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| Sergey Alexandrov, Khalil Bendriss. Mock modularity of Calabi-Yau threefolds. 2024. hal-04809892

HAL Id: hal-04809892

<https://hal.science/hal-04809892v1>

Preprint submitted on 28 Nov 2024

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Mock modularity of Calabi-Yau threefolds

Sergei Alexandrov and Khalil Bendriss

*Laboratoire Charles Coulomb (L2C), Université de Montpellier, CNRS,
F-34095, Montpellier, France*

e-mail: sergey.alexandrov@umontpellier.fr, khalil.bendriss@umontpellier.fr

ABSTRACT: Generating functions $h_r(\tau)$ of D4-D2-D0 BPS indices, appearing in Calabi-Yau compactifications of type IIA string theory and identical to rank 0 Donaldson-Thomas invariants, are known to be higher depth mock modular forms satisfying a specific modular anomaly equation, with depth determined by the D4-brane charge r . We develop a method to solve the anomaly equation for arbitrary charges, in terms of indefinite theta series. This allows us to find the generating functions up to modular forms that can be fixed by computing just a finite number of Fourier coefficients of h_r .

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1. Introduction

The indices counting BPS states in compactifications of type II strings on Calabi-Yau (CY) threefolds \mathfrak{Y} play a prominent role both in physics and mathematics. On the physics side, they represent degeneracies of BPS black holes and encode weights of instanton corrections to the low energy effective action. On the mathematics side, they coincide with the generalized Donaldson-Thomas (DT) invariants whose importance for understanding geometry of the CY threefolds can hardly be overestimated.

For non-compact CYs, there are various techniques to compute these BPS indices, which are based on localization, quivers, spectral networks and their generalizations, relations to topological and gauge theories, etc., see e.g. [1, 2, 3, 4, 5, 6]. However, for compact threefolds most of these techniques cannot be applied and the problem becomes much more complicated.

There are actually two classes of BPS indices which, at least in principle, can be systematically calculated. First, for D6-brane charge equal to ± 1 , the BPS indices (at large volume) coincide with the ordinary DT (respectively, PT (due to Padharipande-Thomas)) invariants. Their generating function is given by the famous MNOP formula [7, 8] in terms of Gopakumar-Vafa (GV) invariants, which in turn can be found by computing the topological string free energy, for example, by the direct integration method [9, 10, 11].

Second, for vanishing D6-brane charge, the BPS indices, known also as rank 0 DT invariants, count D4-D2-D0 BPS states and can be organized in generating functions $h_p(\tau)$ where p^a (with $a = 1, \dots, b_2(\mathfrak{Y})$) is the D4-brane charge which geometrically corresponds to a divisor

$\mathcal{D}_p = p^a \gamma_a$ of \mathfrak{Y} where γ_a is a basis of $H_4(\mathfrak{Y}, \mathbb{Z})$.¹ These functions turn out to possess nice modular properties [12, 13, 14, 15] which severely restrict $h_p(\tau)$ and, again at least in principle, can be used to fix them up to a finite number of coefficients.

The precise modular properties of $h_p(\tau)$ strongly depend on properties of the divisor \mathcal{D}_p . If the divisor is *irreducible*, the generating function must be a weakly holomorphic modular form of weight $-\frac{1}{2}b_2 - 1$ [12], i.e. it has the expansion

$$h_p(\tau) = \sum_{n \geq n_{\min}} c_n q^n, \quad q = e^{2\pi i \tau}, \quad (1.1)$$

where $n_{\min} < 0$, and transforms in the usual way under the standard $SL(2, \mathbb{Z})$ transformations acting on τ . The space of such modular forms is finite dimensional and its dimension is bounded from above by the number of *polar* terms, i.e. terms with $n < 0$. This is why in this case it is enough to compute only the polar terms in (1.1) to completely fix $h_p(\tau)$. This idea was applied long ago to a few one-parameters CY threefolds in [16, 17, 18, 19] and revised recently in [20, 21]. In particular, in [21] a systematic way to compute first terms in the expansion (1.1) has been suggested which is based on new wall-crossing relations between PT and rank 0 DT invariants [22]. Combined with the MNOP formula, they allow to express D4-D2-D0 BPS indices in terms of GV invariants so that, if the latter are known up to sufficiently high genus, all polar terms (and not only) can be computed.

If the divisor \mathcal{D}_p is *reducible*, i.e. $p^a = \sum_{i=1}^r p_i^a$ with positive p_i^a and $r > 1$, the modular properties of $h_p(\tau)$ are more involved. It was shown in [14, 15] that the generating functions are *mock* modular forms of depth $r - 1$ with a specific modular anomaly. A convenient way to characterize the anomaly is to consider a *modular completion* that is a non-holomorphic function $\widehat{h}_p(\tau, \bar{\tau})$ that transforms as a usual modular form and differs from $h_p(\tau)$ only by terms suppressed in the limit $\text{Im } \tau \rightarrow \infty$. An exact expression for the completion is given below in section 2.2 (see (2.8)) in a simplified form found recently in [23]. An important feature of this formula is that $\widehat{h}_p(\tau, \bar{\tau})$ is determined by the generating functions $h_{p_i}(\tau)$ of the constituents.

Although for mock modular forms the polar terms alone are not sufficient anymore to fix the function uniquely, the missing information can be recovered from the modular anomaly. Namely, one can follow the two-step strategy. First, one finds *any* mock modular form $h_p^{(\text{an})}(\tau)$ having the given modular anomaly. Obviously, the generating function h_p can differ from $h_p^{(\text{an})}$ at most by a modular form $h_p^{(0)}$, i.e.

$$h_p = h_p^{(\text{an})} + h_p^{(0)}. \quad (1.2)$$

Given this representation, at the second step, the modular ambiguity $h_p^{(0)}$ can be fixed in the usual way by computing its polar terms given by the difference of the polar terms of h_p and $h_p^{(\text{an})}$.

¹In fact, the generating functions are vector valued so that their components $h_{p,\mu}(\tau)$ are labeled by residue class μ_a taking values in the discriminant group Λ^*/Λ where $\Lambda = H_4(\mathfrak{Y}, \mathbb{Z})$. For simplicity of exposition, we drop the vector index in the Introduction.

For one-parameter CYs with the triple intersection number equal to a power of a prime number and D4-brane charge $r = 2$,² the first step (solution of the modular anomaly) has been realized in [20]. Then for two CYs known as decanctic X_{10} and octic X_8 , the second step (computing the polar terms) has been done in [24], which resulted in explicit mock modular generating functions h_2 for this pair of threefolds.

The goal of this paper is, still restricting to the one-parameter case $b_2 = 1$, to find a solution of the modular anomaly, i.e. the functions $h_r^{(\text{an})}$, for higher charges. Thus, we reduce the problem of finding the generating functions h_r to just the problem of computing their polar terms. This last problem is left for future research.

The immediate question which arises when one solves the modular anomaly for h_r is how this can be done given that the anomaly depends on the generating functions of lower charges that remain unknown because their polar terms are not fixed yet? To address this issue, we disentangle the anomalous parts of all generating functions from their modular ambiguities fixed by the polar terms. Namely, we express each h_r as a polynomial in $h_{r_i}^{(0)}$ with $r_i \leq r$ (see (3.2)) and show that the coefficients $g^{(\mathbf{r})}$, where $\mathbf{r} = (r_1, \dots, r_n)$ such that $\sum_{i=1}^n r_i = r$, are themselves mock modular forms of depth $n - 1$ satisfying an appropriate anomaly equation (3.3). Thus, the problem of solving the modular anomaly for h_r is reformulated as the problem of solving the modular anomaly equations for the holomorphic functions $g^{(\mathbf{r})}(\tau)$ parametrized by n charges r_i . We call these functions *anomalous coefficients*.

It turns out that it is relatively easy to give a solution for two infinite families of the anomalous coefficients. First, in the $n = 2$ case with arbitrary r_1 and r_2 , the anomaly is characterized by a simple theta series depending on a single combination of all parameters which we denote by κ_{12} . A partial solution for such $g^{(r_1, r_2)}$ (when κ_{12} is a power of a prime number) has already been given in [20]. But, in fact, a solution for generic κ_{12} is also known and provided by *mock modular forms of optimal growth* introduced in [25]. They are constructed by applying certain Hecke-like operators to a set of “seed” mock modular functions $\mathcal{G}^{(d)}$ defined for each d that is a square-free positive integer with an even number of prime factors. In particular, $\mathcal{G}^{(1)}$ coincides with the generating series of Hurwitz class numbers, which is also known to be the normalized generating function of $SU(2)$ Vafa-Witten (VW) invariants on \mathbb{P}^2 [26], consistently with the results of [20]. In fact, this is a particular case of the second family of solutions. Namely, we show that the anomaly equations for $g^{(\mathbf{r})}$ with all $r_i = 1$ form a closed system which, for the intersection number of \mathfrak{Y} equal to one, coincides with a similar system of anomaly equations for the normalized generating functions of $SU(n)$ VW invariants on \mathbb{P}^2 , with n equal to the number of charges. Thus, these two sets of functions can be simply identified. Since the generating functions of VW invariants on \mathbb{P}^2 are by now well-known for any rank of the gauge group [27, 28] (see also [29, 30, 31, 4]), this identification provides a solution for the subset of anomalous coefficients.

Unfortunately, neither of these solutions seems to have a simple generalization to other cases. Therefore, we follow an alternative strategy which is meant to work for an arbitrary set of charges and is based on the use of indefinite theta series. This however requires two

²For one-parameter CYs, the D4-brane charge p^1 coincides with the degree of reducibility of the divisor \mathcal{D}_p and therefore will be denoted by r in the rest of this paper.

preliminary steps. First, the system of anomaly equations should be extended to include a *refinement* encoded by an elliptic parameter z . This allows to simplify both the equations and their solution, but most importantly it provides a regularization of certain singularities which would otherwise plague the theta series. Second, one should artificially extend the relevant charge lattice (which is achieved by multiplication by an appropriate combination of Jacobi theta series) to ensure that the resulting lattice possesses a set of null vectors necessary to write down a general solution. Such solution is then given by a combination of indefinite theta series and holomorphic modular functions $\phi^{(r)}(\tau, z)$ (see Theorem 5.1) which ensure that the unrefined limit $z \rightarrow 0$ is non-singular. In fact, it is the proper choice of these functions and the explicit evaluation of the unrefined limit that are the most non-trivial elements of our construction.

In this paper we perform the construction in detail and derive the final form of $g^{(r)}$ for the cases of two and three charges, while in generic case we find the general form of the refined solution, obtain the functions $\phi^{(r)}$ ensuring the existence of the unrefined limit, but leave the limit itself non-evaluated since it appears quite hard to do this analytically. Besides, we check that the solutions based on the indefinite theta series are consistent with the ones obtained by Hecke-like operators and from VW theory.

The organization of the paper is as follows. In the next section we recall properties of the generating series of D4-D2-D0 BPS indices, including their behavior under modular transformations. In section 3 we introduce the anomalous coefficients that disentangle the mock modular parts of the generating series, which are fixed by the modular anomaly equations, from their modular ambiguities fixed by computing the polar terms. In section 4 we establish relations to the mock modular forms of optimal growth introduced in [25] and to the normalized generating functions of VW invariants on \mathbb{P}^2 . In section 5 we present our main construction of the anomalous coefficients in terms of indefinite theta series. Finally, in section 6 we discuss our results and their possible extensions. Several appendices contain some useful information on various building blocks of the construction, details of our calculations and some explicit q -series. For the reader's convenience, the last appendix J includes an index of notations.

2. BPS indices and their modular anomaly

In this section we introduce our main objects of interest, the generating series of D4-D2-D0 BPS indices, and describe their main properties. We restrict ourselves from the very beginning to the case of $b_2(\mathfrak{Y}) = 1$. A more complete discussion of BPS indices can be found, e.g., in [21].

2.1 D4-D2-D0 BPS indices and their generating series

In type IIA string theory compactified on a CY threefold \mathfrak{Y} , BPS indices depend on the electromagnetic charge γ which labels elements in the even cohomology of \mathfrak{Y} and can be represented by a vector (p^0, r, q, q_0) where different components correspond to D6, D4, D2 and D0-charges, respectively. In this paper we are interested in D4-D2-D0 BPS states for which $p^0 = 0$, which are also known as rank 0 DT invariants.

In general, BPS indices also depend in a piece-wise linear way on the Kähler modulus due to the wall-crossing phenomenon and thus they take different values in different chambers of the moduli space. Here we take them to be evaluated in the large volume attractor chamber [32] containing the point

$$z_\infty^a(\gamma) = \lim_{\lambda \rightarrow +\infty} \left(-\frac{q}{\kappa r} + i\lambda r \right), \quad (2.1)$$

where κ is the intersection number of \mathfrak{Y} . In this chamber the BPS indices are invariant under the so-called spectral flow transformation acting on the charge vector. It gives rise to the following decomposition of the D2-brane charge

$$q = \mu + \frac{1}{2} \kappa r^2 + \kappa r \epsilon, \quad (2.2)$$

where $\epsilon \in \mathbb{Z}$ is the parameter shifted by the spectral flow, while $\mu \in \Lambda^*/\Lambda$ with $\Lambda = \kappa r \mathbb{Z}$ is the so-called *residue class* taking κr values and staying invariant. Thus, D4-D2-D0 BPS indices depend only on r , μ and an invariant combination of D2 and D0-charges

$$\hat{q}_0 \equiv q_0 - \frac{q^2}{2\kappa r}, \quad (2.3)$$

and will be denoted by $\Omega_{r,\mu}(\hat{q}_0)$.

An important fact, known as the Bogomolov-Gieseker bound [33], is that $\Omega_{r,\mu}(\hat{q}_0)$ vanishes unless the invariant charge \hat{q}_0 satisfies

$$\hat{q}_0 \leq \hat{q}_0^{\max} = \frac{1}{24} (\kappa r^3 + c_2 r), \quad (2.4)$$

where c_2 is the second Chern class of \mathfrak{Y} . It allows to define the generating series

$$h_{r,\mu}(\tau) = \sum_{\hat{q}_0 \leq \hat{q}_0^{\max}} \bar{\Omega}_{r,\mu}(\hat{q}_0) q^{-\hat{q}_0}, \quad (2.5)$$

where $q = e^{2\pi i \tau}$ and the bar denotes *rational* BPS indices defined for any charge as $\bar{\Omega}(\gamma) = \sum_{d|\gamma} \Omega(\gamma/d)/d^2$. Only generating series of rational BPS indices are expected to possess nice modular properties [34]. Although it does not lead to conceptual simplifications, sometimes it is useful to use the symmetry³ $h_{r,-\mu} = h_{r,\mu}$ where we extended the range of μ from $[0, \kappa r - 1]$ by periodicity.

2.2 Modular symmetry

The most important feature of the generating series $h_{r,\mu}(\tau)$ is that they transform as depth $r - 1$ vector valued (VV) mock modular forms under the standard $SL(2, \mathbb{Z})$ transformations $\tau \mapsto \frac{a\tau + b}{c\tau + d}$. While a mathematical proof of this modular behavior is still absent, it was derived using duality symmetries of string theory [12, 13, 14, 15] and obtained recently a striking confirmation by verifying predictions of modularity against a direct calculation of DT invariants [21, 24].

³Mathematically, this symmetry follows from dualization of the coherent sheaf induced by the D4-brane.

More precisely, in the simplest case $r = 1$, $h_{1,\mu}$ is expected to be a weakly holomorphic VV modular form of weight $-3/2$ with the multiplier system closely related to the Weil representation attached to the lattice $\kappa r \mathbb{Z}$ with quadratic form $Q(x) = \kappa r x^2$ and determined by the following two matrices for T and S-transformations [35, Eq.(2.10)] (see also [16, 36, 37, 38, 13])

$$\begin{aligned} M_{\mu\nu}^{(h_r)}(T) &= e^{\frac{\pi i}{\kappa r}(\mu + \frac{1}{2}\kappa r^2)^2 + \frac{\pi i}{12} c_2 r} \delta_{\mu\nu}, \\ M_{\mu\nu}^{(h_r)}(S) &= \frac{(-1)^{\chi_r}}{\sqrt{\kappa r}} e^{-\frac{\pi i}{4} - 2\pi i \frac{\mu\nu}{\kappa r}}, \end{aligned} \quad (2.6)$$

where χ_r is the arithmetic genus of the divisor \mathcal{D}_r given by

$$\chi_r = \frac{1}{6} \kappa r^3 + \frac{1}{12} c_2 r \in \mathbb{Z}. \quad (2.7)$$

We wrote the multiplier system (2.6) for generic r because for $r > 1$ it also enters the modular transformation of $h_{r,\mu}$. However, in this case the transformation has a modular anomaly so that the generating series is only mock modular [14, 15]. This means that $h_{r,\mu}$ can be promoted to a non-holomorphic modular completion $\widehat{h}_{r,\mu}(\tau, \bar{\tau})$ constructed from iterated integrals of some modular forms and transforming itself as a true modular form of the same weight $-3/2$ and multiplier system (2.6). The fact that $h_{r,\mu}$ has depth $r - 1$ means that the $\bar{\tau}$ -derivative of its completion, which is known as *shadow* of the mock modular form, is itself a completion of a mock modular form of depth $r - 2$ (see [39] for the precise definition).

The explicit form of the completion has been found in [15] and then slightly simplified in [23]. As a result, it reads as⁴

$$\widehat{h}_{r,\mu}(\tau, \bar{\tau}) = h_{r,\mu}(\tau) + \sum_{n=2}^r \sum_{\sum_{i=1}^n r_i=r} \sum_{\boldsymbol{\mu}} R_{\boldsymbol{\mu},\boldsymbol{\mu}}^{(r)}(\tau, \bar{\tau}) \prod_{i=1}^n h_{r_i,\mu_i}(\tau), \quad (2.8)$$

where we use the bold script to denote tuples of n variables like $\boldsymbol{r} = (r_1, \dots, r_n)$. Note that the first term can be included into the second by setting $R_{\boldsymbol{\mu},\boldsymbol{\mu}'}^{(r)} = \delta_{\boldsymbol{\mu},\boldsymbol{\mu}'}$. The other coefficients $R_{\boldsymbol{\mu},\boldsymbol{\mu}}^{(r)}$ can be represented as non-holomorphic theta series defined on a $n - 1$ -dimensional lattice

$$R_{\boldsymbol{\mu},\boldsymbol{\mu}}^{(r)}(\tau, \bar{\tau}) = \sum_{\substack{\sum_{i=1}^n q_i = \mu + \frac{1}{2}\kappa r^2 \\ q_i \in \kappa r_i \mathbb{Z} + \mu_i + \frac{1}{2}\kappa r_i^2}} \text{Sym} \left\{ (-1)^{\sum_{i<j} \gamma_{ij}} \mathcal{Z}_n(\hat{\boldsymbol{\gamma}}; \tau_2) \right\} e^{\pi i \tau Q_n(\hat{\boldsymbol{\gamma}})}, \quad (2.9)$$

where $\hat{\boldsymbol{\gamma}}$ is the n -tuple of reduced charge vectors $\hat{\gamma}_i = (r_i, q_i)$ with q_i decomposed as in (2.2) with fixed residue classes μ_i . Besides, Sym denotes symmetrization (with weight $1/n!$) with respect to charges $\hat{\gamma}_i$, γ_{ij} is the anti-symmetric Dirac product of charges

$$\gamma_{ij} = r_j q_i - r_i q_j, \quad (2.10)$$

and Q_n denotes the quadratic form, originating from the quadratic term in the definition (2.3) of the invariant charge \hat{q}_0 ,

$$Q_n(\hat{\boldsymbol{\gamma}}) = \frac{1}{\kappa r} \left(\sum_{i=1}^n q_i \right)^2 - \sum_{i=1}^n \frac{q_i^2}{\kappa r_i}. \quad (2.11)$$

⁴We always assume that r_i are positive integers and do not write explicitly this condition in the sum over decompositions of D4-brane charge.

Finally, the coefficients \mathcal{R}_n determining the kernel of the theta series are constructed as suitable combinations of derivatives of the so-called generalized error functions introduced in [40, 41]. We relegate the precise definitions of all these functions to appendix D.

The equation for the completion (2.8) specifies the modular anomaly of $h_{r,\mu}$. Equivalently, one can talk about the holomorphic anomaly for $\widehat{h}_{r,\mu}$. One can check that this anomaly given by $\partial_\tau \widehat{h}_{r,\mu}$ is manifestly modular since it can be expressed through the completions \widehat{h}_{r_i,μ_i} , see [15, Eq.(5.35)]. The important feature of all these anomaly equations is that the r.h.s. is expressed through the generating series for charges $r_i < r$. Thus, (2.8) can be seen as a recursive system of equations determining the anomalous parts of the generating functions.

2.3 Redefinition

Before we turn to the main goal of this paper, which is to solve the anomaly equations (2.8), let us make a slight redefinition of the generating series $h_{r,\mu}$ by shifting their vector index and multiplying by a sign factor. This will allow us to avoid some annoying shifts and signs in what follows. More precisely, we set

$$\widetilde{h}_{r,\mu}(\tau) = (-1)^{(r-1)\mu} h_{r,\widetilde{\mu}(r)}(\tau), \quad (2.12)$$

where

$$\widetilde{\mu}(r) = \mu - \frac{\kappa r(r-1)}{2}. \quad (2.13)$$

This redefinition leads to two simplifications. First, the shift of μ replaces the quadratic term in the spectral flow decomposition (2.2) by a linear one so that now it reads

$$q = \mu + \frac{1}{2} \kappa r + \kappa r \epsilon. \quad (2.14)$$

As a result, all such terms cancel in the condition on the sum over q_i in (2.9) and, using the decomposition (2.14), it can be rewritten as

$$\kappa \sum_{i=1}^n r_i \epsilon_i = \Delta \mu, \quad \Delta \mu = \mu - \sum_{i=1}^n \mu_i. \quad (2.15)$$

Second, the sign factor in (2.12) cancels the sign factor in (2.9). Indeed, substituting the decomposition (2.14) into the Dirac product (2.10), one finds

$$\begin{aligned} \sum_{i<j} \gamma_{ij} &= \sum_{i<j} \left(r_j \left(\kappa r_i \epsilon_i + \mu_i + \frac{\kappa r_i}{2} \right) - r_i \left(\kappa r_j \epsilon_j + \mu_i + \frac{\kappa r_j}{2} \right) \right) \\ &= \sum_{i<j} \left(\kappa r_i r_j (\epsilon_i + \epsilon_j) + r_j \mu_i + r_i \mu_j \right) \pmod{2} \\ &= \sum_i \left(\kappa r_i (r - r_i) \epsilon_i + (r - r_i) \mu_i \right) \pmod{2} \\ &= (r-1) \sum_i (\kappa r_i \epsilon_i + \mu_i) - \sum_i (r_i - 1) \mu_i \pmod{2} \\ &= (r-1) \mu - \sum_i (r_i - 1) \mu_i \pmod{2}, \end{aligned} \quad (2.16)$$

where at the last step we used (2.15).

Given the above observations, the anomaly equation (2.8) reformulated in terms of the redefined generating series takes the following form

$$\widehat{h}_{r,\mu}(\tau, \bar{\tau}) = \sum_{n=1}^r \sum_{\sum_{i=1}^n r_i=r} \sum_{\mu} \tilde{R}_{\mu,\mu}^{(r)}(\tau, \bar{\tau}) \prod_{i=1}^n \tilde{h}_{r_i,\mu_i}(\tau), \quad (2.17)$$

where

$$\tilde{R}_{\mu,\mu}^{(r)}(\tau, \bar{\tau}) = \sum_{\substack{\sum_{i=1}^n q_i = \mu + \kappa r / 2 \\ q_i \in \kappa r_i \mathbb{Z} + \mu_i + \kappa r_i / 2}} \text{Sym} \left\{ \mathcal{R}_n(\hat{\gamma}; \tau_2) \right\} e^{\pi i \tau Q_n(\hat{\gamma})}. \quad (2.18)$$

The redefinition (2.12) also affects the multiplier system which is now given by (B.2).

3. Anomalous coefficients

We expect that for each D4-brane charge r , the anomaly equation fixes the generating function $\tilde{h}_{r,\mu}$ up to a modular ambiguity which in turn can be fixed by other means, e.g. by computing first few terms in the Fourier expansion of $\tilde{h}_{r,\mu}$. In other words, we can represent

$$\tilde{h}_{r,\mu} = \tilde{h}_{r,\mu}^{(\text{an})} + \tilde{h}_{r,\mu}^{(0)}, \quad (3.1)$$

where $\tilde{h}_{r,\mu}^{(\text{an})}$ is a depth $r-1$ mock modular form satisfying (2.17), while $\tilde{h}_{r,\mu}^{(0)}$ is pure modular. The problem however is that the r.h.s. of (2.17) depends on the full generating functions \tilde{h}_{r_i,μ_i} with $r_i < r$ and hence on all $\tilde{h}_{r_i,\mu_i}^{(0)}$ which remain unknown at this point. Therefore, $\tilde{h}_{r,\mu}^{(\text{an})}$ must also depend on them, and what we can do at best is to find $\tilde{h}_{r,\mu}^{(\text{an})}$ up to these modular functions. To achieve this goal, we first parametrize the dependence of $\tilde{h}_{r,\mu}$ on $\tilde{h}_{r_i,\mu_i}^{(0)}$ by holomorphic functions $g_{\mu,\mu}^{(r)}(\tau)$ which we call anomalous coefficients, characterize them by anomaly equations similar to (2.17), and then solve these equations. In this section we perform the first two steps and leave the third one to the subsequent sections. The main result is captured by the following

Theorem 3.1. *Let $g_{\mu,\mu}^{(r)} = \delta_{\mu,\mu}$ and $\tilde{h}_{r,\mu}^{(0)}$ be a set of holomorphic modular forms. Then*

$$\tilde{h}_{r,\mu}(\tau) = \sum_{n=1}^r \sum_{\sum_{i=1}^n r_i=r} \sum_{\mu} g_{\mu,\mu}^{(r)}(\tau) \prod_{i=1}^n \tilde{h}_{r_i,\mu_i}^{(0)}(\tau), \quad (3.2)$$

is a depth $r-1$ modular form with completion of the form (2.17) provided $g_{\mu,\mu}^{(r)}$ are depth $n-1$ mock modular forms (where n is the number of charges r_i) with completions satisfying

$$\widehat{g}_{\mu,\mu}^{(r)} = \text{Sym} \left\{ \sum_{m=1}^n \sum_{\sum_{k=1}^m n_k=n} \sum_{\nu} \tilde{R}_{\mu,\nu}^{(s)} \prod_{k=1}^m g_{\nu_k, \mathfrak{m}_k}^{(r_k)} \right\}, \quad (3.3)$$

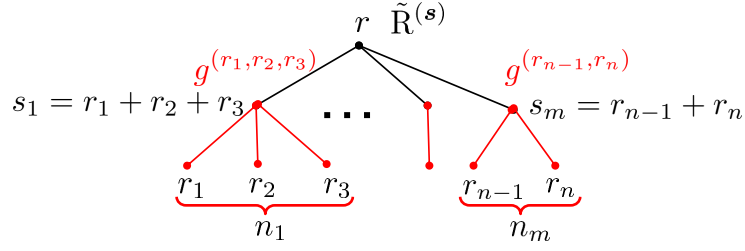


Figure 1: A representation of contributions to the r.h.s. of (3.3) in terms of rooted trees of depth 2.

where⁵

$$j_k = \sum_{l=1}^{k-1} n_l, \quad s_k = \sum_{i=1}^{n_k} r_{j_k+i}, \quad \mathbf{r}_k = (r_{j_k+1}, \dots, r_{j_{k+1}}), \quad (3.4)$$

$$\mathbf{m}_k = (\mu_{j_k+1}, \dots, \mu_{j_{k+1}}).$$

Proof. To prove the theorem, we must show that (3.2) and (3.3) give rise to the same modular completions $\widehat{h}_{r,\mu}$. On one hand, this completion is obtained by substituting the ansatz (3.2) into the r.h.s. of (2.17) which gives

$$\widehat{h}_{r,\mu} = \sum_{n=1}^r \sum_{\sum_{k=1}^n s_k=r} \sum_{\boldsymbol{\mu}} \widetilde{R}_{\boldsymbol{\mu},\boldsymbol{\mu}}^{(s)} \prod_{k=1}^n \left(\sum_{n_k=1}^{s_k} \sum_{\sum_{l=1}^{n_k} r_{k,l}=s_k} \sum_{\boldsymbol{\nu}_k} g_{\boldsymbol{\mu}_k,\boldsymbol{\nu}_k}^{(\mathbf{r}_k)} \prod_{l=1}^{n_k} \widetilde{h}_{r_{k,l},\nu_{k,l}}^{(0)} \right). \quad (3.5)$$

On the other hand, it is obtained by completing each term in (3.2) and then using the equation (3.3). This leads to the following expression

$$\begin{aligned} \widehat{h}_{r,\mu} &= \sum_{n=1}^r \sum_{\sum_{i=1}^n r_i=r} \sum_{\boldsymbol{\mu}} \widehat{g}_{\boldsymbol{\mu},\boldsymbol{\mu}}^{(r)} \prod_{i=1}^n \widetilde{h}_{r_i,\mu_i}^{(0)} \\ &= \sum_{n=1}^r \sum_{\sum_{i=1}^n r_i=r} \sum_{\boldsymbol{\mu}} \sum_{m=1}^n \sum_{\sum_{k=1}^m n_k=n} \sum_{\boldsymbol{\nu}} \widetilde{R}_{\boldsymbol{\mu},\boldsymbol{\nu}}^{(s)} \prod_k g_{\boldsymbol{\nu}_k,\mathbf{m}_k}^{(\mathbf{r}_k)} \prod_{i=1}^n \widetilde{h}_{r_i,\mu_i}^{(0)}, \end{aligned} \quad (3.6)$$

where we omitted the sign of symmetrization from (3.3) because it is ensured by the sums over decompositions $r = \sum_{i=1}^n r_i$ and residue classes μ_i . Clearly, the two equations coincide if one can identify $(r_{k,l}, \mu_k, \nu_{k,l})$ in (3.5) with (r_i, ν_k, μ_i) in (3.6) and claim that

$$\sum_{n=1}^r \sum_{\sum_{k=1}^n s_k=r} \sum_{n_k=1}^{s_k} \sum_{\sum_{l=1}^{n_k} r_{k,l}=s_k} = \sum_{n=1}^r \sum_{\sum_{i=1}^n r_i=r} \sum_{m=1}^n \sum_{\sum_{k=1}^m n_k=n} . \quad (3.7)$$

Here on the l.h.s. the sum goes over double decompositions: first, decompose r into s_k and then each s_k into $r_{k,l}$, while on the r.h.s. one first sums over decompositions of r into r_i and

⁵Note that while the sets \mathbf{r} and $\boldsymbol{\mu}$ have n elements, the sets \mathbf{s} and $\boldsymbol{\nu}$ have only $m \leq n$ elements. To comprehend the structure of the equation (3.3), it might be useful to use the fact that the sum on its r.h.s. is equivalent to the sum over rooted trees of depth 2 with leaves labelled by charges r_i and other vertices labelled by the sum of charges of their children. Using this labelling, we assign the function $\widetilde{R}_{\boldsymbol{\mu},\boldsymbol{\nu}}^{(s)}$ to the root vertex and the anomalous coefficients to the vertices of depth 1 with arguments determined by the charges of their children. Then the contribution of a tree is given by the product of the weights of its vertices. See Fig. 1.

then over various groupings of the indices i into sets preserving their ordering. It is obvious that the two sums are identical and hence the two representations of the completion, (3.5) and (3.6), coincide. \square

This theorem defines a family of holomorphic functions $g_{\mu,\mu}^{(r)}(\tau)$ restricted to have a modular anomaly determined by (3.3). Their weight and multiplier system follow from that of $\tilde{h}_{r,\mu}$ (see (B.2)) and are given in (B.3). As usual, together with the anomaly these data fix the anomalous coefficients up to a modular ambiguity. However, in contrast to the case of the generating functions, there are no any restrictions on this ambiguity and therefore we can choose any solution of (3.3). It is only important that, once a solution has been chosen for small charges, it is this solution that is used in the r.h.s. of (3.3) to determine $g_{\mu,\mu}^{(r)}$ for higher charges. In the rest of the paper, our goal will be to find explicit solutions for these functions.

An important observation is that the parameter κ and charges r_i enter the functions $\kappa^{n-1}\tilde{R}_{\mu,\mu}^{(r)}$ and the multiplier system (B.3) of the anomalous coefficients $g_{\mu,\mu}^{(r)}$ always in the form of the product κr_i . Indeed, this is true for the quadratic form (2.11), the bilinear form (D.9) and the charges (2.14). On the other hand, the vectors \mathbf{v}_{ij} (D.5) are linear in r_i . Since the generalized error functions (D.2) are independent of the overall scale of the matrix of parameters, this implies that the functions (D.7) and hence \mathcal{E}_n are homogeneous of degree $n - 1$ in r_i . Then (D.13) immediately leads to the same property for \mathcal{R}_n , which in turn confirms the claim for $\tilde{R}_{\mu,\mu}^{(r)}$. This implies that we can (although do not have to) choose a solution for all $g_{\mu,\mu}^{(r)}$ satisfying the following property

$$g_{\mu,\mu}^{(\kappa;r)}(\tau) = \kappa^{1-n} g_{\mu,\mu}^{(1;\kappa r)}(\tau), \quad (3.8)$$

where n is the number of charges and we explicitly indicated the dependence on κ in the upper index. Using this property would allow to reduce the problem of finding the anomalous coefficients to the case of $\kappa = 1$. Note that for this feature to hold it was crucial to perform the redefinition of section 2.3.

4. Partial solutions

In this section we provide a solution for two infinite families of anomalous coefficients.

4.1 Two charges and Hecke-like operators

Let us first consider the case of two arbitrary charges r_1 and r_2 . In this case the formula for the modular completion $\widehat{g}_{\mu,\mu_1,\mu_2}^{(r_1,r_2)}$ (3.3), representing the anomaly equation, takes the simple form

$$\widehat{g}_{\mu,\mu_1,\mu_2}^{(r_1,r_2)}(\tau, \bar{\tau}) = g_{\mu,\mu_1,\mu_2}^{(r_1,r_2)}(\tau) + \tilde{R}_{\mu,\mu_1,\mu_2}^{(r_1,r_2)}(\tau, \bar{\tau}), \quad (4.1)$$

and $g_{\mu,\mu_1,\mu_2}^{(r_1,r_2)}$ is required to be a mock modular form of weight $3/2$ with the multiplier system (B.3) specialized to $n = 2$. The function $\tilde{R}_{\mu,\mu_1,\mu_2}^{(r_1,r_2)}$ determining the completion is easily computable, but for our purposes it is sufficient to consider its derivative with respect to $\bar{\tau}$ which

specifies the shadow of $g_{\mu,\mu_1,\mu_2}^{(r_1,r_2)}$. It is given in (D.27) and suggests to look for a solution of the form

$$g_{\mu,\mu_1,\mu_2}^{(r_1,r_2)}(\tau) = r_0 \delta_{\Delta\mu}^{(\kappa r_0)} G_{\mu_{12}}^{(\kappa_{12})}(\tau), \quad (4.2)$$

where $r_0 = \gcd(r_1, r_2)$, $\Delta\mu$ was defined in (2.15), κ_{12} and μ_{12} are effective parameters introduced in (D.24), μ_{12} runs over $2\kappa_{12}$ values, and $\delta_x^{(n)}$ is the mod- n Kronecker delta defined by

$$\delta_x^{(n)} = \begin{cases} 1 & \text{if } x = 0 \pmod n, \\ 0 & \text{otherwise.} \end{cases} \quad (4.3)$$

In particular, the ansatz (4.2) is consistent with the property (3.8). If $G_\mu^{(\kappa)}$ is a VV mock modular form of weight $3/2$ with a modular completion satisfying

$$\tau_2^{3/2} \partial_{\bar{\tau}} \widehat{G}_\mu^{(\kappa)}(\tau, \bar{\tau}) = \frac{\sqrt{\kappa}}{16\pi i} \overline{\theta}_\mu^{(\kappa)}(\bar{\tau}), \quad (4.4)$$

where $\theta_\mu^{(\kappa)}(\tau)$ is the theta series (C.7) at $z = 0$, then it is trivial to see that (4.2) solves the anomaly equation (4.1). The only non-trivial fact to check is that it has the correct multiplier system. But this follows directly from Proposition D.1 because the relation (4.4) ensures that $G_\mu^{(\kappa)}$ has the multiplier system $M_{\mu\nu}^{(\kappa)}$ (D.28) conjugate to that of $\theta_\mu^{(\kappa)}$.

As a result, we have reduced the problem of finding the anomalous coefficients for arbitrary two charges to exactly the same problem that was studied in [20] for charges $r_1 = r_2 = 1$, in which case $\kappa_{12} = \kappa$. It was found that for any κ equal to a power of a prime integer, $G_\mu^{(\kappa)}$ is determined by the generating series H_μ ($\mu = 0, 1$) of Hurwitz class numbers⁶ through the action on it by a certain modification of the Hecke-like operator introduced in [45, 46]. However, it turns out that a solution of this problem for generic κ has already been found in the seminal paper [25]. More precisely, that paper looked for mock modular forms with shadow proportional to $\theta_\mu^{(\kappa)}$ and further restricted to have the slowest possible asymptotic growth of their Fourier coefficients. Such functions have been called mock modular forms of optimal growth. In our case we do not have to impose any restrictions on the asymptotic growth. But since any solution of (4.4) is equally suitable, we can take the one provided by [25]. All other solutions should differ just by a pure modular form.

In the rest of this subsection we present the formula for the mock modular forms of optimal growth found in [25] adjusting (and correcting) some normalization factors.⁷ To this end, let

- $\omega(\kappa)$ be the number of distinct prime divisors of κ , i.e.

$$\omega(\kappa) = \#\{p : p \text{ is prime and } p|\kappa\}; \quad (4.5)$$

- $\mu(d)$ be the Möbius function given by

$$\mu(d) = \begin{cases} +1 & \text{if } d \text{ is a square-free with an even number of prime factors,} \\ -1 & \text{if } d \text{ is a square-free with an odd number of prime factors,} \\ 0 & \text{if } d \text{ has a squared prime factor;} \end{cases} \quad (4.6)$$

⁶An explicit formula for the generating series can be found in [42, Eq.(1.12)] and its mock modular properties have been established in [43, 44].

⁷Strictly speaking, [25] worked in terms of *mock Jacobi forms* rather than mock modular forms. However, the latter can be easily extracted from the former by means of the theta expansion (E.1).

- $\hat{\mathcal{T}}_r^{(d)}$ be a Hecke-like operator given, when acting on modular forms (not necessarily holomorphic) of weight (w, \bar{w}) and multiplier system $M_{\mu\nu}^{(d)}$, by

$$\hat{\mathcal{T}}_r^{(d)} = \sum_{\substack{s^2|r \\ \gcd(s,d)=1}} \mu(s) s^{w+\bar{w}-1/2} U_s \mathcal{T}_r/s^2 \quad (4.7)$$

with

$$(U_s[h^{(\kappa)}])_\mu(\tau) = \delta_\mu^{(s)} h_{\mu/s}^{(\kappa)}(\tau), \quad \mu = 0, \dots, 2s^2\kappa - 1, \quad (4.8)$$

and

$$(\mathcal{T}_r[h^{(\kappa)}])_\mu(\tau) = \sum_{\substack{a,d>0 \\ ad=r}} \left(\frac{\sqrt{r}}{d}\right)^{w+\bar{w}+\frac{1}{2}} \sum_{b=0}^{d-1} \delta_{\mu/a}^{(1)} e^{\frac{\pi ib}{2\kappa ar}} \mu^2 h_{\mu/a}^{(\kappa)}\left(\frac{a\tau+b}{d}\right), \quad \mu = 0, \dots, 2r\kappa - 1. \quad (4.9)$$

We relate these operators to the ones defined in [25] and acting on Jacobi forms in appendix E. One can also check that for r prime power, $\hat{\mathcal{T}}_r^{(1)}$ coincides with the modification of \mathcal{T}_r introduced in [20].

In terms of these quantities, the mock modular forms of optimal growth are given by

$$G_\mu^{(\kappa)} = 2^{-\omega(\kappa)} \sum_{\substack{d|\kappa \\ \mu(d)=1}} \left(\hat{\mathcal{T}}_{\kappa/d}^{(d)} [\mathcal{G}^{(d)}]\right)_\mu, \quad (4.10)$$

where $\mathcal{G}^{(d)}$ are VV mock modular forms of weight $3/2$ with multiplier system $M_{\mu\nu}^{(d)}$. Thus, for each square-free integer with an even number of prime factors, such as 1, 6, 10, 14, 15, etc., one needs to provide such a mock modular form. The first two of them turn out to be well-known functions: for $d = 1$ it is (the doublet of) the generating series of Hurwitz class numbers,

$$\mathcal{G}_\mu^{(1)}(\tau) = H_\mu(\tau), \quad (4.11)$$

and for $d = 6$ it has the following explicit expression

$$\mathcal{G}_\mu^{(6)}(\tau) = \frac{\chi_{12}(\mu)}{12} h^{(6)}(\tau), \quad (4.12)$$

where

$$\chi_{12}(\mu) = \begin{cases} +1 & \text{if } \mu = \pm 1 \pmod{12}, \\ -1 & \text{if } \mu = \pm 5 \pmod{12}, \\ 0 & \text{if } \gcd(\mu, 12) > 0, \end{cases} \quad (4.13)$$

and

$$h^{(6)}(\tau) = \frac{12F_2^{(6)}(\tau) - E_2(\tau)}{\eta(\tau)} \quad (4.14)$$

is a mock modular form of weight $3/2$ with shadow proportional to the Dedekind eta function $\eta(\tau)$, which is defined in terms of the quasimodular Eisenstein series $E_2(\tau)$ and the function

$$F_2^{(6)}(\tau) = - \sum_{r>s>0} \chi_{12}(r^2 - s^2) s q^{rs/6}. \quad (4.15)$$

For many other functions $\mathcal{G}^{(d)}$, [25] determined their first Fourier coefficients, however we are not aware about any explicit expressions for their generating series.

4.2 Unit charges and Vafa-Witten

Let us now consider the case of n charges r_i all equal to 1. In addition, we also restrict ourselves to CYs with the intersection number $\kappa = 1$. A crucial simplification in this case is that one can drop all indices μ_i because they take only $\kappa r_i = 1$ value. Therefore, the corresponding anomalous coefficients can be denoted simply as $g_{n,\mu} \equiv g_{\mu}^{(1,\dots,1)}$. Another feature of this set of anomalous coefficients is that the anomaly equations for $g_{n,\mu}$ form a closed system and do not involve other anomalous coefficients. Moreover, it is easy to see that in this sector the anomaly equation (3.3) becomes identical to (2.17) under the identification $g_{n,\mu} \leftrightarrow \tilde{h}_{n,\mu}$ and thus takes the form

$$\widehat{g}_{n,\mu} = \sum_{m=1}^n \sum_{\sum_{k=1}^m n_k = n} \sum_{\mu} \tilde{R}_{\mu,\mu}^{(n)} \prod_{k=1}^m g_{n_k,\mu_k}. \quad (4.16)$$

The case $n = 2$ has already been analyzed in the previous subsection. It follows from the results presented there, and in agreement with [20], that

$$g_{2,\mu} = H_{\mu}, \quad \mu = 0, 1. \quad (4.17)$$

The vector valued function H_{μ} appearing here is known not only as the generating series of Hurwitz class numbers, but also as the (normalized) generating series of $SU(2)$ Vafa-Witten invariants on \mathbb{P}^2 , namely [26]

$$h_{2,\mu}^{\text{VW}} = 3(h_1^{\text{VW}})^2 H_{\mu}, \quad (4.18)$$

where $h_{n,\mu}^{\text{VW}}$ denotes the generating series of $SU(n)$ VW invariants and $h_1^{\text{VW}} = \eta^{-3}$. Combining the two relations, one obtains

$$g_{2,\mu} = \frac{1}{3} \mathfrak{g}_{2,\mu}, \quad (4.19)$$

where we introduced the normalized generating series

$$\mathfrak{g}_{n,\mu}(\tau) = \eta^{3n}(\tau) h_{n,\mu}^{\text{VW}}(\tau). \quad (4.20)$$

As we show below, the relation (4.19) is not an accident, but a particular case of a more general relation between $g_{n,\mu}$ and $\mathfrak{g}_{n,\mu}$.

Let us recall that the VW invariants count the Euler characteristic of moduli spaces of instantons in a topological supersymmetric gauge theory on a complex surface S obtained from the usual $N = 4$ super-Yang-Mills by a topological twist [26]. The partition function of the theory reduces to the generating series of VW invariants and one could expect that it must be a modular form as a consequence of S-duality of the $N = 4$ super-Yang-Mills. However, it turns out that on surfaces with $b_2^+(S) = 1$, which includes $S = \mathbb{P}^2$, there is a modular anomaly [26, 47]. Its precise form can be established from the fact that the VW invariants on S coincide with the D4-D2-D0 BPS indices on the non-compact CY given by the canonical bundle over S [48, 49, 50], which in turn can be obtained from a compact CY given by an elliptic fibration over S in the limit of large fiber. Since the modularity of the D4-D2-D0 BPS indices on such compact CY is governed by a generalization of (2.8) or (2.17) to $b_2 > 1$, the generating series of VW invariants are subject to the same anomaly equation [35].

Furthermore, since in the local limit where the elliptic fiber becomes large the only divisor which remains finite is $[S]$, the D4-brane charges belong to the one-dimensional lattice, and if $b_2(S) = 1$, as is the case for \mathbb{P}^2 , the lattice of D2-brane charges is also one-dimensional. Thus, for $S = \mathbb{P}^2$ one reduces to the “one-dimensional” case captured by the anomaly equation (2.17) with $\kappa = [H]^2 = 1$ where $[H]$ is the hyperplane class of \mathbb{P}^2 . However, the fact that the anomaly equation arises as a limit of a compact CY with $b_2 > 1$ does lead to two modifications: the second term in the spectral flow decomposition (2.14) and the Dirac product of charges (2.10) both get an additional factor of -3 , which can be traced back to the value of the first Chern class $c_1(\mathbb{P}^2) = 3[H]$ [35, Ap.F].⁸ Thus, if one denotes the functions $\tilde{\mathbf{R}}_{\mu,\mu}^{(n)}$ with these two modifications implemented by $\tilde{\mathfrak{R}}_{\mu,\mu}^{(n)}$, then the normalized generating series of VW invariants satisfy

$$\widehat{\mathbf{g}}_{n,\mu} = \sum_{m=1}^n \sum_{\sum_{k=1}^m n_k = n} \sum_{\mu} \tilde{\mathfrak{R}}_{\mu,\mu}^{(n)} \prod_{k=1}^m \mathbf{g}_{n_k, \mu_k}. \quad (4.21)$$

The first modification can actually be undone by a simple shift of the spectral flow parameter. On the other hand, the second one is equivalent to multiplying the vectors \mathbf{v}_{ij} (D.5) by -3 . Under this rescaling of parameters, the functions (D.7) simply get an overall factor $(-3)^{n-1}(-1)^{n-1} = 3^{n-1}$. Thus, one concludes that

$$\tilde{\mathfrak{R}}_{\mu,\mu}^{(n)} = 3^{m-1} \tilde{\mathbf{R}}_{\mu,\mu}^{(n)}, \quad (4.22)$$

where m is the number of charges which the functions depend on. Substituting this into (4.21) and comparing to (4.16), one finds that the two equations become identical provided one identifies⁹

$$g_{n,\mu} = 3^{1-n} \mathbf{g}_{n,\mu}(\tau). \quad (4.23)$$

This result is consistent with (4.19) and provides an explicit solution for the anomalous coefficients with $r_i = \kappa = 1$.

5. Solution via indefinite theta series

5.1 Motivation and strategy

In the previous section we have found solutions for two infinite families of anomalous coefficients: with two arbitrary charges and with arbitrary number of charges, but all set to 1 together with the intersection number. It is natural to try extending these solutions to more general cases. In particular, one could expect that a solution for the case with $r_i = 1$ but κ an arbitrary prime number should be described by the action of Hecke-like operators similar to (4.7) on the normalized generating functions of $SU(n)$ VW invariants on \mathbb{P}^2 . However, we

⁸Strictly speaking, [35] analyzed generating functions of *refined* VW invariants (see §5.2) which count Betti numbers of moduli spaces of instantons. However, the presented results are easily recovered in the unrefined limit.

⁹The freedom to include in this relation a constant factor c^n allowed by the equations is fixed by the normalization conditions $g_1 = \mathbf{g}_1 = 1$.

have not been able to show this and all other attempts to extend the above constructions also failed.¹⁰ Therefore, we change the strategy and suggest an approach which works in general. It is based on the use of indefinite theta series and is similar to the solution of the same kind of modular anomaly equation for the generating functions of VW invariants constructed in [51, 28].

An indefinite theta series is defined as a sum over a lattice Λ endowed with a quadratic form \mathbb{k}^2 of indefinite signature,¹¹

$$\vartheta_{\mu}(\tau) = \sum_{\mathbb{k} \in \Lambda + \mu} \Phi(\mathbb{k}; \tau_2) e^{-\pi i \tau \mathbb{k}^2}, \quad (5.1)$$

where $\mu \in \Lambda^*/\Lambda$ labels its different components. The kernel $\Phi(\mathbb{k})$ can be a non-trivial function of $\tau_2 = \text{Im } \tau$ and must ensure convergence of the sum. In fact, it is very natural to use such theta series to represent solutions of our modular anomaly equations because for $b_2 > 1$ the functions analogous to $R_{\mu, \mu}^{(r)}$ have precisely the form (5.1). This is also true for $b_2 = 1$ (cf. (2.9) or (2.18)), but in this case the relevant quadratic form coincides with $-Q_n$ (2.11) and has a definite signature. But since it is *positive* definite, this case also calls for the use of indefinite theta series.

The anomalous coefficients we are looking for, and hence the indefinite theta series representing them, must be holomorphic in τ . The only way to make (5.1) holomorphic and convergent simultaneously is to restrict the sum to the negative definite cone of the lattice, which can be done by choosing the kernel $\Phi(\mathbb{k})$ to be an appropriate combination of sign functions. An example of such kernel is provided by Theorem C.1 and is characterized by two sets of vectors $\{\mathbf{v}_{s,i}\}$, $s = 1, 2$. As we will see below, one set is determined by the same vectors \mathbf{v}_e (D.4) that define the functions $R_{\mu, \mu}^{(r)}$, while the second set must consist of *null* vectors, i.e. satisfying $\mathbf{v}_{2,i}^2 = 0$. This immediately implies that the lattice Λ cannot be the one that appears in the definition of $R_{\mu, \mu}^{(r)}$ (2.9) and will be denoted below by $\Lambda^{(r)}$, since the numbers of positive and negative eigenvalues of the quadratic form must be both non-vanishing for null vectors to exist.

This can be achieved by the so-called *lattice extension*, which is a standard trick in the theory of mock modular forms [52]. The idea is that the original problem defined on a lattice Λ is reformulated on a larger lattice $\Lambda_{\text{ext}} = \Lambda \oplus \Lambda_{\text{ad}}$ that admits a solution in terms of indefinite theta series and, because Λ_{ext} is a direct sum, such solution is expected to be reducible to a solution on Λ . However, if the discriminant group $D_{\text{ad}} = \Lambda_{\text{ad}}^*/\Lambda_{\text{ad}}$ is non-trivial, the reduction to the original lattice is possible only if the solution on the extended lattice satisfies certain identities ensuring that components of the solution labelled by different elements of D_{ad} reduce

¹⁰At technical level, there are two main complications appearing for $n > 2$. First, $g_{\mu, \mu}^{(r)}$ do not reduce to a vector-like object and keep a non-trivial tensor structure (cf. (4.2)). Second, the action of Hecke-like operators on a product of functions is not factorized, so that applying them to the r.h.s. of (3.3), for terms with $1 < m < n$, one cannot proceed in an iterative way.

¹¹Note the unusual minus sign in the exponential. The same minus sign appears also in (C.1). This convention follows the conventions used in the previous works on this topic and can be traced back to the natural quadratic form induced on D4 and D2-brane charges on a compact CY. In this convention, the usual convergent theta series with a trivial kernel correspond to *negative* definite quadratic forms.

to the same functions. For example, this is the case for the generating functions of VW invariants where the invariants on \mathbb{P}^2 can be obtained from those on the Hirzebruch surface \mathbb{F}_1 because the latter satisfy the so-called blow-up identities [53, 28]. However, for a general solution on Λ_{ext} this is not the case. Therefore, we should require triviality of D_{ad} , which in turn requires that, if $\Lambda_{\text{ad}} = \mathbb{Z}^{d_{\text{ad}}}$, then the corresponding quadratic form is given by (minus) the identity matrix.

In our case $\Lambda = \Lambda^{(r)}$ with quadratic form $-Q_n$ and Λ_{ad} should be chosen so that to ensure the existence of a null vector on $\Lambda = \Lambda_{\text{ext}}$. One could think that it is sufficient to take $\Lambda_{\text{ad}} = \mathbb{Z}$ with the quadratic form $-x^2$. But actually it is not because, for a theta series to converge, the null vector appearing in its kernel (possibly after rescaling) should *belong to the lattice*. Otherwise, the indefinite theta series would diverge due to accumulation of lattice points near the null cone leading to an infinite number of terms of the same strength (see [54, §B.3] for an illustrative example). Thus, typically, the dimension of Λ_{ad} must be non-trivial. The simplest possibility would be to take $d_{\text{ad}} = \mathbf{v}^2$ where $\mathbf{v} \in \Lambda$ is a vector with the minimal norm. This would ensure that $(\mathbf{v}, \vec{1}) \in \Lambda_{\text{ext}}$, where the vector $\vec{1}$ has d_{ad} components all equal to 1, is a null vector. However, this is not always the optimal choice and in our case it is actually inconsistent with the iterative structure of the equations (3.3). Below in §5.3 we propose a lattice extension satisfying all the requirements discussed above and adapted to our system of equations.

But this is not the end of the story. The problem is that even if the null vector belongs to the lattice, this does not ensure the convergence yet. The additional divergence comes from the sum over the sublattice $\mathbb{Z}\mathbf{v} \subset \Lambda$, where $\mathbf{v} \in \Lambda$ is the null vector. This is easy to see for (5.1) with quadratic form of signature $(1, n_-)$ and the kernel $\Phi(\mathbf{k}) = \text{sgn}(\mathbf{v} * \mathbf{k}) - \text{sgn}(\mathbf{v}' * \mathbf{k})$. A way out is to consider *Jacobi forms* instead of the usual modular forms. They depend on an additional elliptic parameter z transforming under $SL(2, \mathbb{Z})$ as $z \mapsto z/(c\tau + d)$. For theta series, the elliptic transformation property of Jacobi forms fixes the dependence on z as shown in (C.1) (with $z = \theta z$). In particular, it shifts the lattice vector \mathbf{k} in the kernel and introduces an exponential z -dependent factor. Together these two changes allow to avoid the divergence due to the null vector, which manifests now as a pole at $z = 0$. Since eventually we are interested in the limit $z = 0$, these poles should be cancelled by combining the indefinite theta series with certain *Jacobi-like modular forms* (see §A for the definition of Jacobi and Jacobi-like forms). The latter have the same modular transformations as Jacobi forms, but they are not required to satisfy the elliptic property, which in our context is irrelevant since we care only about the behavior near $z = 0$. Note that, apart from relaxing the elliptic property, exactly the same strategy to combine indefinite theta series constructed from null vectors with Jacobi forms cancelling poles has been used in [51] to produce the generating functions of VW invariants on Hirzebruch and del Pezzo surfaces as solutions of a modular anomaly equation similar to (4.21).

The extension to Jacobi forms is known as the so-called *refinement* which has a physical interpretation as switching on an Ω -background [55, 56] and has been investigated in the context of modularity of BPS indices in [35]. A quite unexpected result of that analysis is that the refinement considerably simplifies the coefficients \mathcal{R}_n determining the modular anomaly. Thus, the necessity to introduce the refinement should be considered not as a shortcoming,

but as a virtue which makes the system of anomaly equations more amenable to solution.

However, new complications arise when the refinement is combined with the lattice extension discussed above. It turns out that for a solution on Λ_{ext} to be reducible to a solution on Λ , it should have zero of order d_{ad} at $z = 0$, which is very difficult to achieve. Fortunately, there is a trick that allows to avoid this problem: one should introduce multiple refinement parameters combined in a vector \mathbb{z} so that the indefinite theta series become multi-variable (mock) Jacobi forms as (C.1). Then, as will be shown below, if one sets $\mathbb{z} = (z\boldsymbol{\theta}, \vec{z})$ where $\boldsymbol{\theta} \in \Lambda$ and \vec{z} has d_{ad} components and is such that $(0, \vec{z})$ is orthogonal to all null vectors, the lattice Λ_{ad} together with the associated refinement parameters \vec{z} decouples and the reduction to Λ crucially simplifies.

To summarize, we need to perform the following steps:

1. introduce refinement,
2. extend the charge lattice so that it possesses a set of null vectors and is consistent with the anomaly equation,
3. associate with the extension a vector of additional refinement parameters satisfying certain orthogonality properties with the null vectors,
4. solve the refined system of anomaly equations on the extended lattice,
5. reduce the solution to the original lattice,
6. take the unrefined limit.

In the next subsection we perform the first step. Steps 2 and 3 are done in §5.3. The last 3 steps are realized in §5.4 in the case of two charges and in §5.5 in the case of three charges. Finally, in §5.6 we consider the generic case for which we perform steps 4 and 5 explicitly, whereas the last step is too cumbersome to be done analytically.

5.2 Refinement

As was mentioned in the previous subsection, a refinement has its physical origin in a non-trivial Ω -background. It introduces a complex parameter $y = e^{2\pi iz}$ which can be thought of as a fugacity conjugate to the angular momentum J_3 in uncompactified dimensions. At the same time, the BPS indices, which from the mathematical point of view (roughly) count the Euler number of the moduli spaces of semi-stable coherent sheaves, are replaced by *refined* BPS indices which are symmetric Laurent polynomials in y constructed from the Betti numbers of these moduli spaces. These refined indices are known to satisfy similar and even simpler wall-crossing relations as the usual ones [57, 58, 59]. But most importantly is that the refinement preserves the modular properties of the generating series of BPS indices [35]. More precisely, after refinement they become *mock Jacobi forms* for which the role of the elliptic argument is played by the refinement parameter z and the formula for their modular completions takes

exactly the same form as in (2.8), but with the coefficients given now by¹²

$$R_{\mu,\mu}^{(r)\text{ref}}(\tau, \bar{\tau}, z) = \sum_{\substack{\sum_{i=1}^n q_i = \mu + \kappa\tau/2 \\ q_i \in \kappa\tau_i\mathbb{Z} + \mu_i + \kappa\tau_i/2}} \text{Sym} \left\{ \mathcal{R}_n^{\text{ref}}(\hat{\gamma}; \tau_2, \beta) y^{\sum_{i<j} \gamma_{ij}} \right\} e^{\pi i \tau Q_n(\hat{\gamma})}, \quad (5.2)$$

where we set $z = \alpha - \tau\beta$ with $\alpha, \beta \in \mathbb{R}$. The main difference here, besides the appearance of a power of y , lies in the form of the coefficients $\mathcal{R}_n^{\text{ref}}$ which we describe in appendix D.3. They turn out to be much simpler than their unrefined version \mathcal{R}_n .¹³ In particular, while the coefficients \mathcal{R}_n involve a sum over two types of trees weighted by generalized error functions and their derivatives, for $\mathcal{R}_n^{\text{ref}}$ one needs only one type of trees and no derivatives.

It should be stressed that the status of the refined BPS indices for compact CY threefolds, the case we are really interested in, is unclear. While in the non-compact case they are well-defined due to a certain \mathbb{C}^\times action carried by the moduli space of semi-stable objects, in its absence it seems impossible to refine DT invariants in a deformation-invariant way (see however [57]). This is not a problem for our construction because we do *not* use the refined BPS indices or their generating functions, but only the coefficients (5.2) characterizing the refined completions. In other words, we use the existence and properties of $R_{\mu,\mu}^{(r)\text{ref}}$ as a mere trick to produce solutions to the anomaly equations (3.3).

In particular, the main property which we need is that in the unrefined limit $R_{\mu,\mu}^{(r)\text{ref}}$ develop a zero of order $n - 1$ with a coefficient given by $\tilde{R}_{\mu,\mu}^{(r)}$:

$$\tilde{R}_{\mu,\mu}^{(r)}(\tau, \bar{\tau}) = \lim_{y \rightarrow 1} (y - y^{-1})^{1-n} R_{\mu,\mu}^{(r)\text{ref}}(\tau, \bar{\tau}, z). \quad (5.3)$$

Therefore, if we define *refined anomalous coefficients* as solutions of the following modular anomaly equation

$$\hat{g}_{\mu,\mu}^{(r)\text{ref}} = \text{Sym} \left\{ \sum_{m=1}^n \sum_{\sum_{k=1}^m n_k = n} \sum_{\nu} R_{\mu,\nu}^{(s)\text{ref}} \prod_{k=1}^m g_{\nu_k, m_k}^{(r_k)\text{ref}} \right\}, \quad (5.4)$$

where $\hat{g}_{\mu,\mu}^{(r)\text{ref}}$ is required to be a VV Jacobi-like form of weight $\frac{1}{2}(n - 1)$, index¹⁴

$$m_{\mathbf{r}} = -\frac{\kappa}{6} \left(r^3 - \sum_{i=1}^n r_i^3 \right), \quad (5.5)$$

and the same multiplier system as $g_{\mu,\mu}^{(r)}$ (see (B.4)), then a solution of (3.3) is obtained from these refined anomalous coefficients as

$$g_{\mu,\mu}^{(r)}(\tau) = \lim_{y \rightarrow 1} (y - y^{-1})^{1-n} g_{\mu,\mu}^{(r)\text{ref}}(\tau, z). \quad (5.6)$$

¹²We give the coefficients *after* performing the same redefinition as in (2.12), so that the formula to compare with is (2.18) rather than (2.9), but we omit the tilde on $R_{\mu,\mu}^{(r)\text{ref}}$ to avoid cluttering.

¹³More precisely, while the formula (D.14) looks exactly as (D.13), these are the functions $\mathcal{E}_n^{(\text{ref})}$ that are much simpler than their unrefined analogues \mathcal{E}_n ((D.6) and (D.7) versus (D.15)).

¹⁴The weight is obtained from the relation (5.6) by taking into account that the y -dependent factor in the limit $y \rightarrow 1$ is proportional to z^{1-n} and thus increases the weight by $n - 1$. The index instead follows from the index of the generating series of refined BPS indices which should be equal to $-\chi_{\mathbf{r}}$ (2.7), as was established in [35].

This is easily checked by multiplying (5.4) by $(y - y^{-1})^{1-n}$ and taking the unrefined limit. As a result, we have reformulated the problem of solving one anomaly equation in terms of solving another equation and subsequent evaluation of the unrefined limit. Importantly, the relation (5.6) implies that the unrefined limit exists only if the refined solution has a zero of order $n - 1$ at $z = 0$. Although it might be non-trivial to ensure this property, for generic set of charges this reformulation makes the problem more feasible.

Finally, we note that the refined anomalous coefficients can be chosen to satisfy a property similar to (3.8), namely,

$$g_{\mu,\mu}^{(\kappa;r)\text{ref}}(\tau, z) = g_{\mu,\mu}^{(1;\kappa r)\text{ref}}(\tau, z/\kappa). \quad (5.7)$$

5.3 Lattice extension

The next step is to reformulate the anomaly equation (5.4) for the refined anomalous coefficients in a way that involves an extended lattice possessing a set of null vectors. To this end, let us introduce:

- integer valued function d_r of the magnetic charge (and intersection number κ) such that $d_r \geq 2$;
- d_r -dimensional vectors $\mathbf{t}^{(r)}$ such that their components are all non-vanishing integers and sum to zero, $\sum_{\alpha=1}^{d_r} \mathbf{t}_{\alpha}^{(r)} = 0$.

Note that if d_r could be equal to 1, it would be impossible to satisfy the last condition on $\mathbf{t}^{(r)}$. The main features of the construction below do not depend on a specific form of d_r and $\mathbf{t}^{(r)}$, and we return to their choice, which is important for the concrete form of the solution, in the end of the subsection.

Let us now consider the anomaly equation

$$\widehat{g}_{\mu,\mu}^{(r)\text{ref}}(\tau, z, \mathbf{z}) = \text{Sym} \left\{ \sum_{m=1}^n \sum_{\sum_{k=1}^m n_k = n} \sum_{\nu} R_{\mu,\nu}^{(s)\text{ref}}(\tau, z) \prod_{k=1}^m \check{g}_{\nu_k, \mathbf{m}_k}^{(\mathbf{t}_k)\text{ref}}(\tau, z, \mathbf{z}_k) \right\}. \quad (5.8)$$

where $\mathbf{z}_k = (z_{j_k+1}, \dots, z_{j_{k+1}})$. Formally it looks the same as (5.4). However, there are two differences. First, the new functions $\check{g}_{\mu,\mu}^{(r)\text{ref}}$ and their completions $\widehat{g}_{\mu,\mu}^{(r)\text{ref}}$ depend on a vector of additional refinement parameters $\mathbf{z} = (z_1, \dots, z_n)$. Second, we change the normalization for the case $n = 1$ which now reads

$$\check{g}_{\mu,\mu'}^{(r)\text{ref}}(\tau, z, z') = \delta_{\mu,\mu'} \prod_{\alpha=1}^{d_r} \theta_1(\tau, \mathbf{t}_{\alpha}^{(r)} z'), \quad (5.9)$$

where $\theta_1(\tau, z)$ is the standard Jacobi theta function (C.9). The additional factor in (5.9) leads to a change in the modular properties of $\check{g}_{\mu,\mu}^{(r)\text{ref}}$ compared to $g_{\mu,\mu}^{(r)\text{ref}}$: they should be higher depth *multi-variable* Jacobi-like forms of the weight, index (which is now a matrix since there are several elliptic arguments) and multiplier system specified in (B.5), which can be easily obtained by combining (B.4) with the modular properties of the Jacobi theta function given in (C.10).

The important property of the system of equations (5.8) is that any solution that is regular at $z = 0$ gives rise to a solution of (5.4) with the required modular properties. The relation between the two solutions is given by

$$g_{\mu,\mu}^{(r)\text{ref}}(\tau, z) = \frac{1}{(-2\pi\eta^3(\tau))^{d_r}} \left(\prod_{i=1}^n \frac{\mathcal{D}^{(d_{r_i})}}{\frac{1}{2}(\mathfrak{t}^{(r_i)})^2(z_i)} \right) \check{g}_{\mu,\mu}^{(r)\text{ref}}|_{z=0}, \quad (5.10)$$

where $d_r = \sum_{i=1}^n d_{r_i}$ and the differential operators $\mathcal{D}_m^{(n)}$ are defined in (A.5). Indeed, due to Proposition A.2 and the fact that $\theta_1(\tau, z)$ and $\eta^3(\tau)$ have identical multiplier systems, the product of the differential operators in (5.10) acting on the completion $\widehat{g}_{\mu,\mu}^{(r)\text{ref}}$ produces a Jacobi-like form with weight, index and multiplier system as in (B.4). Then to see that $g_{\mu,\mu}^{(r)\text{ref}}$ defined by (5.10) satisfies the anomaly equation (5.4), it is sufficient to apply this product of the differential operators to (5.8) and use the fact that each differential operator acts only on one of the functions $\check{g}^{(\mathfrak{t}_k)\text{ref}}$ on the r.h.s. of this equation.¹⁵ Finally, the standard normalization for the case $n = 1$ is reproduced due to the property

$$\frac{\mathcal{D}^{(d_r)}}{\frac{1}{2}(\mathfrak{t}^{(r)})^2} \prod_{\alpha=1}^{d_r} \theta_1(\tau, \mathfrak{t}_\alpha^{(r)} z)|_{z=0} = (\partial_z \theta_1(\tau, 0))^{d_r} = (-2\pi\eta^3(\tau))^{d_r}. \quad (5.11)$$

The main advantage of the new system of equations (5.8) compared to (5.4) is that it corresponds to a lattice extension of the latter. To see how it comes about, first note that the lattice which one sums over in the definition of $R_{\mu,\mu}^{(r)\text{ref}}$ (5.2) can be defined as (see appendix F.1 for details)

$$\Lambda^{(r)} = \left\{ \mathbf{k} \in \mathbb{Z}^n : \sum_{i=1}^n r_i k_i = 0 \right\} \quad (5.12)$$

and carries the bilinear form

$$\mathbf{x} \cdot \mathbf{y} = \kappa \sum_{i=1}^n r_i x_i y_i. \quad (5.13)$$

The new normalization (5.9) then effectively gives rise to an additional sum over the lattice $\mathbb{Z}^{d_{r_i}}$ with quadratic form $-\text{diag}(1, \dots, 1)$ associated to each magnetic charge r_i . This is especially easy to see for the term in (5.8) with $m = n$ which contains the product of n functions like (5.9). As a result, the overall effect is that the lattice $\Lambda^{(r)}$ is extended to

$$\mathbb{A}^{(r)} = \Lambda^{(r)} \oplus \mathbb{Z}^{d_r} \quad (5.14)$$

and the extended lattice carries the bilinear form

$$\mathbb{x} * \mathbb{y} = \sum_{i=1}^n \left(\kappa r_i x_i y_i - \sum_{\alpha=1}^{d_{r_i}} x_{i,\alpha} y_{i,\alpha} \right), \quad (5.15)$$

where $\mathbb{x} = \{x_i, x_{i,\alpha}\}$ with $i = 1, \dots, n$ and $\alpha = 1, \dots, d_{r_i}$.

¹⁵It was to ensure this factorization property that was the main reason for introducing the additional refinement parameters z_i for each magnetic charge.

To discuss null vectors on $\Lambda^{(r)}$, one should first specify the function d_r . To motivate its choice, let us consider the case of two charges. It is easy to see that

$$\Lambda^{(r_1, r_2)} = \{(n\hat{r}_2, -n\hat{r}_1), n \in \mathbb{Z}\}, \quad (5.16)$$

where $\hat{r}_i = r_i / \gcd(r_1, r_2)$, and hence is identical to \mathbb{Z} with quadratic form $\kappa r \hat{r}_1 \hat{r}_2 x^2$. Therefore, the norm of the vector $(1; a_1, \dots, a_1; a_2, \dots, a_2) \in \mathbb{Z} \oplus \mathbb{Z}^{d_{r_1}} \oplus \mathbb{Z}^{d_{r_2}} \simeq \Lambda^{(r_1, r_2)}$ is equal to $\kappa(r_1 \hat{r}_2^2 + r_2 \hat{r}_1^2) - d_{r_1} a_1^2 - d_{r_2} a_2^2$. Thus, the most natural choice is to take $d_r = \kappa r$ which ensures that the above vector is null for $a_1 = \pm \hat{r}_2$ and $a_2 = \pm \hat{r}_1$. However, this choice fails to satisfy the condition $d_r \geq 2$ for $r = \kappa = 1$. This introduces a complication that this particular case should be treated differently. There are two natural ways to do this by setting

$$a) d_r = \begin{cases} 4r, & \kappa = 1, \\ \kappa r, & \kappa > 1, \end{cases} \quad b) d_r = \begin{cases} 4, & \kappa = r = 1, \\ \kappa r, & \kappa r > 1. \end{cases} \quad (5.17)$$

The advantage of the second choice is that it preserves the property (5.7) and allows to work with lattices of smaller dimensions. On the other hand, it is more involved at the computational level. Therefore, in the following we proceed with the first choice (despite it spoils the property (5.7)).¹⁶

For the vectors $\mathfrak{t}^{(r)}$ there are plenty of possible choices. The following two seem to be the most ‘‘canonical’’:

$$a) \mathfrak{t}_\alpha^{(r)} = \begin{cases} 1, & \alpha \leq \lceil d_r/2 \rceil, \\ -1, & \lceil d_r/2 \rceil < \alpha \leq 2 \lfloor d_r/2 \rfloor, \\ -2, & \alpha = d_r \text{ if } d_r \text{ is odd,} \end{cases} \quad b) \mathfrak{t}_\alpha^{(r)} = \begin{cases} 1, & \alpha < d_r, \\ 1 - d_r, & \alpha = d_r. \end{cases} \quad (5.18)$$

In our calculations we will mostly use the first choice.

In the following we will use two sets of vectors belonging to the extended lattice $\Lambda^{(r)}$. Both of them are extensions of the vectors $\mathbf{v}_{ij} \in \Lambda^{(r)}$ defined as in (D.5)

$$(\mathbf{v}_{ij})_k = \delta_{ki} r_j - \delta_{kj} r_i \quad (5.19)$$

and are given by

$$\begin{aligned} (\mathbb{V}_{ij})_k &= (\mathbf{v}_{ij})_k, & (\mathbb{V}_{ij})_{k,\alpha} &= 0, \\ (\mathbb{W}_{ij})_k &= 2^\epsilon (\mathbf{v}_{ij})_k, & (\mathbb{W}_{ij})_{k,\alpha} &= (\mathbf{v}_{ij})_k, \end{aligned} \quad (5.20)$$

where $\epsilon = \delta_{\kappa-1}$. Here the factor of 2^ϵ compensates the factor of 4 appearing in (5.17) for $\kappa = 1$ and ensures that $\mathbb{w}_{ij}^2 = 0$. We will also extensively use their normalized versions

$$\hat{\mathbb{V}}_{ij} = \mathbb{V}_{ij}/r_{ij}, \quad \hat{\mathbb{W}}_{ij} = \mathbb{W}_{ij}/r_{ij}, \quad (5.21)$$

¹⁶Another possibility would be to restrict to the case $\kappa = 1$ and use the property (5.7), or its unrefined analogue (3.8), to obtain other cases. We prefer to proceed with generic κ because, as we will see, due to the additional factor of 4 in the definition of d_r , the case $\kappa = 1$ appears to be more complicated than $\kappa > 1$. Besides, it leads to a larger extended lattice which decreases efficiency of numerical computations.

where $r_{ij} = \gcd(r_i, r_j)$. Their scalar products with respect to the bilinear form (5.15) are found to be

$$\begin{aligned} \mathbb{V}_{ij}^2 &= 2^{-\epsilon} \mathbb{V}_{ij} * \mathbb{W}_{ij} = \kappa r_i r_j (r_i + r_j), & \mathbb{V}_{ij} * \mathbb{V}_{jk} &= 2^{-\epsilon} \mathbb{V}_{ij} * \mathbb{W}_{jk} = -\kappa r_i r_j r_k, \\ \mathbb{V}_{i+j,k} * \mathbb{V}_{ij} &= \mathbb{V}_{i+j,k} * \mathbb{W}_{ij} = \mathbb{W}_{ij} * \mathbb{W}_{kl} = 0, & \text{for } \forall i, j, k, l, \\ \mathbb{V}_{ij} * \mathbb{V}_{kl} &= 0, & \text{if } \{i, j\} \cap \{k, l\} = \emptyset, \end{aligned} \quad (5.22)$$

where $\mathbb{V}_{i+j,k} = \mathbb{V}_{ik} + \mathbb{V}_{jk}$ is an extension of $\mathbf{v}_{i+j,k} = \mathbf{v}_{ik} + \mathbf{v}_{jk}$. We will also often use the notations

$$\kappa_i = \frac{\kappa r r_i r_{i+1}}{r_0 r_{i,i+1}}, \quad \kappa_{ij} = \frac{\kappa r_i r_j}{2r_{ij}^2} (r_i + r_j), \quad \kappa_{ijk} = \frac{\kappa r_i r_j r_k}{r_{ij} r_{jk}}, \quad (5.23)$$

where $r_0 = \gcd(r_1, \dots, r_n)$, encoding various scalar products of the normalized vectors. Finally, it is useful to note the property $r_k \mathbf{v}_{ij} + r_i \mathbf{v}_{jk} + r_j \mathbf{v}_{ki} = 0$ which implies

$$\begin{aligned} r_k \mathbb{V}_{ij} + r_i \mathbb{V}_{jk} + r_j \mathbb{V}_{ki} &= 0, \\ r_k \mathbb{W}_{ij} + r_i \mathbb{W}_{jk} + r_j \mathbb{W}_{ki} &= 0. \end{aligned} \quad (5.24)$$

Below we will see how the existence of the null vectors \mathbb{w}_{ij} gives the possibility to construct holomorphic theta series associated with the extended lattice and satisfying the anomaly equation (5.8).

5.3.1 Lattice factorization

Before we proceed with solving the extended anomaly equation (5.8), let us perform an important technical step which will be crucial for determining a solution that has a well-defined unrefined limit. Namely, let us decompose the extended lattice $\mathbb{A}^{(r)}$ into two orthogonal sublattices which we denote by $\mathbb{A}_{\parallel}^{(r)}$ and $\mathbb{A}_{\perp}^{(r)}$. The former is taken to be the span of the vectors $\hat{\mathbf{w}}_{ij}$ and $\hat{\mathbf{w}}_{ij}$ introduced in (5.21), i.e. all their linear combinations with integer coefficients. It is clear that it is a direct sum of two lattices

$$\mathbb{A}_{\parallel}^{(r)} = \mathbf{\Lambda}^{(r)} \oplus \tilde{\mathbf{\Lambda}}^{(r)}, \quad (5.25)$$

where $\mathbf{\Lambda}^{(r)}$ is generated by $\hat{\mathbf{w}}_{ij}$ and is the same as (5.12), while $\tilde{\mathbf{\Lambda}}^{(r)}$ is the span of $\hat{\mathbf{u}}_{ij} = \hat{\mathbf{w}}_{ij} - 2^\epsilon \hat{\mathbf{w}}_{ij}$ and embedded into \mathbb{Z}^{d_r} .¹⁷ The embedding is given by

$$(k_1, \dots, k_n) \mapsto (k_1^{[d_{r_1}]}, \dots, k_n^{[d_{r_n}]}), \quad (5.26)$$

where $x^{[n]}$ denotes the n -dimensional vector with all components equal to x , and the resulting lattice is actually isomorphic to $\mathbf{\Lambda}^{(r)}$ with quadratic form rescaled by -4^ϵ . The lattice $\mathbb{A}_{\perp}^{(r)}$ is taken to be generated by the vectors $\mathbf{e}_0, \mathbf{e}_{i,\alpha}$, with $i = 1, \dots, n$ and $\alpha = 1, \dots, d_{r_i} - 1$, given by

$$\begin{aligned} (\mathbf{e}_0)_k &= 0, & (\mathbf{e}_0)_{k,\beta} &= 1, \\ (\mathbf{e}_{i,\alpha})_k &= 0, & (\mathbf{e}_{i,\alpha})_{k,\beta} &= \delta_{ik} (\delta_{\alpha+1,\beta} - \delta_{\alpha\beta}). \end{aligned} \quad (5.27)$$

¹⁷Note that for our choice $d_r = 4^\epsilon \kappa r$, one has $d_r = d_r$ where $r = \sum_i r_i$, which is not generally true for choice b) in (5.17). This is one of several complications of the second choice.

Using the bilinear form (5.15), it is easy to check that these vectors are indeed orthogonal to \hat{w}_{ij} and \hat{w}_{ij} . Moreover, each of the sets $\{e_{i,\alpha}\}_{\alpha=1}^{d_{r_i}}$ generates a lattice isomorphic to the A_{N-1} root lattice with $N = d_{r_i}$, and all of them are mutually orthogonal as well as to the vector e_0 . Therefore, we have in addition the following orthogonal decomposition

$$\mathbb{A}_{\perp}^{(r)} = \mathbb{Z} \oplus \mathbb{A}_{d_{r_1}-1} \oplus \cdots \oplus \mathbb{A}_{d_{r_n}-1}. \quad (5.28)$$

In contrast, the full lattice $\mathbb{A}^{(r)}$ is *not* a direct sum of $\mathbb{A}_{\parallel}^{(r)}$ and $\mathbb{A}_{\perp}^{(r)}$ because some of its elements require rational coefficients being decomposed in the basis of the two sublattices. In such situation, to get the full lattice from the sublattices, one has to introduce the so called *glue vectors*.

According to the general theory [60], if $\oplus_{a=1}^n \mathbb{A}^{(a)}$ is a sublattice of \mathbb{A} of the same dimension, the corresponding glue vectors are given by the sum of representatives of the discriminant groups $\mathbb{D}^{(a)} = (\mathbb{A}^{(a)})^*/\mathbb{A}^{(a)}$ which at the same time belongs to the original lattice, i.e. $\mathfrak{g}_{\mathbb{A}} = \oplus_{a=1}^n \mathfrak{g}_{\mathbb{A}}^{(a)} \in \mathbb{A}$ where $\mathfrak{g}_{\mathbb{A}}^{(a)} \in \mathbb{D}^{(a)}$. The number of glue vectors is equal to

$$N_g = \left| \frac{\prod_{a=1}^n \det \mathbb{A}^{(a)}}{\det \mathbb{A}} \right|^{1/2}, \quad (5.29)$$

where $\det \mathbb{A} = |\mathbb{A}^*/\mathbb{A}|$ is the order of the discriminant group and is equal to the determinant of the matrix of scalar products of the basis elements. The decomposition formula of the lattice \mathbb{A} then reads

$$\mathbb{A} = \bigcup_{\mathbb{A}=0}^{N_g-1} \left[\bigoplus_{a=1}^n \left(\mathbb{A}^{(a)} + \mathfrak{g}_{\mathbb{A}}^{(a)} \right) \right]. \quad (5.30)$$

In our case it takes the form

$$\mathbb{A}^{(r)} = \bigcup_{\mathbb{A}=0}^{N_g-1} \left[\left(\mathbb{A}_{\parallel}^{(r)} + \mathfrak{g}_{\mathbb{A}}^{\parallel} \right) \oplus \left(\mathbb{A}_{\perp}^{(r)} + \mathfrak{g}_{\mathbb{A}}^{\perp} \right) \right]. \quad (5.31)$$

One finds the following lattice determinants

$$\begin{aligned} \det \mathbb{A}^{(r)} &= (-1)^{d_r} \det \mathbf{\Lambda}^{(r)}, \\ \det \mathbb{A}_{\parallel}^{(r)} &= (-4^\epsilon)^{n-1} (\det \mathbf{\Lambda}^{(r)})^2, \\ \det \mathbb{A}_{\perp}^{(r)} &= (-1)^{d_r+1} d_r \prod_{i=1}^n d_{r_i}, \end{aligned} \quad (5.32)$$

where $\det \mathbf{\Lambda}^{(r)}$ is evaluated in (F.6). Substituting this into (5.29), one finds that the number of glue vectors is given by

$$N_g = \frac{r}{r_0} \prod_{i=1}^n d_{r_i}. \quad (5.33)$$

There is a natural choice of glue vectors for the decomposition (5.31). Let us fix a n -tuple $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n)$ such that $\sum_{i=1}^n \rho_i r_i = r_0$. Then we represent glue vectors as a sum of several terms

$$\mathfrak{g}_{\mathbb{A}} = \mathfrak{a}_0 \mathfrak{g}_0 + \sum_{i=1}^n \mathfrak{g}_{i,\mathfrak{a}_i}, \quad (5.34)$$

where

$$(\mathfrak{g}_{i,\mathbf{a}})_k = 0, \quad (\mathfrak{g}_{i,\mathbf{a}})_{k,\alpha} = \delta_{ik} \sum_{\beta=1}^{\mathbf{a}} \delta_{\alpha\beta}, \quad \mathfrak{g}_0 = \sum_{i=1}^n \rho_i \mathfrak{g}_i, \quad \mathfrak{g}_i \equiv \mathfrak{g}_{i,d_{r_i}}. \quad (5.35)$$

Thus, a glue vector is labelled by the set $\mathbf{A} = \{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n\}$ and the indices take values in the following ranges: $\mathbf{a}_0 = 0, \dots, r/r_0 - 1$ and $\mathbf{a}_i = 0, \dots, d_{r_i} - 1$. It is trivial to see that the cardinality of the resulting set agrees with the required number (5.33).¹⁸

The main application of the lattice decomposition (5.30) is a factorization of theta series. Let us consider a general indefinite theta series as in (C.1) with a kernel having a factorized form $\Phi(\mathbb{x}) = \prod_{a=1}^n \Phi_a(\mathbb{x}^{(a)})$ where the upper index (a) on a vector denotes its projection to $\mathbb{A}^{(a)}$. Then the lattice factorization formula (5.30) implies that one can split the sum in the definition of the theta series into n sums coupled by the additional sum over the glue vectors so that one arrives at the following identity for theta series

$$\vartheta_{\mathbb{P}}(\tau, \mathbb{z}; \mathbb{A}, \Phi, \mathbb{P}) = \sum_{\mathbf{A}=0}^{N_g-1} \prod_{a=1}^n \vartheta_{\mathbb{P}^{(a)} + \mathfrak{g}_{\mathbf{A}}^{(a)}}(\tau, \mathbb{z}^{(a)}; \mathbb{A}^{(a)}, \Phi_a, \mathbb{P}^{(a)}). \quad (5.36)$$

In this paper we are interested in theta series associated with the extended lattice $\mathbb{A}^{(r)}$ and with other ingredients given by

$$\begin{aligned} \mathbb{P} &= (\hat{\boldsymbol{\mu}}; 0, \dots, 0), & \hat{\mu}_i &= \frac{\mu_i}{\kappa r_i} - \frac{\mu}{\kappa r} + \frac{\rho_i \Delta \mu}{\kappa r_0}, \\ \mathbb{Z} &= (\boldsymbol{\theta} \mathbb{z}; -\mathbf{t}^{(r_1)} z_1; \dots; -\mathbf{t}^{(r_n)} z_n), & \boldsymbol{\theta} &= \sum_{i < j} \mathbf{v}_{ij}, \\ \mathbb{P} &= (\mathbf{0}; -1, \dots, -1) = -\mathbb{E}_0, \end{aligned} \quad (5.37)$$

where $\Delta \mu = \mu - \sum_{i=1}^n \mu_i$ as in (2.15). Note that one has the relations

$$\boldsymbol{\theta}^2 = -2m_r, \quad \boldsymbol{\theta} \cdot \mathbf{q} = \sum_{i < j} \gamma_{ij}, \quad (5.38)$$

where $\mathbf{q} = \left(\frac{q_1}{\kappa r_1}, \dots, \frac{q_n}{\kappa r_n}\right)$ in terms of the physical charges. They ensure that the factor $e^{2\pi i \mathbb{z} \cdot \mathbf{k}}$ in the theta series reproduces the y -dependent factor in (5.2) and gives rise to the index (5.5).

¹⁸The geometric origin of these glue vectors is as follows. First, let us combine $\tilde{\mathbb{A}}^{(r)}$ with the one-dimensional lattice generated by \mathbb{E}_0 using the glue vectors $\mathbf{a}_0 \mathfrak{g}_0$. It is easy to see that the result is the lattice \mathbb{Z}^n with the same embedding into \mathbb{Z}^{d_r} as in (5.26), i.e.

$$\bigcup_{\mathbf{a}_0=0}^{r/r_0-1} \bigcup_{\ell \in \mathbb{Z}} \left(\tilde{\mathbb{A}}^{(r)} + \ell \mathbb{E}_0 + \mathbf{a}_0 \mathfrak{g}_0 \right) = \mathbb{Z}^n \subset \mathbb{Z}^{d_r}.$$

Next, one combines the i th factor \mathbb{Z} with $\mathbb{A}_{d_{r_i}-1}$ generated by $\mathbb{E}_{i,\alpha}$ using the glue vectors $\mathfrak{g}_{i,\mathbf{a}_i}$, which gives

$$\bigcup_{\mathbf{a}_i=0}^{d_{r_i}-1} \left(\mathbb{A}_{d_{r_i}-1} + \mathbb{Z} + \mathfrak{g}_{i,\mathbf{a}_i} \right) = \mathbb{Z}^{d_{r_i}}.$$

Summing over i , one obtains \mathbb{Z}^{d_r} which together with $\mathbb{A}^{(r)}$ produces the full extended lattice $\mathbb{A}^{(r)}$.

Let us also mention here another useful relation. The argument of the kernel in the theta series (C.1) is $\mathbb{x} = \sqrt{2\tau_2}(\mathbb{k} + \beta)$ where \mathbb{k} runs over the (shifted) lattice. Therefore, it is useful to introduce $\mathbb{x}_\beta = \mathbb{x} - \sqrt{2\tau_2}\beta$ which in our case takes the form $\mathbb{x}_\beta = \sqrt{2\tau_2}(\mathbf{q}; k_{1,1}, \dots, k_{n,d_{r_n}})$.¹⁹ Therefore, with respect to the bilinear form (5.15), one finds that

$$\mathbb{x}_\beta * \mathbb{v}_{ij} = \sqrt{2\tau_2}\gamma_{ij}. \quad (5.39)$$

Let us now assume that a kernel Φ does not depend on the summation along $\Lambda_\perp^{(r)}$, i.e. it satisfies $\Phi(\mathbb{x}) = \Phi(\mathbb{x}_{||})$, where the indices $||$ and \perp denote projections on $\Lambda_{||}^{(r)}$ and $\Lambda_\perp^{(r)}$, respectively. This property is the main motivation to decompose the lattice into these two orthogonal sublattices. Applying (5.36) to our case, one obtains the following factorization property

$$\vartheta_{\mathbb{A}}(\tau, \mathbf{z}; \Lambda^{(r)}, \Phi, \mathbb{P}) = \sum_{\mathbb{A}} \vartheta_{\mathbb{A}, \mathbb{A}}^{(r)||}(\tau, z) \vartheta_{\mathbb{A}}^{(r)\perp}(\tau, \mathbf{z}), \quad (5.40)$$

where we introduced

$$\begin{aligned} \vartheta_{\mathbb{A}, \mathbb{A}}^{(r)||}(\tau, z) &= \vartheta_{\mathbb{g}_{\mathbb{A}}^{||+\mathbb{A}}}(\tau, z_{||}; \Lambda_{||}^{(r)}, \Phi, 0), \\ \vartheta_{\mathbb{A}}^{(r)\perp}(\tau, \mathbf{z}) &= \vartheta_{\mathbb{g}_{\mathbb{A}}^\perp}(\tau, z_\perp; \Lambda_\perp^{(r)}, 1, -\mathbb{e}_0) \end{aligned} \quad (5.41)$$

and took into account that z_\perp is independent of z , while $z_{||}$ is independent of z_i due to $\mathbb{w}_{ij} * \mathbb{z} = 2^\epsilon(\mathbf{v}_{ij} \cdot \boldsymbol{\theta})z$. It is to achieve this property, we required that the components of $\mathbf{t}^{(r_i)}$ sum to zero, which in turn was the reason to introduce the additional factors of 2 in (5.17) and (5.20).

Furthermore, due to (5.28), the second theta series in (5.41) has itself a factorized form. To write it explicitly, let us represent the summation variable as $\mathbb{k}_\perp = \ell_0 \mathbb{e}_0 + \sum_{i=1}^n \sum_{\alpha=1}^{d_{r_i}-1} \ell_{i,\alpha} \mathbb{e}_{i,\alpha}$ with $\ell_0 \in \mathbb{Z} + \frac{\nu_0}{d_r} + \frac{1}{2}$ and $\ell_{i,\alpha} \in \mathbb{Z} + \frac{\nu_{i,\alpha}}{d_{r_i}}$. The variables ν_0 and $\nu_{i,\beta}$ determining the fractional parts depend on the index $\mathbb{A} = \{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n\}$ of glue vectors. The precise dependence can be determined by expanding the glue vectors $\mathbb{g}_{\mathbb{A}}^\perp$ in the basis (5.27). Starting from (5.34), one can then arrive at the following expressions (see (F.21))

$$\nu_0 = 4^\epsilon \kappa r_0 \mathbf{a}_0 + \sum_{i=1}^n \mathbf{a}_i, \quad \nu_{i,\alpha} = \alpha \mathbf{a}_i. \quad (5.42)$$

As a result, one obtains

$$\vartheta_{\mathbb{A}}^{(r)\perp}(\tau, \mathbf{z}) = \vartheta_{\nu_0(\mathbb{A})}^{(d_r)}(\tau) \prod_{i=1}^n \Theta_{\mathbf{a}_i}^{(d_{r_i})}(\tau, z_i; \mathbf{t}^{(r_i)}), \quad (5.43)$$

where

$$\vartheta_{\nu_0}^{(d)}(\tau) = \vartheta_{\nu_0}^{(d,1)}(\tau, 0) = \sum_{\ell_0 \in \mathbb{Z} + \frac{\nu_0}{d} + \frac{1}{2}} (-1)^{d\ell_0} q^{\frac{d}{2} \ell_0^2}, \quad (5.44)$$

¹⁹Note that the components of \mathbf{q} do not sum to zero. Therefore, to see it as an element of $(\Lambda^{(r)})^*$, one should use the identification (F.15) silently assumed here.

$$\Theta_a^{(N)}(\tau, z; \mathbf{t}) = \left(\prod_{\alpha=1}^{N-1} \sum_{\ell_\alpha \in \mathbb{Z} + \frac{\alpha a}{N}} \right) q^{\sum_{\alpha=1}^{N-1} (\ell_\alpha^2 - \ell_\alpha \ell_{\alpha+1})} y^{\sum_{\alpha=1}^{N-1} (\mathbf{t}_{\alpha+1} - \mathbf{t}_\alpha) \ell_\alpha}, \quad (5.45)$$

$\vartheta_\mu^{(d,p)}(\tau, z)$ is the theta series (C.5), and in the last equation we used the convention $\ell_N = 0$. A nice feature of this representation is that z_i appears only in the theta series defined by the corresponding A_{N-1} lattice. This significantly simplifies recovering the refined anomalous coefficients $g_{\mu, \mu}^{(r)\text{ref}}(\tau, z)$ by means of (5.10) because each differential operator acts only on one theta function (see (5.69), (5.98) and (5.127) below).

Finally, let us note that $n(n-1)/2$ vectors $\hat{\mathbf{v}}_{ij}$ (or $\hat{\mathbf{v}}_{ij} = \mathbf{v}_{ij}/r_{ij}$) form an overcomplete basis of the $n-1$ dimensional lattice $\Lambda^{(r)}$. On the other hand, if we restrict to the set $\{\hat{\mathbf{v}}_{12}, \dots, \hat{\mathbf{v}}_{n-1, n}\}$, in general, it is *not* a basis and its span is only a sublattice in $\Lambda^{(r)}$. A relation between the two lattices can be described using the same formalism as above based on glue vectors. More precisely, one can show that

$$\Lambda^{(r)} = \bigcup_{\mathbf{b}_{ij}=0}^{N_{ij}-1} \left(\text{Span} \{ \hat{\mathbf{v}}_{k, k+1} \}_{k=1}^{n-1} + \sum_{\substack{i, j=1 \\ j-i > 1}}^n \mathbf{b}_{ij} \hat{\mathbf{v}}_{ij} \right), \quad (5.46)$$

where the number of *self-glue vectors* N_{ij} is given in (F.7). The idea is that linear combinations with integer coefficients of $\hat{\mathbf{v}}_{i, i+1}$ allow to get only multiples of $N_{i, i+2} \hat{\mathbf{v}}_{i, i+2}$ and one needs to add $N_{i, i+2}$ glue vectors to get arbitrary multiples of $\hat{\mathbf{v}}_{i, i+2}$. Similarly, all these vectors can be used to get only multiples of $N_{i, i+3} \hat{\mathbf{v}}_{i, i+3}$, etc. Proceeding by iterations, one recovers all vectors of the lattice $\Lambda^{(r)}$. It is obvious that a formula similar to (5.46) holds for $\tilde{\Lambda}^{(r)}$ with $\hat{\mathbf{v}}_{ij}$ replaced by $\hat{\mathbf{u}}_{ij}$.

Although below we will present a general solution to the extended anomaly equation (5.8) which will be the subject of the factorization developed in this subsection, it is instructive first to consider the cases of two and three charges.

5.4 Two charges

5.4.1 General solution

Let us first analyze in detail the case of two charges. We will use the notations introduced in §D.4.1: $r_0 = \text{gcd}(r_1, r_2)$, $\hat{r}_i = r_i/r_0$, $\hat{r} = r/r_0$ and κ_{12} , μ_{12} defined in (D.24).²⁰ Specifying the anomaly equation (5.8) to $n=2$ and substituting the result (D.25) for $R_{\mu, \mu_1, \mu_2}^{(r_1, r_2)\text{ref}}$, one finds

$$\begin{aligned} \hat{g}_{\mu, \mu_1, \mu_2}^{(r_1, r_2)\text{ref}} &= \check{g}_{\mu, \mu_1, \mu_2}^{(r_1, r_2)\text{ref}} + \prod_{i=1}^2 \left(\prod_{\alpha=1}^{d_{r_i}} \theta_1(\tau, \mathbf{t}_\alpha^{(d_{r_i})} z_i) \right) R_{\mu, \mu_1, \mu_2}^{(r_1, r_2)\text{ref}} \\ &= \check{g}_{\mu, \mu_1, \mu_2}^{(r_1, r_2)\text{ref}} + \frac{1}{4} \delta_{\Delta\mu}^{(\kappa r_0)} \sum_{\sigma=\pm 1} \sum_{k \in \mathbb{Z} + \frac{\mu_{12}}{2\kappa_{12}}} \left(\prod_{i=1}^2 \prod_{\alpha=1}^{d_{r_i}} \sum_{k_{i, \alpha} \in \mathbb{Z} + \frac{1}{2}} \right) \left[E_1(2\sqrt{\kappa_{12} \tau_2}(\sigma k + r_0 \beta)) \right] \end{aligned}$$

²⁰This definition of κ_{12} valid for $n=2$ obviously agrees with the general one in (5.23).

$$- \operatorname{sgn}(\sigma k) \Big] (-1)^{\mathbb{P} * \mathbb{k}} \mathfrak{q}^{-\frac{1}{2} \mathbb{k}^2} e^{2\pi i \mathbb{z}_\sigma * \mathbb{k}}, \quad (5.47)$$

where $\Delta\mu$ is the difference of residue classes defined in (2.15) and we introduced $d_r + 1$ -dimensional vectors $\mathbb{k} = (k; k_{1,1}, \dots, k_{1,d_{r_1}}; k_{2,1}, \dots, k_{2,d_{r_2}})$, $\mathbb{P} = (0; -1, \dots, -1)$ and $\mathbb{z}_\sigma = (\sigma r_0 z; -\mathfrak{t}^{(r_1)} z_1; -\mathfrak{t}^{(r_2)} z_2)$, which are contracted using the bilinear form

$$\mathbb{k} * \mathbb{k}' = 2\kappa_{12} k k' - \sum_{i=1}^2 \sum_{\alpha=1}^{d_{r_i}} k_{i,\alpha} k'_{i,\alpha}. \quad (5.48)$$

This bilinear form is the image of (5.15) upon the isomorphism $\mathbb{A}^{(r_1, r_2)} \simeq \mathbb{Z} \oplus \mathbb{Z}^{d_{r_1}} \oplus \mathbb{Z}^{d_{r_2}}$ implied by (5.16). Under the same isomorphism, the vectors (5.21) become

$$\hat{\mathbb{v}}_{12} = (1; 0^{[d_{r_1}]}; 0^{[d_{r_2}]}) , \quad \hat{\mathbb{w}}_{12} = (2^\epsilon; \hat{r}_2^{[d_{r_1}]}; -\hat{r}_1^{[d_{r_2}]}) , \quad (5.49)$$

where we used the same notation as in (5.26). Using $\hat{\mathbb{v}}_{12}$, or $\mathbb{v}_{12} = r_0 \hat{\mathbb{v}}_{12}$, the argument of the error function can be rewritten as $2\sqrt{\kappa_{12}\tau_2}(k + \sigma r_0 \beta) = \sqrt{2\tau_2}(\mathbb{k} + \beta_\sigma) * \mathbb{v}_{12} / \sqrt{\mathbb{v}_{12}^2}$ where we have done the usual decomposition $\mathbb{z}_\sigma = \alpha_\sigma - \tau\beta_\sigma$. As a result, the second term in (5.47), up to a σ -dependent factor and a β -dependent shift in the argument of the sign function, acquires the form of the theta series (C.1) associated with the lattice $\mathbb{A}^{(r_1, r_2)}$, residue class $\mathbb{p} = \frac{\mu_{12}}{2\kappa_{12}} \hat{\mathbb{v}}_{12}$ and kernel

$$\Phi_{\mathbb{R}}^{(r_1, r_2)}(\mathbb{x}) = E_1 \left(\frac{\mathbb{x} * \mathbb{v}_{12}}{\|\mathbb{v}_{12}\|} \right) - \operatorname{sgn}(\mathbb{x}_\beta * \mathbb{v}_{12}), \quad (5.50)$$

where $\|\mathbb{v}\| = \sqrt{\mathbb{v}^2}$ is the norm of a vector and $\mathbb{x}_\beta = \mathbb{x} - \sqrt{2\tau_2} \beta_\sigma$. More precisely, we get

$$\hat{g}_{\mu, \mu_1, \mu_2}^{(r_1, r_2)\text{ref}} = \check{g}_{\mu, \mu_1, \mu_2}^{(r_1, r_2)\text{ref}} + \frac{1}{4} \delta_{\Delta\mu}^{(\kappa r_0)} \sum_{\sigma=\pm 1} \sigma \vartheta_{\mathbb{p}}(\tau, \mathbb{z}_\sigma; \mathbb{A}^{(r_1, r_2)}, \Phi_{\mathbb{R}}^{(r_1, r_2)}, \mathbb{p}). \quad (5.51)$$

The theta series is not modular because the kernel $\Phi_{\mathbb{R}}^{(r_1, r_2)}(\mathbb{x})$ fails to satisfy the Vignéras equation (C.2) due to the presence of the sign function and its weird argument. This is supposed to be cured by the first term in (5.47), which therefore should also be taken in the form of a theta series. However, since it must be holomorphic, its kernel must be a difference of two sign functions, as required by convergence. The first of them can be taken exactly as the one in (5.50) so that it cancels the sign function spoiling modularity in $\Phi_{\mathbb{R}}^{(r_1, r_2)}$. But then one remains with the second sign function, say $\operatorname{sgn}(\mathbb{x} * \mathbb{v})$. It also spoils modularity unless the vector \mathbb{v} is null and, in particular, can be identified with \mathbb{w}_{12} ! This is due to the property (D.3) of the (generalized) error functions which is easy to see in the present example: the error function E_1 that satisfies the Vinéras equation depends on the normalized vector (see (5.50)) and when its norm goes to zero, the argument becomes large and E_1 reduces to the sign function.

This reasoning implies that a general solution to (5.51) is given by

$$\check{g}_{\mu, \mu_1, \mu_2}^{(r_1, r_2)\text{ref}} = \check{\phi}_{\mu, \mu_1, \mu_2}^{(r_1, r_2)} + \frac{1}{4} \delta_{\Delta\mu}^{(\kappa r_0)} \sum_{\sigma=\pm 1} \sigma \vartheta_{\mathbb{p}}(\tau, \mathbb{z}_\sigma; \mathbb{A}^{(r_1, r_2)}, \Phi^{(r_1, r_2)}, \mathbb{p}), \quad (5.52)$$

where

$$\Phi^{(r_1, r_2)}(\mathbb{x}) = \operatorname{sgn}(\mathbb{x}_\beta * \mathbb{v}_{12}) - \operatorname{sgn}(\mathbb{x} * \mathbb{w}_{12}) \quad (5.53)$$

and $\check{\phi}_{\mu,\mu_1,\mu_2}^{(r_1,r_2)}(\tau, z, \mathbf{z})$ is a holomorphic Jacobi-like form with the same modular properties as $\widehat{g}_{\mu,\mu_1,\mu_2}^{(r_1,r_2)\text{ref}}$. It represents an inherent ambiguity of solution of the anomaly equation and will be fixed later by requiring the correct unrefined limit. The convergence of the theta series is ensured by Theorem C.1 and the fact that $\mathfrak{v}_{12} * \mathfrak{w}_{12} > 0$ (see (5.22)), while using (C.3) it is straightforward to check that the weight, index and multiplier system agree with (B.5).

5.4.2 Holomorphic modular ambiguity

In contrast to the original problem (3.3), not every solution for $\check{g}_{\mu,\mu}^{(r)\text{ref}}$ suits our purposes. The restriction to be imposed is that it must have a well-defined unrefined limit. More precisely, $\check{g}_{\mu,\mu_1,\mu_2}^{(r_1,r_2)\text{ref}}$ must be regular at $z_i = 0$ and have a first order zero at $z = 0$. It is this condition that should be used to fix the holomorphic modular ambiguity $\check{\phi}_{\mu,\mu_1,\mu_2}^{(r_1,r_2)}$. As we will see below, the second term in (5.52) is finite at small z_i , but has a pole at small z , so that $\check{\phi}_{\mu,\mu_1,\mu_2}^{(r_1,r_2)}$ does have to be non-trivial. To extract the pole explicitly, we proceed in several steps.

Factorization and split

First, note that all ingredients of the theta series in (5.52) are as in (5.37) subject to the isomorphism (5.16) and the kernel $\Phi^{(r_1,r_2)}$ depends only on the projection \mathfrak{x}_{\parallel} . This allows to apply the factorization property (5.40). To this end, note that the theta series $\vartheta_{\mu,\mathbf{A}}^{(r)\parallel}$ (5.41) is given by a sum over a two-dimensional lattice $\mathbb{k}_{\parallel} = \ell \widehat{\mathfrak{v}}_{12} + \tilde{\ell} \widehat{\mathfrak{u}}_{12}$ with $\ell \in \mathbb{Z} + \frac{\nu}{2\kappa_{12}}$ and $\tilde{\ell} \in \mathbb{Z} + \frac{\tilde{\nu}}{2\kappa_{12}}$ where the dependence of the variables ν and $\tilde{\nu}$ on the index $\mathbf{A} = \{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2\}$ of glue vectors follows from (F.23) and is given by

$$\nu = \mu_{12}, \quad \tilde{\nu} = \kappa r_0 \hat{r}_1 \hat{r}_2 (\rho_1 - \rho_2) \mathbf{a}_0 + 4^{-\epsilon} (\hat{r}_2 \mathbf{a}_1 - \hat{r}_1 \mathbf{a}_2). \quad (5.54)$$

Therefore, (5.52) can be written as

$$\check{g}_{\mu,\mu_1,\mu_2}^{(r_1,r_2)\text{ref}} = \check{\phi}_{\mu,\mu_1,\mu_2}^{(r_1,r_2)} + \frac{1}{4} \delta_{\Delta\mu}^{(\kappa r_0)} \sum_{\mathbf{A}} \left(\sum_{\sigma=\pm 1} \sigma \vartheta_{\mu_{12},\tilde{\nu}(\mathbf{A})}^{(\kappa_{12})\parallel}(\tau, \sigma r_0 z) \right) \vartheta_{\mathbf{A}}^{(r)\perp}(\tau, \mathbf{z}). \quad (5.55)$$

where $\vartheta_{\mathbf{A}}^{(r)\perp}$ is given by (5.43) and

$$\vartheta_{\nu,\tilde{\nu}}^{(\kappa)\parallel}(\tau, z) = \sum_{\ell \in \mathbb{Z} + \frac{\nu}{2\kappa}} \sum_{\tilde{\ell} \in \mathbb{Z} + \frac{\tilde{\nu}}{2\kappa}} \left(\text{sgn}(\ell) - \text{sgn}(\ell - 2^\epsilon \tilde{\ell} + \beta) \right) \mathfrak{q}^{\kappa(4^\epsilon \tilde{\ell}^2 - \ell^2)} y^{2\kappa\ell}. \quad (5.56)$$

Next, we split the theta series (5.56) into two parts, $\vartheta_{\nu,\tilde{\nu}}^{(\kappa)\parallel} = \overset{\circ}{\vartheta}_{\nu,\tilde{\nu}}^{(\kappa)} + \tilde{\vartheta}_{\nu,\tilde{\nu}}^{(\kappa)}$, where in the first term one sums only over $(\ell, \tilde{\ell})$ satisfying the condition $\ell = 2^\epsilon \tilde{\ell}$, which can also be written in geometric terms as

$$\mathbb{k}_{\parallel} * \mathfrak{w}_{12} = 0, \quad (5.57)$$

while in the second the sum goes over the rest of the lattice. Then in $\tilde{\vartheta}_{\nu,\tilde{\nu}}^{(\kappa)}$, for sufficiently small z one can drop the shift by β in the second sign function and one obtains

$$\sum_{\sigma=\pm 1} \sigma \tilde{\vartheta}_{\nu,\tilde{\nu}}^{(\kappa)\parallel}(\tau, \sigma r_0 z) = \sum_{\substack{\ell \in \mathbb{Z} + \frac{\nu}{2\kappa} \\ \tilde{\ell} \in \mathbb{Z} + \frac{\tilde{\nu}}{2\kappa} : \ell \neq 2^\epsilon \tilde{\ell}}} \left(\text{sgn}(\ell) - \text{sgn}(\ell - 2^\epsilon \tilde{\ell}) \right) \mathfrak{q}^{\kappa(4^\epsilon \tilde{\ell}^2 - \ell^2)} \left(y^{2r_0\kappa\ell} - y^{-2r_0\kappa\ell} \right). \quad (5.58)$$

This theta series is not only convergent for all z , but also vanishes at $z = 0$. Thus, it has a well-defined unrefined limit and it remains to analyze only the function $\vartheta_{\nu, \tilde{\nu}}^{(\kappa)}$ which we call “zero mode contribution”.

Pole evaluation

The zero mode contribution is characterized by the condition $\ell = 2^\epsilon \tilde{\ell}$. Importantly, it also restricts the set of glue vectors by requiring $\nu - 2^\epsilon \tilde{\nu} \in 2\kappa_{12}\mathbb{Z}$ where ν and $\tilde{\nu}$ are given in (5.54). We denote the set of the glue vectors satisfying this condition by $\mathcal{A}_0^{(r)}(\mu_{12})$ and find it explicitly in appendix F.3.

Implementing the zero mode condition in (5.56), one finds

$$\sum_{\sigma=\pm 1} \sigma \vartheta_{\nu, \tilde{\nu}}^{(\kappa_{12})}(\tau, \sigma r_0 z) = - \sum_{\sigma=\pm 1} \sum_{\tilde{\ell} \in \mathbb{Z} + \frac{\tilde{\nu}}{2\kappa_{12}}} \left(\text{sgn}(\beta) - \sigma \text{sgn}(\tilde{\ell}) \right) y^{\sigma 2^{1+\epsilon} r_0 \kappa_{12} \tilde{\ell}}. \quad (5.59)$$

This is just a simple geometric progression. Assuming that $\beta > 0$, so that $\text{Im } z < 0$ and $|y| > 1$, it evaluates to

$$- \sum_{\sigma=\pm 1} \left(2 \frac{y^{\sigma 2^{1+\epsilon} r_0 \kappa_{12} \left(\frac{\tilde{\nu}}{2\kappa_{12}} - \left\lfloor \frac{\tilde{\nu}}{2\kappa_{12}} \right\rfloor + \frac{1}{2}(1-\sigma) \right)}}{1 - y^{-2^{1+\epsilon} r_0 \kappa_{12}}} - \sigma \delta_{\tilde{\nu}}^{(2\kappa_{12})} \right) = -2 \frac{y^{2^{1+\epsilon} r_0 \kappa_{12} \lambda_{12}} + y^{-2^{1+\epsilon} r_0 \kappa_{12} \lambda_{12}}}{y^{2^\epsilon r_0 \kappa_{12}} - y^{-2^\epsilon r_0 \kappa_{12}}}, \quad (5.60)$$

where we defined

$$\lambda_{12} = \left\lfloor \frac{\tilde{\nu}}{2\kappa_{12}} \right\rfloor - \frac{\tilde{\nu}}{2\kappa_{12}} - \frac{1}{2}, \quad (5.61)$$

which depends on the glue vector index \mathbf{A} through (5.54).²¹ The same result actually holds for $\beta < 0$ as well. Thus, the zero mode contribution to (5.55) is given by

$$\begin{aligned} & - \frac{1}{2} \delta_{\Delta\mu}^{(\kappa r_0)} \sum_{\mathbf{A} \in \mathcal{A}_0^{(r)}} \frac{y^{2^{1+\epsilon} r_0 \kappa_{12} \lambda_{12}} + y^{-2^{1+\epsilon} r_0 \kappa_{12} \lambda_{12}}}{y^{2^\epsilon r_0 \kappa_{12}} - y^{-2^\epsilon r_0 \kappa_{12}}} \vartheta_{\mathbf{A}}^{(r)\perp}(\tau, \mathbf{z}) \\ & = - \frac{\delta_{\Delta\mu}^{(\kappa r_0)}}{2^{2+\epsilon} \pi i r_0 \kappa_{12} z} \sum_{\mathbf{A} \in \mathcal{A}_0^{(r)}} \left(1 + \frac{2}{3} (1 - 12\lambda_{12}^2) (2^\epsilon \pi r_0 \kappa_{12} z)^2 + O(z^4) \right) \vartheta_{\mathbf{A}}^{(r)\perp}(\tau, \mathbf{z}) \end{aligned} \quad (5.62)$$

and confirms our claim that it has a pole at $z = 0$ which needs to be cancelled by a proper choice of the Jacobi-like form $\check{\phi}_{\mu, \mu_1, \mu_2}^{(r_1, r_2)}$.

Fixing the ambiguity

The result (5.62) for the singular contribution of the indefinite theta series suggests that the VV Jacobi-like form representing the holomorphic modular ambiguity can be chosen in a similar form:

$$\check{\phi}_{\mu, \mu_1, \mu_2}^{(r_1, r_2)} = \frac{1}{2} \delta_{\Delta\mu}^{(\kappa r_0)} \phi^{(\kappa_{12})}(\tau, r_0 z) \sum_{\mathbf{A} \in \mathcal{A}_0^{(r)}} \vartheta_{\mathbf{A}}^{(r)\perp}(\tau, \mathbf{z}), \quad (5.63)$$

²¹In fact, due to the zero mode condition, for $\epsilon = 0$, i.e. $\kappa > 1$, $\tilde{\nu}$ is uniquely fixed by the residue class $\tilde{\nu} = \mu_{12}$, while for $\epsilon = 1$ it can take two values $\tilde{\nu} = \mu_{12}/2$ and $\mu_{12}/2 + \kappa_{12}$.

where $\phi^{(\kappa_{12})}(\tau, z)$ is already a scalar valued Jacobi-like form whose modular properties can be obtained from (B.5) and (5.63). Using that $\frac{\kappa}{6}(r^3 - r_1^3 - r_2^3) = r_0^2 \kappa_{12}$, one finds that it should have weight 1, index $-\kappa_{12}$ and a trivial multiplier system. The last fact follows from the observation that the leading coefficient in the small z expansion of a VV Jacobi-like form has the same multiplier system as the form itself.

It is easy to find a function with the required modular properties cancelling the pole in (5.62). The simplest solution is to take

$$\phi^{(\kappa_{12})}(\tau, z) = \frac{e^{\frac{\pi^2}{3} \kappa_{12} E_2(\tau) z^2}}{2^{1+\epsilon} \pi i \kappa_{12} z}, \quad (5.64)$$

where $E_2(\tau)$ is the quasimodular Eisenstein series whose modular anomaly (A.3) ensures the right index for $\phi^{(\kappa_{12})}$. Expanding this function at small z , one gets

$$\phi^{(\kappa_{12})}(\tau, z) = \frac{1}{2^{1+\epsilon} \pi i \kappa_{12} z} + \frac{\pi}{2^{\epsilon} 6i} E_2(\tau) z + O(z^3). \quad (5.65)$$

Combining (5.55) with (5.63), we arrive at the following result

$$\check{g}_{\mu, \mu_1, \mu_2}^{(r_1, r_2) \text{ref}}(\tau, z, \mathbf{z}) = \frac{1}{2} \delta_{\Delta \mu}^{(\kappa r_0)} \sum_{\mathbf{A}} \left[\frac{1}{2} \sum_{\sigma=\pm 1} \sigma \vartheta_{\mu_{12}, \tilde{\nu}(\mathbf{A})}^{(\kappa_{12}) \parallel}(\tau, \sigma r_0 z) + \delta_{\mathbf{A} \in \mathcal{A}_0^{(r)}} \phi^{(\kappa_{12})}(\tau, r_0 z) \right] \vartheta_{\mathbf{A}}^{(r) \perp}(\tau, \mathbf{z}). \quad (5.66)$$

For what follows, it will be useful to undo the lattice factorization for the second term in (5.66) and rewrite it as a theta series associated to the extended lattice. This can be done at the price of having a kernel that does not combine dependence on τ_2 , \mathbb{k} and $\mathbf{z} = \alpha - \tau\beta$ into a single argument $\mathbf{x} = \sqrt{2\tau_2}(\mathbb{k} + \beta)$ as in (C.1). Namely, one finds

$$\phi^{(\kappa_{12})}(\tau, r_0 z) \sum_{\mathbf{A} \in \mathcal{A}_0^{(r)}} \vartheta_{\mathbf{A}}^{(r) \perp}(\tau, \mathbf{z}) = \vartheta_{\mathbb{A}}(\tau, \mathbf{z}; \mathbb{A}^{(r_1, r_2)}, \Phi_{\delta}^{(r_1, r_2)}, \mathbb{P}), \quad (5.67)$$

where

$$\Phi_{\delta}^{(r_1, r_2)} = \delta_{\mathbb{k} * \mathbb{w}_{12}} \phi^{(\kappa_{12})}(\tau, r_0 z) \sum_{\eta=1}^{2^{1+\epsilon} \kappa_{12}} \delta_{\mathbb{k} * \hat{\mathbf{v}}_{12} - \eta \mathbf{y}}^{-r_0 \eta}. \quad (5.68)$$

The presence of two Kronecker symbols in the kernel ensures that there is no summation along $\mathbb{A}_{\parallel}^{(r_1, r_2)}$ and implies that $\mathbf{A} \in \mathcal{A}_0^{(r)}(\mu_{12})$, while the sum over η allows for an arbitrary residue class along $\hat{\mathbf{v}}_{12}$ and takes into account the factor 2^{ϵ} in the zero mode condition $\ell = 2^{\epsilon} \tilde{\ell}$.²²

5.4.3 The unrefined limit

Let us now reduce the solution (5.66) on the extended lattice to the anomalous coefficient $g_{\mu, \mu_1, \mu_2}^{(r_1, r_2)}(\tau)$ we are really interested in. At the first step we obtain the refined anomalous coefficient $g_{\mu, \mu_1, \mu_2}^{(r_1, r_2) \text{ref}}(\tau, z)$ using the relation (5.10). As was already mentioned, the absence of

²²In fact, as follows from (5.54), the residue class along $\hat{\mathbf{v}}_{12}$ is not arbitrary but equal to μ_{12} . This means that the Kronecker symbol is non-vanishing only for $\eta = \mu_{12}$ and, if $\epsilon = 1$, for $\eta = \mu_{12} + 2\kappa_{12}$. However, to cover more general cases considered below, it is convenient to write the sum over all possible range of η .

z_i -dependence in ϑ^{\parallel} and the factorized form (5.43) of ϑ^{\perp} makes the application of (5.10) almost trivial: one should simply apply each of the differential operators to the corresponding A_{N-1} lattice theta series $\Theta_{\mathbf{a}_i}^{(d_{r_i})}(\tau, z_i)$. This gives

$$g_{\mu, \mu_1, \mu_2}^{(r_1, r_2)\text{ref}}(\tau, z) = \frac{1}{2} \delta_{\Delta\mu}^{(\kappa r_0)} \sum_{\mathbf{A}=\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2\}} \left[\frac{1}{2} \sum_{\sigma=\pm 1} \sigma \vartheta_{\mu_{12}, \tilde{\nu}(\mathbf{A})}^{(\kappa_{12})\parallel}(\tau, \sigma r_0 z) \right. \\ \left. + \delta_{\mathbf{A} \in \mathcal{A}_0^{(r)}} \phi^{(\kappa_{12})}(\tau, r_0 z) \right] \vartheta_{\nu_0(\mathbf{A})}^{(d_r)}(\tau) \prod_{i=1}^2 \mathcal{D}\Theta_{\mathbf{a}_i}^{(d_{r_i})}(\tau; \mathbf{t}^{(r_i)}), \quad (5.69)$$

where

$$\mathcal{D}\Theta_{\mathbf{a}}^{(N)}(\tau; \mathbf{t}) = \frac{\mathcal{D}_{\mathbf{t}^2/2}^{(N)} \Theta_{\mathbf{a}}^{(N)}(\tau, z; \mathbf{t})|_{z_i=0}}{N! \left(\prod_{\alpha=1}^N \mathbf{t}_{\alpha} \right) (-2\pi\eta^3(\tau))^N}. \quad (5.70)$$

Finally, we take the unrefined limit $z \rightarrow 0$ according to (5.6). To this end, we split ϑ^{\parallel} into the zero mode and the non-zero mode parts. Their contributions are evaluated in (5.62) and (5.58), respectively, where in the latter equation κ should be replaced by κ_{12} . Using the expansion (5.65), we then get

$$g_{\mu, \mu_1, \mu_2}^{(r_1, r_2)}(\tau) = \frac{r_0 \kappa_{12}}{2} \delta_{\Delta\mu}^{(\kappa r_0)} \sum_{\mathbf{A}=\{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2\}} \left[\vartheta_{\mu_{12}, \tilde{\nu}(\mathbf{A})}^{(\kappa_{12})0}(\tau) \right. \\ \left. + \delta_{\mathbf{A} \in \mathcal{A}_0^{(r)}} \frac{2^\epsilon}{12} \left(1 - 12\lambda_{12}^2 - \frac{E_2(\tau)}{2^{1+2\epsilon}\kappa_{12}} \right) \right] \vartheta_{\nu_0(\mathbf{A})}^{(d_r)}(\tau) \prod_{i=1}^2 \mathcal{D}\Theta_{\mathbf{a}_i}^{(d_{r_i})}(\tau; \mathbf{t}^{(r_i)}), \quad (5.71)$$

where λ_{12} is defined in (5.61) and

$$\vartheta_{\nu, \tilde{\nu}}^{(\kappa)0}(\tau) = \sum_{\substack{\ell \in \mathbb{Z} + \frac{\nu}{2\kappa} \\ \tilde{\ell} \in \mathbb{Z} + \frac{\tilde{\nu}}{2\kappa} : \ell \neq 2\epsilon \tilde{\ell}}} \ell \left(\text{sgn}(\ell) - \text{sgn}(\ell - 2^\epsilon \tilde{\ell}) \right) \mathfrak{q}^{\kappa(4^\epsilon \tilde{\ell}^2 - \ell^2)}. \quad (5.72)$$

In appendix H.1 we verified for several values of r_1 , r_2 and κ that the solution (5.71) is consistent with the one constructed in section 4.1 using Hecke-like operators, which amounts to showing that their difference is a VV modular form.

5.5 Three charges

5.5.1 General solution

Next, we analyze the case of three charges. The r.h.s. of the anomaly equation (5.8) now gets three contributions

$$\widehat{g}_{\mu, \mu}^{(\mathbf{r})\text{ref}} = \check{g}_{\mu, \mu}^{(\mathbf{r})\text{ref}} + 2 \text{Sym} \left\{ \prod_{\alpha=1}^{d_{r_3}} \theta_1(\tau, \mathbf{t}_{\alpha}^{(d_{r_3})} z_3) \sum_{\nu} \mathbb{R}_{\mu, \nu, \mu_3}^{(r_1+r_2, r_3)\text{ref}} \check{g}_{\nu, \mu_1, \mu_2}^{(r_1, r_2)\text{ref}} \right\} \\ + \prod_{i=1}^3 \left(\prod_{\alpha=1}^{d_{r_i}} \theta_1(\tau, \mathbf{t}_{\alpha}^{(d_{r_i})} z_i) \right) \mathbb{R}_{\mu, \mu}^{(\mathbf{r})\text{ref}}. \quad (5.73)$$

The second contribution is fixed by (D.25), (5.52), (5.63) and (5.67), while the third one can be written explicitly using (D.29), (D.30) and (D.33). A crucial observation is that the quadratic form (2.11) satisfies the property

$$Q_2(\hat{\gamma}_1, \hat{\gamma}_2) + Q_2(\hat{\gamma}_1 + \hat{\gamma}_2, \hat{\gamma}_3) = Q_3(\hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3). \quad (5.74)$$

As a result, the second and third terms in (5.73) can be written as a theta series associated with the extended lattice $\Lambda^{(r)}$ (5.14) defined by three charges r_i , $i = 1, 2, 3$. More precisely, one obtains

$$\widehat{g}_{\mu, \mu}^{(r)\text{ref}} = \check{g}_{\mu, \mu}^{(r)\text{ref}} + \frac{1}{4} \delta_{\Delta\mu}^{(\kappa r_0)} \text{Sym} \left\{ \vartheta_{\mathfrak{p}}(\tau, z; \Lambda^{(r)}, \Phi_{\mathbb{R}}^{(r)}, \mathfrak{p}) \right\}, \quad (5.75)$$

where the variables \mathfrak{p} , z and \mathfrak{p} are defined in (5.37), while the kernel is given by

$$\begin{aligned} \Phi_{\mathbb{R}}^{(r)}(\mathbb{x}) &= \Phi_2^E(\mathbb{v}_{1,2+3}, \mathbb{v}_{1+2,3}; \mathbb{x}) - \text{sgn}(\mathbb{x}_{\beta} * \mathbb{v}_{1,2+3}) \text{sgn}(\mathbb{x}_{\beta} * \mathbb{v}_{1+2,3}) - \frac{1}{3} \delta_{1,2+3} \delta_{1+2,3} \\ &- \Phi_{\mathbb{R}}^{(r_1+r_2, r_3)}(\mathbb{x}) \left(\text{sgn}(\mathbb{x} * \mathbb{w}_{12}) - \Phi_{\delta}^{(r_1, r_2)} \right) - \Phi_{\mathbb{R}}^{(r_1, r_2+r_3)}(\mathbb{x}) \left(\text{sgn}(\mathbb{x} * \mathbb{w}_{23}) - \Phi_{\delta}^{(r_2, r_3)} \right). \end{aligned} \quad (5.76)$$

Here we replaced the vectors \mathbf{v}_{ij} and \mathbf{x} appearing as arguments of Φ_2^E in (D.33) by their extended versions, abbreviated $\delta_{ij} = \delta_{\mathbb{x}_{\beta} * \mathbf{v}_{ij}}$ and used \mathbb{x}_{β} introduced above (5.39) as well as the functions $\Phi_{\mathbb{R}}^{(r_i, r_j)}(\mathbb{x})$ and $\Phi_{\delta}^{(r_i, r_j)}$, which are the same as (5.50) and (5.68), respectively, but with indices 12 replaced by ij .²³

The result (5.75) suggests to take

$$\check{g}_{\mu, \mu}^{(r)\text{ref}} = \check{\phi}_{\mu, \mu}^{(r)} + \frac{1}{4} \delta_{\Delta\mu}^{(\kappa r_0)} \text{Sym} \left\{ \vartheta_{\mathfrak{p}}(\tau, z; \Lambda^{(r)}, \Phi^{(r)}, \mathfrak{p}) \right\}, \quad (5.77)$$

where

$$\begin{aligned} \Phi^{(r)}(\mathbb{x}) &= \left(\text{sgn}(\mathbb{x}_{\beta} * \mathbb{v}_{1,2+3}) - \text{sgn}(\mathbb{x} * \mathbb{w}_{12}) \right) \left(\text{sgn}(\mathbb{x}_{\beta} * \mathbb{v}_{1+2,3}) - \text{sgn}(\mathbb{x} * \mathbb{w}_{23}) \right) + \frac{1}{3} \delta_{1,2+3} \delta_{1+2,3} \\ &+ \left(\text{sgn}(\mathbb{x}_{\beta} * \mathbb{v}_{1+2,3}) - \text{sgn}(\mathbb{x} * \mathbb{w}_{1+2,3}) \right) \Phi_{\delta}^{(r_1, r_2)} + \left(\text{sgn}(\mathbb{x}_{\beta} * \mathbb{v}_{1,2+3}) - \text{sgn}(\mathbb{x} * \mathbb{w}_{1,2+3}) \right) \Phi_{\delta}^{(r_2, r_3)} \end{aligned} \quad (5.78)$$

and $\check{\phi}_{\mu, \mu}^{(r)}(\tau, z, \mathbf{z})$ is a holomorphic Jacobi-like form with the same modular properties as $\widehat{g}_{\mu, \mu}^{(r)\text{ref}}$ representing the ambiguity of solution. Indeed, the sum of the kernels (5.76) and (5.78) involves only generalized error functions, sign functions of scalar products with null vectors and $\Phi_{\delta}^{(r_i, r_i)}$, so that the corresponding theta series transforms as a modular form without anomaly. It is also easy to check that the first term in $\Phi^{(r)}$ satisfies the conditions of Theorem C.1 which ensures the convergence of the theta series. Finally, the weight, index and multiplier system follow from (C.3) and agree with (B.5).

5.5.2 Holomorphic modular ambiguity

To fix a solution for $\check{g}_{\mu, \mu}^{(r)\text{ref}}$, it remains to determine the holomorphic modular ambiguity $\check{\phi}_{\mu, \mu}^{(r)}$ by requiring the existence of a well-defined unrefined limit. This is equivalent to the condition that $\check{g}_{\mu, \mu}^{(r)\text{ref}}$ is regular at $z_i = 0$ and has a second order zero at $z = 0$.

²³In (5.68) one should also replace r_0 by r_{ij} .

To investigate the behavior of the theta series in (5.77) at small refinement parameters, we first apply the factorization property (5.40), which is possible since the kernel (5.78) again depends only on the projection \mathbb{x}_{\parallel} . This immediately implies the absence of divergences at small z_i . Furthermore, as the theta series $\vartheta_{\mu, \mathbf{A}}^{(r)\perp}$ (5.43) and the glue vectors $\mathbf{g}_{\mathbf{A}}$ (5.34) are symmetric under permutations of charges, we can write

$$\check{g}_{\mu, \mu}^{(r)\text{ref}} = \check{\phi}_{\mu, \mu}^{(r)} + \frac{1}{4} \delta_{\Delta\mu}^{(\kappa r_0)} \sum_{\mathbf{A}} \text{Sym} \{ \vartheta_{\mu, \mathbf{A}}^{(r)\parallel}(\tau, z) \} \vartheta_{\mathbf{A}}^{(r)\perp}(\tau, z). \quad (5.79)$$

Next, we split the theta series $\vartheta_{\mu, \mathbf{A}}^{(r)\parallel}$ into several parts determined by the vanishing of $\mathbb{k} * \mathbf{w}_{ij}$. While in §5.4.2 there was just one null vector \mathbf{w}_{12} leading to the split of the theta series into two parts, now there are three different null vectors. Hence, we define *zero mode order* of a contribution as the number of linearly independent vanishing scalar products $\mathbb{k} * \mathbf{w}_{ij}$ and split $\vartheta_{\mu, \mathbf{A}}^{(r)\parallel}$ into parts with different zero mode order. In our case this order can be 0, 1 or 2 because due to (5.24) the three null vectors are linearly dependent and the vanishing of two scalar products implies the vanishing of the third. Note also that each Φ_{δ} factor contains one of the vanishing conditions and thus adds 1 to the zero mode order.

In appendix G.1 we demonstrate that the symmetrization ensures that the contributions of zero mode order equal to 0 and 1 both have a zero of second order at $z = 0$. Thus, they have a well-defined unrefined limit and it remains to analyze only the zero modes of order 2. To this end, we decompose the lattices $\Lambda^{(r)}$ and $\tilde{\Lambda}^{(r)}$ in $\Lambda_{\parallel}^{(r)} = \Lambda^{(r)} \oplus \tilde{\Lambda}^{(r)}$ as in (5.46) using $\hat{\mathbf{w}}_{ij}$ and $\tilde{\mathbf{u}}_{ij}$, respectively. In other words, we expand

$$\mathbb{k}_{\parallel} = \sum_{i=1}^{n-1} (\ell_i \hat{\mathbf{w}}_{i, i+1} + \tilde{\ell}_i \tilde{\mathbf{u}}_{i, i+1}), \quad (5.80)$$

where in our case $n = 3$. The coefficients satisfy $\ell_i \in \mathbb{Z} + \frac{\nu_i}{\kappa_i}$ and $\tilde{\ell}_i \in \mathbb{Z} + \frac{\tilde{\nu}_i}{\kappa_i}$ with κ_i defined in (5.23). The variables ν_i and $\tilde{\nu}_i$ determining the rational parts are fixed by μ and by the glue vectors labelled by $\mathbf{A} = \{\mathbf{a}_0, \mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\}$ and $\mathbf{B} = \{\mathbf{b}_{13}, \tilde{\mathbf{b}}_{13}\}$. Their explicit expressions can be obtained from (F.21), (F.22) and by evaluating scalar products of the vectors (F.20) with μ . In terms of the variables $\hat{r} = r/r_0$ and $\hat{r}_i = r_i/r_0$, this gives

$$\begin{aligned} \nu_1 &= \hat{r}\mu_1 - \hat{r}_1\mu + \hat{r}\hat{r}_1\rho_1\Delta\mu + \kappa\hat{r} \frac{r_1 r_3}{r_{13}} \mathbf{b}_{13}, \\ \nu_2 &= \hat{r}_3\mu - \hat{r}\mu_3 - \hat{r}\hat{r}_3\rho_3\Delta\mu + \kappa\hat{r} \frac{r_1 r_3}{r_{13}} \mathbf{b}_{13}, \\ \tilde{\nu}_1 &= \kappa\hat{r} \frac{r_1 r_3}{r_{13}} \tilde{\mathbf{b}}_{13} - \kappa r_1 (1 - \hat{r}\rho_1) \mathbf{a}_0 + 4^{-\epsilon} ((\hat{r}_2 + \hat{r}_3) \mathbf{a}_1 - \hat{r}_1 (\mathbf{a}_2 + \mathbf{a}_3)), \\ \tilde{\nu}_2 &= \kappa\hat{r} \frac{r_1 r_3}{r_{13}} \tilde{\mathbf{b}}_{13} + \kappa r_3 (1 - \hat{r}\rho_3) \mathbf{a}_0 + 4^{-\epsilon} (\hat{r}_3 (\mathbf{a}_1 + \mathbf{a}_2) - (\hat{r}_1 + \hat{r}_2) \mathbf{a}_3). \end{aligned} \quad (5.81)$$

In terms of the coefficients $\ell_i, \tilde{\ell}_i$, the second order zero mode condition $\mathbb{k} * \mathbf{w}_{12} = \mathbb{k} * \mathbf{w}_{23} = 0$ is equivalent to $\ell_1 - 2^\epsilon \tilde{\ell}_1 = \ell_2 - 2^\epsilon \tilde{\ell}_2 = 0$. Therefore, when one rewrites the expression for the contribution of second order zero modes to $\text{Sym} \{ \vartheta_{\mu, \mathbf{A}}^{(r)\parallel} \}$ in terms of these coefficients, the sum over ℓ_i 's disappears. Furthermore, \mathbb{k}_{\parallel} becomes null and one remains with

$$\text{Sym} \left\{ \sum_{\mathbf{B}} \Delta_0^{(r)}(\mathbf{A}, \mathbf{B}) \vartheta_{\tilde{\nu}_1, \tilde{\nu}_2}^{\circ(r)}(z) \right\}, \quad (5.82)$$

where $\Delta_0^{(r)} = \delta_{\nu_1 - 2^\epsilon \tilde{\nu}_1}^{(\kappa_1)} \delta_{\nu_2 - 2^\epsilon \tilde{\nu}_2}^{(\kappa_2)}$ is the Kronecker symbol imposing the second order zero mode condition and

$$\begin{aligned} \vartheta_{\tilde{\nu}_1, \tilde{\nu}_2}^{(r)}(z) = & \sum_{\tilde{\ell}_1 \in \mathbb{Z} + \frac{\tilde{\nu}_1}{\kappa_1}} \sum_{\tilde{\ell}_2 \in \mathbb{Z} + \frac{\tilde{\nu}_2}{\kappa_2}} \left[\left(\left(\text{sgn}(\tilde{\ell}_1) - \text{sgn}(\beta) \right) \left(\text{sgn}(\tilde{\ell}_2) - \text{sgn}(\beta) \right) + \frac{1}{3} \delta_{\tilde{\ell}_1} \delta_{\tilde{\ell}_2} \right) y^{2^{1+\epsilon}(r_{12}\kappa_{12}\tilde{\ell}_1 + r_{23}\kappa_{23}\tilde{\ell}_2)} \right. \\ & \left. + \phi^{(\kappa_{12})}(\tau, r_{12}z) \sum_{\eta=1}^{2^{1+\epsilon}\kappa_{12}} \delta_{2^\epsilon(2\kappa_{12}\tilde{\ell}_1 - \kappa_{123}\tilde{\ell}_2) - \eta} \sum_{\sigma=\pm 1} \left(\sigma \text{sgn}(\tilde{\ell}_2) - \text{sgn}(\beta) \right) y^{\sigma 2^\epsilon r_0 \kappa_2 \tilde{\ell}_2} \right]. \end{aligned} \quad (5.83)$$

To get this expression, we applied the permutation $1 \leftrightarrow 3$ to the last term in (5.78) before substituting (5.80). The sum over $\tilde{\ell}_i$'s can be evaluated explicitly. First, we note that

$$\sum_{\tilde{\ell}_1 \in \mathbb{Z} + \frac{\tilde{\nu}_1}{\kappa_1}} \sum_{\eta=1}^{2^{1+\epsilon}\kappa_{12}} \delta_{2^\epsilon(2\kappa_{12}\tilde{\ell}_1 - \kappa_{123}\tilde{\ell}_2) - \eta} = \delta_{\frac{2\kappa_{12}}{\kappa_1}\nu_1 - \frac{\kappa_{123}}{\kappa_2}\nu_2}^{(1)} = 1, \quad (5.84)$$

where in the first equality we used that $2^\epsilon \tilde{\nu}_i = \nu_i \pmod{\kappa_i}$, as required by the second order zero mode condition, while the last equality follows by substituting the explicit expressions for ν_i (5.81). As a result, all sums in (5.83) become simple geometric progressions and for $\beta > 0$ one finds²⁴

$$\begin{aligned} \vartheta_{\tilde{\nu}_1, \tilde{\nu}_2}^{(r)}(z) = & \frac{1}{3} \delta_{\tilde{\nu}_1}^{(\kappa_1)} \delta_{\tilde{\nu}_2}^{(\kappa_2)} + \left(\delta_{\tilde{\nu}_1}^{(\kappa_1)} - \frac{2y^{-2^{1+\epsilon}r_{12}\kappa_{12}\lambda_1}}{y^{2^\epsilon r_{12}\kappa_{12}} - y^{-2^\epsilon r_{12}\kappa_{12}}} \right) \left(\delta_{\tilde{\nu}_2}^{(\kappa_2)} - \frac{2y^{-2^{1+\epsilon}r_{23}\kappa_{23}\lambda_2}}{y^{2^\epsilon r_{23}\kappa_{23}} - y^{-2^\epsilon r_{23}\kappa_{23}}} \right) \\ & - 2\phi_{12}(\tau, r_{12}z) \frac{y^{2^\epsilon r_0 \kappa_2 \lambda_2} + y^{-2^\epsilon r_0 \kappa_2 \lambda_2}}{y^{2^{\epsilon-1}r_0 \kappa_2} - y^{-2^{\epsilon-1}r_0 \kappa_2}}, \end{aligned} \quad (5.85)$$

where we defined

$$\lambda_i = \left\lfloor \frac{\tilde{\nu}_i}{\kappa_i} \right\rfloor - \frac{\tilde{\nu}_i}{\kappa_i} - \frac{1}{2}. \quad (5.86)$$

In fact, it turns out to be convenient to symmetrize this expression with respect to the permutation $1 \leftrightarrow 3$. This could be done before performing the above calculations, but it can also be done directly for (5.85) because under this permutation the basis vectors in (5.80) are mapped to each with a flip of sign, $\hat{\nu}_{12} \leftrightarrow -\hat{\nu}_{23}$ and $\hat{u}_{12} \leftrightarrow -\hat{u}_{23}$, while the glue vectors just flip their sign. This implies $\tilde{\nu}_1 \leftrightarrow -\tilde{\nu}_2$ and $\lambda_1 \leftrightarrow -\lambda_2 - \delta_{\tilde{\nu}_2}^{(\kappa_2)}$, so that in the second term one should flip the relative signs inside the brackets and the signs in the power of y in the numerators, whereas in the last term in (5.85) one should just replace the indices 12 by 23 and 2 by 1. As a result, if one expands at small z the symmetrized expression, one obtains

$$\frac{1}{2} \left(\vartheta_{\tilde{\nu}_1, \tilde{\nu}_2}^{(r)} + \sigma_{13} [\vartheta_{\tilde{\nu}_1, \tilde{\nu}_2}^{(r)}] \right) = \mathcal{C}_{-2}^{(r)} z^{-2} + \mathcal{C}_0^{(r)}(\tau) + \mathcal{C}_2^{(r)}(\tau) z^2 + O(z^4), \quad (5.87)$$

where we introduced functions of τ

$$\begin{aligned} \mathcal{C}_0^{(r)}(\tau) = & \mathcal{C}_0^{(r)} + \frac{r_{12}r_{23}(r_1 + r_3)}{4^\epsilon 6\kappa r r_1 r_2 r_3} E_2(\tau), \\ \mathcal{C}_2^{(r)}(\tau) = & \mathcal{C}_2^{(r)} + \frac{\pi^2 r_0}{36} \left(r_{23}\kappa_1(1 - 12\lambda_1^2) + r_{12}\kappa_2(1 - 12\lambda_2^2) \right) E_2(\tau) \\ & + \frac{\pi^2 r_{12}r_{23}}{4^\epsilon 72 r r_1 r_3} \left(r_1^2(r_1 + r_2) + r_3^2(r_2 + r_3) \right) E_2^2(\tau) \end{aligned} \quad (5.88)$$

²⁴Although (5.83) does depend on the sign of β , its symmetrized version (5.82) does not. So all expressions written below starting from (5.87) will be valid for both signs.

and constant coefficients

$$\begin{aligned}
\mathcal{C}_{-2}^{(r)} &= \frac{r_2}{4^{1+\epsilon}\pi^2 r r_{12} r_{23} \kappa_{12} \kappa_{23}}, \\
\mathcal{C}_0^{(r)} &= \frac{1}{3} \delta_{\tilde{\nu}_1}^{(\kappa_1)} \delta_{\tilde{\nu}_2}^{(\kappa_2)} + \left(2\lambda_1 + \delta_{\tilde{\nu}_1}^{(\kappa_1)}\right) \left(2\lambda_2 + \delta_{\tilde{\nu}_2}^{(\kappa_2)}\right) + \frac{1}{6} \left(\frac{r_{23} r_1 (1 - 12\lambda_1^2)}{r_{12}(r_2 + r_3)} + \frac{r_{12} r_3 (1 - 12\lambda_2^2)}{r_{23}(r_1 + r_2)} \right), \\
\mathcal{C}_2^{(r)} &= 4^\epsilon \pi^2 \left[\frac{(\kappa r_2)^2}{360} \left(\frac{r_{23} r_1^3 (r^3 - (r_1 + r_2)^3)}{r_{12}^3 r_3 (r_2 + r_3)} (7 - 120\lambda_1^2 + 240\lambda_1^4) \right. \right. \\
&\quad \left. \left. + \frac{r_{12} r_3^3 (r^3 - (r_2 + r_3)^3)}{r_{23}^3 r_1 (r_1 + r_2)} (7 - 120\lambda_2^2 + 240\lambda_2^4) \right) - \frac{1}{9} r_{12} r_{23} \kappa_{12} \kappa_{23} (1 - 12\lambda_1^2)(1 - 12\lambda_2^2) \right. \\
&\quad \left. + \frac{4}{3} \left((r_{12} \kappa_{12})^2 \lambda_1 (1 - 4\lambda_1^2) \left(2\lambda_2 + \delta_{\tilde{\nu}_2}^{(\kappa_2)}\right) + (r_{23} \kappa_{23})^2 \lambda_2 (1 - 4\lambda_2^2) \left(2\lambda_1 + \delta_{\tilde{\nu}_1}^{(\kappa_1)}\right) \right) \right]. \tag{5.89}
\end{aligned}$$

The main conclusion of all this analysis is that the only contribution of the theta series term in (5.79) that does not have a zero of second order at $z = 0$ and needs to be cancelled by the holomorphic modular ambiguity originates from the second order zero modes and is given by

$$\frac{1}{4} \delta_{\Delta\mu}^{(\kappa r_0)} \sum_{\mathbf{A}} \text{Sym} \left\{ \sum_{\mathbf{B}} \Delta_0^{(r)}(\mathbf{A}, \mathbf{B}) \left(\mathcal{C}_{-2}^{(r)} z^{-2} + \mathcal{C}_0^{(r)}(\tau) \right) \right\} \vartheta_{\mathbf{A}}^{(r)\perp}(\tau, \mathbf{z}). \tag{5.90}$$

Furthermore, it turns out that

$$\text{Sym} \left\{ \sum_{\mathbf{B}} \Delta_0^{(r)} \mathcal{C}_0^{(r)} \right\} = 0, \tag{5.91}$$

while the sum over \mathbf{B} in the first term can be evaluated using Corollary F.1. Thus, one remains with

$$-\frac{1}{4} \delta_{\Delta\mu}^{(\kappa r_0)} \text{Sym} \{c_{\mathbf{r}}\} \left(\frac{1}{z^2} - \frac{\pi^2}{3} m_{\mathbf{r}} E_2(\tau) \right) \sum_{\mathbf{A} \in \mathcal{A}_0^{(r)}} \vartheta_{\mathbf{A}}^{(r)\perp}(\tau, \mathbf{z}), \tag{5.92}$$

where $\mathcal{A}_0^{(r)}$ is the set characterized by the conditions (F.30) implementing the second order zero mode condition on \mathbf{A} indices, and

$$c_{\mathbf{r}} = -\frac{r_0 r_2}{r_{12} r_{23}} \mathcal{C}_{-2}^{(r)} = -\frac{r_0}{4^\epsilon \pi^2 \kappa^2 r r_1 r_3 (r_1 + r_2)(r_2 + r_3)}. \tag{5.93}$$

We have also taken into account that

$$\frac{r_{12} r_{23} (r_1 + r_3)}{4^\epsilon 6 \kappa r r_1 r_2 r_3} = -\frac{\pi^2}{3} m_{\mathbf{r}} \mathcal{C}_{-2}^{(r)}, \tag{5.94}$$

where

$$m_{\mathbf{r}} = -\frac{\kappa}{6} \left(r^3 - \sum_{i=1}^3 r_i^3 \right) = -\frac{\kappa}{2} (r_1 + r_2)(r_1 + r_3)(r_2 + r_3). \tag{5.95}$$

We do not provide here a proof of the vanishing property (5.91) (which has been extensively checked on a computer) because we will prove its generalization to arbitrary number of charges in §5.6. As we will see, it turns out to be a direct consequence of modularity.

The contribution (5.92) can be cancelled by the holomorphic modular ambiguity chosen as

$$\check{\phi}_{\mu,\mu}^{(r)} = \frac{1}{4} \delta_{\Delta\mu}^{(\kappa r_0)} \phi^{(r)}(\tau, z) \sum_{A \in \mathcal{A}_0^{(r)}} \vartheta_A^{(r)\perp}(\tau, z), \quad (5.96)$$

where

$$\phi^{(r)}(\tau, z) = \frac{\text{Sym}\{c_r\}}{z^2} e^{-\frac{\pi^2}{3} m_r E_2(\tau) z^2}. \quad (5.97)$$

Indeed, $\phi^{(r)}$ is a Jacobi-like form of weight 2 and index m_r , so that the weight and index of $\check{\phi}_{\mu,\mu}^{(r)}$ agree with (B.5) specialized to $n = 3$. The multiplier system must also agree because it is the same as the one of the leading terms in the small z expansion of a theta series with the right multiplier system. Finally, the first two non-trivial terms in the small z expansion of $\check{\phi}_{\mu,\mu}^{(r)}$ cancel (5.92), which ensures that (5.79) has a zero of second order at $z = 0$.

5.5.3 The unrefined limit

To get the anomalous coefficient $g_{\mu,\mu}^{(r)}$ from the solution (5.79) on the extended lattice, we again proceed in two steps. First, we apply the relation (5.10) which gives the following refined anomalous coefficient

$$g_{\mu,\mu}^{(r)\text{ref}}(\tau, z) = \frac{1}{4} \delta_{\Delta\mu}^{(\kappa r_0)} \sum_A \left(\phi^{(r)}(\tau, z) \delta_{A \in \mathcal{A}_0^{(r)}} + \text{Sym}\{\vartheta_{\mu,A}^{(r)}(\tau, z)\} \right) \vartheta_{\nu_0(A)}^{(d_r)}(\tau) \prod_{i=1}^3 \mathcal{D}\Theta_{a_i}^{(d_{r_i})}(\tau; \mathbf{t}^{(r_i)}), \quad (5.98)$$

where $\mathcal{D}\Theta_a^{(d_r)}$ was defined in (5.70).

To evaluate the remaining unrefined limit, we use the results of our analysis which showed the existence of a zero of second order. Thus, we represent $\text{Sym}\{\vartheta_{\mu,A}^{(r)}\}$ as a sum of three contributions corresponding to different orders of zero modes: the vanishing order with kernel given in (G.6), the first order with kernel given by the sum of (G.10) and (G.13), and the second order given by (5.82) and (5.85) or its expansion (5.87). The last contribution is to be combined with the first term in (5.98). Applying the relation (5.6), one then finds

$$g_{\mu,\mu}^{(r)}(\tau) = \frac{1}{4} \delta_{\Delta\mu}^{(\kappa r_0)} \sum_A \left[\vartheta_{\mu,A}^{(r)0}(\tau) + \vartheta_{\mu,A}^{(r)1}(\tau) - \text{Sym}\left\{ \frac{1}{16\pi^2} \sum_B \Delta_0^{(r)} \mathcal{C}_2^{(r)}(\tau) \right\} \vartheta_{\nu_0(A)}^{(d_r)}(\tau) \prod_{i=1}^3 \mathcal{D}\Theta_{a_i}^{(d_{r_i})}(\tau; \mathbf{t}^{(r_i)}), \right] \quad (5.99)$$

where the three terms in the square brackets correspond to the three contributions described above. For the first two given by theta series we provide explicit expressions in appendix G.2, while the function $\mathcal{C}_2^{(r)}$ determining the third term in (5.99) is obtained by combining the $O(z^2)$ -terms in (5.87) and in the expansion of (5.97):

$$\begin{aligned} \mathcal{C}_2^{(r)}(\tau) &= \mathcal{C}_2^{(r)}(\tau) - \frac{\pi^4}{18} \mathcal{C}_{-2} m_r^2 E_2^2(\tau) \\ &= \mathcal{C}_2^{(r)} + \frac{\pi^2 r_0}{36} \left(r_{23} \kappa_1 (1 - 12\lambda_1^2) + r_{12} \kappa_2 (1 - 12\lambda_2^2) \right) E_2(\tau) - \frac{\pi^2 r_{12} r_{23}}{4^e 72 r_2} (r + r_2) E_2^2(\tau). \end{aligned} \quad (5.100)$$

The formula (5.99) represents an explicit expression for the anomalous coefficients with arbitrary three charges. In appendix H.2 we verified that for κ and all r_i equal to 1, it is consistent with the solution proportional to the normalized generating function of $SU(3)$ VV invariants on \mathbb{P}^2 presented in section 4.2.

5.6 General case

5.6.1 General solution

Now we turn to the most general case and, as usual, we start by presenting a solution of the anomaly equation (5.8). Of course, for any set of charges this solution involves a holomorphic modular ambiguity parametrized by a Jacobi-like form $\phi_{\mu,\mu}^{(r)}$. From the very beginning we will take into account that it can be chosen in the factorized form (cf. (5.96))

$$\check{\phi}_{\mu,\mu}^{(r)} = \frac{\delta^{(\kappa r_0)}}{2^{n-1}} \sum_{\mathbb{A}} \phi_{\mathfrak{g}_{\mathbb{A}}^{\parallel} + \mu}^{(r)}(\tau, z) \vartheta_{\mathbb{A}}^{(r)\perp}(\tau, \mathbf{z}), \quad (5.101)$$

where $\phi_{\nu}^{(r)}(\tau, z)$ is a VV Jacobi-like form labelled by $\nu \in \mathbb{D}_{\parallel}^{(r)} = (\mathbb{A}_{\parallel}^{(r)})^*/\mathbb{A}_{\parallel}^{(r)}$, and characterized by weight $n-1$, index $m_{\mathbf{r}}$ (5.5), and the multiplier system given by the Weil representation (C.3) associated with the lattice $\mathbb{A}_{\parallel}^{(r)}$. As we did before for $n=2$, formally one can rewrite (5.101) as a theta series over the full extended lattice $\mathbb{A}^{(r)}$

$$\check{\phi}_{\mu,\mu}^{(r)} = \frac{\delta^{(\kappa r_0)}}{2^{n-1}} \vartheta_{\mu}(\tau, \mathbb{Z}; \mathbb{A}^{(r)}, \Phi_{\delta}^{(r)}, \mathbb{P}), \quad (5.102)$$

where μ, \mathbb{Z} and \mathbb{P} are as in (5.37), and the kernel is given by

$$\Phi_{\delta}^{(r)}(\mathbb{x}, \tau, z) = q^{\frac{1}{2}\mathbb{k}_{\parallel}^2} y^{-\boldsymbol{\theta} \cdot \mathbf{k}} \sum_{\nu \in \mathbb{D}_{\parallel}^{(r)}} \delta_{\mathbb{k}_{\parallel} - \nu} \phi_{\nu}^{(r)}(\tau, z), \quad (5.103)$$

where \mathbb{k}_{\parallel} is the projection of $\mathbb{k} = \mathbb{x}/\sqrt{2\tau_2} - \beta$ on $\mathbb{A}_{\parallel}^{(r)}$ and \mathbf{k} is the component of \mathbb{k} along $\mathbb{A}^{(r)}$. It is chosen to ensure that

$$\vartheta_{\nu}(\tau, \mathbb{Z}_{\parallel}; \mathbb{A}_{\parallel}^{(r)}, \Phi_{\delta}^{(r)}, 0) = \phi_{\nu}^{(r)}(\tau, z). \quad (5.104)$$

Although the restriction to (5.101) may not describe the most general solution of the anomaly equation, which is not our goal anyway, it allows us to represent a solution in terms of a theta series on the extended lattice. To this end, let us define

$$\Phi^{(r)}(\mathbb{x}; \{\mathcal{F}^{(\mathbf{s})}\}) = \sum_{m=2}^n \sum_{\sum_{k=1}^m n_k = n} \mathcal{F}^{(\mathbf{s})}(\mathbb{x}^{(0)}) \prod_{k=1}^m \Phi_{\delta}^{(\mathbf{r}_k)}(\mathbb{x}^{(k)}, \tau, z), \quad (5.105)$$

where \mathbf{s} are \mathbf{r}_k are the notations from (3.4), the upper indices (0) and (k) denote projections to $\mathbb{A}^{(\mathbf{s})}$ and $\mathbb{A}^{(\mathbf{r}_k)}$, respectively, and for a single charge we set $\Phi_{\delta}^{(\mathbf{r})} = 1$. Then we have the following

Theorem 5.1. *A solution of the anomaly equation (5.8) and its modular completion can be expressed as*

$$\begin{aligned}\check{g}_{\mu,\mu}^{(r)\text{ref}} &= \check{\phi}_{\mu,\mu}^{(r)} + \frac{\delta_{\Delta\mu}^{(\kappa r_0)}}{2^{n-1}} \text{Sym} \left\{ \vartheta_{\mu}(\tau, \mathbb{Z}; \Lambda^{(r)}, \Phi^{(r)}(\{\mathcal{F}^{(s)}\}), \mathbb{P}) \right\}, \\ \widehat{g}_{\mu,\mu}^{(r)\text{ref}} &= \check{\phi}_{\mu,\mu}^{(r)} + \frac{\delta_{\Delta\mu}^{(\kappa r_0)}}{2^{n-1}} \text{Sym} \left\{ \vartheta_{\mu}(\tau, \mathbb{Z}; \Lambda^{(r)}, \Phi^{(r)}(\{\widehat{\mathcal{F}}^{(s)}\}), \mathbb{P}) \right\},\end{aligned}\tag{5.106}$$

where the functions $\mathcal{F}^{(r)}$ and $\widehat{\mathcal{F}}^{(r)}$ are given by

$$\begin{aligned}\mathcal{F}^{(r)}(\mathbb{x}) &= \sum_{\mathcal{J} \subseteq \mathcal{L}_{n-1}} e_{|\mathcal{J}|} \delta_{\mathcal{J}} \prod_{\ell \in \mathcal{L}_{n-1} \setminus \mathcal{J}} \left(\text{sgn}(\mathbb{x}_{\beta} * \mathbb{v}_{\ell}) - \text{sgn}(\mathbb{x} * \mathbb{w}_{\ell, \ell+1}) \right), \\ \widehat{\mathcal{F}}^{(r)}(\mathbb{x}) &= \sum_{\mathcal{J} \subseteq \mathcal{L}_{n-1}} \Phi_{|\mathcal{J}|}^E(\{\mathbb{v}_l\}_{l \in \mathcal{J}}; \mathbb{x}) \prod_{\ell \in \mathcal{L}_{n-1} \setminus \mathcal{J}} (-\text{sgn}(\mathbb{x} * \mathbb{w}_{\ell, \ell+1})).\end{aligned}\tag{5.107}$$

Here $\mathbb{x}_{\beta} = \mathbb{x} - \sqrt{2\tau_2} \beta$, $\mathcal{L}_n = \{1, \dots, n\}$,

$$e_m = \begin{cases} 0 & \text{if } m \text{ is odd,} \\ \frac{1}{m+1} & \text{if } m \text{ is even,} \end{cases} \quad \delta_{\mathcal{J}} = \prod_{\ell \in \mathcal{J}} \delta_{\mathbb{x}_{\beta} * \mathbb{v}_{\ell}}, \quad \text{and} \quad \mathbb{v}_{\ell} = \sum_{i=1}^{\ell} \sum_{j=\ell+1}^n \mathbb{v}_{ij}.\tag{5.108}$$

Although the functions (5.107) might seem to be complicated, their structure is easy to understand. First, if all scalar products $\mathbb{x}_{\beta} * \mathbb{v}_{\ell}$ are non-vanishing, then the function $\mathcal{F}^{(r)}(\mathbb{x})$ simplifies to

$$\mathcal{F}^{(r)}(\mathbb{x}) = \prod_{\ell=1}^{n-1} \left(\text{sgn}(\mathbb{x}_{\beta} * \mathbb{v}_{\ell}) - \text{sgn}(\mathbb{x} * \mathbb{w}_{\ell, \ell+1}) \right),\tag{5.109}$$

which is the standard kernel ensuring convergence of indefinite theta series with quadratic form having $n - 1$ positive eigenvalues (see Theorem C.1). If however some of the scalar products vanish, it is not sufficient to set the corresponding sign functions to zero. Instead, one gets additional contributions manifestly visible in (D.12) (for $n = 3$ this the term $\frac{1}{3} \delta_{1,2+3} \delta_{1+2,3}$ in (5.78)). In the presence of refinement only the linear tree is relevant (see (D.17)) and one can apply a simple recipe that $\text{sgn}(0)^m \rightarrow e_m$ [35]. This gives rise to the expression in (5.107). It is useful to note that, using (5.39), the notation (G.1) and the function (D.12), it can also be rewritten as

$$\mathcal{F}^{(r)}(\mathbb{x}) = \sum_{\mathcal{J} \subseteq \mathcal{L}_{n-1}} S_{\mathcal{T}_{\text{lin}}}(\{\hat{\gamma}_l\}_{l \in \mathcal{J}}) \prod_{\ell \in \mathcal{L}_{n-1} \setminus \mathcal{J}} (-\text{sgn}(\omega_{\ell, \ell+1}^{(\beta)})).\tag{5.110}$$

Now it should become obvious where the second function $\widehat{\mathcal{F}}^{(r)}(\mathbb{x})$ comes from: it is obtained from $\mathcal{F}^{(r)}(\mathbb{x})$ by applying the recipe to construct modular completions of indefinite theta series explained in §D.1 which amounts to replacing each product of sign functions²⁵ by the (boosted) generalized error function with parameters determined by the vectors entering the sign functions. If some of the vectors are null, in addition one applies the property (D.3).

²⁵Since the large τ_2 limit of the generalized error function is precisely the function $S_{\mathcal{T}}$ and not the simple product of signs [23], it is actually this function that should be replaced by Φ_{n-1}^E .

The proof of Theorem 5.1 is completely analogous to the proof of Theorem 1 in [51]. The similarity between these theorems become particularly obvious if one applies the factorization property (5.40). It allows to rewrite the solution given in the theorem as

$$\check{g}_{\mu, \boldsymbol{\mu}}^{(\mathbf{r})\text{ref}} = \frac{\delta^{(\kappa r_0)}_{\Delta \mu}}{2^{n-1}} \sum_{\mathbb{A}} \left(\phi_{\mathbb{g}_{\mathbb{A}}^{\parallel} + \boldsymbol{\mu}}^{(\mathbf{r})}(\tau, z) + \text{Sym} \{ \vartheta_{\boldsymbol{\mu}, \mathbb{A}}^{(\mathbf{r})\parallel}(\tau, z) \} \right) \vartheta_{\mathbb{A}}^{(\mathbf{r})\perp}(\tau, z), \quad (5.111)$$

where $\vartheta_{\mathbb{A}}^{(\mathbf{r})\perp}$ has the universal form (5.43). The key point is that $\vartheta_{\boldsymbol{\mu}, \mathbb{A}}^{(\mathbf{r})\parallel}$ can be significantly simplified due to the property (5.104) of the kernels $\Phi_{\delta}^{(\mathbf{r})}$. However, to be able to use this property, one should somehow obtain theta series with a kernel given by only one $\Phi_{\delta}^{(\mathbf{r})}$. This can be achieved by factorizing the lattice $\mathbb{A}_{\parallel}^{(\mathbf{r})}$ according to the factorized form of the kernel $\Phi^{(\mathbf{r})}$ in (5.105), which in turn will induce the corresponding factorization of theta series. Since $\mathbb{A}_{\parallel}^{(\mathbf{r})} = \mathbb{A}^{(\mathbf{r})} \oplus \tilde{\mathbb{A}}^{(\mathbf{r})}$, this amounts to using (F.9) for both factors, which leads to a generalization of (F.19) where the theta series are associated to the product of two lattices. As a consequence, the indices labelling them are doubled and can be represented as $(\mu, \boldsymbol{\mu}; \tilde{\mu}, \tilde{\boldsymbol{\mu}})$. This set of indices can be thought of as a label of the elements of the discriminant group $\nu \in \mathbb{D}_{\parallel}^{(\mathbf{r})}$ which itself can be seen as a pair of vectors $(\hat{\mu}, \hat{\boldsymbol{\mu}})$. A relation between $(\mu, \boldsymbol{\mu})$ and $\hat{\mu}$ is given in (5.37), and to get its tilded version it is enough to replace κ by $4^{\epsilon} \kappa$. In particular, using (F.21), one finds that the element $\mathbb{g}_{\mathbb{A}}^{\parallel} + \boldsymbol{\mu} \in \mathbb{D}_{\parallel}^{(\mathbf{r})}$ corresponds to $(\mu, \boldsymbol{\mu}; \tilde{\mu}, \tilde{\boldsymbol{\mu}})$ with

$$\tilde{\mu} = 4^{\epsilon} \kappa r_0 \mathbf{a}_0 + \sum_{i=1}^n \mathbf{a}_i, \quad \tilde{\mu}_i = \mathbf{a}_i. \quad (5.112)$$

As a result, one arrives at the following expression

$$\vartheta_{\boldsymbol{\mu}, \mathbb{A}}^{(\mathbf{r})\parallel}(\tau, z) = \sum_{m=2}^n \sum_{\sum_{k=1}^m n_k = n} \sum_{\boldsymbol{\nu}, \tilde{\boldsymbol{\nu}}} \vartheta_{\mu, \boldsymbol{\nu}; \tilde{\mu}, \tilde{\boldsymbol{\nu}}}(\tau, z_{\parallel}^{(0)}; \mathbb{A}_{\parallel}^{(\mathbf{s})}, \mathcal{F}^{(\mathbf{s})}, 0) \prod_{k=1}^m \phi_{\nu_k, \mathbf{m}_k; \tilde{\nu}_k, \tilde{\mathbf{m}}_k}^{(\mathbf{r}_k)}(\tau, z). \quad (5.113)$$

It is a simple generalization of [51, Eq.(3.11)] which is twofold: the doubling of indices discussed above and arbitrary set of charges instead of n charges all equal to 1. Furthermore, the kernels (5.107) are exactly the same as the ones given in Eqs. (3.13) and (3.14) of that paper. Given this similarity, we refrain from repeating the proof.

5.6.2 Holomorphic modular ambiguity

As usual, it remains to fix the holomorphic modular ambiguities which are now encoded in the VV Jacobi-like forms $\phi_{\nu}^{(\mathbf{r})}$ entering the construction via (5.111) and (5.113). To this end, as in §5.5.2, we split the theta series $\vartheta_{\boldsymbol{\mu}, \mathbb{A}}^{(\mathbf{r})\parallel}$ into contributions with different zero mode order equal to the number of linearly independent vanishing scalar products $\mathbb{k} * \mathbf{w}_{ij}$. On the basis of our results for $n = 3$ (and some checks done at $n = 4$ which are too cumbersome to be presented here) we make the following

Conjecture 5.1. *Let us fix an integer n_0 , and assume that for all sets of charges \mathbf{r} with the number of charges $n < n_0$, the functions $\phi_{\nu}^{(\mathbf{r})}(\tau, z)$ are Jacobi-like forms that ensure the existence of the unrefined limit for all $\check{g}_{\mu, \boldsymbol{\mu}}^{(\mathbf{r})\text{ref}}$ so that they behave as $O(z^{n-1})$ at small z . Then for $n = n_0$, the contributions to $\text{Sym} \{ \vartheta_{\boldsymbol{\mu}, \mathbb{A}}^{(\mathbf{r})\parallel}(\tau, z) \}$ of any zero mode order different from the maximal one, given by $n - 1$, behave as $O(z^{n-1})$.*

For the contributions of vanishing zero mode order this conjecture is equivalent to Conjecture 1 in [51], which has been extensively tested numerically, and there is a simple argument why it is expected to hold for other zero mode orders as well. Note that due to Proposition A.1, the expansion coefficients at small z of the function $e^{\frac{m_{\mathbf{r}}}{3}\pi^2 E_2(\tau)z^2} \widehat{g}_{\mu,\mu}^{(\mathbf{r})\text{ref}}$, where $m_{\mathbf{r}}$ is the index (5.5) of the refined anomalous coefficient, transform as modular forms without any anomaly. On the other hand, due to the induction hypothesis, all terms in the refined anomaly equation (5.4), except the one with $m = 1$, behave as $O(z^{n-1})$. Thus, the only term that can spoil this behavior is $g_{\mu,\mu}^{(\mathbf{r})\text{ref}}$. Combining the two conclusions, one obtains that if

$$\mathcal{L}_{n-2} \left[e^{\frac{m_{\mathbf{r}}}{3}\pi^2 E_2(\tau)z^2} g_{\mu,\mu}^{(\mathbf{r})\text{ref}} \right] \quad (5.114)$$

is the part of the Laurent series in z truncated at z^{n-2} , then all its coefficients must be modular forms. Furthermore, from the representation (5.111) of our solution, it is clear that the same is true for the coefficients of

$$\mathcal{L}_{n-2} \left[e^{\frac{m_{\mathbf{r}}}{3}\pi^2 E_2(\tau)z^2} \text{Sym} \left\{ \vartheta_{\mu,\mathbf{A}}^{(\mathbf{r})\parallel}(\tau, z) \right\} \right]. \quad (5.115)$$

But all contributions to $\text{Sym} \left\{ \vartheta_{\mu,\mathbf{A}}^{(\mathbf{r})\parallel}(\tau, z) \right\}$ of any zero mode order different from the maximal one involve holomorphic indefinite theta series. They are (higher depth) mock modular forms whose anomalies cannot be cancelled by quasimodular forms. Therefore, they cannot generate pure modular forms and should vanish. For the zero modes of maximal order the situation is different because, as we will see below, their contribution does not involve indefinite theta series. Of course, this argument is far from being a proof and we hope to return to this issue in a future work.

The above conjecture reduces the problem to evaluating the zero mode contribution of maximal order. To get it, let us first analyze the zero mode contribution of maximal order to $\vartheta_{\mathbf{v}}(\tau, \mathbb{Z}_{\parallel}; \mathbb{A}_{\parallel}^{(\mathbf{r})}, \mathcal{F}^{(\mathbf{r})}, 0)$ which we will denote by $\mathcal{I}_{\mathbf{v}}^{(\mathbf{r})}$. The zero mode condition implies that $n-1$ scalar products $\mathbb{k} * \mathbb{w}_{ij}$ are vanishing, but since among vectors \mathbb{w}_{ij} there are only $n-1$ linearly independent, actually all such scalar products are vanishing. Under this condition, $\mathbb{k}_{\parallel}^2 = 0$ so that $\mathcal{I}_{\mathbf{v}}^{(\mathbf{r})}$ does not depend on τ and is a function of z only. To find it explicitly, we perform the lattice decomposition (5.46) for the two factors in $\mathbb{A}_{\parallel}^{(\mathbf{r})} = \mathbb{A}^{(\mathbf{r})} \oplus \tilde{\mathbb{A}}^{(\mathbf{r})}$ and substitute the expansion (5.80) with $\ell_i \in \mathbb{Z} + \frac{\nu_i}{\kappa_i}$ and $\tilde{\ell}_i \in \mathbb{Z} + \frac{\tilde{\nu}_i}{\kappa_i}$ where ν_i and $\tilde{\nu}_i$ are determined by the residue class \mathbf{v} . As a result, we end up with the following expression (cf. (5.82))

$$\mathcal{I}_{\mathbf{v}}^{(\mathbf{r})}(z) = \sum_{\mathbf{B}} \Delta_0^{(\mathbf{r})}(\mathbf{A}, \mathbf{B}) \vartheta_{\tilde{\mathbf{v}}(\mathbf{A}, \mathbf{B})}^{\circ(\mathbf{r})}(z), \quad (5.116)$$

where is the Kronecker symbol imposing the maximal order zero mode condition and

$$\vartheta_{\tilde{\mathbf{v}}}^{\circ(\mathbf{r})}(z) = \left(\prod_{i=1}^n \sum_{\tilde{\ell}_i \in \mathbb{Z} + \frac{\tilde{\nu}_i}{\kappa_i}} \right) \left[\sum_{\mathcal{J} \subseteq \mathcal{Z}_{n-1}} e_{|\mathcal{J}|} \prod_{i \in \mathcal{J}} \delta_{\tilde{\ell}_i} \prod_{i \in \mathcal{Z}_{n-1} \setminus \mathcal{J}} (\text{sgn}(\tilde{\ell}_i) - 1) \right] y^{2^{1+\epsilon} \sum_{i=1}^{m-1} r_{i,i+1} \kappa_{i,i+1} \tilde{\ell}_i}. \quad (5.117)$$

Here for simplicity we restricted to $\beta > 0$ (as usual, the final result after symmetrization will be valid for both signs of β). The remaining sums produce geometric progressions resulting in

$$\vartheta_{\nu}^{(r)}(z) = \sum_{\mathcal{J} \subseteq \mathcal{Z}_{n-1}} e_{|\mathcal{J}|} \prod_{i \in \mathcal{J}} \delta_{\nu_i}^{(\kappa_i)} \prod_{i \in \mathcal{Z}_{n-1} \setminus \mathcal{J}} \left(\delta_{\nu_i}^{(\kappa_i)} - \frac{2y^{-2^{1+\epsilon} r_{i,i+1} \kappa_{i,i+1} \lambda_i}}{y^{2^\epsilon r_{i,i+1} \kappa_{i,i+1}} - y^{-2^\epsilon r_{i,i+1} \kappa_{i,i+1}}} \right), \quad (5.118)$$

where we used the notation λ_i from (5.86).

Let us now concentrate on the leading singular contribution in the small z limit which will allow us to fix the most singular term in the holomorphic modular ambiguity. From (5.118), one finds that

$$\vartheta_{\nu}^{(r)}(z) = \frac{(-2^{1+\epsilon} \pi i z)^{1-n}}{\prod_{i=1}^{n-1} r_{i,i+1} \kappa_{i,i+1}} + O(z^{2-n}). \quad (5.119)$$

Importantly, the leading term does not depend on the glue vector indices \mathbf{B} . Therefore, due to Corollary F.1, the leading term in (5.116) is given by

$$\mathcal{I}_{\nu}^{(r)}(z) = \frac{(-2^\epsilon \pi i \kappa z)^{1-n} r_0}{\prod_{i=1}^n r_i \prod_{i=1}^{n-1} (r_i + r_{i+1})} \delta_{\mathbf{A} \in \mathcal{A}_0^{(r)}} + O(z^{2-n}), \quad (5.120)$$

where $\mathcal{A}_0^{(r)}$ is the set from Proposition F.1 implementing the maximal order zero mode condition on \mathbf{A} indices. Next, we assume that the leading term of the Jacobi-like form $\phi_{\nu}^{(r)}(\tau, z)$ also depends on the residue class ν only through the zero mode condition. Namely, representing $\nu = (\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\mu}})$ and defining $\Delta \hat{\boldsymbol{\mu}} = \hat{\boldsymbol{\mu}} - 2^\epsilon \hat{\boldsymbol{\mu}}$ (cf. §F.4) we take

$$\phi_{\nu}^{(r)}(\tau, z) = \frac{\text{Sym}\{c_{\mathbf{r}}\}}{z^{n-1}} \prod_{i=1}^n \delta_{\Delta \hat{\boldsymbol{\mu}}_i}^{(1)} + O(z^{2-n}). \quad (5.121)$$

For $\nu = \mathfrak{g}_{\mathbf{A}}^{\parallel} + \boldsymbol{\mu}$, the product of Kronecker symbols is nothing but $\delta_{\mathbf{A} \in \mathcal{A}_0^{(r)}}$. Substituting (5.120) and (5.121) into (5.113), one obtains

$$\vartheta_{\boldsymbol{\mu}, \mathbf{A}}^{(r)\parallel}(\tau, z) = \frac{\delta_{\mathbf{A} \in \mathcal{A}_0^{(r)}}}{z^{n-1}} \sum_{m=2}^n \sum_{\sum_{k=1}^m n_k = n} \sum_{\boldsymbol{\nu}} \frac{\text{gcd}(s_1, \dots, s_m) \prod_{k=1}^m c_{\mathbf{r}_k}}{(-2^\epsilon \pi i \kappa)^{n-1} \prod_{k=1}^m s_k \prod_{k=1}^{m-1} (s_k + s_{k+1})} + O(z^{2-n}). \quad (5.122)$$

Since the summand does not depend on $\boldsymbol{\nu}$, the sum over these indices produces just a numerical factor equal to the number of their independent values. In appendix F.1 we showed that these indices can be identified with the glue vectors of the lattice decomposition (F.9) and therefore the factor is given by (F.13). Finally, the consistency requires that the resulting singular contribution should be exactly cancelled by adding the leading singular term (5.121) of the holomorphic modular ambiguity with $\nu = \mathfrak{g}_{\mathbf{A}}^{\parallel} + \boldsymbol{\mu}$ as in (5.111). Thus, the coefficients $c_{\mathbf{r}}$ should be solutions of the following system of equations

$$\sum_{m=1}^n \sum_{\sum_{k=1}^m n_k = n} \left(-\frac{1}{2^\epsilon \pi i \kappa} \right)^{m-1} \frac{\prod_{k=1}^m c_{\mathbf{r}_k}}{\prod_{k=1}^m \text{gcd}(\mathbf{r}_k) \prod_{k=1}^{m-1} (s_k + s_{k+1})} = 0. \quad (5.123)$$

We claim that this system is solved by

$$c_{\mathbf{r}} = \frac{r_0}{(2^\epsilon \pi i \kappa)^{n-1} r} \prod_{k=1}^{n-1} \left(\sum_{i=1}^k r_i \sum_{j=n-k+1}^n r_j \right)^{-1}, \quad (5.124)$$

which agrees with (5.65) and (5.93). Although we were not able to find an analytic proof, we have checked on a computer that this formula does solve the equations for arbitrary charges up to $n = 8$, which leaves no doubt that this is the right solution.

Having found the leading term in the expansion of $\phi_{\mathbf{v}}^{(\mathbf{r})}$, we can now provide the full function. It is given by the following

Theorem 5.2. *Provided Conjecture 5.1 holds, the holomorphic modular ambiguity given by (5.101) with*

$$\phi_{\mathbf{v}}^{(\mathbf{r})}(\tau, z) = \frac{\text{Sym}\{c_{\mathbf{r}}\}}{z^{n-1}} e^{-\frac{\pi^2}{3} m_{\mathbf{r}} E_2(\tau) z^2} \prod_{i=1}^n \delta_{\Delta \hat{\mu}_i}^{(1)} \quad (5.125)$$

ensures the existence of the unrefined limit.

Proof. The proof will follow the logic of the argument used to justify Conjecture 5.1. Namely, let us consider the function

$$\begin{aligned} & e^{\frac{m_{\mathbf{r}}}{3} \pi^2 E_2(\tau) z^2} \left(\phi_{\mathfrak{g}_{\Lambda}^{\parallel} + \mathfrak{p}}^{(\mathbf{r})}(\tau, z) + \text{Sym}\{\vartheta_{\mathfrak{p}, \Lambda}^{(\mathbf{r}) \parallel}(\tau, z)\} \right) \\ &= \delta_{A \in A_0^{(\mathbf{r})}} \text{Sym} \left\{ \sum_{m=1}^n \frac{1}{z^{n-m}} \sum_{\sum_{k=1}^m n_k = n} e^{\frac{\pi^2}{3} \left(m_{\mathbf{r}} - \sum_{k=1}^m m_{\tau_k} \right) E_2(\tau) z^2} \prod_{k=1}^m c_{\tau_k} \sum_{\mathbf{v}, \bar{\mathbf{v}}} \mathcal{I}_{\mu, \nu; \bar{\mu}, \bar{\nu}}^{(s)}(z) \right\} + O(z^{n-1}), \end{aligned} \quad (5.126)$$

where to get the second line, we used Conjecture 5.1, the representation (5.113) and the proposed form of $\phi_{\mathbf{v}}^{(\mathbf{r})}$. To prove the theorem, one needs to show that the first term is also $O(z^{n-1})$. Let us assume that this is not the case and there are terms $\sim z^k$ with $k < n - 1$. Here k must be large than $1 - n$ because we have already shown the cancellation of the leading singularity $\sim z^{1-n}$ for $c_{\mathbf{r}}$ given by (5.124). Then the same chain of reasoning as above (5.115) allows to conclude that the coefficient of the z^k -term must be a modular form of weight $n - 1 + k > 0$. However, as follows from (5.126), the only dependence of τ of these coefficients is through the polynomial dependence on the quasimodular form $E_2(\tau)$. But no polynomial of $E_2(\tau)$ can produce a modular form of positive weight. Hence all of them must vanish, which proves the statement of the theorem. \square

5.6.3 The unrefined limit

The two theorems 5.1 and 5.2 provide a solution for the functions $\check{g}_{\mu, \mu}^{(\mathbf{r}) \text{ref}}$ satisfying the anomaly equation (5.8) and having a well-defined unrefined limit. It remains just to reduce it to the original anomalous coefficients $g_{\mu, \mu}^{(\mathbf{r})}$. The first step, the reduction to the refined anomalous coefficients $g_{\mu, \mu}^{(\mathbf{r}) \text{ref}}$, is trivial and done by applying the relation (5.10) to the expression (5.111). This affects only the A_{N-1} lattice theta series and results in

$$g_{\mu, \mu}^{(\mathbf{r}) \text{ref}}(\tau, z) = \frac{1}{4} \delta_{\Delta \mu}^{(\kappa r_0)} \sum_{\Lambda} \left(\phi_{\mathfrak{g}_{\Lambda}^{\parallel} + \mathfrak{p}}^{(\mathbf{r})}(\tau, z) + \text{Sym}\{\vartheta_{\mathfrak{p}, \Lambda}^{(\mathbf{r}) \parallel}(\tau, z)\} \right) \vartheta_{\nu_0(\Lambda)}^{(d_r)}(\tau) \prod_{i=1}^n \mathcal{D} \Theta_{\mathfrak{a}_i}^{(d_{r_i})}(\tau; \mathfrak{t}^{(r_i)}). \quad (5.127)$$

The last step is to evaluate the unrefined limit $z \rightarrow 0$. Unfortunately, we cannot accomplish it analytically in full generality because this would require rewriting $\text{Sym}\{\vartheta_{\mathfrak{p}, \Lambda}^{(\mathbf{r}) \parallel}(\tau, z)\}$ in a form

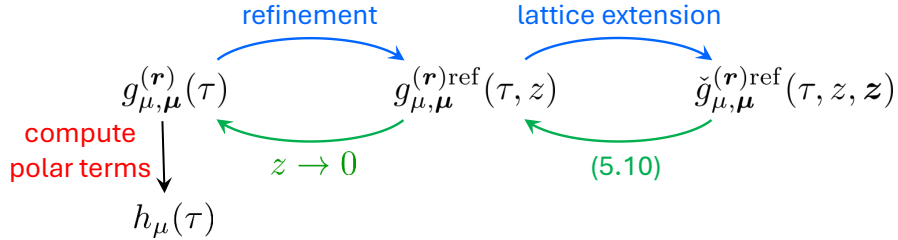


Figure 2: Construction of the refined anomalous coefficients through the refinement and lattice extension and their relation to the generating functions of BPS indices.

which makes manifest the existence of zero of order $n - 1$ at small z for all contributions except the zero modes of maximal order and, in particular, would automatically provide a proof of Conjecture 5.1. However, since evaluating a limit of a function should certainly be simpler than solving non-trivial anomaly equations, we see this problem as just a technical obstacle and hope to return to it elsewhere.

6. Conclusions

In this paper we solved the modular anomaly equation for the generating functions $h_{r,\mu}(\tau)$ of D4-D2-D0 BPS indices, the same as rank 0 DT invariants of Calabi-Yau threefolds, restricting to the case of threefolds with one Kähler modulus. Since for a fixed D4-brane charge r , the anomaly equation fixes the generating function only up to a modular form $h_{r,\mu}^{(0)}$ and involves all generating functions h_{r_i,μ_i} with $r_i < r$, the solution is expressed as a polynomial in $h_{r_i,\mu_i}^{(0)}$, which at this stage remain unknown functions. The coefficients of this polynomial, called anomalous coefficients $g_{\mu,\mu}^{(r)}(\tau)$, satisfy their own anomaly equation (Theorem 3.1), and it is solving this equation that takes the main part of our work.

In two particular cases (arbitrary two charges $\mathbf{r} = (r_1, r_2)$ or arbitrary number of charges but all, together with the intersection number κ , equal to 1), it is immediate to write a solution for $g_{\mu,\mu}^{(r)}(\tau)$ given by the mock modular forms of optimal growth from [25] and the normalized generating functions of $SU(n)$ VW invariants on \mathbb{P}^2 , respectively. The generic case is treated using indefinite theta series. In fact, this requires introducing auxiliary functions, $g_{\mu,\mu}^{(r)\text{ref}}(\tau, z)$ and $\check{g}_{\mu,\mu}^{(r)\text{ref}}(\tau, z, z)$, depending on additional variables and satisfying a proper set of anomaly equations. The former is a refined version of the anomalous coefficients and the latter is obtained from the former by a lattice extension. As a result, we have solved the anomaly equation for $\check{g}_{\mu,\mu}^{(r)\text{ref}}$ (Theorems 5.1 and 5.2), reduced it $g_{\mu,\mu}^{(r)\text{ref}}$ (see (5.127)), and evaluated its unrefined limit $z \rightarrow 0$ producing $g_{\mu,\mu}^{(r)}$ in the cases of two and three charges. Unfortunately, the last step turns out to be too complicated to be done analytically in generic case. The realized strategy is schematically presented in Fig. 2.

In fact, the existence of the unrefined limit of our solution for generic charges remains conjectural since it relies on Conjecture 5.1. Although we gave a strong argument why we expect it to be true, it leaves a gap in our construction. It would certainly be desirable to fill this gap and to find a way to rigorously prove the conjecture. This might also suggest a way

to explicitly evaluate the unrefined limit in generic case, the other missing step for having a complete result.

For a few cases of small charges and small intersection number, we presented first terms in q -series of the anomalous coefficients in appendix I. Besides, in appendix H, we have shown that our solution in terms of indefinite theta series is consistent with the other solutions mentioned above, provided by mock modular forms of optimal growth and generating functions of VW invariants. To this end, we constructed a combination of the solutions that is required to be a Jacobi form and then explicitly evaluated it confirming that the requirement is indeed satisfied.

This calculation opens an interesting possibility. The point is that the mock modular forms of optimal growth introduced in [25] are constructed in terms of certain seed functions $\mathcal{G}^{(d)}$ (see (4.10)), but only for the first two of them, $\mathcal{G}^{(1)}$ and $\mathcal{G}^{(6)}$, one knows analytic expressions. Our results can be used to obtain such analytic expressions for other seed functions.

It is useful to note that our construction resembles a lot the solution of a similar modular anomaly equation for the generating functions of refined VW invariants in [51]. But there is an important difference that here we could fix the holomorphic modular ambiguity for the anomalous coefficients in terms of a Jacobi-like form, whereas in the VW case a similar function had to be a Jacobi form. It is this possibility to use Jacobi-like forms that is responsible for a very simple form of the solution (5.125) for the ambiguity.

Our solution for the anomalous coefficients reduces the problem of finding the generating functions $h_{r,\mu}$ to the problem of finding just a finite number of Fourier coefficients. For example, it would be sufficient to compute their polar terms, which allows us to fix the modular ambiguities $h_{r,\mu}^{(0)}$ and thereby the whole generating functions. Moreover, for $r > 1$, typically, the polar terms must satisfy non-trivial constraints to produce a mock modular form [38, 61]. Therefore, such computation would provide an extremely strong test of mock modularity, which at the physical level appears as a consequence of S-duality, whereas at the mathematical level remains completely mysterious.

The problem however is that the existing techniques are likely insufficient to do this. For example, the naive extension of the approach based on wall-crossing, as outlined in [24, §3.3], would require knowledge of GV invariants up to genus 232 to compute all polar terms of $h_{3,\mu}$ for CY X_{10} , which is given by the degree 10 hypersurface in weighted projective space $\mathbb{P}^{5,2,1,1,1}$ (one of the two CYs for which $h_{2,\mu}$ has been explicitly found). This seems unrealistic in the current state of affairs. Therefore, one needs to look for alternative methods to compute polar terms which would not be limited to extremely small charges.

Finally, although we restricted in this paper to the one-modulus case, our construction seems to be easily generalizable to CYs with multiple moduli. In this case CYs can also have additional structures, such as elliptic or K3 fibrations, and it would be interesting to study an interplay between these structures and the mock modularity of rank 0 DT invariants exploited here.

Acknowledgements

The authors are grateful to Abhiram Kidambi for valuable discussions. SA thanks the Galileo Galilei Institute for Theoretical Physics for the hospitality and the INFN for partial support during the completion of this work. Besides, SA would like to thank the Isaac Newton Institute for Mathematical Sciences, Cambridge, for support and hospitality during the programme “Twistor theory”, supported by EPSRC grant no EP/R014604/1, where work on this paper was undertaken.

A. Jacobi and Jacobi-like forms

A.1 Jacobi forms

Jacobi forms have been introduced and studied in detail by Eichler and Zagier in [62]. Here we provide a definition which generalizes the original one in several aspects: it allows the function to have multiple elliptic arguments, to be non-holomorphic, to be vector valued and to possess a non-trivial multiplier system. All these generalizations are well-known and play an important role in numerous physical problems.

Let $\varphi_\mu(\tau, \mathbf{z})$ be a finite set of (in general, non-holomorphic) functions, labelled by μ , on $\mathbb{H} \times \mathbb{C}^n$, and $\mathbf{x} \cdot \mathbf{y} = \sum_{i,j=1}^n Q_{ij} x_i y_j$ denotes a bilinear form on \mathbb{C}^n . Then $\varphi_\mu(\tau, \mathbf{z})$ is a vector valued (multi-variable) Jacobi form of weight (w, \bar{w}) and (matrix valued) index mQ_{ij} if it satisfies the following transformation properties

$$\varphi_\mu(\tau, \mathbf{z} + \mathbf{a}\tau + \mathbf{b}) = e^{-2\pi i m(\mathbf{a}^2 \tau + 2\mathbf{a} \cdot \mathbf{z})} \varphi_\mu(\tau, \mathbf{z}), \quad \mathbf{a}, \mathbf{b} \in \mathbb{Z}^n, \quad (\text{A.1a})$$

$$\varphi_\mu \left(\frac{a\tau + b}{c\tau + d}, \frac{\mathbf{z}}{c\tau + d} \right) = (c\tau + d)^w (c\bar{\tau} + d)^{\bar{w}} e^{\frac{2\pi i m c \mathbf{z}^2}{c\tau + d}} \sum_{\nu} M_{\mu\nu}(\rho) \varphi_\nu(\tau, \mathbf{z}), \quad (\text{A.1b})$$

where $\rho = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z})$ and $M_{\mu\nu}(\rho)$ is a multiplier system.

Setting the elliptic variables $\mathbf{z} = 0$, (A.1) reduces to the definition of a vector valued modular form. Note that since $M_{\mu\nu}(\rho)$ must furnish a representation of the group $SL(2, \mathbb{Z})$ generated by two transformations, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, to define the multiplier system, it is enough to specify it for $\rho = T$ and S . Thus, to characterize the modular behavior of a Jacobi form, it is sufficient to provide its modular weight (w, \bar{w}) , index m and two matrices $M_{\mu\nu}(T)$ and $M_{\mu\nu}(S)$.

A.2 Jacobi-like forms

Jacobi-like forms have been first introduced in [63, 64] and further studied in the mathematical literature (see, e.g., [65, 66, 67]). In the simplest case they are defined as formal power series in X , with coefficients in functions on \mathbb{H} , satisfying the following transformation property under $SL(2, \mathbb{Z})$

$$\Phi \left(\frac{a\tau + b}{c\tau + d}, \frac{X}{(c\tau + d)^2} \right) = e^{\frac{cX}{c\tau + d}} \Phi(\tau, X). \quad (\text{A.2})$$

Given a Jacobi-like form, let us consider the function $X^{-w}\Phi(\tau, X)$. Then it is easy to see that under the identification $X = 2\pi imz^2$, one gets a function $\varphi(\tau, z)$ which satisfies the modular transformation property (A.1b) of a Jacobi form of weight w and index m . However, the elliptic property (A.1a) is in general missing. This justifies the name ‘‘Jacobi-like’’.

In the applications relevant to our work, it is more convenient to work directly with functions of z that have a definite weight and index and to allow for all the generalizations (multiple variables, non-holomorphicity, vector valuedness) that we allowed for Jacobi forms. Therefore, for the purposes of this paper, we will call Jacobi-like form any function $\varphi_\mu(\tau, \mathbf{z})$ that satisfies (A.1b).²⁶

Next, we are interested in modular properties of the expansion coefficients of a Jacobi-like form around a point where one of the (would be) elliptic variables, say z_1 , vanishes. For simplicity, we restrict ourselves to the case $n = 1$ of only one elliptic variable and set $Q_{11} = 1$, but the propositions below are trivially generalized to $n > 1$ provided the quadratic form is factorized, i.e. $Q_{1i} = 0$ for $i > 1$.

In fact, it is well-known that the coefficients of the expansion in $X \sim z^2$ of a Jacobi-like form are in one-to-one correspondence with modular forms which can be constructed as linear combinations of the τ -derivatives of the coefficients [63, 64]. In particular, the coefficient of the leading term, say, z^{n_0} in the expansion is a modular form of weight $w + n_0$.

It is also known that Jacobi-like forms are closely related to quasimodular forms [67], the simplest example of which is the Eisenstein series $E_2(\tau) = 1 - 24 \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi in\tau}$ satisfying the transformation property

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 \left(E_2(\tau) + \frac{6}{\pi i} \frac{c}{c\tau + d} \right). \quad (\text{A.3})$$

In particular, one can note that the anomalous term in this transformation has the same form as the logarithm of the exponential factor in the transformation of a Jacobi-like form. This immediately implies the following

Proposition A.1. *Let $\varphi_\mu(\tau, z)$ be a Jacobi-like form of modular weight w and index m . Then*

$$\tilde{\varphi}_\mu(\tau, z) = e^{\frac{m}{3}\pi^2 E_2(\tau) z^2} \varphi_\mu(\tau, z) \quad (\text{A.4})$$

is a Jacobi-like form of the same weight and vanishing index, and the coefficients of its Laurent expansion $\tilde{\varphi}_\mu(\tau, z) = \sum_{n=n_0}^{\infty} h_n(\tau) z^n$ are modular forms of weight $w + n$.

This simple observation can be used to prove

Proposition A.2. *Let $\varphi_\mu(\tau, z)$ be a Jacobi-like form of modular weight w and index m , and having a smooth limit at $z \rightarrow 0$. We define the following differential operator*

$$\mathcal{D}_m^{(n)} = \sum_{k=0}^{\lfloor n/2 \rfloor} c_{n,k} E_2^k(\tau) \partial_z^{n-2k}, \quad c_{n,k} = \frac{n! \left(\frac{2m}{3}\pi^2\right)^k}{(2k)!!(n-2k)!}. \quad (\text{A.5})$$

²⁶The original definition of Jacobi-like forms implies that they have an expansion in *even* powers of z . However, once one allows for a non-trivial multiplier system, there is no much sense keeping this condition. In practice, the functions appearing in the main text are functions of τ, z and $\mathbf{z} = (z_1, \dots, z_n)$ which behave as Jacobi-like forms with respect to z with an expansion in even powers, up to an overall shift in the power, and as usual Jacobi forms with respect to \mathbf{z} .

Then

$$\phi_\mu^{(n)}(\tau) \equiv \mathcal{D}_m^{(n)} \varphi_\mu(\tau, z)|_{z=0}, \quad (\text{A.6})$$

is a vector valued modular form of weight $w + n$.

Proof. If ϕ_μ is smooth at small z , the same is true for the function $\tilde{\varphi}_\mu$ (A.4) and hence its expansion coefficients are given by the derivatives with respect to z evaluated at $z = 0$. According to Proposition A.1, such derivatives $\partial_z^n \tilde{\varphi}_\mu(\tau, 0)$ transform as modular forms of weight $w + n$. On the other hand, we have

$$\partial_z^n \tilde{\varphi}_\mu(\tau, 0) = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n!}{(2k)!(n-2k)!} \left(e^{-\frac{x^2}{2}} \frac{d^{2k}}{d^{2k}x} e^{\frac{x^2}{2}} \right) \Big|_{x=0} \left(\frac{2m}{3} \pi^2 E_2(\tau) \right)^k \partial_z^{n-2k} \varphi_\mu(\tau, 0). \quad (\text{A.7})$$

Taking into account that $\left(e^{-\frac{x^2}{2}} \frac{d^{2k}}{d^{2k}x} e^{\frac{x^2}{2}} \right) \Big|_{x=0} = (2k-1)!!$, we conclude that

$$\partial_z^n \tilde{\varphi}_\mu(\tau, 0) = \mathcal{D}_m^{(n)} \varphi_\mu(\tau, z)|_{z=0}, \quad (\text{A.8})$$

which proves the statement of the proposition. \square

B. Summary of modular properties

In this appendix we collect the modular weights, indices and multiplier systems of the main modular functions appearing in the paper.

Generating functions of BPS indices:

$$\begin{aligned} w(h_r) &= -3/2, \\ M_{\mu\nu}^{(h_r)}(T) &= e^{\frac{\pi i}{\kappa r} (\mu + \frac{1}{2} \kappa r^2)^2 + \frac{\pi i}{12} c_2 r} \delta_{\mu\nu}, \\ M_{\mu\nu}^{(h_r)}(S) &= \frac{(-1)^{\chi_r}}{\sqrt{\kappa r}} e^{-\frac{\pi i}{4} - 2\pi i \frac{\mu\nu}{\kappa r}}, \end{aligned} \quad (\text{B.1})$$

where χ_r is defined in (2.7).

Redefined generating functions:

$$\begin{aligned} w(\tilde{h}_r) &= -3/2, \\ M_{\mu\nu}^{(\tilde{h}_r)}(T) &= e^{\frac{\pi i}{\kappa r} (1 - \kappa r) \mu^2 + \frac{\pi i}{4} (\kappa + \frac{c_2}{3}) r} \delta_{\mu\nu}, \\ M_{\mu\nu}^{(\tilde{h}_r)}(S) &= \frac{e^{\frac{\pi i}{4} ((2\kappa + c_2)r - 1)}}{\sqrt{\kappa r}} e^{-2\pi i \frac{\mu\nu}{\kappa r}}. \end{aligned} \quad (\text{B.2})$$

Anomalous coefficients:

$$\begin{aligned} w(g^{(r)}) &= 3(n-1)/2, \\ M_{\mu, \mu, \nu, \nu}^{(g^{(r)})}(T) &= e^{\pi i (\mu - \sum_i \mu_i) + \pi i \left(\frac{\mu^2}{\kappa r} - \sum_i \frac{\mu_i^2}{\kappa r_i} \right)} \delta_{\mu\nu} \delta_{\mu\nu}, \\ M_{\mu, \mu, \nu, \nu}^{(g^{(r)})}(S) &= \frac{e^{\frac{\pi i}{4} (n-1)}}{\sqrt{\kappa^{n+1} r \prod_i r_i}} e^{-2\pi i \left(\frac{\mu\nu}{\kappa r} - \sum_i \frac{\mu_i \nu_i}{\kappa r_i} \right)}. \end{aligned} \quad (\text{B.3})$$

Refined anomalous coefficients:

$$\begin{aligned}
w(g^{(r)\text{ref}}) &= (n-1)/2, \\
m(g^{(r)\text{ref}}) &= -\frac{\kappa}{6} \left(r^3 - \sum_{i=1}^n r_i^3 \right), \\
M_{\mu, \mu, \nu, \nu}^{(g^{(r)\text{ref}})}(T) &= e^{\pi i (\mu - \sum_i \mu_i) + \pi i \left(\frac{\mu^2}{\kappa r} - \sum_i \frac{\mu_i^2}{\kappa r_i} \right)} \delta_{\mu\nu} \delta_{\mu\nu}, \\
M_{\mu, \mu, \nu, \nu}^{(g^{(r)\text{ref}})}(S) &= \frac{e^{\frac{\pi i}{4}(n-1)}}{\sqrt{\kappa^{n+1} r \prod_i r_i}} e^{-2\pi i \left(\frac{\mu\nu}{\kappa r} - \sum_i \frac{\mu_i \nu_i}{\kappa r_i} \right)}.
\end{aligned} \tag{B.4}$$

Extended refined anomalous coefficients:

$$\begin{aligned}
w(\check{g}^{(r)\text{ref}}) &= (n-1 + d_r)/2, \\
m(\check{g}^{(r)\text{ref}}) &= \frac{1}{2} \text{diag} \left(-\frac{\kappa}{3} \left(r^3 - \sum_{i=1}^n r_i^3 \right), (\mathfrak{t}^{(r_1)})^2, \dots, (\mathfrak{t}^{(r_n)})^2 \right), \\
M_{\mu, \mu, \nu, \nu}^{(\check{g}^{(r)\text{ref}})}(T) &= e^{\pi i (\mu - \sum_i \mu_i) + \pi i \left(\frac{\mu^2}{\kappa r} - \sum_i \frac{\mu_i^2}{\kappa r_i} \right) + \frac{\pi i}{4} d_r} \delta_{\mu\nu} \delta_{\mu\nu}, \\
M_{\mu, \mu, \nu, \nu}^{(\check{g}^{(r)\text{ref}})}(S) &= \frac{e^{\frac{\pi i}{4}(n-1-3d_r)}}{\sqrt{\kappa^{n+1} r \prod_i r_i}} e^{-2\pi i \left(\frac{\mu\nu}{\kappa r} - \sum_i \frac{\mu_i \nu_i}{\kappa r_i} \right)},
\end{aligned} \tag{B.5}$$

where $(\mathfrak{t}^{(r)})^2 = \sum_{\alpha=1}^{d_r} (\mathfrak{t}_\alpha^{(r)})^2$ and $d_r = \sum_{i=1}^n d_{r_i}$.

C. Theta series

In this appendix we define some useful theta series and describe their modular properties.

C.1 Generalized theta series and modularity condition

Let us define

$$\vartheta_\mu(\tau, \mathfrak{z}; \mathbb{A}, \Phi, \mathfrak{p}) = \sum_{\mathfrak{k} \in \mathbb{A} + \mu + \frac{1}{2}\mathfrak{p}} (-1)^{\mathbb{P} * \mathfrak{k}} \Phi(\sqrt{2\tau_2}(\mathfrak{k} + \beta)) q^{-\frac{1}{2}\mathfrak{k}^2} e^{2\pi i \mathfrak{z} * \mathfrak{k}}, \tag{C.1}$$

where $q = e^{2\pi i \tau}$, \mathbb{A} is a d -dimensional lattice equipped with a bilinear form $\mathfrak{x} * \mathfrak{y}$ such that the associated quadratic form has signature $(n, d-n)$ and is integer valued, \mathfrak{p} is a characteristic vector satisfying $\mathfrak{k} * (\mathfrak{k} + \mathfrak{p}) = 0 \pmod{2}$ for $\forall \mathfrak{k} \in \mathbb{A}$, $\mu \in \mathbb{A}^*/\mathbb{A}$, and $\mathfrak{z} = \omega - \tau\beta \in \mathbb{C}^d$ with $\omega, \beta \in \mathbb{R}^d$. (We follow the convention to denote d -dimensional quantities by blackboard letters.) The Vignéras theorem [68] asserts that if the kernel $\Phi(\mathfrak{x})$ satisfies suitable decay properties as well as the following differential equation

$$\left[\partial_{\mathfrak{x}}^2 + 2\pi(\mathfrak{x} * \partial_{\mathfrak{x}} - \lambda) \right] \Phi(\mathfrak{x}) = 0, \tag{C.2}$$

where λ is an integer parameter, then the theta series is a vector valued (multi-variable) Jacobi form²⁷ with the following modular properties:

$$\begin{aligned} w(\vartheta) &= \left(\frac{1}{2}(d + \lambda), -\frac{1}{2}\lambda \right), & m(\vartheta) &= -\frac{1}{2}*, \\ M_{\mu\nu}^{(\vartheta)}(T) &= e^{-\pi i(\mu + \frac{1}{2}\mathfrak{p})^2} \delta_{\mu\nu}, & M_{\mu\nu}^{(\vartheta)}(S) &= \frac{e^{(2n-d)\frac{\pi i}{4}}}{\sqrt{|\mathbb{A}^*/\mathbb{A}|}} e^{\frac{\pi i}{2}\mathfrak{p}^2} e^{2\pi i\mu*\nu}, \end{aligned} \quad (\text{C.3})$$

where by $*$ in the formula for the index we mean the matrix representing the bilinear form. The multiplier system here forms the Weil representation of the modular group defined by the lattice \mathbb{A} . A particularly interesting case is when the multi-variable Jacobi form is reduced to the usual Jacobi form by choosing $z = \Theta z$ where $\Theta \in \mathbb{A}$. Then the index is a scalar and is given by

$$m(\vartheta) = -\frac{1}{2}\Theta^2. \quad (\text{C.4})$$

C.2 Unary theta series

Let us specialize (C.1) to the case where $d = 1$, $n = 0$ and $\mathbb{A} = m\mathbb{Z}$ so that the bilinear form is $\mathfrak{x} * \mathfrak{y} = -\mathfrak{x}\mathfrak{y}/m$. We also take $\mathfrak{p} = -mp$ where p is odd for odd m and arbitrary integer otherwise, $z = -mz$ and $\Phi = 1$ (hence $\lambda = 0$). Then the theta series reduces to

$$\vartheta_{\mu}^{(m,p)}(\tau, z) = \sum_{k \in \mathbb{Z} + \frac{\mu}{m} + \frac{p}{2}} (-1)^{mpk} q^{mk^2/2} y^{mk}, \quad (\text{C.5})$$

where we introduced $y = e^{2\pi iz}$. Its modular properties follow from (C.3) and are given by

$$\begin{aligned} w(\vartheta^{(m,p)}) &= 1/2, & m(\vartheta^{(m,p)}) &= m/2, \\ M_{\mu\nu}^{(m,p)}(T) &= e^{\frac{\pi i}{m}(\mu + \frac{mp}{2})^2} \delta_{\mu\nu}, & M_{\mu\nu}^{(m,p)}(S) &= \frac{e^{-\frac{\pi i}{2}mp^2}}{\sqrt{im}} e^{-2\pi i\frac{\mu\nu}{m}}. \end{aligned} \quad (\text{C.6})$$

For even $m = 2\kappa$, we can choose $p = 0$. Then (C.5) gives

$$\theta_{\mu}^{(\kappa)}(\tau, z) \equiv \vartheta_{\mu}^{(2\kappa,0)}(\tau, z) = \sum_{k \in 2\kappa\mathbb{Z} + \mu} q^{\frac{k^2}{4\kappa}} y^{2\kappa k}. \quad (\text{C.7})$$

If $z = 0$, we will simply drop the last argument and write $\theta_{\mu}^{(\kappa)}(\tau)$. The multiplier system (C.6) reduces to

$$M_{\mu\nu}^{(\theta^{(\kappa)})}(T) = e^{\frac{\pi i}{2\kappa}\mu^2} \delta_{\mu\nu}, \quad M_{\mu\nu}^{(\theta^{(\kappa)})}(S) = \frac{1}{\sqrt{2i\kappa}} e^{-\frac{\pi i}{\kappa}\mu\nu}. \quad (\text{C.8})$$

On the other hand, specifying $m = p = 1$ in (C.5), we reproduce the standard Jacobi theta function

$$\theta_1(\tau, z) = \vartheta_0^{(1,1)}(\tau, z) = \sum_{k \in \mathbb{Z} + \frac{1}{2}} q^{k^2/2} (-y)^k \quad (\text{C.9})$$

²⁷More precisely, the elliptic transformation (A.1a) can generate an additional sign factor $(-1)^{\mathfrak{p}*(\mathfrak{a}+\mathfrak{b})}$.

whose modular properties are

$$\begin{aligned} w(\theta_1) &= 1/2, & m(\theta_1) &= 1/2, \\ M^{(\theta_1)}(T) &= e^{\pi i/4}, & M^{(\theta_1)}(S) &= e^{-3\pi i/4}. \end{aligned} \tag{C.10}$$

It has the following expansion around $z = 0$

$$\theta_1(\tau, z) = -2\pi\eta(\tau)^3 z - 4\pi^2 i \eta'(\tau) \eta(\tau)^2 z^3 + O(z^5). \tag{C.11}$$

C.3 Convergence of indefinite theta series

Let us now consider theta series with a quadratic form of indefinite signature. In this case the kernel $\Phi(\mathbf{x})$ cannot be trivial anymore since otherwise the theta series would be divergent. On the other hand, a non-trivial kernel would spoil holomorphicity in τ unless $\Phi(\mathbf{x})$ is a piece-wise constant function.²⁸ Thus, the only way to get a convergent and holomorphic theta series is to take $\Phi(\mathbf{x})$ to be a combination of sign functions. The following theorem from [51] (generalizing results of [41, 54, 69]) provides the simplest choice of such kernel

Theorem C.1. *Let the signature of the quadratic form be $(n, d - n)$ and*

$$\Phi(\mathbf{x}) = \prod_{i=1}^n \left(\text{sgn}(\mathbb{v}_{1,i} * \mathbf{x}) - \text{sgn}(\mathbb{v}_{2,i} * \mathbf{x}) \right). \tag{C.12}$$

Then the theta series (C.1) is convergent provided:

1. for all $i \in \mathcal{I}_n = \{1, \dots, n\}$, $\mathbb{v}_{1,i}^2, \mathbb{v}_{2,i}^2 \geq 0$;
2. for any subset $\mathcal{I} \subseteq \mathcal{I}_n$ and any set of $s_i \in \{1, 2\}$, $i \in \mathcal{I}$,

$$\Delta_{\mathcal{I}}(\{s_i\}) \equiv \det_{i,j \in \mathcal{I}} (\mathbb{v}_{s_i,i} * \mathbb{v}_{s_j,j}) \geq 0; \tag{C.13}$$

3. for all $\ell \in \mathcal{I}_n$ and any set of $s_i \in \{1, 2\}$, $i \in \mathcal{I}_n \setminus \{\ell\}$,

$$\mathbb{v}_{1,\ell \perp \{s_i\}} * \mathbb{v}_{2,\ell \perp \{s_i\}} > 0, \tag{C.14}$$

where $\perp_{\{s_i\}}$ denotes the projection on the subspace orthogonal to the span of $\{\mathbb{v}_{s_i,i}\}_{i \in \mathcal{I}_n \setminus \{\ell\}}$;

4. if $\mathbb{v}_{s,i}^2 = 0$, then $\exists \alpha_{s,i} \in \mathbb{R}$ such that $\alpha_{s,i} \mathbb{v}_{s,i} \in \mathbb{A}$.

Note that the last condition requiring that the (rescaled) null vectors, i.e. satisfying $\mathbb{v}_{s,i}^2 = 0$, that appear in the definition of the kernel belong to the lattice is important. If such a null vector is present, it is also important to keep the elliptic variable z generic because the theta series has poles at the points where $\exists \mathbb{k} \in \mathbb{A} + \mu + \frac{1}{2} \mathbb{p}$ such that $\mathbb{v}_{s,i} * (\mathbb{k} + \beta) = 0$. In particular, theta series involving null vectors are typically divergent in the limit $z \rightarrow 0$.

²⁸It is possible also to multiply it by a homogeneous polynomial in \mathbf{x} since the non-holomorphic dependence can then be canceled by multiplying by a power of τ_2 .

D. Functions determining completions

D.1 Generalized error functions

In section C.3, we provided a class of functions $\Phi(\mathbf{x})$ that define convergent and holomorphic indefinite theta series. However, in contrast to the usual theta series with negative²⁹ definite quadratic form, they are not modular. This can be seen, for example, from the fact that the discontinuities of the signs spoil the Vignéras equation (C.3). Nevertheless, there is a simple recipe to construct their modular completions [52, 40, 41].

This is achieved with help of the generalized error functions introduced in [40, 41] (see also [70]). They are defined by

$$E_n(\mathcal{M}; \mathbf{u}) = \int_{\mathbb{R}^n} d\mathbf{u}' e^{-\pi \sum_{i=1}^n (u_i - u'_i)^2} \prod_{i=1}^n \text{sgn}(\mathcal{M}^{\text{tr}} \mathbf{u}')_i, \quad (\text{D.1})$$

where $\mathbf{u} = (u_1, \dots, u_n)$ is n -dimensional vector and \mathcal{M} is $n \times n$ matrix of parameters. To get kernels of indefinite theta series, we need however functions depending on a d -dimensional vector rather than n -dimensional one. To define such functions, let \mathcal{V} be $d \times n$ matrix which can be viewed as a collection of n vectors, $\mathcal{V} = (\mathbb{v}_1, \dots, \mathbb{v}_n)$, and it is assumed that these vectors span a positive definite subspace in \mathbb{R}^d endowed with the quadratic form $*$, i.e. $\mathcal{V}^{\text{tr}} * \mathcal{V}$ is positive definite. We also introduce a $n \times d$ matrix \mathcal{B} whose rows define an orthonormal basis for this subspace. Then we set

$$\Phi_n^E(\mathcal{V}; \mathbf{x}) = E_n(\mathcal{B} * \mathcal{V}; \mathcal{B} * \mathbf{x}). \quad (\text{D.2})$$

The detailed properties of these functions can be found in [41]. Most importantly, they do not depend on \mathcal{B} , solve the Vignéras equation (C.2) with $\lambda = 0$ and at large \mathbf{x} reduce to $\prod_{i=1}^n \text{sgn}(\mathbb{v}_i * \mathbf{x})$. Thus, to construct a completion of the theta series whose kernel is a combination of sign functions, it is sufficient to replace each product of n sign functions by Φ_n^E with matrix of parameters \mathcal{V} given by the corresponding vectors \mathbb{v}_i .

Finally, if one of the vectors is null, it reduces the rank of the generalized error function. Namely, for $\mathbb{v}_\ell^2 = 0$, one has

$$\Phi_n^E(\{\mathbb{v}_i\}; \mathbf{x}) = \text{sgn}(\mathbb{v}_\ell * \mathbf{x}) \Phi_{n-1}^E(\{\mathbb{v}_i\}_{i \in \mathcal{Z}_n \setminus \{\ell\}}; \mathbf{x}). \quad (\text{D.3})$$

In other words, for such vectors the completion is not required.

D.2 Coefficients \mathcal{R}_n

The generalized error functions defined in the previous subsection play the role of building blocks in the definition of the coefficients $\mathcal{R}_n(\hat{\gamma}; \tau_2)$ appearing in (2.18). Here we provide their expression found in [23].

²⁹Usually, the convergent case corresponds to positive definite quadratic forms. In our conventions it is negative due to the minus sign in the power of q in (C.1). See also footnote 11.

The construction proceeds in two steps. At the first step, we introduce functions of τ_2 parametrized by n reduced charges $\hat{\gamma}_i = (r_i, q_i)$. To this end, let \mathbb{T}_n^ℓ be the set of unrooted labelled trees with n vertices decorated by charges from the set $\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_n)$. Given a tree $\mathcal{T} \in \mathbb{T}_n^\ell$, we denote the set of its edges by $E_{\mathcal{T}}$, the set of vertices by $V_{\mathcal{T}}$, the source and target vertex³⁰ of an edge e by $s(e)$ and $t(e)$, respectively, and the two disconnected trees obtained from \mathcal{T} by removing the edge e by \mathcal{T}_e^s and \mathcal{T}_e^t . Furthermore, to each edge we assign the vector

$$\mathbf{v}_e = \sum_{i \in V_{\mathcal{T}_e^s}} \sum_{j \in V_{\mathcal{T}_e^t}} \mathbf{v}_{ij}, \quad (\text{D.4})$$

where \mathbf{v}_{ij} are n -dimensional vectors with the following components

$$(\mathbf{v}_{ij})_k = \delta_{ki} r_j - \delta_{kj} r_i. \quad (\text{D.5})$$

Using these notations, we define

$$\mathcal{E}_n(\hat{\gamma}; \tau_2) = \frac{\Phi_n^{\mathcal{E}}(\sqrt{2\tau_2} \mathbf{q})}{(\sqrt{2\tau_2})^{n-1}}, \quad (\text{D.6})$$

where $\mathbf{q} = \left(\frac{q_1}{\kappa r_1}, \dots, \frac{q_n}{\kappa r_n} \right)$,

$$\Phi_n^{\mathcal{E}}(\mathbf{x}) = \frac{1}{n!} \sum_{\mathcal{T} \in \mathbb{T}_n^\ell} \left[\prod_{e \in E_{\mathcal{T}}} \mathcal{D}(\mathbf{v}_{s(e)t(e)}, \mathbf{y}) \right] \Phi_{n-1}^E(\{\mathbf{v}_e\}; \mathbf{x}) \Big|_{\mathbf{y}=\mathbf{x}}, \quad (\text{D.7})$$

and

$$\mathcal{D}(\mathbf{v}, \mathbf{y}) = \mathbf{v} \cdot \left(\mathbf{y} + \frac{1}{2\pi} \partial_{\mathbf{x}} \right). \quad (\text{D.8})$$

The dot in (D.8) denotes the bilinear form

$$\mathbf{x} \cdot \mathbf{y} = \kappa \sum_{i=1}^n r_i x_i y_i. \quad (\text{D.9})$$

In particular, this implies that $\mathbf{v}_{ij} \cdot \mathbf{q} = \gamma_{ij}$.

Importantly, each function $\mathcal{E}_n(\hat{\gamma}; \tau_2)$ defined by (D.6) has a canonical decomposition

$$\mathcal{E}_n(\hat{\gamma}; \tau_2) = \mathcal{E}_n^{(0)}(\hat{\gamma}) + \mathcal{E}_n^{(+)}(\hat{\gamma}; \tau_2), \quad (\text{D.10})$$

where the first term $\mathcal{E}_n^{(0)}$ does not depend on τ_2 , whereas the second term $\mathcal{E}_n^{(+)}$ is exponentially suppressed as $\tau_2 \rightarrow \infty$ keeping the charges $\hat{\gamma}_i$ fixed. In [23] it was shown that

$$\mathcal{E}_n^{(0)}(\hat{\gamma}) = \frac{1}{n!} \sum_{\mathcal{T} \in \mathbb{T}_n^\ell} S_{\mathcal{T}}(\hat{\gamma}) \prod_{e \in E_{\mathcal{T}}} \gamma_{s(e)t(e)}, \quad (\text{D.11})$$

where

$$S_{\mathcal{T}}(\hat{\gamma}) = \sum_{\mathcal{J} \subseteq E_{\mathcal{T}}} e_{\mathcal{T}_{\mathcal{J}}} \prod_{e \in \mathcal{J}} \delta_{\Gamma_e} \prod_{e \in E_{\mathcal{T}} \setminus \mathcal{J}} \text{sgn}(\Gamma_e), \quad \Gamma_e = \sum_{i \in V_{\mathcal{T}_e^s}} \sum_{j \in V_{\mathcal{T}_e^t}} \gamma_{ij}, \quad (\text{D.12})$$

³⁰The orientation of edges on a given tree can be chosen arbitrarily, the final result does not depend on this choice.

Here $\mathcal{T}_{\mathcal{J}}$ denotes the tree obtained from \mathcal{T} by contracting the edges $e \in E_{\mathcal{T}} \setminus \mathcal{J}$ and $e_{\mathcal{T}}$ are some rational numbers depending only on topology of \mathcal{T} . In particular, they vanish for trees with even number of vertices, $e_{\bullet} = 1$, $e_{\bullet\bullet\bullet} = 1/3$, and an iterative formula for generic tree can be found in [23]. If all Γ_e are non-vanishing, $S_{\mathcal{T}}$ reduces to the product of their signs. This shows that the functions \mathcal{E}_n have a meaning of kernels providing completions for holomorphic theta series constructed from signs of Dirac products of charges.

At the second step, we introduce another type of trees, the so-called Schröder trees. They are defined as rooted planar trees such that all vertices $v \in V_T$ (the set of vertices of T excluding the leaves) have $k_v \geq 2$ children. The set of such trees with n leaves will be denoted by \mathbb{T}_n^S . Besides, we take n_T to be the number of elements in V_T and v_0 to denote the root vertex. The vertices of T are labelled by charges so that the leaves carry charges $\hat{\gamma}_i$, whereas the charges assigned to other vertices are given recursively by the sum of charges of their children, $\hat{\gamma}_v \in \sum_{v' \in \text{Ch}(v)} \hat{\gamma}_{v'}$. Then, given a Schröder tree T , we set $\mathcal{E}_v \equiv \mathcal{E}_{k_v}(\{\hat{\gamma}_{v'}\})$ (and similarly for $\mathcal{E}_v^{(0)}, \mathcal{E}_v^{(+)}$) where $v' \in \text{Ch}(v)$ runs over the k_v children of the vertex v . In terms of these notations, the coefficients \mathcal{R}_n are given by

$$\mathcal{R}_n(\hat{\gamma}; \tau_2) = \frac{1}{2^{n-1}} \sum_{T \in \mathbb{T}_n^S} (-1)^{n_T-1} \mathcal{E}_{v_0}^{(+)} \prod_{v \in V_T \setminus \{v_0\}} \mathcal{E}_v^{(0)}. \quad (\text{D.13})$$

D.3 Coefficients $\mathcal{R}_n^{\text{ref}}$

The refined version of the coefficients \mathcal{R}_n has been introduced in [35]. It is given by the same sum over Schröder trees as in (D.13),

$$\mathcal{R}_n^{\text{ref}}(\hat{\gamma}; \tau_2, \beta) = \frac{1}{2^{n-1}} \sum_{T \in \mathbb{T}_n^S} (-1)^{n_T-1} \mathcal{E}_{v_0}^{(+)\text{ref}} \prod_{v \in V_T \setminus \{v_0\}} \mathcal{E}_v^{(0)\text{ref}}, \quad (\text{D.14})$$

but now with the weights assigned to vertices determined by new functions $\mathcal{E}_n^{(\text{ref})}(\hat{\gamma}; \tau_2, \beta)$. Although they depend on an additional parameter β , they are actually much simpler than their unrefined analogues \mathcal{E}_n because in their definition there is no any sum over trees. Namely, they are given by

$$\mathcal{E}_n^{(\text{ref})}(\hat{\gamma}; \tau_2, \beta) = \Phi_{n-1}^E(\{\mathbf{v}_\ell\}; \sqrt{2\tau_2}(\mathbf{q} + \beta\boldsymbol{\theta})), \quad (\text{D.15})$$

where

$$\mathbf{v}_\ell = \sum_{i=1}^{\ell} \sum_{j=\ell+1}^n \mathbf{v}_{ij}, \quad \boldsymbol{\theta} = \sum_{i < j} \mathbf{v}_{ij}. \quad (\text{D.16})$$

As in the unrefined case, $\mathcal{E}_n^{(+)\text{ref}} = \mathcal{E}_n^{(\text{ref})} - \mathcal{E}_n^{(0)\text{ref}}$, while $\mathcal{E}_n^{(0)\text{ref}}$ is the large τ_2 limit of $\mathcal{E}_n^{(\text{ref})}$. However, before taking the limit, one should first set $\beta = 0$, i.e.

$$\mathcal{E}_n^{(0)\text{ref}}(\hat{\gamma}) \equiv \lim_{\tau_2 \rightarrow \infty} \mathcal{E}_n^{(\text{ref})}(\{\tilde{\gamma}_i\}; \tau_2, 0) = S_{\mathcal{T}_{\text{lin}}}(\hat{\gamma}), \quad (\text{D.17})$$

where $\mathcal{T}_{\text{lin}} = \bullet - \bullet - \dots - \bullet - \bullet$ is the simplest linear tree. The last relation follows from the observation that the vectors \mathbf{v}_ℓ can be seen as the vectors (D.4) assigned to edges of \mathcal{T}_{lin} . Note that for the linear tree $e_{\mathcal{T}_{\text{lin}}} = \delta_{n-1}^{(2)}/n$ where n is the number of vertices.

D.4 Explicit expressions

Let us now compute explicitly the functions $R_{\mu,\mu}^{(r)\text{ref}}$ (5.2) for $n = 2$ and 3.

D.4.1 Two charges

In this case we have

$$R_{\mu,\mu_1,\mu_2}^{(r_1,r_2)\text{ref}}(\tau, \bar{\tau}, z) = \frac{1}{2} \sum_{q_1+q_2=\mu+\frac{\kappa r}{2}} \left(\mathcal{R}_2^{\text{ref}}(\check{\gamma}_1, \check{\gamma}_2) y^{\gamma_{12}} + \mathcal{R}_2^{\text{ref}}(\check{\gamma}_2, \check{\gamma}_1) y^{-\gamma_{12}} \right) q^{\frac{1}{2}Q_2(\check{\gamma}_1, \check{\gamma}_2)}, \quad (\text{D.18})$$

where, using the restriction on $q_1 + q_2$, one easily finds from (2.11) that

$$Q_2(\hat{\gamma}_1, \hat{\gamma}_2) = -\frac{\gamma_{12}^2}{\kappa r r_1 r_2}, \quad (\text{D.19})$$

while the function $\mathcal{R}_2^{\text{ref}}$ follows from the definitions in §D.3 to be

$$\mathcal{R}_2^{\text{ref}}(\check{\gamma}_1, \check{\gamma}_2) = \frac{1}{2} \mathcal{E}_2^{(+)}(\check{\gamma}_1, \check{\gamma}_2) = \frac{1}{2} \left[E_1 \left(\frac{\sqrt{2\tau_2} (\gamma_{12} + \kappa r r_1 r_2 \beta)}{\sqrt{\kappa r r_1 r_2}} \right) - \text{sgn}(\gamma_{12}) \right]. \quad (\text{D.20})$$

Note also that the function E_1 coincides with the usual error function: $E_1(u) = \text{Erf}(\sqrt{\pi} u)$.

Next, let us rewrite the sum over electric charges in (D.18) as an unconstrained sum. Upon substituting the spectral flow decomposition (2.14), the condition on the sum of charges becomes

$$r_1 \epsilon_1 + r_2 \epsilon_2 = \Delta \mu / \kappa. \quad (\text{D.21})$$

Let us define $r_0 = \text{gcd}(r_1, r_2)$, $\hat{r}_i = r_i / r_0$ and $\hat{r} = r / r_0$. Then the condition (D.21) is solvable for $\epsilon_i \in \mathbb{Z}$ only if $\Delta \mu = 0 \pmod{\kappa r_0}$. If this is the case, let ρ_i be integers such that $\hat{r}_1 \rho_1 + \hat{r}_2 \rho_2 = 1$. Then a general solution to (D.21) is given by

$$\epsilon_1 = \frac{\Delta \mu}{\kappa r_0} \rho_1 + \hat{r}_2 \ell, \quad \epsilon_2 = \frac{\Delta \mu}{\kappa r_0} \rho_2 - \hat{r}_1 \ell, \quad \ell \in \mathbb{Z}. \quad (\text{D.22})$$

Using this solution in the formula for the Dirac product (2.10) after plugging there the spectral flow decomposition, one finds that

$$\gamma_{12} = r_0 (2\kappa_{12} \ell + \mu_{12}), \quad (\text{D.23})$$

where we introduced

$$\begin{aligned} \kappa_{12} &= \frac{1}{2} \kappa r \hat{r}_1 \hat{r}_2, \\ \mu_{12} &= \hat{r}_2 \mu_1 - \hat{r}_1 \mu_2 + \hat{r}_1 \hat{r}_2 (\rho_1 - \rho_2) \Delta \mu \\ &= \hat{r} \mu_1 - \hat{r}_1 \mu + \hat{r} \hat{r}_1 \rho_1 \Delta \mu. \end{aligned} \quad (\text{D.24})$$

Thus, we arrive at the following result

$$R_{\mu,\mu_1,\mu_2}^{(r_1,r_2)\text{ref}} = \frac{1}{4} \delta_{\Delta \mu}^{(\kappa r_0)} \sum_{\sigma=\pm 1} \sum_{k \in \mathbb{Z} + \frac{\sigma \mu_{12}}{2\kappa_{12}}} \left[E_1(2\sqrt{\kappa_{12} \tau_2} (k + r_0 \beta)) - \text{sgn}(k) \right] q^{-\kappa_{12} k^2} y^{2r_0 \kappa_{12} k}. \quad (\text{D.25})$$

The corresponding holomorphic anomaly is found to be³¹

$$\partial_{\bar{\tau}} R_{\mu, \mu_1, \mu_2}^{(r_1, r_2)\text{ref}} = \frac{i}{4} \sqrt{\frac{\kappa_{12}}{\tau_2}} \delta_{\Delta\mu}^{(\kappa r_0)} \sum_{\sigma=\pm 1} \sum_{k \in \mathbb{Z} + \frac{\sigma \mu_{12}}{2\kappa_{12}}} (k - r_0\beta) (\text{q}\bar{\text{q}})^{\kappa_{12}(k+r_0\beta)^2} \text{q}^{-\kappa_{12}k^2} y^{2r_0\kappa_{12}k}. \quad (\text{D.26})$$

The unrefined function $\tilde{R}_{\mu, \mu_1, \mu_2}^{(r_1, r_2)}$ can be computed either from (2.18) and equations in §D.2, or by taking the unrefined limit of (D.25) and using (5.3). In this paper we will not use the resulting function, which has a form similar to (D.25) but with a little bit more complicated summand. Instead, we give here its holomorphic anomaly, which turns out to be simpler than its refined counterpart (D.26) and is proportional to the complex conjugate of the theta series $\theta_{\mu}^{(\kappa)}(\tau)$ introduced in (C.7) and evaluated at $z = 0$:

$$\partial_{\bar{\tau}} \tilde{R}_{\mu, \mu_1, \mu_2}^{(r_1, r_2)}(\tau, \bar{\tau}) = \frac{r_0 \sqrt{\kappa_{12}}}{16\pi i \tau_2^{3/2}} \delta_{\Delta\mu}^{(\kappa r_0)} \overline{\theta_{\mu_{12}}^{(\kappa_{12})}(\tau)}. \quad (\text{D.27})$$

Note that $\partial_{\bar{\tau}} \tilde{R}_{\mu, \mu_1, \mu_2}^{(r_1, r_2)}$ must be a modular form of weight $(3/2, 2)$ with the same multiplier system as $g_{\mu, \mu_1, \mu_2}^{(r_1, r_2)}$ given in (B.3). Therefore, the result (D.27) immediately implies

Proposition D.1. *If $G_{\mu}^{(\kappa)}$ ($\mu = 0, \dots, 2\kappa - 1$) transforms with the multiplier system*

$$M_{\mu\nu}^{(\kappa)}(T) = e^{-\frac{\pi i}{2\kappa} \mu^2} \delta_{\mu\nu}, \quad M_{\mu\nu}^{(\kappa)}(S) = \frac{e^{\frac{\pi i}{4}}}{\sqrt{2\kappa}} e^{\frac{\pi i}{\kappa} \mu\nu}, \quad (\text{D.28})$$

which is the complex conjugate of the multiplier system (C.8) of $\theta_{\mu}^{(\kappa)}$, then $\delta_{\Delta\mu}^{(\kappa r_0)} G_{\mu_{12}}^{(\kappa_{12})}$ transforms with the multiplier system (B.3) specified for $n = 2$.

D.4.2 Three charges

In this case we have

$$R_{\mu, \mu}^{(r)\text{ref}}(\tau, \bar{\tau}, z) = \sum_{\sum_{i=1}^3 q_i = \mu + \kappa r / 2} \text{Sym} \left\{ \mathcal{R}_3^{\text{ref}}(\hat{\gamma}; \tau_2, \beta) y^{\gamma_{1+2,3} + \gamma_{12}} \right\} e^{\pi i r Q_3(\hat{\gamma})}, \quad (\text{D.29})$$

where the quadratic form can be written as

$$Q_3(\hat{\gamma}) = -\frac{r_1 \gamma_{23}^2 + r_2 \gamma_{13}^2 + r_3 \gamma_{12}^2}{\kappa r r_1 r_2 r_3}, \quad (\text{D.30})$$

while the coefficient $\mathcal{R}_3^{\text{ref}}$ follows from (D.14) and is given by the sum of three Schröder trees resulting to

$$\begin{aligned} \mathcal{R}_3^{\text{ref}}(\hat{\gamma}; \tau_2, \beta) &= \frac{1}{4} \left[\mathcal{E}_3^{(+)\text{ref}}(\hat{\gamma}) - \mathcal{E}_2^{(+)\text{ref}}(\hat{\gamma}_{1+2}, \hat{\gamma}_3) \mathcal{E}_2^{(0)\text{ref}}(\hat{\gamma}_1, \hat{\gamma}_2) \right. \\ &\quad \left. - \mathcal{E}_2^{(+)\text{ref}}(\hat{\gamma}_1, \hat{\gamma}_{2+3}) \mathcal{E}_2^{(0)\text{ref}}(\hat{\gamma}_2, \hat{\gamma}_3) \right]. \end{aligned} \quad (\text{D.31})$$

³¹To obtain the holomorphic anomaly, one takes the derivative with respect to $\bar{\tau}$ while keeping fixed τ , y and \bar{y} (and not α , β).

The function $\mathcal{E}_2^{(+)\text{ref}}$ was computed in (D.20) which implies that $\mathcal{E}_2^{(0)\text{ref}}(\hat{\gamma}_1, \hat{\gamma}_2) = \text{sgn}(\gamma_{12})$. Finally, $\mathcal{E}_3^{(+)\text{ref}}$ is found to be

$$\mathcal{E}_3^{(+)\text{ref}}(\hat{\gamma}) = \Phi_2^E(\mathbf{v}_{1,2+3}, \mathbf{v}_{1+2,3}; \mathbf{x}) - \text{sgn}(\gamma_{1,2+3}) \text{sgn}(\gamma_{1+2,3}) - \frac{1}{3} \delta_{\gamma_{1,2+3}} \delta_{\gamma_{1+2,3}}, \quad (\text{D.32})$$

where $\mathbf{x} = \sqrt{2\tau_2}(\mathbf{q} + \beta\boldsymbol{\theta})$ and $\Phi_2^E(\mathbf{x})$ can be expressed through the generalized error function E_2 . Substituting these results into (D.31), one obtains

$$\begin{aligned} \mathcal{R}_3^{\text{ref}}(\hat{\gamma}; \tau_2, \beta) = & \frac{1}{4} \left[\Phi_2^E(\mathbf{v}_{1,2+3}, \mathbf{v}_{1+2,3}; \mathbf{x}) - \text{sgn}(\gamma_{1,2+3}) \text{sgn}(\gamma_{1+2,3}) - \frac{1}{3} \delta_{\gamma_{1,2+3}} \delta_{\gamma_{1+2,3}} \right. \\ & - \left(E_1 \left(\frac{\sqrt{2\tau_2}(\gamma_{1+2,3} + \kappa r r_{1+2} r_3 \beta)}{\sqrt{\kappa r r_{1+2} r_3}} \right) - \text{sgn}(\gamma_{1+2,3}) \right) \text{sgn}(\gamma_{12}) \\ & \left. - \left(E_1 \left(\frac{\sqrt{2\tau_2}(\gamma_{1,2+3} + \kappa r r_1 r_{2+3} \beta)}{\sqrt{\kappa r r_1 r_{2+3}}} \right) - \text{sgn}(\gamma_{1,2+3}) \right) \text{sgn}(\gamma_{23}) \right]. \end{aligned} \quad (\text{D.33})$$

where $r_{i+j} = r_i + r_j$.

E. Hecke-like operators

In this appendix we define two operators acting on Jacobi forms and then derive their induced action on modular forms appearing as coefficients in the theta expansion, which for a (mock) Jacobi form of index m reads [62]

$$\varphi(\tau, z) = \sum_{\mu=0}^{2m-1} h_{\mu}^{(m)}(\tau) \theta_{\mu}^{(m)}(\tau, z), \quad (\text{E.1})$$

where $\theta_{\mu}^{(m)}$ is the index m theta function (C.7). Although this will not be needed in this paper, we will not assume that φ is holomorphic in τ , but will omit the complex conjugate argument.

The first Hecke-like operator is defined by a simple rescaling of the elliptic argument [25, Eq.(4.36)]:

$$(U_s[\varphi])(\tau, z) = \varphi(\tau, sz). \quad (\text{E.2})$$

The action on the modular forms follows from the following property of the theta functions

$$\theta_{\mu}^{(m)} \left(\frac{a\tau + b}{d}, az \right) = \sum_{\nu=0}^{d-1} e^{\frac{\pi i b}{2m d} (\mu + 2m\nu)^2} \theta_{a(\mu + 2m\nu)}^{(adm)}(\tau, z) \quad (\text{E.3})$$

valid for $a, d \in \mathbb{N}$ and $b \in \mathbb{Z}$. In particular, choosing $a = d = s$ and $b = 0$, one obtains

$$\theta_{\mu}^{(m)}(\tau, sz) = \sum_{\nu=0}^{s-1} \theta_{s(\mu + 2m\nu)}^{(ms^2)}(\tau, z). \quad (\text{E.4})$$

This result immediately implies that the action of U_s on $h_{\mu}^{(m)}$ is given by (4.8).

The second operator is defined by [46, Def. 3.1]³²

$$(\mathcal{T}_r[\varphi])(\tau, z) = \sum_{\substack{a, d > 0 \\ ad=r}} \left(\frac{\sqrt{r}}{d} \right)^{w+\bar{w}} \sum_{b=0}^{d-1} \varphi \left(\frac{a\tau + b}{d}, az \right), \quad (\text{E.5})$$

where (w, \bar{w}) is the weight of the Jacobi form. To compare with [25], let us restrict to holomorphic Jacobi forms of weight w and perform a Fourier expansion of $\varphi(\tau, z)$ in both variables

$$\varphi(\tau, z) = \sum_{n, m} c(n, m) \mathfrak{q}^n y^m, \quad \mathfrak{q} = e^{2\pi i \tau}, \quad y = e^{2\pi i z}. \quad (\text{E.6})$$

Then, using the identity

$$d^{-1} \sum_{b=0}^{d-1} e^{2\pi i s b/d} = \delta_{n/d}^{(1)}, \quad (\text{E.7})$$

one finds

$$\begin{aligned} (\mathcal{T}_r[\varphi])(\tau, z) &= r^{1-w/2} \sum_{a|r} a^{w-1} \sum_{n, m} \delta_{an/r}^{(1)} c(n, m) \mathfrak{q}^{a^2 n/r} y^{am} \\ &= r^{1-w/2} \sum_{a|r} a^{w-1} \sum_{n, m} c\left(\frac{rn}{a}, m\right) \mathfrak{q}^{an} y^{am}. \end{aligned} \quad (\text{E.8})$$

As a result, we arrive at the following action on the Fourier coefficients

$$\mathcal{T}_r : c(n, m) \mapsto r^{1-w/2} \sum_{d|(n, m, r)} d^{w-1} c\left(\frac{nr}{d^2}, \frac{m}{d}\right), \quad (\text{E.9})$$

which coincides up to the factor $r^{1-w/2}$ (see footnote 32) with [25, Eq.(4.37)]. The action of (E.5) on the VV modular forms $h_\mu^{(m)}$ follows from Definition 3.8 and Theorem 3.9 of [46] and coincides with the one given in (4.9).³³

F. Lattices, glue vectors and zero modes

In this appendix we present various results about our main lattice $\Lambda^{(r)}$, about the glue vectors appearing in decompositions of $\Lambda^{(r)}$ and the extended lattice $\mathbb{A}^{(r)}$, and an analysis of the set of zero modes. Throughout the appendix we will use the following convenient notations: $\hat{r} = r/r_0$, $\hat{r}_i = r_i/r_0$ and $r_{i_1 \dots i_m} = \gcd(r_{i_1}, \dots, r_{i_m})$. If the indices are consecutive, i.e. $i_k = i_1 + k - 1$ and $i_m = j$, we will use a shorthand notation $r_{i::j}$ instead of $r_{i_1 \dots i_m}$. Note that $r_{1::n} = r_0$.

³²We multiplied the operator defined in [25, 46] by the factor $r^{1-\frac{1}{2}(w+\bar{w})}$ in order to make it commuting with the simple operations changing the weight such as ∂_τ or multiplication by τ_2 , but leaving it intact for $w + \bar{w} = 2$. This does not affect the property

$$\mathcal{T}_r \mathcal{T}_s = \mathcal{T}_{rs}, \quad \text{for } \gcd(r, s) = 1$$

and little bit simplifies the property (cf. [46, Lemma 3.7])

$$\mathcal{T}_{p^l} = \mathcal{T}_p \mathcal{T}_{p^{l-1}} - p U_p \mathcal{T}_{p^{l-2}}, \quad \text{for } l \geq 2 \text{ and } p \text{ prime.}$$

³³Note that w in (4.9) is equal to $w - 1/2$ in terms of the weight appearing in (E.5). Hence the additional normalization factor discussed in footnote 32 now reads $r^{\frac{3}{4}-\frac{1}{2}(w+\bar{w})}$.

F.1 Lattice $\Lambda^{(r)}$

The lattice $\Lambda^{(r)}$ is defined as

$$\Lambda^{(r)} = \left\{ \mathbf{k} \in \mathbb{Z}^n : \sum_{i=1}^n r_i k_i = 0 \right\} \quad (\text{F.1})$$

and carries the bilinear form

$$\mathbf{x} \cdot \mathbf{y} = \kappa \sum_{i=1}^n r_i x_i y_i. \quad (\text{F.2})$$

To describe its dual lattice, let us fix a n -tuple $\boldsymbol{\rho} = (\rho_1, \dots, \rho_n)$ such that $\sum_{i=1}^n \rho_i r_i = r_0$ and introduce

$$\hat{\boldsymbol{\mu}}_i^{(r)}(\boldsymbol{\mu}, \lambda) = \frac{\mu_i}{\kappa r_i} - \frac{\sum_{i=1}^n \mu_i}{\kappa r} + \left(\rho_i - \frac{r_0}{r} \right) \lambda. \quad (\text{F.3})$$

Note that it satisfies $\sum_{i=1}^n r_i \hat{\boldsymbol{\mu}}_i^{(r)} = 0$. Then the dual lattice is the following subset of $\text{Span}(\Lambda^{(r)})$

$$(\Lambda^{(r)})^* = \left\{ \mathbf{k} \in \mathbb{Z}^n + \hat{\boldsymbol{\mu}}^{(r)}(\boldsymbol{\mu}, \lambda) : \sum_{i=1}^n r_i k_i = 0, \mu_i \in \mathbb{Z}_{\kappa r_i}, \lambda \in \mathbb{Z}_{\hat{r}} \right\}. \quad (\text{F.4})$$

Since the overall shift $\mu_i \rightarrow \mu_i + \hat{r}_i$ for all i leaves $\hat{\boldsymbol{\mu}}^{(r)}$ invariant, the discriminant group is

$$\mathbf{D}^{(r)} = \mathbb{Z}_{\hat{r}} \otimes \frac{\prod_{i=1}^n \mathbb{Z}_{\kappa r_i}}{\mathbb{Z}_{\kappa r_0}} \quad (\text{F.5})$$

and hence its order, which coincides with the determinant of the bilinear form (F.2), is given by

$$|\mathbf{D}^{(r)}| = \det \Lambda^{(r)} = \frac{\kappa^{n-1} r}{r_0^2} \prod_{i=1}^n r_i. \quad (\text{F.6})$$

In the special case $\kappa = r_1 = \dots = r_n = 1$, the lattice coincides with the standard A_{n-1} root lattice whose discriminant group is \mathbb{Z}_n .

There are two decompositions of $\Lambda^{(r)}$ which play an important role in our story. One is the decomposition (5.46) on the sublattice generated by the vectors $\hat{\mathbf{v}}_{i,i+1} = \mathbf{v}_{i,i+1}/r_{i,i+1}$, $i = 1, \dots, n-1$. In this case the glue vectors are given by linear combinations of other normalized vectors $\hat{\mathbf{v}}_{ij}$ with $j-i > 1$. The number of values taken by the coefficients of $\hat{\mathbf{v}}_{ij}$ that generate independent glue vectors is given by

$$N_{ij} \equiv \frac{r_{i+1::j-1} r_{i::j}}{r_{i::j-1} r_{i+1::j}}. \quad (\text{F.7})$$

The total number of glue vectors is

$$\prod_{i=1}^{n-2} \prod_{j=i+2}^n N_{ij} = \frac{r_0 \prod_{j=2}^{n-1} r_i}{\prod_{i=1}^{n-1} r_{i,i+1}}, \quad (\text{F.8})$$

which agrees with the formula (5.29).

The other important decomposition corresponds to a split of the set of charges into m subsets induced by a decomposition $n = \sum_{k=1}^m n_k$. This gives rise to the following lattice factorization

$$\mathbf{\Lambda}^{(r)} = \bigcup_{\mathbf{C}=0}^{N_g-1} \left[\left(\mathbf{\Lambda}^{(s)} + \mathbf{g}_{\mathbf{C}}^{(0)} \right) \oplus \left(\bigoplus_{k=1}^m \left(\mathbf{\Lambda}^{(\mathbf{r}_k)} + \mathbf{g}_{\mathbf{C}}^{(k)} \right) \right) \right], \quad (\text{F.9})$$

where we used notations from (3.4). To write the glue vectors, we introduce $s_0 = \text{gcd}(s_1, \dots, s_m)$, $\hat{s}_k = s_k / \text{gcd}(\mathbf{r}_k)$ and

$$\begin{aligned} \boldsymbol{\rho}^{(0)} &= (\rho_1^{(0)[n_1]}, \dots, \rho_m^{(0)[n_m]}), & \sum_{k=1}^m s_k \rho_k^{(0)} &= s_0, \\ \boldsymbol{\rho}^{(k)} &= (0^{[j_k]}, \rho_1^{(k)}, \dots, \rho_{n_k}^{(k)}, 0^{[n-j_{k+1}]}), & \sum_{i=1}^{n_k} r_{j_k+i} \rho_i^{(k)} &= \text{gcd}(\mathbf{r}_k). \end{aligned} \quad (\text{F.10})$$

Then the glue vectors are given by

$$\mathbf{g}_{\mathbf{C}} = \sum_{k=1}^m \mathbf{c}_k \boldsymbol{\rho}^{(k)} - \frac{1}{s_0} \left(\sum_{\ell=1}^m \text{gcd}(\mathbf{r}_k) \mathbf{c}_{\ell} \right) \boldsymbol{\rho}^{(0)}. \quad (\text{F.11})$$

and are labelled by the following set of indices

$$\mathbf{C} = \left\{ (\mathbf{c}_1, \dots, \mathbf{c}_m) : \mathbf{c}_k \in \mathbb{Z}_{\hat{s}_k}, \sum_{k=1}^m \text{gcd}(\mathbf{r}_k) \mathbf{c}_k \in s_0 \mathbb{Z} \right\}. \quad (\text{F.12})$$

Thus, the indices are not free, but should satisfy a constraint. Taking this into account, the total number of glue vectors equals

$$N_g = \frac{r_0}{s_0} \prod_{k=1}^m \hat{s}_k. \quad (\text{F.13})$$

It is useful to note that since $\mathbf{g}_{\mathbf{C}}^{(k)} \in \mathbf{D}^{(\mathbf{r}_k)}$ and $\mathbf{g}_{\mathbf{C}}^{(0)} \in \mathbf{D}^{(s)}$, they should be particular cases of the vector defined in (F.3). And indeed, it is easy to check that

$$\mathbf{g}_{\mathbf{C}}^{(k)} = \hat{\boldsymbol{\mu}}^{(\mathbf{r}_k)}(0, \mathbf{c}_k), \quad \mathbf{g}_{\mathbf{C}}^{(0)} = \hat{\boldsymbol{\mu}}^{(s)}(\boldsymbol{\nu}, -\frac{\sum_{k=1}^m \nu_k}{\kappa s_0}), \quad \text{where } \nu_k = \kappa \text{gcd}(\mathbf{r}_k) \mathbf{c}_k. \quad (\text{F.14})$$

Let us now establish links to the sum over charges appearing in all anomaly equations of this paper. First, we note that the vector $\hat{\boldsymbol{\mu}}$ defined in (5.37) coincides with $\hat{\boldsymbol{\mu}}^{(r)}(\boldsymbol{\mu}, \frac{\Delta\mu}{\kappa r_0})$ from (F.3) where the integer valuedness of the second argument is the usual condition imposed by $\delta_{\Delta\mu}^{(\kappa r_0)}$ (see, e.g., Theorem 5.1) and following from the condition $\sum_{i=1}^n q_i = \mu + \kappa r / 2$ in the sum over D2-brane charges (see, e.g., (5.2)). This makes obvious the fact that $(\mu, \boldsymbol{\mu})$, subject to the above condition and the identification $(\mu, \boldsymbol{\mu}) \simeq (\mu + \hat{r}, \boldsymbol{\mu} + \hat{\mathbf{r}})$, labels elements of the discriminant group $\mathbf{D}^{(r)}$. To see that the whole sum over D2-brane charges corresponds to the sum over $\mathbf{\Lambda}^{(r)}$, it is sufficient to identify

$$q_i = \kappa r_i \left(k_i + \frac{\mu}{\kappa r} + \frac{1}{2} \right) = \kappa r_i \left(\epsilon_i + \frac{\rho_i \Delta\mu}{\kappa r_0} \right) + \mu_i + \frac{1}{2} \kappa r_i, \quad (\text{F.15})$$

where $k_i = \epsilon_i + \hat{\mu}_i^{(r)}(\boldsymbol{\mu}, \frac{\Delta\boldsymbol{\mu}}{\kappa r_0})$, with $\epsilon_i \in \mathbb{Z}$, is the component of the vector \mathbf{k} belonging to the dual lattice (F.4). Due to the constraint on $\Delta\boldsymbol{\mu}$, the expression in the round brackets in the first term is an integer, so that $q_i \in \kappa r_i \mathbb{Z} + \mu_i + \kappa r_i/2$, as required. It is also easy to check that the identification (F.15) maps the quadratic form $-Q_n(\hat{\boldsymbol{\gamma}})$ (2.11) to the one in (F.2).

Finally, we observe that the sum over $\boldsymbol{\nu}$ in the anomaly equation for the anomalous coefficients or their refined version (see, e.g., (5.4)) is nothing but the sum over the glue vectors of the lattice factorization (F.9). To see this explicitly, let us introduce another m -tuple $(\hat{\rho}_1, \dots, \hat{\rho}_m)$ such that $\sum_{k=1}^m \text{gcd}(\mathbf{r}_k) \hat{\rho}_k = r_0$. It allows to take $\boldsymbol{\rho} = \sum_{k=1}^m \hat{\rho}_k \boldsymbol{\rho}^{(k)}$. Then it is straightforward to verify that, identifying $\mathbf{c}_k = \frac{\nu_k - \sum_{i=1}^{n_k} \mu_{j_k+i}}{\kappa \text{gcd}(\mathbf{r}_k)} - \frac{\hat{\rho}_k \Delta\boldsymbol{\mu}}{\kappa r_0}$, one gets the relation

$$\hat{\boldsymbol{\mu}} + \mathbf{g}_C = \hat{\boldsymbol{\mu}}^{(s)}\left(\boldsymbol{\nu}, \frac{\mu - \sum_{k=1}^m \nu_k}{\kappa s_0}\right) + \sum_{k=1}^m \hat{\boldsymbol{\mu}}^{(\mathbf{r}_k)}\left(\boldsymbol{\mu}, \frac{\nu_k - \sum_{i=1}^{n_k} \mu_{j_k+i}}{\kappa \text{gcd}(\mathbf{r}_k)}\right), \quad (\text{F.16})$$

where the two vectors on the r.h.s. belong to $\mathbf{D}^{(s)}$ and $\mathbf{D}^{(\mathbf{r}_k)}$, respectively. This demonstrates that the lattice factorization (F.9) gives rise to factors labelled by $(\boldsymbol{\mu}, \boldsymbol{\nu})$ and (ν_k, \mathbf{m}_k) , as expected. The ranges of summations over ν_k also agree, while the conditions restricting the values of ν_k originating from (F.12) and the above identification, namely

$$\mu - \sum_{k=1}^m \nu_k \in \kappa s_0 \mathbb{Z}, \quad \nu_k - \sum_{i=1}^{n_k} \mu_{j_k+i} \in \kappa \text{gcd}(\mathbf{r}_k) \mathbb{Z}, \quad (\text{F.17})$$

in equations like (5.4) arise from the conditions on the sum over charges defining each of the factors.

Let us apply what we have just shown to theta series. To this end, let

$$\Phi(\mathbf{x}) = \Phi_0(\mathbf{x}^{(0)}) \prod_{k=1}^m \Phi_k(\mathbf{x}^{(k)}), \quad (\text{F.18})$$

where the upper indices (0) and (k) on a vector denote its projections to $\boldsymbol{\Lambda}^{(s)}$ and $\boldsymbol{\Lambda}^{(\mathbf{r}_k)}$, respectively. We also assume that theta series associated to lattices $\boldsymbol{\Lambda}^{(r)}$ and labelled by $(\boldsymbol{\mu}, \boldsymbol{\mu})$ implicitly contain the factor $\delta_{\Delta\boldsymbol{\mu}}^{(\kappa r_0)}$. Then our result implies that

$$\vartheta_{\boldsymbol{\mu}, \boldsymbol{\mu}}(\tau, \boldsymbol{\theta} z; \boldsymbol{\Lambda}^{(r)}, \Phi, 0) = \left(\prod_{k=1}^m \sum_{\nu_k=0}^{\kappa s_k - 1} \right) \vartheta_{\boldsymbol{\mu}, \boldsymbol{\nu}}(\tau, \boldsymbol{\theta}^{(0)} z; \boldsymbol{\Lambda}^{(s)}, \Phi_0, 0) \prod_{k=1}^m \vartheta_{\nu_k, \mathbf{m}_k}(\tau, \boldsymbol{\theta}^{(k)} z; \boldsymbol{\Lambda}^{(\mathbf{r}_k)}, \Phi_k, 0). \quad (\text{F.19})$$

F.2 Basis expansion of glue vectors

Here we obtain an expansion of the glue vectors appearing in the lattice decompositions (5.31) and (5.46) in the basis $\mathbf{e}_\lambda = (\mathbf{e}_0, \mathbf{e}_{i,\alpha}, \hat{\mathbf{v}}_{i,i+1}, \hat{\mathbf{u}}_{i,i+1})$. To this end, we note that the dual basis, i.e. such that $\mathbf{e}_\lambda * \mathbf{e}_{\lambda'}^* = \delta_{\lambda\lambda'}$, is given by

$$\begin{aligned} \mathbf{e}_0^* &= -\frac{1}{d_r} \mathbf{e}_0, & \mathbf{e}_{i,\alpha}^* &= \mathbf{g}_{i,\alpha} - \frac{\alpha}{d_{r_i}} \mathbf{g}_i, \\ \hat{\mathbf{v}}_{i,i+1}^* &= \frac{1}{r_0 \kappa_i} \sum_{k=1}^i \sum_{l=i+1}^n r_{kl} \hat{\mathbf{v}}_{kl}, & \hat{\mathbf{u}}_{i,i+1}^* &= -\frac{1}{4^e r_0 \kappa_i} \sum_{k=1}^i \sum_{l=i+1}^n r_{kl} \hat{\mathbf{u}}_{kl}, \end{aligned} \quad (\text{F.20})$$

where we used the notation κ_i introduced in (5.23). Then evaluating the scalar products of the vectors appearing in (5.34) with the dual basis, one finds the following expansions

$$\begin{aligned} \mathfrak{g}_0 &= \sum_{l=1}^{n-1} \frac{\kappa_l r_0}{\kappa_l} \left(\sum_{j=1}^l \sum_{k=l+1}^n \hat{r}_j \hat{r}_k (\rho_j - \rho_k) \right) \hat{\mathbf{u}}_{l,l+1} + \frac{1}{\hat{r}} \mathbb{e}_0, \\ \mathfrak{g}_{i,\mathbf{a}} &= 4^{-\epsilon} \mathbf{a} \left[\sum_{l=i}^{n-1} \frac{1}{\kappa_l} \left(\sum_{k=l+1}^n \hat{r}_k \right) \hat{\mathbf{u}}_{l,l+1} - \sum_{l=1}^{i-1} \frac{1}{\kappa_l} \left(\sum_{k=1}^l \hat{r}_k \right) \hat{\mathbf{u}}_{l,l+1} \right] + \frac{\mathbf{a}}{d_r} \mathbb{e}_0 \\ &\quad - \frac{1}{d_{r_i}} \left[\sum_{\alpha=1}^{\mathbf{a}} (d_{r_i} - \mathbf{a}) \alpha \mathbb{e}_{i,\alpha} + \sum_{\alpha=\mathbf{a}+1}^{d_{r_i}-1} (d_{r_i} - \alpha) \mathbf{a} \mathbb{e}_{i,\alpha} \right], \end{aligned} \quad (\text{F.21})$$

while for $\hat{\mathbf{v}}_{ij}$ and $\hat{\mathbf{u}}_{ij}$, one gets

$$\hat{\mathbf{v}}_{ij} = \frac{r_i r_j}{r_{ij}} \sum_{l=i}^{j-1} \frac{\kappa_l \hat{r}}{\kappa_l} \hat{\mathbf{v}}_{l,l+1}, \quad \hat{\mathbf{u}}_{ij} = \frac{r_i r_j}{r_{ij}} \sum_{l=i}^{j-1} \frac{\kappa_l \hat{r}}{\kappa_l} \hat{\mathbf{u}}_{l,l+1}. \quad (\text{F.22})$$

F.3 Zero modes for two charges

In the case $n = 2$, (F.21) takes the form

$$\begin{aligned} \mathfrak{g}_0 &= \frac{1}{\hat{r}} ((\rho_1 - \rho_2) \hat{\mathbf{u}}_{12} + \mathbb{e}_0), \\ \mathfrak{g}_{i,\mathbf{a}} &= \frac{\mathbf{a}}{d_r} \left(\frac{(-1)^{i-1}}{\hat{r}_i} \hat{\mathbf{u}}_{12} + \mathbb{e}_0 \right) - \frac{1}{d_{r_i}} \left(\sum_{\alpha=1}^{\mathbf{a}} (d_{r_i} - \mathbf{a}) \alpha \mathbb{e}_{i,\alpha} + \sum_{\alpha=\mathbf{a}+1}^{d_{r_i}-1} (d_{r_i} - \alpha) \mathbf{a} \mathbb{e}_{i,\alpha} \right). \end{aligned} \quad (\text{F.23})$$

which leads to the dependence of the indices of theta series, ν and $\tilde{\nu}$, on the glue vectors given in (5.54).

Let us find a manifest description of the set of glue vectors ensuring the existence of zero modes which is defined as

$$\mathcal{A}_0^{(r)}(\mu_{12}) = \{ \mathbf{A} : 2^\epsilon \tilde{\nu}(\mathbf{A}) - \mu_{12} \in 2\kappa_{12} \mathbb{Z} \}. \quad (\text{F.24})$$

Taking into account (5.54), explicitly the zero mode condition reads

$$2^\epsilon \kappa r_0 \hat{r}_1 \hat{r}_2 (\rho_1 - \rho_2) \mathbf{a}_0 + 2^{-\epsilon} (\hat{r}_2 \mathbf{a}_1 - \hat{r}_1 \mathbf{a}_2) - \mu_{12} \in 2\kappa_{12} \mathbb{Z}. \quad (\text{F.25})$$

In the following, to treat the case $\epsilon = 0$, we will have to distinguish two cases whether \hat{r} is even or not. Therefore, we define $\epsilon' = \epsilon \delta_{\hat{r}}^{(2)}$ and $\epsilon'_i = \epsilon \delta_{\hat{r}_i}^{(2)}$. We then take $\mathbf{a}_i \in [0, d_{r_i} - 1]$ such that

$$\mathbf{a}_i = (-1)^{i-1} 2^{\epsilon+\epsilon'} \kappa r_0 \hat{r}_1 \hat{r}_2 \left\{ \frac{\mu_{12}}{2^{\epsilon'} \kappa r_0 \hat{r}_1 \hat{r}_2} \right\} \rho_{3-i} + \ell \hat{r}_i + \epsilon'_i (1 - \epsilon') 2^{1+\epsilon'} \kappa r_0 \hat{r}_1 \hat{r}_2 \varepsilon \pmod{d_{r_i}}, \quad (\text{F.26})$$

where $\{x\}$ denotes the fractional part of x , $\ell = 0, \dots, 4^\epsilon \kappa r_0 - 1$ and $\varepsilon = 0, 2^\epsilon - 1$. Note that the last term is non-vanishing only for $\kappa = 1$, odd \hat{r} and even \hat{r}_i . We also observe that the ℓ -dependent term cancels in (F.25) and the transformation $\mathbf{a}_i \rightarrow \mathbf{a}_i + d_{r_i}$ induces a shift of $2^{-\epsilon} (\hat{r}_2 \mathbf{a}_1 - \hat{r}_1 \mathbf{a}_2)$ by $\pm 2^\epsilon \kappa r_0 \hat{r}_1 \hat{r}_2$. Therefore, \mathbf{a}_i specified in (F.26) satisfy

$$2^{-\epsilon} (\hat{r}_2 \mathbf{a}_1 - \hat{r}_1 \mathbf{a}_2) = \mu_{12} + 2^{\epsilon'} \kappa r_0 \hat{r}_1 \hat{r}_2 m, \quad m \in \mathbb{Z}. \quad (\text{F.27})$$

Substituting this into (F.25), one reduces the condition to

$$2^{\epsilon-\epsilon'}(\rho_1 - \rho_2)\mathbf{a}_0 - 2^{-\epsilon'}\hat{r}n = m, \quad n \in \mathbb{Z}. \quad (\text{F.28})$$

Note that $2^{-\epsilon'}\hat{r}$ is integer and, due to the possibility to rewrite the condition on ρ_i as $(\rho_1 - \rho_2)\hat{r}_1 + \rho_2\hat{r} = 1$, one has $\gcd(\rho_1 - \rho_2, \hat{r}) = 1$, which also implies that $\gcd(2^{\epsilon-\epsilon'}(\rho_1 - \rho_2), 2^{-\epsilon'}\hat{r}) = 1$. Thus, if we restrict \mathbf{a}_0 to belong to the interval $[0, \hat{r}-1]$, for $\epsilon' = 0$ and given m , there is a unique pair (\mathbf{a}_0, n) satisfying (F.28), while for $\epsilon' = 1$ there are two such pairs $(\mathbf{a}_0 + \frac{1}{2}\hat{r}\varepsilon, n + (\rho_1 - \rho_2)\varepsilon)$, $\varepsilon = \{0, 2^\epsilon - 1\}$. Thus, the equations (F.26) and (F.28) encode $2^{3\epsilon}\kappa r_0$ solutions to the condition (F.25) parametrized by (ℓ, ε) and provide an explicit description of the set $\mathcal{A}_0^{(r)}(\mu_{12})$.

F.4 Zero modes of maximal order

For n charges, the maximal order of zero modes is $n - 1$ and their set is determined by the conditions $w_{ij} * \mathbb{k} = 0$ for all i, j and $\mathbb{k} \in \Lambda^{(r)} + \mu$. In practice we work with the lattice decompositions (5.31) and (5.46). Therefore, \mathbb{k} can be replaced by $\mathbb{k}_{\parallel} \in \Lambda_{\parallel}^{(r)} + \mathfrak{g}_{\mathbb{A}}^{\parallel} + \mu$ which is expanded as in (5.80) with coefficients whose fractional parts are determined by the glue vectors of the two decompositions, labelled by $\mathbf{A} = \{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n\}$ and $\mathbf{B} = \{\mathbf{b}_{ij}, \tilde{\mathbf{b}}_{ij}\}$, respectively. What we are interested in is the set of glue vectors for which the space of solutions to the above conditions is non-empty.

To describe the resulting set, let us define $\Delta\hat{\mu} = \hat{\mu} - 2^\epsilon\tilde{\mu}$ where $\hat{\mu}$ is the vector from (5.37), while $\tilde{\mu}$ is the same vector with κ multiplied by 4^ϵ and μ, μ_i replaced by $\tilde{\mu}, \tilde{\mu}_i$ computed in (5.112). This latter vector can be seen as the projection of $\mathfrak{g}_{\mathbb{A}}$ on $\tilde{\Lambda}^{(r)}$. Using these definitions, one finds that the components of $\Delta\hat{\mu}$ have the following explicit expressions

$$\Delta\hat{\mu}_i = \frac{\mu_i - 2^{-\epsilon}\mathbf{a}_i}{\kappa r_i} - \frac{\mu - 2^{-\epsilon}\sum_{j=1}^n \mathbf{a}_j}{\kappa r} + \frac{2^\epsilon r_0 \mathbf{a}_0}{r} + \left(\frac{\Delta\mu}{\kappa r_0} - 2^\epsilon \mathbf{a}_0 \right) \rho_i. \quad (\text{F.29})$$

Then we summarize the main facts about the desired set in the following

Proposition F.1. *The set of glue vectors of zero modes of maximal order can be characterized by two sets of conditions:*

1. *the first specifies a set $\mathcal{A}_0^{(r)}$ of \mathbf{A} indices and is independent of \mathbf{B} :*

$$\Delta\hat{\mu}_i(\mathbf{A}) \in \mathbb{Z}, \quad i = 1, \dots, n; \quad (\text{F.30})$$

2. *the second restricts \mathbf{B} indices:*

$$\mathbf{b}_{ij} - 2^\epsilon \tilde{\mathbf{b}}_{ij} = -c_{ij}(\mathbf{A}) \pmod{N_{ij}}, \quad i < j - 1, \quad (\text{F.31})$$

where N_{ij} is given in (F.7) and $c_{ij}(\mathbf{A})$ are the coefficients in the expansion

$$\Delta\hat{\mu}(\mathbf{A}) = \sum_{i < j} c_{ij}(\mathbf{A}) \hat{\mathbf{v}}_{ij}, \quad (\text{F.32})$$

which is unique provided $\mathbf{A} \in \mathcal{A}_0^{(r)}$ and $c_{ij}(\mathbf{A})$ are required to be integers ranging from 0 to $N_{ij} - 1$.

Proof. Let $\mathbb{k}_{\parallel} = \lambda + \mathfrak{g}_{\mathbf{A}}^{\parallel} + \mu$ where $\lambda = (\boldsymbol{\lambda}, \tilde{\boldsymbol{\lambda}}) \in \mathbb{A}_{\parallel}^{(r)}$. Given the form of the null vectors w_{ij} (5.20), the maximal order zero mode condition can be written as

$$\mathbf{v}_{ij} \cdot (\boldsymbol{\lambda} - 2^{\epsilon} \tilde{\boldsymbol{\lambda}} + \hat{\boldsymbol{\mu}} - 2^{\epsilon} \hat{\boldsymbol{\mu}}) = 0 \quad \text{for all } i, j, \quad (\text{F.33})$$

where we used the bilinear form (5.13) and took into account that the bilinear form on $\tilde{\boldsymbol{\Lambda}}^{(r)}$ differs by the factor -4^{ϵ} . It is clear that the condition (F.33) is equivalent to

$$\boldsymbol{\lambda} - 2^{\epsilon} \tilde{\boldsymbol{\lambda}} = -\Delta \hat{\boldsymbol{\mu}}, \quad (\text{F.34})$$

which in turn implies that $\Delta \hat{\boldsymbol{\mu}} \in \boldsymbol{\Lambda}^{(r)}$. Since the components (F.29) automatically satisfy the condition $\sum_{i=1}^n r_i \Delta \hat{\mu}_i = 0$, the only remaining condition is that they must be integer. This is precisely the condition (F.30) which provides the definition of the set $\mathcal{A}_0^{(r)}$.

Next, each vector in (F.34) has a unique decomposition according to the lattice decomposition in (5.46). In particular, $\boldsymbol{\lambda} = \boldsymbol{\lambda}' + \sum_{\substack{i,j=1 \\ j-i>1}}^n \mathbf{b}_{ij} \hat{\mathbf{v}}_{ij}$ and $\tilde{\boldsymbol{\lambda}} = \tilde{\boldsymbol{\lambda}}' + \sum_{\substack{i,j=1 \\ j-i>1}}^n \tilde{\mathbf{b}}_{ij} \hat{\mathbf{v}}_{ij}$ where $\boldsymbol{\lambda}', \tilde{\boldsymbol{\lambda}}' \in \text{Span}\{\hat{\mathbf{v}}_{k,k+1}\}_{k=1}^{n-1}$, while $\Delta \hat{\boldsymbol{\mu}}$ can be written as in (F.32) where c_{ij} with $j-i > 1$ play the role of the glue vector indices. Then the second set of conditions (F.31) claimed by the proposition is a direct consequence of (F.34). \square

Remark F.1. *In fact, due to $\sum_{i=1}^n r_i \Delta \hat{\mu}_i = 0$, a stronger form of the condition (F.30) holds: $\Delta \hat{\mu}_i \in \text{gcd}(\{\hat{r}_j\}_{j \neq i})\mathbb{Z}$. In particular, this should be taken into account to reproduce the condition (F.25) in the $n = 2$ case.*

Corollary F.1. *Each pair $(\mathbf{b}_{ij}, \tilde{\mathbf{b}}_{ij})$ contributes N_{ij} solutions so that their total number is given by (F.8) and is \mathbf{A} -independent. As a result, if $\Delta_0^{(r)}(\mathbf{A}, \mathbf{B})$ is the Kronecker symbol implementing the maximal zero mode condition, then*

$$\sum_{\mathbf{B}} \Delta_0^{(r)}(\mathbf{A}, \mathbf{B}) = \frac{r_0 \prod_{j=2}^{n-1} r_j}{\prod_{i=1}^{n-1} r_{i,i+1}} \delta_{\mathbf{A} \in \mathcal{A}_0^{(r)}}. \quad (\text{F.35})$$

Although Proposition F.1 gives a formula (F.31) for the \mathbf{B} indices corresponding to the zero modes, it does not tell us how to compute the coefficients c_{ij} encoding the solution. Let us explain a simple recipe how this can be done. We start with the first component of the expansion (F.32) multiplied by the factor $\frac{r_{1n}}{r_{2::n}}$ so that, according to Remark F.1, the result is still an integer. It is found to be

$$\frac{r_{1n}}{r_{2::n}} \Delta \hat{\mu}_1 = \sum_{j=2}^n \frac{r_{1n} r_j}{r_{1j} r_{2::n}} c_{1j} = \frac{r_n}{r_{2::n}} c_{1n} \quad \text{mod } N_{1n}, \quad (\text{F.36})$$

where we used the fact that all $\frac{r_{1n} r_j}{r_{1j} r_{2::n}}$ with $2 \leq j \leq n-1$ are divisible by N_{1n} , as follows from its expression in (F.7). Furthermore, it is easy to see that $\text{gcd}(\frac{r_n}{r_{2::n}}, N_{1n}) = 1$. This implies that there exist $a_{1n}, b_{1n} \in \mathbb{Z}$ such that $a_{1n} \frac{r_n}{r_{2::n}} + b_{1n} N_{1n} = 1$, i.e.

$$a_{1n} \frac{r_{1n}}{r_{2::n}} \Delta \hat{\mu}_1 = c_{1n} \quad \text{mod } N_{1n}. \quad (\text{F.37})$$

Then one subtracts the already found part of the expansion (F.32) from $\Delta\hat{\boldsymbol{\mu}}$ and repeats the procedure, first by lowering the second index and then by raising the first index. Thus, to determine c_{ij} , one chooses $a_{ij}, b_{ij} \in \mathbb{Z}$ such that $a_{ij} \frac{r_j}{r_{i+1::j}} + b_{ij} N_{ij} = 1$ and uses the fact that

$$\begin{aligned} & a_{ij} \frac{r_{ij}}{r_{i+1::j}} \left(\Delta\hat{\boldsymbol{\mu}}_i - \sum_{k=1}^{i-1} \sum_{l=k+1}^n c_{kl} (\hat{\boldsymbol{v}}_{kl})_i - \sum_{l=j+1}^n c_{il} (\hat{\boldsymbol{v}}_{kl})_i \right) \\ &= a_{ij} \sum_{l=i+1}^j \frac{r_{ij} r_l}{r_{il} r_{i+1::j}} c_{il} = a_{ij} \left(\frac{r_j}{r_{i+1::j}} c_{ij} \bmod N_{ij} \right) = c_{ij} \bmod N_{ij}. \end{aligned} \quad (\text{F.38})$$

G. Computations for three charges

G.1 Contributions with zero mode order equal 0 and 1

For three charges, the kernel of the theta series $\vartheta_{\mu, \mathbf{A}}^{(\mathbf{r})||}$ defined in (5.41) is given in (5.78) in terms of γ_{ij} . The scalar products $\mathbb{x}_\beta * \mathbb{v}_{ij}$ appearing in the kernel are given by the relation (5.39). Similarly, we define for $i < j$

$$\begin{aligned} \omega_{ij} &= \frac{1}{\sqrt{2\tau_2}} \mathbb{w}_{ij} * \mathbb{x}_\beta = \mathbb{w}_{ij} * \mathbb{k}, \\ \omega_{ij}^{(\beta)} &= \frac{1}{\sqrt{2\tau_2}} \mathbb{w}_{ij} * \mathbb{x} = \mathbb{w}_{ij} * (\mathbb{k} + \beta) \\ &= \omega_{ij} + \begin{cases} 2^\epsilon \kappa(r_i + r_j) r_i r_j \beta, & (ij) = (12), (23), \\ 2^\epsilon \kappa(r + r_2) r_1 r_3 \beta, & (ij) = (13). \end{cases} \end{aligned} \quad (\text{G.1})$$

For $i > j$, ω_{ij} and $\omega_{ij}^{(\beta)}$ are defined by applying the permutation $i \leftrightarrow j$ to (G.1). More generally, permutations leave \mathbb{k} invariant and permute the indices of all vectors like \mathbb{v}_{ij} and \mathbb{w}_{ij} . Note that the component of \mathbb{z} along $\boldsymbol{\Lambda}^{(\mathbf{r})}$ is proportional to $\boldsymbol{\theta}$ (see (5.37)) defined in terms of the vectors \boldsymbol{v}_{ij} and therefore it is affected by the permutations so as β . This explains why ω_{ij} are anti-symmetric, whereas this is not true for $\omega_{ij}^{(\beta)}$ as follows from the last line in (G.1). We also have $\omega_{i+j,k} = \omega_{ik} + \omega_{jk}$ and similarly for $\omega_{1+2,3}^{(\beta)}$ and $\omega_{1,2+3}^{(\beta)}$. Using these properties, one can write

$$\begin{aligned} \text{Sym} \left\{ \vartheta_{\mu, \mathbf{A}}^{(\mathbf{r})||}(\tau, z) \right\} &= \sum_{\mathbb{k}_{||} \in \Lambda_{||}^{(\mathbf{r})} + \mathfrak{g}_{\mathbf{A}}^{||} + \mu} q^{-\frac{1}{2} \mathbb{k}_{||}^2} \text{Sym} \left\{ y^{\gamma_{1,2+3} + \gamma_{23}} \left[\frac{1}{3} \delta_{\gamma_{1,2+3}} \delta_{\gamma_{1+2,3}} \right. \right. \\ &+ \left. \left. \left(\text{sgn}(\gamma_{1,2+3}) - \text{sgn}(\omega_{12}^{(\beta)}) \right) \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{23}^{(\beta)}) \right) \right] \right. \\ &+ \delta_{\omega_{12}} \phi^{(\kappa_{12})}(\tau, r_{12} z) \left[y^{\gamma_{1+2,3}} \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{1+2,3}^{(\beta)}) \right) \right. \\ &\left. \left. + y^{-\gamma_{1+2,3}} \left(\text{sgn}(\gamma_{3,1+2}) - \text{sgn}(\omega_{3,1+2}^{(\beta)}) \right) \right] \sum_{\eta=1}^{2^{1+\epsilon} \kappa_{12}} \delta_{\frac{\gamma_{12}}{r_{12}} - \eta} \right\}, \end{aligned} \quad (\text{G.2})$$

where we applied the permutation $1 \leftrightarrow 3$ to the last term in (5.78).

Let us extract the contribution of vanishing zero mode order. It is given by the first two lines in (G.2) where all ω_{ij} are non-vanishing. Under this condition, for small β one can replace $\text{sgn}(\omega_{ij}^{(\beta)})$ by $\text{sgn}(\omega_{ij})$ so that this term becomes

$$\text{Sym} \left\{ y^{\gamma_{1,2+3}+\gamma_{23}} \left[\frac{1}{3} \delta_{\gamma_{1,2+3}} \delta_{\gamma_{1+2,3}} + \left(\text{sgn}(\gamma_{1,2+3}) - \text{sgn}(\omega_{12}) \right) \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{23}) \right) \right] \right\}. \quad (\text{G.3})$$

We expand the product and apply the sign identity

$$\text{sgn}(x_1) \text{sgn}(x_2) = 1 - \delta_{x_1} \delta_{x_2} - \text{sgn}(x_1 - x_2) (\text{sgn}(x_1) - \text{sgn}(x_2)) \quad (\text{G.4})$$

to $\frac{1}{3}$ of the terms with two γ 's and two ω 's. This results in

$$\begin{aligned} \text{Sym} \left\{ y^{\gamma_{1,2+3}+\gamma_{23}} \left[\frac{2}{3} \text{sgn}(\gamma_{1,2+3}) \text{sgn}(\gamma_{1+2,3}) - \frac{1}{3} \text{sgn}(\gamma_{1+3,2}) \left(\text{sgn}(\gamma_{1,2+3}) - \text{sgn}(\gamma_{1+2,3}) \right) \right. \right. \\ \left. \left. + \frac{2}{3} \text{sgn}(\omega_{12}) \text{sgn}(\omega_{23}) + \frac{1}{3} \text{sgn}(\omega_{13}) \left(\text{sgn}(\omega_{12}) + \text{sgn}(\omega_{23}) \right) \right. \right. \\ \left. \left. - \text{sgn}(\gamma_{1,2+3}) \text{sgn}(\omega_{23}) - \text{sgn}(\omega_{12}) \text{sgn}(\gamma_{1+2,3}) \right] \right\}, \quad (\text{G.5}) \end{aligned}$$

where we used $\gamma_{1,2+3} - \gamma_{1+2,3} = \gamma_{1+3,2}$ and $r_3\omega_{12} + r_1\omega_{23} = r_2\omega_{13}$ following from (5.24). Then we use the symmetrization to bring the products of sign functions in each line to a single expression. This gives

$$\begin{aligned} \text{Sym} \left\{ \left(\frac{1}{3} \text{sgn}(\gamma_{1,2+3}) \text{sgn}(\gamma_{1+2,3}) + \frac{1}{3} \text{sgn}(\omega_{12}) \text{sgn}(\omega_{23}) - \frac{1}{2} \text{sgn}(\gamma_{1,2+3}) \text{sgn}(\omega_{23}) \right) \right. \\ \left. \times \left(y^{\gamma_{1,2+3}} - y^{-\gamma_{1,2+3}} \right) \left(y^{\gamma_{23}} - y^{-\gamma_{23}} \right) \right\}. \quad (\text{G.6}) \end{aligned}$$

This result makes it manifest that the contribution of vanishing zero mode order has a zero of second order at $z = 0$.

Next, we consider zero mode contributions of order 1. They are characterized by vanishing of only one of ω_{ij} 's. This will be indicated by insertion of the corresponding Kronecker symbol $\delta_{\omega_{ij}}$, but we will omit the factors $1 - \delta_{\omega_{jk}}$ ensuring that other variables are non-vanishing. Thus, the relevant contribution to (G.2) reads

$$\begin{aligned} \text{Sym} \left\{ y^{\gamma_{1,2+3}+\gamma_{23}} \left[\frac{1}{3} \delta_{\gamma_{1,2+3}} \delta_{\gamma_{1+2,3}} (\delta_{\omega_{12}} + \delta_{\omega_{23}} + \delta_{\omega_{13}}) \right. \right. \\ \left. \left. + \left(\text{sgn}(\gamma_{1,2+3}) - \text{sgn}(\beta) \right) \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{23}) \right) \delta_{\omega_{12}} \right. \right. \\ \left. \left. + \left(\text{sgn}(\gamma_{1,2+3}) - \text{sgn}(\omega_{12}) \right) \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\beta) \right) \delta_{\omega_{23}} \right. \right. \\ \left. \left. + \left(\text{sgn}(\gamma_{1,2+3}) + \text{sgn}(\omega_{23}) \right) \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{23}) \right) \delta_{\omega_{13}} \right] \right. \\ \left. + \delta_{\omega_{12}} \left(y^{\gamma_{1+2,3}} - y^{-\gamma_{1+2,3}} \right) \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{1+2,3}) \right) \phi^{(\kappa_{12})} \sum_{\eta=1}^{2^{1+\epsilon_{\kappa_{12}}}} \delta_{\frac{\gamma_{12}}{r_{12}} - \eta} \right\}, \quad (\text{G.7}) \end{aligned}$$

where in the fourth line we used that $\text{sgn}(\omega_{12}) = -\text{sgn}(\omega_{23})$ provided $\omega_{13} = 0$. Then we use the symmetrization to transform all $\delta_{\omega_{ij}}$ to $\delta_{\omega_{12}}$, and add and subtract a term proportional to $\text{sgn}(\gamma_{12})$, which is recombined with the terms coming from the second and third lines of (G.7). Taking also into account that $\text{sgn}(\omega_{23}) = \text{sgn}(\omega_{1+2,3})$ provided $\omega_{12} = 0$, we end up with the following result

$$\begin{aligned}
& \text{Sym} \left\{ \delta_{\omega_{12}} \left[\frac{1}{3} \left(y^{\gamma_{1,2+3}+\gamma_{23}} + y^{-\gamma_{1,2+3}-\gamma_{23}} + y^{\gamma_{1,2+3}-\gamma_{23}} \right) \delta_{\gamma_{1,2+3}} \delta_{\gamma_{1+2,3}} \right. \right. \\
& \quad + \left(y^{\gamma_{1,2+3}+\gamma_{23}} + y^{-\gamma_{1,2+3}-\gamma_{23}} \right) \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{23}) \right) \left(\text{sgn}(\gamma_{1,2+3}) - \text{sgn}(\gamma_{12}) \right) \\
& \quad + y^{\gamma_{1,2+3}-\gamma_{23}} \left(\text{sgn}(\gamma_{1,2+3}) - \text{sgn}(\omega_{23}) \right) \left(\text{sgn}(\gamma_{1+3,2}) + \text{sgn}(\omega_{23}) \right) \\
& \quad + \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{1+2,3}) \right) \left(\left(y^{\gamma_{1+2,3}} - y^{-\gamma_{1+2,3}} \right) \phi^{(\kappa_{12})} \sum_{\eta=1}^{2^{1+\epsilon\kappa_{12}}} \delta_{\frac{\gamma_{12}}{\tau_{12}}-\eta} \right. \\
& \quad \left. \left. + y^{\gamma_{1+2,3}+\gamma_{12}} \left(\text{sgn}(\gamma_{12}) - \text{sgn}(\beta) \right) + y^{-\gamma_{1+2,3}-\gamma_{12}} \left(\text{sgn}(\gamma_{12}) + \text{sgn}(\beta) \right) \right) \right] \right\}. \tag{G.8}
\end{aligned}$$

Expanding the product in the third line and applying the identity (G.4) to the product of two $\text{sgn}(\gamma_{i+j,k})$, one finds that it is equal to

$$\text{Sym} \left\{ \delta_{\omega_{12}} y^{\gamma_{1,2+3}-\gamma_{23}} \left[\left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{23}) \right) \left(\text{sgn}(\gamma_{1+3,2}) - \text{sgn}(\gamma_{1,2+3}) \right) - \delta_{\gamma_{1,2+3}} \delta_{\gamma_{1+3,2}} \right] \right\}. \tag{G.9}$$

The last term cancels the first line in (G.8) (to see this, one should use the vanishing condition imposed by the Kronecker symbols and the symmetrization $1 \leftrightarrow 2$), while the first term can be combined with the second line in that equation and, after the same symmetrization and the replacement of $\text{sgn}(\omega_{23})$ by $\text{sgn}(\omega_{1+2,3})$, gives

$$\begin{aligned}
& \text{Sym} \left\{ \delta_{\omega_{12}} \left(y^{\gamma_{1,2+3}} - y^{-\gamma_{1,2+3}} \right) \left(y^{\gamma_{23}} - y^{-\gamma_{23}} \right) \right. \\
& \quad \left. \times \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{1+2,3}) \right) \left(\text{sgn}(\gamma_{1,2+3}) - \text{sgn}(\gamma_{12}) \right) \right\}. \tag{G.10}
\end{aligned}$$

Thus, this contribution also has a zero of second order at $z = 0$.

It remains to compute the contribution of the last two lines in (G.8), which after symmetrization takes the form

$$\begin{aligned}
& \text{Sym} \left\{ \delta_{\omega_{12}} \left(y^{\gamma_{1+2,3}} - y^{-\gamma_{1+2,3}} \right) \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{1+2,3}) \right) \right. \\
& \quad \left. \times \left(\phi^{(\kappa_{12})} \sum_{\eta=1}^{2^{1+\epsilon\kappa_{12}}} \delta_{\frac{\gamma_{12}}{\tau_{12}}-\eta} + y^{\gamma_{12}} \left(\text{sgn}(\gamma_{12}) - \text{sgn}(\beta) \right) \right) \right\}. \tag{G.11}
\end{aligned}$$

We observe that it factorizes into two parts: the second line depends only on γ_{12} , while the first line together with the factor $q^{-\frac{1}{2}k_{\parallel}^2}$ can be shown to depend on γ_{12} only mod $2^{1+\epsilon\tau_{12}\kappa_{12}}$. Indeed, it is easy to verify that, provided $\omega_{12} = 0$, the transformation $\mathbb{k}_{\parallel} \mapsto \mathbb{k}_{\parallel} + \hat{w}_{12}$ affects

neither $\mathbb{k}_{||}^2$, nor the quantities appearing in the first line: $\omega_{12}, \omega_{1+2,3}, \gamma_{1+2,3}$. On the other hand, it maps $\gamma_{12} \rightarrow \gamma_{12} + 2^{1+\epsilon} r_{12} \kappa_{12}$, which proves the claim. This means that, substituted into the theta series, the contribution (G.11) is equivalent to

$$\begin{aligned} & \text{Sym} \left\{ \delta_{\omega_{12}} \left(y^{\gamma_{1+2,3}} - y^{-\gamma_{1+2,3}} \right) \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{1+2,3}) \right) \right. \\ & \quad \times \left. \sum_{\eta=1}^{2^{1+\epsilon} \kappa_{12}} \delta_{\frac{\gamma_{12}}{r_{12}} - \eta} \left(\phi^{(\kappa_{12})} + \sum_{\tilde{\ell} \in \mathbb{Z}} \left(\text{sgn}(\eta + 2^{1+\epsilon} \kappa_{12} \tilde{\ell}) - \text{sgn}(\beta) \right) y^{r_{12}(\eta + 2^{1+\epsilon} \kappa_{12} \tilde{\ell})} \right) \right\}. \end{aligned} \quad (\text{G.12})$$

The sum over $\tilde{\ell}$ is identical to the one in (5.59) (where the sum over σ is equivalent to the symmetrization with respect to $1 \leftrightarrow 2$ in (G.12)). Thus, we can borrow the result (5.60), which gives

$$\begin{aligned} & \text{Sym} \left\{ \delta_{\omega_{12}} \left(y^{\gamma_{1+2,3}} - y^{-\gamma_{1+2,3}} \right) \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{1+2,3}) \right) \right. \\ & \quad \times \left. \sum_{\eta=1}^{2^{1+\epsilon} \kappa_{12}} \delta_{\frac{\gamma_{12}}{r_{12}} - \eta} \left(\phi^{(\kappa_{12})} - \frac{y^{2^{1+\epsilon} r_{12} \kappa_{12} \lambda_{12}} + y^{-2^{1+\epsilon} r_{12} \kappa_{12} \lambda_{12}}}{y^{2^\epsilon r_{12} \kappa_{12}} - y^{-2^\epsilon r_{12} \kappa_{12}}} \right) \right\}, \end{aligned} \quad (\text{G.13})$$

where λ_{12} is given by (5.61) with $\tilde{\nu}$ replaced by $2^{-\epsilon} \eta$. Since $\phi^{(\kappa_{12})}$ was chosen precisely to cancel the pole coming from the geometric progression, the last bracket behaves as $\mathcal{O}(z)$ and the contribution (G.13) has a zero of second order at $z = 0$. As a result, the same conclusion applies to the total zero mode contribution of order 1 to the summand in (G.2), given by the sum of (G.10) and (G.13).

G.2 The unrefined limit

We define the theta series appearing in (5.99) as

$$\vartheta_{\mu, \mathbf{A}}^{(\mathbf{r})k}(\tau) = \lim_{z \rightarrow 0} \frac{[\text{Sym} \{ \vartheta_{\mu, \mathbf{A}}^{(\mathbf{r})||}(\tau, z) \}]_k}{(y - y^{-1})^2}, \quad (\text{G.14})$$

where $[\cdot]_k$ denotes the contribution of zero mode order equal k . In particular, the kernel of the contribution of vanishing order is given by (G.6), and the one of order 1 is given by the sum of (G.10) and (G.13).

First, we represent the theta series (G.14) as a sum over the lattice $\Lambda_{||}^{(\mathbf{r})}$ with the kernel expressed through γ_{ij} and ω_{ij} as in (G.2). One easily finds

$$\begin{aligned} \vartheta_{\mu, \mathbf{A}}^{(\mathbf{r})0}(\tau) = & \sum_{\mathbb{k}_{||} \in \Lambda_{||}^{(\mathbf{r})} + \mathfrak{g}_{\mathbf{A}}^{||} + \mu} \overline{\Delta}_0^{(\mathbf{r})} \text{Sym} \left\{ \gamma_{23} \gamma_{1,2+3} \left(\frac{1}{3} \text{sgn}(\gamma_{1,2+3}) \text{sgn}(\gamma_{1+2,3}) + \frac{1}{3} \text{sgn}(\omega_{12}) \text{sgn}(\omega_{23}) \right. \right. \\ & \left. \left. - \frac{1}{2} \text{sgn}(\gamma_{1,2+3}) \text{sgn}(\omega_{23}) \right) \right\} \mathfrak{q}^{-\frac{1}{2} \mathbb{k}_{||}^2}, \end{aligned} \quad (\text{G.15})$$

$$\vartheta_{\mu, \mathbf{A}}^{(\mathbf{r})1}(\tau) = \sum_{\mathbb{k}_{||} \in \Lambda_{||}^{(\mathbf{r})} + \mathfrak{g}_{\mathbf{A}}^{||} + \mu} \text{Sym} \left\{ \delta_{\omega_{12}} (1 - \delta_{\omega_{23}}) \gamma_{1,2+3} \left(\text{sgn}(\gamma_{1+2,3}) - \text{sgn}(\omega_{1+2,3}) \right) \right\}$$

$$\times \left[\gamma_{23} \left(\text{sgn}(\gamma_{1,2+3}) - \text{sgn}(\gamma_{12}) \right) + \frac{2^\epsilon r_{12}}{12} \left(1 - 12\lambda_{12}^2 - \frac{E_2(\tau)}{2^{1+2\epsilon}\kappa_{12}} \right) \sum_{\eta=1}^{2^{1+\epsilon}\kappa_{12}} \delta_{\frac{\gamma_{12}}{r_{12}} - \eta} \right] \mathfrak{q}^{-\frac{1}{2}\mathbf{k}_{\parallel}^2},$$

where the factor $\overline{\Delta}_0^{(\mathbf{r})} = (1 - \delta_{\omega_{12}})(1 - \delta_{\omega_{23}})(1 - \delta_{\omega_{13}})$ implements the condition of vanishing zero mode order. Note that in the first theta series the symmetrization over charges must be implemented *before* performing the sum. The reason is that in the process of getting the kernel that makes manifest the existence of a second order zero and allows to compute the unrefined limit, the conditions of convergence ceased to be satisfied. This happened due to the use of various permutations which spoiled the original structure of the kernel satisfying Theorem C.1. Of course, one can do them backwards to recover the manifest convergence. We omit the details of manipulations, which are similar to the ones in the previous subsection, and present just the final result

$$\vartheta_{\mu, \mathbf{A}}^{(\mathbf{r})0}(\tau) = \frac{1}{6} \sum_{\mathbf{k}_{\parallel} \in \Lambda_{\parallel}^{(\mathbf{r})} + \mathbf{g}_{\mathbf{A}} + \mu} \overline{\Delta}_0^{(\mathbf{r})} (1 + \sigma_{13}) \left[\gamma_{23} \gamma_{1,2+3} \left(\text{sgn}(\gamma_{1,2+3}) - \text{sgn}(\omega_{13}) \right) \left(\text{sgn}(\gamma_{2,1+3}) - \text{sgn}(\omega_{23}) \right) \right] \mathfrak{q}^{-\frac{1}{2}\mathbf{k}_{\parallel}^2}, \quad (\text{G.16})$$

where σ_{ij} denotes the permutation $i \leftrightarrow j$. Note that although the symmetry of this expression under all permutations of charges is not manifest, it is actually symmetric being obtained from a symmetrized expression. For the second theta series in (G.15) the issue of convergence is absent since it is ensured by the kernel so that the symmetrization can be put outside the sum.

For computer evaluation it might be useful also to rewrite the above expressions as a sum over an unconstrained lattice. To this end, we can substitute the expansion (5.80) of the lattice vector and rewrite the sum over the lattice $\Lambda_{\parallel}^{(\mathbf{r})}$ as a sum over the coefficients ℓ_i and $\tilde{\ell}_i$ at the price of introducing new glue vectors $\mathbf{B} = \{\mathbf{b}_{13}, \tilde{\mathbf{b}}_{13}\}$. This leads to more explicit, although more cumbersome expressions:

$$\begin{aligned} \vartheta_{\mu, \mathbf{A}}^{(\mathbf{r})0}(\tau) &= \frac{r_0}{6} (1 + \sigma_{13}) \sum_{\mathbf{B}} \sum_{\ell_i \in \mathbb{Z} + \frac{\nu_i}{\kappa_i}} \sum_{\tilde{\ell}_i \in \mathbb{Z} + \frac{\tilde{\nu}_i}{\kappa_i}} \overline{\Delta}_0^{(\mathbf{r})} \left[r_{23} \kappa_1 \ell_1 (2\kappa_{23} \ell_2 - \kappa_{123} \ell_1) \left(\text{sgn}(\ell_1) - \text{sgn}\left(\frac{\Delta \ell_1}{r_{12}} + \frac{\Delta \ell_2}{r_{23}}\right) \right) \right. \\ &\quad \times \left. \left(\text{sgn}\left(\frac{r_3}{r_{23}} \ell_2 - \frac{r_1}{r_{12}} \ell_1\right) - \text{sgn}\left(\frac{r_2+r_3}{r_{23}} \Delta \ell_2 - \frac{r_1}{r_{12}} \Delta \ell_1\right) \right) \right] \mathfrak{q}^{\kappa_{12}(4^\epsilon \tilde{\ell}_1^2 - \ell_1^2) + \kappa_{23}(4^\epsilon \tilde{\ell}_2^2 - \ell_2^2) - \kappa_{123}(4^\epsilon \tilde{\ell}_1 \tilde{\ell}_2 - \ell_1 \ell_2)}, \\ \vartheta_{\mu, \mathbf{A}}^{(\mathbf{r})1}(\tau) &= r_0 \text{Sym} \left\{ \kappa_1 \sum_{\mathbf{B}} \sum_{\ell_i \in \mathbb{Z} + \frac{\nu_i}{\kappa_i}} \sum_{\tilde{\ell}_i \in \mathbb{Z} + \frac{\tilde{\nu}_i}{\kappa_i}} \delta_{\frac{r_1+r_2}{r_{12}} \Delta \ell_1 - \frac{r_3}{r_{23}} \Delta \ell_2} (1 - \delta_{\Delta \ell_1}) \ell_1 \left(\text{sgn}(\ell_2) - \text{sgn}(\Delta \ell_2) \right) \right. \\ &\quad \times \left[r_{23} (2\kappa_{23} \ell_2 - \kappa_{123} \ell_1) \left(\text{sgn}(\ell_1) - \text{sgn}\left(\frac{r_1+r_2}{r_{12}} \ell_1 - \frac{r_3}{r_{23}} \ell_2\right) \right) \right. \\ &\quad \left. \left. + \frac{2^\epsilon r_{12}}{12} \left(1 - 12\lambda_{12}^2 - \frac{E_2(\tau)}{2^{1+2\epsilon}\kappa_{12}} \right) \sum_{\eta=1}^{2^{1+\epsilon}\kappa_{12}} \delta_{2\kappa_{12}\ell_1 - \kappa_{123}\ell_2 - \eta} \right] \mathfrak{q}^{\frac{r_0 r_2 \kappa_2}{2(r_1+r_2)} (4^\epsilon \tilde{\ell}_2^2 - \ell_2^2)} \right\}, \quad (\text{G.17}) \end{aligned}$$

where $\Delta \ell_i = \ell_i - 2^\epsilon \tilde{\ell}_i$. The variables ν_i and $\tilde{\nu}_i$ determining the rational parts are given in (5.81), and the factor implementing the condition of vanishing zero mode order reads as

$$\overline{\Delta}_0^{(\mathbf{r})} = \left(1 - \delta_{\frac{r_1+r_2}{r_{12}} \Delta \ell_1 - \frac{r_3}{r_{23}} \Delta \ell_2} \right) \left(1 - \delta_{\frac{r_2+r_3}{r_{23}} \Delta \ell_2 - \frac{r_1}{r_{12}} \Delta \ell_1} \right) \left(1 - \delta_{\frac{\Delta \ell_1}{r_{12}} + \frac{\Delta \ell_2}{r_{23}}} \right).$$

H. Consistency of different solutions

H.1 Two charges

The solution (5.71) obtained using indefinite theta series can be compared with the one constructed in section 4.1 using Hecke-like operators. The consistency requires that their difference is a VV modular form. To verify whether this is the case, let us write the function (5.71) as in (4.2)

$$g_{\mu,\mu_1,\mu_2}^{(r_1,r_2)} = r_0 \delta_{\Delta\mu}^{(\kappa r_0)} \check{G}_{\mu_{12}}^{(\kappa;r_1,r_2)} \quad (\text{H.1})$$

and define

$$\varphi_{r_1,r_2}^{(\kappa)}(\tau, z) = \sum_{\mu=0}^{2\kappa_{12}-1} \left(\check{G}_{\mu}^{(\kappa;r_1,r_2)}(\tau) - G_{\mu}^{(\kappa_{12})}(\tau) \right) \theta_{\mu}^{(\kappa_{12})}(\tau, z), \quad (\text{H.2})$$

where $G^{(\kappa)}$ is given by (4.10) and $\theta_{\mu}^{(\kappa)}(\tau, z)$ is the theta series (C.7). The modularity condition is equivalent to the requirement that $\varphi_{r_1,r_2}^{(\kappa)}$ is a Jacobi form of weight 2, index κ_{12} and trivial multiplier system. Using Mathematica, for the choice a) in (5.18) of the vectors $\mathbf{t}^{(r)}$, we have found the following results for several low values of charges and κ :

$$\varphi_{1,1}^{(1)}(\tau, z) = \frac{7}{5971968} \frac{E_4(\tau)E_6(\tau)}{\Delta(\tau)} \left(E_4(\tau) \varphi_{0,1} + E_6(\tau) \varphi_{-2,1} \right) - \frac{11}{3456} E_4(\tau) \varphi_{-2,1}, \quad (\text{H.3a})$$

$$\varphi_{1,1}^{(2)}(\tau, z) = -\frac{E_4(\tau)}{2304} \varphi_{-2,1} \varphi_{0,1}, \quad (\text{H.3b})$$

$$\varphi_{1,1}^{(3)}(\tau, z) = \frac{1}{9216} \left(E_4(\tau) \varphi_{-2,1} \varphi_{0,1}^2 + 2E_6(\tau) \varphi_{-2,1}^2 \varphi_{0,1} + E_4^2(\tau) \varphi_{-2,1}^3 \right), \quad (\text{H.3c})$$

$$\begin{aligned} \varphi_{1,1}^{(4)}(\tau, z) = \varphi_{2,2}^{(2)}(\tau, z) = & \frac{1}{497664} \left(-5E_4(\tau) \varphi_{-2,1} \varphi_{0,1}^3 + 14E_6(\tau) \varphi_{-2,1}^2 \varphi_{0,1}^2 \right. \\ & \left. + 7E_4^2(\tau) \varphi_{-2,1}^3 \varphi_{0,1} + 2E_4(\tau)E_6(\tau) \varphi_{-2,1}^4 \right), \end{aligned} \quad (\text{H.3d})$$

$$\begin{aligned} \varphi_{1,1}^{(5)}(\tau, z) = & \frac{1}{19906560} \left(29E_4(\tau) \varphi_{-2,1} \varphi_{0,1}^4 + 130E_6(\tau) \varphi_{-2,1}^2 \varphi_{0,1}^3 + 195E_4^2(\tau) \varphi_{-2,1}^3 \varphi_{0,1}^2 \right. \\ & \left. + 116E_4(\tau)E_6(\tau) \varphi_{-2,1}^4 \varphi_{0,1} + \left(112E_6^2(\tau) - 90E_4^3(\tau) \right) \varphi_{-2,1}^5 \right), \end{aligned} \quad (\text{H.3e})$$

$$\begin{aligned} \varphi_{1,1}^{(6)}(\tau, z) = & \frac{1}{597196800} \left(-125E_4(\tau) \varphi_{-2,1} \varphi_{0,1}^5 + 300E_6(\tau) \varphi_{-2,1}^2 \varphi_{0,1}^4 - \frac{3}{2} E_4^2(\tau) \varphi_{-2,1}^3 \varphi_{0,1}^3 \right. \\ & \left. + 149E_4(\tau)E_6(\tau) \varphi_{-2,1}^4 \varphi_{0,1}^2 - \left(\frac{2529}{8} E_4^3(\tau) - 264E_6^2(\tau) \right) \varphi_{-2,1}^5 \varphi_{0,1} \right. \\ & \left. + \frac{63}{2} E_4^2(\tau)E_6(\tau) \varphi_{-2,1}^5 \right), \end{aligned} \quad (\text{H.3f})$$

$$\begin{aligned} \varphi_{1,2}^{(2)}(\tau, z) = & -\frac{1}{71663616} \left(19E_4(\tau) \varphi_{-2,1} \varphi_{0,1}^5 + 30E_6(\tau) \varphi_{-2,1}^2 \varphi_{0,1}^4 + 74E_4^2(\tau) \varphi_{-2,1}^3 \varphi_{0,1}^3 \right. \\ & \left. + 148E_4(\tau)E_6(\tau) \varphi_{-2,1}^4 \varphi_{0,1}^2 + 99E_4^3(\tau) \varphi_{-2,1}^5 \varphi_{0,1} + 14E_4^2(\tau)E_6(\tau) \varphi_{-2,1}^5 \right), \end{aligned} \quad (\text{H.3g})$$

while for the choice b) in (5.18), we have

$$\begin{aligned} \varphi_{1,1}^{(6)}(\tau, z) = & \frac{1}{1194393600} \left(-1250E_4(\tau) \varphi_{-2,1} \varphi_{0,1}^5 + 12600E_6(\tau) \varphi_{-2,1}^2 \varphi_{0,1}^4 - 19487E_4^2(\tau) \varphi_{-2,1}^3 \varphi_{0,1}^3 \right. \\ & \left. + 259558E_4(\tau)E_6(\tau) \varphi_{-2,1}^4 \varphi_{0,1}^2 - \left(\frac{23409}{4} E_4^3(\tau) + 851952E_6^2(\tau) \right) \varphi_{-2,1}^5 \varphi_{0,1} \right) \end{aligned}$$

$$+9457E_4^2(\tau)E_6(\tau)\varphi_{-2,1}^5). \quad (\text{H.4})$$

Here we used the standard Jacobi forms defined in [25]

$$\begin{aligned} \Delta(\tau) &= \eta^{24}(\tau) = \frac{E_4^3(\tau) - E_6^2(\tau)}{1728}, \\ \varphi_{-2,1}(\tau, z) &= \frac{\theta_1^2(\tau, z)}{\eta^6(\tau)}, \\ \varphi_{0,1}(\tau, z) &= 4 \left(\frac{\theta_2(\tau, z)^2}{\theta_2(\tau)^2} + \frac{\theta_3(\tau, z)^2}{\theta_3(\tau)^2} + \frac{\theta_4(\tau, z)^2}{\theta_4(\tau)^2} \right), \end{aligned} \quad (\text{H.5})$$

and $\theta_2, \theta_3, \theta_4$ are the Jacobi theta functions. All functions in (H.3) and (H.4) have the right weight and index, so that the two solutions are indeed mutually consistent.

H.2 Three unit charges

Here we compare the solution (5.99) for three charges specified to the case $r_i = \kappa = 1$ with the one given by the normalized generating function of $SU(3)$ VW invariants on \mathbb{P}^2 (4.23). An explicit expression for this generating function can be found, e.g., in [71, §A]. For convenience of the reader we copy it here. Let $T_3(\tau) = \Theta_0^{(3)}(\tau, 0)$ be the theta series associated with the A_2 lattice and

$$\begin{aligned} S_{1,\mu}(k; q) &= \frac{(-1 + E_2(\tau))(k - \mu + 1)}{2(1 - q^{3k-\mu})} + \frac{9(k - \mu)^2 + 33(k - \mu) + 31 - E_2(\tau)}{2(1 - q^{3k-\mu})^2} \\ &\quad - \frac{15(k - \mu) + 34}{(1 - q^{3k-\mu})^3} + \frac{19}{(1 - q^{3k-\mu})^4}, \\ S_2(A, B; q) &= \frac{4q^B}{(1 - q^A)(1 - q^B)^3} + \frac{4q^A}{(1 - q^A)^3(1 - q^B)} + \frac{4}{(1 - q^A)(1 - q^B)^2} \\ &\quad - \frac{2(A + B + 1)q^B}{(1 - q^A)(1 - q^B)^2} - \frac{2(A + B + 1)q^A}{(1 - q^A)^2(1 - q^B)} + \frac{(A + B - 2)^2 - 8}{2(1 - q^A)(1 - q^B)}. \end{aligned} \quad (\text{H.6})$$

Then the normalized generating function has the following components

$$\begin{aligned} \mathfrak{g}_{3,0} &= \frac{1}{T_3(\tau)} \left[\frac{13}{240} + \frac{1}{24} E_2(\tau) + \frac{1}{72} E_2(\tau)^2 + \frac{1}{720} E_4(\tau) - \frac{9}{2} \sum_{k \in \mathbb{Z}} k^2 q^{3k^2} \right. \\ &\quad \left. + \frac{1}{6} \sum_{k_1, k_2 \in \mathbb{Z}} (k_1 + 2k_2)^2 q^{k_1^2 + k_2^2 + k_1 k_2} + \sum_{\substack{k \in \mathbb{Z} \\ k \neq 0}} S_{1,0}(k; q) q^{3k^2} \right. \\ &\quad \left. + \sum_{\substack{k_1, k_2 \in \mathbb{Z} \\ 2k_1 + k_2 \neq 0, k_2 \neq k_1}} S_2(2k_1 + k_2, k_2 - k_1; q) q^{k_1^2 + k_2^2 + k_1 k_2 + 2k_1 + k_2} \right], \\ \mathfrak{g}_{3,\pm 1} &= \frac{1}{T_3(\tau)} \left[\sum_{k \in \mathbb{Z}} S_{1,1}(k; q) q^{3k^2 - \frac{1}{3}} + \sum_{\substack{k_1, k_2 \in \mathbb{Z} \\ 2k_1 + k_2 \neq 1, k_2 \neq k_1}} S_2(2k_1 + k_2 - 1, k_2 - k_1; q) q^{k_1^2 + k_2^2 + k_1 k_2 - \frac{1}{3}} \right]. \end{aligned} \quad (\text{H.7})$$

In contrast to the case of two charges, the difference of two solutions for three charges should not be modular. Instead, taking the difference of two anomaly equations, one finds that

$$\delta \tilde{g}_\mu^{(1,1,1)} = \delta g_\mu^{(1,1,1)} + 2 \sum_\nu \tilde{R}_{\mu,\nu}^{(1,2)} \delta g_\nu^{(1,1)}, \quad (\text{H.8})$$

where $\delta g_\mu^{(\dots)}$ denotes the difference of two solutions. The non-holomorphic function $\tilde{R}_{\mu,\nu}^{(1,2)}$ has a modular anomaly opposite to the one of the anomalous coefficient $g_{\mu,\nu}^{(1,2)}$. This implies that the function

$$\varphi_{1,1,1}^{(1)}(\tau, z; \mathbf{t}) = \sum_{\mu=0}^2 \left(\delta g_\mu^{(1,1,1)}(\tau) - 2 \sum_{\nu=0}^1 g_{\mu,\nu}^{(1,2)}(\tau) \delta g_\nu^{(1,1)}(\tau) \right) \Theta_\mu^{(3)}(\tau, z; \mathbf{t}), \quad (\text{H.9})$$

where $\Theta_\mu^{(3)}$ is the theta series for the A_2 lattice defined in (5.45), must be a Jacobi form of weight 4, index $t^2/2$ and trivial multiplier system, for any vector \mathbf{t} . In our case $\delta g_\mu^{(1,1,1)}$ is the difference between (5.99) and $\frac{1}{9}\mathfrak{g}_{3,\mu}$, $\delta g_\mu^{(1,1)}$ is the difference of (5.71) and $\frac{1}{3}\mathfrak{g}_{2,\mu} = H_\mu$, and $g_{\mu,\nu}^{(1,2)}$ is the solution (5.71) specialized to $(r_1, r_2) = (1, 2)$ and $\kappa = 1$. Using Mathematica, for the choice a) in (5.18) of the vectors $\mathbf{t}^{(r)}$ and for \mathbf{t} equal to any of the three vectors: $(1, -1, 0)$, $(0, 1, -1)$ or $(1, 0, -1)$, we have found that

$$\begin{aligned} \varphi_{1,1,1}^{(1)}(\tau, z; \mathbf{t}) = & \frac{1}{156031757844480\Delta(\tau)^2} \left[\left(34638E_4(\tau)^6 E_6(\tau) - 82938E_4(\tau)^3 E_6(\tau)^3 \right. \right. \\ & \left. \left. + 21840E_6(\tau)^5 \right) \varphi_{-2,1} + \left(22559E_4(\tau)^7 - 99247E_4(\tau)^4 E_6(\tau)^2 + 40428E_4(\tau) E_6(\tau)^4 \right) \varphi_{0,1} \right], \end{aligned} \quad (\text{H.10})$$

which is indeed a Jacobi form of the right weight and index. We have checked that the same conclusion holds also for $\mathbf{t} = (1, 1, -2)$ and $\mathbf{t} = (2, 0, -2)$. These checks confirm the consistency of the two solutions.

I. Expansion of anomalous coefficients for small charges

In this appendix we provide explicit q -series of the anomalous coefficients for a few sets of small charges. For two charges, the result is presented in terms of the VV function $r_0 \check{G}_{\mu_{12}}^{(\kappa; r_1, r_2)}$ introduced in (H.1) since it efficiently encodes independent components of the anomalous coefficients. The index μ_{12} is related to the indices of $g_{\mu,\mu_1,\mu_2}^{(r_1, r_2)}$ via (D.24) and takes $2\kappa_{12}$ values. However, the symmetry $\mu_{12} \rightarrow -\mu_{12}$ reduces the number of independent components to $\kappa_{12} + 1$. For three charges, we present directly the non-vanishing components of $g_{\mu,\mu_1,\mu_2,\mu_3}^{(\kappa; r_1, r_2, r_3)}$ where we added κ to the set of charges to distinguish different cases.

$$\begin{aligned} \check{G}_0^{(1;1,1)} &= q^{-1} \left(\frac{7}{497664} - \frac{7573}{82944} q - \frac{11993}{3456} q^2 - \frac{6147187}{15552} q^3 - \frac{417892013}{20736} q^4 - \frac{2669990303}{4608} q^5 \right. \\ &\quad \left. - \frac{3236466331}{288} q^6 - \frac{141840373163}{864} q^7 - \frac{106915932005927}{55296} q^8 + \dots \right), \\ \check{G}_1^{(1;1,1)} &= q^{-1/4} \left(\frac{247}{62208} + \frac{2441}{2592} q - \frac{685847}{6912} q^2 - \frac{60354863}{7776} q^3 - \frac{1794183169}{6912} q^4 - \frac{4761308023}{864} q^5 \right. \\ &\quad \left. - \frac{890009700749}{10368} q^6 - \frac{688179765559}{648} q^7 - \frac{25273195362785}{2304} q^8 + \dots \right), \end{aligned} \quad (\text{I.1})$$

$$\begin{aligned} \check{G}_0^{(1;2,1)} &= q^{-1} \left(\frac{23}{248832} - \frac{588457}{2903040} q - \frac{4374197}{181440} q^2 - \frac{1405439843}{435456} q^3 - \frac{45080954077}{241920} q^4 - \frac{18356399752001}{2903040} q^5 \right. \\ &\quad \left. - \frac{326985343617523}{2177280} q^6 - \frac{1998756778032397}{725760} q^7 - \frac{13340225185984769}{322560} q^8 + \dots \right), \\ \check{G}_1^{(1;2,1)} &= q^{-13/12} \left(\frac{7}{559872} + \frac{254521}{19595520} q + \frac{21208469}{19595520} q^2 - \frac{2141274337}{3265920} q^3 - \frac{436455080941}{9797760} q^4 - \frac{8043883306813}{6531840} q^5 \right. \\ &\quad \left. - \frac{70596332554177}{3919104} q^6 - \frac{289466108764979}{2799360} q^7 + \frac{307669392722141}{181440} q^8 + \dots \right), \\ \check{G}_2^{(1;2,1)} &= q^{-4/3} \left(\frac{5}{4478976} + \frac{12343}{2612736} q + \frac{20174263}{39191040} q^2 - \frac{22053267949}{78382080} q^3 - \frac{578517733229}{39191040} q^4 - \frac{678670136857}{2449440} q^5 \right. \end{aligned} \quad (\text{I.2})$$

$$\check{G}_3^{(1;2,1)} = q^{-3/4} \left(\frac{221}{725760} + \frac{12463}{72576} q - \frac{37963483}{2177280} q^2 - \frac{863218933}{362880} q^3 - \frac{3226760741}{20160} q^4 - \frac{6663319581967}{1088640} q^5 \right. \\ \left. - \frac{38738609259101}{241920} q^6 - \frac{115345074567433}{36288} q^7 - \frac{110739444675550361}{2177280} q^8 + \dots \right),$$

$$2\check{G}_0^{(1;2,2)} = q^{-2} \left(\frac{5}{60466176} + \frac{761731}{9876142080} q - \frac{5964113}{20995200} q^2 - \frac{2481864301319}{74071065600} q^3 - \frac{2872361748037}{411505920} q^4 \right. \\ \left. - \frac{2209512975931771}{2743372800} q^5 - \frac{41509779007262023}{617258880} q^6 - \frac{49227475374711403543}{12345177600} q^7 \right. \\ \left. - \frac{19464785233233206843}{114307200} q^8 + \dots \right),$$

$$2\check{G}_1^{(1;2,2)} = q^{-9/8} \left(\frac{277841}{8230118400} + \frac{1059416423}{74071065600} q + \frac{74795669047}{24690355200} q^2 - \frac{1801948967597}{1371686400} q^3 \right. \\ \left. - \frac{16603156407869}{55987200} q^4 - \frac{416033291771649169}{12345177600} q^5 - \frac{2105420048303835229}{914457600} q^6 \right. \\ \left. - \frac{313599714564738170867}{2962842624} q^7 - \frac{88220836719674177597447}{24690355200} q^8 + \dots \right), \quad (I.3)$$

$$2\check{G}_2^{(1;2,2)} = q^{-1/2} \left(\frac{39977803}{14814213120} + \frac{3589837921}{6172588800} q + \frac{129752549}{5443200} q^2 - \frac{22102735798051}{1543147200} q^3 \right. \\ \left. - \frac{114275375877392837}{24690355200} q^4 - \frac{936409392998996809}{2057529600} q^5 - \frac{4796069474036495173}{188956800} q^6 \right. \\ \left. - \frac{1501044707018652100451}{1543147200} q^7 - \frac{77342357414981744924971}{2743372800} q^8 + \dots \right),$$

$$\check{G}_0^{(1;3,1)} = q^{-2} \left(\frac{47}{318504960} + \frac{4247129}{45984153600} q - \frac{68036588023}{183936614400} q^2 - \frac{55422609269}{696729600} q^3 - \frac{337075127071841}{26276659200} q^4 \right. \\ \left. - \frac{5507214757771}{4490640} q^5 - \frac{7981897203311365099}{91968307200} q^6 - \frac{1110685079693008997}{239500800} q^7 - \frac{17226825559762052679301}{91968307200} q^8 + \dots \right),$$

$$\check{G}_1^{(1;3,1)} = q^{-25/24} \left(\frac{3981871}{55180984320} + \frac{2648895629}{137952460800} q + \frac{41222942111}{13138329600} q^2 - \frac{285856288266173}{91968307200} q^3 - \frac{2814997287076607}{5109350400} q^4 \right. \\ \left. - \frac{1347624999852051079}{25082265600} q^5 - \frac{935913250177725865183}{275904921600} q^6 - \frac{20586251240400924448327}{137952460800} q^7 \right. \\ \left. - \frac{270196785882281507783201}{55180984320} q^8 + \dots \right),$$

$$\check{G}_2^{(1;3,1)} = q^{-7/6} \left(\frac{2971}{107775360} + \frac{5544628433}{551809843200} q + \frac{115442040149}{13795246080} q^2 + \frac{23452076233427}{68976230400} q^3 - \frac{61420667950261}{638668800} q^4 \right. \\ \left. - \frac{5663089816965736121}{275904921600} q^5 - \frac{9916439642198445779}{5748019200} q^6 - \frac{2995144716151528141301}{34488115200} q^7 \right. \\ \left. - \frac{37830819493668472285}{12317184} q^8 + \dots \right),$$

$$\check{G}_3^{(1;3,1)} = q^{-11/8} \left(\frac{19}{2949120} + \frac{738683593}{91968307200} q - \frac{2206279697}{2874009600} q^2 - \frac{31913899958429}{91968307200} q^3 - \frac{1052975439005939}{18393661440} q^4 \right. \\ \left. - \frac{870826166392452223}{91968307200} q^5 - \frac{24710897250831165317}{30656102400} q^6 - \frac{3872950410038690100437}{91968307200} q^7 \right. \\ \left. - \frac{17796943645723753462087}{11496038400} q^8 + \dots \right), \quad (I.4)$$

$$\check{G}_4^{(1;3,1)} = q^{-2/3} \left(\frac{398074393}{220723937280} + \frac{7709928847}{19707494400} q - \frac{1124847101051}{110361968640} q^2 - \frac{56937410997937}{3284582400} q^3 - \frac{1198914724901166557}{367873228800} q^4 \right. \\ \left. - \frac{3539458767899738927}{12541132800} q^5 - \frac{8440035595252062897707}{551809843200} q^6 - \frac{4516265260602369778967}{7664025600} q^7 \right. \\ \left. - \frac{16911256238023424894329}{973209600} q^8 + \dots \right),$$

$$\check{G}_5^{(1;3,1)} = q^{-25/24} \left(\frac{1507663}{55180984320} + \frac{8390729309}{137952460800} q + \frac{138045501791}{13138329600} q^2 - \frac{108275799214493}{91968307200} q^3 - \frac{6633302879487181}{15328051200} q^4 \right. \\ \left. - \frac{1253534988377466919}{25082265600} q^5 - \frac{914235335203953688063}{275904921600} q^6 - \frac{20417088939750658199287}{137952460800} q^7 \right. \\ \left. - \frac{269351350642042364835521}{55180984320} q^8 + \dots \right),$$

$$\check{G}_6^{(1;3,1)} = q^{-3/2} \left(\frac{11}{159252480} + \frac{47213977}{22992076800} q + \frac{24794030699q^{1/2}}{22992076800} q^2 - \frac{107364499601}{821145600} q^3 - \frac{3501922956080341}{183936614400} q^4 \right. \\ \left. - \frac{100084012971526063}{22992076800} q^5 - \frac{286767755453058307}{656916480} q^6 - \frac{52241700229979368001}{2090188800} q^7 \right. \\ \left. - \frac{2134111773571338037187}{2189721600} q^8 + \dots \right),$$

$$\begin{aligned}
\check{G}_0^{(2;1,1)} &= -\frac{17}{128} - \frac{113}{72}q - \frac{4139}{64}q^2 - \frac{4249}{4}q^3 - \frac{3279635}{288}q^4 - \frac{370057}{4}q^5 - \frac{9852609}{16}q^6 \\
&\quad - \frac{63375745}{18}q^7 - \frac{1142163379}{64}q^8 + \dots, \\
\check{G}_1^{(2;1,1)} &= q^{-1/8} \left(\frac{1}{288} + \frac{47}{32}q + \frac{1369}{32}q^2 + \frac{109447}{144}q^3 + \frac{274299}{32}q^4 + \frac{575603}{8}q^5 + \frac{15679485}{32}q^6 \right. \\
&\quad \left. + \frac{91227113}{32}q^7 + \frac{234427805}{16}q^8 + \dots \right), \\
\check{G}_2^{(2;1,1)} &= q^{-1/2} \left(\frac{1}{2304} + \frac{5}{16}q - \frac{281}{32}q^2 - \frac{9563}{36}q^3 - \frac{913121}{256}q^4 - \frac{531815}{16}q^5 - \frac{70096291}{288}q^6 \right. \\
&\quad \left. - \frac{5984707}{4}q^7 - \frac{2056205767}{256}q^8 + \dots \right),
\end{aligned} \tag{I.5}$$

$$\begin{aligned}
\check{G}_0^{(2;2,1)} &= -\frac{6775}{20736} - \frac{7157}{576}q - \frac{361297}{768}q^2 - \frac{46696309}{5184}q^3 - \frac{600613969}{5184}q^4 - \frac{329780519}{288}q^5 \\
&\quad - \frac{12090114491}{1296}q^6 - \frac{169489988459}{2592}q^7 - \frac{467398870843}{1152}q^8 + \dots, \\
\check{G}_1^{(2;2,1)} &= q^{-1/24} \left(\frac{65}{5184} + \frac{6353}{864}q + \frac{1386517}{5184}q^2 + \frac{27324275}{5184}q^3 + \frac{87346705}{1296}q^4 + \frac{3384145273}{5184}q^5 \right. \\
&\quad \left. + \frac{26915024329}{5184}q^6 + \frac{46072770397}{1296}q^7 + \frac{559857514087}{2592}q^8 + \dots \right), \\
\check{G}_2^{(2;2,1)} &= q^{-1/6} \left(\frac{7}{2592} - \frac{319}{2304}q - \frac{178241}{5184}q^2 - \frac{793715}{1728}q^3 - \frac{470705}{2592}q^4 + \frac{5047214}{81}q^5 \right. \\
&\quad \left. + \frac{2610260809}{2592}q^6 + \frac{13190202073}{1296}q^7 + \frac{718155005}{9}q^8 + \dots \right), \\
\check{G}_3^{(2;2,1)} &= q^{-3/8} \left(\frac{11}{5184} + \frac{8129}{5184}q + \frac{3797}{144}q^2 - \frac{84545}{5184}q^3 - \frac{44772229}{5184}q^4 - \frac{98195357}{576}q^5 \right. \\
&\quad \left. - \frac{10553330059}{5184}q^6 - \frac{94932443735}{5184}q^7 - \frac{78048945335}{576}q^8 + \dots \right), \\
\check{G}_4^{(2;2,1)} &= q^{-2/3} \left(\frac{7}{41472} + \frac{3869}{10368}q - \frac{77077}{3456}q^2 - \frac{652597}{1296}q^3 - \frac{2914033}{648}q^4 - \frac{81599053}{5184}q^5 \right. \\
&\quad \left. + \frac{2596870511}{20736}q^6 + \frac{4497526727}{1728}q^7 + \frac{180081086629}{6912}q^8 + \dots \right), \\
\check{G}_5^{(2;2,1)} &= q^{-1/24} \left(\frac{281}{5184} + \frac{5093}{864}q + \frac{1358437}{5184}q^2 + \frac{27265307}{5184}q^3 + \frac{87314575}{1296}q^4 + \frac{3383929057}{5184}q^5 \right. \\
&\quad \left. + \frac{26914617169}{5184}q^6 + \frac{46072612447}{1296}q^7 + \frac{559856986399}{2592}q^8 + \dots \right), \\
\check{G}_6^{(2;2,1)} &= q^{-1/2} \left(\frac{55}{20736} + \frac{649}{576}q - \frac{228775}{5184}q^2 - \frac{2115335}{1296}q^3 - \frac{65662657}{2304}q^4 - \frac{1760666641}{5184}q^5 \right. \\
&\quad \left. - \frac{16293026827}{5184}q^6 - \frac{3481650799}{144}q^7 - \frac{1113729358001}{6912}q^8 + \dots \right),
\end{aligned} \tag{I.6}$$

$$\begin{aligned}
\check{G}_0^{(3;1,1)} &= -\frac{13}{96} + \frac{71}{48}q + \frac{103}{16}q^2 + \frac{139}{8}q^3 + \frac{2243}{48}q^4 + 104q^5 + \frac{3693}{16}q^6 + \frac{3667}{8}q^7 + \frac{14403}{16}q^8 + \dots, \\
\check{G}_1^{(3;1,1)} &= q^{-1/12} \left(-\frac{1}{64} + \frac{27}{32}q + \frac{211}{64}q^2 + \frac{139}{16}q^3 + \frac{179}{8}q^4 + \frac{1585}{32}q^5 + \frac{6979}{64}q^6 + \frac{6993}{32}q^7 + \frac{27263}{64}q^8 + \dots \right), \\
\check{G}_2^{(3;1,1)} &= q^{2/3} \left(-\frac{1}{32} - \frac{13}{16}q - \frac{157}{32}q^2 - \frac{233}{16}q^3 - \frac{309}{8}q^4 - \frac{677}{8}q^5 - \frac{2925}{16}q^6 - 355q^7 - \frac{10867}{16}q^8 + \dots \right), \\
\check{G}_3^{(3;1,1)} &= q^{1/4} \left(-\frac{5}{96} - \frac{37}{32}q - \frac{611}{96}q^2 - \frac{641}{32}q^3 - \frac{869}{16}q^4 - \frac{3991}{32}q^5 - \frac{25955}{96}q^6 - \frac{4341}{8}q^7 - \frac{16719}{16}q^8 + \dots \right),
\end{aligned} \tag{I.7}$$

$$\begin{aligned}
\check{G}_0^{(4;1,1)} &= -\frac{25891}{124416} - \frac{45463}{5184}q - \frac{852205}{1296}q^2 - \frac{32943085}{1296}q^3 - \frac{13379780347}{20736}q^4 - \frac{46060126277}{3888}q^5 \\
&\quad - \frac{12147774365}{72}q^6 - \frac{158959400096}{81}q^7 - \frac{66913439964953}{3456}q^8 + \dots, \\
\check{G}_1^{(4;1,1)} &= q^{-1/16} \left(-\frac{37}{7776} + \frac{743}{144}q + \frac{186041}{432}q^2 + \frac{49307035}{2592}q^3 + \frac{250278505}{486}q^4 + \frac{25390442633}{2592}q^5 \right. \\
&\quad \left. + \frac{1109071859713}{7776}q^6 + \frac{2183125435445}{1296}q^7 + \frac{21784639784605}{1296}q^8 + \dots \right), \\
\check{G}_2^{(4;1,1)} &= q^{-1/4} \left(\frac{247}{62208} + \frac{2441}{2592}q - \frac{685847}{6912}q^2 - \frac{60354863}{7776}q^3 - \frac{1794183169}{6912}q^4 - \frac{4761308023}{864}q^5 \right. \\
&\quad \left. - \frac{890009700749}{10368}q^6 - \frac{688179765559}{648}q^7 - \frac{25273195362785}{2304}q^8 + \dots \right), \\
\check{G}_3^{(4;1,1)} &= q^{-9/16} \left(\frac{5}{7776} + \frac{667}{2592}q + \frac{9853}{7776}q^2 + \frac{4114013}{2592}q^3 + \frac{208989733}{2592}q^4 + \frac{16164572585}{7776}q^5 \right. \\
&\quad \left. + \frac{31419487859}{864}q^6 + \frac{78893525585}{162}q^7 + \frac{41486136304309}{7776}q^8 + \dots \right), \\
\check{G}_4^{(4;1,1)} &= q^{-1} \left(\frac{7}{248832} + \frac{793}{31104}q + \frac{2371}{1296}q^2 - \frac{1033957}{7776}q^3 - \frac{51449111}{3456}q^4 - \frac{2662533095}{5184}q^5 \right. \\
&\quad \left. - \frac{10331116165}{972}q^6 - \frac{68953726973}{432}q^7 - \frac{157973370319477}{82944}q^8 + \dots \right),
\end{aligned} \tag{I.8}$$

$$\begin{aligned}
\check{G}_0^{(6;1,1)} &= -\frac{465535}{1769472} - \frac{525963}{25600} q - \frac{4385292809}{1638400} q^2 - \frac{43459913743}{230400} q^3 - \frac{95409288460249}{11059200} q^4 - \frac{716734134057}{2560} q^5 \\
&\quad - \frac{19031436950174249}{2764800} q^6 - \frac{46566304858176491}{345600} q^7 - \frac{357424842056971557}{163840} q^8 + \dots \\
\check{G}_1^{(6;1,1)} &= q^{-1/24} \left(-\frac{128371}{5529600} + \frac{40650107}{3686400} q + \frac{2207272423}{1228800} q^2 + \frac{799482612769}{5529600} q^3 + \frac{15579841901093}{2211840} q^4 \right. \\
&\quad \left. + \frac{3410305584319}{14400} q^5 + \frac{16460186000332049}{2764800} q^6 + \frac{130847802327022001}{1105920} q^7 + \frac{793328412584550589}{409600} q^8 + \dots \right) \\
\check{G}_2^{(6;1,1)} &= q^{-1/6} \left(\frac{1052509}{176947200} + \frac{209941}{138240} q - \frac{5204324219}{11059200} q^2 - \frac{2193309727}{34560} q^3 - \frac{67263775905863}{17694720} q^4 - \frac{49340829606167}{345600} q^5 \right. \\
&\quad \left. - \frac{1416482503757447}{368640} q^6 - \frac{6909229953035321}{86400} q^7 - \frac{239502419541351089947}{176947200} q^8 + \dots \right) \\
\check{G}_3^{(6;1,1)} &= q^{-3/8} \left(\frac{3059}{1228800} + \frac{1240819}{2211840} q + \frac{6103987}{409600} q^2 + \frac{40005301093}{2764800} q^3 + \frac{14584022769029}{11059200} q^4 + \frac{12408048600401}{204800} q^5 \right. \\
&\quad \left. + \frac{2031688897405031}{1105920} q^6 + \frac{14301651369906251}{345600} q^7 + \frac{30365831720828173}{40960} q^8 + \dots \right) \quad (I.9) \\
\check{G}_4^{(6;1,1)} &= q^{-2/3} \left(\frac{13157}{29491200} + \frac{202109}{1382400} q + \frac{117034913}{22118400} q^2 - \frac{13051427}{10800} q^3 - \frac{51184196269}{184320} q^4 - \frac{4080532598711}{230400} q^5 \right. \\
&\quad \left. - \frac{28426573177934497}{44236800} q^6 - \frac{748741181058329}{46080} q^7 - \frac{13975691481395331721}{44236800} q^8 + \dots \right) \\
\check{G}_5^{(6;1,1)} &= q^{-25/24} \left(\frac{19}{442368} + \frac{204533}{11059200} q + \frac{12303371}{11059200} q^2 - \frac{176760667}{2764800} q^3 + \frac{115535286821}{3686400} q^4 + \frac{19874511607}{5760} q^5 \right. \\
&\quad \left. + \frac{1786765849460017}{11059200} q^6 + \frac{52850388569884871}{11059200} q^7 + \frac{76411388946994309}{737280} q^8 + \dots \right) \\
\check{G}_6^{(6;1,1)} &= q^{-4/2} \left(\frac{11}{7077888} + \frac{877}{691200} q + \frac{108537}{409600} q^2 + \frac{1579361}{172800} q^3 - \frac{42904604779}{35389440} q^4 - \frac{10934623253}{25600} q^5 \right. \\
&\quad \left. - \frac{315649065581971}{11059200} q^6 - \frac{178841598736999}{172800} q^7 - \frac{8452543304385033}{327680} q^8 + \dots \right),
\end{aligned}$$

$$\begin{aligned}
g_0^{(1;1,1,1)} &= q^{-1} \left(-\frac{307}{53084160} + \frac{248189}{10616832} q + \frac{19861423}{8847360} q^2 + \frac{53650187}{737280} q^3 + \frac{1952990383}{8847360} q^4 - \frac{1384556270653}{17694720} q^5 \right. \\
&\quad \left. - \frac{5992150958009}{1769472} q^6 - \frac{36863885478071}{442368} q^7 - \frac{5287836198627985}{3538944} q^8 + \dots \right), \quad (I.10) \\
g_1^{(1;1,1,1)} &= q^{-4/3} \left(-\frac{67}{8599633920} + \frac{621023}{4299816960} q + \frac{145699391}{1074954240} q^2 - \frac{9013190993}{859963392} q^3 - \frac{4987262966201}{4299816960} q^4 \right. \\
&\quad \left. - \frac{29007521629433}{537477120} q^5 - \frac{779386701680657}{537477120} q^6 - \frac{56215493359702901}{2149908480} q^7 - \frac{49271213570980177}{143327232} q^8 + \dots \right),
\end{aligned}$$

$$\begin{aligned}
g_{0,0}^{(1;2,1,1)} &= q^{-2} \left(-\frac{17}{6449725440} - \frac{767269}{13544423424} q + \frac{112089181661}{1015831756800} q^2 + \frac{1043317980281}{56435097600} q^3 + \frac{27231789142567}{48372940800} q^4 \right. \\
&\quad \left. - \frac{9482261170891493}{253957939200} q^5 - \frac{96727150075903997}{20316635136} q^6 - \frac{35202935902746029581}{126978969600} q^7 - \frac{1159782609474435130523}{101583175680} q^8 + \dots \right) \\
g_{0,1}^{(1;2,1,1)} &= q^{-3/4} \left(-\frac{131899}{9069926400} + \frac{9311573}{220449600} q - \frac{3649155569}{21163161600} q^2 - \frac{1965367009399}{661348800} q^3 - \frac{583179799365883}{1984046400} q^4 \right. \\
&\quad \left. - \frac{1301972583520723}{75582720} q^5 - \frac{7019402825382382613}{9069926400} q^6 - \frac{56099092574472434849}{1984046400} q^7 - \frac{308436599879131398913}{362797056} q^8 + \dots \right) \\
g_{1,0}^{(1;2,1,1)} &= q^{-11/8} \left(-\frac{631}{1074954240} + \frac{1410085567}{1015831756800} q + \frac{16470146659}{28217548800} q^2 - \frac{19337664631871}{338610585600} q^3 - \frac{8998136290013387}{1015831756800} q^4 \right. \\
&\quad \left. - \frac{696106138300623817}{1015831756800} q^5 - \frac{8514447105692797999}{203166351360} q^6 - \frac{2131628379429727232249}{1015831756800} q^7 - \frac{4631805605272789952699}{56435097600} q^8 + \dots \right) \\
g_{1,1}^{(1;2,1,1)} &= q^{-9/8} \left(\frac{352537}{507915878400} + \frac{113751559}{18811699200} q + \frac{34152804511}{24186470400} q^2 - \frac{17760737685259}{42326323200} q^3 - \frac{2380349828670577}{56435097600} q^4 \right. \\
&\quad \left. - \frac{62194762125935}{2418647040} q^5 - \frac{67386472957097802719}{507915878400} q^6 - \frac{982660446399115373719}{169305292800} q^7 - \frac{11579230105257202681697}{56435097600} q^8 + \dots \right) \\
g_{2,0}^{(1;2,1,1)} &= q^{-3/2} \left(-\frac{253}{29023764480} + \frac{4062503}{18811699200} q + \frac{197690483}{846526464} q^2 - \frac{1374141989333}{31744742400} q^3 - \frac{153753168177467}{37623398400} q^4 \right. \\
&\quad \left. - \frac{5101538961903499}{24186470400} q^5 - \frac{846083267062775029}{63489484800} q^6 - \frac{3977241004532831147}{4534963200} q^7 - \frac{222532070896474737197}{5290790400} q^8 + \dots \right) \quad (I.11) \\
g_{2,1}^{(1;2,1,1)} &= q^{-5/4} \left(-\frac{1}{1866240} + \frac{24730109}{10581580800} q + \frac{2530558951}{15872371200} q^2 - \frac{510086582701}{3527193600} q^3 - \frac{160843599445}{70543872} q^4 \right. \\
&\quad \left. + \frac{1902396378056567}{31744742400} q^5 - \frac{5180315221699661}{317447424} q^6 - \frac{7375174671318611383}{3968092800} q^7 - \frac{3036698723644858704937}{31744742400} q^8 + \dots \right) \\
g_{3,0}^{(1;2,1,1)} &= q^{-11/8} \left(-\frac{631}{1074954240} + \frac{1410085567}{1015831756800} q + \frac{16470146659}{28217548800} q^2 - \frac{19337664631871}{338610585600} q^3 - \frac{8998136290013387}{1015831756800} q^4 \right. \\
&\quad \left. - \frac{232035378553067779}{338610585600} q^5 - \frac{2838149068202368037}{67722117120} q^6 - \frac{710542834607711853043}{338610585600} q^7 - \frac{41686249629140403636691}{507915878400} q^8 + \dots \right) \\
g_{3,1}^{(1;2,1,1)} &= q^{-9/8} \left(\frac{352537}{507915878400} + \frac{113751559}{18811699200} q + \frac{34152804511}{24186470400} q^2 - \frac{17760737685259}{42326323200} q^3 - \frac{2380349828670577}{56435097600} q^4 \right.
\end{aligned}$$

$$\begin{aligned}
& -\frac{230350970736869}{89579520} q^5 - \frac{22462157789819916613}{169305292800} q^6 - \frac{2947981349665353464437}{507915878400} q^7 - \frac{104213053588139844142793}{507915878400} q^8 + \dots), \\
g_{0,0,0,0}^{(2;1,1,1)} &= \frac{62959}{1658880} + \frac{226289}{207360} q + \frac{9809837}{552960} q^2 + \frac{13343}{69120} q^3 - \frac{2629441427}{829440} q^4 \\
&\quad - \frac{20201197}{360} q^5 - \frac{85538444479}{138240} q^6 - \frac{36356182001}{6912} q^7 - \frac{2584999288199}{69120} q^8 + \dots \\
g_{0,1,1,0}^{(2;1,1,1)} &= q^{-1/2} \left(-\frac{73}{995328} + \frac{6793}{829440} q + \frac{1989169}{829440} q^2 + \frac{24402883}{497664} q^3 + \frac{759624757}{1658880} q^4 \right. \\
&\quad \left. + \frac{662585173}{829440} q^5 - \frac{17919291181}{497664} q^6 - \frac{501969016519}{829440} q^7 - \frac{3398716509241}{552960} q^8 + \dots \right) \\
g_{1,1,0,0}^{(2;1,1,1)} &= q^{-1/6} \left(\frac{101}{995328} + \frac{824453}{4976640} q + \frac{476039}{207360} q^2 + \frac{54842903}{829440} q^3 + \frac{2494364191}{1658880} q^4 \right. \\
&\quad \left. + \frac{26617621331}{1244160} q^5 + \frac{110997670277}{497664} q^6 + \frac{1163782557059}{622080} q^7 + \frac{33182101157969}{2488320} q^8 + \dots \right) \\
g_{1,1,1,1}^{(2;1,1,1)} &= q^{-2/3} \left(-\frac{23}{9953280} + \frac{16607}{1244160} q - \frac{284833}{995328} q^2 - \frac{4913885}{124416} q^3 - \frac{1967897171}{2488320} q^4 \right. \\
&\quad \left. - \frac{1152163793}{124416} q^5 - \frac{378680060593}{4976640} q^6 - \frac{97855125631}{207360} q^7 - \frac{5549175976751}{2488320} q^8 + \dots \right).
\end{aligned} \tag{I.12}$$

J. Index of notations

In most of the paper we use boldface letters to denote vectors consisting of n or $n - 1$ components, the blackboard script for the extended lattice as well as for generic d -dimensional lattices and their vectors, the sans-serif script for glue vectors, and the mathfrak script for vectors with n_k components resulting from the decomposition $n = \sum_{k=1}^m n_k$ and for the vectors $\mathbf{t}_\alpha^{(r)}$ determining the dependence on the refinement parameters.

Symbol	Description	Appears or defined in
$\mathbf{A} = \{\mathbf{a}_0, \mathbf{a}_1, \dots, \mathbf{a}_n\}$	glue vector indices for the lattice decomposition (5.31)	(5.34)
$\mathcal{A}_0^{(r)}$	set of glue vectors of the lattice decomposition (5.31)	(F.30)
	corresponding to the maximal order zero modes	
$\beta = -\text{Im}(z)/\tau_2$	real variable parametrizing z	(5.2)
$\beta = -\text{Im}(z)/\tau_2$	real vector parametrizing z	(C.1)
$b_2 = b_2(\mathfrak{Y})$	second Betti number of \mathfrak{Y}	p.2
$\mathbf{B} = \{\mathbf{b}_{ij}, \tilde{\mathbf{b}}_{ij}\}$	self-glue vector indices induced by the basis (5.46)	below (5.80)
c_2	the second Chern class of \mathfrak{Y}	(2.4)
c_r	coefficient of the leading pole of the modular ambiguity	(5.124)
$\mathbf{C} = \{\mathbf{c}_k\}$	glue vector indices for the lattice decomposition (F.9)	(F.12)
$\delta_x^{(n)}$	mod- n Kronecker delta symbol	(4.3)
d_r	dimension of the lattice extension factor \mathbb{Z}^{d_r} associated to charge r	(5.17)
$\mathbf{D}^{(r)}, \mathbb{D}_{\parallel}^{(r)}$	discriminant groups of the lattices $\mathbf{\Lambda}^{(r)}, \mathbb{\Lambda}_{\parallel}^{(r)}$	§F.1, §5.6.1
$\mathcal{D}_m^{(n)}$	differential operator on Jacobi forms	(A.5)
$\mathcal{D}\Theta_a^{(N)}(\tau; \mathbf{t})$	modular derivative of the A_{N-1} lattice theta series	(5.70)
$\Delta_0^{(r)}(\mathbf{A}, \mathbf{B})$	Kronecker symbol ensuring the maximal order zero mode condition	(5.116)
e_m	weights of degenerate charge configurations	(5.108)
$\mathbb{e}_0, \mathbb{e}_{i,\alpha}$	basis vectors of $\mathbb{\Lambda}_{\perp}^{(r)}$	(5.27)

$E_n(\tau)$	Eisenstein series	(A.3), §H
$E_n(\mathcal{M}; \mathbf{u})$	generalized error function on \mathbb{R}^n	(D.1)
$\mathcal{E}_n = \mathcal{E}_n^{(0)} + \mathcal{E}_n^{(+)}$	function encoding the modular completion	(D.10)
$\check{\phi}_{\mu, \mu}^{(r)}(\tau, z, \mathbf{z})$	modular ambiguity of anomalous coefficients	(5.101)
$\phi_v^{(r)}(\tau, z)$	Jacobi-like form encoding the modular ambiguity	(5.101)
$\Phi_n^E(\{\mathbb{V}_i\}; \mathbb{X})$	boosted error function	(D.2)
$\Phi_n^{\mathcal{E}}(\mathbf{x})$	kernel corresponding to the function \mathcal{E}_n	(D.7)
$\Phi^{(r)}(\mathbb{X}; \{\mathcal{F}^{(s)}\})$	kernel of the theta series representing $\check{g}_{\mu, \mu}^{(r)\text{ref}}$	(5.105)
$\Phi_{\delta}^{(r)}(\mathbb{X}, \tau, \mathbf{z})$	kernel representing the modular ambiguity $\phi_v^{(r)}$	(5.103)
$\mathcal{F}^{(r)}(\mathbb{X})$	theta series kernel encoding $\Phi^{(r)}$	(5.107)
$\gamma = (p^0, r, q, q_0)$	charge vector of a generic D6-D4-D2-D0 bound state	p.5
$\hat{\gamma} = (r, q)$	reduced charge vector	(2.9)
$\hat{\gamma} = (\hat{\gamma}_1, \dots, \hat{\gamma}_n)$	vector of reduced charges	(2.9)
γ_{ij}	Dirac-Schwinger-Zwanziger product	(2.10)
$g_{\mu, \mu}^{(r)}(\tau)$	anomalous coefficient	Thm 3.1
$g_{\mu, \mu}^{(r)\text{ref}}(\tau, z)$	refined anomalous coefficient	(5.4)
$\check{g}_{\mu, \mu}^{(r)\text{ref}}(\tau, z, \mathbf{z})$	auxiliary anomalous coefficient associated to the extended lattice	(5.8)
$\widehat{g}_{\mu, \mu}^{(r)}, \widehat{g}_{\mu, \mu}^{(r)\text{ref}}, \widetilde{g}_{\mu, \mu}^{(r)\text{ref}}$	modular completions for various versions of anomalous coefficients	(3.3)
$\mathfrak{g}_{n, \mu}(\tau)$	normalized generating function of $SU(n)$ VW invariants on \mathbb{P}^2	(4.20)
\mathfrak{g}_A	glue vectors for the lattice decomposition (5.31)	(5.34)
$G_{\mu}^{(\kappa)}, \check{G}_{\mu}^{(\kappa; r_1, r_2)}$	vectorial versions of the anomalous coefficient for two charges	(4.2), (H.1)
$\mathcal{G}_{\mu}^{(d)}(\tau)$	seed functions for the mock modular forms of optimal growth	(4.10)
$h_{r, \mu}(\tau)$	generating function of D4-D2-D0 BPS indices (rank 0 DT invariants)	(2.5)
$\widehat{h}_{r, \mu}(\tau)$	modular completion of $h_{r, \mu}(\tau)$	(2.8)
$\widetilde{h}_{r, \mu}(\tau)$	redefined version of $h_{r, \mu}(\tau)$	(2.12)
$\widetilde{h}_{r, \mu}^{(\text{an})}(\tau)$	anomalous part of $\widetilde{h}_{r, \mu}(\tau)$	(3.1)
$\widetilde{h}_{r, \mu}^{(0)}(\tau)$	modular ambiguity in $\widetilde{h}_{r, \mu}(\tau)$	(3.1)
$\mathcal{I}_v^{(r)}(z)$	maximal order zero mode contribution to $\vartheta_{\mu, \mathbf{A}}^{(r)\parallel}$	(5.116)
j_k	labels appearing in anomaly equations	(3.4)
κ	intersection number of CY	(2.1)
$\kappa_i, \kappa_{ij}, \kappa_{ijk}$	κ rescaled by magnetic charges r_i	(5.23)
\mathbb{k}	summation variable for theta series	(C.1)
$\Lambda^{(r)}$	lattice of D2-brane charges of constituents	(5.12)
$\widetilde{\Lambda}^{(r)}$	sublattice of $\Lambda_{\parallel}^{(r)}$ isomorphic to $\Lambda^{(r)}$	(5.25)
$\mathbb{A}^{(r)}$	extended lattice	(5.14)
$\mathbb{A}_{\parallel}^{(r)}$	sublattice of $\mathbb{A}^{(r)}$ containing null-vectors	(5.25)

$\mathbb{A}_\perp^{(r)}$	sublattice of $\mathbb{A}^{(r)}$ orthogonal to $\mathbb{A}_\parallel^{(r)}$	(5.28)
λ_i	convenient combination of residue classes	(5.86)
μ	residue class of q modulo spectral flow	(2.2)
μ_{12}	effective residue class for two charges	(D.24)
$\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$	vector of residue classes	(2.8)
$\hat{\boldsymbol{\mu}} = (\hat{\mu}_1, \dots, \hat{\mu}_n)$	vector of residue classes belonging to $\mathbf{D}^{(r)}$	(5.37)
$\hat{\boldsymbol{\mu}}^{(r)}(\boldsymbol{\mu}, \lambda)$	a different parametrization of $\hat{\boldsymbol{\mu}}$	(F.3)
\sqcup	vector of residue classes of the extended lattice	(5.37)
$\Delta\mu$	difference of residue classes	(2.15)
m_r	index of $g_{\mu, \mu}^{(r)\text{ref}}$	(5.5)
\mathbf{m}_k	subset of residue classes appearing in anomaly equations	(3.4)
$\nu_0(\mathbf{A}), \nu_{i, \alpha}(\mathbf{A})$	residue classes determined by glue vectors	(5.42)
$\nu = (\hat{\boldsymbol{\mu}}, \hat{\boldsymbol{\mu}})$	residue class of $\mathbb{A}_\parallel^{(r)}$	below (5.101)
n_k	number of charges in the k -th subset	(3.3)
N_{ij}	range of the glue vector index \mathbf{b}_{ij}	(F.7)
\mathbb{P}	characteristic vector of the extended lattice	(5.37)
$q = e^{2\pi i \tau}$	expansion parameter of generating series	(2.5)
q	D2-brane charge	(2.2)
\hat{q}_0	invariant D0-brane charge	(2.3)
$\mathbf{q} = (\frac{q_1}{\kappa r_1}, \dots, \frac{q_n}{\kappa r_n})$	vector of D2-brane charges	(5.38)
$Q_n(\gamma_i)$	quadratic form in anomaly equations	(2.11)
$\boldsymbol{\rho} = (\rho_1, \dots, \rho_n)$	vector of Bezout integers for \mathbf{r}	above (5.34)
$\boldsymbol{\rho}^{(0)}, \boldsymbol{\rho}^{(k)}$	vector of Bezout integers for \mathbf{s} and \mathbf{r}_k	(F.10)
r	total D4-brane charge	p.5
r_0	gcd of (r_1, \dots, r_n)	below (5.23)
r_i	D4-brane charge of the i -th constituent	(2.8)
$\hat{r}_i = r_i/r_0$	normalized D4-brane charge	(5.16)
r_{ij}	gcd of (r_i, r_j)	(5.21)
$\mathbf{r} = (r_1, \dots, r_n)$	vector of D4-brane charges	(2.8)
\mathbf{r}_k	k -th subset of D4-brane charges appearing in anomaly equations	(3.4)
$R_{\mu, \boldsymbol{\mu}}^{(r)}(\tau, \bar{\tau})$	coefficients of the anomaly equation (2.8)	(2.9)
$\tilde{R}_{\mu, \boldsymbol{\mu}}^{(r)}(\tau, \bar{\tau})$	redefined coefficients of the anomaly equation	(2.18)
$R_{\mu, \boldsymbol{\mu}}^{(r)\text{ref}}(\tau, \bar{\tau}, z)$	coefficients of the refined anomaly equation	(5.2)
$\mathcal{R}_n(\hat{\boldsymbol{\gamma}}; \tau_2)$	contribution to $R_{\mu, \boldsymbol{\mu}}^{(r)}$ of a given charge decomposition	(D.13)
$\mathcal{R}_n^{\text{ref}}(\hat{\boldsymbol{\gamma}}; \tau_2, \beta)$	contribution to $R_{\mu, \boldsymbol{\mu}}^{(r)\text{ref}}$ of a given charge decomposition	(D.14)
$\mathbf{s} = (s_1, \dots, s_m)$	vector of D4-brane charges of subsets of constituents	(3.4)
$S_{\mathcal{T}}(\hat{\boldsymbol{\gamma}})$	large τ_2 limit of the generalized error function $\Phi_n^E(\{\mathbf{v}_e\}; \mathbf{x})$ defined by the tree \mathcal{T}	(D.12)
Sym	symmetrization with respect to charges	(2.9)
$\mathfrak{t}^{(r_i)}$	vector determining the dependance on z_i	(5.18)
$\theta_1(\tau, z)$	Jacobi theta series	(C.9)

$\theta_\mu^{(\kappa)}(\tau)$	unary theta series with even quadratic form	(C.7)
$\vartheta_{\nu_0}^{(d)}(\tau)$	unary theta series appearing in (5.43)	(5.44)
$\vartheta_{\mathfrak{p}}(\tau, \mathbb{Z}; \mathbb{A}, \Phi, \mathbb{P})$	indefinite theta series associated with the lattice \mathbb{A} and kernel Φ	(C.1)
$\vartheta_{\vec{\nu}(\mathbb{A}, \mathbb{B})}^{(\mathbf{r})}(z)$	contribution to $\mathcal{I}_{\vec{\nu}}^{(\mathbf{r})}$ for a fixed glue vector of index \mathbb{B}	(5.118)
$\vartheta_{\mathfrak{p}, \mathbb{A}}^{(\mathbf{r})k}(\tau)$	unrefined limit of the contribution of order k zero modes	(G.14)
$\vartheta_{\mathfrak{p}, \mathbb{A}}^{(\mathbf{r})\parallel}(\tau, z)$	theta series associated with the lattice $\mathbb{A}_{\parallel}^{(\mathbf{r})}$	(5.41)
$\vartheta_{\mathbb{A}}^{(\mathbf{r})\perp}(\tau, z)$	theta series associated with the lattice $\mathbb{A}_{\perp}^{(\mathbf{r})}$	(5.41)
$\Theta_a^{(N)}(\tau, z; \mathbf{t})$	theta series associated with A_{N-1} root lattice	(5.45)
θ	vector of $\mathbf{\Lambda}^{(\mathbf{r})}$ determining the index	(5.37)
$\mathbb{T}_n^{\mathbb{S}}$	set of Schröder trees with n leaves	p.54
$\hat{\mathbf{u}}_{ij}$	normalized vectors in $\tilde{\mathbf{\Lambda}}^{(\mathbf{r})}$	below (5.25)
$\mathbf{v}_{ij}, \mathbf{v}_e, \mathbf{v}_\ell$	vectors in $\mathbf{\Lambda}^{(\mathbf{r})}$	(5.19), §D
$\mathbb{V}_{ij}, \mathbb{V}_\ell$	vectors in $\mathbb{A}^{(\mathbf{r})}$ extending $\mathbf{v}_{ij}, \mathbf{v}_\ell$	(5.20), (5.108)
$\hat{\mathbb{W}}_{ij}$	normalized vectors in $\mathbb{A}^{(\mathbf{r})}$	(5.21)
ω_{ij}	scalar product of the lattice vector with null vectors	(G.1)
\mathbb{W}_{ij}	null vectors in $\mathbb{A}^{(\mathbf{r})}$	(5.20)
$\hat{\mathbb{W}}_{ij}$	normalized null vectors	(5.21)
χ_r	arithmetic genus of the divisor \mathcal{D}_r	(2.7)
$\mathbb{x} = \sqrt{2\tau_2}(\mathbb{k} + \beta)$	d -dimensional vector, argument of kernels of theta series	(C.1)
$\mathbb{x}_\beta = \sqrt{2\tau_2}\mathbb{k}$	shifted vector \mathbb{x}	above (5.39)
$y = e^{2\pi iz}$	exponential of the refinement parameter	§5.2
z	refinement parameter	§5.2
$\mathbf{z} = (z_1, \dots, z_n)$	refinement parameters associated to the extended lattice	(5.8)
\mathbb{Z}	vector of all refinement parameters	(5.37)
\mathcal{Z}_n	set of integers from 1 to n	(5.107)
$\cdot, *$	bilinear forms on $\mathbf{\Lambda}^{(\mathbf{r})}$ and $\mathbb{A}^{(\mathbf{r})}$	(5.13), (5.15)

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