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Julian Le Clainche

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Homological smoothness of Hopf-Galois extensions

Julian Le Clainche

Abstract

We show that if H is a Hopf algebra with bijective antipode and $B \subset A$ is a faithfully flat H -Galois extension, then A is homologically smooth if H and B are.

1 Introduction

An algebra A is said to be homologically smooth if A admits, as an A -bimodule, a finite length resolution by finitely generated projective A -bimodules. Homological smoothness is an appropriate analogue of regularity for (noncommutative) algebras, and is the basic condition involved in homological duality questions for algebras, see [3, 17, 18, 8]. The aim of this paper is to provide a result that produces new examples of homologically smooth algebras, in the setting of Hopf-Galois extensions, the analogue of principal bundles in noncommutative algebra.

Recall [10] that if H is a Hopf algebra and A is a H -comodule algebra, the algebra extension $A^{\text{co}H} \subset A$ is said to be an H -Galois extension if a certain canonical map $\beta : A \otimes_B A \rightarrow A \otimes H$ is bijective, see Section 3. This framework includes examples of various nature such as Hopf crossed products [12] or exact sequences of Hopf algebras [1] (in particular exact sequences of groups).

In this setting, our main result is as follows:

Theorem 1.1. *Let H be a Hopf algebra with bijective antipode and let $B \subset A$ be an H -Galois extension such that A is faithfully flat as left and right B -module. If H and B are homologically smooth algebras, then A is homologically smooth as well.*

Similar results were known to be true in some particular cases, notably in the setting of twisted Calabi-Yau algebras: the case of Galois objects (i.e. for $B = k$) was done by Yu [21] and smash products of algebras by Hopf algebras were studied by Le Meur [11].

The main ingredients to prove Theorem 1.1 are

1. Stefan's spectral sequences [16] for Hopf-Galois extensions;
2. a smoothness criterion using Tor due to Bieri-Eckmann [2], which seems to have been slightly forgotten in the recent literature.

The paper is organized as follows. Section 2 consists of preliminaries about homological finiteness of modules and reminders about Hochschild cohomology. In section 3 we recall the definition of Hopf-Galois extensions and we introduce Stefan's spectral sequences. Section 4 is devoted to the proof of Theorem 1.1 and in the final Section 5, we present an illustrating example of Theorem 1.1.

Notations. Throughout the paper, we work over a field k , and all algebras are (unital) k -algebras. If A is an algebra, the opposite algebra is denoted A^{op} and the enveloping algebra $A \otimes A^{op}$ is denoted A^e . The category of right (resp. left) A -modules is denoted \mathcal{M}_A (resp. ${}_A\mathcal{M}$). We have $\mathcal{M}_A = {}_{A^{op}}\mathcal{M}$, hence any definition we give for left modules has an obvious analogue for right modules, that we will not necessarily give. An A -bimodule structure is equivalent to a left (or right) A^e -module structure. Indeed, if M is an A -bimodule, a left (resp. right) A^e -module structure on M is defined by

$$(a \otimes b) \cdot m = amb \text{ (resp. } m \cdot (a \otimes b) = bma \text{) for } a, b \in A \text{ and } m \in M.$$

The category of A -bimodule is thus identified with the category \mathcal{M}_{A^e} , or with the category ${}_{A^e}\mathcal{M}$.

If M and N are left A -modules, Ext spaces ([20]) are denoted $\text{Ext}_A^\bullet(M, N)$ and if P is a right A -module Tor spaces are denoted $\text{Tor}_A^\bullet(P, M)$. We will also need to consider Ext spaces in categories of right modules, which we denote $\text{Ext}_{A^{op}}^\bullet(-, -)$.

If H is a Hopf algebra, its comultiplication, counit and antipode are denoted Δ , ε and S and we will use Sweedler notation in the usual way, i.e. for $h \in H$, we write $\Delta(h) = h_{(1)} \otimes h_{(2)}$. See [12].

2 Homological preliminaries

In this section we recall the various homological ingredients needed in the paper.

Definition 2.1. Let A be an algebra. The *projective dimension* of an A -module M is defined by

$$\text{pdim}_A(M) := \min \{n \in \mathbb{N}, M \text{ admits a length } n \text{ resolution by projective } A\text{-modules}\} \in \mathbb{N} \cup \{\infty\}.$$

The projective dimension can as well be characterized by

$$\begin{aligned} \text{pdim}_A(M) &= \min \{n \in \mathbb{N}, \text{Ext}_A^{n+1}(M, N) = \{0\} \text{ for any } A\text{-module } N\} \\ &= \max \{n \in \mathbb{N}, \text{Ext}_A^n(M, N) \neq \{0\} \text{ for some } A\text{-module } N.\} \end{aligned}$$

We now recall various finiteness conditions on modules.

Definition 2.2. Let A be an algebra and let M be an A -module.

1. The A -module M is said to be *of type FP_∞* if it admits a projective resolution $P_\bullet \rightarrow M$ with P_i finitely generated for all i .
2. The A -module M is said to be *of type FP* if it admits a finitely generated projective resolution of finite length.

The following result characterizes modules of type FP among those of type FP_∞ , see e.g [5, Chapter VIII].

Proposition 2.3. *Let A be an algebra. An A -module M is of type FP if and only if it is of type FP_∞ and $\text{pdim}_A(M)$ is finite. In this case we have*

$$\text{pdim}_A(M) = \max \{n \in \mathbb{N}, \text{Ext}_A^n(M, A) \neq \{0\}\}$$

The following result is [2, Corollary 1.6]. The implication (iii) \Rightarrow (i) is due to Bieri-Eckmann [4], and gives an effective condition to check that a module is of type FP_∞ and will be useful in section 4.

Proposition 2.4. *Let A be an algebra and M a left (resp. right) A -module. The following conditions are equivalent:*

- (i) *The left (resp. right) A -module M is of type FP_∞ ;*
- (ii) *For any direct product $\prod_{i \in I} N_i$ of right (resp. left) A -modules, the natural map $\text{Tor}_k^A(\prod_{i \in I} N_i, M) \rightarrow \prod_{i \in I} \text{Tor}_k^A(N_i, M)$ (resp. $\text{Tor}_k^A(M, \prod_{i \in I} N_i) \rightarrow \prod_{i \in I} \text{Tor}_k^A(M, N_i)$) is an isomorphism for all $k \geq 0$;*
- (iii) *For any direct product $\prod A$ of arbitrary many copies of A , we have $\text{Tor}_i^A(\prod A, M) = 0$ (resp. $\text{Tor}_i^A(N, \prod A) = 0$) for $i \geq 1$ and the natural map $(\prod A) \otimes_A M \rightarrow \prod M$ (resp. $M \otimes_A (\prod A) \rightarrow \prod M$) is an isomorphism.*

There are similar results for Ext functors, but will only use the following particular instance.

Proposition 2.5. *Let A be an algebra and M a right (resp. left) A -module, if M is of type FP_∞ then $\text{Ext}_{A^{op}}(M, -)$ (resp. $\text{Ext}_A(M, -)$) commutes with direct sums.*

Those above general homological finiteness notions specify to the case of bimodules which is our case of interest when considering Hochschild (co)homology.

Definition 2.6. Let A be an algebra.

1. The *cohomological dimension* of A is defined to be $\text{cd}(A) := \text{pdim}_{A^e}(A)$.
2. The algebra A is said to be *homologically smooth* if A is of type FP as a left A^e -module.
3. Let M an A -bimodule. The *Hochschild cohomology* spaces of A with coefficients in M are the vector spaces $\text{HH}^\bullet(A, M) := \text{Ext}_{A^e}^\bullet(A, M)$. The *Hochschild homology* spaces of A with coefficients in M are the vector spaces $\text{HH}_\bullet(A, M) := \text{Tor}_\bullet^{A^e}(M, A)$.

Notice that if A is a homologically smooth algebra, then $\text{cd}(A) = \max\{n \in \mathbb{N}, \text{HH}^n(A, A^e) \neq 0\}$, by Proposition 2.3.

We finally record that if $A = H$ is a Hopf algebra, homological smoothness can be expressed using H -modules rather than H -bimodules.

Theorem 2.7. *[19, Proposition A.2] Let H be a Hopf algebra. The following assertions are equivalent:*

- (i) *The algebra H is homologically smooth;*
- (ii) *The right H -module k_ε is of type FP ;*
- (iii) *The left H -module ${}_\varepsilon k$ is of type FP .*

3 Hopf-Galois extensions

3.1 Definitions and examples

Recall that if H is a Hopf algebra, an H -comodule algebra is an algebra A together with an algebra map $\rho : A \rightarrow A \otimes H$ making it into an H -comodule.

Definition 3.1. Let H be a Hopf algebra. An H -comodule algebra A is said to be an H -Galois extension of $B := A^{\text{co}H} = \{a \in A, a_{(0)} \otimes a_{(1)} = a \otimes 1\}$ if the (canonical) map

$$\begin{aligned} \beta : A \otimes_B A &\longrightarrow A \otimes H \\ a \otimes_B b &\longmapsto ab_{(0)} \otimes b_{(1)} \end{aligned}$$

is bijective. An H -Galois extension of the base field k is called an H -Galois object.

An overview of the theory of Hopf-Galois extensions can be found in [13]. We now list a number of important examples.

Example 3.2. Let H be a Hopf algebra. Then H is an H -Galois object, with the right H -comodule structure given by the comultiplication.

Example 3.3. Let G be a group and let A a G -graded algebra, i.e. an algebra with a decomposition $A = \bigoplus_{g \in G} A_g$ such that $A_g A_h \subset A_{gh}$ for any $g, h \in G$ and $1 \in A_e$, where e denotes the neutral element of G . The G -grading makes A into a kG -comodule algebra, and $A_e \subset A$ is a kG -Galois extension if and only if A is G -strongly graded, which means that $A_g A_h = A_{gh}$ for any $g, h \in G$.

In that case, the inverse of β is given by

$$\beta^{-1}(a \otimes g) = \sum_i ac_i \otimes_{A_e} d_i \text{ for } a \in A, g \in G$$

where $c_i \in A_{g^{-1}}$ and $d_i \in A_g$ are elements such that $\sum_i c_i d_i = 1$.

Example 3.4. Let H be a Hopf algebra and let B be a left H -module algebra, i.e. B is a left H -module with $h \cdot (ab) = (h_{(1)} \cdot a)(h_{(2)} \cdot b)$ and $h \cdot 1 = \varepsilon(h)1$ for any $h \in H, a, b \in B$. Let A be the smash product algebra $B \# H$. Then A is an H -Galois extension of B where the H -comodule structure on A is given by

$$a \# h \mapsto (a \# h_{(1)}) \otimes h_{(2)}$$

and the inverse of β is

$$\begin{aligned} \beta^{-1} : (B \# H) \otimes H &\longrightarrow (B \# H) \otimes_B (B \# H) \\ (a \# h) \otimes k &\longmapsto (a \# hS(k_{(1)})) \otimes_B (1 \# k_{(2)}) \end{aligned}$$

Example 3.5. Let $p : A \rightarrow H$ be a surjective Hopf algebra map. Then p induces a right H -comodule algebra structure on A , given by $a \mapsto a_{(1)} \otimes p(a_{(2)})$. Let $B = A^{\text{co}H}$. If $B^+ A = \text{Ker}(p)$, then $B \subset A$ is an H -Galois extension, with the inverse of the canonical map β being given by

$$\begin{aligned} \beta^{-1} : A \otimes H &\longrightarrow A \otimes_B A \\ a \otimes p(a') &\longmapsto aS(a'_{(1)}) \otimes_B a'_{(2)} \end{aligned}$$

Recall [1] that a sequence of Hopf algebra maps

$$k \longrightarrow B \xrightarrow{i} A \xrightarrow{p} H \longrightarrow k$$

is said to be exact if the following conditions hold:

- (1) i is injective and p is surjective,
- (2) $\text{Ker}(p) = i(B)^+ A = Ai(B)^+$,

$$(3) \quad i(B) = A^{\text{co}H} = {}^{\text{co}H}A,$$

where $A^{\text{co}H} = \{a \in A, a_{(1)} \otimes p(a_{(2)}) = a \otimes 1\}$ and ${}^{\text{co}H}A = \{a \in A, p(a_{(1)}) \otimes a_{(2)} = 1 \otimes a\}$. Exact sequences of groups correspond to exact sequences of their group algebras.

Hence, if $k \longrightarrow B \xrightarrow{i} A \xrightarrow{p} H \longrightarrow k$ is an exact sequence of Hopf algebras, then by the previous consideration we have that A is an H -Galois extension of $i(B)$ where the H -comodule structure on A is given by $a \mapsto a_{(1)} \otimes p(a_{(2)})$.

Remark 3.6. Let $B \subset A$ be an H -Galois extension. Since the canonical map is left A -linear, its inverse is uniquely determined by $\kappa : H \rightarrow A \otimes_B A$ defined by $\kappa(h) = \beta^{-1}(1 \otimes h)$. Denoting $\kappa(h) = h^{(1)} \otimes_B h^{(2)}$, the following identities, which we will not use directly but are essential for some constructions in the next subsection, are [15, Remark 3.4]:

$$\begin{aligned} bh^{(1)} \otimes_B h^{(2)} &= h^{(1)} \otimes_B h^{(2)}b, & a_{(0)}a_{(1)}^{(1)} \otimes_B a_{(1)}^{(2)} &= 1 \otimes_B a, & h^{(1)}h^{(2)} &= \varepsilon(h) \\ h^{(1)} \otimes_B (h^{(2)})_{(0)} \otimes (h^{(2)})_{(1)} &= h_{(1)}^{(1)} \otimes_B h_{(1)}^{(2)} \otimes h_{(2)}, \\ (h^{(1)})_{(0)} \otimes_B h^{(2)} \otimes (h^{(1)})_{(1)} &= h_{(2)}^{(1)} \otimes_B h_{(2)}^{(2)} \otimes S(h_{(1)}) \\ (hk)^{(1)} \otimes_B (hk)^{(2)} &= k^{(1)}h^{(1)} \otimes_B h^{(2)}k^{(2)} \end{aligned}$$

3.2 The Stefan spectral sequences

The fundamental tool that we will use in the proof of Theorem 1.1 is the Stefan spectral sequence, constructed in [16] by using the Grothendieck spectral sequence.

It involves a right (resp. left) H -module structures on the cohomology spaces $\text{HH}^q(B, M)$ (resp. on the homology spaces $\text{HH}_q(B, M)$) for $q \geq 0$ and any A -bimodule M defined in [16]. For $q = 0$, the H -action on $\text{HH}^0(B, M) \simeq M^B$ is given by $m \cdot h = h^{(1)}mh^{(2)}$ for $m \in M^B$ and $h \in H$ and it is extended to $q \geq 0$ using the machinery of cohomological functors [5, Theorem 7.5]. In [7], explicit formulas are given for $q \geq 0$ in the case of a smash product.

For $q = 0$, the H -action on $\text{HH}_0(B, M) \simeq M/[M, B]$ is given by $h \cdot \pi_M(m) = \pi_M(h^{(2)}mh^{(1)})$ for $m \in M$ and $h \in H$, where π_M denotes the natural projection of M onto $M/[M, B]$.

Theorem 3.7 ([16]). *Let H be a Hopf algebra, let $B \subset A$ be an H -Galois extension and let M be an A -bimodule.*

(i) *Assume that A is flat as left and right B -module. Then there is a spectral sequence*

$$E_2^{p,q} = \text{Ext}_{H^{\text{op}}}^p(k_\varepsilon, \text{HH}^q(B, M)) \implies \text{HH}^{p+q}(A, M)$$

which is natural in M .

(ii) *Assume that A is projective as left and right B -module. Then there is a spectral sequence*

$$E_{p,q}^2 = \text{Tor}_p^H(k_\varepsilon, \text{HH}_q(B, M)) \implies \text{HH}_{p+q}(A, M)$$

which is natural in M .

As explained in [16], the above spectral sequences are natural generalizations of the usual Lyndon-Hochschild-Serre spectral sequences in group or Lie algebra (co)homology.

Remark 3.8. It is shown in [14, Theorem 4.10] that if H is a Hopf algebra with bijective antipode, and $B \subset A$ an H -Galois extension with A faithfully flat as a left and right B -module, then A is projective as a left and right B -module. Thus, in order to invoke one of the above spectral sequences, we can indifferently assume projectivity or faithful flatness for an H -Galois extension $B \subset A$.

Remark 3.9. In [16], the homology spaces $\mathrm{HH}_\bullet(A, M)$ are defined as the spaces $\mathrm{Tor}_\bullet^{A^e}(A, M)$ for A an algebra and M an A -bimodule. This is equivalent to Definition 2.6, indeed, if P is an A -bimodule, we have $M \otimes_{A^e} P \simeq P \otimes_{A^e} M$ hence, the complexes defining $\mathrm{Tor}_\bullet^{A^e}(M, A)$ and $\mathrm{Tor}_\bullet^{A^e}(A, M)$ are isomorphic, hence their homology are and $\mathrm{Tor}_\bullet^{A^e}(M, A) \simeq \mathrm{Tor}_\bullet^{A^e}(A, M)$.

4 Smoothness of Hopf-Galois extensions

This section is dedicated to the proof of our main theorem. We begin with a result on finiteness of the cohomological dimension.

Proposition 4.1. *Let H be a Hopf algebra with bijective antipode and let $B \subset A$ be an H -Galois extension such that A is flat as left and right B -module. We have $\mathrm{cd}(A) \leq \mathrm{cd}(B) + \mathrm{cd}(H)$, and hence if $\mathrm{cd}(B)$ and $\mathrm{cd}(H)$ are finite, so is $\mathrm{cd}(A)$.*

Proof. Under those assumptions, we can use Theorem 3.7. Thus, for every A -bimodule M , we get a spectral sequence

$$E_2^{pq} = \mathrm{Ext}_{H^{op}}^p(k_\varepsilon, \mathrm{HH}^q(B, M)) \implies \mathrm{HH}^{p+q}(A, M).$$

If $\mathrm{cd}(B)$ or $\mathrm{cd}(H)$ is infinite, there is nothing to show, so we assume that these are finite. For $p > \mathrm{cd}(H)$ or $q > \mathrm{cd}(B)$, we have $E_2^{pq} = \{0\}$. Now, if we denote $d_2^{p,q} : E_2^{pq} \rightarrow E_2^{p+2, q-1}$ the differential on the second page of the spectral sequence, we get that for $p > \mathrm{cd}(H)$ or $q > \mathrm{cd}(B)$ the maps $d_2^{p,q} : E_2^{p,q} \rightarrow E_2^{p+2, q-1}$ and $d_2^{p-2, q+1} : E_2^{p-2, q+1} \rightarrow E_2^{p, q}$ are both 0 hence $\{0\} = E_2^{p, q} \simeq E_3^{p, q} \simeq \dots \simeq \mathrm{HH}^{p+q}(A, M)$. We obtain that $\mathrm{HH}^n(A, M) = \{0\}$ for $n > \mathrm{cd}(B) + \mathrm{cd}(H)$, and hence $\mathrm{cd}(A) \leq \mathrm{cd}(H) + \mathrm{cd}(B)$. \square

Proof of Theorem 1.1. Let H be a Hopf algebra with bijective antipode and let $B \subset A$ be an H -Galois extension. We assume that A is faithfully flat as left and right B -module and that H and B are homologically smooth algebras. We know from Proposition 4.1 that A has finite cohomological dimension, hence to prove that A is homologically smooth, it remains to prove, by Proposition 2.3, that A is of type FP_∞ as an A -bimodule.

To prove that A is of type FP_∞ as an A -bimodule, we will use the characterization of Proposition 2.4, and hence we consider the A -bimodule $M = \prod A^e$ where \prod is an arbitrary direct product. We have to show that for $n \geq 1$, one has $\mathrm{HH}_n(A, M) = 0$ and that $\mathrm{HH}_0(A, M) \simeq \mathrm{Tor}_0^{A^e}(M, A) \simeq (\prod A^e) \otimes_{A^e} A \simeq \prod A \simeq \prod \mathrm{Tor}_0^{A^e}(A^e, A) \simeq \prod \mathrm{HH}_0(A, A^e)$.

Under the hypothesis that A is faithfully flat as a right and left B -module and regarding Remark 3.8, we can use Theorem 3.7 (ii). Thus, for every A -bimodule M we get a spectral sequence

$$E_{pq}^2 = \mathrm{Tor}_p^H(k_\varepsilon, \mathrm{HH}_q(B, M)) \implies \mathrm{HH}_{p+q}(A, M).$$

The algebra B is of type FP_∞ as a left B^e -module hence $\mathrm{HH}_q(B, -) = \mathrm{Tor}_q^{B^e}(-, B)$ commutes with direct products. Moreover, A^e is flat as a left and right B^e -module [16, Lemma 2.1] thus for $q \geq 1$, we get that

$$\mathrm{HH}_q(B, \prod A^e) \simeq \prod \mathrm{HH}_q(B, A^e) = \{0\}.$$

Hence we have $E_{pq}^2 = \{0\}$ for $q > 0$.

If π_M (resp. π_{A^e}) denotes the natural projection of M (resp. A^e) onto $\mathrm{HH}_0(B, M) \simeq M/[M, B]$ (resp. $\mathrm{HH}_0(B, A^e) \simeq A^e/[A^e, B]$), the natural isomorphism $\mu : \mathrm{HH}_0(B, M) \rightarrow \prod \mathrm{HH}_0(B, A^e)$ is given by $\mu(\pi_M((x_i \otimes y_i)_i)) = (\pi_{A^e}(x_i \otimes y_i))_i$ for any $x_i, y_i \in A$. Hence, for $h \in H$, we have

$$\mu(h \cdot \pi_M((x_i \otimes y_i)_i)) = \mu(\pi_M((h^{(2)}x_i \otimes y_i h^{(1)}))_i) = (\pi_{A^e}(h^{(2)}x_i \otimes y_i h^{(1)}))_i = h \cdot \left((\pi_{A^e}(x_i \otimes y_i))_i \right).$$

The natural isomorphism $\mathrm{HH}_0(B, M) \simeq \prod \mathrm{HH}_0(B, A^e)$ is thus an isomorphism of H -modules. Therefore, since the algebras B and H are homologically smooth, we get, using Proposition 2.4, for $p \geq 0$

$$E_{p0}^2 = \mathrm{Tor}_p^H(k_\varepsilon, \mathrm{HH}_0(B, M)) \simeq \mathrm{Tor}_p^H(k_\varepsilon, \prod \mathrm{HH}_0(B, A^e)) \simeq \prod \mathrm{Tor}_p^H(k_\varepsilon, \mathrm{HH}_0(B, A^e)).$$

The A^e -module A^e is projective, hence [16, Proposition 4.4] ensures that

$$\mathrm{Tor}_p^H(k_\varepsilon, \mathrm{HH}_0(B, A^e)) = \{0\} \text{ for } p \geq 1.$$

Hence for $p \geq 1$ we have

$$E_{p0}^2 = \mathrm{Tor}_p^H(k_\varepsilon, \mathrm{HH}_0(B, M)) \simeq \prod \mathrm{Tor}_p^H(k_\varepsilon, \mathrm{HH}_0(B, A^e)) = \{0\}$$

We get that $E_{p,q}^2 = \mathrm{Tor}_p^H(k_\varepsilon, \mathrm{HH}_q(B, M)) = \{0\}$ for $(p, q) \neq (0, 0)$, and hence the spectral sequence ensures that $\mathrm{HH}_n(A, M) = \{0\}$ for $n > 0$.

Finally, using [16, Proposition 4.2], that k_ε is of type FP_∞ as a right H -module and that B is smooth, we obtain

$$\begin{aligned} \mathrm{HH}_0(A, \prod A^e) &\simeq \mathrm{Tor}_0^H(k_\varepsilon, \mathrm{HH}_0(B, \prod A^e)) \\ &\simeq \mathrm{Tor}_0^H(k_\varepsilon, \prod \mathrm{HH}_0(B, A^e)) \\ &\simeq \prod \mathrm{Tor}_0^H(k_\varepsilon, \mathrm{HH}_0(B, A^e)) \\ &\simeq \prod \mathrm{HH}_0(A, A^e). \end{aligned}$$

It is not difficult to check that the above isomorphism is the natural map μ in the third item of Proposition 2.4, the first and last isomorphism being explicit from [16, Proposition 4.2], and the second and third one being obtained from the natural respective maps μ as well. We thus conclude from Proposition 2.4 that A is homologically smooth. \square

Remark 4.2. In the situation of Theorem 1.1, we have seen that $\mathrm{cd}(A) \leq \mathrm{cd}(B) + \mathrm{cd}(H)$. We have not been able to show that the equality $\mathrm{cd}(A) = \mathrm{cd}(B) + \mathrm{cd}(H)$ holds in general, although we suspect it does. It holds in the case of Galois objects [21], of smash products [11], and in the case of exact sequences of Hopf algebras, as we will show in a forthcoming paper.

5 An example

We finish the paper by presenting an illustrative example.

Definition 5.1. Let B be a commutative k -algebra, let $b \in B$ and let $q \in k^\times$ with $q^2 \neq 1$. The k -algebra $U_q^{B,b}$ is defined by

$$U_q^{B,b} = B \left\langle g, g^{-1}, e, f \mid eg = q^{-2}ge, gf = q^{-2}fg, ef - fe = bg - \frac{1}{q - q^{-1}}g^{-1} \right\rangle$$

When $B = k$ and $b = (q + q^{-1})^{-1}$, the algebra $U_q^{B,b}$ is the quantized enveloping algebra $U_q(\mathfrak{sl}_2)$.

Proposition 5.2. *If B is an homologically smooth commutative algebra, then $U_q^{B,b}$ is as well homologically smooth, with $\text{cd}(U_q^{B,b}) \leq \text{cd}(B) + 3$.*

Proof. Recall that denoting $E = e$, $K = g$ and $F = f$, the algebra $U_q(\mathfrak{sl}_2)$ has a Hopf algebra structure defined by

$$\Delta(E) = 1 \otimes E + E \otimes K, \quad \Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \Delta(K) = K \otimes K,$$

$$S(E) = -EK^{-1}, \quad S(F) = -KK, \quad S(K) = K^{-1}, \quad \varepsilon(E) = \varepsilon(F) = 0, \quad \varepsilon(K) = 1.$$

More generally, $U_q^{B,b}$ is a right $U_q(\mathfrak{sl}_2)$ -comodule algebra with $\rho(a) = a \otimes 1$ for any $a \in B$, $\rho(g) = g \otimes K$, $\rho(e) = e \otimes K + 1 \otimes E$, $\rho(f) = f \otimes 1 + g^{-1} \otimes F$. It is shown in [9, Lemma 16] that $B \subset U_q^{B,b}$ is a cleft (hence in particular free) $U_q(\mathfrak{sl}_2)$ -Galois extension. Hence combining [6, Proposition 3.2.1] (for $\mathfrak{g} = \mathfrak{sl}_2$), Proposition 4.1 and Theorem 1.1, we obtain the announced result. \square

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Université Clermont Auvergne, CNRS, LMBP, F-63000 CLERMONT-FERRAND, FRANCE