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THE NAVIER-STOKES LIMIT OF KINETIC EQUATIONS FOR LOW REGULARITY DATA

KLEBER CARRAPATOSO, ISABELLE GALLAGHER, AND ISABELLE TRISTANI

Abstract. In this paper, we investigate the link between kinetic equations (including Boltzmann with or without cutoff assumption and Landau equations) and the incompressible Navier-Stokes equation. We work with strong solutions and we treat all the cases in a unified framework. The main purpose of this work is to be as accurate as possible in terms of functional spaces. More precisely, it is well-known that the Navier-Stokes equation can be solved in a lower regularity setting (in the space variable) than kinetic equations. Our main result allows to get a rigorous link between solutions to the Navier-Stokes equation with such low regularity data and kinetic equations.

Contents

1. Introduction

In this paper, we are interested in a problem in the theory of hydrodynamical limits. Our goal is to obtain a rigorous result of convergence of solutions to various kinetic equations towards solutions to the incompressible Navier-Stokes equation. This problem can be seen as a part of the program initiated by the 6th problem of Hilbert in 1900 at the International Congress of Mathematicians. Indeed, the question is to understand the link between microscopic and macroscopic descriptions of a fluid, and deriving macroscopic equations from mesoscopic ones can be seen as an intermediate step of this program. We refer for instance to the book by Saint-Raymond [52] for a detailed presentation of the

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subject and for mathematical results in the field. More specifically, in this paper, we seek to get a result on the convergence of sequences of strong solutions to the rescaled mesoscopic equations in which the connection between the kinetic and the fluid equations is as accurate as possible in terms of functional spaces.

1.1. **Kinetic equations.** At the kinetic level, we shall consider Boltzmann or Landau type equations for not too soft potentials. We denote by $F = F(t,x,v)$ the density of particles, which depends on time $t \in \mathbb{R}^+$, position $x \in \mathbb{T}^3$ (the unit periodic box) and velocity $v \in \mathbb{R}^3$. The dimensionless version of our kinetic equation reads

$$\mathrm{St}\,\partial_t F + v \cdot \nabla_x F = \frac{1}{\mathrm{Kn}} Q(F,F)\,,$$

where the Strouhal number St and the Knudsen number Kn are dimensionless parameters which are natural in kinetic problems. Here and below, Q can be the Boltzmann (with or without cutoff) collision operator or the Landau collision operator. The Boltzmann collision operator is an integral operator defined as

$$Q_B(g,f) := \int_{\mathbb{R}^3\times\mathbb{S}^2} B(v-v_*,\sigma)\,(g'_* f' - g_* f)\,\mathrm{d}\sigma\,\mathrm{d}v_*\,. \tag{1.1}$$

Here and below, we are using the shorthand notations $f = f(v)$, $g_* = g(v_*)$, $f' = f(v')$ and $g'_* = g(v'_*)$. In this expression, v, v_* and v', v'_* are the velocities of a pair of particles after and before collision. We make a choice of parametrization of the set of solutions to the conservation of momentum and energy (physical laws of elastic collisions):

$$\begin{aligned} v + v_* &= v' + v'_* \\ |v|^2 + |v_*|^2 &= |v'|^2 + |v'_*|^2 \end{aligned} \tag{1.2}$$

so that the pre-collisional velocities are given by

$$v' = \frac{v+v_*}{2} + \frac{|v-v_*|}{2}\sigma\,, \quad v'_* = \frac{v+v_*}{2} - \frac{|v-v_*|}{2}\sigma\,, \qquad \sigma \in \mathbb{S}^2\,.$$

The Boltzmann collision kernel $B = B(v-v_*,\sigma)$ only depends on the relative velocity $|v-v_*|$ and on the deviation angle ϑ through $\cos\vartheta = \langle v-v_*,\sigma\rangle/|v-v_*|$ where $\langle\cdot,\cdot\rangle$ is the usual scalar product in $\mathbb{R}^3$. The form of the collision kernel depends on the type of collisions that occur between particles. In dimension 3 in the case where particles behave as billiard balls, known as the hard-spheres case, the collision kernel is proportional to the norm of the relative velocity, namely

$$B(v-v_*,\sigma) = C|v-v_*|\,, \qquad C > 0\,.$$

When particles interact through inverse power law potentials of type

$$\phi(r) = r^{-(p-1)} \qquad \text{with} \qquad p \in (2,+\infty)\,, \tag{1.3}$$

the collision kernel cannot be computed explicitly but Maxwell [46] has shown that the collision kernel can be computed in terms of the interaction potential ϕ. More precisely, in dimension 3, the kernel B satisfies the following properties.

– It takes product form in its arguments as

$$B(v-v_*,\sigma) = \Phi(|v-v_*|)\,b(\cos\vartheta)\,. \tag{1.4}$$

– The angular function b is locally smooth, and has a nonintegrable singularity for $\vartheta \to 0$: it satisfies for some $c_b > 0$ and any $\vartheta \in (0,\pi/2]$,

$$\frac{c_b}{\vartheta^{1+2s}} \leqslant \sin\vartheta\, b(\cos\vartheta) \leqslant \frac{1}{c_b\,\vartheta^{1+2s}} \qquad \text{with} \qquad s := \frac{1}{p-1} \in (0,1)\,. \tag{1.5}$$

– The kinetic factor Φ satisfies

$$\Phi(|v-v_*|) = |v-v_*|^\gamma \qquad \text{with} \qquad \gamma := \frac{p-5}{p-1} \in (-3,1)\,. \tag{1.6}$$

One traditionally calls hard potentials the case $p > 5$ (for which $0 < \gamma < 1$), Maxwell molecules the case $p = 5$ (for which $\gamma = 0$), moderately soft potentials the case corresponding with $3 \leqslant p < 5$ (for which $-2s \leqslant \gamma < 0$) and very soft potentials the case $2 < p < 3$ (for which $-3 < \gamma < -2s$). In this paper, we shall not consider the very soft potentials case, meaning we shall restrict to $\gamma \geqslant -2s$ (see Remark 2 for a discussion on this restriction).

Grad's cut-off assumption consists in additionally supposing that the angular kernel b is integrable on the sphere by removing its singularity for small deviation angles ϑ (see (1.5)). In that case, the Boltzmann collision operator is thus of the form (1.1) with

$$B(v - v_*, \sigma) = b(\cos\vartheta)|v - v_*|^\gamma \quad \text{with} \quad \int_{\mathbb{S}^2} b(\cos\vartheta)\,\mathrm{d}\sigma < \infty \quad \text{and} \quad \gamma \in (-3, 1]\,.$$

Notice that this in particular includes the case of hard-spheres collisions by taking the angular kernel to be constant. Here again, we do not consider the very soft potentials case, that is we restrict ourselves to $\gamma \geqslant 0$.

In the case of the Coulomb potential ($s = 1$ and thus $\gamma = -3$), the Boltzmann operator does not make any sense (see [55] for example). The Boltzmann operator has then to be replaced by the Landau one which can be obtained in the so-called grazing collision limit after having made a cut-off on the Coulomb interaction. The Landau operator, defined in 1936 by Landau [41] (independently of the Boltzmann operator), is used in plasma physics and is an integro-differential operator given by

$$Q_L(g, f)(v) := \partial_{v_i} \int_{\mathbb{R}^d} a_{ij}(v - v_*) \Big(g(v_*)\partial_{v_j} f(v) - f(v)\partial_{v_j} g(v_*) \Big)\,\mathrm{d}v_*\,, \tag{1.7}$$

where we use the convention of summation of repeated indices. The matrix a_{ij} is symmetric, semi-positive and is given by

$$a_{ij}(v) := |v|^{\gamma+2} \left(\delta_{ij} - \frac{v_i v_j}{|v|^2} \right), \quad -3 \leqslant \gamma \leqslant 1\,. \tag{1.8}$$

Similarly to the Boltzmann equation, we have the following classification according to the values of γ: interactions are referred to as hard potentials if $\gamma \in (0, 1]$, Maxwellian molecules if $\gamma = 0$, moderately soft potentials if $\gamma \in [-2, 0)$, very soft potentials if $\gamma \in (-3, -2)$ and Coulomb potential if $\gamma = -3$. We mention that only the case $\gamma = -3$ is relevant from a physical viewpoint and is the one that has been derived by Landau in [41]. Once more, we shall only consider not too soft potentials, which correspond to $\gamma \geqslant -2$.

In the three cases (Boltzmann with and without cut-off assumption and Landau), weak formulations of the collision operators allow to obtain the following conservation laws:

$$\int_{\mathbb{R}^3} Q(g, g)(v)\,\varphi(v)\,\mathrm{d}v = 0 \qquad \text{for} \qquad \varphi(v) = 1, v\,, |v|^2, \tag{1.9}$$

as well as Boltzmann's H-theorem that asserts that Boltzmann's entropy of solutions to these equations, namely $\int f \log f$, is non-increasing along time. Moreover, the second part of the theorem states that any distribution minimizing the entropy is a local Maxwellian distribution in velocity.

1.2. **Hydrodynamic limit.** All kinetic models leading to incompressible models are based on a regime in which both the Strouhal and the Knudsen numbers are small. In order to reach the incompressible Navier-Stokes equation, we shall work with $\mathrm{St} = \mathrm{Kn} = \varepsilon \ll 1$ (see for example [7]). Our kinetic equation then reads

$$\begin{cases} \partial_t F^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x F^\varepsilon = \varepsilon^{-2} Q(F^\varepsilon, F^\varepsilon) & \text{in} \quad \mathbb{R}^+ \times \mathbb{T}^3 \times \mathbb{R}^3 \\ F^\varepsilon_{|t=0} = F^\varepsilon_{\text{in}} & \text{in} \quad \mathbb{T}^3 \times \mathbb{R}^3\,. \end{cases} \tag{1.10}$$

The Knudsen number is actually proportional to the inverse of the average number of collisions for each particle per unit of time. Taking ε small has thus the effect of enhancing

the role of collisions. To relate our kinetic models to the incompressible Navier-Stokes equation, we then look at equation (1.10) under the following linearization of order ε:

$$F^\varepsilon = \mu + \varepsilon\sqrt{\mu} f^\varepsilon \,,$$

where μ is the global Maxwellian defined by

$$\mu(v) := \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{|v|^2}{2}} \,.$$

The equation we are going to study on the fluctuation f^ε is thus the following:

$$\left\{ \begin{array}{ll} \partial_t f^\varepsilon + \varepsilon^{-1} v \cdot \nabla_x f^\varepsilon = \varepsilon^{-2} L f^\varepsilon + \varepsilon^{-1} \Gamma(f^\varepsilon, f^\varepsilon) & \text{in} \quad \mathbb{R}^+ \times \mathbb{T}^3 \times \mathbb{R}^3 \\ f^\varepsilon_{|t=0} = f^\varepsilon_{\rm in} := \varepsilon^{-1}(F^\varepsilon_{\rm in} - \mu)\, \mu^{-\frac{1}{2}} & \text{in} \quad \mathbb{T}^3 \times \mathbb{R}^3 \end{array} \right. \tag{1.11}$$

with

$$\Gamma(f_1, f_2) := \mu^{-\frac{1}{2}} Q(\sqrt{\mu} f_1, \sqrt{\mu} f_2)$$

and

$$L f := \Gamma(\sqrt{\mu}, f) + \Gamma(f, \sqrt{\mu}) \,. \tag{1.12}$$

We say that a distribution $f = f(x,v)$ has global mass, momentum and energy when it satisfies

$$\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} f(x,v)\, \varphi(v) \sqrt{\mu}(v)\, {\rm d}v\, {\rm d}x = 0 \,, \quad \text{for} \quad \varphi(v) = 1, v, |v|^2 \,. \tag{1.13}$$

Conservation laws (1.9) imply that the perturbation f^ε satisfies (1.13) for all times $t \geqslant 0$ if $F^\varepsilon_{\rm in}$ satisfies

$$\int_{\mathbb{T}^3} \int_{\mathbb{R}^3} F^\varepsilon_{\rm in}(x,v)\, \varphi(v)\, {\rm d}v\, {\rm d}x = \int_{\mathbb{R}^3} \mu(v)\, \varphi(v)\, {\rm d}v \,, \quad \text{for} \quad \varphi(v) = 1, v, |v|^2 \,.$$

For every $f = f(x,v)$ we write the decomposition

$$f = \mathbf{P}_0^\perp f + \mathbf{P}_0 f \,, \qquad \mathbf{P}_0^\perp = {\rm Id} - \mathbf{P}_0 \,,$$

where $\mathbf{P}_0$ is the orthogonal projection onto

$$\mathrm{Ker}\, L = \left\{ \sqrt{\mu}(v), v_1 \sqrt{\mu}(v), v_2 \sqrt{\mu}(v), v_3 \sqrt{\mu}(v), |v|^2 \sqrt{\mu}(v) \right\} \tag{1.14}$$

given by the so-called hydrodynamic modes

$$\mathbf{P}_0 f(x,v) := \left\{ \rho[f](x) + u[f](x) \cdot v + \theta[f](x) \frac{|v|^2 - 3}{2} \right\} \sqrt{\mu}(v) \tag{1.15}$$

where

$$\begin{aligned} \rho[f](x) &:= \int_{\mathbb{R}^3} f(x,v) \sqrt{\mu}(v)\, {\rm d}v \,, \\ u[f](x) &:= \int_{\mathbb{R}^3} f(x,v) v \sqrt{\mu}(v)\, {\rm d}v \,, \\ \theta[f](x) &:= \int_{\mathbb{R}^3} f(x,v) \frac{|v|^2 - 3}{3} \sqrt{\mu}(v)\, {\rm d}v \,. \end{aligned}$$

Returning to (1.11), it is expected that as ε goes to zero, the solution f^ε should converge to an element of $\mathrm{Ker}(L)$. This is actually proved in many situations (see Paragraph 1.4 below), and in particular the hydrodynamic modes of the limit satisfy the incompressible Navier-Stokes Fourier system

$$\left\{ \begin{array}{r} \partial_t u + u \cdot \nabla u - \nu_{\rm NS} \Delta u = -\nabla p \\ \partial_t \theta + u \cdot \nabla \theta - \nu_{\rm heat} \Delta \theta = 0 \\ {\rm div}\, u = 0 \\ \nabla(\rho + \theta) = 0 \,. \end{array} \right. \tag{NSF}$$

To define the viscosity coefficients, we introduce the two unique functions Φ (which is a matrix function) and Ψ (which is a vectorial function) in $(\mathrm{Ker}\, L)^\perp$ such that

$$\mu^{-\frac{1}{2}} L(\mu^{\frac{1}{2}}\Phi) = \frac{|v|^2}{3}\mathrm{Id} - v\otimes v \quad \text{and} \quad \mu^{-\frac{1}{2}} L(\mu^{\frac{1}{2}}\Psi) = v\Big(\frac{5}{2} - \frac{|v|^2}{2}\Big)\,.$$

The viscosity coefficients are then defined by

$$\nu_{\mathrm{NS}} := \frac{1}{10}\int \Phi : L(\mu^{\frac{1}{2}}\Phi)\mu^{\frac{1}{2}}\,\mathrm{d}v \quad \text{and} \quad \nu_{\mathrm{heat}} := \frac{2}{15}\int \Psi\cdot L(\mu^{\frac{1}{2}}\Psi)\mu^{\frac{1}{2}}\,\mathrm{d}v\,.$$

1.3. **Functional framework and notation.** We first define weighted Lebesgue spaces as follows. For any nonnegative measurable weight function $m : \mathbb{R}^3 \to \mathbb{R}^+$ (notice that all the weights we consider will depend only on velocity, i.e. $m = m(v)$), we define the Lebesgue space $L^p_v(m)$ for $1 \leqslant p \leqslant +\infty$ through its norm

$$\|g\|_{L^p_v(m)} := \|gm\|_{L^p_v}\,.$$

In order to treat the three cases (Boltzmann with and without cut-off assumption and Landau equations) in a unified framework, we introduce the space $H^{s,\star}_v$ with $s\in[0,1]$ by: for $s=0$ (corresponding to the Boltzmann operator with cutoff)

$$\|f\|_{H^{0,\star}_v} = \|f\|_{L^2_v\left(\langle v\rangle^{\frac{\gamma}{2}}\right)}\,, \tag{1.16}$$

for $s\in(0,1)$ (corresponding to the Boltzmann operator without cutoff)

$$\begin{aligned}\|f\|^2_{H^{s,\star}_v} :=& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\vartheta)|v-v_*|^\gamma \mu(v_*)[f(v')-f(v)]^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\\ &+\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} b(\cos\vartheta)|v-v_*|^\gamma f(v_*)^2[\sqrt{\mu}(v')-\sqrt{\mu}(v)]^2\,\mathrm{d}\sigma\,\mathrm{d}v_*\,\mathrm{d}v\,,\end{aligned} \tag{1.17}$$

and finally for $s=1$ (corresponding to the Landau operator) we define

$$\|f\|^2_{H^{1,\star}_v} := \big\|\langle v\rangle^{\frac{\gamma}{2}+1} f\big\|^2_{L^2_v} + \big\|\langle v\rangle^{\frac{\gamma}{2}}\,\mathrm{pr}_v\,\nabla_v f\big\|^2_{L^2_v} + \big\|\langle v\rangle^{\frac{\gamma}{2}+1}(\mathrm{Id}-\mathrm{pr}_v)\nabla_v f\big\|^2_{L^2_v}\,, \tag{1.18}$$

where pr_v stands for the projection on v, namely

$$\forall w\in\mathbb{R}^3\,,\quad \mathrm{pr}_v\, w = \left(w\cdot\frac{v}{|v|}\right)\frac{v}{|v|}\,.$$

For every $s\in[0,1]$, we also define the dual space $(H^{s,\star}_v)'$ endowed with the norm

$$\|\phi\|_{(H^{s,\star}_v)'} := \sup_{\|f\|_{H^{s,\star}_v}\leqslant 1} \langle \phi, f\rangle\,.$$

It is worth mentioning that for $s\in[0,1]$ there holds (see [1, 34] for the case $s\in(0,1)$, the other cases being immediate),

$$\big\|\langle v\rangle^{\frac{\gamma}{2}+s} f\big\|_{L^2_v} + \big\|\langle v\rangle^{\frac{\gamma}{2}} f\big\|_{H^s_v} \lesssim \|f\|_{H^{s,\star}_v} \lesssim \big\|\langle v\rangle^{\frac{\gamma}{2}+s} f\big\|_{H^s_v}\,.$$

We recall that if $\ell > 3/2$, then $H^\ell_x \subset L^\infty_x$. For $m\geqslant 0$ and when E_v is a Lebesgue or Sobolev space in velocity, we define the space $\widetilde{L}^\infty\big([0,T], H^m_x E_v\big)$ (with the notation introduced in [19]) through its norm

$$\|f\|^2_{\widetilde{L}^\infty\left([0,T],H^m_x E_v\right)} := \sum_{k\in\mathbb{Z}^3}\langle k\rangle^{2m}\big\|\widehat{f}(\cdot,k,\cdot)\big\|^2_{L^\infty([0,T],E_v)}\,.$$

We have denoted by $(\widehat{f}(k))_{k\in\mathbb{Z}^3}$ the Fourier coefficients of f in the space variable. When more convenient, we will sometimes use the notation $\mathcal{F}_x$. To lighten notation, we shall often write $L^p_I H^m_x E_v$ for $L^p(I, H^m_x E_v)$ and similarly $\widetilde{L}^\infty_I H^m_x E_v$ for $\widetilde{L}^\infty\big(I, H^m_x E_v\big)$ when I is an interval of $\mathbb{R}^+$. If $I=[0,T]$ we will simply write $L^p_T H^m_x E_v$ and $\widetilde{L}^\infty_T H^m_x E_v$. Finally if $T=\infty$ and in the absence of ambiguity we write $L^p_t H^m_x E_v$ for $L^p(\mathbb{R}^+, H^m_x E_v)$.

We will use the notation $\mathbb{P}$ for the Leray projector onto divergence free vector fields. For any triplet $(\rho_{\rm in}, u_{\rm in}, \theta_{\rm in})$ defined on $\mathbb{T}^3$ (considered as initial data, whence the subscript "in") we denote their projection onto incompressible/Boussinesq modes by

$$\bar\rho_{\rm in} := \frac{2}{5}\rho_{\rm in} - \frac{3}{5}\theta_{\rm in}\,, \qquad \bar u_{\rm in} := \mathbb{P}u_{\rm in} \qquad \text{and} \qquad \bar\theta_{\rm in} := -\rho_{\rm in}\,. \tag{1.19}$$

The kinetic counterpart of $(\bar\rho_{\rm in}, \bar u_{\rm in}, \bar\theta_{\rm in})$ will be denoted

$$\bar g_{\rm in}(x,v) := \Big\{\bar\rho_{\rm in}(x) + \bar u_{\rm in}(x)\cdot v + \bar\theta_{\rm in}(x)\frac{|v|^2-3}{2}\Big\}\sqrt{\mu}(v) \tag{1.20}$$

and if (ρ, u, θ) solves (NSF) with the initial data $(\bar\rho_{\rm in}, \bar u_{\rm in}, \bar\theta_{\rm in})$ then we will write

$$g(t,x,v) := \Big\{\rho(t,x) + u(t,x)\cdot v + \theta(t,x)\frac{|v|^2-3}{2}\Big\}\sqrt{\mu}(v)\,. \tag{1.21}$$

Note that if $(\rho_{\rm in}, u_{\rm in}, \theta_{\rm in})$ lie in $H^{\ell-1}(\mathbb{T}^3)$ with $\ell > 3/2$, then g belongs to $\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v \cap L^2_T H^\ell_x H^{s,\star}_v$ for some $T>0$ (the life span of the Navier-Stokes-Fourier system).

1.4. **State of the art.** We here give a short overview of the existing literature on the problem of deriving fluid equations from kinetic ones.

The first justifications of the link between kinetic and fluid equations were formal and based on asymptotic expansions by Hilbert [38], Chapman, Enskog [17] and Grad [32]. The first rigorous convergence proofs based also on asymptotic expansions were given by Caflisch [12] (see also [40] and [20]). In those papers, the limit is justified up to the first singular time for the fluid equation. By using his nonlinear energy method, Guo [37] justified the limit towards the Navier-Stokes equation and beyond in Hilbert's expansion from Boltzmann and Landau equations.

There have also been some convergence proofs based on spectral analysis in the framework of strong solutions close to equilibrium introduced by Grad [33] and Ukai [54] for the Boltzmann equation. In this respect, we refer to the works by Nishida [50], Bardos and Ukai [8]. These results use the description of the spectrum of the linearized Boltzmann equation in Fourier space in the space variable performed in [49, 16, 25] by respectively Nicolaenko; Cercignani, Illner and Pulvirenti; Ellis and Pinsky. The approach in the present paper as well as in [26, 15, 27, 28, 29, 13] are reminiscent of these ones.

Finally, let us mention that this problem has been extensively studied in the framework of weak solutions, the goal being to obtain solutions for the fluid models from renormalized solutions introduced by DiPerna and Lions in [22] for the Boltzmann equation. We shall not make an extensive presentation of this program as it is out of the realm of this paper, but let us mention that it was started by Bardos, Golse and Levermore at the beginning of the nineties in [7, 6] and was continued by those authors, Saint-Raymond, Masmoudi, Lions among others. We mention here a (non exhaustive) list of papers which are part of this program [30, 31, 44, 45, 52].

More recently, some uniform in ε estimates on kinetic equations have allowed to prove (at least) weak convergence towards the Navier-Stokes equation. Let us mention [39, 51] in which the cases of the Boltzmann equation without cut-off and the Landau equations are treated by Jiang, Xu and Zhao on the one hand and by Rachid on the other hand. In [10, 11], Briant and Briant, Merino-Aceituno and Mouhot have obtained convergence to equilibrium results for the rescaled Boltzmann equation (and also the Landau equation in [10]) uniformly in the rescaling parameter using respectively hypocoercivity and enlargement methods. In [11], the authors are able to weaken the assumptions on the data down to Sobolev spaces with polynomial weights (see also [3] for the inelastic Boltzmann equation). Notice that Briant [10] has combined this with the Ellis and Pinsky result [25] to recover strong convergence in the case of the elastic Boltzmann equation. To end this part, we mention the works [15, 13] in which the authors also obtain uniform in ε estimates on the Landau equation and Boltzmann equation without cutoff respectively and also obtain a result of strong convergence towards the incompressible Navier-Stokes equation.

Finally, let us bring up more recent works that have inspired the present paper. First, the paper [26] in which the second and third authors proved that the life span of the solutions to the rescaled Boltzmann equation (for hard-spheres collisions) is bounded from below by that of the Navier-Stokes equation for ε small enough. The main feature of the proof was to perform a fixed point argument by using information on the limit system since the starting point is the solutions of the Navier-Stokes system (which is not the most common viewpoint). Gervais [27, 28] extended the functional framework in which this result holds. He proved a similar result in polynomially weighted spaces, his strategy is a combination of [26] and of the one used in [11] by Briant, Merino-Aceituno and Mouhot in order to get uniform in ε estimates on solutions in polynomially weighted spaces. We also point out the paper by Gervais and Lods [29] in which a unified framework is also provided, which encompasses a large class of kinetic equations (including in particular the result in [26]).

1.5. **Main result.** All the results mentioned in the previous paragraph concerning the convergence of strong solutions are stated in functional spaces which are usual for the study of strong solutions to nonlinear kinetic problems, namely in which there is an algebra structure in the space variable, typically H^ℓ_x with $\ell > 3/2$ (more generally $\ell > d/2$ in dimension d). Indeed the collision operator Q_B involves the product of $f(x,v)$ and $f(x,v')$ at the same point x, so continuity of f seems to be required to make sense of the product. However it is well-known that the Navier-Stokes equations can be solved for initial data with less regularity, namely $H^{\frac12}_x$ ($H^{\frac d2-1}_x$ in dimension d). Our goal in this work is to analyze to what extent the assumptions one makes on the initial data $f^\varepsilon_{\rm in}$ to the kinetic equation (1.11) can reflect this discrepancy between the kinetic and the fluid frameworks.

The main goal of our analysis is thus to show that given an initial data in $H^{\frac12}_x$ (actually we will consider $H^{\ell-1}_x$ with $\ell > 3/2$) for the incompressible (NSF) system, the associate solution to (NSF) is the limit, as long as it exists, of a sequence of solutions to the rescaled Boltzmann or Landau equation. More precisely we are able to construct, on the same life span as the solution to (NSF), a sequence of solutions to the kinetic equation associated with initial data whose hydrodynamic part converges in $H^{\ell-1}_x$ to the given hydrodynamic profile, and whose microscopic part converges to zero in $H^{\ell-1}_x$ and belongs to H^ℓ_x but is allowed to blow up (in a controled way) in that norm. Let us also underline that there is no smallness assumption on the initial data of the fluid system, and we are able to treat the cases of non-global and global solutions to the fluid system in a unified framework.

Theorem 1. *Let $\ell \in (3/2, 5/2)$ be given. Consider $(\rho_{\rm in}, u_{\rm in}, \theta_{\rm in}) \in H^{\ell-1}(\mathbb{T}^3)$ that are mean-free. With the notation (1.19), let (ρ, u, θ) be the unique solution to (NSF) associated with the initial data $(\bar\rho_{\rm in}, \bar u_{\rm in}, \bar\theta_{\rm in})$ in the space $\widetilde{L}^\infty_T H^{\ell-1}_x \cap L^2_T H^\ell_x$, for some $T > 0$. We also consider $\bar g_{\rm in}$ and g as defined in (1.20) and (1.21).*

Given $\alpha \in (0, 1/4)$ and $\beta \in (\alpha, 1/2)$, let $f^\varepsilon_{\rm in}$ be a family of functions such that

$$\mathbf{P}_0 f^\varepsilon_{\rm in} = \chi(\varepsilon^\alpha |D_x|)\bar g_{\rm in} \quad \text{and} \quad \|\mathbf{P}_0^\perp f^\varepsilon_{\rm in}\|_{H^{\ell-1}_x L^2_v} + \varepsilon^\beta \|\mathbf{P}_0^\perp f^\varepsilon_{\rm in}\|_{H^\ell_x L^2_v} \xrightarrow[\varepsilon\to 0]{} 0$$

for some smooth, compactly supported function χ. Then, there is $\varepsilon_0 > 0$ such that for any $\varepsilon \leqslant \varepsilon_0$, there exists a unique solution f^ε to the kinetic equation (1.11) with initial data $f^\varepsilon_{\rm in}$, which belongs to the space $\widetilde{L}^\infty_T H^\ell_x L^2_v \cap L^2_T H^\ell_x H^{s,\star}_v$, and it moreover satisfies

$$\|f^\varepsilon - g\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} + \|f^\varepsilon - g\|_{L^2_T H^\ell_x H^{s,\star}_v} \xrightarrow[\varepsilon\to 0]{} 0\,.$$

Remark 1. The restriction $\ell < 5/2$ is purely technical, the result would hold for any $\ell > 3/2$ up to some adaptations in the nonlinear estimates. The threshold value $1/4$ for the truncation parameter α comes from technical considerations that appear throughout the proof. The additional parameter β quantifies the possible blow up of the $H^\ell_x H^{s,\star}_v$ norm of the "microscopic" part of the initial data. Such an assumption is reminiscent of the setting chosen in [24] in the context of the incompressible limit.

Remark 2. Throughout this paper, we only consider the case of well-prepared data in the torus and also only the case of not too soft potentials for the kinetic equations. We believe that using the same method of proof combined with arguments and estimates of [26, 13], our analysis could be extended to a more general setting by considering the problem in the whole space (also including ill-prepared data) and very soft potentials for the kinetic equations.

1.6. **Sketch of the proof and plan of the paper.** The idea of the proof follows the method of [26], consisting in solving by a fixed point argument the equation obtained by taking the difference between the kinetic and hydrodynamic equations, written in Duhamel form. The main interest of this equation is that it no longer involves the kinetic unknown but writes schematically as

$$\delta^\varepsilon(t) = \mathcal{D}^\varepsilon(t) + \mathcal{S}^\varepsilon(t) + \mathcal{L}^\varepsilon[\delta^\varepsilon](t) + \Psi^\varepsilon[\delta^\varepsilon, \delta^\varepsilon](t)\,, \tag{1.22}$$

where $\mathcal{D}^\varepsilon(t)$ depends only on the initial data, $\mathcal{S}^\varepsilon(t)$ is a source term depending only on the hydrodynamical solution, $\mathcal{L}^\varepsilon[\delta^\varepsilon]$ is a linear operator depending on the hydrodynamic solution, and $\Psi^\varepsilon[\delta^\varepsilon, \delta^\varepsilon]$ is the usual, Boltzmann bilinear operator (see (3.8) below). The difficulty then consists in proving that $\mathcal{D}^\varepsilon(t)$ and $\mathcal{S}^\varepsilon(t)$ are small, and that Ψ^ε is bilinear continuous, in a low regularity framework. An additional difficulty comes from the fact that $\mathcal{L}^\varepsilon$ is not small if $\bar{g}_{\rm in}$ is not small : smallness is necessary for the fixed-point to work, but a Gronwall-type argument allows to get round this difficulty (in this regard, the proof differs from the one presented in [26]).

In Section 2, we give some useful tools to estimate each part of equation (1.22) (spectral decomposition, semi-group and nonlinear estimates). In Section 3, we make the strategy of our proof more precise and reduce the proof to a number of intermediate estimates. Finally, in Section 5, we provide all the necessary estimates to conclude the proof of our main result.

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2. Preliminaries

Our approach heavily relies on previous results on the spectral analysis of the linearized kinetic operator

$$\Lambda^\varepsilon := \frac{1}{\varepsilon^2} L - \frac{1}{\varepsilon} v \cdot \nabla_x$$

in Fourier space for the space variable x (see [49, 25, 27, 29]), where we recall that L is defined in (1.12). We denote by $U^\varepsilon(t)$ the semi-group associated to Λ^ε.

Taking the Fourier transform in the space variable, we denote, for all $k \in \mathbb{Z}^3$,

$$\widehat{\Lambda}^\varepsilon(k) := \frac{1}{\varepsilon^2} L - \frac{1}{\varepsilon} i v \cdot k$$

and $\widehat{U}^\varepsilon(t,k) := e^{t\widehat{\Lambda}^\varepsilon(k)}$, so that $U^\varepsilon(t) = \mathcal{F}_x^{-1} \widehat{U}^\varepsilon(t,\cdot)\mathcal{F}_x$. We also denote

$$\Psi^\varepsilon[f_1, f_2](t) := \frac{1}{\varepsilon} \int_0^t U^\varepsilon(t-t') \Gamma_{\rm sym}(f_1(t'), f_2(t'))\, {\rm d}t'\,, \tag{2.1}$$

where $\Gamma_{\rm sym}(f_1, f_2) := \frac{1}{2}(\Gamma(f_1, f_2) + \Gamma(f_2, f_1))$ denotes the symmetrized form of Γ, so that (1.11) takes the Duhamel form

$$f^\varepsilon(t) = U^\varepsilon(t) f^\varepsilon_{\rm in} + \Psi^\varepsilon[f^\varepsilon, f^\varepsilon](t)\,. \tag{2.2}$$

In Fourier space we have

$$\widehat{\Psi}^\varepsilon[f_1, f_2](t,k) = \frac{1}{\varepsilon} \int_0^t \widehat{U}^\varepsilon(t-t', k) \widehat{\Gamma}_{\rm sym}(f_1(t'), f_2(t'))(k)\, {\rm d}t'\,,$$

where

$$\widehat{\Gamma}_{\rm sym}(f_1,f_2)(k) := \sum_{k'\in\mathbb{Z}^3} \Gamma_{\rm sym}(\widehat{f}_1(k-k'),\widehat{f}_2(k'))\,.$$

Observe that

$$\Psi^\varepsilon[f_1,f_2](t) = \mathcal{F}_x^{-1}\widehat{\Psi}^\varepsilon[f_1,f_2](t,\cdot)\mathcal{F}_x\,.$$

It turns out that there is a complete description of the operator U^ε: this goes back to [49, 25] for the Boltzmann hard-spheres kernel, [56] for the Boltzmann non-cutoff (resp. Landau) kernels with hard and moderately soft potentials $\gamma+2s\geqslant 0$ (resp. $\gamma+2\geqslant 0$), and [57] for the Boltzmann non-cutoff (resp.Landau) kernels with very soft potentials $\gamma+2s<0$ (resp. $\gamma+2<0$). For the not too soft potentials, we also refer to the paper [29] in which the authors provide a more modern spectral approach.

Let us start by noticing that

$$\widehat{U}^\varepsilon(t,k) = \widehat{U}^1\Big(\frac{t}{\varepsilon^2},\varepsilon k\Big)\,. \tag{2.3}$$

Roughly speaking, for $|k|\leqslant\kappa$ small enough, the operator $\widehat{\Lambda}^1(k) := L - iv\cdot k$ can be seen as a perturbation of L. In particular it can be proved (see [25]) that the 5-dimensional kernel of L recalled in (1.14) splits into 4 eigenvalues (the first one below is double) that satisfy for all $|k|\leqslant\kappa$

$$\begin{aligned}
&\lambda_{\rm NS}(k) := -\nu_{\rm NS}|k|^2+\gamma_{\rm NS}(k)\,,\quad \nu_{\rm NS}>0\,,\quad |\gamma_{\rm NS}(k)|\leqslant\frac{\nu_{\rm NS}}{2}|k|^2\\
&\lambda_{\rm heat}(k) := -\nu_{\rm heat}|k|^2+\gamma_{\rm heat}(k)\,,\quad \nu_{\rm heat}>0\,,\quad |\gamma_{\rm heat}(k)|\leqslant\frac{\nu_{\rm heat}}{2}|k|^2
\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}
\lambda_{\rm wave\pm}(k) &:= \pm ic|k|-\nu_{\rm wave\pm}|k|^2+\gamma_{\rm wave\pm}(k)\,,\\
&\qquad c>0\,,\ \nu_{\rm wave\pm}>0\,,\quad |\gamma_{\rm wave\pm}(k)|\leqslant\frac{\nu_{\rm wave\pm}}{2}|k|^2\,.
\end{aligned} \tag{2.5}$$

Moreover, the associate projectors $\mathcal{P}_\star$ can be written (where $\star$ stands for NS, heat, or wave$\pm$)

$$\mathcal{P}_\star = \mathcal{P}^0_\star\left(\frac{k}{|k|}\right)+|k|\mathcal{P}^1_\star\left(\frac{k}{|k|}\right)+|k|^2\mathcal{P}^2_\star(k)\,,$$

with $\mathcal{P}^n_\star$ bounded linear operators on L^2_v with operator norms uniform for $|k|\leqslant\kappa$. We even have that $\mathcal{P}^0_\star(k/|k|)$, $\mathcal{P}^1_\star(k/|k|)$ and $\mathcal{P}^2_\star(k)$ are bounded from $(H^{s,\star}_v)'$ into $H^{s,\star}_v$ uniformly in $|k|\leqslant\kappa$. We refer to [29, Theorem 1.6-(2)] for this property (note the following correspondance of notation $H^\bullet = H^{s,\star}_v$ and $H^\circ = (H^{s,\star}_v)'$). We also have that if $\star\neq\star'$, then $\mathcal{P}^0_\star\mathcal{P}^0_{\star'}=0$ and the orthogonal projector $\mathbf{P}_0$ onto $\operatorname{Ker}L$ satisfies

$$\mathbf{P}_0 = \sum_{\star\in\{\rm NS,heat,wave\pm\}}\mathcal{P}^0_\star\Big(\frac{k}{|k|}\Big)\,.$$

Actually $\mathcal{P}^0_{\rm NS}(k/|k|)$ is the projection onto the 2-dimensional space spanned by $v-\mathrm{pr}_k\,v$ for any k (this corresponds to the divergence free condition), and

$$\mathcal{P}^0_{\rm heat}\Big(\frac{k}{|k|}\Big)\hat{f}(k,v) = \frac{2}{5}\Big(-1+\frac{1}{2}(|v|^2-3)\Big)\sqrt{\mu}(v)\int_{\mathbb{R}^3}\Big(-1+\frac{1}{2}(|w|^2-3)\Big)\sqrt{\mu}(w)\widehat{f}(k,w)\,\mathrm{d}w\,.$$

Finally

$$\begin{aligned}
&\mathcal{P}^0_{\rm wave\pm}\Big(\frac{k}{|k|}\Big)\hat{f}(k,v)\\
&= \frac{3}{10}\Big(1\pm\frac{k}{|k|}\cdot v+\frac{1}{3}(|v|^2-3)\Big)\sqrt{\mu}(v)\int_{\mathbb{R}^3}\Big(1\pm\frac{k}{|k|}\cdot w+\frac{1}{3}(|w|^2-3)\Big)\sqrt{\mu}(w)\widehat{f}(k,w)\,\mathrm{d}w\,.
\end{aligned}$$

Thanks to (2.3) and to this spectral study, we deduce as in [8, 26] that U^ε can be decomposed as follows:

$$U^\varepsilon(t) = U^{\varepsilon,\flat}(t)+U^{\varepsilon,\sharp}(t) \tag{2.6}$$

where $U^{\varepsilon,\flat}(t)$ corresponds to the contribution of the low frequencies in the right part of the plane:

$$\widehat{U}^{\varepsilon,\flat}(t,k) := \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) \sum_{\star\in\{\mathrm{NS},\mathrm{heat},\mathrm{wave}\pm\}} e^{\lambda_\star(\varepsilon k)\frac{t}{\varepsilon^2}}\mathcal{P}_\star(\varepsilon k)\,, \tag{2.7}$$

where χ is a fixed smooth, compactly supported function. Moreover, since we consider not too soft potentials, there is $\lambda_0>0$ such that uniformly in $k\in\mathbb{Z}^3$

$$\|\widehat{U}^{\varepsilon,\sharp}(t,k)\|_{L^2_v\to L^2_v} \lesssim e^{-\lambda_0\frac{t}{\varepsilon^2}}\,, \qquad \forall t\geqslant 0\,. \tag{2.8}$$

In the study of the limit $\varepsilon\to 0$ of (1.11), it will be useful to decompose $U^{\varepsilon,\flat}(t)$ into a part independent of ε and a remainder, which will be shown to go to zero, in a sense to be made precise later:

$$U^{\varepsilon,\flat} = U_{\mathrm{NSF}} + U^{\varepsilon,\flat}_{\mathrm{wave}} + \widetilde{U}^{\varepsilon}\,, \tag{2.9}$$

where in Fourier variables

$$\begin{aligned}
\widehat{U}_{\mathrm{NSF}}(t,k) &:= e^{-\nu_{\mathrm{NS}}|k|^2 t}\mathcal{P}^0_{\mathrm{NS}}\Big(\frac{k}{|k|}\Big) + e^{-\nu_{\mathrm{heat}}|k|^2 t}\mathcal{P}^0_{\mathrm{heat}}\Big(\frac{k}{|k|}\Big)\\
\widehat{U}^{\varepsilon,\flat}_{\mathrm{wave}}(t,k) &:= \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\sum_{\pm} e^{\lambda_{\mathrm{wave}\pm}(\varepsilon k)\frac{t}{\varepsilon^2}}\mathcal{P}_{\mathrm{wave}\pm}(\varepsilon k)\,.
\end{aligned} \tag{2.10}$$

According to (2.1) and (2.6), we can also decompose

$$\Psi^{\varepsilon} = \Psi^{\varepsilon,\flat} + \Psi^{\varepsilon,\sharp} \tag{2.11}$$

where

$$\mathcal{F}_x\Big(\Psi^{\varepsilon,\sharp}[f_1,f_2](t)\Big)(k) := \frac{1}{\varepsilon}\int_0^t \widehat{U}^{\varepsilon,\sharp}(t-t')\widehat{\Gamma}_{\mathrm{sym}}\big(f_1(t'),f_2(t')\big)(k)\,\mathrm{d}t' \tag{2.12}$$

and

$$\mathcal{F}_x\Big(\Psi^{\varepsilon,\flat}[f_1,f_2](t)\Big)(k) := \frac{1}{\varepsilon}\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\sum_{\star}\int_0^t e^{\lambda_\star(\varepsilon k)\frac{t-t'}{\varepsilon^2}}\mathcal{P}_\star(\varepsilon k)\widehat{\Gamma}_{\mathrm{sym}}\big(f_1(t'),f_2(t')\big)(k)\,\mathrm{d}t'\,, \tag{2.13}$$

where the sum runs over $\{\mathrm{NS},\mathrm{heat},\mathrm{wave}\pm\}$. In the interest of the limit $\varepsilon\to 0$, this can be again decomposed as in (2.9), as follows:

$$\Psi^{\varepsilon,\flat} = \Psi_{\mathrm{NSF}} + \Psi^{\varepsilon,\flat}_{\mathrm{wave}} + \widetilde{\Psi}^{\varepsilon} \tag{2.14}$$

where writing $\widehat{\Psi}_\star[f_1,f_2](t) = \mathcal{F}_x(\Psi_\star[f,f](t))$ and recalling that $\mathbf{P}_0\Gamma_{\mathrm{sym}}=0$,

$$\begin{aligned}
\widehat{\Psi}_{\mathrm{NSF}}[f_1,f_2](t,k) &:= \sum_{\star\in\{\mathrm{NS},\mathrm{heat}\}}\int_0^t e^{-\nu_\star(t-t')|k|^2}|k|\mathcal{P}^1_\star\Big(\frac{k}{|k|}\Big)\widehat{\Gamma}_{\mathrm{sym}}\big(f_1(t'),f_2(t')\big)(k)\,\mathrm{d}t'\,,\\
\widehat{\Psi}^{\varepsilon,\flat}_{\mathrm{wave}}[f_1,f_2](t,k) &:= \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\\
&\quad\times\sum_{\pm}\int_0^t e^{\lambda_{\mathrm{wave}\pm}(\varepsilon k)\frac{t-t'}{\varepsilon^2}}\mathcal{P}_{\mathrm{wave}\pm}(\varepsilon k)\widehat{\Gamma}_{\mathrm{sym}}\big(f_1(t'),f_2(t')\big)(k)\,\mathrm{d}t'\,.
\end{aligned}$$

3. Proof of the theorem

Let us start by presenting the functional framework in which we shall develop our proof. We fix a parameter $\beta\in(\alpha,1/2)$ where we recall that $\alpha\in(0,1/4)$ has been introduced in Theorem 1 and define for any interval I of $\mathbb{R}^+$

$$\mathcal{X}^\varepsilon_I := \Big\{f\in\widetilde{L}^\infty_I H^{\ell-1}_x L^2_v\cap L^2_I H^\ell_x H^{s,\star}_v\cap\widetilde{L}^\infty_I H^\ell_x L^2_v\;\Big|\;\|f\|_{\mathcal{X}^\varepsilon_I}<+\infty\Big\} \tag{3.1}$$

which we endow with the norm

$$\|f\|_{\mathcal{X}^\varepsilon_I} = \|f\|_{\widetilde{L}^\infty_I H^{\ell-1}_x L^2_v} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}_0^\perp f\|_{L^2_I H^\ell_x H^{s,\star}_v} + \|\mathbf{P}_0 f\|_{L^2_I H^\ell_x H^{s,\star}_v} + \varepsilon^\beta\|f\|_{\widetilde{L}^\infty_I H^\ell_x L^2_v}\,. \tag{3.2}$$

In the following we write $\mathcal{X}^\varepsilon_T := \mathcal{X}^\varepsilon_{[0,T]}$.

Remark 3. If $f = f(x,v)$ is a function in $H^{\ell-1}_x L^2_v$ and if χ is a smooth, compactly supported function on $\mathbb{R}^3$, then the sequence $f^\varepsilon := \chi(\varepsilon^\alpha|D_x|)f$ goes to zero in $\varepsilon^\beta H^\ell_x L^2_v$, in the sense that

$$\varepsilon^\beta \|f^\varepsilon\|_{H^\ell_x L^2_v} \lesssim \varepsilon^{\beta-\alpha}\|f\|_{H^{\ell-1}_x L^2_v}\,.$$

Recall that we consider well-prepared initial data $\bar g_{\rm in}$ in $H^{\ell-1}_x L^2_v$ and the associated fluid solution $g \in \widetilde{L}^\infty_T H^{\ell-1}_x L^2_v \cap L^2_T H^\ell_x L^2_v$ for $T < T^\star$, where the maximal life span $T^\star > 0$ is such that

$$\lim_{T\to T^\star} \|g\|_{L^2_T H^\ell_x L^2_v} = \infty\,.$$

This solution satisfies

$$\|g\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} + \|g\|_{L^2_T H^\ell_x L^2_v} \lesssim \|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}. \tag{3.3}$$

We then build a family of initial data $f^\varepsilon_{\rm in}$ to (1.11) such that $\mathbf{P}_0 f^\varepsilon_{\rm in} = \chi(\varepsilon^\alpha|D_x|)\bar g_{\rm in}$ for some smooth, compactly supported function χ. We assume moreover that $\mathbf{P}_0^\perp f^\varepsilon_{\rm in}$ goes to zero in $H^{\ell-1}_x L^2_v$ and $\varepsilon^\beta \mathbf{P}_0^\perp f^\varepsilon_{\rm in}$ goes to zero in $H^\ell_x L^2_v$. Then as pointed out in Remark 3, $\mathbf{P}_0 f^\varepsilon_{\rm in}$ goes to 0 in $\varepsilon^\beta H^\ell_x L^2_v$ (since $\beta > \alpha$). Our goal is to prove that the solution f^ε of (1.11) with data $f^\varepsilon_{\rm in}$ converges to g as stated in Theorem 1, on the same life span as g.

The first step consists in replacing g by a smooth solution to (NSF) in the following way: let us define

$$g^\varepsilon(t,x,v) := \Big\{\rho^\varepsilon(t,x) + u^\varepsilon(t,x)\cdot v + \theta^\varepsilon(t,x)\frac{|v|^2-3}{2}\Big\}\sqrt{\mu}(v)$$

where $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ solves (NSF) with the initial data $\chi(\varepsilon^\alpha|D_x|)(\bar\rho_{\rm in}, \bar u_{\rm in}, \bar\theta_{\rm in})$. It is classical (see for instance [26, Proposition B.5], and [18, 4, 42] for more), that for ε small enough, $(\rho^\varepsilon, u^\varepsilon, \theta^\varepsilon)$ has a life span at least T and there holds

$$\|g^\varepsilon - g\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} + \|g^\varepsilon - g\|_{L^2_T H^\ell_x H^{s,\star}_v} \xrightarrow[\varepsilon\to 0]{} 0\,. \tag{3.4}$$

Note that in particular

$$\|g^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} \xrightarrow[\varepsilon\to 0]{} \|g\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} \quad\text{and}\quad \|g^\varepsilon\|_{L^2_T H^\ell_x H^{s,\star}_v} \xrightarrow[\varepsilon\to 0]{} \|g\|_{L^2_T H^\ell_x H^{s,\star}_v}\,. \tag{3.5}$$

To prove Theorem 1, it thus suffices to prove that

$$\|g^\varepsilon - f^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} + \|g^\varepsilon - f^\varepsilon\|_{L^2_T H^\ell_x H^{s,\star}_v} \xrightarrow[\varepsilon\to 0]{} 0\,.$$

Note that by propagation of regularity (see again [26, Proposition B.5]) there holds

$$\begin{aligned}\|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} + \|g^\varepsilon\|_{L^2_T H^{\ell+1}_x H^{s,\star}_v} &\lesssim \|\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{H^\ell_x L^2_v} \exp\Big(C\|g^\varepsilon\|^2_{L^2_T H^\ell_x H^{s,\star}_v}\Big)\\ &\lesssim \|\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{H^\ell_x L^2_v} \exp\Big(C\|g\|^2_{L^2_T H^\ell_x H^{s,\star}_v}\Big)\\ &\lesssim \|\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{H^\ell_x L^2_v} \exp\Big(C\|\bar g_{\rm in}\|^2_{H^{\ell-1}_x L^2_v}\Big)\end{aligned} \tag{3.6}$$

due to (3.5) and (3.3). In particular g^ε satisfies

$$\begin{aligned}\varepsilon^\beta\|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} &\lesssim \varepsilon^\beta\|\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{H^\ell_x L^2_v} \exp\Big(C\|g\|^2_{L^2_T H^\ell_x H^{s,\star}_v}\Big)\\ &\lesssim \varepsilon^{\beta-\alpha}\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v} \exp\Big(C\|\bar g_{\rm in}\|^2_{H^{\ell-1}_x L^2_v}\Big) \xrightarrow[\varepsilon\to 0]{} 0\end{aligned} \tag{3.7}$$

because $\beta > \alpha$.

It is also worth mentioning that $g^\varepsilon = \mathbf{P}_0 g^\varepsilon$ so that $\|g^\varepsilon(t,x,\cdot)\|_{H^{s,\star}_v} \lesssim \|g^\varepsilon(t,x,\cdot)\|_{L^2_v}$.

In what follows, we shall look for a solution f^ε to (1.11) under the form $f^\varepsilon = g^\varepsilon + \delta^\varepsilon$. Since

$$g^\varepsilon(t) = U_{\rm NSF}(t)\mathbf{P}_0 f^\varepsilon_{\rm in} + \Psi_{\rm NSF}[g^\varepsilon, g^\varepsilon](t)\,,$$

elementary algebraic computations lead to the following equation on δ^ε:

$$\begin{aligned}\delta^\varepsilon(t) &= (U^\varepsilon(t) - U_{\rm NSF}(t))\mathbf{P}_0 f_{\rm in}^\varepsilon + U^\varepsilon(t)\mathbf{P}_0^\perp f_{\rm in}^\varepsilon \\ &\quad + \Psi^\varepsilon[g^\varepsilon, g^\varepsilon](t) - \Psi_{\rm NSF}[g^\varepsilon, g^\varepsilon](t) \\ &\quad + \Psi^\varepsilon[g^\varepsilon, \delta^\varepsilon](t) + \Psi^\varepsilon[\delta^\varepsilon, \delta^\varepsilon](t)\,.\end{aligned} \tag{3.8}$$

As we shall see, the main point is to be able to solve the equation on δ^ε, with an H^ℓ_x norm which is finite but blows up (in a controled way) as $\varepsilon \to 0$. Our method of proof will allow us to prove that the equation has a unique solution on the same time interval as g^ε hence as g, at least for ε small enough. In doing so we shall also prove that δ^ε converge to 0.

The method will rely on the following fixed point lemma.

Lemma 3.1. *There is a constant $C_0 > 0$ such that the following holds. Let X be a Banach space, $\mathcal{L}$ be a continuous linear map from X to X, and $\mathcal{B}$ be a bilinear map from $X \times X$ to X. Let us define*

$$\|\mathcal{L}\| := \sup_{\|x\|=1} \|\mathcal{L}x\| \quad \text{and} \quad \|\mathcal{B}\| := \sup_{\|x\|=\|y\|=1} \|\mathcal{B}(x,y)\|\,.$$

If $\|\mathcal{L}\| < 1$, then for any x_0 in X such that

$$\|x_0\|_X < \frac{(1-\|\mathcal{L}\|)^2}{4\|\mathcal{B}\|} \tag{3.9}$$

the equation $x = x_0 + \mathcal{L}x + \mathcal{B}(x,x)$ has a unique solution in the ball of center 0 and radius $\dfrac{1-\|\mathcal{L}\|}{2\|\mathcal{B}\|}$ and $\|x\| \leqslant C_0\|x_0\|$.

In the next sections, we shall provide all the necessary estimates in order to implement this fixed-point argument to solve (3.8), which we re-write in the following form:

$$\delta^\varepsilon(t) = \mathcal{D}^\varepsilon(t) + \mathcal{S}^\varepsilon(t) + \mathcal{L}^\varepsilon[\delta^\varepsilon](t) + \Psi^\varepsilon[\delta^\varepsilon, \delta^\varepsilon](t)\,,$$

where the data $\mathcal{D}^\varepsilon$, source $\mathcal{S}^\varepsilon$ and linear $\mathcal{L}^\varepsilon[\delta^\varepsilon]$ terms are defined by

$$\begin{aligned}\mathcal{D}^\varepsilon(t) &:= (U^\varepsilon(t) - U_{\rm NSF}(t))\mathbf{P}_0 f_{\rm in}^\varepsilon + U^\varepsilon(t)\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\,, \\ \mathcal{S}^\varepsilon(t) &:= \Psi^\varepsilon[g^\varepsilon, g^\varepsilon](t) - \Psi_{\rm NSF}[g^\varepsilon, g^\varepsilon](t)\,, \\ \mathcal{L}^\varepsilon[\delta^\varepsilon](t) &:= \Psi^\varepsilon[g^\varepsilon, \delta^\varepsilon](t)\,.\end{aligned} \tag{3.10}$$

Paragraphs 5.1 to 5.5 will be devoted to the proof of the following result.

Proposition 3.2. *Under the assumptions of Theorem 1, the following holds.*

(1) *For any $\tau \in (0,T)$ there holds*

$$\|U^\varepsilon(\cdot - \tau)F(\tau)\|_{\mathcal{X}^\varepsilon_{[\tau,T]}} \lesssim \|F\|_{\mathcal{X}^\varepsilon_{[0,\tau]}}.$$

(2) *The data term goes to zero globally in time: for any $t \geqslant 0$*

$$\|\mathcal{D}^\varepsilon\|_{\mathcal{X}^\varepsilon_\infty} \lesssim \varepsilon^{\frac12-\alpha}\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v} + \|\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{H^{\ell-1}_x L^2_v} + \varepsilon^\beta \|\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{H^\ell_x L^2_v} \xrightarrow[\varepsilon\to 0]{} 0\,.$$

(3) *The source term goes to zero in $\mathcal{X}^\varepsilon_T$: there is an increasing function Φ such that for any $t \in [0,T]$*

$$\|\mathcal{S}^\varepsilon\|_{\mathcal{X}^\varepsilon_T} \leqslant \varepsilon^{\frac12-2\alpha}\,\Phi\left(\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\right) \xrightarrow[\varepsilon\to 0]{} 0\,.$$

(4) *The linear term satisfies the following continuity estimate for ε small enough:*

$$\|\mathcal{L}^\varepsilon[f]\|_{\mathcal{X}^\varepsilon_T} \lesssim \|f\|_{\mathcal{X}^\varepsilon_T}\left(\varepsilon^\beta\|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} + \sqrt{\varepsilon}\|g^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} + \|g^\varepsilon\|_{L^2_T H^\ell_x L^2_v}\right).$$

(5) *The nonlinear term satisfies the following continuity estimate:*

$$\|\Psi^\varepsilon[f_1, f_2]\|_{\mathcal{X}^\varepsilon_T} \lesssim \|f_1\|_{\mathcal{X}^\varepsilon_T}\|f_2\|_{\mathcal{X}^\varepsilon_T}\,.$$

Let us investigate how Proposition 3.2 ensures the wellposedness of (3.8) in $\mathcal{X}_T^\varepsilon$ and the convergence of δ^ε to zero, thus proving Theorem 1.

End of the proof of Theorem 1. We shall check that (3.8) takes the form required by Lemma 3.1. On the one hand thanks to Proposition 3.2 we have

$$\|\mathcal{D}^\varepsilon\|_{\mathcal{X}_\infty^\varepsilon} + \|\mathcal{S}^\varepsilon\|_{\mathcal{X}_T^\varepsilon} \xrightarrow[\varepsilon\to 0]{} 0$$

so (3.9) will be satisfied as soon as we have a hold on the continuity constant on $\mathcal{L}^\varepsilon$: we need the linear operator $\mathcal{L}^\varepsilon$ to be a a contraction in $\mathcal{X}_T^\varepsilon$. As can be seen from Proposition 3.2–(4), for that to be the case one needs $\|g\|_{L^2_T H^\ell_x L^2_v}$ to be small, which unfortunately is not the case (unless $\bar g_{\rm in}$ is small, which we do not assume here).

In order to get around this difficulty, we shall apply Lemma 3.1 iteratively on small time intervals. Note that due to Proposition 3.2–(4) and the estimates (3.3) and (3.7), there is a constant $C>0$ and $\varepsilon_0>0$ such that for all $\varepsilon\leqslant\varepsilon_0$

$$\|\mathcal{L}^\varepsilon[f]\|_{\mathcal{X}_T^\varepsilon} \leqslant C\|f\|_{\mathcal{X}_T^\varepsilon}\Big(\frac{1}{4C} + \|g\|_{L^2_T H^\ell_x H^{s,\star}_v}\Big)\,.$$

Now there exists $K>0$ and a succession of times $t_1:=0<t_2<\cdots<t_K$ such that

$$\forall 1\leqslant i\leqslant K-1\,,\quad \|g\|_{L^2([t_i,t_{i+1}];H^\ell_x H^{s,\star}_v)} \leqslant \frac{1}{4C}\cdot$$

Then

$$\|\mathcal{L}^\varepsilon[f]\|_{\mathcal{X}_{t_2}^\varepsilon} \leqslant \frac12\|f\|_{\mathcal{X}_{t_2}^\varepsilon}\cdot \tag{3.11}$$

Applying Lemma 3.1 on $[0,t_2]$ then implies that there is a unique solution δ^ε to (3.8) in $\mathcal{X}_{t_2}^\varepsilon$, which satisfies

$$\|\delta^\varepsilon\|_{\mathcal{X}_{t_2}^\varepsilon} \leqslant C_0\Big(D^\varepsilon_{\rm in} + \|\mathcal{S}^\varepsilon\|_{\mathcal{X}_{t_2}^\varepsilon}\Big) \xrightarrow[\varepsilon\to 0]{} 0\,, \tag{3.12}$$

with thanks to Proposition 3.2–(2)

$$D^\varepsilon_{\rm in} := \varepsilon^{\frac12-\alpha}\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v} + \|\mathbf{P}_0^\perp f^\varepsilon_{\rm in}\|_{H^{\ell-1}_x L^2_v} + \varepsilon^\beta\|\mathbf{P}_0^\perp f^\varepsilon_{\rm in}\|_{H^\ell_x L^2_v}\,. \tag{3.13}$$

Then we solve (3.8) on $[t_2,t_3]$. We recall that (3.8) writes

$$\forall t\in[t_2,t_3]\,,\quad \delta^\varepsilon(t) = \mathcal{D}^\varepsilon(t) + \mathcal{S}^\varepsilon(t) + \mathcal{L}^\varepsilon[\delta^\varepsilon](t) + \Psi^\varepsilon[\delta^\varepsilon,\delta^\varepsilon](t)\,,$$

with $\mathcal{D}^\varepsilon$, $\mathcal{S}^\varepsilon$ and $\mathcal{L}^\varepsilon$ defined in (3.10). We want to recast this equation in a form suited to a fixed point on $[t_2,t_3]$. According to (2.1) and since U^ε is a semigroup, we can write for all $t\geqslant t_2$

$$\begin{aligned}\Psi^\varepsilon[f,g](t) &= \frac{1}{\varepsilon}U^\varepsilon(t-t_2)\int_0^{t_2} U^\varepsilon(t_2-t')\Gamma_{\rm sym}(f(t'),g(t'))\,{\rm d}t' \\ &\qquad + \frac{1}{\varepsilon}\int_{t_2}^t U^\varepsilon(t-t')\Gamma_{\rm sym}(f(t'),g(t'))\,{\rm d}t' \\ &=: \frac{1}{\varepsilon}U^\varepsilon(t-t_2)\int_0^{t_2} U^\varepsilon(t_2-t')\Gamma_{\rm sym}(f(t'),g(t'))\,{\rm d}t' + \Psi^\varepsilon[f,g](t_2;t)\,.\end{aligned}$$

We also define the operator

$$\mathcal{L}^\varepsilon[f](t_2;t) := \Psi^\varepsilon[g^\varepsilon,f](t_2;t)$$

and we set

$$\mathcal{D}_2^\varepsilon(t) := \mathcal{D}^\varepsilon(t) - U^\varepsilon(t-t_2)\mathcal{D}^\varepsilon(t_2)$$

and

$$\mathcal{S}_2^\varepsilon(t) := \mathcal{S}^\varepsilon(t) - U^\varepsilon(t-t_2)\mathcal{S}^\varepsilon(t_2)\,.$$

Then (3.8) can be recast on $[t_2,t_3]$ as follows:

$$\delta^\varepsilon(t) = U^\varepsilon(t-t_2)\delta^\varepsilon(t_2) + \mathcal{D}_2^\varepsilon(t) + \mathcal{S}_2^\varepsilon(t) + \mathcal{L}^\varepsilon[\delta^\varepsilon](t_2;t) + \Psi^\varepsilon[\delta^\varepsilon,\delta^\varepsilon](t_2;t)\,.$$

Thanks to Proposition 3.2–(1), (2) and (3), $\mathcal{D}_2^\varepsilon$ and $\mathcal{S}_2^\varepsilon$ go to zero in $\mathcal{X}^\varepsilon_{[t_2,t_3]}$, with for some universal constant C_1

$$\|\mathcal{D}_2^\varepsilon\|_{\mathcal{X}^\varepsilon_{[t_2,t_3]}} \leqslant C_1 D^\varepsilon_{\rm in}$$

with notation (3.13), and

$$\|\mathcal{S}_2^\varepsilon\|_{\mathcal{X}^\varepsilon_{[t_2,t_3]}} \leqslant C_1 \varepsilon^{\frac12-2\alpha}\Phi\left(\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\right).$$

The linear operator $\mathcal{L}^\varepsilon[\delta^\varepsilon](t_2;t)$ is dealt with exactly as above to produce similarly to (3.11), for ε small enough,

$$\|\mathcal{L}^\varepsilon[f]\|_{\mathcal{X}^\varepsilon_{[t_2,t_3]}} \leqslant \frac12 \|f\|_{\mathcal{X}^\varepsilon_{[t_2,t_3]}}.$$

Finally thanks to Proposition 3.2–(1) and (3.12), we have for some universal constant $C_2 > 0$ that

$$\begin{aligned}\|U^\varepsilon(t-t_2)\delta^\varepsilon(t_2)\|_{\mathcal{X}^\varepsilon_{[t_2,t_3]}} &\lesssim \|\mathcal{D}^\varepsilon\|_{\mathcal{X}^\varepsilon_\infty} + \|\mathcal{S}^\varepsilon\|_{\mathcal{X}^\varepsilon_{t_2}}\\ &\leqslant C_2\left[D^\varepsilon_{\rm in} + \varepsilon^{\frac12-2\alpha}\Phi\left(\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\right)\right].\end{aligned}$$

We can therefore apply Lemma 3.1 which implies that

$$\begin{aligned}\|\delta^\varepsilon\|_{\mathcal{X}^\varepsilon_{[t_2,t_3]}} &\leqslant C_0\Big(\|U^\varepsilon(\cdot-t_2)\delta^\varepsilon(t_2)\|_{\mathcal{X}^\varepsilon_{[t_2,t_3]}} + \|\mathcal{D}_2^\varepsilon\|_{\mathcal{X}^\varepsilon_{[t_2,t_3]}} + \|\mathcal{S}_2^\varepsilon\|_{\mathcal{X}^\varepsilon_{[t_2,t_3]}}\Big)\\ &\leqslant C_0(C_1+C_2)\left[D^\varepsilon_{\rm in} + \varepsilon^{\frac12-2\alpha}\Phi\left(\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\right)\right].\end{aligned}$$

Iterating this argument K times and noticing that K is of the order of $\|g\|_{L^2_T H^\ell_x H^{s,\star}_v}$, we find that there is a unique solution $\delta^\varepsilon \in \mathcal{X}^\varepsilon_T$ to (3.8) on $[0,T]$ which satisfies for some universal constant $C \geqslant 2$

$$\|\delta^\varepsilon\|_{\mathcal{X}^\varepsilon_T} \lesssim C^{\|g\|_{L^2_T H^\ell_x H^{s,\star}_v}}\left[D^\varepsilon_{\rm in} + \varepsilon^{\frac12-2\alpha}\Phi\left(\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\right)\right] \xrightarrow[\varepsilon\to0]{} 0.$$

Theorem 1 is proved. □

4. Some results on the operators U^ε and Ψ^ε

In this section, we provide useful continuity estimates on U^ε and Ψ^ε. We also refine decomposition (2.11) on Ψ^ε using the spectral properties introduced in Section 2.

4.1. Estimates on U^ε and Ψ^ε.

The first series of estimates (Propositions 4.1, 4.2 and Corollary 4.3) are very close to the ones established in [13] (and in [29]) and are based on hypocoercive energy estimates (see Appendix A for a presentation of hypocoercivity results). Since the functional framework is a bit different, we reformulate them in our functional setting. Some key elements of proofs are provided in Appendix A.

Proposition 4.1. *Let $m \geqslant 0$ and $T > 0$. There holds:*

(1) *Let $f \in H^m_x L^2_v$ and assume f verifies (1.13). Then*

$$\|U^\varepsilon(\cdot)f\|_{\widetilde{L}^\infty_T H^m_x L^2_v} + \frac1\varepsilon\|\mathbf{P}_0^\perp U^\varepsilon(\cdot)f\|_{L^2_T H^m_x H^{s,\star}_v} + \|\mathbf{P}_0 U^\varepsilon(\cdot)f\|_{L^2_T H^m_x L^2_v} \lesssim \|f\|_{H^m_x L^2_v},$$

and moreover $U^\varepsilon(t)f$ verifies (1.13) for all $t \geqslant 0$.

(2) *Let $S = S(t,x,v)$ satisfy $\mathbf{P}_0 S = 0$ and $S \in L^2_T H^m_x (H^{s,\star}_v)'$, then for any $t \leqslant T$,*

$$\begin{aligned}&\left\|\int_0^t U^\varepsilon(t-t')S(t')\,\mathrm{d}t'\right\|_{\widetilde{L}^\infty_T H^m_x L^2_v} + \frac1\varepsilon\left\|\mathbf{P}_0^\perp\int_0^t U^\varepsilon(t-t')S(t')\,\mathrm{d}t'\right\|_{L^2_T H^m_x H^{s,\star}_v}\\ &\qquad + \left\|\mathbf{P}_0\int_0^t U^\varepsilon(t-t')S(t')\,\mathrm{d}t'\right\|_{L^2_T H^m_x L^2_v} \lesssim \varepsilon\|S\|_{L^2_T H^m_x (H^{s,\star}_v)'}.\end{aligned}$$

From [29, Lemmas 4.8 and 4.9] we also have estimates for the semi-group $U^{\varepsilon,\sharp}$.

Proposition 4.2. *Let $m \geqslant 0$ and $T > 0$. There holds:*

(1) *Let $f \in H^m_x L^2_v$, then*

$$\|U^{\varepsilon,\sharp}(\cdot)f\|_{\widetilde{L}^\infty_T H^m_x L^2_v} + \frac{1}{\varepsilon}\|U^{\varepsilon,\sharp}(\cdot)f\|_{L^2_T H^m_x H^{s,\star}_v} \lesssim \|f\|_{H^m_x L^2_v}\,.$$

(2) *Let $S = S(t,x,v)$ satisfy $S \in L^2_T H^m_x (H^{s,\star}_v)'$. Then for any $t \leqslant T$,*

$$\left\|\int_0^t U^{\varepsilon,\sharp}(t-t')S(t')\,\mathrm{d}t'\right\|_{\widetilde{L}^\infty_T H^m_x L^2_v} + \frac{1}{\varepsilon}\left\|\int_0^t U^{\varepsilon,\sharp}(t-t')S(t')\,\mathrm{d}t'\right\|_{L^2_T H^m_x H^{s,\star}_v} \lesssim \varepsilon\|S\|_{L^2_T H^m_x (H^{s,\star}_v)'}\,.$$

From the two previous propositions, since Γ_{sym} is such that $\mathbf{P}_0\Gamma_{\mathrm{sym}} = 0$, it is straightforward to deduce the following result.

Corollary 4.3. *Consider $m \geqslant 0$, f_1, f_2 such that $\Gamma_{\mathrm{sym}}(f_1,f_2) \in L^2_T H^m_x (H^{s,\star}_v)'$ for some given $T > 0$. Then, there holds:*

$$\|\Psi^\varepsilon[f_1,f_2]\|_{\widetilde{L}^\infty_T H^m_x L^2_v} + \frac{1}{\varepsilon}\left\|\mathbf{P}_0^\perp \Psi^\varepsilon[f_1,f_2]\right\|_{L^2_T H^m_x H^{s,\star}_v} + \|\mathbf{P}_0\Psi^\varepsilon[f,g]\|_{L^2_T H^m_x L^2_v} \lesssim \|\Gamma_{\mathrm{sym}}(f_1,f_2)\|_{L^2_T H^m_x (H^{s,\star}_v)'}$$

and

$$\|\Psi^{\varepsilon,\sharp}[f_1,f_2]\|_{\widetilde{L}^\infty_T H^m_x L^2_v} + \frac{1}{\varepsilon}\|\Psi^{\varepsilon,\sharp}[f_1,f_2]\|_{L^2_T H^m_x H^{s,\star}_v} \lesssim \|\Gamma_{\mathrm{sym}}(f_1,f_2)\|_{L^2_T H^m_x (H^{s,\star}_v)'}\,.$$

We finally give a straighforward estimate for $U^{\varepsilon,\flat}$.

Proposition 4.4. *Let $m \geqslant 0$ and $T > 0$. For any $f \in H^m_x L^2_v$ there holds*

$$\|U^{\varepsilon,\flat}(\cdot)f\|_{L^2_T H^{m+1}_x H^{s,\star}_v} \lesssim \|f\|_{H^m_x L^2_v}\,.$$

4.2. **Nonlinear estimates.** We now provide nonlinear estimates that are central to estimate the nonlinear collisional operator Γ in various functional spaces. It is well-known (see [36] for cutoff Boltzmann, [34, 2] for non-cutoff Boltzmann, [35] for Landau) that

$$\left|\langle \Gamma(f_1,f_2), f_3\rangle_{L^2_v}\right| \lesssim \|f_1\|_{L^2_v}\|f_2\|_{H^{s,\star}_v}\|f_3\|_{H^{s,\star}_v}\,,$$

from which we obtain

$$\|\Gamma(f_1,f_2)\|_{(H^{s,\star}_v)'} := \sup_{\|\phi\|_{H^{s,\star}_v}\leqslant 1} \langle \Gamma(f_1,f_2),\phi\rangle_{L^2_v} \lesssim \|f_1\|_{L^2_v}\|f_2\|_{H^{s,\star}_v}\,. \tag{4.1}$$

Proposition 4.5. *Let $m \geqslant 0$. For any $r_1, r_2 \neq 3/2$, any $p_1,q_1,p_2,q_2 \in [1,\infty]$ that are such that $1/p_1 + 1/q_1 = 1/p_2 + 1/q_2 = 1/2$, and any smooth enough functions f_1, f_2 there holds:*

$$\|\Gamma(f_1,f_2)\|_{L^2_T H^m_x (H^{s,\star}_v)'} \lesssim \|f_1\|_{\widetilde{L}^{p_1}_T H^{m+(3/2-r_1)_+}_x L^2_v}\|f_2\|_{\widetilde{L}^{q_1}_T H^{r_1}_x H^{s,\star}_v} + \|f_1\|_{\widetilde{L}^{p_2}_T H^{r_2}_x L^2_v}\|f_2\|_{\widetilde{L}^{q_2}_T H^{m+(3/2-r_2)_+}_x H^{s,\star}_v}\,.$$

Proof. To simplify we write $F_1(t,k) = \|\hat{f}_1(t,k,\cdot)\|_{L^2_v}$ and $F_2(t,k) = \|\hat{f}_2(t,k,\cdot)\|_{H^{s,\star}_v}$. By (4.1) we have, for any $k \in \mathbb{Z}^3$,

$$\|\widehat{\Gamma}(f_1,f_2)(k)\|_{L^2_T (H^{s,\star}_v)'} \lesssim \left\{\int_0^T \left[\sum_{n\in\mathbb{Z}^3} F_1(t,k-n)F_2(t,n)\right]^2 \mathrm{d}t\right\}^{\frac12},$$

and applying Minkowski's inequality yields

$$\|\widehat{\Gamma}(f_1,f_2)(k)\|_{L^2_T (H^{s,\star}_v)'} \lesssim \sum_{n\in\mathbb{Z}^3}\left\{\int_0^T |F_1(t,k-n)|^2|F_2(t,n)|^2\,\mathrm{d}t\right\}^{\frac12}.$$

We now follow [43, Lemma 7.3]. We first split

$$\|\widehat{\Gamma}(f_1,f_2)(k)\|_{L^2_T (H^{s,\star}_v)'} \lesssim I_1(k) + I_2(k)$$

with

$$I_1(k) = \sum_{n\in\mathbb{Z}^3} \mathbf{1}_{|n|<|k-n|} \left\{ \int_0^T |F_1(t,k-n)|^2 |F_2(t,n)|^2 \,\mathrm{d}t \right\}^{\frac{1}{2}}$$

and

$$I_2(k) = \sum_{n\in\mathbb{Z}^3} \mathbf{1}_{|k-n|\geqslant|n|} \left\{ \int_0^T |F_1(t,n)|^2 |F_2(t,k-n)|^2 \,\mathrm{d}t \right\}^{\frac{1}{2}}.$$

We now estimate the term I_1. Thanks to Hölder's inequality in time, we obtain

$$I_1(k) \lesssim \sum_{n\in\mathbb{Z}^3} \mathbf{1}_{|n|<|k-n|} \|F_1(\cdot,k-n)\|_{L^{p_1}_T} \|F_2(\cdot,n)\|_{L^{q_1}_T}, \tag{4.2}$$

where $1/p_1 + 1/q_1 = 1/2$, and to simply notation we introduce $\mathcal{F}_1(k) = \|F_1(\cdot,k)\|_{L^{p_1}_T}$ and $\mathcal{F}_2(k) = \|F_2(\cdot,k)\|_{L^{q_1}_T}$. By the Cauchy-Schwarz inequality it follows that

$$I_1(k) \lesssim \|\langle\cdot\rangle^{r_1}\mathcal{F}_2\|_{\ell^2(\mathbb{Z}^3)} \left\{ \sum_{n\in\mathbb{Z}^3} \mathbf{1}_{|n|<|k-n|} \langle n\rangle^{-2r_1} \mathcal{F}_1(k-n)^2 \right\}^{\frac{1}{2}}$$

where we recall that $r_1 \neq 3/2$. Multypliying $I_1(k)$ by $\langle k\rangle^m$ then taking the square and summing it gives

$$\sum_{k\in\mathbb{Z}^3} \langle k\rangle^{2m} I_1(k)^2 \lesssim \|\langle\cdot\rangle^{r_1}\mathcal{F}_2\|^2_{\ell^2(\mathbb{Z}^3)} \sum_{k\in\mathbb{Z}^3}\sum_{n\in\mathbb{Z}^3} \mathbf{1}_{|n|<|k-n|} \langle k\rangle^{2m}\langle n\rangle^{-2r_1}\mathcal{F}_1(k-n)^2.$$

Using that $\mathbf{1}_{|n|<|k-n|}\langle k\rangle^{2m} \lesssim \mathbf{1}_{|n|<|k-n|}\langle k-n\rangle^{2m}$, the above sum can be bounded by

$$\sum_{n'\in\mathbb{Z}^3} \left\{ \sum_{n\in\mathbb{Z}^3} \mathbf{1}_{|n|<|n'|}\langle n\rangle^{-2r_1} \right\} \langle n'\rangle^{2m}\mathcal{F}_1(n')^2$$

and we observe by standard arguments that

$$\sum_{n\in\mathbb{Z}^3} \mathbf{1}_{|n|<|n'|}\langle n\rangle^{-2r_1} \lesssim \begin{cases} \langle n'\rangle^{3-2r_1} & \text{if} \quad r_1 < \frac{3}{2}, \\ 1 & \text{if} \quad r_1 > \frac{3}{2}. \end{cases}$$

This implies

$$\begin{aligned} \sum_{k\in\mathbb{Z}^3} \langle k\rangle^{2m} I_1(k)^2 &\lesssim \|\langle\cdot\rangle^{r_1}\mathcal{F}_2\|^2_{\ell^2(\mathbb{Z}^3)} \|\langle\cdot\rangle^{m+(3/2-r_1)_+}\mathcal{F}_1\|^2_{\ell^2(\mathbb{Z}^3)} \\ &= \|f_1\|^2_{\widetilde{L}^{p_1}_T H^{m+(3/2-r_1)_+}_x L^2_v} \|f_2\|^2_{\widetilde{L}^{q_1}_T H^{r_1}_x H^{s,\star}_v}. \end{aligned}$$

The term I_2 can be estimated in a similar fashion, by exchanging the role of f_1 and f_2. Indeed, we first apply to Hölder's inequality in time with $1/p_2 + 1/q_2 = 1/2$ to obtain

$$I_2(k) \lesssim \sum_{n\in\mathbb{Z}^3} \mathbf{1}_{|k-n|\geqslant|n|} \|F_1(\cdot,n)\|_{L^{p_2}_T} \|F_2(\cdot,k-n)\|_{L^{q_2}_T}. \tag{4.3}$$

Denoting $\mathcal{F}_1'(k) = \|F_1(\cdot,k)\|_{L^{p_2}_T}$ and $\mathcal{F}_2'(k) = \|F_2(\cdot,k)\|_{L^{q_2}_T}$, the Cauchy-Schwarz inequality yields

$$I_2(k) \lesssim \|\langle\cdot\rangle^{r_2}\mathcal{F}_1'\|_{\ell^2(\mathbb{Z}^3)} \left\{ \sum_{n\in\mathbb{Z}^3} \mathbf{1}_{|k-n|\geqslant|n|} \langle n\rangle^{-2r_2} \mathcal{F}_2'(k-n)^2 \right\}^{\frac{1}{2}}.$$

Arguing as above, it follows

$$\sum_{k\in\mathbb{Z}^3} \langle k\rangle^{2m} I_2(k)^2 \lesssim \|f_1\|^2_{\widetilde{L}^{p_2}_T H^{r_2}_x L^2_v} \|f_2\|^2_{\widetilde{L}^{q_2}_T H^{m+(3/2-r_2)_+}_x H^{s,\star}_v},$$

which completes the proof. □

We give another estimate on Γ in the specific case where both entries are macroscopic (which in particular implies that there is no loss of regularity in the velocity variable).

Proposition 4.6. *Let $m \geqslant 0$. For any $r_1, r_2 \neq 3/2$ and any smooth enough functions f_1, f_2 there holds:*

$$\|\Gamma(\mathbf{P}_0 f_1, \mathbf{P}_0 f_2)\|_{\widetilde{L}^\infty_T H^m_x L^2_v} \lesssim \|\mathbf{P}_0 f_1\|_{\widetilde{L}^\infty_T H^{m+(3/2-r_1)_+}_x L^2_v} \|\mathbf{P}_0 f_2\|_{\widetilde{L}^\infty_T H^{r_1}_x L^2_v} + \|\mathbf{P}_0 f_1\|_{\widetilde{L}^\infty_T H^{r_2}_x L^2_v} \|\mathbf{P}_0 f_2\|_{\widetilde{L}^\infty_T H^{m+(3/2-r_2)_+}_x L^2_v} .$$

Proof. Using the regularization properties of $\mathbf{P}_0$, thanks to [53, 15] respectively for the noncutoff Boltzmann and Landau equations, and the fact that $\|\langle v\rangle^p \mathbf{P}_0\phi\|_{H^q_v} \lesssim \|\mathbf{P}_0\phi\|_{L^2_v}$ for all $p, q \geqslant 0$, we have

$$\|\Gamma(\mathbf{P}_0 f_1, \mathbf{P}_0 f_2)\|_{L^2_v} \lesssim \|\mathbf{P}_0 f_1\|_{L^2_v} \|\mathbf{P}_0 f_2\|_{L^2_v} . \tag{4.4}$$

Therefore we have for any $k \in \mathbb{Z}^3$,

$$\|\widehat{\Gamma}(\mathbf{P}_0 f_1, \mathbf{P}_0 f_2)(k)\|_{L^\infty_T L^2_v} \lesssim \sum_{n\in\mathbb{Z}^3} \|\widehat{\mathbf{P}_0 f_1}(k-n)\|_{L^\infty_T L^2_v} \|\widehat{\mathbf{P}_0 f_2}(n)\|_{L^\infty_T L^2_v} .$$

We then conclude as in the proof of Proposition 4.5. □

4.3. **Refined estimates on Ψ^ε.** We recall that as introduced in Section 2

$$\Psi^\varepsilon = \Psi^{\varepsilon,\flat} + \Psi^{\varepsilon,\sharp} \tag{4.5}$$

and

$$\Psi^{\varepsilon,\flat} = \Psi_{\rm NSF} + \Psi^{\varepsilon,\flat}_{\rm wave} + \widetilde{\Psi}^\varepsilon .$$

We can further expand $\Psi^{\varepsilon,\flat}$ by writing

$$\Psi^{\varepsilon,\flat} = \Psi_{\rm NSF} + \Psi^{\varepsilon 1,\sharp}_{\rm NSF} + \Psi^{\varepsilon 1,\flat}_{\rm wave} + \Psi^{\varepsilon 1,\flat} + \Psi^{\varepsilon 2,\flat} \tag{4.6}$$

where writing $\widehat{\Psi}_\star[f_1, f_2](t) = \mathcal{F}_x\big(\Psi_\star[f, f](t)\big)$ and recalling that $\mathbf{P}_0\Gamma_{\rm sym} = 0$,

$$\widehat{\Psi}_{\rm NSF}[f_1, f_2](t,k) := \sum_{\star\in\{\rm NS,heat\}} \int_0^t e^{-\nu_\star(t-t')|k|^2} |k| \mathcal{P}^1_\star\Big(\frac{k}{|k|}\Big) \widehat{\Gamma}_{\rm sym}\big(f_1(t'), f_2(t')\big)(k)\,{\rm d}t' ,$$

$$\widehat{\Psi}^{\varepsilon 1,\sharp}_{\rm NSF}[f_1, f_2](t,k) := \Big(\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) - 1\Big) \times \sum_{\star\in\{\rm NS,heat\}} \int_0^t e^{-\nu_\star|k|^2(t-t')} |k| \mathcal{P}^1_\star\Big(\frac{k}{|k|}\Big) \widehat{\Gamma}_{\rm sym}\big(f_1(t'), f_2(t')\big)(k)\,{\rm d}t' ,$$

$$\widehat{\Psi}^{\varepsilon 1,\flat}_{\rm wave}[f_1, f_2](t,k) := \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) \times \sum_{\pm} \int_0^t e^{(\pm ic\varepsilon|k| - \nu_{\rm wave}\varepsilon^2|k|^2)\frac{t-t'}{\varepsilon^2}} |k| \mathcal{P}^1_{\rm wave\pm}\Big(\frac{k}{|k|}\Big) \widehat{\Gamma}_{\rm sym}\big(f_1(t'), f_2(t')\big)(k)\,{\rm d}t' ,$$

$$\widehat{\Psi}^{\varepsilon 1,\flat}[f_1, f_2](t,k) := \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) \sum_{\star\in\{\rm NS,heat,wave\pm\}} \int_0^t e^{(\pm ic_\star\varepsilon|k| - \nu_\star\varepsilon^2|k|^2)\frac{t-t'}{\varepsilon^2}} \times \Big(e^{(t-t')\frac{\gamma_\star(\varepsilon|k|)}{\varepsilon^2}} - 1\Big) |k| \mathcal{P}^1_\star\Big(\frac{k}{|k|}\Big) \widehat{\Gamma}_{\rm sym}\big(f_1(t'), f_2(t')\big)(k)\,{\rm d}t' ,$$

$$\widehat{\Psi}^{\varepsilon 2,\flat}[f_1, f_2](t,k) := \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) \times \sum_{\star\in\{\rm NS,heat,wave\pm\}} \int_0^t e^{\lambda_\star(\varepsilon k)\frac{t-t'}{\varepsilon^2}} \varepsilon|k|^2 \mathcal{P}^2_\star(\varepsilon k) \widehat{\Gamma}_{\rm sym}\big(f_1(t'), f_2(t')\big)(k)\,{\rm d}t' .$$

We have used the notation $c_\star := 0$ if $\star \in \{\rm NS, heat\}$.

In what follows, we give estimates on specific parts of Ψ^ε coming from the decomposition (4.5)–(4.6) which will imply continuity estimates on Ψ^ε in various functional spaces.

Proposition 4.7. *Consider $T > 0$ and $m \geqslant 0$. For any smooth enough functions f_1 and f_2, we have*

$$\big\|\Psi^{\varepsilon,\flat}[f_1, f_2]\big\|_{L^2_T H^{m+1}_x H^{s,\star}_v} \lesssim \|\Gamma_{\rm sym}(f_1, f_2)\|_{L^2_T H^m_x (H^{s,\star}_v)'} . \tag{4.7}$$

Proof. Recalling (2.13), for any $k \in \mathbb{Z}^3$, there holds

$$\widehat{\Psi^{\varepsilon,\flat}}[f_1,f_2](t,k) = \frac{1}{\varepsilon}\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) \sum_{\star\in\{\mathrm{NS},\mathrm{heat},\mathrm{wave}\pm\}} \int_0^t e^{\lambda_\star(\varepsilon k)\frac{t-t'}{\varepsilon^2}} \mathcal{P}_\star(\varepsilon k)\widehat{\Gamma}_{\mathrm{sym}}(f_1(t'),f_2(t'))(k)\,\mathrm{d}t' .$$

Due to the form (2.4)-(2.5) of $\lambda_\star$ and to the fact that $\mathbf{P}_0\Gamma_{\mathrm{sym}} = 0$, there is a constant $\lambda_1 > 0$ such that

$$\begin{aligned}
&\Big\|\widehat{\Psi^{\varepsilon,\flat}}[f_1,f_2](t,k)\Big\|_{H_v^{s,\star}} \\
&\quad\lesssim |k|\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\int_0^t e^{-\lambda_1|k|^2(t-t')}\Big\|\mathcal{P}^1\Big(\frac{k}{|k|}\Big)\widehat{\Gamma}_{\mathrm{sym}}(f_1,f_2)(t',k)\Big\|_{H_v^{s,\star}}\,\mathrm{d}t' \\
&\qquad+\varepsilon|k|^2\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\int_0^t e^{-\lambda_1|k|^2(t-t')}\Big\|\mathcal{P}^2(\varepsilon k)\widehat{\Gamma}_{\mathrm{sym}}(f_1,f_2)(t',k)\Big\|_{H_v^{s,\star}}\,\mathrm{d}t' \\
&\quad\lesssim |k|\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\int_0^t e^{-\lambda_1|k|^2(t-t')}\Big\|\mathcal{P}^1\Big(\frac{k}{|k|}\Big)\widehat{\Gamma}_{\mathrm{sym}}(f_1,f_2)(t',k)\Big\|_{H_v^{s,\star}}\,\mathrm{d}t' \\
&\qquad+|k|\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\int_0^t e^{-\lambda_1|k|^2(t-t')}\Big\|\mathcal{P}^2(\varepsilon k)\widehat{\Gamma}_{\mathrm{sym}}(f_1,f_2)(t',k)\Big\|_{H_v^{s,\star}}\,\mathrm{d}t'
\end{aligned}$$

where we used that εk lies in a compact set to get the last inequality and where $\mathcal{P}^1$ and $\mathcal{P}^2$ are bounded from $(H_v^{s,\star})'$ into $H_v^{s,\star}$ uniformly in $\varepsilon|k| \leqslant \kappa$. We then have

$$\begin{aligned}
\Big\|\mathcal{P}^1\Big(\frac{k}{|k|}\Big)\widehat{\Gamma}_{\mathrm{sym}}(f_1,f_2)(t',k)\Big\|_{H_v^{s,\star}} + \Big\|\mathcal{P}^2(\varepsilon k)\widehat{\Gamma}_{\mathrm{sym}}(f_1,f_2)(t',k)\Big\|_{H_v^{s,\star}} \\
\lesssim \|\widehat{\Gamma}_{\mathrm{sym}}(f_1,f_2)(t',k,\cdot)\|_{(H_v^{s,\star})'} .
\end{aligned}$$

We denote $A(t',k) := \|\widehat{\Gamma}_{\mathrm{sym}}(f_1,f_2)(t',k,\cdot)\|_{(H_v^{s,\star})'}$, and we use Young's inequality in time $L^2_T\star L^1_T \subset L^2_T$ to estimate

$$\|\widehat{\Psi^{\varepsilon,\flat}}[f_1,f_2](k)\|_{L^2_T H_v^{s,\star}} \lesssim \Big\|\int_0^t |k|^2 e^{-\lambda_1(t-t')|k|^2}|k|^{-1}A(t',\cdot)\,\mathrm{d}t'\Big\|_{L^2_T} \lesssim \||k|^{-1}A(\cdot,k)\|_{L^2_T} .$$

Therefore we obtain

$$\|\Psi^{\varepsilon,\flat}[f_1,f_2]\|_{L^2_T H_x^{m+1} H_v^{s,\star}} \lesssim \|\langle k\rangle^m A\|_{\ell^2_k L^2_T} \lesssim \|\Gamma_{\mathrm{sym}}(f_1,f_2)\|_{L^2_T H_x^m (H_v^{s,\star})'} ,$$

which concludes the proof. □

We now give an estimate on $\Psi^{\varepsilon,\sharp}$ when both entries are macroscopic.

Proposition 4.8. *Consider $T > 0$ and $m \geqslant 0$. For any smooth enough functions f_1 and f_2 we have:*

$$\|\Psi^{\varepsilon,\sharp}[\mathbf{P}_0 f_1,\mathbf{P}_0 f_2]\|_{\widetilde{L}^\infty_T H_x^m L_v^2} \lesssim \varepsilon\|\Gamma_{\mathrm{sym}}(\mathbf{P}_0 f_1,\mathbf{P}_0 f_2)\|_{\widetilde{L}^\infty_T H_x^m L_v^2} .$$

Proof. We first write for any $k \in \mathbb{Z}^3$,

$$\Big\|\widehat{\Psi}^{\varepsilon,\sharp}[\mathbf{P}_0 f_1,\mathbf{P}_0 f_2](k)\Big\|_{L^\infty_T L^2_v} \lesssim \frac{1}{\varepsilon}\Big\|\int_0^t \Big\|\widehat{U}^{\varepsilon,\sharp}(t-t',k)\widehat{\Gamma}_{\mathrm{sym}}(\mathbf{P}_0 f_1,\mathbf{P}_0 f_2)(t',k)\Big\|_{L^2_v}\mathrm{d}t'\Big\|_{L^\infty_T} .$$

Using then (2.8), we deduce that

$$\Big\|\widehat{\Psi}^{\varepsilon,\sharp}[\mathbf{P}_0 f_1,\mathbf{P}_0 f_2](k)\Big\|_{L^\infty_T L^2_v} \lesssim \frac{1}{\varepsilon}\Big\|\int_0^t e^{-\lambda_0\frac{t-t'}{\varepsilon^2}}\Big\|\widehat{\Gamma}_{\mathrm{sym}}(\mathbf{P}_0 f_1,\mathbf{P}_0 f_2)(t',k)\Big\|_{L^2_v}\mathrm{d}t'\Big\|_{L^\infty_T} ,$$

and thus Young's inequality in time yields

$$\Big\|\widehat{\Psi}^{\varepsilon,\sharp}[\mathbf{P}_0 f_1,\mathbf{P}_0 f_2](k)\Big\|_{L^\infty_T L^2_v} \lesssim \varepsilon\Big\|\widehat{\Gamma}_{\mathrm{sym}}(\mathbf{P}_0 f_1,\mathbf{P}_0 f_2)(k)\Big\|_{L^\infty_T L^2_v} .$$

Thsi concludes the proof of Proposition 4.8. □

5. The equation on δ^ε: proof of Proposition 3.2

This section is devoted to the proof of Proposition 3.2.

5.1. **Continuity estimate for U^ε.** We start with a continuity estimate for U^ε. From Proposition 4.1 we have

$$\varepsilon^\beta \|U^\varepsilon(\cdot-\tau)F(\tau)\|_{\widetilde{L}^\infty_{[\tau,T]}H^\ell_x L^2_v} \lesssim \varepsilon^\beta \|F(\tau)\|_{H^\ell_x L^2_v}\,,$$

as well as

$$\|U^\varepsilon(\cdot-\tau)F(\tau)\|_{\widetilde{L}^\infty_{[\tau,T]}H^{\ell-1}_x L^2_v} \lesssim \|F(\tau)\|_{H^{\ell-1}_x L^2_v}$$

and

$$\frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}_0^\perp U^\varepsilon(\cdot-\tau)F(\tau)\|_{L^2_{[\tau,T]}H^\ell_x H^{s,\star}_v} \lesssim \sqrt{\varepsilon}\|F(\tau)\|_{H^\ell_x L^2_v}\,.$$

We now decompose $\mathbf{P}_0 U^\varepsilon = \mathbf{P}_0 U^{\varepsilon,\sharp} + \mathbf{P}_0 U^{\varepsilon,\flat}$ as in (2.6). By Proposition 4.2 we get

$$\|\mathbf{P}_0 U^{\varepsilon,\sharp}(\cdot-\tau)F(\tau)\|_{L^2_{[\tau,T]}H^\ell_x H^{s,\star}_v} \lesssim \varepsilon\|F(\tau)\|_{H^\ell_x L^2_v}\,.$$

Furthermore, Proposition 4.4 yields

$$\|\mathbf{P}_0 U^{\varepsilon,\flat}(\cdot-\tau)F(\tau)\|_{L^2_{[\tau,T]}H^\ell_x H^{s,\star}_v} \lesssim \|F(\tau)\|_{H^{\ell-1}_x L^2_v}.$$

Gathering previous estimates and using that $\beta < 1/2$, it follows that

$$\begin{aligned}\|U^\varepsilon(\cdot-\tau)F(\tau)\|_{\mathcal{X}^\varepsilon_{[\tau,T]}} &\lesssim \varepsilon^\beta\|F(\tau)\|_{H^\ell_x L^2_v} + \|F(\tau)\|_{H^{\ell-1}_x L^2_v}\\ &\lesssim \varepsilon^\beta\|F\|_{\widetilde{L}^\infty_{[0,\tau]}H^\ell_x L^2_v} + \|F\|_{\widetilde{L}^\infty_{[0,\tau]}H^{\ell-1}_x L^2_v} \lesssim \|F\|_{\mathcal{X}^\varepsilon_{[0,\tau]}}\,.\end{aligned}$$

This concludes the proof of Proposition 3.2–(1).

5.2. **Contribution of the data $\mathcal{D}^\varepsilon$.** Recall that

$$\mathcal{D}^\varepsilon(t) := (U^\varepsilon(t) - U_{\rm NSF}(t))\mathbf{P}_0 f^\varepsilon_{\rm in} + U^\varepsilon(t)\mathbf{P}_0^\perp f^\varepsilon_{\rm in}\,. \tag{5.1}$$

Let us first prove that

$$\big\|(U^\varepsilon(t) - U_{\rm NSF}(t))\mathbf{P}_0 f^\varepsilon_{\rm in}\big\|_{\mathcal{X}^\varepsilon_\infty} \lesssim \varepsilon^{\frac12-\alpha}\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\,. \tag{5.2}$$

We start that by recalling that by (2.6) and (2.9) there holds

$$(U^\varepsilon(t) - U_{\rm NSF}(t))\mathbf{P}_0 f^\varepsilon_{\rm in} = (\widetilde{U}^\varepsilon + U^{\varepsilon,\flat}_{\rm wave} + U^{\varepsilon,\sharp})(t)\mathbf{P}_0 f^\varepsilon_{\rm in}\,.$$

We notice that since $\mathbf{P}_0 f^\varepsilon_{\rm in}$ is well-prepared, then $U^{\varepsilon,\flat}_{\rm wave}(t)\mathbf{P}_0 f^\varepsilon_{\rm in} \equiv 0$ so

$$(\widetilde{U}^\varepsilon + U^{\varepsilon,\flat}_{\rm wave} + U^{\varepsilon,\sharp})(t)\mathbf{P}_0 f^\varepsilon_{\rm in} = (\widetilde{U}^\varepsilon + U^{\varepsilon,\sharp})(t)\mathbf{P}_0 f^\varepsilon_{\rm in}\,.$$

Let us start by considering $U^{\varepsilon,\sharp}(t)\mathbf{P}_0 f^\varepsilon_{\rm in}$. Thanks to Proposition 4.2–(1) we have

$$\|U^{\varepsilon,\sharp}(\cdot)\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{L^2_t H^m_x H^{s,\star}_v} \lesssim \varepsilon\|\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{H^m_x L^2_v}$$

for any $m \geqslant 0$. We shall now follow the arguments of [26] (see in particular the proofs of Lemmas 3.3 and 3.5). We notice as in [8, Lemma 6.2] that

$$U^{\varepsilon,\sharp}(t)f = U^\varepsilon(t)U^{\varepsilon,\sharp}(0)f = U^\varepsilon(t)\Big[\mathcal{F}_x^{-1}\Big(\mathrm{Id} - \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\sum_{\star\in\{\rm NS,heat,wave\pm\}}\mathcal{P}_\star(\varepsilon k)\widehat{f}(k)\Big]$$

so in particular

$$U^{\varepsilon,\sharp}(t)\mathbf{P}_0 f^\varepsilon_{\rm in} = U^\varepsilon(t)\left[\mathcal{F}_x^{-1}\left(\Big(\mathrm{Id}-\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\Big) - \varepsilon|k|\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\sum_{\star\in\{\rm NS,heat,wave\pm\}}\mathcal{P}_\star(\varepsilon k)\right)\widehat{\mathbf{P}_0 f^\varepsilon_{\rm in}}(k)\right].$$

The $H^m_x L^2_v$-norm of the first term in the right-hand side can be estimated using

$$\Big|\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) - 1\Big| \lesssim \varepsilon|k|\,, \tag{5.3}$$

and thanks to the fact that the projectors $\mathcal{P}_\star$ are bounded from L^2_v to L^2_v. The same holds for the terms coming from the second part of the right-hand side, so we find that

$$\|U^{\varepsilon,\sharp}(\cdot)\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{\widetilde{L}^\infty_t H^m_x L^2_v} \lesssim \varepsilon \|\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{H^{m+1}_x L^2_v}\,.$$

Using Remark 3, we deduce that

$$\varepsilon^\beta \|U^{\varepsilon,\sharp}(\cdot)\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{\widetilde{L}^\infty_t H^\ell_x L^2_v} \lesssim \varepsilon^{1+\beta} \|\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{H^{\ell+1}_x L^2_v} \lesssim \varepsilon^{1+\beta-2\alpha} \|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\,,$$

as well as

$$\begin{aligned}\|U^{\varepsilon,\sharp}(\cdot)\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{\widetilde{L}^\infty_t H^{\ell-1}_x L^2_v} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}_0^\perp U^{\varepsilon,\sharp}(\cdot)\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{L^2_t H^\ell_x H^{s,\star}_v} + \|\mathbf{P}_0 U^{\varepsilon,\sharp}(\cdot)\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{L^2_t H^\ell_x H^{s,\star}_v}\\ \lesssim (\sqrt{\varepsilon}+\varepsilon)\,\|\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{H^\ell_x L^2_v} \lesssim \varepsilon^{\frac12-\alpha}\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\,.\end{aligned}$$

We conclude that

$$\big\|U^{\varepsilon,\sharp}(\cdot)\mathbf{P}_0 f^\varepsilon_{\rm in}\big\|_{\mathcal{X}^\varepsilon_\infty} \lesssim \varepsilon^{\frac12-\alpha}\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v} \xrightarrow[\varepsilon\to 0]{} 0\,. \tag{5.4}$$

Now let us turn to $\widetilde{U}^\varepsilon(t)\mathbf{P}_0 f^\varepsilon_{\rm in}$ as defined in (2.10). By construction it is made of three terms, defined in Fourier variables by

$$\begin{aligned}\widehat{\widetilde{U}^\varepsilon}(t,k) &:= \widehat{\widetilde{U}^{\varepsilon,\sharp}_{\rm NSF}}(t,k) + \widehat{\widetilde{U}^{\varepsilon 1,\flat}}(t,k) + \widehat{\widetilde{U}^{\varepsilon 2,\flat}}(t,k)\\ &:= \Big(1-\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\Big)\sum_{\star\in\{\rm NS,heat\}} e^{-\nu_\star|k|^2 t}\,\mathcal{P}^0_\star\Big(\frac{k}{|k|}\Big)\\ &\quad + \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\sum_{\star\in\{\rm NS,heat\}}\Big(e^{\lambda_\star(\varepsilon k)\frac{t}{\varepsilon^2}} - e^{-\nu_\star|k|^2 t}\Big)\mathcal{P}^0_\star\Big(\frac{k}{|k|}\Big)\\ &\quad + \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\sum_{\star\in\{\rm NS,heat\}} e^{\lambda_\star(\varepsilon k)\frac{t}{\varepsilon^2}}\Big[\varepsilon|k|\mathcal{P}^1_\star\big(\tfrac{k}{|k|}\big) + \varepsilon^2|k|^2\mathcal{P}^2_\star(\varepsilon k)\Big]\,.\end{aligned}$$

We shall study the three contributions in turn, starting with $\widetilde{U}^{\varepsilon,\sharp}_{\rm NSF}(t)$. We recall again that the projectors $\mathcal{P}^j_\star$ are bounded from L^2_v to L^2_v and from $H^{s,\star}_v$ to $H^{s,\star}_v$. Using the fact that since $\mathbf{P}_0 f^\varepsilon_{\rm in}$ is mean free which implies that there is no contribution to $k=0$, we have for any $m\geqslant 0$,

$$\|\widetilde{U}^{\varepsilon,\sharp}_{\rm NSF}(\cdot)\mathbf{P}_0 f^\varepsilon_{\rm in}\|^2_{L^2_t H^m_x H^{s,\star}_v} \lesssim \sum_{\star\in\{\rm NS,heat\}}\sum_{k\in\mathbb{Z}^3\setminus\{0\}}\langle k\rangle^{2m}|k|^{-2}\Big(1-\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\Big)^2\Big\|\widehat{\mathbf{P}_0 f^\varepsilon_{\rm in}}(k,\cdot)\Big\|^2_{H^{s,\star}_v}\,.$$

Using (5.3) and the fact that $\|\mathbf{P}_0 f\|_{H^{s,\star}_v} \lesssim \|\mathbf{P}_0 f\|_{L^2_v}$, we get

$$\|\widetilde{U}^{\varepsilon,\sharp}_{\rm NSF}(\cdot)\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{L^2_t H^m_x H^{s,\star}_v} \lesssim \varepsilon\left(\sum_{k\in\mathbb{Z}^3}\langle k\rangle^{2m}\Big\|\widehat{\mathbf{P}_0 f^\varepsilon_{\rm in}}(k,\cdot)\Big\|^2_{L^2_v}\right)^{\frac12} \lesssim \varepsilon\|\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{H^m_x L^2_v}\,.$$

Similar computations give

$$\|\widetilde{U}^{\varepsilon,\sharp}_{\rm NSF}(\cdot)\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{\widetilde{L}^\infty_t H^m_x L^2_v} \lesssim \varepsilon\|\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{H^m_x L^2_v}\,.$$

Therefore, arguing similarly as for obtaining estimate (5.4) for the term $U^{\varepsilon,\sharp}$, we also get

$$\begin{aligned}\|\widetilde{U}^{\varepsilon,\sharp}_{\rm NSF}(\cdot)\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{\mathcal{X}^\varepsilon_\infty} &\lesssim (\varepsilon^{1+\beta}+\sqrt{\varepsilon}+\varepsilon)\|\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{H^\ell_x L^2_v} + \varepsilon\|\mathbf{P}_0 f^\varepsilon_{\rm in}\|_{H^{\ell-1}_x L^2_v}\\ &\lesssim \Big(\varepsilon^{1+\beta-\alpha}+\varepsilon^{\frac12-\alpha}\Big)\,\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\,.\end{aligned} \tag{5.5}$$

Next we turn to $\widetilde{U}^{\varepsilon 1,\flat}(t)$. We write

$$\begin{aligned}\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)\Big|e^{\lambda_\star(\varepsilon k)\frac{t}{\varepsilon^2}} - e^{-\nu_\star|k|^2 t}\Big| &\lesssim \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big)e^{-\frac{\nu_\star}{2}|k|^2 t}t\varepsilon|k|^3\\ &\lesssim e^{-\frac{\nu_\star}{4}|k|^2 t}\varepsilon|k|\,.\end{aligned}$$

Compared to the computations above we have gained a factor ε but lost a power of k. The same argument gives

$$\|\widetilde{U}^{\varepsilon 1,\flat}(\cdot)\mathbf{P}_0 f_{\rm in}^\varepsilon\|_{\widetilde{L}^\infty_t H^m_x L^2_v} + \|\widetilde{U}^{\varepsilon 1,\flat}(\cdot)\mathbf{P}_0 f_{\rm in}^\varepsilon\|_{L^2_t H^m_x L^2_v} \lesssim \varepsilon \|\mathbf{P}_0 f_{\rm in}^\varepsilon\|_{H^m_x L^2_v}.$$

We can then again argue as for obtaining the bound (5.5) to deduce

$$\|\widetilde{U}^{\varepsilon 1,\flat}(\cdot)\mathbf{P}_0 f_{\rm in}^\varepsilon\|_{\mathcal{X}^\varepsilon_\infty} \lesssim \left(\varepsilon^{1+\beta-\alpha} + \varepsilon^{\frac12-\alpha}\right)\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\,. \tag{5.6}$$

Finally the computations for $\widetilde{U}^{\varepsilon 2,\flat}(t)$ are very similar: We find

$$\|\widetilde{U}^{\varepsilon 2,\flat}(\cdot)\mathbf{P}_0 f_{\rm in}^\varepsilon\|_{\widetilde{L}^\infty_t H^m_x L^2_v} + \|\widetilde{U}^{\varepsilon 2,\flat}(\cdot)\mathbf{P}_0 f_{\rm in}^\varepsilon\|_{L^2_t H^m_x L^2_v} \lesssim \varepsilon\|\mathbf{P}_0 f_{\rm in}^\varepsilon\|_{H^m_x L^2_v} + \varepsilon^2\|\mathbf{P}_0 f_{\rm in}^\varepsilon\|_{H^{m+1}_x L^2_v},$$

so that arguing as before

$$\|\widetilde{U}^{\varepsilon 2,\flat}(\cdot)\mathbf{P}_0 f_{\rm in}^\varepsilon\|_{\mathcal{X}^\varepsilon_\infty} \lesssim \left(\varepsilon^{1+\beta-\alpha} + \varepsilon^{\frac12-\alpha} + \varepsilon^{2+\beta-2\alpha} + \varepsilon^{3/2-2\alpha}\right)\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\,. \tag{5.7}$$

Putting together the estimates on $\widetilde{U}^{\varepsilon,\sharp}_{\rm NSF}(t)\mathbf{P}_0 f_{\rm in}^\varepsilon$, $\widetilde{U}^{\varepsilon 1,\flat}(t)\mathbf{P}_0 f_{\rm in}^\varepsilon$ and $\widetilde{U}^{\varepsilon 2,\flat}(t)\mathbf{P}_0 f_{\rm in}^\varepsilon$ gives (5.2).

Recalling (5.1), it remains to prove that

$$\|U^\varepsilon(t)\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{\mathcal{X}^\varepsilon_\infty} \lesssim \varepsilon^\beta\|\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{H^\ell_x L^2_v} + \|\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{H^{\ell-1}_x L^2_v}\,. \tag{5.8}$$

From Proposition 4.1–(1), we have

$$\|U^\varepsilon(t)\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} \lesssim \|\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{H^{\ell-1}_x L^2_v}\,,$$

as well as

$$\frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}_0^\perp U^\varepsilon(t)\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{L^2_T H^\ell_x H^{s,\star}_v} \lesssim \sqrt{\varepsilon}\|\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{H^\ell_x L^2_v}\,,$$

and also

$$\varepsilon^\beta\|U^\varepsilon(t)\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} \lesssim \varepsilon^\beta\|\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{H^\ell_x L^2_v}\,.$$

From the decomposition (2.6), for any $k\in\mathbb{Z}^3$ and $t\geqslant 0$, we have

$$\|\widehat{U}^\varepsilon(t,k)\mathbf{P}_0^\perp\|_{L^2_v\to L^2_v} \lesssim \varepsilon|k|e^{-t|k|^2} + e^{-\lambda_0\frac{t}{\varepsilon^2}}\,.$$

Using that $\mathbf{P}_0$ is bounded from $H^{s,\star}_v$ into L^2_v and that $\mathbf{P}_0^\perp\mathbf{P}_0^\perp = \mathbf{P}_0^\perp$, we obtain

$$\|\mathbf{P}_0\widehat{U}^\varepsilon(\cdot,k)\mathbf{P}_0^\perp\widehat{f}^\varepsilon_{\rm in}(k)\|_{L^2_t H^{s,\star}_v} \lesssim \varepsilon\|\mathbf{P}_0^\perp\widehat{f}^\varepsilon_{\rm in}(k)\|_{L^2_v}\,,$$

from which we deduce

$$\|\mathbf{P}_0 U^\varepsilon(\cdot)\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{L^2_t H^\ell_x H^{s,\star}_v} \lesssim \varepsilon\|\mathbf{P}_0^\perp f_{\rm in}^\varepsilon\|_{H^\ell_x L^2_v}\,.$$

Gathering previous estimates ends the proof of (5.8). The estimates (5.2) and (5.8) together complete the proof of Proposition 3.2–(2).

5.3. **Contribution of the source term $\mathcal{S}^\varepsilon$.** We recall that

$$\mathcal{S}^\varepsilon(t) = \Psi^\varepsilon[g^\varepsilon,g^\varepsilon](t) - \Psi_{\rm NSF}[g^\varepsilon,g^\varepsilon](t)\,,$$

and we want to prove that

$$\|\mathcal{S}^\varepsilon\|_{\mathcal{X}^\varepsilon_T} \leqslant \varepsilon^{(1-4\alpha)/2}\Phi\left(\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\right).$$

Due to decompositions (4.5)-(4.6) one can write

$$\mathcal{S}^\varepsilon(t) = \left(\Psi^{\varepsilon,\sharp} + \Psi^{\varepsilon 1}_{\rm wave} + \Psi^{\varepsilon 1,\sharp}_{\rm NSF} + \Psi^{\varepsilon 1,\flat} + \Psi^{\varepsilon 2,\flat}\right)[g^\varepsilon,g^\varepsilon](t)\,, \tag{5.9}$$

and we estimate each term separately.

We start with the first term in (5.9).

Lemma 5.1. *There holds*

$$\begin{aligned}\|\Psi^{\varepsilon,\sharp}[g^\varepsilon,g^\varepsilon]\|_{\mathcal{X}^\varepsilon_T} \lesssim{}& \varepsilon^{1+\beta}\|\Gamma(g^\varepsilon,g^\varepsilon)\|_{\widetilde{L}^\infty_T H^\ell_x (H^{s,\star}_v)'} + \varepsilon\|\Gamma(g^\varepsilon,g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v}\\ &+ \sqrt{\varepsilon}\|\Gamma(g^\varepsilon,g^\varepsilon)\|_{L^2_T H^\ell_x (H^{s,\star}_v)'}\,.\end{aligned}$$

Proof. Since $g^\varepsilon = \mathbf{P}_0 g^\varepsilon$, from Proposition 4.8 we obtain

$$\|\Psi^{\varepsilon,\sharp}[g^\varepsilon, g^\varepsilon](t)\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} \lesssim \varepsilon \|\Gamma(g^\varepsilon, g^\varepsilon)\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v},$$

and also

$$\|\Psi^{\varepsilon,\sharp}[g^\varepsilon, g^\varepsilon](t)\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} \lesssim \varepsilon \|\Gamma(g^\varepsilon, g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v}.$$

Moreover thanks to Corollary 4.3 we know that

$$\|\Psi^{\varepsilon,\sharp}[g^\varepsilon, g^\varepsilon]\|_{L^2_T H^\ell_x H^{s,\star}_v} \lesssim \varepsilon \|\Gamma(g^\varepsilon, g^\varepsilon)\|_{L^2_T H^\ell_x (H^{s,\star}_v)'}.$$

We conclude the proof by gathering previous estimates. □

Before looking at the other contributions, let us remark that from (4.6), for $k = 0$, we have

$$\widehat{\Psi}^\varepsilon[g^\varepsilon, g^\varepsilon](t,0) = \widehat{\Psi}^{\varepsilon,\sharp}[g^\varepsilon, g^\varepsilon](t,0),$$

it is thus enough to analyze the other contributions in (5.9) for $k \in \mathbb{Z}^3 \setminus \{0\}$ i.e. for $|k| \geqslant 1$.

For the second term $\Psi^{\varepsilon 1}_{\rm wave}$ in (5.9), we follow the arguments of [26, 13]: one needs to exploit the oscillations of the phase by integrations by parts in time. Thus with notation inspired from [26, 13] we define

$$H^\varepsilon_\pm(t,t',x) := \mathcal{F}^{-1}_x \Big(\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) e^{-\nu_{\rm wave}\pm(t-t')|k|^2} |k| \mathcal{P}^1_{\rm wave\pm}\Big(\frac{k}{|k|}\Big) \widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t',k) \Big)$$

so that after an integration by parts in time

$$\begin{aligned}
&\widehat{\Psi}^{\varepsilon 1}_{\rm wave}[g^\varepsilon, g^\varepsilon](t,k) \\
&\quad = \sum_\pm \frac{\varepsilon}{ic|k|} \Big(\int_0^t e^{\pm ic|k|\frac{t-t'}{\varepsilon}} \partial_{t'} \widehat{H}^\varepsilon_\pm(t,t',k)\,\mathrm{d}t' - \widehat{H}^\varepsilon_\pm(t,t,k) + e^{\pm ic|k|\frac{t}{\varepsilon}} \widehat{H}^\varepsilon_\pm(t,0,k) \Big).
\end{aligned}$$

Let us define

$$\widehat{J^\varepsilon_\pm}(t,k) := \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) \frac{\varepsilon}{ic} \int_0^t e^{\pm ic|k|\frac{t-t'}{\varepsilon} - \nu_{\rm wave}\pm(t-t')|k|^2} \mathcal{P}^1_{\rm wave\pm}\Big(\frac{k}{|k|}\Big) \partial_{t'} \widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t',k)\,\mathrm{d}t' \tag{5.10}$$

and

$$\widehat{I^\varepsilon_\pm}(t,k) := \widehat{\Psi}^{\varepsilon 1}_{\rm wave}[g^\varepsilon, g^\varepsilon](t,k) - \widehat{J^\varepsilon_\pm}(t,k), \tag{5.11}$$

which will be estimated separately.

For the term $I^\varepsilon_\pm$ we have the following result.

Lemma 5.2. *There holds*

$$\begin{aligned}
\|I^\varepsilon_\pm\|_{\mathcal{X}^\varepsilon_T} \lesssim{}& \varepsilon^{1+\beta} \|\Gamma(g^\varepsilon, g^\varepsilon)\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} + \varepsilon \|\Gamma(g^\varepsilon, g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} \\
&+ \sqrt{\varepsilon} \|\Gamma(g^\varepsilon, g^\varepsilon)\|_{L^2_T H^\ell_x (H^{s,\star}_v)'} + \sqrt{\varepsilon} \|\Gamma(g^\varepsilon_{\rm in}, g^\varepsilon_{\rm in})\|_{H^\ell_x (H^{s,\star}_v)'}.
\end{aligned}$$

Proof. Since $\mathcal{P}^1_{\rm wave\pm}$ is bounded from L^2_v into L^2_v as well as from $(H^{s,\star}_v)'$ into $H^{s,\star}_v$, we obtain from [13, Proof of Lemma 6.5] that, for all $t \in [0,T]$ and $k \in \mathbb{Z}^3 \setminus \{0\}$,

$$\begin{aligned}
\|\widehat{I^\varepsilon_\pm}(t,k)\|_{L^2_v} \lesssim{}& \varepsilon \int_0^t |k|^2 e^{-\nu_{\rm wave}\pm(t-t')|k|^2} \|\widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t',k)\|_{L^2_v}\,\mathrm{d}t' \\
&+ \varepsilon \|\widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t,k)\|_{L^2_v} + \varepsilon e^{-\nu_{\rm wave}\pm t|k|^2} \|\widehat{\Gamma}(g^\varepsilon_{\rm in}, g^\varepsilon_{\rm in})(k)\|_{L^2_v},
\end{aligned}$$

and

$$\begin{aligned}
\|\widehat{I^\varepsilon_\pm}(t,k)\|_{H^{s,\star}_v} \lesssim{}& \varepsilon \int_0^t |k|^2 e^{-\nu_{\rm wave}\pm(t-t')|k|^2} \|\widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t',k)\|_{(H^{s,\star}_v)'}\,\mathrm{d}t' \\
&+ \varepsilon \|\widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t,k)\|_{(H^{s,\star}_v)'} + \varepsilon e^{-\nu_{\rm wave}\pm t|k|^2} \|\widehat{\Gamma}(g^\varepsilon_{\rm in}, g^\varepsilon_{\rm in})(k)\|_{(H^{s,\star}_v)'}.
\end{aligned}$$

Applying Young's convolution in time $L^1_T * L^\infty_T \subset L^\infty_T$ and, respectively, $L^1_T * L^2_T \subset L^2_T$, we therefore obtain

$$\|\widehat{I^\varepsilon_\pm}(k)\|_{L^\infty_T L^2_v} \lesssim \varepsilon \|\Gamma(g^\varepsilon, g^\varepsilon)(k)\|_{L^\infty_T L^2_v}$$

and

$$\|\widehat{I^\varepsilon_\pm}(k)\|_{L^2_T H^{s,\star}_v} \lesssim \varepsilon \|\Gamma(g^\varepsilon, g^\varepsilon)(k)\|_{L^2_T (H^{s,\star}_v)'} + \varepsilon \|\widehat{\Gamma}(g^\varepsilon_{\rm in}, g^\varepsilon_{\rm in})(k)\|_{(H^{s,\star}_v)'}.$$

This implies

$$\varepsilon^\beta \|I^\varepsilon_\pm\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} \lesssim \varepsilon^{1+\beta} \|\Gamma(g^\varepsilon, g^\varepsilon)\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v},$$

as well as

$$\begin{aligned}
&\|I^\varepsilon_\pm\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} + \frac{1}{\sqrt{\varepsilon}} \|\mathbf{P}_0^\perp I^\varepsilon_\pm\|_{L^2_T H^\ell_x H^{s,\star}_v} + \|\mathbf{P}_0 I^\varepsilon_\pm\|_{L^2_T H^\ell_x H^{s,\star}_v} \\
&\quad \lesssim \varepsilon \|\Gamma(g^\varepsilon, g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} + \left(\sqrt{\varepsilon} + \varepsilon\right) \left(\|\Gamma(g^\varepsilon, g^\varepsilon)\|_{L^2_T H^\ell_x (H^{s,\star}_v)'} + \|\Gamma(g^\varepsilon_{\rm in}, g^\varepsilon_{\rm in})\|_{H^\ell_x (H^{s,\star}_v)'} \right).
\end{aligned}$$

Lemma 5.2 is proved. □

Recalling the definition of $J^\varepsilon_\pm$ in (5.10), we have:

Lemma 5.3. *There holds*

$$\begin{aligned}
\|J^\varepsilon_\pm\|_{\mathcal{X}^\varepsilon_T} \lesssim\ & \varepsilon^{1+\beta} \|\Gamma_{\rm sym}(g^\varepsilon, \partial_t g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-2}_x L^2_v} + \varepsilon \|\Gamma_{\rm sym}(g^\varepsilon, \partial_t g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-3}_x L^2_v} \\
& + \sqrt{\varepsilon} \|\Gamma_{\rm sym}(g^\varepsilon, \partial_t g^\varepsilon)\|_{L^2_T H^{\ell-2}_x (H^{s,\star}_v)'}.
\end{aligned}$$

Proof. Starting from (5.10), we use the fact that $\mathcal{P}^1_{\rm wave\pm}$ is bounded from L^2_v into L^2_v as well as from $(H^{s,\star}_v)'$ into $H^{s,\star}_v$ to obtain that, for all $t \in [0,T]$ and $k \in \mathbb{Z}^3 \setminus \{0\}$,

$$\|\widehat{J^\varepsilon_\pm}(t,k)\|_{L^2_v} \lesssim \varepsilon \int_0^t |k|^2 e^{-\nu_{\rm wave\pm}(t-t')|k|^2} |k|^{-2} \|\partial_t \widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t',k)\|_{L^2_v} \, {\rm d}t'$$

and

$$\|\widehat{J^\varepsilon_\pm}(t,k)\|_{H^{s,\star}_v} \lesssim \varepsilon \int_0^t |k|^2 e^{-\nu_{\rm wave\pm}(t-t')|k|^2} |k|^{-2} \|\partial_t \widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t',k)\|_{(H^{s,\star}_v)'} \, {\rm d}t'.$$

Using Young's inequality for convolutions as in the proof of Lemma 5.2 together with the fact that $\partial_t \widehat{\Gamma}(g^\varepsilon, g^\varepsilon) = \widehat{\Gamma}(\partial_t g^\varepsilon, g^\varepsilon) + \widehat{\Gamma}(g^\varepsilon, \partial_t g^\varepsilon)$, we thus deduce

$$\|\widehat{J^\varepsilon_\pm}(t,k)\|_{L^\infty_T L^2_v} \lesssim \varepsilon |k|^{-2} \|\widehat{\Gamma}_{\rm sym}(g^\varepsilon, \partial_t g^\varepsilon)(k)\|_{L^\infty_T L^2_v}$$

and

$$\|\widehat{J^\varepsilon_\pm}(t,k)\|_{L^2_T H^{s,\star}_v} \lesssim \varepsilon |k|^{-2} \|\widehat{\Gamma}_{\rm sym}(g^\varepsilon, \partial_t g^\varepsilon)(k)\|_{L^2_T (H^{s,\star}_v)'}.$$

These two inequalities imply

$$\varepsilon^\beta \|J^\varepsilon_\pm\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} \lesssim \varepsilon^{1+\beta} \|\Gamma_{\rm sym}(g^\varepsilon, \partial_t g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-2}_x L^2_v},$$

and also

$$\begin{aligned}
&\|J^\varepsilon_\pm\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} + \frac{1}{\sqrt{\varepsilon}} \|\mathbf{P}_0^\perp J^\varepsilon_\pm\|_{L^2_T H^\ell_x H^{s,\star}_v} + \|\mathbf{P}_0 J^\varepsilon_\pm\|_{L^2_T H^\ell_x H^{s,\star}_v} \\
&\quad \lesssim \varepsilon \|\Gamma_{\rm sym}(g^\varepsilon, \partial_t g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-3}_x L^2_v} + \left(\sqrt{\varepsilon} + \varepsilon\right) \|\Gamma_{\rm sym}(g^\varepsilon, \partial_t g^\varepsilon)\|_{L^2_T H^{\ell-2}_x (H^{s,\star}_v)'}.
\end{aligned}$$

Lemma 5.3 is proved. □

The contributions of the terms $\Psi^{\varepsilon 1,\sharp}_{\rm NSF}$, $\Psi^{\varepsilon 1,\flat}$ and $\Psi^{\varepsilon 2,\flat}$ in (5.9) are easier to be obtained.

Lemma 5.4. *There holds*

$$\begin{aligned}
&\|\Psi^{\varepsilon 1,\sharp}_{\rm NSF}[g^\varepsilon, g^\varepsilon]\|_{\mathcal{X}^\varepsilon_T} + \|\Psi^{\varepsilon 1,\flat}[g^\varepsilon, g^\varepsilon]\|_{\mathcal{X}^\varepsilon_T} + \|\Psi^{\varepsilon 2,\flat}[g^\varepsilon, g^\varepsilon]\|_{\mathcal{X}^\varepsilon_T} \\
&\quad \lesssim \varepsilon^{1+\beta} \|\Gamma(g^\varepsilon, g^\varepsilon)\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} + \varepsilon \|\Gamma(g^\varepsilon, g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} + \sqrt{\varepsilon} \|\Gamma(g^\varepsilon, g^\varepsilon)\|_{L^2_T H^\ell_x (H^{s,\star}_v)'}.
\end{aligned}$$

Proof. Recall that $\mathcal{P}^1_{\rm wave\pm}$ is bounded from L^2_v into L^2_v as well as from $(H^{s,\star}_v)'$ into $H^{s,\star}_v$. Let $t \in [0,T]$ and $k \in \mathbb{Z}^3 \setminus \{0\}$. Remarking that

$$\left| \chi\left(\frac{\varepsilon|k|}{\kappa}\right) - 1 \right| \lesssim \min\left(1, \varepsilon|k|\right),$$

we obtain that denoting $\nu_0 = \min(\nu_{\rm NS}, \nu_{\rm heat}) > 0$,

$$\|\widehat{\Psi}^{\varepsilon 1,\sharp}_{\rm NSF}[g^\varepsilon, g^\varepsilon](t,k)\|_{L^2_v} \lesssim \varepsilon \int_0^t |k|^2 e^{-\nu_0(t-t')|k|^2} \|\widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t',k)\|_{L^2_v}\, {\rm d}t',$$

and

$$\|\widehat{\Psi}^{\varepsilon 1,\sharp}_{\rm NSF}[g^\varepsilon, g^\varepsilon](t,k)\|_{H^{s,\star}_v} \lesssim \varepsilon \int_0^t |k|^2 e^{-\nu_0(t-t')|k|^2} \|\widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t',k)\|_{(H^{s,\star}_v)'}\, {\rm d}t'.$$

Using that

$$\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) e^{-\nu_\star t|k|^2} \Big| e^{t\frac{\gamma_\star(\varepsilon|k|)}{\varepsilon^2}} - 1\Big| |k| \lesssim \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) e^{-\frac{\nu_\star}{2} t|k|^2} t\varepsilon|k|^4 \lesssim \varepsilon|k|^2 \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) e^{-\frac{\nu_\star}{4} t|k|^2},$$

we also get, denoting $\nu_1 = \min(\nu_{\rm NS}, \nu_{\rm heat}, \nu_{\rm wave\pm}) > 0$,

$$\|\widehat{\Psi}^{\varepsilon 1,\flat}[g^\varepsilon, g^\varepsilon](t,k)\|_{L^2_v} \lesssim \varepsilon \int_0^t |k|^2 e^{-\frac{\nu_1}{4}(t-t')|k|^2} \|\widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t',k)\|_{L^2_v}\, {\rm d}t',$$

and

$$\|\widehat{\Psi}^{\varepsilon 1,\flat}[g^\varepsilon, g^\varepsilon](t,k)\|_{H^{s,\star}_v} \lesssim \varepsilon \int_0^t |k|^2 e^{-\frac{\nu_1}{4}(t-t')|k|^2} \|\widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t',k)\|_{(H^{s,\star}_v)'}\, {\rm d}t'.$$

Finally, observing that

$$\chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) e^{-\nu_\star t|k|^2 + t\frac{\gamma_\star(\varepsilon|k|)}{\varepsilon^2}} \varepsilon|k|^2 \lesssim \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) e^{-\frac{\nu_\star}{2} t|k|^2} \varepsilon|k|^2 \lesssim \varepsilon|k|^2 \chi\Big(\frac{\varepsilon|k|}{\kappa}\Big) e^{-\frac{\nu_\star}{2} t|k|^2},$$

we also deduce

$$\|\widehat{\Psi}^{\varepsilon 2,\flat}[g^\varepsilon, g^\varepsilon](t,k)\|_{L^2_v} \lesssim \varepsilon \int_0^t |k|^2 e^{-\frac{\nu_1}{2}(t-t')|k|^2} \|\widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t',k)\|_{L^2_v}\, {\rm d}t',$$

and

$$\|\widehat{\Psi}^{\varepsilon 2,\flat}[g^\varepsilon, g^\varepsilon](t,k)\|_{H^{s,\star}_v} \lesssim \varepsilon \int_0^t |k|^2 e^{-\frac{\nu_1}{2}(t-t')|k|^2} \|\widehat{\Gamma}(g^\varepsilon, g^\varepsilon)(t',k)\|_{(H^{s,\star}_v)'}\, {\rm d}t'.$$

We can then conclude by arguing as in the proof of Lemma 5.2. Lemma 5.4 is proved. □

We can now gather the contributions of Lemmas 5.1, 5.2, 5.3 and 5.4 to conclude the proof of Proposition 3.2–(3). We first observe that from Proposition 4.6, since $\mathbf{P}_0 g^\varepsilon = g^\varepsilon$, we obtain

$$\|\Gamma(g^\varepsilon, g^\varepsilon)\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} \lesssim \|g^\varepsilon\|^2_{\widetilde{L}^\infty_T H^\ell_x L^2_v},$$

as well as

$$\|\Gamma(g^\varepsilon, g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} \lesssim \|g^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} \|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v}.$$

Furthermore from Proposition 4.5 we have

$$\|\Gamma(g^\varepsilon, g^\varepsilon)\|_{L^2_T H^\ell_x (H^{s,\star}_v)'} \lesssim \|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} \|g^\varepsilon\|_{L^2_T H^\ell_x L^2_v}.$$

By Proposition 4.6 there holds

$$\|\Gamma(g^\varepsilon_{\rm in}, g^\varepsilon_{\rm in})\|_{H^\ell_x (H^{s,\star}_v)'} \lesssim \|g^\varepsilon_{\rm in}\|^2_{H^\ell_x L^2_v}.$$

Therefore estimates (3.3) and (3.7) together with Lemma 5.1, Lemma 5.2 and Lemma 5.4 yield

(5.12)
$$\begin{aligned}\|\Psi^{\varepsilon,\sharp}[g^\varepsilon, g^\varepsilon]\|_{\mathcal{X}^\varepsilon_T} + \|I^\varepsilon_\pm\|_{\mathcal{X}^\varepsilon_T} + \|\Psi^{\varepsilon 1,\sharp}_{\rm NSF}[g^\varepsilon, g^\varepsilon]\|_{\mathcal{X}^\varepsilon_T} + \|\Psi^{\varepsilon 1,\flat}[g^\varepsilon, g^\varepsilon]\|_{\mathcal{X}^\varepsilon_T} + \|\Psi^{\varepsilon 2,\flat}[g^\varepsilon, g^\varepsilon]\|_{\mathcal{X}^\varepsilon_T}\\ \lesssim \varepsilon^{\frac12-\alpha} \|\bar g_{\rm in}\|^2_{H^{\ell-1}_x L^2_v} \exp\Big(2C\|\bar g_{\rm in}\|^2_{H^{\ell-1}_x L^2_v}\Big) + \varepsilon^{\frac12-2\alpha} \|\bar g_{\rm in}\|^2_{H^{\ell-1}_x L^2_v}.\end{aligned}$$

It only remains to investigate the contribution of the term $J^\varepsilon_\pm$. Since $3/2 < \ell < 3$, thanks to Proposition 4.6 we write

$$\begin{aligned}\|\Gamma_{\rm sym}(g^\varepsilon, \partial_t g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-3}_x L^2_v} &\lesssim \|\Gamma_{\rm sym}(g^\varepsilon, \partial_t g^\varepsilon)\|_{\widetilde{L}^\infty_T L^2_x L^2_v}\\ &\lesssim \|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} \|\partial_t g^\varepsilon\|_{\widetilde{L}^\infty_T L^2_x L^2_v}.\end{aligned}$$

In order to estimate the other terms we split into two cases. If $\ell-2\geqslant 0$ then Proposition 4.6 and Proposition 4.5 imply, respectively,

$$\|\Gamma_{\rm sym}(g^\varepsilon,\partial_t g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-2}_x L^2_v}\lesssim \|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v}\|\partial_t g^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-2}_x L^2_v},$$

and

$$\|\Gamma_{\rm sym}(g^\varepsilon,\partial_t g^\varepsilon)\|_{L^2_T H^{\ell-2}_x (H^{s,\star}_v)'}\lesssim \|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v}\|\partial_t g^\varepsilon\|_{L^2_T H^{\ell-2}_x L^2_v}.$$

Otherwise if $\ell-2<0$ then we write

$$\begin{aligned}\|\Gamma_{\rm sym}(g^\varepsilon,\partial_t g^\varepsilon)\|_{\widetilde{L}^\infty_T H^{\ell-2}_x L^2_v}&\lesssim \|\Gamma_{\rm sym}(g^\varepsilon,\partial_t g^\varepsilon)\|_{\widetilde{L}^\infty_T L^2_x L^2_v}\\&\lesssim \|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v}\|\partial_t g^\varepsilon\|_{\widetilde{L}^\infty_T L^2_x L^2_v},\end{aligned}$$

thanks to Proposition 4.6, and also

$$\begin{aligned}\|\Gamma_{\rm sym}(g^\varepsilon,\partial_t g^\varepsilon)\|_{L^2_T H^{\ell-2}_x (H^{s,\star}_v)'}&\lesssim \|\Gamma_{\rm sym}(g^\varepsilon,\partial_t g^\varepsilon)\|_{L^2_T L^2_x (H^{s,\star}_v)'}\\&\lesssim \|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v}\|\partial_t g^\varepsilon\|_{L^2_T L^2_x L^2_v},\end{aligned}$$

using Proposition 4.5. We now observe from (NSF) that, for all $t\geqslant 0$ and $k\in\mathbb{Z}^3$,

$$|\partial_t\widehat{g^\varepsilon}(t,k)|\lesssim |k|^2|\widehat{g^\varepsilon}(t,k)|+|k|\left|\left(\widehat{g^\varepsilon}(t,\cdot)*\widehat{g^\varepsilon}(t,\cdot)\right)(k)\right|.$$

Hence when $\ell-2\geqslant 0$ we compute

$$\|\partial_t g^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-2}_x L^2_v}\lesssim \|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v}+\|(g^\varepsilon)^2\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v}.$$

Arguing as in the proof of Proposition 4.5, we have

$$\|(g^\varepsilon)^2\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v}\lesssim \|g^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v}\|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v},$$

and thus

$$\|\partial_t g^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-2}_x L^2_v}\lesssim \left(1+\|g^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v}\right)\|g^\varepsilon\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v}.$$

Similarly we obtain

$$\|\partial_t g^\varepsilon\|_{L^2_T H^{\ell-2}_x L^2_v}\lesssim \|g^\varepsilon\|_{L^2_T H^\ell_x L^2_v}+\|(g^\varepsilon)^2\|_{L^2_T H^{\ell-1}_x L^2_v},$$

then we get, as in the proof of Proposition 4.5,

$$\|(g^\varepsilon)^2\|_{L^2_T H^{\ell-1}_x L^2_v}\lesssim \|g^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v}\|g^\varepsilon\|_{L^2_T H^\ell_x L^2_v},$$

and finally

$$\|\partial_t g^\varepsilon\|_{L^2_T H^{\ell-2}_x L^2_v}\lesssim \left(1+\|g^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v}\right)\|g^\varepsilon\|_{L^2_T H^\ell_x L^2_v}.$$

Furthermore, arguing in a similar fashion, we also have

$$\begin{aligned}\|\partial_t g^\varepsilon\|_{\widetilde{L}^\infty_T L^2_x L^2_v}&\lesssim \|g^\varepsilon\|_{\widetilde{L}^\infty_T H^2_x L^2_v}+\|(g^\varepsilon)^2\|_{\widetilde{L}^\infty_T H^1_x L^2_v}\\&\lesssim \left(1+\|g^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v}\right)\|g^\varepsilon\|_{\widetilde{L}^\infty_T H^2_x L^2_v},\end{aligned}$$

as well as

$$\begin{aligned}\|\partial_t g^\varepsilon\|_{L^2_T L^2_x L^2_v}&\lesssim \|g^\varepsilon\|_{L^2_T H^2_x L^2_v}+\|(g^\varepsilon)^2\|_{L^2_T H^1_x L^2_v}\\&\lesssim \left(1+\|g^\varepsilon\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v}\right)\|g^\varepsilon\|_{L^2_T H^2_x L^2_v}.\end{aligned}$$

Therefore estimates (3.3) and (3.7) together with Lemma 5.3 yield: if $\ell-2\geqslant 0$ we have

$$\begin{aligned}\|J^\varepsilon_\pm\|_{\mathcal{X}^\varepsilon_T}\lesssim{}&\left(\varepsilon^{1+\beta-2\alpha}+\varepsilon^{1-\alpha(3-\ell)}+\varepsilon^{\frac12-\alpha}\right)\\&\times\|\bar g_{\rm in}\|^2_{H^{\ell-1}_x L^2_v}\left(1+\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\right)\exp\left(2C\|\bar g_{\rm in}\|^2_{H^{\ell-1}_x L^2_v}\right).\end{aligned}\tag{5.13}$$

Otherwise if $\ell-2<0$ we have

$$\begin{aligned}\|J^\varepsilon_\pm\|_{\mathcal{X}^\varepsilon_T}\lesssim{}&\left(\varepsilon^{1+\beta-\alpha(3-\ell)}+\varepsilon^{1-\alpha(3-\ell)}+\varepsilon^{\frac12-\alpha(3-\ell)}\right)\\&\times\|\bar g_{\rm in}\|^2_{H^{\ell-1}_x L^2_v}\left(1+\|\bar g_{\rm in}\|_{H^{\ell-1}_x L^2_v}\right)\exp\left(2C\|\bar g_{\rm in}\|^2_{H^{\ell-1}_x L^2_v}\right).\end{aligned}\tag{5.14}$$

We finally gather (5.12), (5.13) and (5.14), and use the fact that $3/2 < \ell < 5/2$, $\alpha \in (0, 1/4)$ and $\beta \in (0, 1/2)$, which yields

$$\|\mathcal{S}^\varepsilon\|_{\mathcal{X}_T^\varepsilon} \leqslant \varepsilon^{\frac{1}{2}-2\alpha}\, \Phi\left(\|\bar g_{\rm in}\|_{H_x^{\ell-1}L_v^2}\right) \tag{5.15}$$

where $\Phi(z) = C(1+z)z^2e^{2Cz^2}$. This completes the proof of Proposition 3.2–(3).

5.4. **Estimates on the linear term $\mathcal{L}^\varepsilon[\cdot]$.** Consider $f \in \mathcal{X}_T^\varepsilon$. First we have, from Corollary 4.3 and Proposition 4.5 and using additionally the fact that $g^\varepsilon = \mathbf{P}_0 g^\varepsilon$ and $H_v^{s,\star} \subset L_v^2$,

$$\begin{aligned}
&\|\Psi^\varepsilon[f,g^\varepsilon]\|_{\widetilde{L}_T^\infty H_x^\ell L_v^2} + \frac{1}{\varepsilon}\|\mathbf{P}_0^\perp \Psi^\varepsilon[f,g^\varepsilon]\|_{L_T^2 H_x^\ell H_v^{s,\star}} + \frac{1}{\varepsilon}\|\Psi^{\varepsilon,\sharp}[f,g^\varepsilon]\|_{L_T^2 H_x^\ell H_v^{s,\star}} \\
&\quad\lesssim \|\Gamma_{\rm sym}[f,g^\varepsilon]\|_{L_T^2 H_x^\ell (H_v^{s,\star})'} \\
&\quad\lesssim \|f\|_{L_T^2 H_x^\ell H_v^{s,\star}} \|g^\varepsilon\|_{\widetilde{L}_T^\infty H_x^\ell L_v^2}\,.
\end{aligned}$$

We then compute, thanks to Corollary 4.3, Proposition 4.7 and Proposition 4.5,

$$\begin{aligned}
&\|\Psi^\varepsilon[f,g^\varepsilon]\|_{\widetilde{L}_T^\infty H_x^{\ell-1} L_v^2} + \|\Psi^{\varepsilon,\flat}[f,g^\varepsilon]\|_{L_T^2 H_x^\ell H_v^{s,\star}} \\
&\quad\lesssim \|\Gamma_{\rm sym}(\mathbf{P}_0^\perp f, g^\varepsilon) + \Gamma_{\rm sym}(\mathbf{P}_0 f, g^\varepsilon)\|_{L_T^2 H_x^{\ell-1}(H_v^{s,\star})'} \\
&\quad\lesssim \|\mathbf{P}_0^\perp f\|_{L_T^2 H_x^\ell H_v^{s,\star}} \|g^\varepsilon\|_{\widetilde{L}_T^\infty H_x^{\ell-1} L_v^2} + \|\mathbf{P}_0 f\|_{\widetilde{L}_T^\infty H_x^{\ell-1} L_v^2} \|g^\varepsilon\|_{L_T^2 H_x^\ell L_v^2}\,.
\end{aligned}$$

Gathering previous estimate and using that $\Psi^\varepsilon = \Psi^{\varepsilon,\sharp} + \Psi^{\varepsilon,\flat}$ finally yield

$$\begin{aligned}
&\varepsilon^\beta \|\Psi^\varepsilon[f,g^\varepsilon]\|_{\widetilde{L}_T^\infty H_x^\ell L_v^2} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}_0^\perp \Psi^\varepsilon[f,g^\varepsilon]\|_{L_T^2 H_x^\ell H_v^{s,\star}} \\
&+ \|\mathbf{P}_0 \Psi^\varepsilon[f,g^\varepsilon]\|_{L_T^2 H_x^\ell H_v^{s,\star}} + \|\Psi^\varepsilon[f,g^\varepsilon]\|_{\widetilde{L}_T^\infty H_x^{\ell-1} L_v^2} \\
&\quad\lesssim \left(\|f\|_{\widetilde{L}_T^\infty H_x^{\ell-1} L_v^2} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}_0^\perp f\|_{L_T^2 H_x^\ell H_v^{s,\star}} + \|\mathbf{P}_0 f\|_{L_T^2 H_x^\ell H_v^{s,\star}}\right) \\
&\qquad\times \left(\left[\varepsilon^\beta + \sqrt{\varepsilon} + \varepsilon\right]\|g^\varepsilon\|_{\widetilde{L}_T^\infty H_x^\ell L_v^2} + \sqrt{\varepsilon}\|g^\varepsilon\|_{\widetilde{L}_T^\infty H_x^{\ell-1} L_v^2} + \|g^\varepsilon\|_{L_T^2 H_x^\ell L_v^2}\right).
\end{aligned} \tag{5.16}$$

This completes the proof of Proposition 3.2–(4).

5.5. **Estimates on the nonlinear term $\Psi^\varepsilon[\cdot,\cdot]$.** Thanks to Corollary 4.3 and Proposition 4.5 we have

$$\begin{aligned}
&\|\Psi^\varepsilon[f_1,f_2]\|_{\widetilde{L}_T^\infty H_x^\ell L_v^2} + \frac{1}{\varepsilon}\|\mathbf{P}_0^\perp \Psi^\varepsilon[f_1,f_2]\|_{L_T^2 H_x^\ell H_v^{s,\star}} + \frac{1}{\varepsilon}\|\Psi^{\varepsilon,\sharp}[f_1,f_2]\|_{L_T^2 H_x^\ell H_v^{s,\star}} \\
&\quad\lesssim \|\Gamma_{\rm sym}(f_1,f_2)\|_{L_T^2 H_x^\ell (H_v^{s,\star})'} \\
&\quad\lesssim \|f_1\|_{\widetilde{L}_T^\infty H_x^\ell L_v^2}\|f_2\|_{L_T^2 H_x^\ell H_v^{s,\star}} + \|f_1\|_{L_T^2 H_x^\ell H_v^{s,\star}}\|f_2\|_{\widetilde{L}_T^\infty H_x^\ell L_v^2}\,,
\end{aligned}$$

as well as

$$\begin{aligned}
\|\Psi^\varepsilon[f_1,f_2]\|_{\widetilde{L}_T^\infty H_x^{\ell-1} L_v^2} &\lesssim \|\Gamma_{\rm sym}(f_1,f_2)\|_{L_T^2 H_x^{\ell-1}(H_v^{s,\star})'} \\
&\lesssim \|f_1\|_{\widetilde{L}_T^\infty H_x^{\ell-1} L_v^2}\|f_2\|_{L_T^2 H_x^\ell H_v^{s,\star}} + \|f_1\|_{L_T^2 H_x^\ell H_v^{s,\star}}\|f_2\|_{\widetilde{L}_T^\infty H_x^{\ell-1} L_v^2}\,.
\end{aligned}$$

Moreover from Proposition 4.7 and Proposition 4.5 it follows

$$\begin{aligned}
\|\Psi^{\varepsilon,\flat}[f_1,f_2]\|_{L_T^2 H_x^\ell H_v^{s,\star}} &\lesssim \|\Gamma_{\rm sym}(f_1,f_2)\|_{L_T^2 H_x^{\ell-1}(H_v^{s,\star})'} \\
&\lesssim \|f_1\|_{\widetilde{L}_T^\infty H_x^{\ell-1} L_v^2}\|f_2\|_{L_T^2 H_x^\ell H_v^{s,\star}} + \|f_1\|_{L_T^2 H_x^\ell H_v^{s,\star}}\|f_2\|_{\widetilde{L}_T^\infty H_x^{\ell-1} L_v^2}\,.
\end{aligned}$$

Gathering previous estimates and using that $\Psi^\varepsilon = \Psi^{\varepsilon,\sharp} + \Psi^{\varepsilon,\flat}$, we deduce

(5.17)
$$\begin{aligned}
&\varepsilon^\beta \|\Psi^\varepsilon[f_1,f_2]\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}_0^\perp \Psi^\varepsilon[f_1,f_2]\|_{L^2_T H^\ell_x H^{s,\star}_v} \\
&+ \|\mathbf{P}_0 \Psi^\varepsilon[f_1,f_2]\|_{L^2_T H^\ell_x H^{s,\star}_v} + \|\Psi^\varepsilon[f_1,f_2]\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} \\
&\lesssim \left([\varepsilon^\beta + \sqrt{\varepsilon} + \varepsilon]\|f_1\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}_0^\perp f_1\|_{L^2_T H^\ell_x H^{s,\star}_v} + \|\mathbf{P}_0 f_1\|_{L^2_T H^\ell_x H^{s,\star}_v} + \|f_1\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} \right) \\
&\times \left([\varepsilon^\beta + \sqrt{\varepsilon} + \varepsilon]\|f_2\|_{\widetilde{L}^\infty_T H^\ell_x L^2_v} + \frac{1}{\sqrt{\varepsilon}}\|\mathbf{P}_0^\perp f_2\|_{L^2_T H^\ell_x H^{s,\star}_v} + \|\mathbf{P}_0 f_2\|_{L^2_T H^\ell_x H^{s,\star}_v} + \|f_2\|_{\widetilde{L}^\infty_T H^{\ell-1}_x L^2_v} \right).
\end{aligned}$$

This completes the proof of Proposition 3.2–(5), since $\beta < 1/2$.

Appendix A. Hypocoercivity

It is well-known, see for instance [21, 35, 48, 5, 47], that the linearized Boltzmann and Landau collision operators satisfy the following coercive-type inequality

$$\langle Lf, f\rangle_{L^2_v} \leqslant -\lambda_2 \|(\mathrm{Id} - \mathbf{P}_0) f\|^2_{H^{s,\star}_v}\,, \tag{A.1}$$

for some $\lambda_2 > 0$, where we recall that $\mathbf{P}_0$ is the orthogonal projection onto $\operatorname{Ker} L$ given by (1.15).

For all $\varepsilon \in (0,1]$ and all $k \in \mathbb{Z}^3$, we recall that $\Lambda^\varepsilon(k)$ is the Fourier transform in space of the full linearized operator

$$\frac{1}{\varepsilon^2} L - \frac{1}{\varepsilon} v \cdot \nabla_x\,,$$

namely

$$\Lambda^\varepsilon(k) := \frac{1}{\varepsilon^2}(L - i\varepsilon v \cdot k)\,. \tag{A.2}$$

We now state a hypocoercive result for $\Lambda^\varepsilon(k)$ (for a detailed presentation of the subject, we refer to [9] and the references therein, we also point out the papers [53] and [23] in the case of the whole space), as presented in [13, 14].

Proposition A.1. *There is an inner product $\langle\!\langle \cdot,\cdot\rangle\!\rangle_{L^2_v}$ on L^2_v (depending on k) such that the associate norm $|\!|\!| \cdot |\!|\!|_{L^2_v}$ is equivalent to the standard norm $\|\cdot\|_{L^2_v}$ on L^2_v with bounds that are independent of k and ε, and there exists $\lambda_3 > 0$ such that for every f satisfiying (1.13) and all $k \in \mathbb{Z}^3$, there holds*

$$\mathrm{Re}\langle\!\langle \Lambda^\varepsilon(k)\widehat{f}(k), \widehat{f}(k)\rangle\!\rangle_{L^2_v} \leqslant -\lambda_3 \left(\frac{1}{\varepsilon^2}\|(\mathrm{Id} - \mathbf{P}_0)\widehat{f}(k)\|^2_{H^{s,\star}_v} + \|\mathbf{P}_0 \widehat{f}(k)\|^2_{L^2_v} \right).$$

Proof. For every $k \in \mathbb{Z}^3$, we define

$$\begin{aligned}
\psi[f_1,f_2](k) := {}& \frac{\delta_1 i}{\langle k\rangle^2} k\theta[\widehat{f}_1(k)] \cdot M[(\mathrm{Id} - \mathbf{P}_0)\widehat{f}_2(k)] + \frac{\delta_1 i}{\langle k\rangle^2} k\theta[\widehat{f}_2(k)] \cdot M[(\mathrm{Id} - \mathbf{P}_0)\widehat{f}_1(k)] \\
&+ \frac{\delta_2 i}{\langle k\rangle^2}(k \otimes u[\widehat{f}_1(k)])^{\mathrm{sym}} : \left\{ \Theta[(\mathrm{I_d} - \mathbf{P}_0)\widehat{f}_2(k)] + \theta[\widehat{g}(k)] I \right\} \\
&+ \frac{\delta_2 i}{\langle k\rangle^2}(k \otimes u[\widehat{f}_2(k)])^{\mathrm{sym}} : \left\{ \Theta[(\mathrm{I_d} - \mathbf{P}_0)\widehat{f}_1(k)] + \theta[\widehat{f}(k)] I \right\} \\
&+ \frac{\delta_3 i}{\langle k\rangle^2} k\rho[\widehat{f}_1(k)] \cdot u[\widehat{f}_2(k)] + \frac{\delta_3 i}{\langle k\rangle^2} k\rho[\widehat{f}_2(k)] \cdot u[\widehat{f}_1(k)]\,,
\end{aligned}$$

with constants $0 < \delta_3 \ll \delta_2 \ll \delta_1 \ll 1$, where $\mathrm{I_d}$ is the 3×3 identity matrix and the moments M and Θ are defined by

$$M[f] := \int_{\mathbb{R}^3} f v\,(|v|^2 - 5)\sqrt{\mu}(v)\,\mathrm{d}v\,, \qquad \Theta[f] := \int_{\mathbb{R}^3} f\,(v \otimes v - \mathrm{I_d})\,\sqrt{\mu}(v)\,\mathrm{d}v\,,$$

and where for vectors $a, b \in \mathbb{R}^3$ and matrices $A, B \in \mathbb{R}^{3\times 3}$, we denote

$$(a \otimes b)^{\mathrm{sym}} = \frac{1}{2}(a_j b_k + a_k b_j)_{1\leqslant j,k\leqslant 3}\,, \qquad A : B = \sum_{j,k=1}^{3} A_{jk}B_{jk}\,.$$

We then define the inner product $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{L^2_v}$ on L^2_v (depending on k) by

$$\langle\!\langle \widehat{f}_1(k), \widehat{f}_2(k) \rangle\!\rangle_{L^2_v} := \langle \widehat{f}_1(k), \widehat{f}_2(k) \rangle_{L^2_v} + \varepsilon\, \psi[f_1, f_2](k)\,, \tag{A.3}$$

and the associated norm

$$|\!|\!| \widehat{f}(k) |\!|\!|^2_{L^2_v} := \langle\!\langle \widehat{f}(k), \widehat{f}(k) \rangle\!\rangle_{L^2_v}\,. \tag{A.4}$$

We then argue as in [53], the only difference being the factor ε in the second term of (A.3). □

Using this hypocoercivity result, we are able to prove Proposition 4.1.

Proof of Proposition 4.1.

- (1) Let $f(t) = U^\varepsilon(t) f_{\mathrm{in}}$ for all $t \geqslant 0$, which satisfies the equation

$$\partial_t f = \frac{1}{\varepsilon^2}(L - \varepsilon v \cdot \nabla_x) f\,, \quad f_{|t=0} = f_{\mathrm{in}}\,. \tag{A.5}$$

We already observe that $f(t)$ verifies (1.13) thanks to the conservation properties of Γ (and hence of L). Taking the Fourier transform in space of the above equation, we obtain that $\widehat{f}$ satisfies

$$\partial_t \widehat{f}(k) = \Lambda^\varepsilon(k)\widehat{f}(k)\,, \quad \widehat{f}(k)_{|t=0} = \widehat{f}_{\mathrm{in}}(k)\,, \tag{A.6}$$

for all $k \in \mathbb{Z}^3$. Applying Proposition A.1 yields, for all $t \geqslant 0$,

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\!|\!| \widehat{f}(k) |\!|\!|^2_{L^2_v} &= \mathrm{Re}\langle\!\langle \Lambda^\varepsilon(k)\widehat{f}(k), \widehat{f}(k) \rangle\!\rangle_{L^2_v} \\ &\leqslant -\lambda_3 \left(\frac{1}{\varepsilon^2} \|\mathbf{P}_0^\perp \widehat{f}(k)\|^2_{H^{s,\star}_v} + \|\mathbf{P}_0 \widehat{f}(k)\|^2_{L^2_v} \right)\,, \end{aligned}$$

which implies

$$\|\widehat{f}(t,k)\|^2_{L^2_v} + \frac{1}{\varepsilon^2}\int_0^t \|(\mathrm{Id} - \mathbf{P}_0)\widehat{f}(t',k)\|^2_{H^{s,\star}_v}\,\mathrm{d}t' + \int_0^t \|\mathbf{P}_0 \widehat{f}(t',k)\|^2_{L^2_v}\,\mathrm{d}t' \lesssim \|\widehat{f}_{\mathrm{in}}(k)\|^2_{L^2_v}\,,$$

where we have used that $|\!|\!|\cdot|\!|\!|_{L^2_v}$ is equivalent to $\|\cdot\|_{L^2_v}$ independently of k and ε. Taking the supremum in time and then multipliyng by $\langle k\rangle^{2m}$ yields

$$\langle k\rangle^{2m}\|\widehat{f}(k)\|^2_{L^\infty_t L^2_v} + \frac{\langle k\rangle^{2m}}{\varepsilon}\|(\mathrm{Id} - \mathbf{P}_0)\widehat{f}(k)\|^2_{L^2_t H^{s,\star}_v} + \langle k\rangle^{2m}\|\mathbf{P}_0\widehat{f}(k)\|^2_{L^2_t L^2_v} \lesssim \langle k\rangle^{2m}\|\widehat{f}_{\mathrm{in}}(k)\|^2_{L^2_v}\,.$$

We conclude by summing in k.

- (2) Denote

$$h(t) = \int_0^t U^\varepsilon(t - t')S(t')\,\mathrm{d}t'$$

which is the solution to

$$\partial_t h = \frac{1}{\varepsilon^2}(L - \varepsilon v \cdot \nabla_x)h + S\,, \quad h_{|t=0} = 0\,. \tag{A.7}$$

Taking the Fourier transform in space gives

$$\partial_t \widehat{h}(k) = \Lambda^\varepsilon(k)\widehat{h}(k) + \widehat{S}(k), \quad \widehat{h}(k)_{|t=0} = 0\,, \tag{A.8}$$

for all $k \in \mathbb{Z}^3$. From the definition of (A.3) and the hypothesis $\mathbf{P}_0 S = 0$, we observe that

$$\begin{aligned}\langle\!\langle \widehat{S}(k), \widehat{h}(k)\rangle\!\rangle_{L^2_v} &= \langle \widehat{S}(k), \widehat{h}(k)\rangle_{L^2_v} + \varepsilon \Psi[S,h](k) \\ &= \langle \widehat{S}(k), (\mathrm{Id} - \mathbf{P}_0)\widehat{h}(k)\rangle_{L^2_v} + \varepsilon \frac{\delta_1 i}{1+|k|^2} k\theta[\widehat{h}(k)] \cdot M[(\mathrm{Id}-\mathbf{P}_0)\widehat{S}(k)] \\ &\quad + \varepsilon \frac{\delta_2 i}{1+|k|^2}(k \otimes u[\widehat{h}(k)])^{\mathrm{sym}} : \Theta[(\mathrm{Id}-\mathbf{P}_0)\widehat{S}(k)]\,.\end{aligned}$$

Observing that for any polynomial $p = p(v)$ we have

$$\left| \int_{\mathbb{R}^3} \widehat{S}(k) p(v) \sqrt{\mu}(v)\, \mathrm{d}v \right| \lesssim \|\widehat{S}(k)\|_{(H^{s,\star}_v)'}\,,$$

we get

$$|\Psi[S,h](k)| \lesssim \|(\mathrm{Id}-\mathbf{P}_0)\widehat{S}(k)\|_{(H^{s,\star}_v)'} \frac{|k|}{\langle k\rangle} \|\mathbf{P}_0 \widehat{h}(k)\|_{L^2_v}\,.$$

By duality, we also have

$$\langle \widehat{S}(k), (\mathrm{Id}-\mathbf{P}_0)\widehat{h}(k)\rangle_{L^2_v} \lesssim \|\widehat{S}(k)\|_{(H^{s,\star}_v)'} \|(\mathrm{Id}-\mathbf{P}_0)\widehat{h}(k)\|_{H^{s,\star}_v}\,,$$

therefore gathering previous estimates yields

$$\langle\!\langle \widehat{S}(k), \widehat{h}(k)\rangle\!\rangle_{L^2_v} \lesssim \varepsilon \|\widehat{S}(k)\|_{(H^{s,\star}_v)'} \left(\frac{1}{\varepsilon} \|(\mathrm{Id}-\mathbf{P}_0)\widehat{h}(k)\|_{H^{s,\star}_v} + \|\mathbf{P}_0 \widehat{h}(k)\|_{L^2_v} \right)\,. \tag{A.9}$$

Using Proposition A.1 and arguing as in the proof of Proposition 4.1–(1) we have, for all $t \geqslant 0$ and all $k \in \mathbb{Z}^3$,

(A.10)

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t} |\!|\!| \widehat{h}(k) |\!|\!|^2_{L^2_v} &\leqslant -\lambda_3 \left(\frac{1}{\varepsilon^2} \|(\mathrm{Id}-\mathbf{P}_0)\widehat{h}(k)\|^2_{H^{s,\star}_v} + \|\mathbf{P}_0 \widehat{h}(k)\|^2_{L^2_v} \right) \\ &\quad + \varepsilon C \|\widehat{S}(k)\|_{(H^{s,\star}_v)'} \left(\frac{1}{\varepsilon} \|(\mathrm{Id}-\mathbf{P}_0)\widehat{h}(k)\|_{H^{s,\star}_v} + \|\mathbf{P}_0 \widehat{h}(k)\|_{L^2_v} \right) \\ &\leqslant -\frac{\lambda_3}{2} \left(\frac{1}{\varepsilon^2} \|(\mathrm{Id}-\mathbf{P}_0)\widehat{h}(k)\|^2_{H^{s,\star}_v} + \|\mathbf{P}_0 \widehat{h}(k)\|^2_{L^2_v} \right) + C\varepsilon^2 \|\widehat{S}(k)\|^2_{(H^{s,\star}_v)'}\,,\end{aligned}$$

where we have used Young's inequality in last line. This implies

$$\begin{aligned}\|\widehat{h}(t,k)\|^2_{L^2_v} + \frac{1}{\varepsilon^2} \int_0^t \|(\mathrm{Id}-\mathbf{P}_0)\widehat{h}(t',k)\|^2_{H^{s,\star}_v}\,\mathrm{d}t' + \int_0^t \|\mathbf{P}_0 \widehat{h}(t',k)\|^2_{L^2_v}\,\mathrm{d}t' \\ \lesssim \varepsilon^2 \int_0^t \|\widehat{S}(t',k)\|^2_{(H^{s,\star}_v)'}\,\mathrm{d}t'\,.\end{aligned}$$

Taking the supremum in time and then multiplying by $\langle k\rangle^{2m}$ yields

$$\begin{aligned}\langle k\rangle^{2m} \|\widehat{h}(k)\|^2_{L^\infty_t L^2_v} + \frac{\langle k\rangle^{2m}}{\varepsilon^2} \|(\mathrm{Id}-\mathbf{P}_0)\widehat{h}(k)\|^2_{L^2_t H^{s,\star}_v} + \langle k\rangle^{2m} \|\mathbf{P}_0 \widehat{h}(k)\|^2_{L^2_t L^2_v} \\ \lesssim \varepsilon^2 \langle k\rangle^{2m} \|\widehat{S}(k)\|^2_{L^2_t (H^{s,\star}_v)'}\,,\end{aligned}$$

and we conclude by summing in k. Proposition 4.1 is proved. □

References

[1] Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., and Yang, T. The Boltzmann equation without angular cutoff in the whole space: I, Global existence for soft potential. *J. Funct. Anal. 262*, 3 (2012), 915–1010.

[2] Alexandre, R., Morimoto, Y., Ukai, S., Xu, C.-J., and Yang, T. Local existence with mild regularity for the Boltzmann equation. *Kinet. Relat. Models 6*, 4 (2013), 1011–1041.

[3] Alonso, R., Lods, B., and Tristani, I. Fluid dynamic limit of Boltzmann equation for granular hard-spheres in a nearly elastic regime. To appear in Mémoires de la SMF, *arXiv preprint arXiv:2008.05173*, 2024.

[4] Bahouri, H., Chemin, J.-Y., and Danchin, R. *Fourier analysis and nonlinear partial differential equations*, vol. 343 of *Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer, Heidelberg, 2011.

[5] Baranger, C., and Mouhot, C. Explicit spectral gap estimates for the linearized Boltzmann and Landau operators with hard potentials. *Rev. Mat. Iberoamericana 21*, 3 (2005), 819–841.

[6] Bardos, C., Golse, F., and Levermore, C. D. Fluid dynamic limits of kinetic equations. II. Convergence proofs for the Boltzmann equation. *Comm. Pure Appl. Math. 46*, 5 (1993), 667–753.

[7] Bardos, C., Golse, F., and Levermore, D. Fluid dynamic limits of kinetic equations. I. Formal derivations. *J. Statist. Phys. 63*, 1-2 (1991), 323–344.

[8] Bardos, C., and Ukai, S. The classical incompressible Navier-Stokes limit of the Boltzmann equation. *Math. Models Methods Appl. Sci. 1*, 2 (1991), 235–257.

[9] Bernou, A., Carrapatoso, K., Mischler, S., and Tristani, I. Hypocoercivity for kinetic linear equations in bounded domains with general Maxwell boundary condition. *Ann. Inst. H. Poincaré C Anal. Non Linéaire 40*, 2 (2023), 287–338.

[10] Briant, M. From the Boltzmann equation to the incompressible Navier-Stokes equations on the torus: a quantitative error estimate. *J. Differential Equations 259*, 11 (2015), 6072–6141.

[11] Briant, M., Merino-Aceituno, S., and Mouhot, C. From Boltzmann to incompressible Navier-Stokes in Sobolev spaces with polynomial weight. *Anal. Appl. (Singap.) 17*, 1 (2019), 85–116.

[12] Caflisch, R. E. The fluid dynamic limit of the nonlinear Boltzmann equation. *Comm. Pure Appl. Math. 33*, 5 (1980), 651–666.

[13] Cao, C., and Carrapatoso, K. Hydrodynamic limit for the non-cutoff Boltzmann equation. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* (2024), published online first.

[14] Carrapatoso, K., and Gervais, P. Noncutoff Boltzmann equation with soft potentials in the whole space. *Pure Appl. Anal. 6*, 1 (2024), 253–303.

[15] Carrapatoso, K., Rachid, M., and Tristani, I. Regularization estimates and hydrodynamical limit for the Landau equation. *J. Math. Pures Appl. (9) 163* (2022), 334–432.

[16] Cercignani, C., Illner, R., and Pulvirenti, M. *The mathematical theory of dilute gases*, vol. 106 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1994.

[17] Chapman, S., and Cowling, T. G. *The mathematical theory of non-uniform gases: An account of the kinetic theory of viscosity, thermal conduction, and diffusion in gases*. Cambridge University Press, New York, 1960.

[18] Chemin, J.-Y. Remarques sur l'existence globale pour le système de Navier-Stokes incompressible. *SIAM J. Math. Anal. 23*, 1 (1992), 20–28.

[19] Chemin, J.-Y., and Lerner, N. Flot de champs de vecteurs non lipschitziens et équations de Navier-Stokes. *J. Differential Equations 121*, 2 (1995), 314–328.

[20] De Masi, A., Esposito, R., and Lebowitz, J. L. Incompressible Navier-Stokes and Euler limits of the Boltzmann equation. *Comm. Pure Appl. Math. 42*, 8 (1989), 1189–1214.

[21] Degond, P., and Lemou, M. Dispersion relations for the linearized Fokker-Planck equation. *Arch. Rational Mech. Anal. 138*, 2 (1997), 137–167.

[22] DiPerna, R. J., and Lions, P.-L. On the Cauchy problem for Boltzmann equations: global existence and weak stability. *Ann. of Math. (2) 130*, 2 (1989), 321–366.

[23] Duan, R. On the Cauchy problem for the Boltzmann equation in the whole space: global existence and uniform stability in $L^2_\xi(H^N_x)$. *J. Differential Equations 244*, 12 (2008), 3204–3234.

[24] Dutrifoy, A., and Hmidi, T. The incompressible limit of solutions of the two-dimensional compressible Euler system with degenerating initial data. *Comm. Pure Appl. Math. 57*, 9 (2004), 1159–1177.

[25] Ellis, R. S., and Pinsky, M. A. The first and second fluid approximations to the linearized Boltzmann equation. *J. Math. Pures Appl. (9) 54* (1975), 125–156.

[26] Gallagher, I., and Tristani, I. On the convergence of smooth solutions from Boltzmann to Navier-Stokes. *Ann. H. Lebesgue 3* (2020), 561–614.

[27] Gervais, P. A spectral study of the linearized boltzmann operator in L^2-spaces with polynomial and Gaussian weights. *Kinetic & Related Models 14*, 4 (2021), 725–747.

[28] Gervais, P. On the convergence from Boltzmann to Navier-Stokes-Fourier for general initial data. *SIAM J. Math. Anal. 55*, 2 (2023), 805–848.

[29] Gervais, P., and Lods, B. Hydrodynamic limits for kinetic equations preserving mass, momentum and energy: a spectral and unified approach in the presence of a spectral gap. *Ann. Henri Lebesgue 7* (2024), 969–1098.

[30] Golse, F., and Saint-Raymond, L. The Navier-Stokes limit of the Boltzmann equation for bounded collision kernels. *Invent. Math. 155*, 1 (2004), 81–161.

[31] Golse, F., and Saint-Raymond, L. The incompressible Navier-Stokes limit of the Boltzmann equation for hard cutoff potentials. *J. Math. Pures Appl. (9) 91*, 5 (2009), 508–552.

[32] Grad, H. Asymptotic theory of the Boltzmann equation. II. In *Rarefied Gas Dynamics (Proc. 3rd Internat. Sympos., Palais de l'UNESCO, Paris, 1962), Vol. I* (1963), Academic Press, New York, pp. 26–59.

[33] Grad, H. Asymptotic equivalence of the Navier-Stokes and nonlinear Boltzmann equations. In *Proc. Sympos. Appl. Math., Vol. XVII* (1965), Amer. Math. Soc., Providence, R.I., pp. 154–183.
[34] Gressman, P. T., and Strain, R. M. Global classical solutions of the Boltzmann equation without angular cut-off. *J. Amer. Math. Soc. 24*, 3 (2011), 771–847.
[35] Guo, Y. The Landau equation in a periodic box. *Comm. Math. Phys. 231*, 3 (2002), 391–434.
[36] Guo, Y. Classical solutions to the Boltzmann equation for molecules with an angular cutoff. *Arch. Ration. Mech. Anal. 169*, 4 (2003), 305–353.
[37] Guo, Y. Boltzmann diffusive limit beyond the Navier-Stokes approximation. *Comm. Pure Appl. Math. 59*, 5 (2006), 626–687.
[38] Hilbert, D. *Sur les problèmes futurs des mathématiques*. Les Grands Classiques Gauthier-Villars. [Gauthier-Villars Great Classics]. Éditions Jacques Gabay, Sceaux, 1990. Les 23 problèmes. [The 23 problems], Translated from the 1900 German original by M. L. Laugel and revised by the author, Reprint of the 1902 French translation.
[39] Jiang, N., Xu, C.-J., and Zhao, H. Incompressible Navier-Stokes-Fourier limit from the Boltzmann equation: classical solutions. *Indiana Univ. Math. J. 67*, 5 (2018), 1817–1855.
[40] Lachowicz, M. On the initial layer and the existence theorem for the nonlinear Boltzmann equation. *Math. Methods Appl. Sci. 9*, 3 (1987), 342–366.
[41] Landau, L. D. Die kinetische Gleichung für den Fall Coulombscher Wechselwirkung. *Phys. Z. Sowjet 10 154* (1936).
[42] Lemarié-Rieusset, P. G. *Recent developments in the Navier-Stokes problem*, vol. 431 of *Chapman & Hall/CRC Research Notes in Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2002.
[43] Lemarié-Rieusset, P. G. *The Navier-Stokes problem in the 21st century*. CRC Press, Boca Raton, FL, 2016.
[44] Levermore, C. D., and Masmoudi, N. From the Boltzmann equation to an incompressible Navier-Stokes-Fourier system. *Arch. Ration. Mech. Anal. 196*, 3 (2010), 753–809.
[45] Lions, P.-L., and Masmoudi, N. From the Boltzmann equations to the equations of incompressible fluid mechanics. I, II. *Arch. Ration. Mech. Anal. 158*, 3 (2001), 173–193, 195–211.
[46] Maxwell, J. C. On the dynamical theory of gases. *Philosophical Transactions of the Royal Society of London 157* (1867), 49–88.
[47] Mouhot, C. Explicit coercivity estimates for the linearized Boltzmann and Landau operators. *Comm. Partial Differential Equations 31*, 7-9 (2006), 1321–1348.
[48] Mouhot, C., and Strain, R. M. Spectral gap and coercivity estimates for linearized Boltzmann collision operators without angular cutoff. *J. Math. Pures Appl. (9) 87*, 5 (2007), 515–535.
[49] Nicolaenko, B. Dispersion Laws for Plane Wave Propagation. *Courant Institute of Mathematical Sciences, New York University* The Boltzmann equation (1971), 125–172.
[50] Nishida, T. Fluid dynamical limit of the nonlinear Boltzmann equation to the level of the compressible Euler equation. *Comm. Math. Phys. 61*, 2 (1978), 119–148.
[51] Rachid, M. Incompressible Navier-Stokes-Fourier limit from the Landau equation. *Kinetic & Related Models 14*, 4 (2021), 599–638.
[52] Saint-Raymond, L. *Hydrodynamic limits of the Boltzmann equation*, vol. 1971 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2009.
[53] Strain, R. M. Optimal time decay of the non cut-off Boltzmann equation in the whole space. *Kinet. Relat. Models 5*, 3 (2012), 583–613.
[54] Ukai, S. Solutions of the Boltzmann equation. *Studies in Mathematics and its Applications 18* (1986), 37–96.
[55] Villani, C. A review of mathematical topics in collisional kinetic theory. In *Handbook of mathematical fluid dynamics, Vol. I*. North-Holland, Amsterdam, 2002, pp. 71–305.
[56] Yang, T., and Yu, H. Spectrum analysis of some kinetic equations. *Arch. Ration. Mech. Anal. 222*, 2 (2016), 731–768.
[57] Yang, T., and Yu, H. Spectrum structure and decay rate estimates on the Landau equation with Coulomb potential. *Sci. China Math. 66*, 1 (2023), 37–78.

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