2.1 and Proposition 3.20 that we have up to Morita equivalence

$$\nu_{X,A,*}^{\lambda'\varepsilon, s''', d''}(x) = [\text{Ad}_{V_{\mathcal{H}_{d',\delta}, A}} u + I_n \otimes \text{Id}_{\mathcal{H}_{d',\delta}} - I_n \otimes V_{\mathcal{H}_{d',\delta}} V_{\mathcal{H}_{d',\delta}}^*]_{\lambda'\varepsilon, s'''} \quad (36)$$

in $K_1^{\lambda'\varepsilon, s'''}(A_X)$. For $i = 1, 2$, let $x^{(i)\circ}$ be the image of $x^{(i)}$ under the map $K_1(P_{d''}(X_\delta^{(i)}), A) \rightarrow K_1(P_{d''}(X_\delta^{(i)}), A)$ induced by the inclusion $P_{d''}(X_\delta^{(i)}) \hookrightarrow P_{d''}(X_\delta^{(i)})$. We deduce from equation (36) that

$$\begin{aligned} \nu_{X,A,*}^{\lambda'\varepsilon, s''', d''}(x - j_{P_{d''}(X_\delta^{(1)})} (x^{(1)}) - j_{P_{d''}(X_\delta^{(2)})} (x^{(2)})) \\ = [\text{Ad}_{V_{\mathcal{H}_{d',\delta}, A}} u + I_n \otimes \text{Id}_{\mathcal{H}_{d',\delta}} - I_n \otimes V_{\mathcal{H}_{d',\delta}} V_{\mathcal{H}_{d',\delta}}^*]_{\lambda'\varepsilon, s'''} - [U^{(1)}]_{\lambda'\varepsilon, s'''} - [U^{(2)}]_{\lambda'\varepsilon, s'''}, \\ = \iota_*^{-, \lambda'\varepsilon, s'''}(y) \end{aligned}$$

where the first equality holds by Lemma 2.1. Thus the condition $QS_{X,A,*}(d'', s, s''', \varepsilon)$ holds for any C^* -algebra A . \square

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