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On Neighbourhood Based T_0 , T_1 & T_2 Separation Axioms In Graphs

Santhosh Kumar C G, Dr Sr Bindhu K Thomas

November 28, 2024

Abstract 1. *This paper explores neighborhood-based T_0 , T_1 & T_2 separation axioms in graph theory. These separation axioms classify graphs based on their structural properties concerning vertex neighborhoods. We examine theoretical concepts, provide proofs for various types of graphs and explore the implications of these axioms. The findings contribute to understanding graph connectivity and separation, with potential applications in materials science and network design.*

Key Words: Separation axioms, T_0 , T_1 & T_2 axioms

Introduction

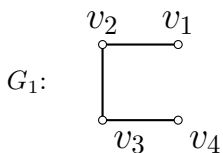
Graph theory plays a crucial role in mathematical modeling, computer science, and network analysis. The T_0 , T_1 & T_2 separation axioms provide a framework for analyzing vertex relationships within graphs. These axioms, derived from topology, help classify graphs based on vertex distinguishability through neighborhood structures. This paper defines and analyzes T_0 , T_1 & T_2 graphs, presenting theorems and proofs to characterize various graph types, including complete, bipartite, and cyclic graphs. Understanding these axioms enhances the study of structural properties and connectivity in complex networks.

Throughout this article, we let G be a simple graph, $V(G)$, $E(G)$ respectively the vertex and edge sets of G .

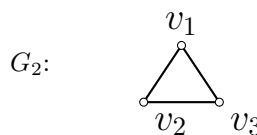
Definition 1. *The neighborhood of a vertex v in $V(G)$ is denoted by $N(v)$ and is defined as $N(v) = \{u \in V(G) : u \text{ adj. } v\}$*

Definition 2. *A graph G is said to be a T_0 graph if for every pair of vertices u and v of G , there exists a neighborhood $N(w)$, where w is distinct from u and v , which contains one of u or v , but not both.*

Examples



G_1 is T_0

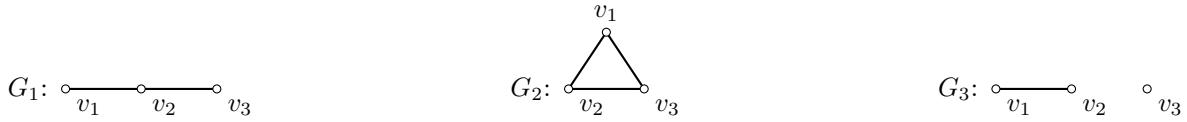


G_2 is not T_0

It is easy to see from the definition that the order of a T_0 graph is greater than or equal to 3, and must have at least one edge. Thus an empty graph is not T_0 .

Proposition 1. *The order of a T_0 graph is greater than 3.*

Proof: Let us consider all possible simple graphs of order 3. For the existence of a neighborhood, there should be at least one edge. Thus the possibilities are :



Not T_0 (Consider v_1, v_3)

Not T_0 (Consider any two vertices)

Not T_0 (Consider v_1, v_2)

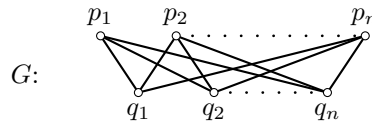
Thus a T_0 graph contains at least 4 vertices.

Theorem 1. *The complete graph K_n is not T_0 for any n .*

Proof: In K_n each vertex is adjacent to all other vertices. So the neighborhood of any vertex contains all other vertices.

Theorem 2. *The complete bipartite graph $G = K_{mn}$ for $m, n \geq 2$ is not T_0 .*

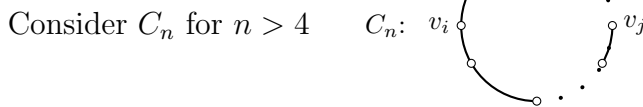
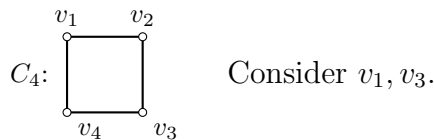
Proof: Let $V(G) = V_1 \cup V_2$. Suppose $V_1 = \{p_1, p_2, \dots, p_m\}$ and $V_2 = \{q_1, q_2, \dots, q_n\}$.



In G , q_i is adjacent to $p_j \forall i, j$. Therefore $N(q_i) = V_1$ and $N(p_j) = V_2$. So K_{mn} is not T_0 .

Theorem 3. *The cycle C_n with n vertices is $T_0 \forall n > 4$.*

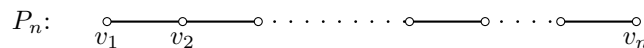
Proof: C_4 is not T_0 .



Consider any two vertices v_i and v_j ($i < j$) of C_n . If v_i adjacent to v_j , then $j = i + 1$ and $N(v_{i-1})$ contains v_i not v_j . If v_i not adjacent to v_j , then $N(v_{i-1})$ or $N(v_{i+1})$ will contain v_i not v_j . Thus C_n is T_0 for $n \geq 5$.

Theorem 4. *The path P_n of n vertices is T_0 for $n > 3$.*

Proof:



Consider any two vertices v_i and v_j ($i < j$) of P_n . If v_i adjacent to v_j , then $j = i + 1$ and $N(v_{i-1})$ contains v_i not v_j . If v_i not adjacent to v_j , then $N(v_{i-1})$ or $N(v_{i+1})$ will contain v_i not v_j . Thus P_n is T_0 for $n \geq 4$.

Theorem 5. *A graph is T_0 iff each of its component is so.*

Proof: Let G be a T_0 graph. If G is connected, done. Suppose not. Let G has r components. Consider an arbitrary component G_k of G . If G_k is not T_0 then there exists vertices u and v of G_k such that the neighborhood $N(w)$ of any vertex $w \neq u, v$ contains both u and v , which in turn makes G not T_0 . Thus G is T_0 implies each of its component is T_0 .

Conversely assume that each component of G is T_0 . Then obviously G is T_0 .

Corollary 1. *Let G be a graph with n vertices. If $K_n, n > 2$ is a component of G , then G is not T_0 .*

Theorem 6. *The T_0 axiom is preserved under isomorphism.*

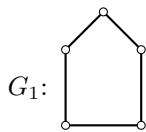
Proof: Let G_1 be a T_0 graph and suppose that $G_1 \approx G_2$. By suitably redrawing G_2 can be identified with G_1 . Thus G_2 is also T_0 .

Theorem 7. *The join of two T_0 graphs is again T_0 .*

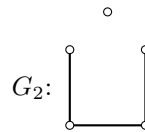
Proof: Let G_1 and G_2 be two T_0 graphs and suppose $G = G_1 \vee G_2$, the join of G_1 and G_2 . Consider any two vertices u_1 and u_2 of G . If u_1 and u_2 are in G_1 (or in G_2), which is T_0 , there exists a vertex v in G_1 (or in G_2) such that $N(v)$ contains either u_1 or u_2 but not both. Suppose u_1 is in G_1 and u_2 is in G_2 . Since G_1 is T_0 there exists a vertex v in G_1 such that $N(v)$ does not contain u_1 , but it contains u_2 in G . Thus G is T_0 .

Definition 3. *A graph G is said to be a T_1 graph if for every pair of vertices u and v of G , there exists a neighborhood $N(w)$, where w is distinct from u and v , which contains u but not v (and hence there exists another $N(w')$ containing v but not u).*

Examples



G_1 is T_1



G_2 is not T_1

Remark: It is obvious that every T_1 graph is T_0 but the converse is not true (the graph G_2 above is T_0 but not T_1).

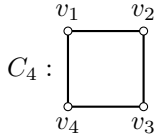
Proposition 2. *Since every T_1 graph is T_0 , we have the following:*

- (a) *The order of a T_1 graph is greater than 3.*
- (b) *The complete graph K_n is not T_1 .*
- (c) *A T_1 graph cannot contain K_n as its component for any $n > 1$.*

(d) The complete bipartite graph K_{mn} is not T_1 .

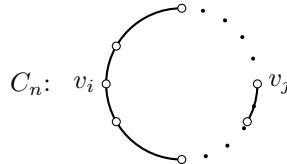
Theorem 8. The cycle C_n with n vertices is $T_1 \forall n \geq 5$.

Proof: C_4 is not T_1 .



Consider the vertices v_1 and v_3 .

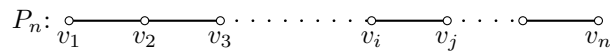
Consider C_n for $n > 4$



Consider any two vertices v_i and v_j ($i < j$) of C_n . If v_i adjacent to v_j , then $j = i + 1$. In this case, $N(v_{i-1})$ contains v_i not v_j and $N(v_{j+1})$ contains v_j not v_i . If v_i not adjacent to v_j , then either $v_i \in N(v_{i-1})$ and $v_j \in N(v_{j+1})$ or $v_i \in N(v_{i+1})$ and $v_j \in N(v_{j-1})$. Thus C_n is T_1 for $n \geq 5$.

Theorem 9. The path P_n of n vertices is not T_1 for any n .

Proof:



Consider the vertices v_1 and v_2 . The only neighborhood of v_1 is v_2 and hence P_n is not T_1 .

Theorem 10. A graph is T_1 iff each of its component is so.

Proof: Let G be a T_1 graph. If G is connected, done. Suppose not. Let G has r components. Consider an arbitrary component G_k of G . If G_k is not T_1 then there exist vertices u and v of G_k such that either every neighborhood $N(w)$ of any vertex $w \neq u, v$ contains both u and v (as in K_n) or at least one of u or v does not have such a neighborhood. In either case G not T_1 . Thus G is T_1 implies each of its component is T_1 .

Conversely assume that each component of G is T_1 . Then obviously G is T_1 .

Theorem 11. In a T_1 graph G , $\delta(v) \geq 2 \forall v \in V(G)$.

Proof: A vertex of degree 0 is not a neighbor of any vertex. If $d(v) = 1$, then v has only one neighbor, say u . Then $\{u, v\}$ cannot be separated by neighborhoods $N(u')$ and $N(v')$ ($u', v' \neq u, v$) as in the definition of a T_1 graph. Thus in a T_1 graph, $\delta(v) \geq 2$.

Theorem 12. The T_1 axiom is preserved under isomorphism.

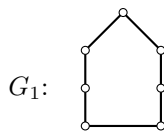
Proof: Let G_1 be a T_1 graph and suppose $G_1 \approx G_2$. By suitably redrawing G_2 can be identified with G_1 . Thus G_2 is also T_1 .

Theorem 13. *The join of two T_1 graphs is never T_1 .*

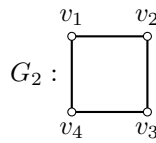
Proof: Let G_1 and G_2 be two T_1 graphs and suppose $G = G_1 \vee G_2$, the join of G_1 and G_2 . Let u and v be any two vertices of G where $u \in V(G_1)$ and $v \in V(G_2)$. G_1 is a T_1 graph and therefore there is a vertex u' in G_1 such that $N(u')$ does not contain u . Also each vertex in G_1 is adjacent to every vertex of G_2 and thus every such $N(u')$ will contain v . So we cannot find a vertex w in $V(G_1)$ or in $V(G_2)$ such that $u \in N(w)$ and $v \notin N(w)$. Hence G is not T_1 .

Definition 4. *A graph G is said to be a T_2 graph if for every pair of vertices u and v of G there exists disjoint neighborhoods $N(u'), N(v')$ (where $u, v \neq u', v'$) such that $u \in N(u')$ and $v \in N(v')$.*

Examples



G_1 is T_2



G_2 is not T_2

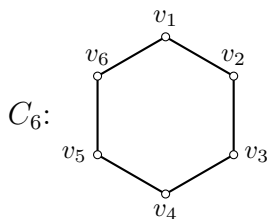
Remark: It is obvious that every T_2 graph is T_1 and therefore T_0 , but the converse is not true (C_5 is T_1 but not T_2).

Proposition 3. *Since every T_2 graph is T_1 , we have the following:*

- (a) *The order of a T_2 graph is > 3 .*
- (b) *The complete graph K_n is not T_2 .*
- (c) *A T_2 graph cannot contain K_n as its component for any $n > 1$.*
- (d) *The complete bipartite graph K_{mn} is not T_2 .*
- (e) *The path P_n of n vertices is not T_2 for any n .*
- (f) *In a T_2 graph G , $\delta(v) \geq 2$ for every $v \in V(G)$*

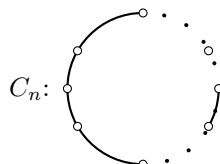
Theorem 14. *The cycle C_n with n vertices is $T_2 \forall n \geq 7$.*

Proof: C_6 is not T_2 .



Consider the vertices v_3 and v_5 .

Consider C_n for $n \geq 7$



Consider any two vertices v_i and v_j ($i < j$) of C_n . If v_i adjacent to v_j , then $j = i + 1$. In this case $N(v_{i-1})$ and $N(v_{j+1})$ are disjoint neighborhoods containing v_i and v_j respectively. If v_i not adjacent to v_j , then either $N(v_{i-1})$ and $N(v_{j+1})$ or $N(v_{i+1})$ and $N(v_{j-1})$ are disjoint neighborhoods containing v_i and v_j . Thus C_n is T_2 for $n \geq 7$.

Theorem 15. *A graph is T_2 iff each of its component is so.*

Proof: Let G be a T_2 graph. If G is connected, done. Suppose not. Let G has r components. Consider an arbitrary component G_k of G . If G_k is not T_2 then there exists a pair of vertices u and v of G_k every neighborhood $N(u')$, $N(v')$ (where $u, v \neq u', v'$) containing u and v has at least a common vertex in them. In any case G not T_2 . Thus G is T_2 implies each of its component is T_2 .

Conversely assume that each component of G is T_2 . Then obviously G is T_2 .

Theorem 16. *Isomorphisms preserve the T_2 axiom.*

Proof: Let G_1 be a T_0 graph and suppose $G_1 \approx G_2$. By suitably redrawing G_2 can be identified with G_1 . Thus G_2 is also T_2 .

Theorem 17. *The join of two T_2 graphs is never T_2 .*

Proof: We know that every T_2 graph is T_1 and the join of two T_1 graphs is never T_1 . Thus the join of T_2 graphs is not T_2 .

Conclusion

Conclusion This paper presents a comprehensive study of neighborhood-based T_0 , T_1 , and T_2 separation axioms in graph theory. Through definitions, examples, and theorems, we demonstrated how these axioms classify different types of graphs and their components. The study enhances the understanding of graph properties, contributing to various applications in science and engineering. Further research could explore the computational aspects of these axioms in larger and more complex networks.

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