Quantitative unique continuation for non-regular perturbations of the Laplacian

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Pedro Caro[†] Sylvain Ervedoza[‡] Lotfi Thabouti[§]

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Abstract

In this work, we investigate the quantitative estimates of the unique continuation property for solutions of an elliptic equation $\Delta u = Vu + W_1 \cdot \nabla u + \operatorname{div}(W_2 u)$ in an open, connected subset of $\mathbb{R}^d$, where $d \geqslant 3$. Here, $V \in L^{q_0}$, $W_1 \in L^{q_1}$, and $W_2 \in L^{q_2}$ with $q_0 > d/2$, $q_1 > d$, and $q_2 > d$. Our aim is to provide an explicit quantification of the unique continuation property with respect to the norms of the potentials. To achieve this, we revisit the Carleman estimates established in [6] and prove a refined version of them, and we combine them with an argument due to T. Wolff introduced in [18] for the proof of unique continuation for solutions of equations of the form $\Delta u = Vu + W_1 \cdot \nabla u$.

Keywords

Carleman estimates, boundary value problem, elliptic equations, Fourier restriction theorems.

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Contents

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1 Introduction

Main result. Our main goal is to prove the following quantitative unique continuation result.

Theorem 1.1. *Let $d \geqslant 3$, $\Omega \subset \mathbb{R}^d$ be a bounded domain, and ω and $\mathcal{O}$ be non-empty open subsets of Ω with $\omega \subset \overline{\omega} \subset \mathcal{O} \subset \overline{\mathcal{O}} \subset \Omega$. Further assume the following geometric condition:*

(GC) *For all $y \in \overline{\mathcal{O}}$, there exist $x_0 \in \omega$, $r_y > 0$ and a smooth path γ_y of finite length such that $\gamma_y(0) = x_0$, $\gamma_y(1) = y$, and $\cup_{s\in[0,1]} B_{\gamma_y(s)}(r_y) \subset \Omega$, where $B_{\gamma_y(s)}(r_y)$ is the ball centered in $\gamma_y(s)$ and of radius r_y.*

Then there exist constants $C = C(\omega, \mathcal{O}, \Omega) > 0$ and $\alpha \in (0, 1)$ depending only on ω, $\mathcal{O}$ and Ω so that for any solution $u \in H^1(\Omega)$ of

$$\Delta u = V u + W_1 \cdot \nabla u + \operatorname{div}(W_2 u) \quad \text{ in } \mathscr{D}'(\Omega), \tag{1.1}$$

where

$$V \in L^{q_0}(\Omega) \text{ with } q_0 \in \left(\frac{d}{2}, \infty\right], \qquad \text{and} \qquad W_j \in L^{q_j}(\Omega; \mathbb{C}^d) \text{ with } q_j \in (d, \infty]\,, \text{ for } j \in \{1, 2\}, \tag{1.2}$$

we have

$$\|u\|_{H^1(\mathcal{O})} \leqslant C e^{C\left(\|V\|_{L^{q_0}(\Omega)}^{\gamma(q_0)} + \|W_1\|_{L^{q_1}(\Omega)}^{\delta(q_1)} + \|W_2\|_{L^{q_2}(\Omega)}^{\delta(q_2)}\right)} \|u\|_{H^1(\omega)}^{\alpha} \|u\|_{H^1(\Omega)}^{1-\alpha}\,, \tag{1.3}$$

with

$$\gamma(q) = \begin{cases} \dfrac{1}{\dfrac{3}{2}\left(1 - \dfrac{d}{2q}\right) + \dfrac{1}{2q}} & \text{if } q \geqslant d, \\[2ex] \dfrac{1}{\left(\dfrac{3}{4} + \dfrac{1}{2d}\right)\left(2 - \dfrac{d}{q}\right)} & \text{if } q \in \left(\dfrac{d}{2}, d\right], \end{cases} \qquad \text{and} \qquad \delta(q) = \frac{2}{1 - \dfrac{d}{q}}, \quad \text{if } q > d. \tag{1.4}$$

Before going further, let us note that the geometric condition **(GC)** assumed in Theorem 1.1 is a very mild technical condition. This condition can be violated in some fractal type sets, but it is certainly satisfied for most geometrical settings. In particular, this geometric condition **(GC)** is satisfied when ω, $\mathcal{O}$ and Ω are concentric balls, case in which (1.3) reduces to the usual 3-balls type estimate, with an explicit quantification with respect to the lower order terms.

Of course, a direct application of Theorem 1.1 yields the following unique continuation property for the Laplace operator: If $u \in H^1(\Omega)$ satisfies (1.1) with potentials V, W_1 and W_2 as in (1.2) and $u = 0$ in ω, then $u = 0$ in any set $\mathcal{O}$ satisfying $\mathcal{O} \Subset \Omega$ and **(GC)**. Theorem 1.1 is a quantification of this property as it states that, if $u \in H^1(\Omega)$ is small in ω and satisfies (1.1) with potentials V, W_1 and W_2 as in (1.2), then u is small in $\mathcal{O}$, with a precise quantification in terms of the norms of the potentials.

It is well-known that unique continuation holds for general $V \in L^{q_0}(\Omega)$, $W_1 \in L^{q_1}(\Omega; \mathbb{C}^d)$, and $W_2 \in L^{q_2}(\Omega; \mathbb{C}^d)$, where $q_0 \geqslant d/2$, $q_1 \geqslant d$, and $q_2 \geqslant d$ (see [19]). Moreover, strong unique continuation has even been proven in cases when $q_0 > d/2$, $q_1 > d$, and $q_2 > d$ (see [9]). These classes of integrability for the potentials are sharp, as shown in [10]. However, establishing unique continuation results requires the use of a Carleman estimate and a delicate argument due to T. Wolff, see [18], as also discussed in [9]. This argument requires to choose a weight function in the Carleman estimate depending on the solution u itself. The question of quantifying the unique continuation property with respect to the norms of the potentials is thus quite delicate, and this is the main novelty of our work.

We also mention the work [12], which quantifies unique continuation properties for the Laplace operator with lower-order terms in the sharp integrability class. However, it does not provide an explicit quantification with respect to the norms of the potentials. Again, since this work builds upon [9], as mentioned earlier, it remains unclear how the proof in [12] can be made quantitative in terms of the norms of the potentials.

When trying to quantify the unique continuation property with respect to the norms of the lower order terms, the known results rely only on the use of a Carleman estimate such as the one presented in [6, Theorem 1.1], which, as pointed out in [3], does not allow to go beyond $W_1 \in L^{\frac{3d-2}{2}}(\Omega; \mathbb{C}^d)$. This corresponds to what is done in [5, 4, 6] using L^p Carleman estimate. The results in [5] describing the maximal order of vanishing of solutions of elliptic equations require V and W_1 respectively in $L^{q_0}(\Omega)$ with $q_0 > d(3d-2)/(5d-2)$ and in $L^{q_1}(\Omega; \mathbb{C}^d)$ with $q_1 > (3d-2)/2$, and $W_2 = 0$. This was improved in [6] using new L^p Carleman estimates allowing to handle V and W_1 respectively in $L^{q_0}(\Omega)$ with $q_0 > d/2$ and in $L^{q_1}(\Omega; \mathbb{C}^d)$ with $q_1 > (3d-2)/2$, exhibiting the same dependence in $\|V\|_{L^{q_0}(\Omega)}$ as in [4], which was limited to the case $W_1 = W_2 = 0$.

We also point out that L^2 Carleman estimates does not allow to reach the sharp integrability class for potentials and lower order terms, see for instance [15] where it is shown that one can obtain quantified unique continuation results for potentials $V \in L^{q_0}(\Omega)$ with $q_0 > 2d/3$. In fact, even if one uses L^p Carleman estimates, one can obtain unique continuation results for potentials V in the sharp class of integrability ($V \in L^{q_0}(\Omega)$ with $q_0 > d/2$), but under restrictive integrability conditions on W_1 and W_2, see for instance [3] and [19].

Theorem 1.1 is an improvement of [6, Theorem 1.3], since Theorem 1.1 allows non-trivial lower order terms $W_1 \in L^{q_1}(\Omega; \mathbb{C}^d)$ and $W_2 \in L^{q_2}(\Omega; \mathbb{C}^d)$ with $q_1 > d$ and $q_2 > d$, while Theorem 1.3 in [6] is restricted to higher integrability class. Indeed, Theorem 1.3 in [6] states that if $W_1 \in L^{q_1}(\Omega; \mathbb{C}^d)$ and $W_2 \in L^{q_2}(\Omega; \mathbb{C}^d)$, $q_1 > (3d-2)/2$, $q_2 > (3d-2)/2$ and $1/q_1 + 1/q_2 < 4(1-1/d)/(3d-2)$, then any solution $u \in H^1(\Omega)$ of (1.1) satisfies

$$\|u\|_{H^1(\mathcal{O})} \leqslant C e^{C\left(\|V\|_{L^{q_0}(\Omega)}^{\gamma(q_0)} + \|W_1\|_{L^{q_1}}^{\widetilde{\delta}(q_1)} + \|W_2\|_{L^{q_2}}^{\widetilde{\delta}(q_2)} + (\|W_1\|_{L^{q_1}} \|W_1\|_{L^{q_2}})^{\gamma(q_1,q_2)}\right)} \|u\|_{H^1(\omega)}^{\alpha} \|u\|_{H^1(\Omega)}^{1-\alpha}, \tag{1.5}$$

with

$$\widetilde{\delta}(q) = \frac{2}{(1 - \frac{(3d-2)}{2q})} \quad \text{and} \quad \gamma(q_1, q_2) = \frac{1}{(1-\frac{1}{d}) - (\frac{3}{4} - \frac{1}{2d})(\frac{d}{q_1} + \frac{d}{q_2})}.$$

Accordingly, the dependence in terms of the norms of the potentials W_1 and W_2 is also weaker in (1.3) than in (1.5), as δ given by (1.4) is smaller than $\widetilde{\delta}$, even for $q > (3d-2)/2$.

In fact, this also suggests to analyze the optimality of the coefficients γ and δ in (1.4), but this question is, to our knowledge, fully open, except when $W_1 = W_2 = 0$ and $V \in L^\infty(\Omega)$. Indeed, in this case, it is known that in dimension $d \geqslant 3$ the dependence of the constant in the quantification of unique continuation is of the form $C \exp(C\|V\|_{L^\infty}^{\frac{2}{3}})$, see [13, 7]. This coincides with our estimate as $\gamma(\infty) = 2/3$.

Our proof does not allow to derive an estimate up to the boundary of Ω. This is due to a technical fact, coming from the use of Wolff's argument on one hand and of the use of the Carleman estimate in [6] on the other hand. Indeed, as we will see, roughly speaking, Wolff's argument requires the possibility to play with (the gradient of) the weight function within the Carleman estimate. But the Carleman estimate in [6], which we will use and revisit within this work, requires the boundary of the domain to be a level set of the weight function. These two conditions are thus not compatible and cause trouble when working in a neighborhood of the boundary.

We also would like to emphasize that the Schrödinger operators in (1.1) also include the consideration of potentials $V \in W^{-1,d+\epsilon}(\Omega)$, where $\epsilon > 0$. Indeed, any such potential can be represented under the form $V \equiv V_0 + \operatorname{div}(W)$, with $V_0 \in L^{d+\epsilon}(\Omega) \subset L^{\frac{d}{2}+\epsilon}(\Omega)$ and $W \in L^{d+\epsilon}(\Omega, \mathbb{C}^d)$ (see, for example, [1, Theorem 3.9]). Consequently, one can then rewrite Vu under the form $Vu = V_0 u + \operatorname{div}(Wu) - W \cdot \nabla u$.

Outline. Let us briefly comment on the structure of the article. In the next section, we first briefly recall an auxiliary result due to T. Wolff in [18] (cf. Lemma 2.1), followed by the presentation of some improved L^p Carleman type estimates (cf. Theorem 2.12 in Section 2) inspired by [6]. Following this, Section 3 is devoted to the proof of these Carleman estimates. Then, in Section 4, we establish a local quantification of the unique continuation within a specific geometric framework (cf. Lemma 4.1). In Section 5, we subsequently employ this to establish a three balls estimate, which directly yields the quantitative unique continuation result stated in Theorem 1.1, after a few classical manipulations.

Notations. Let us finally introduce some of the notation that we will use throughout the article:

- For every $x \in \mathbb{R}^d$, $x = (x_1, .., x_d)$, we set $x = (x_1, x')$, where $x' = (x_2, .., x_d) \in \mathbb{R}^{d-1}$.
- For $x \in \mathbb{R}^d$ and $r > 0$, $B_x(r)$ denotes the ball centered at $x \in \mathbb{R}^d$ and of radius $r > 0$.
- The notations ∇ and Δ respectively stand for the gradient and the Laplacian with respect to $x = (x_1, .., x_d)$, and $\nabla' = (\partial_2, .., \partial_d)$ and $\Delta' = \sum_{j=2}^d \partial_j^2$ are, respectively, the vertical gradient and Laplacian operators.
- The Fourier transform is always taken to be the Fourier transform with respect to $x' = (x_2, .., x_d)$, and then its dual variable $\xi' \in \mathbb{R}^{d-1}$ is indexed by $\xi' = (\xi_2, .., \xi_d)$. Note that for a function f defined on $\mathbb{R}^d$

(or a vertical strip $(X_0, X_1) \times \mathbb{R}^{d-1}$) such that $f(x_1, \cdot) \in \mathscr{S}(\mathbb{R}^{d-1})$, $\widehat{f}(x_1, \cdot)$ denotes the partial Fourier transform with respect to x', that is:

$$\widehat{f}(x_1, \xi') = \frac{1}{(2\pi)^{\frac{d-1}{2}}} \int_{\mathbb{R}^{d-1}} e^{-\mathrm{i}x'\cdot\xi'} f(x_1, x')\, dx', \qquad \xi' \in \mathbb{R}^{d-1}. \tag{1.6}$$

- For a measurable subset E of $\mathbb{R}^d$, we denote by $|E|$ its Lebesgue measure.

2 Auxiliary Results

In this section, we begin by recalling Wolff's lemma, and we present a refined version of the L^p Carleman estimates obtained in [6]. These elements are the main points in the proof of Theorem 1.1.

2.1 Wolff's Lemma

We hereby present Wolff's argument, introduced in [18]:

Lemma 2.1 ([18, Lemma 1])**.** *Suppose μ is a positive measure in $\mathbb{R}^d$ which has faster than exponential decay in the following sense*

$$\lim_{T\to\infty} T^{-1} \log(\mu\{x \in \mathbb{R}^d\, , |x| \geqslant T\}) = -\infty. \tag{2.1}$$

For $k \in \mathbb{R}^d$, define the measure μ_k by $d\mu_k(x) = e^{k\cdot x} d\mu(x)$. Suppose $\mathcal{C} \subset \mathbb{R}^d$ is a compact convex set. Then there is a family $(k_j)_{j\in J}$ of elements of $\mathcal{C}$ and a family of two by two disjoint convex sets $(E_{k_j})_{j\in J}$ included in $\mathbb{R}^d$ so that the measures $d\mu_{k_j}$ are concentrated in E_{k_j},

$$\mu_{k_j}(\mathbb{R}^d \setminus (1+T)E_{k_j}) \leqslant \frac{1}{2} e^{-T/C_W} \|\mu_{k_j}\|, \quad \forall T \geqslant 0. \tag{2.2}$$

and such that

$$\sum_j |E_{k_j}|^{-1} \geqslant C_W^{-1} |\mathcal{C}|, \tag{2.3}$$

where C_W is a positive constant depending only on d, and $(1+T)E_{k_j}$ is the dilation of E_{k_j} around its barycentre by a factor of $1+T$.

Lemma 2.1 is the main argument in [18], and the basis for the proof of unique continuation in [18] for potentials V and W_1 in the sharp class of integrability (in [18], the term W_2 is not considered).

2.2 Carleman type estimates.

We first present the Carleman estimates obtained in [6, Theorem 1.1]:

Theorem 2.2 ([6, Theorem 1.1])**.** *Let $d \geqslant 3$. Consider a bounded domain $\Omega \subset \mathbb{R}^d$ of class C^3, and non-empty open subsets ω_0 and ω of Ω with $\omega_0 \Subset \omega \Subset \Omega$. Let $\varphi \in C^3(\overline{\Omega})$ be such that*

$$\forall x \in \partial\Omega,\ \varphi(x) = 0 \text{ and } \partial_n \varphi(x) < 0, \tag{2.4}$$

and there exist $\alpha, \beta > 0$ for which

$$\inf_{\overline{\Omega}\setminus\omega_0} |\nabla\varphi| > \alpha, \tag{2.5}$$

and

$$\forall x \in \overline{\Omega} \setminus \omega_0,\ \forall \xi \in \mathbb{R}^d \text{ with } |\nabla\varphi(x)| = |\xi| \text{ and } \nabla\varphi(x)\cdot\xi = 0,$$
$$(\operatorname{Hess}\varphi(x))\nabla\varphi(x)\cdot\nabla\varphi(x) + (\operatorname{Hess}\varphi(x))\xi\cdot\xi \geqslant \beta|\nabla\varphi(x)|^2, \tag{2.6}$$

where $\operatorname{Hess}\varphi$ denotes the Hessian matrix of φ.

Then there exist $C>0$ and $\tau_0 \geqslant 1$ (depending only on α, β, $\|\varphi\|_{C^3(\overline{\Omega})}$, and the geometric configuration of Ω, ω, and ω_0) such that for all $u \in H_0^1(\Omega)$ satisfying

$$-\Delta u = f_2 + f_{2*'} + \mathit{div}\, F \quad \text{in } \mathscr{D}'(\Omega), \tag{2.7}$$

with $(f_2, f_{2*'}, F)$ satisfying

$$f_2 \in L^2(\Omega), \quad f_{2*'} \in L^{\frac{2d}{d+2}}(\Omega), \quad F \in L^2(\Omega;\mathbb{C}^d), \tag{2.8}$$

we have, for all $\tau \geqslant \tau_0$,

$$\tau^{\frac{3}{2}}\|e^{\tau\varphi}u\|_{L^2(\Omega)} + \tau^{\frac{1}{2}}\|e^{\tau\varphi}\nabla u\|_{L^2(\Omega)} \leqslant C\Big(\|e^{\tau\varphi}f_2\|_{L^2(\Omega)} + \tau\|e^{\tau\varphi}F\|_{L^2(\Omega)} + \tau^{\frac{3}{4}-\frac{1}{2d}}\|e^{\tau\varphi}f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^{\frac{3}{2}}\|e^{\tau\varphi}u\|_{L^2(\omega)} + \tau^{\frac{3}{4}}\|e^{\tau\varphi}u\|_{L^{\frac{2d}{d-2}}(\omega)}\Big), \tag{2.9}$$

and

$$\tau^{\frac{3}{4}+\frac{1}{2d}}\|e^{\tau\varphi}u\|_{L^{\frac{2d}{d-2}}(\Omega)} \leqslant C\Big(\|e^{\tau\varphi}f_2\|_{L^2(\Omega)} + \tau\|e^{\tau\varphi}F\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\|e^{\tau\varphi}f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^{\frac{3}{2}}\|e^{\tau\varphi}u\|_{L^2(\omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\|e^{\tau\varphi}u\|_{L^{\frac{2d}{d-2}}(\omega)}\Big). \tag{2.10}$$

Remark 2.3. *The notations $2*$ and $2*'$ stem from the Sobolev's embedding $H^1(\Omega) \subset L^{2*}(\Omega)$, with $2* = 2d/(d-2)$ and $L^{2*'}(\Omega) \subset H^{-1}(\Omega)$, with $2*' = 2d/(d+2)$.*

In order to get Theorem 1.1, we prove the following refined version of Theorem 2.2, whose proof will be given in Section 3.

Theorem 2.4. *Let $d \geqslant 3$. Consider a bounded domain $\Omega \subset \mathbb{R}^d$, and non-empty open subsets ω_0 and ω of Ω with $\omega_0 \Subset \omega \Subset \Omega$. Let $\varphi \in C^3(\overline{\Omega})$ satisfying conditions (2.5)–(2.6).*

Then, for all compact subset K of Ω, there exist $C>0$ and $\tau_0 \geqslant 1$ (depending only on α, β, $\|\varphi\|_{C^3(\overline{\Omega})}$, and the geometric configuration of Ω, ω, ω_0 and K) such that for all $u \in H^1(\Omega)$ satisfying $\operatorname{supp} u \subset K$ *and (2.7) with $(f_2, f_{2*'}, F = F_2 + F_{2*'})$ satisfying*

$$f_2 \in L^2(\Omega), \quad f_{2*'} \in L^{\frac{2d}{d+2}}(\Omega), \quad F_2 \in L^2(\Omega;\mathbb{C}^d), \quad \text{and} \quad F_{2*'} \in L^{\frac{2d}{d+2}}(\Omega;\mathbb{C}^d) \cap L^2(\Omega;\mathbb{C}^d), \tag{2.11}$$

we have, for all $\tau \geqslant \tau_0$,

$$\tau^{\frac{3}{2}}\|e^{\tau\varphi}u\|_{L^2(\Omega)} + \tau^{\frac{1}{2}}\|e^{\tau\varphi}\nabla u\|_{L^2(\Omega)} \leqslant C\Big(\|e^{\tau\varphi}f_2\|_{L^2(\Omega)} + \tau\|e^{\tau\varphi}F_2\|_{L^2(\Omega)} + \tau^{\frac{1}{2}}\|e^{\tau\varphi}F_{2*'}\|_{L^2(\Omega)} + \tau^{\frac{3}{4}-\frac{1}{2d}}\Big(\|e^{\tau\varphi}f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau\|e^{\tau\varphi}F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}\Big) + \tau^{\frac{3}{2}}\|e^{\tau\varphi}u\|_{H^1(\omega)}\Big), \tag{2.12}$$

and, for all measurable sets E of Ω,

$$\begin{aligned}&\tau^{\frac{3}{4}+\frac{1}{2d}}\|e^{\tau\varphi}u\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\min\left\{\frac{1}{\tau|E|^{\frac{1}{d}}}, 1\right\}\left(\tau\|e^{\tau\varphi}u\|_{L^2(E)} + \|e^{\tau\varphi}\nabla u\|_{L^2(E)}\right)\\ &\leqslant C\Big(\|e^{\tau\varphi}f_2\|_{L^2(\Omega)} + \tau\|e^{\tau\varphi}F_2\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\Big(\|e^{\tau\varphi}f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau\|e^{\tau\varphi}F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \|e^{\tau\varphi}F_{2*'}\|_{L^2(\Omega)}\Big)\\ &\qquad + \tau^{\frac{7}{4}+\frac{1}{2d}}\|e^{\tau\varphi}u\|_{H^1(\omega)}\Big).\end{aligned} \tag{2.13}$$

Remark 2.5. *Theorem 2.4 presents several new features compared to Theorem 2.2:*

- *One of the main difference between Theorem 2.4 and Theorem 2.2 is that Theorem 2.4 allows a source term of the form $\mathit{div}(F_{2*'})$ with $F_{2*'} \in L^{\frac{2d}{d+2}}(\Omega;\mathbb{C}^d) \cap L^2(\Omega;\mathbb{C}^d)$, and quantifies the estimate on the solution u of (2.7) in terms of the $L^{\frac{2d}{d+2}}(\Omega;\mathbb{C}^d) \cap L^2(\Omega;\mathbb{C}^d)$-norm of $F_{2*'}$. Note that the $L^2(\Omega;\mathbb{C}^d)$-norm of $F_{2*'}$ appearing in the right hand sides of (2.12)–(2.13) appears with a power of the Carleman parameter which is strictly smaller than the one appearing for the $L^2(\Omega;\mathbb{C}^d)$-norms of F_2. In some sense, for $F_{2*'}$, the loss of ellipticity of the conjugated Laplace operator $e^{\tau\varphi}\Delta(e^{-\tau\varphi}\cdot)$ appears within the $L^{\frac{2d}{d+2}}(\Omega;\mathbb{C}^d)$-norm of $F_{2*'}$ and not within its $L^2(\Omega;\mathbb{C}^d)$-norm.*

- *Estimate* (2.13) *also presents an estimate of the H^1-norm of u on measurable sets E. Similar improvements appear in the work [18, Section 6]. Note that these estimates are particularly relevant when $|E| \lesssim \tau^{-d}$, that is on small measurable subsets, for which we get from* (2.13) *an estimate on $\tau^{\frac{7}{4}+\frac{1}{2d}}\|e^{\tau\varphi}u\|_{L^2(E)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\|e^{\tau\varphi}\nabla u\|_{L^2(E)}$. This estimate is reasonable and should be compared with the term $\tau^{\frac{3}{4}+\frac{1}{2d}}\|e^{\tau\varphi}u\|_{L^{\frac{2d}{d-2}}(\Omega)}$ since, by Hölder's estimate,*

$$\tau^{\frac{7}{4}+\frac{1}{2d}}\|e^{\tau\varphi}u\|_{L^2(E)} \leqslant \tau^{\frac{3}{4}+\frac{1}{2d}}\|e^{\tau\varphi}u\|_{L^{\frac{2d}{d-2}}(E)}(\tau|E|^{\frac{1}{d}}) \lesssim \tau^{\frac{3}{4}+\frac{1}{2d}}\|e^{\tau\varphi}u\|_{L^{\frac{2d}{d-2}}(\Omega)}.$$

  Note however that for large sets E, the estimate in (2.13) *of the L^2 norm of $e^{\tau\varphi}u$ is worse than the estimate given by* (2.12).
- *Regarding the observation terms, i.e. the terms involving norms on ω, the estimates* (2.12) *and* (2.13) *involve the $H^1(\omega)$-norm of u instead of weaker norms as in* (2.9) *and* (2.10). *In fact, we write it for convenience, as it will be of no impact on the result stated in Theorem 1.1.*
- *In this work, our focus is on deriving local Carleman estimates (in other words, we consider only functions u which are compactly supported). Nevertheless, let us point that, following the proof of Theorem 2.2 in [6] (in particular Subsection 6.4), Theorem 2.4 can be extended to functions $u \in H^1(\Omega)$ with possibly non-homogeneous Dirichlet boundary conditions.*

3 Improved Carleman estimates: Proof of Theorem 2.4

This section is devoted to the proof of Theorem 2.4.

Although Theorem 2.4 looks rather close to Theorem 2.2, we will need to revise in depth the proof of Theorem 2.2 given in [6] and incorporate some changes along the way.

The key step in the proof of Theorem 2.2 is to prove suitable estimates on the inverse of an operator of the form

$$\Delta - x_1 \sum_{j=2}^{d} \lambda_j \partial_j^2 - 2\tau\partial_1 + \tau^2, \tag{3.1}$$

in a vertical strip

$$\Omega = (X_0, X_1) \times \mathbb{R}^{d-1}, \text{ with } X_0 < 0 < X_1 \text{ and } \max\{|X_0|, X_1|\} \leqslant 1, \tag{3.2}$$

where the coefficients $(\lambda_j)_{j\in\{1,\cdots,d\}} \in \mathbb{R}^d$ satisfy $\lambda_1 = 0$,

$$\exists c_0 > 0, \quad \forall x_1 \in [X_0, X_1], \forall \xi \in \mathbb{R}^d, \quad \frac{1}{c_0^2}|\xi|^2 \leqslant \sum_{j=1}^{d}(1 - x_1\lambda_j)|\xi_j|^2 \leqslant c_0^2|\xi|^2, \tag{3.3}$$

and

$$0 < m_* \leqslant \min_{j\in\{2,\cdots,d\}} \lambda_j \leqslant \max_{j\in\{2,\cdots,d\}} \lambda_j \leqslant M_*. \tag{3.4}$$

Here, $\tau \geqslant 1$ plays the role of the Carleman parameter, as the operator in (3.1) coincides with

$$e^{\tau x_1}\left(\Delta - x_1 \sum_{j=2}^{d} \lambda_j \partial_j^2\right)(e^{-\tau x_1}\cdot).$$

This really is the main step of the proof of Theorem 2.2, as one can then use the local character of Carleman estimates to recover Theorem 2.2 in its full generality (see [6] for details; this strategy will be briefly recalled in Section 3.2).

Our proof of Theorem 2.4 follows the same path, and the key estimate is a refined estimate on the inverse of the operator (3.1) in a strip, which will be done in Subsection 3.1. Once this will be done, the proof of Theorem 2.4 can be deduced similarly as in [6] by a suitable localization process, which is rapidly explained in Subsection 3.2 for the convenience of the reader.

3.1 Main step: Carleman estimates for solutions in a strip.

In all this section, Ω is a strip of the form $(X_0, X_1) \times \mathbb{R}^{d-1}$ as in (3.2).

Our goal in this subsection is to prove the following Carleman estimate, which is a refined version of [6, Theorem 2.2].

Theorem 3.1. *Let $\Omega = (X_0, X_1)\times\mathbb{R}^{d-1}$ as in (3.2), with $d \geqslant 3$, and assume that the coefficients $(\lambda_j)_{j\in\{1,\cdots,d\}} \in \mathbb{R}^d$ satisfy $\lambda_1 = 0$, (3.3) and (3.4). Then there exist constants $C > 0$ and $\tau_0 \geqslant 1$ depending on c_0, m_* and M_*, all independent of X_0, X_1, such that for all $\tau \geqslant \tau_0$, for all $w \in H^1(\Omega)$ compactly supported in Ω and satisfying*

$$\Delta w - x_1 \sum_{j=2}^{d} \lambda_j \partial_j^2 w - 2\tau\partial_1 w + \tau^2 w = f_2 + f_{2*'} + div\,(F_2 + F_{2*'}), \qquad in\ \Omega, \tag{3.5}$$

with

$$f_2 \in L^2(\Omega), \quad f_{2*'} \in L^{\frac{2d}{d+2}}(\Omega), \quad F_2 \in L^2(\Omega;\mathbb{C}^d), \quad and \quad F_{2*'} \in L^{\frac{2d}{d+2}}(\Omega;\mathbb{C}^d) \cap L^2(\Omega;\mathbb{C}^d), \tag{3.6}$$

we have

$$\tau^{\frac{3}{2}}\|w\|_{L^2(\Omega)} + \tau^{\frac{1}{2}}\|\nabla w\|_{L^2(\Omega)} \leqslant C\left(\|f_2\|_{L^2(\Omega)} + \tau\|F_2\|_{L^2(\Omega)} + \tau^{\frac{3}{4}-\frac{1}{2d}}(\|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau\|F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}) + \tau^{\frac{1}{2}}\|F_{2*'}\|_{L^2(\Omega)}\right), \tag{3.7}$$

and, and for all measurable subsets E of Ω,

$$\tau^{\frac{3}{4}+\frac{1}{2d}}\|w\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\min\left\{\frac{1}{\tau|E|^{\frac{1}{d}}}, 1\right\}(\tau\|w\|_{L^2(E)} + \|\nabla w\|_{L^2(E)}) \leqslant C\left(\|f_2\|_{L^2(\Omega)} + \tau\|F_2\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\left(\|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau\|F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}\right) + \tau^{\frac{3}{4}+\frac{1}{2d}}\|F_{2*'}\|_{L^2(\Omega)}\right). \tag{3.8}$$

The proof of Theorem 3.1 is of course very close to the one of [6, Theorem 2.2.], as Theorem 3.1 is an improved version of Theorem 2.2 in [6] with respect to the same points as in Remark 2.5.

It will require several steps:

1. Construction of an explicit parametrix giving w solving (3.5) in terms of the source terms $(f_2, f_{2*'}, F = F_2 + F_{2*'})$, based on [6, Proposition 3.1], that we will recall hereafter in Subsubsection 3.1.1.
2. Estimates on the operators involved in the parametrix, which will be also mainly corresponding to the ones in Section 6 of [6], which we also recall. New estimates will also be needed to get the complete estimates of Theorem 3.1, relying on the decomposition of $F_{2*'}$ in its low and high frequency components, see Subsubsection 3.1.2.
3. A suitable combination of the estimates on the various operators involved in the parametrix, see Subsubsection 3.1.3.

In particular, we will rely upon the following explicit parametrix giving w solution of (3.5) in terms of the source terms $(f_2, f_{2*'}, F = F_2 + F_{2*'})$.

3.1.1 An explicit parametrix

We use the parametrix constructed in the work [6]. To do so, we introduce the function $\psi : \overline{\Omega} \to \mathbb{R}$ defined as follows:

$$\psi(x_1, \xi') = \sqrt{\sum_{j=2}^{d}(1 - x_1\lambda_j)\xi_j^2}, \qquad x_1 \in [X_0, X_1],\ \xi' \in \mathbb{R}^{d-1}. \tag{3.9}$$

According to [6, Proposition 3.1], we then have the following explicit parametrix:

Proposition 3.2 ([6, Proposition 3.1])**.** *Under the same setting of Theorem 3.1. For all $\tau \geqslant 1$, if w is compactly supported and satisfies (3.5) with source terms $(f_2, f_{2*'}, F_2, F_{2*'})$ as in (3.6), then*

$$w = K_{\tau,0}(f_2 + f_{2*'}) + \sum_{j=1}^{d} K_{\tau,j}((F_2 + F_{2*'}) \cdot e_j) + R_\tau(w), \tag{3.10}$$

where the family of vectors $(e_j)_{j\in\{1,\cdots,d\}}$ is the canonical basis of $\mathbb{R}^d$ and, using the partial Fourier transform, the operators $K_{\tau,j}$, for $j\in\{0,\cdots,d\}$, and R_τ are defined for f depending on $(x_1,x')\in\Omega$ by

$$\widehat{K_{\tau,j}f}(x_1,\xi')=\int_{y_1\in(X_0,X_1)}k_{\tau,j}(x_1,y_1,\xi')\widehat{f}(y_1,\xi')\,dy_1,\qquad (x_1,\xi')\in\Omega, \tag{3.11}$$

$$\widehat{R_\tau f}(x_1,\xi')=\int_{y_1\in(X_0,X_1)}r_\tau(x_1,y_1,\xi')\widehat{f}(y_1,\xi')\,dy_1,\qquad (x_1,\xi')\in\Omega, \tag{3.12}$$

with kernels given, for $(x_1,y_1,\xi')\in[X_0,X_1]^2\times\mathbb{R}^{d-1}$, by

$$\begin{aligned} k_{\tau,0}(x_1,y_1,\xi') &= -1_{\psi(x_1,\xi')>\tau}\int_{X_0}^{\min\{x_1,y_1\}} e^{-\tau(y_1-x_1)-\int_{\tilde{x}_1}^{x_1}\psi(\tilde{y}_1,\xi')\,d\tilde{y}_1-\int_{\tilde{x}_1}^{y_1}\psi(\tilde{y}_1,\xi')\,d\tilde{y}_1}\,d\tilde{x}_1 \\ &\quad + 1_{\psi(x_1,\xi')\leqslant\tau}1_{y_1>x_1}\int_{x_1}^{y_1} e^{-\tau(y_1-x_1)+\int_{x_1}^{\tilde{x}_1}\psi(\tilde{y}_1,\xi')\,d\tilde{y}_1-\int_{\tilde{x}_1}^{y_1}\psi(\tilde{y}_1,\xi')\,dy_1}\,d\tilde{x}_1, \end{aligned} \tag{3.13}$$

$$\begin{aligned} k_{\tau,1}(x_1,y_1,\xi') &= -1_{\psi(x_1,\xi')\leqslant\tau}1_{y_1>x_1}e^{-\tau(y_1-x_1)+\int_{x_1}^{y_1}\psi(\tilde{y}_1,\xi')\,d\tilde{y}_1} \\ &\quad + 1_{\psi(x_1,\xi')>\tau}1_{y_1<x_1}e^{\tau(x_1-y_1)-\int_{y_1}^{x_1}\psi(\tilde{y}_1,\xi')\,d\tilde{y}_1}+k_{\tau,0}(x_1,y_1,\xi')(\tau+\psi(y_1,\xi')), \end{aligned} \tag{3.14}$$

$$k_{\tau,j}(x_1,y_1,\xi') = \mathbf{i}\xi_j k_{\tau,0}(x_1,y_1,\xi'),\qquad j\in\{2,\cdots,d\}, \tag{3.15}$$

$$r_\tau(x_1,y_1,\xi) = k_{\tau,0}(x_1,y_1,\xi')\partial_1\psi(y_1,\xi'). \tag{3.16}$$

The key to prove Proposition 3.2 is to remark that the partial Fourier transform of the operator in (3.1) is of the form

$$(\partial_1-\tau)^2-\sum_{j=2}^d|\xi_j|^2(1-x_1\lambda_j)=(\partial_1-\tau-\psi(x_1,\xi'))(\partial_1-\tau+\psi(x_1,\xi'))-\partial_1\psi(x_1,\xi').$$

It is then clear that the set in which the estimates may degenerate is the set $\{(x_1,\xi')\in(X_0,X_1)\times\mathbb{R}^{d-1},\text{ with }\psi(x_1,\xi')=\tau\}$. Accordingly, it is interesting to use a kind of projection operator $P_{hf,\tau}$ on the high-frequency components acting on $L^2(\Omega)$ and given in Fourier, for $f\in L^2(\Omega)$, by the formula

$$\widehat{P_{hf,\tau}f}(x_1,\xi')=\eta\left(\frac{\psi(X_1,\xi')}{\tau}\right)\widehat{f}(x_1,\xi'),\qquad (x_1,\xi')\in(X_0,X_1)\times\mathbb{R}^{d-1}, \tag{3.17}$$

where η is a smooth function in $\mathscr{C}^\infty([0,\infty),\mathbb{R})$, taking value 0 in $[0,2]$, taking value 1 outside $[0,3]$, and bounded by 1.

Note that since the operator $P_{hf,\tau}$ corresponds to a convolution in the x' variable with a L^1 kernel $x'\mapsto\tau^{d-1}\eta_1(\tau x')$ where $\eta_1(z')=\frac{1}{(2\pi)^{(d-1)/2}}\int_{\mathbb{R}^{d-1}}e^{\mathbf{i}z'\cdot\xi'}\eta(\psi(X_1,\xi'))\,d\xi'$, one can check through a simple scaling argument that, for all $p\in[1,\infty]$, there exists C_p such that for all $f\in L^p(\Omega)$,

$$\|P_{hf,\tau}f\|_{L^p(\Omega)}\leqslant C_p\|f\|_{L^p(\Omega)}.$$

This operator presents the advantage of localizing in the frequencies ξ' such that for all $x_1\in(X_0,X_1)$, $\psi(x_1,\xi')\geqslant\psi(X_1,\xi')\geqslant 2\tau$, and thus far away from the critical set $\{(x_1,\xi')\in(X_0,X_1)\times\mathbb{R}^{d-1},\text{ with }\psi(x_1,\xi')=\tau\}$. It is also easy to check that it commutes with all the operators $(K_{\tau,j})_{j\in\{0,\cdots,d\}}$ and R_τ.

This operator can be used in particular on $F_{2*'}$, that we will write as

$$F_{2*'}=P_{hf,\tau}F_{2*'}+(I-P_{hf,\tau})F_{2*'}.$$

Using the notations $F'_{2*'}$ to denote the last $d-1$ components of $F_{2*'}$ and div' to denote the divergence operator on $\mathbb{R}^{d-1}_{x'}$, it is easy to check that $\operatorname{div}'((I-P_{hf,\tau})F'_{2*'})$ belongs to $L^2(\Omega)$, and thus the identity (3.10) can be written as

$$\begin{aligned} w &= K_{\tau,0}(f_2+f_{2*'}+\operatorname{div}'((I-P_{hf,\tau})F'_{2*'}))+K_{\tau,1}((F_2+P_{hf,\tau}F_{2*'})\cdot e_1) \\ &\quad + K_{\tau,1}(((I-P_{hf,\tau})F_{2*'})\cdot e_1)+\sum_{j=2}^d K_{\tau,j}((F_2+P_{hf,\tau}F_{2*'})\cdot e_j)+R_\tau(w). \end{aligned} \tag{3.18}$$

This is one of the formula that we will use next. Note that it involves all the operators $K_{\tau,j}$ and R_τ appearing in Proposition 3.2, so that Theorem 3.1 will be derived using the known estimates on these operators obtained in [6] and the suitable gains that we will have by considering how these operators act at low and high-frequencies.

3.1.2 Boundedness of the operators $K_{\tau,j}$

Our proof will be based on estimates for each of the operators $(K_{\tau,j})_{j\in\{0,\cdots,d\}}$.

Known estimates. Several estimates have already been obtained in [6] and are recalled here:

Proposition 3.3 (Proposition 6.2 in [6]). *Under the setting of Theorem 3.1. There exist $C>0$ and $\tau_0 \geqslant 1$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (3.3) and (3.4)) such that for all $\tau \geqslant \tau_0$, for all $f \in L^{\frac{2d}{d+2}}(\Omega)$,*

$$\|K_{\tau,0}f\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\|K_{\tau,0}f\|_{L^2(\Omega)} + \|\partial_1 \widehat{K_{\tau,0}f}\|_{L^2(\Omega_{1,\tau})} + \tau^{-\frac{1}{4}+\frac{1}{2d}}\|\nabla' K_{\tau,0}f\|_{L^2(\Omega)} \leqslant C\|f\|_{L^{\frac{2d}{d+2}}(\Omega)}, \tag{3.19}$$

and, for all $f \in L^2(\Omega)$,

$$\tau^{\frac{3}{4}+\frac{1}{2d}}\|K_{\tau,0}f\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{3}{2}}\|K_{\tau,0}f\|_{L^2(\Omega)} + \tau\|\partial_1 \widehat{K_{\tau,0}f}\|_{L^2(\Omega_{1,\tau})} + \tau^{\frac{1}{2}}\|\nabla' K_{\tau,0}f\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}, \tag{3.20}$$

with

$$\Omega_{1,\tau} = \{(x_1,\xi') \in (X_0,X_1)\times\mathbb{R}^{d-1}, \text{ with } \psi(x_1,\xi') \neq \tau\}. \tag{3.21}$$

Proposition 3.4 (Proposition 6.7 and 6.10 in [6]). *Under the setting of Theorem 3.1. There exist $C>0$ and $\tau_0 \geqslant 1$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (3.3) and (3.4)) such that for all $j \in \{1,\cdots,d\}$, for all $\tau \geqslant \tau_0$ and for all $f \in L^2(\Omega)$,*

$$\tau^{-\frac{1}{4}+\frac{1}{2d}}\|K_{\tau,j}f\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{1}{2}}\|K_{\tau,j}f\|_{L^2(\Omega)} + \|\partial_1 \widehat{K_{\tau,j}f}\|_{L^2(\Omega_{1,\tau})} + \tau^{-\frac{1}{2}}\|\nabla' K_{\tau,j}f\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}.$$

Proposition 3.5 (Proposition 6.13 in [6]). *Under the setting of Theorem 3.1. There exist $C>0$ and $\tau_0 \geqslant 1$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (3.3) and (3.4)) such that for all $\tau \geqslant \tau_0$ and for all $f \in H^1(\Omega)$,*

$$\tau^{\frac{3}{4}+\frac{1}{2d}}\|R_\tau f\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{3}{2}}\|R_\tau f\|_{L^2(\Omega)} + \tau\|\partial_1 \widehat{R_\tau f}\|_{L^2(\Omega_{1,\tau})} + \tau^{\frac{1}{2}}\|\nabla' R_{\tau,0}f\|_{L^2(\Omega)} \leqslant C\|\nabla' f\|_{L^2(\Omega)}.$$

It is natural to obtain better estimates for the operators $(P_{hf,\tau}K_{\tau,j})_{j\in\{0,\cdots,d\}}$ than for the operators $(K_{\tau,j})_{j\in\{0,\cdots,d\}}$, since the high-frequency projection operator $P_{hf,\tau}$ is a projection which projects on the part in which the conjugated operator in (3.1) is elliptic. Although such estimates are known and rather classical in the Hilbertian setting, this needs to be made precise when trying to get estimates on these operators from $L^p(\Omega)$ to $L^q(\Omega)$ when p or q is different from 2 (we refer to [9] for estimates of that kind in a closely related context). This is precisely our next goal.

High-frequency estimates. We list below the new estimates we obtain on the operators $(K_{\tau,j})_{j\in\{0,\cdots,d\}}$ at high frequencies, to be compared with the ones in Propositions 3.3 and 3.4. These will be proved next.

Proposition 3.6. *Under the setting of Theorem 3.1. There exist $C>0$ and $\tau_0 \geqslant 1$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (3.3) and (3.4)) such that for all $\tau \geqslant \tau_0$, for all $f \in L^{\frac{2d}{d+2}}(\Omega)$,*

$$\tau\|P_{hf,\tau}K_{\tau,0}f\|_{L^2(\Omega)} + \|\nabla P_{hf,\tau}K_{\tau,0}f\|_{L^2(\Omega)} \leqslant C\|f\|_{L^{\frac{2d}{d+2}}(\Omega)}, \tag{3.22}$$

and, for all $f \in L^2(\Omega)$,

$$\tau\|P_{hf,\tau}K_{\tau,0}f\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^2\|P_{hf,\tau}K_{\tau,0}f\|_{L^2(\Omega)} + \tau\|\nabla P_{hf,\tau}K_{\tau,0}f\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}. \tag{3.23}$$

Remark 3.7. *It is interesting to compare the estimates in Proposition 3.6 to the ones in Proposition 3.3. In particular, one sees that for the Hilbertian estimates, i.e. for the $\mathscr{L}(L^2(\Omega), H^1(\Omega))$ norm of $P_{hf}K_{\tau,0}$, the estimates (3.23) are better than the ones in (3.23) for $K_{\tau,0}$ by a factor $\tau^{\frac{1}{2}}$. This improvement is weaker for the estimates in the $\mathscr{L}(H^1(\Omega), L^{\frac{2d}{d-2}}(\Omega))$ and $\mathscr{L}(L^{\frac{2d}{d+2}}(\Omega), H^1(\Omega))$-norms of $P_{hf}K_{\tau,0}$, which still gains a factor $\tau^{\frac{1}{4}-\frac{1}{2d}}$ compared to the $\mathscr{L}(H^1(\Omega), L^{\frac{2d}{d-2}}(\Omega))$ and $\mathscr{L}(L^{\frac{2d}{d+2}}(\Omega), H^1(\Omega))$-norms of $K_{\tau,0}$.*

Proposition 3.8. *Under the setting of Theorem 3.1. There exist $C > 0$ and $\tau_0 \geqslant 1$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (3.3) and (3.4)), such that for all $j \in \{1, \cdots, d\}$, for all $\tau \geqslant \tau_0$ and for all $f \in L^2(\Omega)$,*

$$\|P_{hf,\tau}K_{\tau,j}f\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau\|P_{hf,\tau}K_{\tau,j}f\|_{L^2(\Omega)} + \|\nabla P_{hf,\tau}K_{\tau,j}f\|_{L^2(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}. \tag{3.24}$$

Remark 3.9. *Here again, as in Remark 3.7, comparing the estimates in Proposition 3.8 to the ones in Proposition 3.4, we see that, for $j \geqslant 1$, there is again of a factor $\tau^{\frac{1}{4}-\frac{1}{2d}}$ when considering the $\mathscr{L}(L^2(\Omega), L^{\frac{2d}{d-2}}(\Omega))$-norm of the operator $P_{hf,\tau}K_{\tau,j}$ compared to the norm of $K_{\tau,j}$, and of a factor $\tau^{\frac{1}{2}}$ when considering the $\mathscr{L}(L^2(\Omega), H^1(\Omega))$-norms.*

Before going into the proofs of Propositions 3.6 and 3.8, we point out that each operator $K_{\tau,j}$ for $j \in \{0, \cdots, d\}$ is a Fourier multiplier operator in the vertical variable. We can therefore use the Stein-Tomas restriction Theorem [16] to estimate their behavior as an operator from $L^p(\mathbb{R}^{d-1})$ to $L^q(\mathbb{R}^{d-1})$ for some values of p and q. This approach is briefly recalled in Appendix A with a suitable parametrization of the phase space $\xi' \in \mathbb{R}^{d-1}$ as $\xi' \mapsto (\psi(a, \xi'), \xi'/\psi(a, \xi'))$ for ψ as in (3.9) and $a \in [X_0, X_1]$ adapted to the kernels appearing in Proposition 3.2. The full details can be found in [6, Section 5].

Proof of Proposition 3.6. In view of the results in Proposition A.1, we first estimate weighted norms of $k_{\tau,0}(x_1, y_1, \cdot)$ for x_1 and y_1 in $[X_0, X_1]$ (recall the definition of $k_{\tau,0}$ in (3.13)). We also identify $\xi' \in \mathbb{R}^{d-1}$ with pairs $(\lambda, \omega') \in \mathbb{R}_+ \times \Sigma_{x_1}$, where $\Sigma_{x_1} = \{\omega' \in \mathbb{R}^{d-1}, \psi(x_1, \omega') = 1\}$, through the formula $\xi' = \lambda\omega'$, or equivalently $\lambda = \psi(x_1, \xi')$ and $\omega' = \xi'/\psi(x_1, \xi')$. With a slight abuse of notations, we denote $k_{\tau,0}$ similarly whether it is written in terms of $\xi' \in \mathbb{R}^{d-1}$ or in terms of $(\lambda, \omega') \in \mathbb{R}_+ \times \Sigma_{x_1}$.

From [6, Lemma 6.4], there exists a constant $C > 0$ such that for all $\tau \geqslant 1$, $\lambda > \tau$, and all $x_1, y_1 \in [X_0, X_1]$, we have

$$\|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})} \leqslant \begin{cases} \dfrac{C}{\lambda} e^{-(\lambda-\tau)|x_1-y_1| - \lambda(x_1-y_1)^2/C_1}, & \text{if } y_1 < x_1, \\ \dfrac{C}{\lambda} e^{-(\lambda/C+\tau)|y_1-x_1|}, & \text{if } y_1 > x_1. \end{cases} \tag{3.25}$$

Arguing as in [6, Lemma 6.6], we deduce

$$\left(\int_{2\tau}^\infty \|k_{\tau,0}(x_1, y_1, \lambda, .)\|^2_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}} d\lambda \right)^{\frac{1}{2}} \leqslant Ce^{-\tau|y_1-x_1|}\tau^{-\frac{1}{d}}. \tag{3.26}$$

Accordingly, using Proposition A.1, Young's inequality and the fact that for all $x_1 \in [X_0, X_1]$, $\lambda = \psi(x_1, \xi') \geqslant \psi(X_1, \xi') \geqslant 2\tau$ due to (3.4), we have, for $f \in L^{\frac{2d}{d+2}}(\Omega)$ and $\tau \geqslant 1$,

$$\begin{aligned}
\|P_{hf,\tau}K_{\tau,0}f\|_{L^2(\Omega)} &\leqslant \left\| \|P_{hf,\tau}K_{\tau,0}f(x_1, \cdot)\|_{L^2_{x'}(\mathbb{R}^{d-1})} \right\|_{L^2_{x_1}(X_0,X_1)} \\
&\leqslant \left\| \int_{X_0}^{X_1} \left(\int_{2\tau}^\infty \|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|^2_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}}\, d\lambda \right)^{\frac{1}{2}} \|f(y_1, \cdot)\|_{L^{\frac{2d}{d+2}}_{y'}(\mathbb{R}^{d-1})} dy_1 \right\|_{L^2_{x_1}(X_0,X_1)} \\
&\leqslant C \left\| \left(y_1 \mapsto e^{-\tau|y_1|}\tau^{-\frac{1}{d}} \right) \star_{y_1} \|f(y_1, \cdot)\|_{L^{\frac{2d}{d+2}}_{y'}(\mathbb{R}^{d-1})} \right\|_{L^2_{x_1}(X_0,X_1)} \\
&\leqslant C \left\| y_1 \mapsto e^{-\tau|y_1|}\tau^{-\frac{1}{d}} \right\|_{L^{\frac{d}{d-1}}(\mathbb{R})} \left\| \|f(y_1, \cdot)\|_{L^{\frac{2d}{d+2}}_{y'}(\mathbb{R}^{d-1})} \right\|_{L^{\frac{2d}{d+2}}_{y_1}(X_0,X_1)} = C\tau^{-1} \|f\|_{L^{\frac{2d}{d+2}}(\Omega)}.
\end{aligned} \tag{3.27}$$

Similarly, we get that for all $f \in L^2(\Omega)$ and $\tau \geqslant 1$,

$$\|P_{hf,\tau} K_{\tau,0} f\|_{L^{\frac{2d}{d-2}}(\Omega)} \leqslant C\tau^{-1} \|f\|_{L^2(\Omega)} . \tag{3.28}$$

Similarly, arguing as in [6, Lemma 6.6], we get

$$\left(\int_{2\tau}^{\infty} \|\lambda \omega' k_{\tau,0}(x_1, y_1, \lambda, \omega')\|^2_{L^\infty_{\omega'}(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}} d\lambda\right)^{\frac{1}{2}} \leqslant C|y_1 - x_1|^{-1+\frac{1}{d}}. \tag{3.29}$$

Using Proposition A.1 and the Hardy-Littlewood-Sobolev theorem in 1-d, we then get, for all $f \in L^{\frac{2d}{d+2}}(\Omega)$ and $\tau \geqslant 1$,

$$\begin{aligned}
\|\nabla' P_{hf,\tau} K_{\tau,0} f\|_{L^2(\Omega)} &\leqslant \left\| \|\nabla' P_{hf,\tau} K_{\tau,0} f(x_1, \cdot)\|_{L^2_{x'}(\mathbb{R}^{d-1})} \right\|_{L^2_{x_1}(X_0, X_1)} \\
&\leqslant \left\| \int_{X_0}^{X_1} \left(\int_{2\tau}^{\infty} \|\lambda k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|^2_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}} \, d\lambda\right)^{\frac{1}{2}} \|f(y_1, \cdot)\|_{L^{\frac{2d}{d+2}}_{y'}(\mathbb{R}^{d-1})} \, dy_1 \right\|_{L^2_{x_1}(X_0, X_1)} \\
&\leqslant C \left\| \left(y_1 \mapsto |y_1|^{-1+\frac{1}{d}}\right) \star_{y_1} \|f(y_1, \cdot)\|_{L^{\frac{2d}{d+2}}_{y'}} (\mathbb{R}^{d-1}) \right\|_{L^2_{x_1}(X_0, X_1)} \\
&\leqslant C \left\| \|f(y_1, \cdot)\|_{L^{\frac{2d}{d+2}}_{y'}(\mathbb{R}^{d-1})} \right\|_{L^{\frac{2d}{d+2}}_{y_1}(X_0, X_1)} = C \|f\|_{L^{\frac{2d}{d+2}}(\Omega)} .
\end{aligned} \tag{3.30}$$

We conclude the estimate (3.22) by combining (3.27), (3.30) and (3.19) for the estimate of $\partial_1 P_{hf,\tau} K_{\tau,0} f$ for $f \in L^{\frac{2d}{d+2}}(\Omega)$.

To conclude (3.23), in view of (3.28), we only need to estimate the $\mathscr{L}(L^2(\Omega))$ and $\mathscr{L}(L^2(\Omega), H^1(\Omega))$-norms of $P_{hf,\tau} K_{\tau,0}$. These are easier since the estimate (3.25) yields that there exists a constant $C > 0$ such that for all $\tau \geqslant 1$, $\lambda > 2\tau$, and all $x_1, y_1 \in [X_0, X_1]$, we have

$$(\tau + \lambda)\|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})} \leqslant Ce^{-\tau|x_1 - y_1|} \tag{3.31}$$

We then immediately get, by Young's inequality, that for $f \in L^2(\Omega)$ and $\tau \geqslant 1$,

$$\begin{aligned}
&\tau\|P_{hf,\tau} K_{\tau,0} f\|_{L^2(\Omega)} + \|\nabla' P_{hf,\tau} K_{\tau,0} f\|_{L^2(\Omega)} \\
&\quad\leqslant \tau \left\| \|P_{hf,\tau} K_{\tau,0} f(x_1, \cdot)\|_{L^2_{x'}(\mathbb{R}^{d-1})} \right\|_{L^2_{x_1}(X_0, X_1)} + \left\| \|\nabla' P_{hf,\tau} K_{\tau,0} f(x_1, \cdot)\|_{L^2_{x'}(\mathbb{R}^{d-1})} \right\|_{L^2_{x_1}(X_0, X_1)} \\
&\quad\leqslant C \left\| \int_{X_0}^{X_1} e^{-\tau|x_1 - y_1|} \|f(y_1, \cdot)\|_{L^2_{y'}(\mathbb{R}^{d-1})} \, dy_1 \right\|_{L^2_{x_1}(X_0, X_1)} \leqslant C\tau^{-1} \|f\|_{L^2(\Omega)} .
\end{aligned} \tag{3.32}$$

This estimate, together with (3.28) and the estimate (3.20) for the estimate of $\partial_1 P_{hf,\tau} K_{\tau,0} f$ for $f \in L^2(\Omega)$, gives the estimate (3.23). □

Proof of Proposition 3.8. We only sketch the proof of Proposition 3.8 since it relies on similar arguments as the ones used in the proof of Proposition 3.6.

Let us explain the main steps to get the estimate (3.24). Using [6, Lemma 6.8], we get a constant $C > 0$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (3.3) and (3.4)), such that for all x_1 and y_1 in $[X_0, X_1]$, for all $\tau \geqslant 1$, and $\lambda > 0$,

$$\|k_{\tau,1}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})} \leqslant Ce^{-|\tau-\lambda||y_1-x_1| - \lambda(y_1-x_1)^2/C} + C(\tau + \lambda)\|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})}. \tag{3.33}$$

and, for $j \in \{2, \cdots, d\}$,

$$\|k_{\tau,j}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})} \leqslant C\lambda\|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})}. \tag{3.34}$$

The estimate (3.29) then yields the existence of a constant $C > 0$ such that for all $\tau \geqslant 1$, $j \in \{2, \cdots, d\}$, and all $x_1, y_1 \in [X_0, X_1]$,

$$\left(\int_{2\tau}^{\infty} \|k_{\tau,j}(x_1, y_1, \lambda, .)\|^2_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}} d\lambda \right)^{\frac{1}{2}} \leqslant C|y_1 - x_1|^{-1+\frac{1}{d}}. \tag{3.35}$$

Similarly, one can derive from (3.33) that this also holds for $j = 1$.

Using Proposition A.1 and the Hardy-Littlewood-Sobolev theorem, we then deduce that there exists $C > 0$ such that for all $j \in \{1, \cdots, d\}$, for all $\tau \geqslant 1$ and for all $f \in L^2(\Omega)$,

$$\|P_{hf,\tau} K_{\tau,j} f\|_{L^{\frac{2d}{d-2}}(\Omega)} \leqslant C\|f\|_{L^2(\Omega)}.$$

The estimates on the $\mathscr{L}(L^2(\Omega))$-norms of $P_{hf,\tau}K_{\tau,j}$ and $\nabla' P_{hf,\tau}K_{\tau,j}$ can be achieved more easily and are left to the reader, and the $\mathscr{L}(L^2(\Omega))$-norm of $\partial_1 P_{hf,\tau}K_{\tau,j}$ follows from Proposition 3.4. $\square$

Low-frequency estimates. In our arguments next, we will also need to understand the behavior of the operator $(I - P_{hf,\tau})K_{\tau,1}$ and show how it acts on $L^{\frac{2d}{d+2}}(\Omega)$:

Proposition 3.10. *Under the setting of Theorem 3.1. There exist $C > 0$ and $\tau_0 \geqslant 1$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (3.3) and (3.4)) such that for all $\tau \geqslant \tau_0$ and for all $f \in L^{\frac{2d}{d+2}}(\Omega)$,*

$$\begin{aligned} &\|(I - P_{hf,\tau})K_{\tau,1} f\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \|(I - P_{hf,\tau})K_{\tau,1} f\|_{L^2(\Omega)} \\ &\quad + \|\partial_1 \widehat{(I - P_{hf,\tau})} K_{\tau,1} f\|_{L^2(\Omega_{1,\tau})} + \tau^{-\frac{1}{4}+\frac{1}{2d}} \|\nabla'(I - P_{hf,\tau})K_{\tau,1} f\|_{L^2(\Omega)} \leqslant C\tau \|f\|_{L^{\frac{2d}{d+2}}(\Omega)}, \end{aligned} \tag{3.36}$$

where $\Omega_{1,\tau}$ is the set defined in (3.21).

Proof. Similarly as in the previous proofs, we will use suitable bounds on the kernel $k_{\tau,1}$. Namely, we will use the bound (3.33), the bound (3.25) and the following bound, obtained in [6, Lemma 6.4]: There exist constants $C > 0$ and $C_1 > 0$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (3.3) and (3.4)) such that for all x_1 and y_1 in $[X_0, X_1]$, for all $\tau \geqslant 1$, and $\lambda \leqslant \tau$, the kernel $k_{\tau,0}$ defined in (3.13) satisfies

$$\|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})} \leqslant \begin{cases} C|y_1 - x_1| e^{-\tau|y_1 - x_1|}, & \text{if } \lambda|y_1 - x_1| \leqslant 1, \\ \dfrac{C}{\lambda} e^{-(\tau-\lambda)|y_1 - x_1| - \lambda(y_1 - x_1)^2/C_1}, & \text{if } \lambda|y_1 - x_1| \geqslant 1. \end{cases} \tag{3.37}$$

Lemma 6.9 in [6] states that there exists a constant $C > 0$ independent of X_0, X_1 (and depending only on c_0, m_* and M_* in (3.3) and (3.4)) such that for all x_1 and y_1 in $[X_0, X_1]$, for all $\tau \geqslant 1$,

$$\begin{gathered} \left(\int_{\lambda > 0} \|k_{\tau,1}(x_1, y_1, \lambda, \cdot)\|^2_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}} d\lambda \right)^{\frac{1}{2}} \leqslant C \frac{1}{|x_1 - y_1|^{1-\frac{1}{d}}} + \widetilde{k}_{\tau,1}(x_1 - y_1), \\ \text{with } \widetilde{k}_{\tau,1} \in L^{\frac{d}{d-1}}(\mathbb{R}) \text{ and } \|\widetilde{k}_{\tau,1}\|_{L^{\frac{d}{d-1}}(\mathbb{R})} \leqslant C\tau^{\frac{1}{4}-\frac{1}{2d}}. \end{gathered} \tag{3.38}$$

Using then Proposition A.1, the Hardy-Littlewood-Sobolev theorem and Young's inequality, we deduce that the $\mathscr{L}(L^{\frac{2d}{d+2}}(\Omega), L^2(\Omega))$-norm of $(I - P_{hf,\tau})K_{\tau,1}$ is bounded by $C\tau^{\frac{1}{4}-\frac{1}{2d}}$.

To get a bound on the $\mathscr{L}(L^{\frac{2d}{d+2}}(\Omega), L^{\frac{2d}{d-2}}(\Omega))$-norm of $(I - P_{hf,\tau})K_{\tau,1}$ (recall that $I - P_{hf,\tau}$ localizes at frequency ξ' such that $\psi(X_1, \xi') \leqslant 3\tau$), we first show that there exists a constant $C > 0$, such that for all x_1 and y_1 in $[X_0, X_1]$, for all $\tau \geqslant 1$,

$$\begin{aligned} &\int_0^{3c_1\tau} \|k_{\tau,1}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}} d\lambda \\ &\leqslant C \int_0^{3c_1\tau} e^{-|\tau-\lambda||y_1 - x_1| - \lambda(y_1 - x_1)^2/C} \lambda^{1-\frac{2}{d}} d\lambda + \tau C \int_0^{3c_1\tau} \|k_{\tau,0}(x_1, y_1, \lambda, \cdot)\|_{L^\infty(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}} d\lambda \\ &\leqslant C\tau |x_1 - y_1|^{-1+\frac{2}{d}}, \end{aligned}$$

where the bound on the first term on the right-hand side is derived through simple calculations similar to [6, Section 6]. For the second term, we have used [6, Lemma 6.6], where c_1 is defined by

$$c_1 = \sup_{x_1\in[X_0,X_1]} \sup\{\psi(x_1,\xi'), \text{ s. t. } \psi(X_1,\xi')=1\}.$$

Using then Proposition A.1 and the Hardy-Littlewood-Sobolev theorem, the $\mathscr{L}(L^{\frac{2d}{d+2}}(\Omega), L^{\frac{2d}{d-2}}(\Omega))$-norm of $(I-P_{hf,\tau})K_{\tau,1}$ is bounded by $C\tau$.

For the estimate on the $\mathscr{L}(L^{\frac{2d}{d+2}}(\Omega), L^{\frac{2d}{d-2}}(\Omega))$-norm $\nabla'(I-P_{hf,\tau})K_{\tau,1}$, the crucial point is to prove that there exists a constant $C>0$ such that for all $x_1, y_1 \in [X_0, X_1]$, and for all $\tau \geqslant 1$,

$$\left(\int_0^{3c_1\tau} \|\lambda\omega' k_{\tau,1}(x_1,y_1,\lambda,\omega')\|^2_{L^\infty_{\omega'}(\Sigma_{x_1})} \lambda^{1-\frac{2}{d}}\, d\lambda\right)^{\frac12} \leqslant C\tau^{\frac54-\frac{1}{2d}}|x_1-y_1|^{-1+\frac1d}. \tag{3.39}$$

If so, using again Proposition A.1 and the Hardy-Littlewood-Sobolev inequality, the $\mathscr{L}(L^{\frac{2d}{d+2}}(\Omega), L^{\frac{2d}{d-2}}(\Omega))$-norm $\nabla'(I-P_{hf,\tau})K_{\tau,1}$ is bounded by $\tau^{\frac54-\frac{1}{2d}}$.

To prove inequality (3.39), we can bound the term on the left-hand side using (3.33) as follows

$$C\left(\int_0^{3c_1\tau} e^{-|\tau-\lambda||y_1-x_1|-\lambda(y_1-x_1)^2/C}\lambda^{3-\frac2d}d\lambda\right)^{\frac12} + C\tau\left(\int_0^{3c_1\tau}\|\lambda\omega' k_{\tau,0}(x_1,y_1,\lambda,\omega')\|^2_{L^\infty_{\omega'}(\Sigma_{x_1})}\lambda^{1-\frac2d}d\lambda\right)^{\frac12}.$$

Then, using [6, Lemma 6.6], we obtain the desired estimate on the second term. A straightforward computation yields the same bound on the first term.

The estimate the $\mathscr{L}(L^{\frac{2d}{d+2}}(\Omega), L^{\frac{2d}{d-2}}(\Omega))$-norm of $\partial_1(I-P_{hf,\tau})K_{\tau,1}$ is more technical. First, we compute the kernel of the operator $\partial_1 K_{\tau,1}$, denoted by $k_{\tau,1,\partial_1}$: for all $(x_1,\xi')\in\Omega_{1,\tau}$ and all $y_1\in(X_0,X_1)$,

$$\begin{aligned} k_{\tau,1,\partial_1}(x_1,y_1,\xi') = &-1_{\psi(x_1,\xi')\leqslant\tau}1_{y_1>x_1}(\tau-\psi(x_1,\xi'))e^{-\tau(y_1-x_1)+\int_{x_1}^{y_1}\psi(\tilde y_1,\xi')\,d\tilde y_1}\\ &+1_{\tau<\psi(x_1,\xi')}1_{y_1<x_1}(\tau-\psi(x_1,\xi'))e^{\tau(x_1-y_1)-\int_{y_1}^{x_1}\psi(\tilde y_1,\xi')\,d\tilde y_1}\\ &+k_{\tau,0,\partial_1}(x_1,y_1,\xi')(\tau+\psi(y_1,\xi')), \end{aligned} \tag{3.40}$$

where $k_{\tau,0,\partial_1}$ is defined for $x_1,\ y_1$ in $[X_0,X_1]$ and $\xi'\in\mathbb{R}^{d-1}$, by

$$\begin{aligned} k_{\tau,0,\partial_1}(x_1,y_1,\xi') = &-1_{x_1<y_1}e^{-\tau(y_1-x_1)-\int_{x_1}^{y_1}\psi(\tilde y_1,\xi')\,d\tilde y_1}\\ &-1_{\psi(x_1,\xi')>\tau}(\tau-\psi(x_1,\xi'))\int_{X_0}^{\min\{x_1,y_1\}} e^{-\tau(y_1-x_1)-\int_{\tilde x_1}^{x_1}\psi(\tilde y_1,\xi')\,d\tilde y_1-\int_{\tilde x_1}^{y_1}\psi(\tilde y_1,\xi')\,d\tilde y_1}d\tilde x_1\\ &+1_{\psi(x_1,\xi')\leqslant\tau}1_{x_1<y_1}(\tau-\psi(x_1,\xi'))\int_{x_1}^{y_1} e^{-\tau(y_1-x_1)+\int_{x_1}^{\tilde x_1}\psi(\tilde y_1,\xi')\,d\tilde y_1-\int_{\tilde x_1}^{y_1}\psi(\tilde y_1,\xi')\,d\tilde y_1}d\tilde x_1. \end{aligned}$$

The kernel of $\partial_1(I-P_{hf,\tau})K_{\tau,1}$, denoted by $k_{\tau,1,\partial_1,lf}$ is then given by

$$k_{\tau,1,\partial_1,lf}(x_1,y_1,\xi') = \left(1-\eta\left(\frac{\psi(X_1,\xi')}{\tau}\right)\right)k_{\tau,1,\partial_1}(x_1,y_1,\xi'). \tag{3.41}$$

A tedious computation (similar to the ones in [6, Lemma 6.4]) then shows that there exists a constant $C>0$ independent of X_0 and X_1, such that for all x_1 and y_1 in $[X_0,X_1]$, for all $\tau\geqslant1$, and $0<\lambda<3c_1\tau$,

$$\|k_{\tau,1,\partial_1,lf}(x_1,y_1,\lambda,\cdot)\|_{L^\infty(\Sigma_{x_1})} \leqslant C\tau\left(e^{-|\tau-\lambda||y_1-x_1|-\lambda(y_1-x_1)^2/C} + \|k_{\tau,0,\partial_1}(x_1,y_1,\lambda,\cdot)\|_{L^\infty(\Sigma_{x_1})}\right).$$

As a consequence, using [6, Lemma 6.6] to bound the second term on the right-hand side of the last inequality, we can prove

$$\left(\int_0^{3c_1\tau}\|k_{\tau,1,\partial_1}(x_1,y_1,\lambda,\cdot)\|^2_{L^\infty(\Sigma_{x_1})}\lambda^{1-\frac2d}\,d\lambda\right)^{\frac12} \leqslant C\tau|x_1-y_1|^{-1+\frac1d}. \tag{3.42}$$

Using the Hardy-Littlewood-Sobolev inequality then yields that the $\mathscr{L}(L^{\frac{2d}{d+2}}(\Omega), L^{\frac{2d}{d-2}}(\Omega))$-norm of $\partial_1(I-P_{hf,\tau})K_{\tau,1}$ is bounded by $C\tau$. This concludes the proof of Proposition 3.10. ☐

3.1.3 Proof of Theorem 3.1

With the various estimates established in the propositions presented in the previous subsection, we are now in a position to prove Theorem 3.1, that is the Carleman estimates in the strip.

Proof of Theorem 3.1. Using Proposition 3.2, if w is compactly supported and satisfies (3.5) for some source terms $(f_2, f_{2*'}, F_2, F_{2*'})$ as in (3.6), then

$$w = K_{\tau,0}(f_2 + f_{2*'}) + \sum_{j=1}^{d} K_{\tau,j}((F_2 + F_{2*'}) \cdot e_j) + R_\tau(w).$$

Recall that the operator $P_{hf,\tau}$ commutes with all the operators $(K_{\tau,j})_{j\in\{0,\cdots,d\}}$ and R_τ. Accordingly, the high-frequency part of $w_{hf,\tau} = P_{hf,\tau} w$ satisfies

$$w_{hf,\tau} = P_{hf,\tau} K_{\tau,0}(f_2 + f_{2*'}) + \sum_{j=1}^{d} P_{hf,\tau} K_{\tau,j}((F_2 + F_{2*'}) \cdot e_j) + R_\tau(w_{hf,\tau}).$$

Using the various estimates in Propositions 3.6, 3.8, and 3.5, at high-frequency, we obtain

$$\tau^{\frac{3}{2}} \|w_{hf,\tau}\|_{L^2(\Omega)} + \tau^{\frac{1}{2}} \|\nabla w_{hf,\tau}\|_{L^2(\Omega)} \\ \leqslant C \left(\tau^{-\frac{1}{2}} \|f_2\|_{L^2(\Omega)} + \tau^{\frac{1}{2}} \|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^{\frac{1}{2}} \|F_2 + F_{2*'}\|_{L^2(\Omega)} \right) + C\|\nabla' w_{hf,\tau}\|_{L^2(\Omega)}, \quad (3.43)$$

and we have also (using also (3.19) and the fact that $\|P_{hf,\tau}\|_{L^p} \leqslant C_p$ for $p = 2d/(d-2)$)

$$\tau^{\frac{3}{4}+\frac{1}{2d}} \|w_{hf,\tau}\|_{L^{\frac{2d}{d-2}}(\Omega)} \\ \leqslant C \left(\tau^{-\frac{1}{4}+\frac{1}{2d}} \|f_2\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \|F_2 + F_{2*'}\|_{L^2(\Omega)} \right) + C\|\nabla' w_{hf,\tau}\|_{L^2(\Omega)}. \quad (3.44)$$

Accordingly, there exists $\tau_0 > 0$ such that for all $\tau \geqslant \tau_0$,

$$\tau^{\frac{3}{2}} \|w_{hf,\tau}\|_{L^2(\Omega)} + \tau^{\frac{1}{2}} \|\nabla w_{hf,\tau}\|_{L^2(\Omega)} \\ \leqslant C \left(\tau^{-\frac{1}{2}} \|f_2\|_{L^2(\Omega)} + \tau^{\frac{1}{2}} \|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^{\frac{1}{2}} \|F_2 + F_{2*'}\|_{L^2(\Omega)} \right), \quad (3.45)$$

and thus

$$\tau^{\frac{3}{4}+\frac{1}{2d}} \|w_{hf,\tau}\|_{L^{\frac{2d}{d-2}}(\Omega)} \leqslant C \left(\tau^{-\frac{1}{4}+\frac{1}{2d}} \|f_2\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \|F_2 + F_{2*'}\|_{L^2(\Omega)} \right). \quad (3.46)$$

We then define the low frequency part $w_{lf,\tau} = (I - P_{hf,\tau})w$ of w: Using (3.18), we get

$$w_{lf,\tau} = K_{\tau,0}((I - P_{hf,\tau})(f_2 + f_{2*'}) + \operatorname{div}'((I - P_{hf,\tau})F'_{2*'})) + K_{\tau,1}((I - P_{hf,\tau})F_2 \cdot e_1) \\ + (I - P_{hf,\tau})K_{\tau,1}(F_{2*'} \cdot e_1) + \sum_{j=2}^{d} K_{\tau,j}((I - P_{hf,\tau})F_2 \cdot e_j) + R_\tau(w_{lf,\tau}).$$

Applying Bernstein's inequality (see, for instance, [2, Lemma 2.1.]), we get that $\operatorname{div}'((I - P_{hf,\tau})F'_{2*'})$ belongs to $L^{\frac{2d}{d+2}}(\Omega)$ and

$$\|\operatorname{div}'((I - P_{hf,\tau})F'_{2*'})\|_{L^{\frac{2d}{d+2}}(\Omega)} \leqslant C\tau \|F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}.$$

We then use Propositions 3.3, 3.4, 3.5, and 3.10 to obtain

$$\tau^{\frac{3}{2}} \|w_{lf,\tau}\|_{L^2(\Omega)} + \tau^{\frac{1}{2}} \|\nabla w_{lf,\tau}\|_{L^2(\Omega)} \\ \leqslant C \left(\|f_2\|_{L^2(\Omega)} + \tau \|F_2\|_{L^2(\Omega)} + \tau^{\frac{3}{4}-\frac{1}{2d}} \left(\|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau \|F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} \right) \right) + C\|\nabla' w_{lf,\tau}\|_{L^2(\Omega)}, \quad (3.47)$$

and

$$\tau^{\frac{3}{4}+\frac{1}{2d}}\|w_{lf,\tau}\|_{L^{\frac{2d}{d-2}}(\Omega)}$$
$$\leqslant C\left(\|f_2\|_{L^2(\Omega)}+\tau\|F_2\|_{L^2(\Omega)}+\tau^{\frac{3}{4}+\frac{1}{2d}}\left(\|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}+\tau\|F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}\right)\right)+C\|\nabla' w_{lf,\tau}\|_{L^2(\Omega)}. \quad (3.48)$$

As before, we can then absorb the term $\|\nabla' w_{lf,\tau}\|_{L^2(\Omega)}$ in the right hand-side of (3.47) by choosing τ large enough: Taking $\tau_0 \geqslant 1$ larger if necessary, we get, for $\tau \geqslant \tau_0$,

$$\tau^{\frac{3}{2}}\|w_{lf,\tau}\|_{L^2(\Omega)}+\tau^{\frac{1}{2}}\|\nabla w_{lf,\tau}\|_{L^2(\Omega)}$$
$$\leqslant C\left(\|f_2\|_{L^2(\Omega)}+\tau\|F_2\|_{L^2(\Omega)}+\tau^{\frac{3}{4}-\frac{1}{2d}}\left(\|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}+\tau\|F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}\right)\right), \quad (3.49)$$

and, consequently,

$$\tau^{\frac{3}{4}+\frac{1}{2d}}\|w_{lf,\tau}\|_{L^{\frac{2d}{d-2}}(\Omega)}\leqslant C\left(\|f_2\|_{L^2(\Omega)}+\tau\|F_2\|_{L^2(\Omega)}+\tau^{\frac{3}{4}+\frac{1}{2d}}\left(\|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}+\tau\|F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}\right)\right). \quad (3.50)$$

Combining estimates (3.45) and (3.49), we deduce the Carleman estimate (3.7). Similarly, combining estimates (3.46) and (3.50), we deduce the Carleman estimate (3.8) except for the localized estimates on the set E.

To proceed with (3.8), we thus focus on the localized estimates within the subset $E \subset \Omega$. In order to do so, on one hand, at low frequencies, we use Hölder and Bernstein estimates:

$$\tau\|w_{lf,\tau}\|_{L^2(E)}+\|\nabla' w_{lf,\tau}\|_{L^2(E)}\leqslant |E|^{\frac{1}{d}}\left(\tau\|w_{lf,\tau}\|_{L^{\frac{2d}{d-2}}(E)}+\|\nabla' w_{lf,\tau}\|_{L^{\frac{2d}{d-2}}(E)}\right)\leqslant C|E|^{\frac{1}{d}}\tau\|w_{lf,\tau}\|_{L^{\frac{2d}{d-2}}(\Omega)}.$$

Accordingly, multiplying the above estimate by $\tau^{\frac{3}{4}+\frac{1}{2d}}$ and using (3.50), we deduce

$$\tau^{\frac{3}{4}+\frac{1}{2d}}\left(\tau\|w_{lf,\tau}\|_{L^2(E)}+\|\nabla' w_{lf,\tau}\|_{L^2(E)}\right)\leqslant C|E|^{\frac{1}{d}}\tau\left(\|f_2\|_{L^2(\Omega)}+\tau\|F_2\|_{L^2(\Omega)}\right.$$
$$\left.+\tau^{\frac{3}{4}+\frac{1}{2d}}(\|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}+\tau\|F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)})\right).$$

On the other hand, the estimates (3.45) give

$$\tau^{\frac{3}{4}+\frac{1}{2d}}\left(\tau\|w_{hf,\tau}\|_{L^2(E)}+\|\nabla' w_{hf,\tau}\|_{L^2(E)}\right)\leqslant \tau^{\frac{3}{4}+\frac{1}{2d}}\left(\tau\|w_{hf,\tau}\|_{L^2(\Omega)}+\|\nabla' w_{hf,\tau}\|_{L^2(\Omega)}\right)$$
$$\leqslant C\left(\tau^{-\frac{1}{4}+\frac{1}{2d}}\|f_2\|_{L^2(\Omega)}+\tau^{\frac{3}{4}+\frac{1}{2d}}\|F_2+F_{2*'}\|_{L^2(\Omega)}+\tau^{\frac{3}{4}+\frac{1}{2d}}\|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}\right).$$

Finally, based on Propositions 3.3–3.5, we can derive the following estimate for the term $\partial_1 w$:

$$\tau^{\frac{3}{4}+\frac{1}{2d}}\|\partial_1 w\|_{L^2(E)}\leqslant \tau^{\frac{3}{4}+\frac{1}{2d}}\|\partial_1 w\|_{L^2(\Omega)}$$
$$\leqslant C\left(\tau^{-\frac{1}{4}+\frac{1}{2d}}\|f_2\|_{L^2(\Omega)}+\tau^{\frac{3}{4}+\frac{1}{2d}}\|F_2+F_{2*'}\|_{L^2(\Omega)}+\tau^{\frac{3}{4}+\frac{1}{2d}}\|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}\right).$$

By combining the last three estimates, we conclude

$$\tau^{\frac{3}{4}+\frac{1}{2d}}\min\left\{\frac{1}{\tau|E|^{\frac{1}{d}}},1\right\}\left(\tau\|w\|_{L^2(E)}+\|\nabla w\|_{L^2(E)}\right)$$
$$\leqslant C\left(\|f_2\|_{L^2(\Omega)}+\tau\|F_2\|_{L^2(\Omega)}+\tau^{\frac{3}{4}+\frac{1}{2d}}\left(\|f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}+\tau\|F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}\right)+\tau^{\frac{3}{4}+\frac{1}{2d}}\|F_{2*'}\|_{L^2(\Omega)}\right),$$

and the proof of Theorem 3.1 is thus completed. $\square$

3.2 Proof of Theorem 2.4

The goal of this section is to deduce Theorem 2.4 in the case of a general geometry from Theorem 3.1 which was considering only the case of a strip.

In order to do so, we rely on two main steps:

- A localization process, which allows through a suitable change of variables, to use Theorem 3.1 to deduce a local Carleman estimate.
- A gluing argument to patch these local estimates.

This is the strategy used in [6, Section 7]. We only sketch it below for the convenience of the reader since it does not involve any new difficulty compared to [6].

3.2.1 Local Carleman estimates

For $\tau \geqslant 1$, we introduce

$$w = e^{\tau\varphi} u, \qquad \widetilde{f}_2 := e^{\tau\varphi}(f_2 - \tau \nabla\varphi \cdot F), \qquad \widetilde{f}_{2*'} := e^{\tau\varphi} f_{2*'}, \qquad \widetilde{F} = \widetilde{F}_2 + \widetilde{F}_{2*'} := e^{\tau\varphi} F_2 + e^{\tau\varphi} F_{2*'},$$

so that the function u solves (2.7) if and only if w solves

$$\Delta w - 2\tau \nabla\varphi \cdot \nabla w + \tau^2 |\nabla\varphi|^2 w - \tau \Delta\varphi w = \widetilde{f}_2 + \widetilde{f}_{2*'} + \operatorname{div}(\widetilde{F}) \quad \text{ in } \Omega, \tag{3.51}$$

We now introduce a local version of (3.51). Namely, for $x_0 \in \overline{\Omega} \setminus \omega$, we introduce $\eta_{x_0}(x)$ a cut-off function defined by

$$\eta_{x_0}(x) = \eta(\tau^{\frac{1}{3}}(x - x_0)), \qquad x \in \mathbb{R}^d, \tag{3.52}$$

where η is a non-negative smooth radial function (in $\mathscr{C}_c^\infty(\mathbb{R}^d)$) such that $\eta(\rho) = 1$ for $|\rho| \leqslant 1/2$ and vanishing outside the unit ball. We set

$$w_{x_0}(x) = \eta_{x_0}(x) w(x), \qquad x \in \Omega,$$

which solves

$$\Delta w_{x_0} - 2\tau \nabla\varphi \cdot \nabla w_{x_0} + \tau^2 |\nabla\varphi|^2 w_{x_0} = f_{2,x_0} + f_{2*',x_0} + \operatorname{div}(F_{x_0}), \text{ in } \Omega, \tag{3.53}$$

where

$$f_{2,x_0} = \eta_{x_0} \widetilde{f}_2 - \nabla\eta_{x_0} \cdot \widetilde{F}_2 + \tau \Delta\varphi w_{x_0} + 2 \nabla\eta_{x_0} \cdot \nabla w + \Delta\eta_{x_0} w - 2\tau \nabla\varphi \cdot \nabla\eta_{x_0} w, \tag{3.54}$$

$$f_{2*',x_0} = \eta_{x_0} \widetilde{f}_{2*'} - \nabla\eta_{x_0} \cdot \widetilde{F}_{2*'}, \qquad F_{x_0} = \eta_{x_0} \widetilde{F}. \tag{3.55}$$

Recall that u is assumed to be compactly supported in some compact set K, such that $K \Subset \Omega$. Accordingly, there exists $\varepsilon > 0$ such that $K_\varepsilon = \{x \in \mathbb{R}^d,\, d(x, K) \leqslant \varepsilon\}$ is a subset of Ω.

We then derive the following lemma, whose proof is similar to the one in [6, Lemma 7.1].

Lemma 3.11. *There exist constants $C > 0$ and $\tau_0 \geqslant 1$ (depending only on α, β, $\|\varphi\|_{C^3(\overline{\Omega})}$, K, ω and Ω) such that for all $\tau \geqslant \tau_0$, for all $x_0 \in K_\varepsilon \setminus \omega_0$, for all $(f_{2,x_0}, f_{2*',x_0}, F_{x_0} = F_{2,x_0} + F_{2*',x_0})$ satisfying*

$$f_{2,x_0} \in L^2(\Omega), \quad f_{2*',x_0} \in L^{\frac{2d}{d+2}}(\Omega), \quad F_{2,x_0} \in L^2(\Omega; \mathbb{C}^d), \quad \text{and} \quad F_{2*',x_0} \in L^{\frac{2d}{d+2}}(\Omega; \mathbb{C}^d) \cap L^2(\Omega; \mathbb{C}^d),$$

and w_{x_0} satisfying (3.53) and supported in $B_{x_0}(\tau^{-\frac{1}{3}})$, we have

$$\begin{aligned} &\tau^{\frac{3}{2}} \|w_{x_0}\|_{L^2(\Omega)} + \tau^{\frac{1}{2}} \|\nabla w_{x_0}\|_{L^2(\Omega)} \\ &\leqslant C \left(\|f_{2,x_0}\|_{L^2(\Omega)} + \tau \|F_{2,x_0}\|_{L^2(\Omega)} + \tau^{\frac{3}{4} - \frac{1}{2d}} \left(\|f_{2*',x_0}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau \|F_{2*',x_0}\|_{L^{\frac{2d}{d+2}}(\Omega)} \right) + \tau^{\frac{1}{2}} \|F_{2*',x_0}\|_{L^2(\Omega)} \right), \end{aligned} \tag{3.56}$$

and, for all measurable sets E of Ω,

$$\begin{aligned} &\tau^{\frac{3}{4} + \frac{1}{2d}} \|w_{x_0}\|_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{3}{4} + \frac{1}{2d}} \min\left\{ \frac{1}{\tau |E|^{\frac{1}{d}}}, 1 \right\} \left(\tau \|w_{x_0}\|_{L^2(E)} + \|\nabla w_{x_0}\|_{L^2(E)} \right) + \tau^{\frac{3}{2}} \|w_{x_0}\|_{L^2(\Omega)} + \tau^{\frac{1}{2}} \|\nabla w_{x_0}\|_{L^2(\Omega)} \\ &\leqslant C \left(\|f_{2,x_0}\|_{L^2(\Omega)} + \tau \|F_{2,x_0}\|_{L^2(\Omega)} + \tau^{\frac{3}{4} + \frac{1}{2d}} \left(\|f_{2*',x_0}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau \|F_{2*',x_0}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \|F_{2*',x_0}\|_{L^2(\Omega)} \right) \right). \end{aligned} \tag{3.57}$$

Sketch of the proof. The main idea is to build a suitable change of coordinates which allows to rewrite the equation (3.53) under the form (3.5), up to some lower order terms which can be handled using the localization properties of w_{x_0}.

Namely, let $x_0 \in K_\varepsilon \setminus \omega_0$, and introduce $L_1 \in \mathbb{R}^d$ and $A_1 \in \mathbb{R}^{d\times d}$ as follows:

$$L_1 = \nabla\varphi(x_0) \in \mathbb{R}^d, \quad A_1 = \operatorname{Hess}\varphi(x_0) \in \mathbb{R}^{d\times d}.$$

The bilinear form

$$\xi \in \mathbb{R}^d \mapsto (\operatorname{Hess}\varphi(x_0))\xi\cdot\xi$$

is symmetric on $\mathbb{R}^d$ and on $\operatorname{Span}\{L_1\}^\perp$. Accordingly, there exists a family of orthogonal vectors $(L_j)_{j\in\{2,\cdots,d\}}$ of $\operatorname{Span}\{L_1\}^\perp$ which diagonalizes this form, that we normalize so that for all $j \in \{2,\cdots,d\}$, $|L_j| = |L_1|$. Since the family $(L_j)_{j\in\{2,\cdots,d\}}$ of $\operatorname{Span}\{L_1\}^\perp$ diagonalizes the form $\xi \mapsto (\operatorname{Hess}\varphi(x_0))\xi\cdot\xi$ in $\operatorname{Span}\{L_1\}^\perp$, for all $j \in \{2,\cdots,d\}$, there exist α_j and μ_j in $\mathbb{R}$ such that

$$(\operatorname{Hess}\varphi(x_0))L_j = \mu_j L_j + \alpha_j L_1, \qquad j \in \{2,\cdots,d\}.$$

Note that by symmetry of $\operatorname{Hess}\varphi(x_0)$, we then necessarily have

$$(\operatorname{Hess}\varphi(x_0))L_1 = \mu_1 L_1 + \sum_{k\geqslant 2}\alpha_k L_k,$$

where

$$\mu_1 = \frac{1}{|L_1|^2}(\operatorname{Hess}\varphi(x_0))L_1\cdot L_1 = \frac{1}{|\nabla\varphi(x_0)|^2}(\operatorname{Hess}\varphi(x_0))\nabla\varphi(x_0)\cdot\nabla\varphi(x_0).$$

For $j \in \{2,\cdots,d\}$, we then introduce the self-adjoint matrix $A_j \in \mathbb{R}^{d\times d}$ defined by

$$\begin{cases} A_j L_1 = -\alpha_j L_1 - \mu_j L_j, \\ A_j L_k = \alpha_k L_j - \alpha_j L_k, \quad \text{if } k \in \{2,\cdots,d\}\setminus\{j\}, \\ A_j L_j = -\mu_j L_1 + \displaystyle\sum_{k\geqslant 2}\alpha_k L_k. \end{cases} \tag{3.58}$$

(It is easy to check that each matrix A_j defined that way is indeed symmetric.)

We then define the following change of coordinates for x in a neighbourhood of x_0:

$$\begin{aligned} y_1(x) &= \varphi(x) - \varphi(x_0), \\ y_j(x) &= L_j\cdot(x-x_0) + \frac{1}{2}A_j(x-x_0)\cdot(x-x_0) \qquad \text{for } j \in \{2,\cdots,d\}. \end{aligned}$$

By construction, there exists a neighbourhood, whose size depends on the C^2 norm of φ only, such that $x \mapsto y(x)$ is a local diffeomorphism between a neighbourhood $\mathcal{V}$ of x_0 in $\overline{\Omega}\setminus\omega$ and a neighbourhood of 0, that we call Ω_y.

For τ large enough, we can ensure that the ball of center x_0 and radius $\tau^{-\frac{1}{3}}$, when intersected with $\overline{\Omega}$, is included in a set on which $x \mapsto y(x)$ is a diffeomorphism, and its image is included in a ball $B_0(C\tau^{-\frac{1}{3}})$.

Therefore, for w_{x_0} solving (3.53), we set

$$\check{w}(y) = w_{x_0}(x) \quad \text{for} \quad y = y(x),$$

Tedious computations, detailed in [6, Section 7.2], show that $\check{w}$ then satisfies

$$\Delta_y\check{w} - y_1\sum_{j=2}^d \lambda_j\partial_{y_j}^2\check{w} - 2\tau\partial_{y_1}\check{w} + \tau^2\check{w} = \check{f}_2 + \check{f}_{2*'} + \operatorname{div}_y\check{F} \qquad \text{in } (Y_0,Y_1)\times\mathbb{R}^{d-1},$$

where the coefficients $(\lambda_j)_{j\in\{2,\cdots,d\}}$ are given by

$$\begin{aligned} \lambda_j &= \frac{2}{|L_1|^2}\left(A_1L_1\cdot L_1 + A_1L_j\cdot L_j\right) \\ &= \frac{2}{|\nabla\varphi(x_0)|^2}\left((\operatorname{Hess}\varphi(x_0))\nabla\varphi(x_0)\cdot\nabla\varphi(x_0) + (\operatorname{Hess}\varphi(x_0))L_j\cdot L_j\right), \end{aligned} \tag{3.59}$$

the source terms are

$$\check{f}_2(y) = \frac{1}{|\nabla\varphi(x(y))|^2} f_{2,x_0}(x(y)) - \sum_{j,k} \partial_{y_j}\rho_{k,j} F_{x_0,2,k}(x(y)) + \check{f}_{2,a}(y) + \check{f}_{2,b}(y) + \check{f}_{2,c}(y),$$

$$\check{f}_{2*'}(y) = \frac{1}{|\nabla\varphi(x(y))|^2} f_{2*',x_0}(x(y)) - \sum_{j,k} \partial_{y_j}\rho_{k,j} F_{x_0,2*',k}(x(y)),$$

$$\check{F}_j(y) = \underbrace{\sum_{k=1}^d \rho_{k,j}(y) F_{x_0,2,k}(x(y)) + \check{F}_{j,a}(y)}_{=:\check{F}_{j,2}} + \underbrace{\sum_{k=1}^d \rho_{k,j}(y) F_{x_0,2*',k}(x(y))}_{=:\check{F}_{j,2*'}}, \qquad j \in \{1,\cdots,d\},$$

in which ρ is defined as

$$\rho_{j,k}(y) = \frac{\partial_{x_k} y_j(x(y))}{|\nabla\varphi(x(y))|^2},$$

and $\check{f}_{2,a}$, $\check{f}_{2,b}$, $\check{f}_{2,c}$ and $\check{F}_a$ satisfy, due to the localization of w in $B_{x_0}(\tau^{-\frac{1}{3}})$ (equivalently, of $\check{w}$ in $B_0(C\tau^{-\frac{1}{3}})$),

$$\|\check{f}_{2,a}\|_{L^2(\Omega_y)} \leqslant C\tau^{-\frac{1}{3}} \|\nabla_y \check{w}\|_{L^2(\Omega_y)}, \quad \|\check{f}_{2,b}\|_{L^2(\Omega_y)} \leqslant C\tau^{\frac{1}{3}} \|\nabla_y \check{w}\|_{L^2(\Omega_y)}, \quad \|\check{f}_{2,c}\|_{L^2(\Omega_y)} \leqslant C\|\nabla_y \check{w}\|_{L^2(\Omega_y)},$$
$$\|\check{F}_a\|_{L^2(\Omega_y)} \leqslant C\tau^{-\frac{2}{3}} \|\nabla_y \check{w}\|_{L^2(\Omega_y)}.$$

Now, the condition (2.6) implies that all the λ_j in (3.59) are positive, i.e. that the condition (3.4) is satisfied. Accordingly, the Carleman estimates in Theorem 3.1 apply, and we can readily deduce the estimates in Lemma 3.11. Let us focus for instance on the proof of the Carleman estimate (3.57) (the proof of the Carleman estimate (3.56) is completely similar and in fact easier and left to the reader).

For $\tau \geqslant \tau_0$ and a measurable set E_y of Ω_y, we have from (3.7) and (3.8) that

$$\begin{aligned}
&\tau^{\frac{3}{4}+\frac{1}{2d}} \|\check{w}\|_{L^{\frac{2d}{d-2}}(\Omega_y)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \min\left\{\frac{1}{\tau|E_y|^{\frac{1}{d}}}, 1\right\} \left(\tau\|\check{w}\|_{L^2(E_y)} + \|\nabla\check{w}\|_{L^2(E_y)}\right) + \tau^{\frac{3}{2}} \|\check{w}\|_{L^2(\Omega_y)} + \tau^{\frac{1}{2}} \|\nabla\check{w}\|_{L^2(\Omega_y)} \\
&\leqslant C\left(\|\check{f}_2\|_{L^2(\Omega_y)} + \tau\|\check{F}_2\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\left(\|\check{f}_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega_y)} + \tau\|\check{F}_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega_y)} + \|\check{F}_{2*'}\|_{L^2(\Omega_y)}\right)\right).
\end{aligned}$$

We then simply remark that, from the expression of $\check{f}_2$, $\check{f}_{2*'}$, and $\check{F} = \check{F}_2 + \check{F}_{2*'}$,

$$\begin{aligned}
&\|\check{f}_2\|_{L^2(\Omega_y)} + \tau\|\check{F}_2\|_{L^2(\Omega_y)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\left(\|\check{f}_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega_y)} + \tau\|\check{F}_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega_y)} + \|\check{F}_{2*'}\|_{L^2(\Omega_y)}\right) \\
&\leqslant C\Bigg(\|f_{2,x_0}\|_{L^2(\Omega)} + \tau\|F_{2,x_0}\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\left(\|f_{2*',x_0}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau\|F_{2*',x_0}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \|F_{2*',x_0}\|_{L^2(\Omega)}\right) \\
&\qquad + \tau^{\frac{1}{3}}\|\nabla_y\check{w}\|_{L^2(\Omega_y)}\Bigg),
\end{aligned}$$

Accordingly, taking $\tau_0 \geqslant 1$ larger if necessary, we get for all $\tau \geqslant \tau_0$,

$$\begin{aligned}
&\tau^{\frac{3}{4}+\frac{1}{2d}} \|\check{w}\|_{L^{\frac{2d}{d-2}}(\Omega_y)} + \tau^{\frac{3}{4}+\frac{1}{2d}} \min\left\{\frac{1}{\tau|E_y|^{\frac{1}{d}}}, 1\right\} \left(\tau\|\check{w}\|_{L^2(E_y)} + \|\nabla\check{w}\|_{L^2(E_y)}\right) + \tau^{\frac{3}{2}} \|\check{w}\|_{L^2(\Omega_y)} + \tau^{\frac{1}{2}} \|\nabla\check{w}\|_{L^2(\Omega_y)} \\
&\leqslant C\left(\|f_{2,x_0}\|_{L^2(\Omega)} + \tau\|F_{2,x_0}\|_{L^2(\Omega)} + \tau^{\frac{3}{4}+\frac{1}{2d}}\left(\|f_{2*',x_0}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \tau\|F_{2*',x_0}\|_{L^{\frac{2d}{d+2}}(\Omega)} + \|F_{2*',x_0}\|_{L^2(\Omega)}\right)\right).
\end{aligned}$$

Undoing the change of variables on the left-hand side, we easily deduce the estimates (3.57) for $\tau \geqslant \tau_0$ and a measurable set E of Ω.

The fact that the constants above do not depend on $x_0 \in K_\varepsilon \setminus \omega_0$ can be tracked in the above proof: it comes from uniformity properties of the diffeomorphism $x \mapsto y$, and relies heavily on the uniform bounds (2.5)–(2.6), on the fact that $\varphi \in C^3(\overline{\Omega})$, and that the constants in Theorem 3.1 depend only on c_0, m_* and M_* in (3.3), and (3.4) for $X_0 < 0 < X_1$ with $|X_0|, |X_1| \leqslant 1$. This ends the proof of Lemma 3.11. □

3.2.2 A gluing argument

Here again, the proof closely follows the one in [6, Section 7.3], and we focus on the proof of the estimate (2.13), as the estimate (2.12) can be done similarly and is thus left to the reader.

We start from (3.57): There exist constants $C > 0$ and $\tau_0 \geqslant 1$ such that for all $x_0 \in K_\varepsilon \setminus \omega_0$ and $\tau \geqslant \tau_0$ and w_{x_0} solution of (3.53),

$$\begin{aligned}&\tau^{\frac{3}{2}+\frac{1}{d}}\|w_{x_0}\|^2_{L^{\frac{2d}{d-2}}(\Omega)}+\tau^{\frac{3}{2}+\frac{1}{d}}\min\left\{\frac{1}{\tau|E|^{\frac{1}{d}}},1\right\}^2\left(\tau^2\|w_{x_0}\|^2_{L^2(E)}+\|\nabla w_{x_0}\|^2_{L^2(E)}\right)+\tau^3\|w_{x_0}\|^2_{L^2(\Omega)}+\tau\|\nabla w_{x_0}\|^2_{L^2(\Omega)}\\&\leqslant C\left(\|f_{2,x_0}\|^2_{L^2(\Omega)}+\tau^2\|F_{2,x_0}\|^2_{L^2(\Omega)}+\tau^{\frac{3}{2}+\frac{1}{d}}\left(\|f_{2*',x_0}\|^2_{L^{\frac{2d}{d+2}}(\Omega)}+\tau^2\|F_{2*',x_0}\|^2_{L^{\frac{2d}{d+2}}(\Omega)}+\|F_{2*',x_0}\|^2_{L^2(\Omega)}\right)\right).\end{aligned} \tag{3.60}$$

Using the explicit expressions of the source terms (3.54) and (3.55), we obtain

$$\begin{aligned}\text{Right-Hand Side of (3.60)} \leqslant\ & C\Big(\|\eta_{x_0}\widetilde{f}_2\|^2_{L^2(\Omega)}+\tau^2\|\eta_{x_0}\widetilde{F}_2\|^2_{L^2(\Omega)}\\&+\tau^{\frac{3}{2}+\frac{1}{d}}\left(\|\eta_{x_0}\widetilde{f}_{2*'}\|^2_{L^{\frac{2d}{d+2}}(\Omega)}+\tau^2\|\eta_{x_0}\widetilde{F}_{2*'}\|^2_{L^{\frac{2d}{d+2}}(\Omega)}+\|\eta_{x_0}\widetilde{F}_{2*'}\|^2_{L^2(\Omega)}\right)\Big)\\&+C\Big(\|\nabla\eta_{x_0}\cdot\widetilde{F}_2\|^2_{L^2(\Omega)}+\tau^2\|w_{x_0}\|^2_{L^2(\Omega)}+\|\nabla\eta_{x_0}\cdot\nabla w\|^2_{L^2(\Omega)}+\|\Delta\eta_{x_0}w\|^2_{L^2(\Omega)}+\tau^2\||\nabla\eta_{x_0}|w\|^2_{L^2(\Omega)}\\&+\tau^{\frac{3}{2}+\frac{1}{d}}\left\|\nabla\eta_{x_0}\widetilde{F}_{2*'}\right\|^2_{L^{\frac{2d}{d+2}}(\Omega)}\Big).\end{aligned}$$

By taking $\tau_0 \geqslant 1$ larger if necessary (which can be done uniformly in $x_0 \in K_\varepsilon \setminus \omega_0$), we can absorb the term $\tau^2\|w_{x_0}\|^2_{L^2(\Omega)}$ by the left hand side of (3.60). Then, by integrating in x_0 on $K_\varepsilon \setminus \omega_0$, using Fubini's identity for the Hilbertian norms, we get

$$\begin{aligned}&\tau^{\frac{3}{2}+\frac{1}{d}}\left\|\|\eta_{x_0}w\|_{L_x^{\frac{2d}{d-2}}(\Omega)}\right\|^2_{L^2_{x_0}(K_\varepsilon\setminus\omega_0)}+\tau^{\frac{3}{2}+\frac{1}{d}}\min\left\{\frac{1}{\tau^2|E|^{\frac{2}{d}}},1\right\}\left(\tau^2\|\rho_0^{\frac12}w\|^2_{L^2(E)}+\|\rho_0^{\frac12}\nabla w\|^2_{L^2(E)}\right)\\&\qquad+\tau^3\|\rho_0^{\frac12}w\|^2_{L^2(\Omega)}+\tau\|\rho_0^{\frac12}\nabla w\|^2_{L^2(\Omega)}\\&\leqslant C\left(\|\rho_0^{\frac12}\widetilde{f}_2\|^2_{L^2(\Omega)}+\|(\tau^2\rho_0+\rho_{r,1})^{\frac12}\widetilde{F}_2\|^2_{L^2(\Omega)}+\tau^{\frac{3}{2}+\frac{1}{d}}\left\|\|\eta_{x_0}\widetilde{F}_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)}\right\|^2_{L^2_{x_0}(K_\varepsilon\setminus\omega_0)}\right)\\&+C\tau^{\frac{3}{2}+\frac{1}{d}}\left(\left\|\|\eta_{x_0}\widetilde{f}_{2*'}\|_{L_x^{\frac{2d}{d+2}}(\Omega)}\right\|^2_{L^2_{x_0}(K_\varepsilon\setminus\omega_0)}+\tau^2\left\|\|\eta_{x_0}\widetilde{F}_{2*'}\|_{L_x^{\frac{2d}{d+2}}(\Omega)}\right\|^2_{L^2_{x_0}(K_\varepsilon\setminus\omega_0)}+\|\rho_0^{\frac12}\widetilde{F}_{2*'}\|^2_{L^2(\Omega)}\right)\\&\qquad+C\left(\|(\rho_{r,2}+\tau^2\rho_{r,1})^{\frac12}w\|^2_{L^2(\Omega)}+\|\rho_{r,1}^{\frac12}\nabla w\|^2_{L^2(\Omega)}\right),\end{aligned}$$

where the weights ρ_0, $\rho_{r,i}$ are defined as follows:

$$\rho_0(x)=\int_{K_\varepsilon\setminus\omega_0}|\eta_{x_0}(x)|^2\,dx_0,\qquad \rho_{r,1}(x)=\int_{K_\varepsilon\setminus\omega_0}|\nabla\eta_{x_0}(x)|^2\,dx_0,\qquad \rho_{r,2}(x)=\int_{K_\varepsilon\setminus\omega_0}|\Delta\eta_{x_0}(x)|^2\,dx_0.$$

Minkowski's integral inequality ([14, p.271]) for the non-Hilbertian norms then gives:

$$\begin{aligned}&\tau^{\frac{3}{2}+\frac{1}{d}}\|\rho_0^{\frac12}w\|^2_{L^{\frac{2d}{d-2}}(\Omega)}+\tau^{\frac{3}{2}+\frac{1}{d}}\min\left\{\frac{1}{\tau^2|E|^{\frac{2}{d}}},1\right\}\left(\tau^2\|\rho_0^{\frac12}w\|^2_{L^2(E)}+\|\rho_0^{\frac12}\nabla w\|^2_{L^2(E)}\right)\\&\qquad+\tau^3\|\rho_0^{\frac12}w\|^2_{L^2(\Omega)}+\tau\|\rho_0^{\frac12}\nabla w\|^2_{L^2(\Omega)}\\&\leqslant C\left(\|\rho_0^{\frac12}\widetilde{f}_2\|^2_{L^2(\Omega)}+\|(\tau^2\rho_0+\rho_{r,1})^{\frac12}\widetilde{F}_2\|^2_{L^2(\Omega)}+\tau^{\frac{3}{2}+\frac{1}{d}}\|\rho_{r,1}^{\frac12}\widetilde{F}_{2*'}\|^2_{L^{\frac{2d}{d+2}}(\Omega)}\right)\\&+C\tau^{\frac{3}{2}+\frac{1}{d}}\left(\|\rho_0^{\frac12}\widetilde{f}_{2*'}\|^2_{L^{\frac{2d}{d+2}}(\Omega)}+\tau^2\|\rho_0^{\frac12}\widetilde{F}_{2*'}\|^2_{L^{\frac{2d}{d+2}}(\Omega)}+\|\rho_0^{\frac12}\widetilde{F}_{2*'}\|^2_{L^2(\Omega)}\right)\\&\qquad+C\left(\|(\rho_{r,2}+\tau^2\rho_{r,1})^{\frac12}w\|^2_{L^2(\Omega)}+\|\rho_{r,1}^{\frac12}\nabla w\|^2_{L^2(\Omega)}\right),\end{aligned}$$

It is easy to check from the choice (3.52) that, for τ sufficiently large,

$$\begin{aligned} &\rho_0(x) = \tau^{-\frac{d}{3}} \|\eta\|_{L^2}^2, \quad \forall x \in K \setminus \omega, && \rho_0(x) \leqslant C\tau^{-\frac{d}{3}}, \quad \forall x \in \Omega, \\ &\rho_{r,1}(x) \leqslant C\tau^{\frac{2}{3}-\frac{d}{3}}, \quad \forall x \in \Omega, \qquad \text{and} && \rho_{r,2}(x) \leqslant C\tau^{\frac{4}{3}-\frac{d}{3}}, \quad \forall x \in \Omega. \end{aligned}$$

Thus, for τ large enough,

$$\begin{aligned} \tau^{\frac{3}{2}+\frac{1}{d}} \|w\|^2_{L^{\frac{2d}{d-2}}(\Omega\setminus\omega)} + \tau^{\frac{3}{2}+\frac{1}{d}} \min\left\{\frac{1}{\tau^2|E|^{\frac{2}{d}}}, 1\right\} \left(\tau^2\|w\|^2_{L^2(E\cap(K\setminus\omega))} + \|\nabla w\|^2_{L^2(E\cap(K\setminus\omega))}\right) \\ + \tau^3\|w\|^2_{L^2(\Omega\setminus\omega)} + \tau\|\nabla w\|^2_{L^2(\Omega\setminus\omega)} \\ \leqslant C\left(\|\widetilde{f}_2\|^2_{L^2(\Omega)} + \tau^2\|\widetilde{F}_2\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}+\frac{1}{d}}\left(\|\widetilde{f}_{2*'}\|^2_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^2\|\widetilde{F}_{2*'}\|^2_{L^{\frac{2d}{d+2}}(\Omega)} + \|\widetilde{F}_{2*'}\|^2_{L^2(\Omega)}\right)\right) \\ + C\left(\tau^{\frac{8}{3}}\|w\|^2_{L^2(\omega)} + \tau^{\frac{2}{3}}\|\nabla w\|^2_{L^2(\omega)}\right), \end{aligned}$$

We then add

$$\tau^{\frac{3}{2}+\frac{1}{d}} \|w\|^2_{L^{\frac{2d}{d-2}}(\omega)} \; + \; \tau^{\frac{3}{2}+\frac{1}{d}} \min\left\{\frac{1}{\tau^2|E|^{\frac{2}{d}}}, 1\right\} \left(\tau^2\|w\|^2_{L^2(\omega)} + \|\nabla w\|^2_{L^2(\omega)}\right) \; + \; \tau^3\|w\|^2_{L^2(\omega)} \; + \; \tau\|\nabla w\|^2_{L^2(\omega)}$$

to both sides of the previous estimate and get

$$\begin{aligned} \tau^{\frac{3}{2}+\frac{1}{d}} \|w\|^2_{L^{\frac{2d}{d-2}}(\Omega)} + \tau^{\frac{3}{2}+\frac{1}{d}} \min\left\{\frac{1}{\tau^2|E|^{\frac{2}{d}}}, 1\right\} \left(\tau^2\|w\|^2_{L^2(E)} + \|\nabla w\|^2_{L^2(E)}\right) + \tau^3\|w\|^2_{L^2(\Omega)} + \tau\|\nabla w\|^2_{L^2(\Omega)} \\ \leqslant C\left(\|\widetilde{f}_2\|^2_{L^2(\Omega)} + \tau^2\|\widetilde{F}_2\|^2_{L^2(\Omega)} + \tau^{\frac{3}{2}+\frac{1}{d}}\left(\|\widetilde{f}_{2*'}\|^2_{L^{\frac{2d}{d+2}}(\Omega)} + \tau^2\|\widetilde{F}_{2*'}\|^2_{L^{\frac{2d}{d+2}}(\Omega)} + \|\widetilde{F}_{2*'}\|^2_{L^2(\Omega)}\right)\right) \\ + C\tau^{\frac{3}{2}+\frac{1}{d}}\left(\tau^2\|w\|^2_{L^2(\omega)} + \|\nabla w\|^2_{L^2(\omega)} + \|w\|^2_{L^{\frac{2d}{d-2}}(\omega)}\right). \end{aligned}$$

This concludes the proof of Theorem 2.4.

4 A specific geometric setting

In this section, we will focus on a specific geometric setting involving a ball with a radius $R > 0$ (recall that $B_0(r)$ denotes the ball centred at 0 and of radius r). Within this context, we aim to prove the following lemma on quantitative unique continuation, as presented in Theorem 1.1.

Lemma 4.1. *Let $R > 0$ and $d \geqslant 3$. We consider the following geometric setting (see Figure 1):*

$$\Omega = B_0(2R) \cap \left\{x_1 < -\frac{R}{4}\right\}, \; \mathcal{O} = B_0\left(\frac{3R}{2}\right) \cap \left\{x_1 < -\frac{R}{3}\right\}, \text{ and } \omega = (B_0(2R) \setminus B_0(R)) \cap \left\{x_1 < -\frac{R}{4}\right\},$$

There exist constants $C = C(R,d) > 0$ and $\alpha \in (0,1)$ depending only on R and d so that any solution $u \in H^1(\Omega)$ of (1.1) with (V, W_1, W_2) as in (1.2) satisfies the quantitative unique continuation estimate (1.3) with γ and δ as in (1.4).

Our goal is to explain how we can combine the Carleman estimates established in Theorem 2.4 and Wolff's argument (Lemma 2.1) in order to obtain the quantitative unique continuation estimate (1.3).

A key remark is that Wolff's lemma applies for linear weight functions of the form $y \mapsto k \cdot y$ for $k \in \mathbb{R}^d$, whereas our Carleman estimates are valid under appropriate subellipticity conditions on the weight function ((2.5)–(2.6)), while the parameter τ is a positive real number.

To employ both tools simultaneously, we construct a family of weight functions that satisfy the subellipticity conditions (2.5)–(2.6) and Wolff's argument.

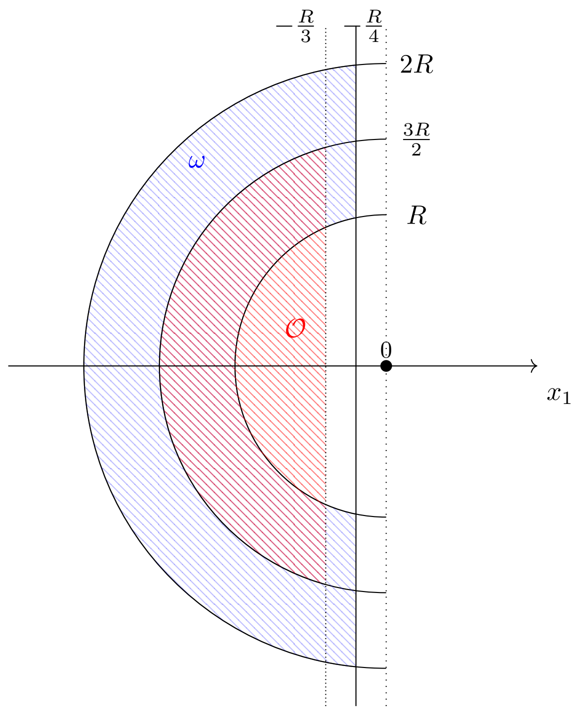

Figure 1: The geometric setting of Lemma 4.1: Propagation of smallness from the left to the right.

Lemma 4.2. *Within the same setting as in Lemma 4.1, for $k \in \mathbb{R}^d$, we set*

$$\varphi_k(x) = k_1 x_1^2 + k' \cdot x', \qquad x \in \Omega. \tag{4.1}$$

Then there exists $\epsilon > 0$ such that

1. *For all $k \in B_{e_1}(\epsilon)$, the function φ_k satisfy (2.5) and (2.6) with some positive constants $\alpha > 0$ and $\beta > 0$ independent of $k \in B_{e_1}(\epsilon)$, and its C^3 norm on $\overline{\Omega}$ is bounded independently on $k \in B_{e_1}(\epsilon)$.*
2. *There exists $\rho > 0$ such that*

$$\inf_{k \in B_{e_1}(\epsilon)} \inf_{x \in \mathcal{O}} \{\varphi_k(x)\} \geqslant (1+\rho) \sup_{k \in B_{e_1}(\epsilon)} \left\{ \sup_{x \in \Omega \cap \{x_1 \in (-\frac{7R}{24}, -\frac{R}{4})\}} \{\varphi_k(x)\} \right\}. \tag{4.2}$$

3. *Setting $\Sigma_\epsilon = \{k \in \mathbb{R}^d \setminus \{0\}$ with $|k/|k| - e_1| \leqslant \epsilon\}$, the family $(\varphi_k)_{k \in \Sigma_\epsilon}$ satisfies the following property: If f is a positive compactly supported function in Ω, we define the family $d\mu_k(x) = e^{\varphi_k(x)} f(x) dx$, then for $\mathcal{C} \subset \Sigma_\varepsilon$ there exist a family $(k_j)_{j \in J}$ of elements of $\mathcal{C}$ and two by two disjoint sets $(E_{k_j})_{j \in J}$ included in Ω so that the measures $d\mu_{k_j}$ satisfy (2.2) with $T = 0$ and the family $(E_{k_j})_{j \in J}$ satisfy (2.3) with C_W a positive constant depending only on d and Ω.*

Proof. Items 1 and 2 can be checked directly using immediate computations, the fact that $\varphi_{e_1}(x) = x_1^2$ satisfies (2.5) and (2.6), and

$$\inf_{x \in \mathcal{O}} \{\varphi_{e_1}(x)\} = \frac{R^2}{9} > \sup_{x \in \Omega \cap \{x_1 \in (-\frac{7R}{24}, -\frac{R}{4})\}} \{\varphi_{e_1}(x)\} = \left(\frac{7}{24}\right)^2 R^2.$$

It remains to check item 3. In order to do so, let us denote by $Y : x \mapsto y$ the diffeomorphism given by $y_1(x) = x_1^2$, and $y' = x'$ (this is clearly a diffeomorphism from Ω to $Y(\Omega)$ since Ω is away from $\{x_1 = 0\}$), and X the inverse of the map Y. Then remark that for all $k \in \mathbb{R}^d$

$$\varphi_k(x) = k_1 x_1^2 + k' \cdot x' = k \cdot Y(x).$$

Let then f be a positive compactly supported function in Ω. Hence, in the new coordinates y, the family of measures $d\mu_k(x) = e^{\varphi_k(x)} f(x)dx$ becomes

$$d\widetilde{\mu}_k(y) = e^{k\cdot y} f(X(y)) |\operatorname{Jac}(X)| dy, \tag{4.3}$$

where $\operatorname{Jac}(X)$ is the Jacobian of the map X. Consequently, by Lemma 2.1, for $\mathcal{C} \subset \mathbb{R}^d$, there is a family $(k_j)_{j\in J}$ of elements of $\mathcal{C}$ and disjoint convex sets $(\widetilde{E}_{k_j})_{j\in J}$ such that the families $(d\widetilde{\mu}_{k_j})_{j\in J}$, $(k_j)_{j\in J}$, and $(\widetilde{E}_{k_j})_{j\in J}$ satisfy (2.2) and (2.3) with $\widetilde{C}_W$ a positive constant depending only on d. Consequently, we consider the sets $E_{k_j} = X(\widetilde{E}_{k_j})$, which are disjoint (not necessarily convex) and satisfy

$$|E_{k_j}| \leqslant \| \operatorname{Jac}(X)\|_\infty |\widetilde{E}_{k_j}|.$$

Using this inequality, the summation property (2.3) for the sets E_{k_j} holds, with the constant $C_W = \| \operatorname{Jac}(X)\|_\infty^{-1} \widetilde{C}_W$. On the other hand, the concentration property (2.2) with $T=0$ on E_{k_j} for each $d\mu_{k_j}$ follows from the concentration property (2.2) for the family $d\widetilde{\mu}_{k_j}$ on $\widetilde{E}_{k_j}$ and the identity (4.3). □

As a consequence of the previous result, let us point that Theorem 2.4 holds for any φ_k with $k \in B(e_1, \epsilon)$, with constants which are uniform with respect to $k \in B_{e_1}(\epsilon)$. Therefore, applying Theorem 2.4, we readily deduce the following result:

Lemma 4.3. *Let $d \geqslant 3$. Let*

$$\Omega = B_0(2R) \cap \left\{ x_1 < -\frac{R}{4} \right\}, \ \mathcal{O} = B_0\left(\frac{3R}{2}\right) \cap \left\{ x_1 < -\frac{R}{3} \right\}, \ \textit{and } \omega = (B_0(2R) \setminus B_0(R)) \cap \left\{ x_1 < -\frac{R}{4} \right\},$$

Then, for all compact subset K of Ω there exist $C > 0$ and $\tau_0 \geqslant 1$ such that for all $u \in H^1(\Omega)$ satisfying $\operatorname{supp} u \subset K$ and (2.7) with $(f_2, f_{2*'}, F = F_2 + F_{2*'})$ as in (2.11), we have, for all $k \in \Sigma_\epsilon$ with $|k| \geqslant \tau_0$, with φ_k as in (4.1),

$$\begin{aligned} |k|^{\frac{3}{2}} \|e^{\varphi_k} u\|_{L^2(\Omega)} + |k|^{\frac{1}{2}} \|e^{\varphi_k} \nabla u\|_{L^2(\Omega)} \leqslant C \Big(\|e^{\varphi_k} f_2\|_{L^2(\Omega)} + |k| \|e^{\varphi_k} F_2\|_{L^2(\Omega)} + |k|^{\frac{1}{2}} \|e^{\varphi_k} F_{2*'}\|_{L^2(\Omega)} \\ + |k|^{\frac{3}{4} - \frac{1}{2d}} \left(\|e^{\varphi_k} f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + |k| \|e^{\varphi_k} F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} \right) + |k|^{\frac{3}{2}} \|e^{\varphi_k} u\|_{H^1(\omega)} \Big), \end{aligned} \tag{4.4}$$

and, for all measurable sets E of Ω,

$$\begin{aligned} |k|^{\frac{3}{4}+\frac{1}{2d}} \|e^{\varphi_k} u\|_{L^{\frac{2d}{d-2}}(\Omega)} + |k|^{\frac{3}{4}+\frac{1}{2d}} \min\left\{ \frac{1}{|k||E|^{\frac{1}{d}}}, 1 \right\} \left(|k| \|e^{\varphi_k} u\|_{L^2(E)} + \|e^{\varphi_k} \nabla u\|_{L^2(E)} \right) \\ \leqslant C \Bigg(\|e^{\varphi_k} f_2\|_{L^2(\Omega)} + |k| \|e^{\varphi_k} F_2\|_{L^2(\Omega)} + |k|^{\frac{3}{4}+\frac{1}{2d}} \left(\|e^{\varphi_k} f_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} + |k| \|e^{\varphi_k} F_{2*'}\|_{L^{\frac{2d}{d+2}}(\Omega)} \right) \\ + |k|^{\frac{3}{4}+\frac{1}{2d}} \|e^{\varphi_k} F_{2*'}\|_{L^2(\Omega)} + |k|^{\frac{7}{4}+\frac{1}{2d}} \|e^{\varphi_k} u\|_{H^1(\omega)} \Bigg). \end{aligned} \tag{4.5}$$

We are now in a position to prove Lemma 4.1.

Proof of Lemma 4.1. For sake of clarity, we divide the proof in several steps.

Step 1: Application of the Carleman estimates. For $u \in H^1(\Omega)$ with

$$\Delta u = Vu + W_1 \cdot \nabla u + \operatorname{div}(W_2 u) \quad \text{in } \Omega,$$

we set $v = \eta u$, where η is a smooth cut-off function that takes 1 in $B_0(3R/2) \cap \{x_1 < -7R/24\}$ and vanishes in a neighbourhood of $\partial\Omega$, so that we have

$$\Delta v = Vv + W_1 \cdot \nabla v + \operatorname{div}(W_2 v) + f_\eta \quad \text{in } \Omega,$$

where f_η is defined by

$$f_\eta = 2\nabla\eta \cdot \nabla u + \Delta\eta u - W_1 \cdot \nabla\eta u - W_2 \cdot \nabla\eta u, \tag{4.6}$$

and thus satisfies

$$\operatorname{Supp} f_\eta \subset \omega \cup (\Omega \cap \{x_1 \in (-7R/24, -R/4)\}). \tag{4.7}$$

Now, for $V \in L^{q_0}(\Omega)$, $W_1 \in L^{q_1}(\Omega;\mathbb{C}^d)$, and $W_2 \in L^{q_2}(\Omega;\mathbb{C}^d)$, with $q_0 > d/2$, $q_1 > d$, and $q_2 > d$, we will perform a decomposition of the form

$$\begin{aligned}
&V = V_{\frac{d}{2}} + V_d + V_\infty, && \text{with } V_{\frac{d}{2}} \in L^{\frac{d}{2}}(\Omega),\ V_d \in L^d(\Omega),\ V_\infty \in L^\infty(\Omega),\\
&W_1 = W_{1,d} + W_{1,\infty}, && \text{with } W_{1,d} \in L^d(\Omega;\mathbb{C}^d),\ W_{1,\infty} \in L^\infty(\Omega);\mathbb{C}^d),\\
&W_2 = W_{2,d} + W_{2,\infty}, && \text{with } W_{2,d} \in L^d(\Omega);\mathbb{C}^d),\ W_{2,\infty} \in L^\infty(\Omega;\mathbb{C}^d).
\end{aligned}$$

We will explain later, in Step 4 of the proof, the precise decomposition we will choose.

We then apply Lemma 4.3. The Carleman estimate (4.4) with $f_{2*'} = V_{\frac{d}{2}}v + V_d v + W_{1,d}\cdot\nabla v$, $f_2 = V_\infty v + W_{1,\infty}\cdot\nabla v + f_\eta$, $F_{2*'} = W_{2,d}v$, and $F_2 = W_{2,\infty}v$ yields that for all $k \in \Sigma_\varepsilon$ with $|k| \geqslant \tau_0$,

$$\begin{aligned}
|k|^{\frac32}\|e^{\varphi_k}v\|_{L^2(\Omega)} + |k|^{\frac12}\|e^{\varphi_k}\nabla v\|_{L^2(\Omega)} \leqslant C\Big(&\|V_\infty\|_{L^\infty(\Omega)}\|e^{\varphi_k}v\|_{L^2(\Omega)} + \|W_{1,\infty}\|_{L^\infty}\|e^{\varphi_k}\nabla v\|_{L^2(\Omega)}\\
&+ \|e^{\varphi_k}f_\eta\|_{L^2(\Omega)} + |k|\|W_{2,\infty}\|_{L^\infty}\|e^{\varphi_k}v\|_{L^2(\Omega)} + |k|^{\frac12}\|W_{2,d}\|_{L^d(\Omega)}\|e^{\varphi_k}v\|_{L^{\frac{2d}{d-2}}(\Omega)}\\
&+ |k|^{\frac32}\|e^{\varphi_k}v\|_{H^1(\omega)} + |k|^{\frac34-\frac{1}{2d}}\left(\|V_d\|_{L^d(\Omega)}\|e^{\varphi_k}v\|_{L^2(\Omega)} + \|V_{\frac{d}{2}}\|_{L^{\frac{d}{2}}(\Omega)}\|e^{\varphi_k}v\|_{L^{\frac{2d}{d-2}}(\Omega)}\right)\\
&+|k|^{\frac34-\frac{1}{2d}}\left(\|e^{\varphi_k}W_{1,d}\cdot\nabla v\|_{L^{\frac{2d}{d+2}}(\Omega)} + |k|\|e^{\varphi_k}W_{2,d}v\|_{L^{\frac{2d}{d+2}}(\Omega)}\right)\Big).
\end{aligned}$$

Accordingly, there exists $c_0 > 0$ such that if

$$\left(\|V_\infty\|_{L^\infty(\Omega)} + |k|^{\frac34-\frac{1}{2d}}\|V_d\|_{L^d(\Omega)}\right) \leqslant c_0|k|^{\frac32}, \quad \text{and} \quad \left(\|W_{1,\infty}\|_{L^\infty(\Omega)} + \|W_{2,\infty}\|_{L^\infty(\Omega)}\right) \leqslant c_0|k|^{\frac12}, \tag{4.8}$$

for all $k \in \Sigma_\varepsilon$ with $|k| \geqslant \tau_0$,

$$\begin{aligned}
|k|^{\frac32}\|e^{\varphi_k}v\|_{L^2(\Omega)} + |k|^{\frac12}\|e^{\varphi_k}\nabla v\|_{L^2(\Omega)} \leqslant C_1\Big(&\|e^{\varphi_k}f_\eta\|_{L^2(\Omega)} + |k|^{\frac32}\|e^{\varphi_k}u\|_{H^1(\omega)}\\
&+ \left(|k|^{\frac12}\|W_{2,d}\|_{L^d(\Omega)} + |k|^{\frac34-\frac{1}{2d}}\|V_{\frac{d}{2}}\|_{L^{\frac{d}{2}}(\Omega)}\right)\|e^{\varphi_k}v\|_{L^{\frac{2d}{d-2}}(\Omega)}\\
&+ |k|^{\frac34-\frac{1}{2d}}\left(\|e^{\varphi_k}W_{1,d}\cdot\nabla v\|_{L^{\frac{2d}{d+2}}(\Omega)} + |k|\|e^{\varphi_k}W_{2,d}v\|_{L^{\frac{2d}{d+2}}(\Omega)}\right)\Big).
\end{aligned} \tag{4.9}$$

Similarly, the Carleman estimate (2.13) with $f_{2*'} = V_{\frac{d}{2}}v + W_{1,d}\cdot\nabla v$, $f_2 = V_\infty v + V_d v + W_{1,\infty}\cdot\nabla v + f_\eta$, $F_{2*'} = W_{2,d}v$, and $F_2 = W_{2,\infty}v$ yields, that for all $k \in \Sigma_\varepsilon$ with $|k| \geqslant \tau_0$, and for all measurable set E;

$$\begin{aligned}
&|k|^{\frac34+\frac{1}{2d}}\|e^{\varphi_k}v\|_{L^{\frac{2d}{d-2}}(\Omega)} + |k|^{\frac34+\frac{1}{2d}}\min\left\{\frac{1}{|k||E|^{\frac1d}},1\right\}\left(|k|\|e^{\varphi_k}v\|_{L^2(E)} + \|e^{\varphi_k}\nabla v\|_{L^2(E)}\right)\\
&\leqslant C\Big(\|V_\infty\|_{L^\infty(\Omega)}\|e^{\varphi_k}v\|_{L^2(\Omega)} + \|V_d\|_{L^d(\Omega)}\|e^{\varphi_k}v\|_{L^{\frac{2d}{d-2}}(\Omega)} + \|W_{1,\infty}\|_{L^\infty(\Omega)}\|e^{\varphi_k}\nabla v\|_{L^2(\Omega)} + \|e^{\varphi_k}f_\eta\|_{L^2(\Omega)}\\
&\quad + |k|^{\frac34+\frac{1}{2d}}\left(\|V_{\frac{d}{2}}\|_{L^{\frac{d}{2}}(\Omega)}\|e^{\varphi_k}v\|_{L^{\frac{2d}{d-2}}(\Omega)} + \|e^{\varphi_k}W_{1,d}\cdot\nabla v\|_{L^{\frac{2d}{d+2}}(\Omega)} + |k|\|e^{\varphi_k}W_{2,d}v\|_{L^{\frac{2d}{d+2}}(\Omega)}\right)\\
&\quad +|k|\|W_{2,\infty}\|_{L^\infty(\Omega)}\|e^{\varphi_k}v\|_{L^2(\Omega)} + |k|^{\frac34+\frac{1}{2d}}\|W_{2,d}\|_{L^d(\Omega)}\|e^{|k|\varphi_k}v\|_{L^{\frac{2d}{d-2}}(\Omega)} + |k|^{\frac74+\frac{1}{2d}}\|e^{\varphi_k}u\|_{H^1(\omega)}\Big).
\end{aligned}$$

Accordingly, there exists $c_1 > 0$ such that if

$$\|V_{\frac{d}{2}}\|_{L^{\frac{d}{2}}(\Omega)} \leqslant c_1, \qquad \|V_d\|_{L^d(\Omega)} \leqslant c_1|k|^{\frac34+\frac{1}{2d}}, \quad \text{and} \quad \|W_{2,d}\|_{L^d(\Omega)} \leqslant c_1, \tag{4.10}$$

for all $k \in \Sigma_\varepsilon$ with $|k| \geqslant \tau_0$, and for all measurable set E;

$$
\begin{aligned}
&|k|^{\frac{3}{4}+\frac{1}{2d}}\|e^{\varphi_k}v\|_{L^{\frac{2d}{d-2}}(\Omega)} + |k|^{\frac{3}{4}+\frac{1}{2d}} \min\left\{\frac{1}{|k||E|^{\frac{1}{d}}}, 1\right\}\left(|k|\|e^{\varphi_k}v\|_{L^2(E)} + \|e^{\varphi_k}\nabla v\|_{L^2(E)}\right) \\
&\quad \leqslant C_2\Big(\left(\|V_\infty\|_{L^\infty(\Omega)} + |k|\|W_{2,\infty}\|_{L^\infty(\Omega)}\right)\|e^{\varphi_k}v\|_{L^2(\Omega)} + \|W_{1,\infty}\|_{L^\infty(\Omega)}\|e^{\varphi_k}\nabla v\|_{L^2(\Omega)} + \|e^{\varphi_k}f_\eta\|_{L^2(\Omega)} \\
&\qquad + |k|^{\frac{3}{4}+\frac{1}{2d}}\left(\|e^{\varphi_k}W_{1,d}\cdot\nabla v\|_{L^{\frac{2d}{d+2}}(\Omega)} + |k|\|e^{\varphi_k}W_{2,d}v\|_{L^{\frac{2d}{d+2}}(\Omega)}\right) + |k|^{\frac{7}{4}+\frac{1}{2d}}\|e^{\varphi_k}u\|_{H^1(\omega)}\Big).
\end{aligned}
\tag{4.11}
$$

From now on, we will assume that conditions (4.8) and (4.10) are satisfied.

Step 2: Application of Wolff's argument. Let $n \in \mathbb{R}$ be larger than $\tau_0/(1-\epsilon)$, with ϵ as in Lemma 4.2. We set $\widetilde{\mathcal{C}_n} = \{k \in \mathbb{R}^d,$ such that $|k - ne_1| \leqslant \epsilon n\}$, so that $\widetilde{\mathcal{C}_n} = nB_{e_1}(\epsilon) \subset \Sigma_\epsilon$. For all $k \in \widetilde{\mathcal{C}_n}$, we define the measure

$$
\begin{aligned}
d\mu_k &= \left(|e^{\varphi_k(x)}W_{1,d}(x)\cdot\nabla v(x)|^{\frac{2d}{d+2}} + |(1+\epsilon)ne^{\varphi_k(x)}W_{2,d}(x)v(x)|^{\frac{2d}{d+2}}\right)dx \\
&= e^{\varphi_{2dk/(d+2)}(x)}\left(|W_{1,d}(x)\cdot\nabla v(x)|^{\frac{2d}{d+2}} + |(1+\epsilon)nW_{2,d}(x)v(x)|^{\frac{2d}{d+2}}\right)dx.
\end{aligned}
$$

Then Lemma 4.2 (applied to $\mathcal{C}_n = 2d\widetilde{\mathcal{C}_n}/(d+2)$) implies the existence of a constant $C_W > 0$ such that for all $n \in \mathbb{N}$, there exists a set of index J_n, a family $(k_{j,n})_{j\in J_n}$ of elements of $\widetilde{\mathcal{C}_n}$ and a corresponding family of pairwise disjoint sets $(E_{k_{j,n}})_{j\in J_n}$ such that for all $j \in J_n$, we have

$$
\begin{aligned}
&\||e^{\varphi_{k_{j,n}}}W_{1,d}\cdot\nabla v|^{\frac{2d}{d+2}} + |n(1+\epsilon)e^{\varphi_{k_{j,n}}}W_{2,d}v|^{\frac{2d}{d+2}}\|_{L^1(\Omega)}^{\frac{d+2}{2d}} \\
&\qquad \leqslant 2\||e^{\varphi_{k_{j,n}}(x)}W_{1,d}(x)\cdot\nabla v(x)|^{\frac{2d}{d+2}} + |n(1+\epsilon)e^{\varphi_{k_{j,n}}(x)}W_{2,d}(x)v(x)|^{\frac{2d}{d+2}}\|_{L^1(E_{k_{j,n}})}^{\frac{d+2}{2d}}
\end{aligned}
\tag{4.12}
$$

and

$$
\sum_{j\in J_n}|E_{k_{j,n}}|^{-1} \geqslant \frac{1}{C_W}\left(\frac{2d}{d+2}\right)^d n^d. \tag{4.13}
$$

Hence, we claim that if the conditions

$$
\left\{
\begin{aligned}
&\|W_{1,d}\|_{L^d(\Omega)}^d + \left(\frac{1+\epsilon}{1-\epsilon}\right)^d\|W_{2,d}\|_{L^d(\Omega)}^d < \frac{1}{C_W(16C_2(1+\epsilon))^d}\left(\frac{2d}{d+2}\right)^d, \\
&8C_2\left(\|W_{1,d}\|_{L^d(\Omega)} + \frac{1+\epsilon}{1-\epsilon}\|W_{2,d}\|_{L^d(\Omega)}\right) \leqslant 1,
\end{aligned}
\right.
\tag{4.14}
$$

(where C_2 is the constant in (4.11)) are satisfied, then for all $n \in \mathbb{N}$, there exists $j_{*,n} \in J_n$ such that

$$
8C_2\left(\|W_{1,d}\|_{L^d(E_{k_{j_*,n}})} + \frac{1+\epsilon}{1-\epsilon}\|W_{2,d}\|_{L^d(E_{k_{j_*,n}})}\right) \leqslant \frac{1}{|k_{j_{*,n},n}||E_{k_{j_{*,n},n}}|^{\frac{1}{d}}}. \tag{4.15}
$$

Indeed, if not, for all $j \in J_n$, we would have

$$
|E_{k_{j,n}}|^{-1} \leqslant (16C_2(1+\epsilon)n)^d\left(\|W_{1,d}\|_{L^d(E_{k_{j,n}})}^d + \left(\frac{1+\epsilon}{1-\epsilon}\right)^d\|W_{2,d}\|_{L^d(E_{k_{j,n}})}^d\right),
$$

(where we use the elementary estimate $(a+b)^d \leqslant 2^d(a^d+b^d)$ for $a, b \geqslant 0$).

By summing these estimates over $j \in J_n$ and taking into account that the sets $(E_{k_{j,n}})_{j\in J_n}$ are pairwise disjoint, we would get

$$
\frac{1}{C_W}\left(\frac{2d}{d+2}\right)^d n^d \leqslant \sum_{j\in J_n}|E_{k_{j,n}}|^{-1} \leqslant (16C_2(1+\epsilon)n)^d\left(\|W_{1,d}\|_{L^d(\Omega)}^d + \left(\frac{1+\epsilon}{1-\epsilon}\right)^d\|W_{2,d}\|_{L^d(\Omega)}^d\right),
$$

which would contradict (4.14).

We thus assume condition (4.14). For $n \geqslant \tau_0/(1-\epsilon)$, we set $k_n = k_{j_*,n}$, where $k_{j_*,n}$ is such that (4.15) holds, and we set $E_n = E_{k_{j_*,n}}$. We then deduce from (4.11) that

$$
\begin{aligned}
&|k_n|^{\frac{3}{4}+\frac{1}{2d}} \min\left\{\frac{1}{|k_n||E_n|^{\frac{1}{d}}}, 1\right\} \left(|k_n|\|e^{\varphi_{k_n}} v\|_{L^2(E_n)} + \|e^{\varphi_{k_n}} \nabla v\|_{L^2(E_n)}\right) \\
&\leqslant C_2 \Big(\left(\|V_\infty\|_{L^\infty(\Omega)} + |k_n|\|W_{2,\infty}\|_{L^\infty(\Omega)}\right) \|e^{\varphi_{k_n}} v\|_{L^2(\Omega)} + \|W_{1,\infty}\|_{L^\infty(\Omega)}\|e^{\varphi_{k_n}} \nabla v\|_{L^2(\Omega)} + \|e^{\varphi_{k_n}} f_\eta\|_{L^2(\Omega)} \\
&\quad + |k_n|^{\frac{3}{4}+\frac{1}{2d}} \left(\|e^{\varphi_{k_n}} W_{1,d}\cdot\nabla v\|_{L^{\frac{2d}{d+2}}(\Omega)} + |k_n|\|e^{\varphi_{k_n}} W_{2,d} v\|_{L^{\frac{2d}{d+2}}(\Omega)}\right) + |k_n|^{\frac{7}{4}+\frac{1}{2d}} \|e^{\varphi_{k_n}} u\|_{H^1(\omega)}\Big). \quad (4.16)
\end{aligned}
$$

Using then the classical estimates $|a|^\alpha + |b|^\alpha \leqslant 2(|a|+|b|)^\alpha$ and $(|a|+|b|)^\alpha \leqslant |a|^\alpha + |b|^\alpha$ for $\alpha \in [0,1]$ and $a, b \in \mathbb{R}$, we obtain that

$$
\begin{aligned}
&\|e^{\varphi_{k_n}} W_{1,d}\cdot\nabla v\|_{L^{\frac{2d}{d+2}}(\Omega)} + |k_n|\|e^{\varphi_{k_n}} W_{2,d} v\|_{L^{\frac{2d}{d+2}}(\Omega)} \\
&\leqslant 2\|e^{\frac{2d}{d+2}\varphi_{k_n}} \left(|W_{1,d}\cdot\nabla v|^{\frac{2d}{d+2}} + (n(1+\epsilon))|W_{2,d} v|^{\frac{2d}{d+2}}\right)\|_{L^1(\Omega)}^{\frac{d+2}{2d}} \\
&\leqslant 4\|e^{\frac{2d}{d+2}\varphi_{k_n}} \left(|W_{1,d}\cdot\nabla v|^{\frac{2d}{d+2}} + (n(1+\epsilon))|W_{2,d} v|^{\frac{2d}{d+2}}\right)\|_{L^1(E_n)}^{\frac{d+2}{2d}} \\
&\leqslant 4\|e^{\varphi_{k_n}} W_{1,d}\cdot\nabla v\|_{L^{\frac{2d}{d+2}}(E_n)} + 4n(1+\epsilon)\|e^{\varphi_{k_n}} W_{2,d} v\|_{L^{\frac{2d}{d+2}}(E_n)} \\
&\leqslant 4\left(\|W_{1,d}\|_{L^d(E_n)}\|e^{\varphi_{k_n}} \nabla v\|_{L^2(E_n)} + n(1+\epsilon)\|W_{2,d}\|_{L^d(E_n)}\|e^{\varphi_{k_n}} v\|_{L^2(E_n)}\right) \\
&\leqslant \frac{1}{2C_2} \min\left\{\frac{1}{|k_n||E_n|^{\frac{1}{d}}}, 1\right\} \left((1-\epsilon)n\|e^{\varphi_{k_n}} v\|_{L^2(E_n)} + \|e^{\varphi_{k_n}} \nabla v\|_{L^2(E_n)}\right),
\end{aligned}
$$

where we used (4.12), and the fact that, from (4.15),

$$
4\|W_{1,d}\|_{L^d(E_n)} \leqslant \frac{1}{2C_2}\frac{1}{|k_n||E_n|^{\frac{1}{d}}}, \quad \text{and} \quad 4(1+\epsilon)\|W_{2,d}\|_{L^d(E_n)} \leqslant \frac{1-\epsilon}{2C_2}\frac{1}{|k_n||E_n|^{\frac{1}{d}}}
$$

and, from $(4.14)_{(2)}$,

$$
4\|W_{1,d}\|_{L^d(E_n)} \leqslant 4\|W_{1,d}\|_{L^d(\Omega)} \leqslant \frac{1}{2C_2}, \quad \text{and} \quad 4(1+\epsilon)\|W_{2,d}\|_{L^d(E_n)} \leqslant 4(1+\epsilon)\|W_{2,d}\|_{L^d(\Omega)} \leqslant \frac{1-\epsilon}{2C_2}.
$$

Accordingly, from (4.16), we deduce that

$$
\begin{aligned}
&|k_n|^{\frac{3}{4}+\frac{1}{2d}} \left(\|e^{\varphi_{k_n}} W_{1,d}\cdot\nabla v\|_{L^{\frac{2d}{d+2}}(\Omega)} + |k_n|\|e^{\varphi_{k_n}} W_{2,d} v\|_{L^{\frac{2d}{d+2}}(\Omega)}\right) \\
&\quad \leqslant 2C_2 \Big(\left(\|V_\infty\|_{L^\infty(\Omega)} + |k_n|\|W_{2,\infty}\|_{L^\infty(\Omega)}\right)\|e^{\varphi_{k_n}} v\|_{L^2(\Omega)} + \|W_{1,\infty}\|_{L^\infty(\Omega)}\|e^{\varphi_{k_n}} \nabla v\|_{L^2(\Omega)} \\
&\qquad + \|e^{\varphi_{k_n}} f_\eta\|_{L^2(\Omega)} + |k_n|^{\frac{7}{4}+\frac{1}{2d}}\|e^{\varphi_{k_n}} u\|_{H^1(\omega)}\Big). \quad (4.17)
\end{aligned}
$$

Step 3: Combining the Carleman estimates (4.9) and (4.11), and the estimate (4.17). Using (4.9), (4.11) and (4.17), we get that there exists a constant $C > 0$ such that for all $n \geqslant \tau_0/(1-\epsilon)$,

$$
\begin{aligned}
&|k_n|^{\frac{3}{2}}\|e^{\varphi_{k_n}} v\|_{L^2(\Omega)} + |k_n|^{\frac{1}{2}}\|e^{\varphi_{k_n}} \nabla v\|_{L^2(\Omega)} \leqslant C\Big(\|e^{\varphi_{k_n}} f_\eta\|_{L^2(\Omega)} + |k_n|^{\frac{3}{2}}\|e^{\varphi_{k_n}} u\|_{H^1(\omega)} \\
&\quad + \left(|k_n|^{\frac{1}{2}}\|W_{2,d}\|_{L^d(\Omega)} + |k_n|^{\frac{3}{4}-\frac{1}{2d}}\|V_{\frac{d}{2}}\|_{L^{\frac{d}{2}}(\Omega)}\right)\|e^{\varphi_{k_n}} v\|_{L^{\frac{2d}{d-2}}(\Omega)} \\
&\quad + |k_n|^{-\frac{1}{d}}\Big(\left(\|V_\infty\|_{L^\infty(\Omega)} + |k_n|\|W_{2,\infty}\|_{L^\infty(\Omega)}\right)\|e^{\varphi_{k_n}} v\|_{L^2(\Omega)} + \|W_{1,\infty}\|_{L^\infty(\Omega)}\|e^{\varphi_{k_n}} \nabla v\|_{L^2(\Omega)} \\
&\quad + \|e^{\varphi_{k_n}} f_\eta\|_{L^2(\Omega)} + |k_n|^{\frac{7}{4}+\frac{1}{2d}}\|e^{\varphi_{k_n}} u\|_{H^1(\omega)}\Big), \quad (4.18)
\end{aligned}
$$

and

$$
\begin{aligned}
&|k_n|^{\frac{3}{4}+\frac{1}{2d}}\|e^{\varphi_{k_n}} v\|_{L^{\frac{2d}{d+2}}(\Omega)} \leqslant C\Big(\left(\|V_\infty\|_{L^\infty(\Omega)} + |k_n|\|W_{2,\infty}\|_{L^\infty(\Omega)}\right)\|e^{\varphi_{k_n}} v\|_{L^2(\Omega)} \\
&\quad + \|W_{1,\infty}\|_{L^\infty(\Omega)}\|e^{\varphi_{k_n}} \nabla v\|_{L^2(\Omega)} + \|e^{\varphi_{k_n}} f_\eta\|_{L^2(\Omega)} + |k_n|^{\frac{7}{4}+\frac{1}{2d}}\|e^{\varphi_{k_n}} u\|_{H^1(\omega)}\Big). \quad (4.19)
\end{aligned}
$$

Note in particular that, in view of the assumptions (4.8), we get from (4.18) that there exists $\tau_1 \geqslant \tau_0/(1-\epsilon)$ such that, for all $n \geqslant \tau_1$,

$$|k_n|^{\frac{3}{2}}\|e^{\varphi_{k_n}}v\|_{L^2(\Omega)} + |k_n|^{\frac{1}{2}}\|e^{\varphi_{k_n}}\nabla v\|_{L^2(\Omega)} \leqslant C\Big(\|e^{\varphi_{k_n}}f_\eta\|_{L^2(\Omega)} + |k_n|^{\frac{7}{4}+\frac{1}{2d}}\|e^{\varphi_{k_n}}u\|_{H^1(\omega)} + \Big(|k_n|^{\frac{1}{2}}\|W_{2,d}\|_{L^d(\Omega)} + |k_n|^{\frac{3}{4}-\frac{1}{2d}}\|V_{\frac{d}{2}}\|_{L^{\frac{d}{2}}(\Omega)}\Big)\|e^{\varphi_{k_n}}v\|_{L^{\frac{2d}{d-2}}(\Omega)}\Big). \quad (4.20)$$

Thus, combining (4.19) and (4.20), we obtain, for all $n \geqslant \tau_1$,

$$\begin{aligned}|k_n|^{\frac{3}{2}}\|e^{\varphi_{k_n}}v\|_{L^2(\Omega)} + |k_n|^{\frac{1}{2}}\|e^{\varphi_{k_n}}\nabla v\|_{L^2(\Omega)} &\leqslant C\Big(\|e^{\varphi_{k_n}}f_\eta\|_{L^2(\Omega)} + |k_n|^{\frac{7}{4}+\frac{1}{2d}}\|e^{\varphi_{k_n}}u\|_{H^1(\omega)}\\ &+ \Big(|k_n|^{\frac{1}{2}}\|W_{2,d}\|_{L^d(\Omega)} + |k_n|^{\frac{3}{4}-\frac{1}{2d}}\|V_{\frac{d}{2}}\|_{L^{\frac{d}{2}}(\Omega)}\Big)|k_n|^{-\frac{3}{4}-\frac{1}{2d}}\Big(\big(\|V_\infty\|_{L^\infty(\Omega)} + |k_n|\|W_{2,\infty}\|_{L^\infty(\Omega)}\big)\|e^{\varphi_{k_n}}v\|_{L^2(\Omega)}\\ &+\|W_{1,\infty}\|_{L^\infty(\Omega)}\|e^{\varphi_{k_n}}\nabla v\|_{L^2(\Omega)} + \|e^{\varphi_{k_n}}f_\eta\|_{L^2(\Omega)} + |k_n|^{\frac{3}{2}}\|e^{\varphi_{k_n}}u\|_{H^1(\omega)}\Big).\end{aligned} \quad (4.21)$$

With the constraints (4.8), (4.10) and (4.14), there exists C such that

$$\Big(|k_n|^{\frac{1}{2}}\|W_{2,d}\|_{L^d(\Omega)} + |k_n|^{\frac{3}{4}-\frac{1}{2d}}\|V_{\frac{d}{2}}\|_{L^{\frac{d}{2}}(\Omega)}\Big)|k_n|^{-\frac{3}{4}-\frac{1}{2d}} \leqslant C|k_n|^{-\frac{1}{d}},$$

and

$$\big(\|V_\infty\|_{L^\infty(\Omega)} + |k_n|\|W_{2,\infty}\|_{L^\infty(\Omega)}\big) \leqslant 2c_0|k_n|^{\frac{3}{2}}, \quad \text{and} \quad \|W_{1,\infty}\|_{L^\infty(\Omega)} \leqslant c_0|k_n|^{\frac{1}{2}}.$$

Accordingly, we deduce from (4.21) that there exists $\tau_2 \geqslant \tau_1$ such that for all $n \geqslant \tau_2$,

$$|k_n|^{\frac{3}{2}}\|e^{\varphi_{k_n}}v\|_{L^2(\Omega)} + |k_n|^{\frac{1}{2}}\|e^{\varphi_{k_n}}\nabla v\|_{L^2(\Omega)} \leqslant C\Big(\|e^{\varphi_{k_n}}f_\eta\|_{L^2(\Omega)} + |k_n|^{\frac{7}{4}+\frac{1}{2d}}\|e^{\varphi_{k_n}}u\|_{H^1(\omega)}\Big). \quad (4.22)$$

Step 4: Quantification. To quantify the unique continuation property, we simply need to choose appropriate values for n (recall that k_n is of the order of n) and suitable decompositions of V, W_1 and W_2 as $V_{\frac{d}{2}} + V_d + V_\infty$, $W_{1,d} + W_{1\infty}$, $W_{2,d} + W_{2\infty}$.

We thus recall the constraints needed so far (see (4.8), (4.10), (4.14)), which we sum up as follows:

$$\|V_{\frac{d}{2}}\|_{L^{\frac{d}{2}}(\Omega)} \lll 1, \qquad |n|^{\frac{3}{4}-\frac{1}{2d}}\|V_d\|_{L^d(\Omega)} \lll n^{\frac{3}{2}}, \qquad \|V_\infty\|_{L^\infty(\Omega)} \lll n^{\frac{3}{2}}, \quad (4.23)$$

$$\|W_{1,d}\|_{L^d(\Omega)} \lll 1, \qquad \|W_{1,\infty}\|_{L^\infty(\Omega)} \lll n^{\frac{1}{2}}, \quad (4.24)$$

$$\|W_{2,d}\|_{L^d(\Omega)} \lll 1, \qquad \|W_{2,\infty}\|_{L^\infty(\Omega)} \lll n^{\frac{1}{2}}. \quad (4.25)$$

Satisfying conditions (4.24)–(4.25). For $W_1 \in L^{q_1}(\Omega)$ and $W_2 \in L^{q_2}(\Omega)$, with q_1 and q_2 in $[d, \infty]$, and for positive numbers λ_1, λ_2 yet to be determined, we set $W_{1,d} = W_1 1_{|W_1|>\lambda_1}$, $W_{1,\infty} = W_1 1_{|W_1|\leqslant\lambda_1}$, $W_{2,d} = W_2 1_{|W_2|>\lambda_2}$, and $W_{2,\infty} = W_2 1_{|W_2|\leqslant\lambda_2}$. Conditions (4.24)–(4.25) then read:

$$\lambda_1^{1-\frac{q_1}{d}}\|W_1\|_{L^{q_1}(\Omega)}^{\frac{q_1}{d}} \lll 1, \qquad \lambda_1 n^{-\frac{1}{2}} \lll 1$$
$$\lambda_2^{1-\frac{q_1}{d}}\|W_2\|_{L^{q_2}(\Omega)}^{\frac{q_1}{d}} \lll 1, \qquad \lambda_2 n^{-\frac{1}{2}} \lll 1.$$

If we choose λ_1 and λ_2 such that $\lambda_j^{1-\frac{q_j}{d}}\|W_j\|_{L^{q_j}(\Omega)}^{\frac{q_j}{d}} = \lambda_j n^{-\frac{1}{2}}$ ($j \in \{1,2\}$), that is

$$\lambda_1 = \|W_1\|_{L^{q_1}(\Omega)} n^{\frac{d}{2q_1}} \quad \text{and} \quad \lambda_2 = \|W_2\|_{L^{q_2}(\Omega)} n^{\frac{d}{2q_2}},$$

then this yields the conditions

$$n^{\frac{1}{2}-\frac{d}{2q_1}} \ggg \|W_1\|_{L^{q_1}(\Omega)}, \quad \text{and} \quad n^{\frac{1}{2}-\frac{d}{2q_2}} \ggg \|W_2\|_{L^{q_2}(\Omega)}. \quad (4.26)$$

Satisfying conditions (4.23). Now, we consider the following two cases for the potential V:

Case $V \in L^{q_0}(\Omega)$ with $q_0 \in [d, \infty]$. For $\lambda_0 > 0$ to be chosen later, we set $V_{\frac{d}{2}} = 0$, and $V_d = V 1_{|V|>\lambda_0}$, $V_\infty = V 1_{|V|\leqslant \lambda_0}$, so that the conditions (4.23) read

$$\lambda_0^{1-\frac{q_0}{d}} \|V\|_{L^{q_0}(\Omega)}^{\frac{q_0}{d}} \lll n^{\frac{3}{4}+\frac{1}{2d}}, \qquad \lambda_0 \lll n^{\frac{3}{2}}.$$

With the choice $\lambda_0 = \|V\|_{L^{q_0}(\Omega)}^{\frac{q_0}{d}} n^{(\frac{3}{4}-\frac{1}{2d})\frac{d}{q_0}}$, this gives

$$n^{(2-\frac{d}{q_0})(\frac{3}{4}-\frac{1}{2d})} \ggg \|V\|_{L^{q_0}(\Omega)}. \tag{4.27}$$

Case $V \in L^{q_0}(\Omega)$ with $q_0 \in (d/2, d]$. For $\lambda_0 > 0$ to be determined later, we set $V_\infty = 0$, and $V_{\frac{d}{2}} = V 1_{|V|>\lambda_0}$, $V_d = V 1_{|V|\leqslant\lambda_0}$, so that the conditions (4.23) read

$$\lambda_0^{1-\frac{2q_0}{d}} \|V\|_{L^{q_0}}^{\frac{2q_0}{d}} \lll 1, \qquad \lambda_0^{1-\frac{q_0}{d}} \|V\|_{L^{q_0}}^{\frac{q_0}{d}} \lll n^{\frac{3}{4}+\frac{1}{2d}}.$$

With the choice $\lambda_0 = \|V\|_{L^{q_0}} n^{(\frac{3}{4}+\frac{1}{2d})\frac{d}{q_0}}$, this gives

$$n^{(2-\frac{d}{q_0})(\frac{3}{4}+\frac{1}{2d})} \ggg \|V\|_{L^{q_0}(\Omega)}. \tag{4.28}$$

In the following, we assume that the conditions (4.26)–(4.28) are satisfied, that is, with the notations (1.4),

$$n \geqslant \tau_3(V, W_1, W_2) := C\left(1 + \|V\|_{L^{q_0}(\Omega)}^{\gamma(q_0)} + \|W_1\|_{L^{q_1}(\Omega)}^{\delta(q_1)} + \|W_2\|_{L^{q_2}(\Omega)}^{\delta(q_2)}\right),$$

for some sufficiently large C, so that in particular the estimate (4.22) holds for all $n \geqslant \tau_3(V, W_1, W_2)$.

Step 5. Getting a stability estimate. We start by estimating the term $\|e^{\varphi_{k_n}} f_\eta\|_{L^2(\Omega)}$ as follows (recall (4.6)–(4.7)): for $n \geqslant \tau_3(V, W_1, W_2)$,

$$\begin{aligned}
\|e^{\varphi_{k_n}} f_\eta\|_{L^2(\Omega)} &\leqslant C\left(1 + \|W_1\|_{L^{q_1}(\Omega)} + \|W_2\|_{L^{q_2}(\Omega)}\right) \|e^{\varphi_{k_n}} u\|_{H^1(\omega)} \\
&\quad + C\left(1 + \|W_1\|_{L^{q_1}(\Omega)} + \|W_2\|_{L^{q_2}(\Omega)}\right) e^{\sup_{x_1 \in \left(-\frac{7R}{24}, -\frac{R}{4}\right)} \{\varphi_{k_n}\}} \|u\|_{H^1(\Omega)} \\
&\leqslant C|n|^{\frac{1}{2}} \|e^{\varphi_{k_n}} u\|_{H^1(\omega)} + C|n|^{\frac{1}{2}} e^{\sup_{x_1 \in \left(-\frac{7R}{24}, -\frac{R}{4}\right)} \{\varphi_{k_n}\}} \|u\|_{H^1(\Omega)},
\end{aligned}$$

where we used the localization properties of the gradient of the cut-off function η and the bound (4.26).

Bounding the weight function $e^{\varphi_{k_n}}$ from above and from below in (4.22), we get for all $n \geqslant \tau_3(V, W_1, W_2)$ such that

$$e^{\inf_{\mathcal{O}}\{\varphi_{k_n}\}} \|v\|_{H^1(\mathcal{O})} \leqslant C|n|^{\frac{5}{4}+\frac{1}{2d}} e^{\sup_\omega \{\varphi_{k_n}\}} \|u\|_{H^1(\omega)} + C e^{\sup_{x_1 \in \left(-\frac{7R}{24}, -\frac{R}{4}\right)} \{\varphi_{k_n}\}} \|u\|_{H^1(\Omega)}.$$

Using then properties (4.2), we deduce that there exist two positive constants A and B such that for all $n \geqslant \tau_3(V, W_1, W_2)$,

$$\|u\|_{H^1(\mathcal{O})} \leqslant C e^{An} \|u\|_{H^1(\omega)} + C e^{-Bn} \|u\|_{H^1(\Omega)}.$$

Optimizing the right hand side with respect to $n \geqslant \tau_3(V, W_1, W_2)$, we obtain

$$\|u\|_{H^1(\mathcal{O})} \leqslant C \|u\|_{H^1(\omega)}^{\frac{B}{A+B}} \|u\|_{H^1(\Omega)}^{\frac{A}{A+B}} \exp(C\tau_3(V, W_1, W_2)).$$

This concludes the proof of Lemma 4.1. □

Remark 4.4. *It is clear from the above proof that, if V is the finite sum of potentials $V_i \in L^{p_i}(\Omega)$, the estimate (1.3) still holds by replacing $\|V\|_{L^p(\Omega)}^{\gamma(p)}$ by $\sum_i \|V_i\|_{L^{p_i}(\Omega)}^{\gamma(p_i)}$.*

5 Other geometries and proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. We will do that using several geometrical settings, up to a quantitative three balls estimate.

5.1 An annulus observed from a neighbourhood of its external boundary

Lemma 5.1. *Let $R > 0$ and $d \geqslant 3$. We consider the following geometric setting (see Figure 2):*

$$\Omega = \mathcal{A}_0\left(\frac{R}{4}, 2R\right), \quad \mathcal{O} = \mathcal{A}_0\left(\frac{R}{2}, 2R\right), \quad \omega = \mathcal{A}_0\left(R, 2R\right).$$

(Here, $\mathcal{A}_0(r_1, r_2)$ denotes the annulus $B_0(r_2) \setminus B_0(r_1)$.)

There exist constants $C = C(R, d) > 0$ and $\alpha \in (0, 1)$ depending only on R and d so that any solution $u \in H^1(\Omega)$ of (1.1) with (V, W_1, W_2) as in (1.2) satisfies the quantitative unique continuation estimate (1.3) with γ and δ as in (1.4).

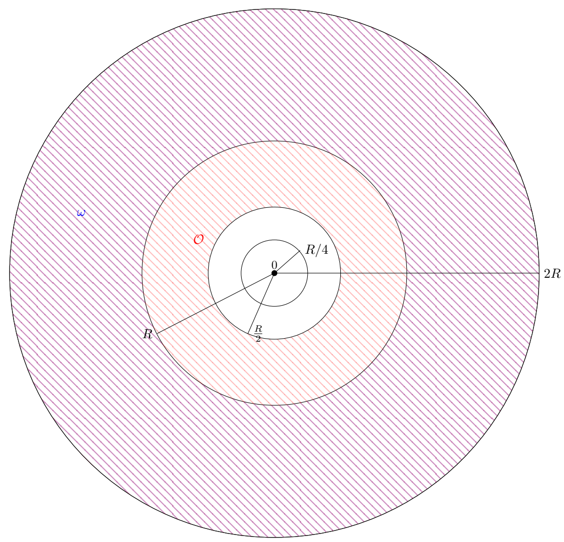

Figure 2: The geometric setting of Lemma 5.1: Propagation of smallness from a neighborhood of a sphere to its interior.

Proof. The proof of Lemma 5.1 directly follows from Lemma 4.1. Let $x_0 \in \mathbb{S}^{d-1}$, and apply Lemma 4.1 with

$$\Omega_{x_0} = B_0(2R) \cap \left\{x \cdot x_0 < -\frac{R}{4}\right\}, \quad \mathcal{O}_{x_0} = B_0\left(\frac{3R}{2}\right) \setminus \left\{x \cdot x_0 < -\frac{R}{3}\right\}, \quad \omega_{x_0} = (B_0(2R) \setminus B_0(R)) \cap \Omega_{x_0}.$$

Accordingly, for all $x_0 \in \mathbb{S}^{d-1}$, there exists a constant $C_{x_0} > 0$ such that

$$\begin{aligned}\|u\|_{H^1(\mathcal{O}_{x_0}))} &\leqslant C_{x_0} e^{C_{x_0}\left(\|V\|^{\gamma(q_0)}_{L^{q_0}(\Omega)} + \|W_1\|^{\delta(q_1)}_{L^{q_1}(\Omega)} + \|W_2\|^{\delta(q_2)}_{L^{q_2}(\Omega)}\right)} \|u\|^{\alpha}_{H^1(\omega_{x_0})} \|u\|^{1-\alpha}_{H^1(\Omega_{x_0})} \\ &\leqslant C_{x_0} e^{C_{x_0}\left(\|V\|^{\gamma(q_0)}_{L^{q_0}(\Omega)} + \|W_1\|^{\delta(q_1)}_{L^{q_1}(\Omega)} + \|W_2\|^{\delta(q_2)}_{L^{q_2}(\Omega)}\right)} \|u\|^{\alpha}_{H^1(\omega)} \|u\|^{1-\alpha}_{H^1(\Omega)}.\end{aligned}$$

The constant C_{x_0} is in fact independent of x_0 due to the invariance by rotation of the problem. Accordingly, we simply denote it by C in the following. Consequently, the right-hand side of the previous estimate does not depend on x_0. Accordingly, taking the square and integrating this inequality with respect to x_0 over the sphere $\mathbb{S}^{d-1}$, we obtain an estimate on

$$\int_{x \in B_0\left(\frac{3R}{2}\right) \setminus B_0\left(\frac{R}{3}\right)} (|u|^2 + |\nabla u|^2) \rho_R(x) dx,$$

where $\rho_R(x) = \int_{x_0 \in \mathbb{S}^{d-1}} 1_{x \cdot x_0 < -\frac{R}{3}}(x_0) dx_0$. It is then easy to check that ρ_R is a radial function, vanishing for $|x| \in (0, R/3)$, and increasing. Consequently, we derive

$$\|u\|_{H^1\left(B_0\left(\frac{3R}{2}\right) \setminus B_0\left(\frac{R}{2}\right)\right)} \leqslant \frac{C}{\sqrt{\rho_R\left(\frac{R}{2}\right)}} e^{C\left(\|V\|^{\gamma(q_0)}_{L^{q_0}(\Omega)} + \|W_1\|^{\delta(q_1)}_{L^{q_1}(\Omega)} + \|W_2\|^{\delta(q_2)}_{L^{q_2}(\Omega)}\right)} \|u\|^{\alpha}_{H^1(\omega)} \|u\|^{1-\alpha}_{H^1(\Omega)}.$$

The estimate on u in $H^1(B_0(2R) \setminus B_0(3R/2))$ is straightforward since $B_0(2R) \setminus B_0(3R/2) \subset \omega \subset \Omega$. $\square$

5.2 A three balls estimate

In this part, we prove a quantitative three balls inequality:

Lemma 5.2 (Three balls estimate)**.** *Let $R > 0$ and $d \geqslant 3$. We consider the following geometric setting (see Figure 3):*

$$\Omega = B_0(4R),\ \mathcal{O} = B_0(2R),\ \text{and } \omega = B_0(R);$$

Then there exist constants $C = C(R,d) > 0$ and $\alpha \in (0,1)$ depending only on R and d so that any solution $u \in H^1(\Omega)$ of (1.1) with (V, W_1, W_2) as in (1.2) satisfies the quantitative unique continuation estimate (1.3) with γ and δ as in (1.4).

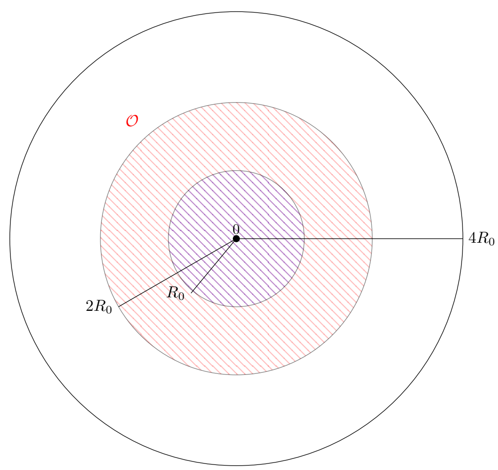

Figure 3: The geometric setting of Lemma 5.2: Propagation of smallness from a ball to its exterior.

Proof. Step 1: Conformal reflection. First, we consider the following geometric setting

$$\Omega_0 = \mathcal{A}_0\left(\frac{R}{2}, 4R\right), \quad \mathcal{O}_0 = \mathcal{A}_0\left(\frac{R}{2}, 2R\right), \quad \omega_0 = \mathcal{A}_0\left(\frac{R}{2}, R\right).$$

We denote by T the conformal reflection with respect to the sphere $S_0(R)$, given by:

$$\mathbb{R}^d \setminus \{0\} \ni x \mapsto \widetilde{x} = T(x) := \frac{R^2}{|x|^2}x, \tag{5.1}$$

The images of the sets Ω_0, $\mathcal{O}_0$ and ω_0 are then given by:

$$\widetilde{\Omega} = T\Omega_0 = \mathcal{A}_0\left(\frac{R}{4}, 2R\right), \quad \widetilde{\mathcal{O}} = T\mathcal{O}_0 = \mathcal{A}_0\left(\frac{R}{2}, 2R\right), \quad \widetilde{\omega} = T\omega_0 = \mathcal{A}_0\left(R, 2R\right). \tag{5.2}$$

Therefore, for $u \in H^1(\Omega)$ a solution of (1.1) with (V, W_1, W_2) as in (1.2), we consider the Kelvin transform of u (see, for example, [17]),

$$\widetilde{u}_R(x) = \left(\frac{R}{|x|}\right)^{(d-2)} u\left(\frac{R^2}{|x|^2}x\right), \qquad x \in \widetilde{\Omega}. \tag{5.3}$$

By a classical computation, using the chain rule, we can verify that for all $i \in \{1, \cdots, d\}$,

$$\partial_{x_i}\widetilde{u}_R(x) = -\frac{(d-2)x_i}{|x|^2}\widetilde{u}_R(x) + \left(\frac{R}{|x|}\right)^d \sum_{j=1}^d \left(\delta_{ij} - 2\frac{x_i x_j}{|x|^2}\right)\partial_{x_j} u\left(\frac{R^2}{|x|^2}x\right), \qquad x \in \widetilde{\Omega},$$

and

$$\Delta \widetilde{u}_R(x) = \left(\frac{R}{|x|}\right)^{(d+2)} \Delta u \left(\frac{R^2}{|x|^2}x\right), \qquad x \in \widetilde{\Omega}.$$

We then consider the following potentials

$$\begin{aligned}
\widetilde{W}_{R,j}(x) &:= \frac{R^2}{|x|^2}\left(W_j\left(\frac{R^2}{|x|^2}x\right) - 2x \cdot W_j\left(\frac{R^2}{|x|^2}x\right)\frac{x}{|x|^2}\right), \quad j \in \{1,2\},\\
\widetilde{V}_R(x) &:= \underbrace{\frac{R^4}{|x|^4}\left(V\left(\frac{R^2}{|x|^2}x\right)\right)}_{=:\widetilde{V}_{R,1}} + \frac{(d-2)}{|x|^2}x \cdot \left(\widetilde{W}_{R,1}(x) + \widetilde{W}_{R,2}(x)\right).
\end{aligned}$$

Consequently, $u \in H^1(\Omega)$ is a solution of (1.1) with (V, W_1, W_2) if and only if u_R given by (5.3) solves

$$\Delta \widetilde{u}_R = \widetilde{V}_R \widetilde{u}_R + \widetilde{W}_{R,1} \cdot \nabla \widetilde{u}_R + \operatorname{div}\left(\widetilde{W}_{R,2}\widetilde{u}_R\right) \quad \text{ in } \widetilde{\Omega}.$$

Step 2: Application of Lemma 5.1. Applying Lemma 5.1 to $\widetilde{u}_R$ with the geometric setting defined in (5.2) (together with Remark 4.4), there exists $C > 0$ depending only on d and R such that

$$\|\widetilde{u}_R\|_{H^1(\widetilde{\mathcal{O}})} \leqslant Ce^{C\left(\|\widetilde{V}_{R,1}\|^{\gamma(q_0)}_{L^{q_0}(\widetilde{\Omega})} + \|\widetilde{W}_{R,1}\|^{\gamma(q_1)}_{L^{q_1}(\widetilde{\Omega})} + \|\widetilde{W}_{R,2}\|^{\gamma(q_2)}_{L^{q_2}(\widetilde{\Omega})} + \|\widetilde{W}_{R,1}\|^{\delta(q_1)}_{L^{q_1}(\widetilde{\Omega})} + \|\widetilde{W}_{R,2}\|^{\delta(q_2)}_{L^{q_2}(\widetilde{\Omega})}\right)} \|\widetilde{u}_R\|^{\alpha}_{H^1(\widetilde{\omega})} \|\widetilde{u}_R\|^{1-\alpha}_{H^1(\widetilde{\Omega})},$$

with γ and δ as defined in (1.4). Then, using the change of variables $y = (R/|x|)^2 x$, one can verify that

$$\begin{aligned}
\left\|\widetilde{W}_{R,j}\right\|_{L^{q_j}(\widetilde{\Omega})} &\simeq \|W_j\|_{L^{q_j}(\Omega_0)}, \quad j \in \{1,2\},\\
\left\|\widetilde{V}_{R,1}\right\|_{L^{q_0}(\widetilde{\Omega})} &\simeq \|V\|_{L^{q_0}(\Omega_0)},\\
\|\widetilde{u}_R\|_{H^1(\widetilde{\Pi})} &\simeq \|u\|_{H^1(T^{-1}\Pi)}, \quad \Pi \in \{\widetilde{\Omega}, \widetilde{\mathcal{O}}, \widetilde{\omega}\},
\end{aligned}$$

where we have used that the Jacobian determinant of the map T^{-1} $(= T)$ is bounded.

Since $\gamma(q) \leqslant \delta(q)$ for all $q \in (d, \infty]$, there exists a positive constant C depending only on d and R, such that

$$\begin{aligned}
\|u\|_{H^1(\mathcal{O}_0)} &\leqslant Ce^{C\left(\|V\|^{\gamma(q_0)}_{L^{q_0}(\Omega_0)} + \|W_1\|^{\delta(q_1)}_{L^{q_1}(\Omega_0)} + \|W_2\|^{\delta(q_2)}_{L^{q_2}(\Omega_0)}\right)} \|u\|^{\alpha}_{H^1(\omega_0)} \|u\|^{1-\alpha}_{H^1(\Omega_0)}\\
&\leqslant Ce^{C\left(\|V\|^{\gamma(q_0)}_{L^{q_0}(\Omega)} + \|W_1\|^{\delta(q_1)}_{L^{q_1}(\Omega)} + \|W_2\|^{\delta(q_2)}_{L^{q_2}(\Omega)}\right)} \|u\|^{\alpha}_{H^1(\omega)} \|u\|^{1-\alpha}_{H^1(\Omega)}.
\end{aligned}$$

To conclude Lemma 5.2, one should also get a similar estimate for $\|u\|_{H^1(B_0(R/4))}$; this latter estimate is straightforward as $B_0(R/4) \subset \omega \subset \Omega$. The proof of Lemma 5.2 is thus completed. □

5.3 The general case: Proof of Theorem 1.1

Proof of Theorem 1.1. The strategy follows the same lines as the one of [11, Theorem 5.6], see also [8, Theorem 1.2], and is based on the classical ideas that three balls estimates allow to propagate the information.

Step 1: Propagation of smallness in neighborhoods of points y in $\overline{\mathcal{O}}$. Recall the geometric condition **(GC)**: For all $y \in \overline{\mathcal{O}}$, there exist $x_0 \in \omega$, $r_y > 0$ and a smooth path γ_y of finite length such that $\gamma_y(0) = x_0$, $\gamma_y(1) = y$, and $\cup_{s\in[0,1]} B_{\gamma_y(s)}(r_y) \subset \Omega$.

Accordingly, for $y \in \overline{\mathcal{O}}$, we take such path γ_y, and define $R_y = \min\{r_y/4, r_0\}$, where r_0 is such that $B_{x_0}(r_0) \subset \omega$.

We define a sequence $(x_{(j)})_j$, for $j \geqslant 0$, by $x_{(j)} = \gamma_y(t_j)$ where $t_0 = 0$ and, for $j \geqslant 1$,

$$t_j = \begin{cases} \inf A_j & \text{if } A_j \neq \emptyset, \\ 1 & \text{if } A_j = \emptyset, \end{cases} \qquad \text{where } A_j = \left\{\sigma \in (t_{j-1}, 1]\,;\, \gamma_y(\sigma) \notin B_{x_{(j-1)}}(R_y)\right\}.$$

The sequence $(x_{(j)})_j$ is finite since the length of γ_y is finite. Let $\big(x_{(0)}, \cdots, x_{(N_y)}\big)$ be such a sequence with $x_{(N_y)} = y$. Note that we have $B_{x_{(j+1)}}(R_y) \subset B_{x_{(j)}}(2R_y) \subset B_{x_{(j)}}(4R_y) \subset \Omega$ for $j = 0, \cdots, N_y - 1$, because of the choice we made for R_y above. By Lemma 5.2 there exist $C > 0$ and $\alpha \in (0,1)$ such that

$$\|u\|_{H^1\left(B_{x_{(j+1)}}(R_y)\right)} \leqslant \|u\|_{H^1\left(B_{x_{(j)}}(2R_y)\right)} \leqslant Ce^{C\left(\|V\|^{\gamma(q_0)}_{L^{q_0}(\Omega)} + \|W_1\|^{\delta(q_1)}_{L^{q_1}(\Omega)} + \|W_2\|^{\delta(q_2)}_{L^{q_2}(\Omega)}\right)} \|u\|^{1-\alpha}_{H^1(\Omega)} \|u\|^{\alpha}_{H^1(B_{x_{(j)}}(R_y))},$$

for $j = 0, \ldots, N-1$. Iterating this estimate we obtain

$$\begin{aligned}\|u\|_{H^1(B_y(R_y))} &\leqslant C^{\sum_{j=0}^{N_y} \alpha^j} e^{C\left(\sum_{j=0}^{N_y} \alpha^j\right)\left(\|V\|^{\gamma(q_0)}_{L^{q_0}(\Omega)} + \|V_2\|^{\gamma_2(p_2)}_{L^{p_2}(\Omega)} + \|W_1\|^{\delta(q_1)}_{L^{q_1}(\Omega)} + \|W_2\|^{\delta(q_2)}_{L^{q_2}(\Omega)}\right)} \|u\|^{1-\alpha^{N_y}}_{H^1(\Omega)} \|u\|^{\alpha^{N_y}}_{H^1\left(B_{x_{(0)}}(R_y)\right)} \\ &\leqslant C_y e^{C_y\left(\|V\|^{\gamma(q_0)}_{L^{q_0}(\Omega)} + \|V_2\|^{\gamma_2(p_2)}_{L^{p_2}(\Omega)} + \|W_1\|^{\delta(q_1)}_{L^{q_1}(\Omega)} + \|W_2\|^{\delta(q_2)}_{L^{q_2}(\Omega)}\right)} \|u\|^{1-\alpha_y}_{H^1(\Omega)} \|u\|^{\alpha_y}_{H^1(\omega)},\end{aligned}$$

for some $C_y > 0$ and $\alpha_y \in (0,1)$.

Step 2: Compactness argument. Because of the compactness of $\overline{\mathcal{O}}$, we can choose a finite number of balls $(B_{y_j}(R_{y_j}/2))_{j\in\{1,\cdots,p\}}$ with $y_j \in \overline{\mathcal{O}}$ and R_{y_j} as above such that $\overline{\mathcal{O}} \subset \cup_{j\in\{1,\cdots,p\}} B_{y_j}(R_{y_j}/2)$. We then construct a partition of unity of $\overline{\mathcal{O}}$ by choosing smooth functions $(\chi_j)_{0\leqslant j\leqslant N}$, each one being supported in $B_{y_j}(R_{y_j})$, such that

$$\sum_{j=1}^{p} \chi_j = 1 \text{ in a neighborhood of } \overline{\mathcal{O}}, \quad 0 \leqslant \chi_j \leqslant 1.$$

Therefore,

$$\begin{aligned}\|u\|_{H^1(\mathcal{O})} &\leqslant C \sum_{j=1}^{p} \|u\|_{H^1(\operatorname{supp}\chi_j)} \leqslant C \sum_{j=1}^{p} \|u\|_{H^1\left(B_{y_j}\left(R_{y_j}\right)\right)} \\ &\leqslant Ce^{C\left(\|V\|^{\gamma(q_0)}_{L^{q_0}(\Omega)} + \|V_2\|^{\gamma_2(p_2)}_{L^{p_2}(\Omega)} + \|W_1\|^{\delta(q_1)}_{L^{q_1}(\Omega)} + \|W_2\|^{\delta(q_2)}_{L^{q_2}(\Omega)}\right)} \|u\|^{1-\alpha}_{H^1(\Omega)} \|u\|^{\alpha}_{H^1(\omega)},\end{aligned}$$

with $\alpha = \min_{j\in\{1,\cdots,p\}}\{\alpha_{y_j}\}$. This concludes the proof of Theorem 1.1. □

A L^p-L^q estimates for Fourier multipliers

In this section, we present the machinery used to get estimates on the operators $(K_{\tau,j})_{j\in\{0,\cdots,d\}}$, and that was developed in [6, Section 5].

Let $n \geqslant 2$. We consider $X_0 < X_1$ and coefficients $(\lambda_j)_{j\in\{1,\cdots,n\}}$ satisfying

$$\exists c_0 > 0, \quad \forall a \in [X_0, X_1], \forall \xi \in \mathbb{R}^n, \quad \frac{1}{c_0}|\xi|^2 \leqslant \sum_{j=1}^{n}(1 - a\lambda_j)|\xi_j|^2 \leqslant c_0|\xi|^2. \tag{A.1}$$

We also introduce the function ψ defined by

$$\psi(a,\xi) = \sqrt{\sum_{j=1}^{n}(1 - a\lambda_j)\xi_j^2}, \qquad a \in [X_0, X_1],\ \xi \in \mathbb{R}^n. \tag{A.2}$$

and Σ_a the ellipsoid defined for $a \in [X_0, X_1]$ by

$$\Sigma_a = \{\xi \in \mathbb{R}^n, \psi(a,\xi) = 1\}. \tag{A.3}$$

For $a \in [X_0, X_1]$ and $k \in L^\infty(\mathbb{R}_+, L^\infty(\Sigma_a))$, we consider operators given as follows:

$$K_{a,k} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n), \text{ given by } \widehat{K_{a,k}(f)}(\xi) = k\left(\psi(a,\xi), \frac{\xi}{\psi(a,\xi)}\right)\widehat{f}(\xi), \qquad \xi \in \mathbb{R}^n. \tag{A.4}$$

In other words, $K_{a,k}$ is defined as a Fourier multiplier, and we look at the multiplier in some kind of radial coordinates associated to Σ_a: $\psi(a,\xi)$ is a positive real number corresponding to a radius, and $\xi/\psi(a,\xi)$ is an element of the ellipsoid Σ_a. Also note that for $a=0$, this coincides with the classical radial coordinates for $\mathbb{R}^n$.

We have the following result:

Proposition A.1 (Proposition 5.3. [6]). *Let $n \in \mathbb{N}$, $n \geqslant 2$. Let $X_0 < X_1$, and the coefficients $(\lambda_j)_{j\in\{1,\cdots,n\}}$ satisfy (A.1). For $a \in [X_0, X_1]$, let ψ and Σ_a be as in (A.3)–(A.2). Then there exists a constant $C > 0$ such that*

- *for all $a \in [X_0, X_1]$, for all $k \in L^\infty(\mathbb{R}_+, L^\infty(\Sigma_a))$, the Fourier multiplier operator $K_{a,k}$ in (A.4) maps $L^2(\mathbb{R}^n)$ to itself and*
$$\|K_{a,k}\|_{\mathscr{L}(L^2(\mathbb{R}^n))} \leqslant \|k\|_{L^\infty(\mathbb{R}_+, L^\infty(\Sigma_a))}. \tag{A.5}$$
- *for all $a \in [X_0, X_1]$, for all $k \in L^\infty(\mathbb{R}_+, L^\infty(\Sigma_a))$ satisfying*
$$\int_0^\infty \|k(\lambda,\cdot)\|_{L^\infty(\Sigma_a)} \lambda^{\frac{n-1}{n+1}} \, d\lambda < \infty,$$
the Fourier multiplier operator $K_{a,k}$ in (A.4) belongs to $\mathscr{L}(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n))$ and
$$\|K_{a,k}\|_{\mathscr{L}(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n))} \leqslant C \int_0^\infty \|k(\lambda,\cdot)\|_{L^\infty(\Sigma_a)} \lambda^{\frac{n-1}{n+1}} \, d\lambda. \tag{A.6}$$
- *for all $a \in [X_0, X_1]$, for all $k \in L^\infty(\mathbb{R}_+, L^\infty(\Sigma_a))$ satisfying*
$$\int_0^\infty \|k(\lambda,\cdot)\|^2_{L^\infty(\Sigma_a)} \lambda^{\frac{n-1}{n+1}} \, d\lambda < \infty,$$
the Fourier multiplier operator $K_{a,k}$ in (A.4) belongs to
$$\mathscr{L}(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^2(\mathbb{R}^n)) \cap \mathscr{L}(L^2(\mathbb{R}^n), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n)),$$
and
$$\|K_{a,k}\|_{\mathscr{L}(L^{\frac{2(n+1)}{(n+3)}}(\mathbb{R}^n), L^2(\mathbb{R}^n))} \leqslant C \sqrt{\int_0^\infty \|k(\lambda,\cdot)\|^2_{L^\infty(\Sigma_a)} \lambda^{\frac{n-1}{n+1}} \, d\lambda}, \tag{A.7}$$
$$\|K_{a,k}\|_{\mathscr{L}(L^2(\mathbb{R}^n), L^{\frac{2(n+1)}{(n-1)}}(\mathbb{R}^n))} \leqslant C \sqrt{\int_0^\infty \|k(\lambda,\cdot)\|^2_{L^\infty(\Sigma_a)} \lambda^{\frac{n-1}{n+1}} \, d\lambda}. \tag{A.8}$$

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