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Extended approximations of nonlinear time-delay control systems

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Abstract—The structure of nonlinear time-delay control systems is analyzed thanks to a sequence of delay free extended systems. Each of these extended systems is viewed as an approximation of the original time-delay system. Whether the trajectories can be recovered via a suitable initialization of the extended system is argued as well.

I. INTRODUCTION

Time-delay systems are frequently used in control theory to describe dynamic processes in many different fields. As an example, it is sufficient to think of a biological process or networks in telecommunications. In recent years, for systems affected by constant and commensurate delays, solutions for different analysis and control problems have been found thanks to the development of an extension of the geometrical theory applied to the nonlinear time-delay systems framework ([1]). This extension involves the definition of a finite-dimensional dynamical system associated to the time-delay system under analysis.

The key idea is that if a nonlinear time-delay system has a solution for a particular period of time, then a delay-free system can be defined and initialized in such a way that its solution will coincide with the instantaneous state of the delayed one during that period of time. Thus, the approximating dynamics will describe the main geometric properties of the original time-delay system. More information about initialization can be found in [2] and geometrical theory for differential equations of retarded type is given in [3].

The approximation using such a finite-dimensional system is useful in the definition of distributions on which it is possible to define Lie brackets. Accordingly, on the associated time-delay system an extended Lie bracket tool is automatically defined [1]. Such a tool allows characterizing the solution to several control problems such as the feedback linearization problem ([4]), the conditions for the equivalence to a dynamics free of delays [5], the input-output injection linearization problem with regular output transformation by means of a bicausal change of coordinates [6], or the characterization of the accessibility conditions [7].

In this paper, the state-space of this extended dynamics on a particular time interval is related, using the so-called step-method [8], with the solution of the nonlinear time-delay system. This relationship states that the initial conditions of the original time-delay system are enough to define the solution of the extended system.

Notations are displayed in Section II. Section III states that, given an initialized time-delay system, its trajectory in some time interval in the future can always be computed from some extended delay-free system. Conversely, its trajectory defined for $t \geq 0$ can be extended in the past only when its initial condition fulfils specific conditions detailed in Section IV.

II. SYSTEM DEFINITION AND MATHEMATICAL SETTINGS

Let us consider the commensurate time-delay systems described by the equations

$$\begin{aligned}\dot{\mathbf{x}}(t) &= f(\mathbf{x}(t), \dots, \mathbf{x}(t - s\tau)) + \\ &\quad \sum_{j=0}^s g_j(\mathbf{x}(t), \dots, \mathbf{x}(t - s\tau))u(t - j\tau) \\ y(t) &= h(\mathbf{x}(t), \dots, \mathbf{x}(t - s\tau)),\end{aligned}\quad (1)$$

where $s \in \{k \in \mathbb{Z} : k \geq 0\}$, in other words, $s \geq 0$ is an integer and the variables $\mathbf{x}(t) \in \mathbb{R}^n$, $y(t), u(t) \in \mathbb{R}$ denote, respectively, the instantaneous values of the state, output, and input functions. The function of initial conditions $\vartheta : [-s\tau, 0] \rightarrow \mathbb{R}^n$ is assumed to be continuous and, in order to simplify the notation, the constant base delay τ can be set equal to 1.

Let us define for $s \geq 0$, by $(\mathbf{x}_{[s]}) = (x(t), \dots, x(t - s))$; $(\mathbf{z}_{[s]})$, and $(\mathbf{u}_{[s]})$, are defined similarly. For simplicity $y(t)$, $x(t)$, and $u(t)$ will stand for $\mathbf{y}_{[0]}$, $\mathbf{x}_{[0]}$ and $\mathbf{u}_{[0]}$. Given the function $f(\mathbf{x}_{[l]})$, we will denote by $f(\mathbf{x}_{[l]}(-j)) := f(\mathbf{x}(t - j), \mathbf{x}(t - j - 1), \dots, \mathbf{x}(t - j - l))$. Then, equation (1) is rewritten as

$$\begin{aligned}\dot{\mathbf{x}}_{[0]} &= f(\mathbf{x}_{[s]}) + \sum_{j=0}^s g_j(\mathbf{x}_{[s]})\mathbf{u}_{[0]}(-j) \\ \mathbf{y}_{[0]} &= h(\mathbf{x}_{[s]}).\end{aligned}\quad (2)$$

III. THE TRAJECTORIES OF A TIME-DELAY DYNAMICAL SYSTEM FOR SOME FUTURE TIME INTERVAL

Consider now the delay-differential system given by the equation

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \bar{f}(\mathbf{x}(t), \mathbf{x}(t - 1)), \quad t > 0 \\ \mathbf{x}(t) &= \vartheta_0(t), \quad -1 \leq t \leq 0,\end{aligned}\quad (3)$$

Where ϑ_0 is a function $\vartheta_0 : [-1, 0] \rightarrow \mathbb{R}^n$. The purpose of this section is to show that the solution of (3) in a suitable future time interval $(\alpha - 1) \leq t \leq \alpha$ is found as the solution of a delay-free system subject to an appropriate initial condition.

- The solution $\vartheta_1(t)$ of the equation (3), computed by the so-called step method, on the interval $[0, 1]$ is given by solving the differential equation

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \bar{f}(\mathbf{x}(t), \vartheta_0(t-1)), & 0 < t \leq 1 \\ \mathbf{x}(\tau) &= \vartheta_0(\tau), & -1 \leq \tau \leq 0.\end{aligned}$$

- Similarly, the solution $\vartheta_2(t)$ of the equation (3) on the interval $[1, 2]$ is given by solving the differential equation

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \bar{f}(\mathbf{x}(t), \vartheta_1(t-1)), & 1 < t \leq 2 \\ \dot{\mathbf{x}}(t-1) &= \bar{f}(\mathbf{x}(t-1), \vartheta_0(t-2)), & 1 < t \leq 2 \\ \mathbf{x}(\tau) &= \vartheta_1(\tau), & 0 \leq \tau \leq 1 \\ \mathbf{x}(\tau) &= \vartheta_0(\tau), & -1 \leq \tau \leq 0.\end{aligned}$$

- More generally, from the knowledge of the previous trajectories $\vartheta_i(t)$, $i = 0, 1, \dots, \alpha-1$, the solution $\vartheta_\alpha(t)$ of the equation (3) on the interval $[(\alpha-1), \alpha]$ is given by solving the differential equation

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \bar{f}(\mathbf{x}(t), \vartheta_{\alpha-1}(t-1)), \\ \dot{\mathbf{x}}(t-1) &= \bar{f}(\mathbf{x}(t-1), \vartheta_{\alpha-2}(t-2)), \\ &\vdots \\ \dot{\mathbf{x}}(t-(\alpha-1)) &= \bar{f}(\mathbf{x}(t-(\alpha-1)), \vartheta_0(t-\alpha)) \\ \mathbf{x}(\tau) &= \vartheta_{\alpha-1}(\tau), & \alpha-2 \leq \tau \leq \alpha-1 \\ &\vdots \\ \mathbf{x}(\tau) &= \vartheta_0(\tau), & -1 \leq \tau \leq 0.\end{aligned}\quad (4)$$

Where ϑ_k are functions $\vartheta_k : [k\tau - 1, k] \rightarrow \mathbb{R}^n$ with $k = 0, \dots, \alpha-1$. Consider now the set of equations defined on the interval $(\alpha-1) \leq t \leq \alpha$ expressed as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \bar{f}(\mathbf{x}(t), \vartheta_{\alpha-1}(t-1)), \\ \dot{\mathbf{x}}(t-1) &= \bar{f}(\mathbf{x}(t-1), \vartheta_{\alpha-2}(t-2)), \\ &\vdots \\ \dot{\mathbf{x}}(t-(\alpha-1)) &= \bar{f}(\mathbf{x}(t-(\alpha-1)), \vartheta_0(t-\alpha)),\end{aligned}\quad (5)$$

with $x(\tau) = \vartheta_0(\tau)$ for $\tau \in [\alpha-1, \alpha)$. The solution on the first α units of time of the equation (3) is identified by the solution of the equation (5) on the interval $(\alpha-1) \leq t \leq \alpha$, with initialization $x(\tau) = \vartheta_0(\tau)$ for $\tau \in [\alpha-1, \alpha)$. Moreover, through a change of variable $x_0(t) = x(t)$, $x_1(t) = x(t-1)$, $x_\alpha(t) = x(t-\alpha)$, system (5) also can be represented as

$$\begin{aligned}\dot{x}_0(t) &= \bar{f}(x_0(t), x_1(t)), \\ \dot{x}_1(t) &= \bar{f}(x_1(t), x_2(t)), \\ &\vdots \\ \dot{x}_{\alpha-1}(t) &= \bar{f}(x_{\alpha-1}(t), \vartheta_0(t-1)),\end{aligned}$$

with initial conditions $x_0(0) = x_1(1)$, $x_1(0) = x_2(1)$, $x_2(0) = x_3(1)$, \dots , $x_{\alpha-1}(0) = \vartheta_0(1)$. The above analysis can be easily extended to the case of multiple delays and with inputs. Consider the following $n(s+\ell)$ -dimensional time-varying

nonlinear system Σ_v :

$$\Sigma_v : \begin{cases} \dot{z}_0(t) = f(z_0(t), \dots, z_s(t)) \\ \quad + \sum_{j=0}^s g_j(z_0(t), \dots, z_s(t)) \bar{\nu}_0(t), \\ \dot{z}_1(t) = f(z_1(t), \dots, z_{s+1}(t)) \\ \quad + \sum_{j=0}^s g_j(z_1(t), \dots, z_{s+1}(t)) \bar{\nu}_1(t), \\ \vdots \\ \dot{z}_\ell(t) = f(z_\ell(t), \dots, z_{\ell+s}(t)) \\ \quad + \sum_{j=0}^s g_j(z_\ell(t), \dots, z_{\ell+s}(t)) \bar{\nu}_\ell(t), \\ \dot{z}_{\ell+1}(t) = \phi_{s-1}(t) \\ \vdots \\ \dot{z}_{\ell+s}(t) = \phi_0(t), \end{cases}$$

where $\bar{\nu}_i(t) = \nu(t-j-i)$ for $i = 0, \dots, \ell$. Now, consider the following hypothesis.

- H) System Σ , under the restriction $u(t) = \nu(t)$ has a unique solution

$$x(t_0 + \theta) = \varphi(\theta, \varphi_0, \nu), \quad \theta \in [0, T] \quad (6)$$

in the interval $[0, T]$.

We are ready to state our main result.

Theorem 1: Consider system (2), with initial conditions $\varphi_0(\theta)$, and subject to the restriction $u(t) = \nu(t)$, $t \geq 0$, and assume that hypothesis H) is satisfied. Then, the solution (6) is obtained from the solution of system Σ_v in the interval $[t_0 + s, t_0 + T]$ if $T > \ell + s$,

$$z_i(t_0 + s) = \varphi(s-i, \varphi_0, \nu), \quad i = 0, \dots, \ell + s,$$

and

$$\phi_i(t) = \dot{\varphi}(t-t_0+i), \quad i = 0, \dots, s-1$$

Proof: Define $t'_0 = t_0 + s$. Note that the solution of the last s equations are copies of the solution of Σ , delayed with respect to t'_0 . \dot{z}_ℓ is a copy of system Σ at time t'_0 , and the solution exists by hypothesis H). Following the same reasoning, one finds that $z_0(t) = \varphi(t-\ell-s, \varphi_0, \nu) = x(t)$ in the interval $t_0 + \ell + s, T]$ ■

Accordingly, the following result can be stated for the single-delay case of (3).

Corollary 1: Given (3), consider the associated extended system (5). There always exists a proper initialization of (5) so that the trajectories of both systems coincide on the time interval $(\alpha-1) \leq t \leq \alpha$.

This means that it is possible to group the solution of (3) on the interval $(\alpha-1) \leq t \leq \alpha$. Moreover, setting $t_0 = (\alpha-1)$, The solution of (3) on $(\alpha-1) \leq t \leq \alpha$ can be computed solving the extended delay-differential equation

Since the extended system described by equation (5) is a system of delay-free differential equations, the conditions for existence and uniqueness of the solution, for the interval $0 < t < \alpha$ are the same as the ones for non-autonomous differential equations.

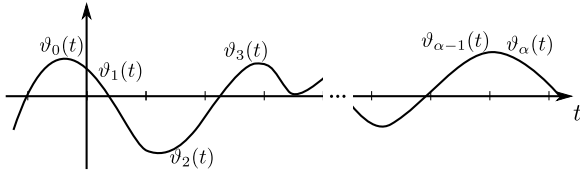


Fig. 1. Solution for a time-delay system.

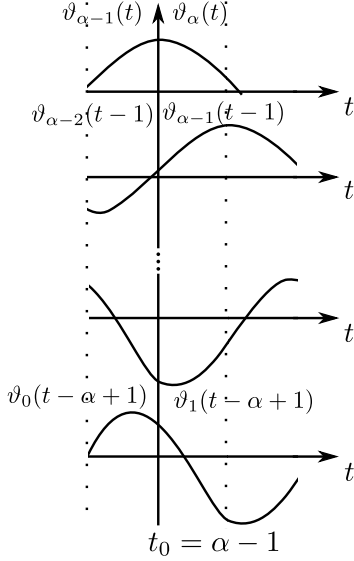


Fig. 2. Solution for an extended time-delay system.

Example 1: Consider the time-delay system described by the differential equation

$$\dot{x}(t) = -ax(t) + bx(t-1), \quad (7)$$

with initial condition $x(t) = 1$ for $t \in [-1, 0]$. Now, associate system

$$\dot{z}_0(t) = -az_0(t) + b\varphi(t) \quad (8)$$

to the system (7). System (8) is a copy of (7) considering $x(t)$ for $t \in [-1, 0]$ as an external signal $\varphi(t+1)$ with $z_0(0) = x(0) = 1$. This approximation is defined for a time window on the interval $0 \leq t < 1$. Computing the solution of (8) for the interval $0 \leq t < 1$ is exactly the first step of the step method to calculate the solution of (7) on the interval $0 \leq t < 1$. This can be observed in Figure 3 which is the simulation of the system (8) compared with the numerical solution of (7). Note that for $t > 1$ the solution of (8) does not match with the one of (7). The next approximation is endowed using an extended system of ordinary differential equations. Let us call $z_1(t) = \varphi_1(t)$ the solution of (8) on the interval $0 \leq t < 1$. Continuing with the step method, the solution of

$$\dot{x}(t) = -ax(t) + b\varphi_1(t-1), \quad (9)$$

for $1 \leq t < 2$. Shifting (9) a units of time to the left we have

$$\dot{x}(t+1) = -ax(t+1) + b\varphi_1(t), \quad (10)$$

is found for $0 \leq t < 1$. Now setting the change of variable $z_1(t) = x(t-1)$ we are able to write the next extended system

$$\begin{aligned} \dot{z}_0(t) &= -az_0(t) + bz_1(t) \\ \dot{z}_1(t) &= -az_1(t) + b\varphi(t) \end{aligned} \quad (11)$$

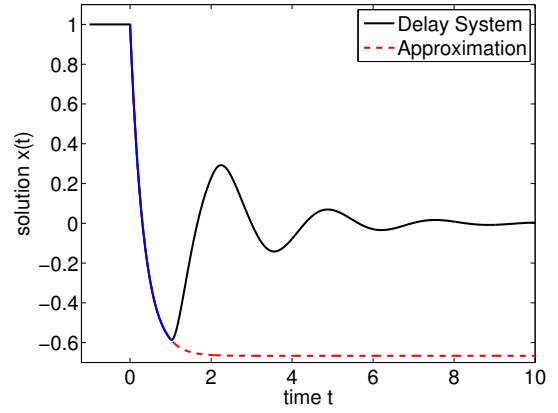


Fig. 3. Extended system solution for $a = 3$, and $b = -2$ for a window $0 \leq t < 1$.

with $z_0(0) = z_1(1)$, $z_1(0) = 1$. Note that doing this, the solution $z_1(t) = \varphi_2(t)$ on the interval $0 \leq t < 1$ is equivalent to the solution of the second step of the step method shifted 1 unit of time to the left. So doing this, the solution of (11) defines the window approximation for $0 \leq t < 2$. The solution of $z_1(t)$ on $0 \leq t < 1$ defines the approximation for the section $0 \leq t < 1$ while the solution of $z_0(t)$ on $0 \leq t < 1$ describes the approximated system on $1 \leq t < 2$. The same

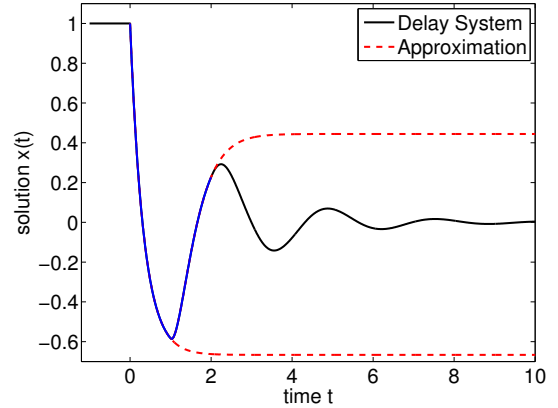


Fig. 4. Extended system solution for $a = 3$, and $b = -2$ for a window $0 \leq t < 2$.

step method approach can be done for larger windows of time. The approximation on the section $0 \leq t < 3$ is defined using the next set of equations

$$\begin{aligned} \dot{z}_0(t) &= -az_0(t) + bz_1(t) \\ \dot{z}_1(t) &= -az_1(t) + bz_2(t) \\ \dot{z}_2(t) &= -az_2(t) + b\varphi(t) \end{aligned} \quad (12)$$

with $z_0(0) = z_1(1)$, $z_1(0) = z_2(1)$, $z_2(0) = 1$. Where the solutions of $z_1(t)$, $z_2(t)$, and $z_3(t)$ on $0 \leq t < 3$ defines the sections $2 \leq t < 3$, $1 \leq t < 2$, and $0 \leq t < 1$ respectively. In the Figure 5 the approximation on the window $0 \leq t < 3$ is represented by the solid blue line. The approximation can be extended until the k -th order given by the ordinary differential

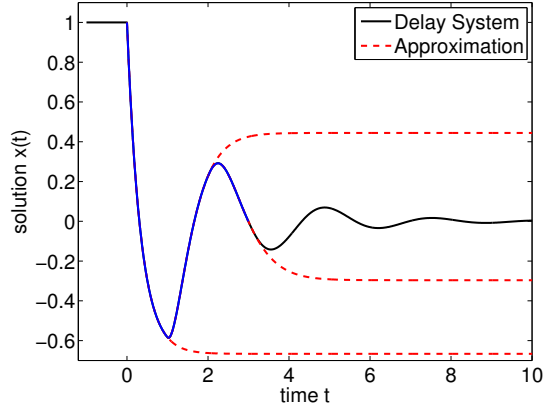


Fig. 5. Extended system solution for $a = 3$, $b = -2$, and $l = 1$ for a window $0 \leq t < 3$.

system

$$\begin{aligned} \dot{z}_0 &= az_0(t) + bz_1(t) \\ \dot{z}_1 &= az_1(t) + bz_2(t) \\ \dot{z}_2 &= az_2(t) + bz_3(t) \\ &\vdots \\ \dot{z}_k &= az_k(t) + b\varphi(t) \end{aligned} \quad (13)$$

with initial conditions $z_0(0) = z_1(1), z_1(0) = z_2(1), \dots, z_k(0) = \varphi(0)$. Where the solutions $z_1(t), \dots, z_2(t)$ on $0 \leq t < 1$ defines the approximation on the sections $k \leq k+1, \dots, 0 \leq t < 1$ respectively. Figure 6 shows the simulation of the approximation for a large value of k . The

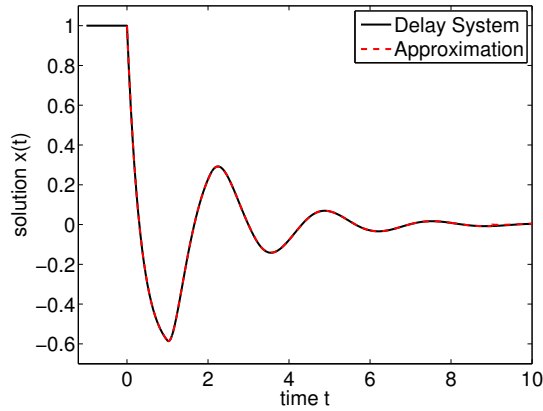


Fig. 6. Extended system solution for $a = 3$, and $b = -2$ for a window $0 \leq t < 9$.

solution on the interval $0 \leq t \leq 1$ of the equation (13) is presented in Figure 7 for $k = 9$. ◀

IV. PAST TIME INITIALIZATION

The purpose of this section is to argue whether the initialization of system (3) can be shifted back onto the extended system (5) so that the solution of both systems are identical for $t \in [0, 1[$.

Example 2: Consider

$$\dot{x} = -x(t-1) \quad (14)$$

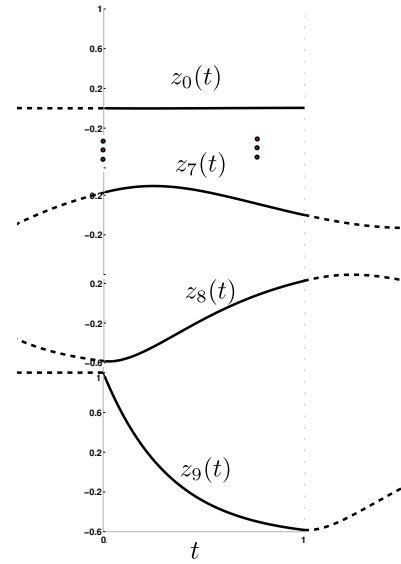


Fig. 7. Solution for a extended time-delay system on the interval $0 \leq t \leq 10$.

with $x(\tau) = 1$ for $\tau \in [-1, 0[$. System (14) is extended to

$$\begin{aligned} \dot{x}(t) &= -x(t-1) \\ \dot{x}(t-1) &= -x(t-2) \end{aligned} \quad (15)$$

There is no initialization function for system (15) over the time interval $[-2, -1]$ so that the trajectory of (15) coincides with the trajectory of (14) for any $t \geq -1$.

Example 3: Consider the system

$$\dot{x}(t) = x(t)x(t-1)$$

with $x(t_0 + \theta) = \varphi_0(\theta)$, and

$$\varphi_0(\theta) = \text{floor}(-3\theta) - 2, \quad \theta \in [-1, 0), \quad \varphi_0(0) = -1 \quad (16)$$

where $\text{floor}(x) := \{\max a \in \mathbb{Z} \mid a < x\}$.

The given initial conditions correspond to the solution of the system $\dot{x}(t-1) = x(t-2)$ in the respective time interval. However, this requires the use of distributions to define the corresponding initial conditions.

However, it is well-known that a time-delay system has a “smoothing” property, in the sense that, if a solution exists, and it can be computed with the *step-by-step* method [8], at step k , the solution will belong to \mathcal{C}^{k-1} . For instance, the instantaneous solution for system Σ , with initial conditions (16), is continuous in the time interval $[0, 1]$, \mathcal{C}^1 in the interval $[1, 2]$, and \mathcal{C}^k in the time interval $[k, k+1]$. Then, this naturally leads to the idea that, instead of trying to analyze backwards in time, look forward instead. This idea has led us to establish local conditions for the existence and uniqueness of solutions for Σ_∞ . Moreover, we have established some relationships with a particular class of time-varying nonlinear systems, which opens new research lines, which are discussed in the next section.

V. EXTENDED SPACE

In this section a general version of the extended system (5) is defined. This, through the step method, associates the

initialization functions of the original system with the initial function of the extended system used in the definition of the so-called extended Lie bracket.

Now, let us define the extended nonlinear time-delay system as

$$\begin{aligned}\dot{\mathbf{x}}(t) &= f(\mathbf{x}_{[s]}) + \sum_{j=0}^s g_j(\mathbf{x}_{[s]})\mathbf{u}(t-j), \\ \dot{\mathbf{x}}(t-1) &= f(\mathbf{x}_{[s]}(-1)) + \sum_{j=0}^s g_j(\mathbf{x}_{[s]}(-1))\mathbf{u}(t-j-1), \\ &\vdots\end{aligned}\quad (17)$$

Note that the functions of initial conditions are defined by $\vartheta : [-s\tau, 0] \rightarrow \mathbb{R}^n$ from (1) and the solutions of (17) on the interval $0 < t < 1$. System (17) is naturally associated with the set of vectors $\mathbf{r}_j(\mathbf{x}, \delta) = \sum_{i=0}^s \mathbf{r}_i^j(\mathbf{x})\delta^j$, $i = 1, \dots, p$ and with the infinite dimensional array

$$\left\{ \begin{array}{cccccccc} \mathbf{r}^1 & \dots & \mathbf{r}^s & 0 & \dots & 0 & \dots & \dots \\ 0 & \mathbf{r}^1(-1) & \dots & \mathbf{r}^s(-1) & 0 & \dots & 0 & \dots \\ \vdots & \ddots & \ddots & \dots & \ddots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & \mathbf{r}^1(-q) & \dots & \mathbf{r}^s(-p) & 0 & \dots \\ \vdots & \dots & \vdots & \ddots & \ddots & \dots & \ddots & \ddots \end{array} \right\}, \quad (18)$$

with $\mathbf{r}^i(-p) = (\mathbf{r}_1^i(\mathbf{x}(-q)), \dots, \mathbf{r}_j^i(\mathbf{x}(-q)))$. This array is constructed by the elements $\mathbf{R}_l^{j_l} = \sum_{i=0}^{j_l} \left(\mathbf{r}_l^{j_l-i}(\mathbf{x}(-i)) \right) \frac{\partial}{\partial \mathbf{x}(t-i)}$ with $l \in \mathbb{Z}^+$.

Let us now state the definition of the extended Lie bracket given in [1].

Definition 1: Consider $\mathbf{R}_l^{j_l} = \sum_{i=0}^{j_l} \left(\mathbf{r}_l^{j_l-i}(\mathbf{x}(-i)) \right) \frac{\partial}{\partial \mathbf{x}(t-i)}$, with $l = 1, 2$. Then, $\forall j_1, j_2 \geq 0$, the operation

$$\left[\mathbf{r}_1^{j_1}(\mathbf{x}), \mathbf{r}_2^{j_2}(\mathbf{x}) \right]_{E_0} = \sum_{j=0}^{\kappa} \left(\left[\mathbf{r}_1^{j_1-j}(\mathbf{x}), \mathbf{r}_2^{j_2-j}(\mathbf{x}) \right]_{E_0} \right)^T \frac{\partial}{\partial \mathbf{x}(t-i)}, \quad (19)$$

where $\kappa = \min(j_1, j_2, i)$, defined on $\mathbb{R}^{(\alpha+\mathbb{K})\times}$, is called extended Lie bracket, where

$$\begin{aligned} \left[\mathbf{r}_1^p(\mathbf{x}), \mathbf{r}_2^q(\mathbf{x}) \right]_{E_0} &= \sum_{i=0}^p \frac{\partial \mathbf{r}_2^q(\mathbf{x})}{\partial \mathbf{x}(t-i)} \mathbf{r}_1^{p-i}(\mathbf{x}(-i)) + \\ &\quad - \sum_{i=0}^q \frac{\partial \mathbf{r}_1^p(\mathbf{x})}{\partial \mathbf{x}(t-i)} \mathbf{r}_2^{q-i}(\mathbf{x}(-i)). \end{aligned} \quad (20)$$

Finally, denoting $j = \min(j_1, j_2)$, the extended Lie bracket can be associated with the definition of the Lie bracket for delay-free system

$$\left[\mathbf{R}_1^{j_1}, \mathbf{R}_2^{j_2} \right] = \left[\mathbf{r}_1^{j_1}(\mathbf{x}), \mathbf{r}_2^{j_2}(\mathbf{x}) \right]_{E_j},$$

over the elements of the infinite-dimensional array (18).

VI. DISCUSSION

In the previous sections, we have established the equivalence of solutions between a time-delay system with a class of time-varying systems without delays. The importance of

such result is twofold: on one hand, it is a contribution for the geometric setting presented in [1], [9], which introduce the so-called *extended Lie bracket*. For that definition, an extended system of sufficiently high dimension is defined. The contribution of this paper gives conditions to guarantee the existence and uniqueness of solutions of such extended system, which is an important technical issue, not considered until now.

Furthermore it shows that the approximating finite dimensional dynamics gives information about the geometric properties of the original time delay system. This allows us to conclude, for instance, that if the approximating system has a sufficiently high dimension (which is linked to the delay appearing in the original system), then the accessibility properties of the approximating system yield the accessibility properties of the original time-delay one. In fact, the operations involved in the computation of the Lie bracket of the extended approximation will be exactly the same as the ones involved for the computation of the extended Lie bracket defined in [9] for (1).

On the other hand, the presented equivalence allows us to transpose some results from one class of systems to the other, or at least give some hints in the search of solutions. As an example, for the class of linear time-varying systems

$$\dot{x}(t) = A(t)x(t) + B(t)u(t),$$

it is known that its controllability is characterized by the rank of the matrix $[B_0 \mid B_1 \mid B_{n-1}]$, where $B_0 = B(t)$ and $B_{i+1} = A(t)B_i - \dot{B}_i$ for $i = 1, \dots, n-1$, which corresponds to the submodules defined in [10].

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