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Global hyperbolicity in higher signatures

Roméo Troubat

November 28, 2024

Abstract

We provide a generalization of global hyperbolicity in pseudo-Riemannian spaces of signature (p, q) for $p \geq q \geq 2$. We then prove the compactness of causal diamonds in globally hyperbolic spaces and deduce the existence of solutions to a Plateau problem for this class of spaces. Finally, we describe the topology of globally hyperbolic spaces in a result generalising Geroch's splitting theorem.

Contents

1	Introduction	2
2	Causality in Lorentzian manifolds	4
2.1	Lorentzian manifolds and time orientation	4
2.2	Causality in a Lorentzian spacetime	6
2.3	Cauchy surfaces and global hyperbolicity	8
3	Causality in higher signature	10
3.1	Time orientation in higher signature	10
3.2	Causality in the pseudo-Euclidian space	12
3.3	Local structure of causal immersions	18
3.4	Future and past of oriented causal spaces of dimension $q - 1$	22
3.5	Causal diamonds in the pseudo-Euclidian space	23
4	Global hyperbolicity in higher signature	27
4.1	Cauchy surfaces	27
4.2	Causal foliations and Cauchy surfaces	29
4.3	Time functions	30
4.4	Cauchy time functions and globally hyperbolic spaces.	31
5	A few results on globally hyperbolic spaces	33
5.1	Global hyperbolicity and causality	33
5.2	Plateau problem in globally hyperbolic spaces	34
5.3	More examples of globally hyperbolic spaces	35
5.4	Topological structure of globally hyperbolic spaces	37
5.5	Closed spacetimes and time functions	39

1 Introduction

The study of Lorentzian geometry on smooth manifolds often takes place through the study the timelike paths, given that a Lorentzian manifold M has a suitable notion of causality. The strongest notion of causality one may ask is that M is *globally hyperbolic*. A relevant characterization is the existence of a smooth map $T : M \rightarrow \mathbb{R}$ such that for each inextendible causal path $c : I \rightarrow M$, the map $T \circ c$ is a bijection from I to \mathbb{R} . Such a map is called a *Cauchy time function*.

Remark 1.1. It is standard to extend the notion of causality to continuous paths by considering their local structure. This gives us more flexibility, allowing us in particular to construct limits of paths, see [10], lemma 6.2.1.

In [5], Geroch showed for a strongly causal Lorentzian manifold M the equivalence between global hyperbolicity and the compactness of the causal diamonds defined as the set of points both in the future of a point x and in the past of another point y . This compactness implies the existence of a length maximizing causal geodesic between any two causally related points by maximizing the length fonctionnal, see [10], section 6.7. More precisely :

Proposition 1.2. *Assume M is a globally hyperbolic Lorentzian manifold and let x, y be two points in M which are the endpoints of at least one future oriented causal path. Then there exists a geodesic γ from x to y such that for any future oriented path c from x to y , $L(\gamma) \leq L(c)$ where*

$$L(c) = \int_I \sqrt{-g(\dot{c}(t))} dt.$$

Instead of a Lorentzian metric, one may want to consider a metric of signature (p, q) where $\min(p, q) \geq 2$, which we shall call a higher signature pseudo-Riemannian metric. Geometry in higher signature has been less studied than its Lorentzian counterpart ; one reason for this is the lack of physical interpretation. Most of the tools developped for Lorentzian geometry fail in this new setting, mostly because of the connectedness the set $(g_{p,q} < 0)$ in $(\mathbb{R}^{p,q}, g_{p,q})$ which does not yield a notion of time orientation for smooth paths.

Pseudo-Riemannian geometry has recently seen some applications in the study of discrete subgroups of $\mathrm{SO}(p, q + 1)$ through the notion of *Anosov representation* (see [4]). Let P_1 be the stabilizer in $\mathrm{SO}(p, q + 1)$ of an isotropic line of $\mathbb{R}^{p,q+1}$ and let Γ be a hyperbolic group of cohomological dimension greater or equal to p . It has been shown, first by Barbot for $q = 1$ in [1] and by Beyrer-Kassel for $q \geq 2$ in [2], that the space of P_1 -Anosov representations from Γ in $\mathrm{SO}(p, q + 1)$ forms a union of connected components, thus giving new examples of higher Teichmüller spaces. The proof of this theorem involves an in-depth study of the actions of these subgroups on the pseudo-hyperbolic space $\mathbb{H}^{p,q}$, a pseudo-Riemannian equivalent to the hyperbolic space. Furthermore, it has been shown by Mess (see [7]) that for $p = 2, q = 1$, those representations are the holonomies of globally hyperbolic $(\mathrm{SO}(2, 2), \mathrm{AdS}_3)$ -spaces with complete Cauchy surfaces. Those results have, in part, motivated the generalizations we attempt in this paper.

Assume (M, g) is a pseudo-Riemannian space of signature (p, q) , let N_0 be a smooth manifold of dimension q and let $f : N_0 \rightarrow M$ be an immersion from N_0 to M . We will say that f is timelike (resp. causal) when for each x in N_0 , the restriction of the metric to the image of df_x is negative (resp. non-positive). Like in Lorentzian geometry, we show in section 3.3 that this notion can be extended to continuous maps from N_0 to M . We can define the notion of inextendible causal map in a manner we shall not expand upon in this introduction (see section 3.2) and we can then generalize the notion of global hyperbolicity to this higher signature.

Definition 1.3. The space M is said to be globally hyperbolic if there exists a smooth manifold N and map $T : M \rightarrow N$ such that for each inextendible causal immersion $f : N_0 \rightarrow M$, the map $T \circ f : N_0 \rightarrow N$ is a homeomorphism. The map T will be called a *Cauchy time function*.

We give a main class of examples of globally hyperbolic spaces.

Proposition 1.4. *Let $(P^p, g_P), (N^q, g_N)$ be two Riemannian manifolds with P complete and N simply connected. Let $M = (P \times N, g_P - g_N)$ be the pseudo-Riemannian product of P by N . Then the projection $\pi : M \rightarrow N$ is a Cauchy time function on M .*

Those examples may be generalized to the setting of complete Riemannian submersions, thus giving more topological diversity to globally hyperbolic spaces.

Proposition 1.5. *Let $T : (M^{p+q}, g_M) \rightarrow (N^q, g_N)$ be a complete Riemannian submersion where N is simply connected and let g be the (p, q) metric defined on M by the splitting $\text{Ker}(dT) \oplus (\text{Ker}(dT))^\perp$ where each fiber is positive and its orthogonal is negative, i.e $g = g_M - 2T^*g_N$. Then the map $T : M \rightarrow N$ is a Cauchy time function.*

Let M be a globally hyperbolic space with time function $T : M \rightarrow N$. For $B \subset N$ a compact submanifold of dimension q with boundary, let $s : \partial B \rightarrow M$ be a causal immersion and let $C(B, s)$ be the set of causal immersions $\hat{s} : B \rightarrow M$ which extend into an inextendible causal map in M and such that $\hat{s}|_{\partial B} = s$.

Theorem 1.6. *The space $C(B, s)$ endowed with the compact-open topology is compact.*

From this compactness result we deduce an application to the Plateau problem in M . Let us first define the area functional for \hat{s} as $\mathcal{A}(\hat{s}) = \text{Vol}(B^\circ, -\hat{s}^*g_M)$.

Theorem 1.7. *There exists an element $\hat{s} \in C(B, s)$ for which the area $\mathcal{A}(\hat{s})$ is maximal.*

Lastly, we prove two results of topological structure on globally hyperbolic spaces. The first one is a direct analogue to Geroch's splitting theorem :

Theorem 1.8. *Let M be a globally hyperbolic space with Cauchy time function $T : M \rightarrow N$, assume that M admits a foliation F by causal spaces of dimension q and let P be a level set of T . Then there exists a diffeomorphism $M \simeq P \times N$ such that T is the projection pr_2 on the second factor and the leaves of F are the fibers of the projection pr_1 on the first factor.*

The second one focuses on the special case in which M must have the structure of a fiber bundle :

Theorem 1.9. *Let M be a globally hyperbolic space with a smooth Cauchy time function $T : M \rightarrow N$. Assume that at least one level set of T is compact. Then $T : M \rightarrow N$ is a smooth fiber bundle over N .*

The structure of the paper will be as follows : In section 2, we re-introduce some basic notions of Lorentzian geometry. In section 3, we adapt some of those notions to the higher signature setting with particular attention given to the pseudo-Euclidian space $M = \mathbb{R}^{p,q}$ and the Einstein space $M = \text{Ein}^{p,q} \simeq \mathbb{S}^p \times \mathbb{S}^q$. In section 4, we introduce globally hyperbolic spaces and give a few examples and immediate properties. In section 5, we give more involved examples and prove the main theorems of the article.

2 Causality in Lorentzian manifolds

2.1 Lorentzian manifolds and time orientation

Let M be a smooth manifold of dimension $p + 1$ and $(g_x)_{x \in M}$ a smooth family of quadratic forms of signature $(p, 1)$ on $T_x M, x \in M$. The couple (M, g) is said to be a *Lorentzian manifold*. Given a point $x \in M$, a tangent vector $v \in T_x M$ can be differentiated by the sign of its norm.

Definition 2.1. A vector $v \in T_x M$ is said to be

- *spacelike* when $g(v) > 0$,
- *lightlike* when $g(v) = 0$,
- *timelike* when $g(v) < 0$.

When $g(v)$ is lightlike or timelike, v is said to be *causal*.

We can extend those infinitesimal definitions to a smooth path of M .

Definition 2.2. Let $c : I \subset \mathbb{R} \rightarrow M$ be a smooth path. It is said to be

- *spacelike* if for all $t \in I, g(\dot{c}(t)) > 0$,
- *lightlike* if for all $t \in I, g(\dot{c}(t)) = 0$,
- *timelike* if for all $t \in I, g(\dot{c}(t)) < 0$.

When for all $t \in I, g(\dot{c}(t))$ is causal, c is said to be *causal*.

Let x be a point of M and let $\mathbb{P}^{<0}(T_x M)$ be the subset of $\mathbb{P}(T_x M)$ made of the timelike lines of $T_x M$. This space can be identified with the hyperbolic space ; in particular, it is contractible. The set of oriented timelike rays $\mathbb{S}^{<0}(T_x M)$ is a double covering of $\mathbb{P}^{<0}(T_x M)$, therefore it must be the disjoint union of two connected components, both homeomorphic to $\mathbb{P}^{<0}(T_x M)$.

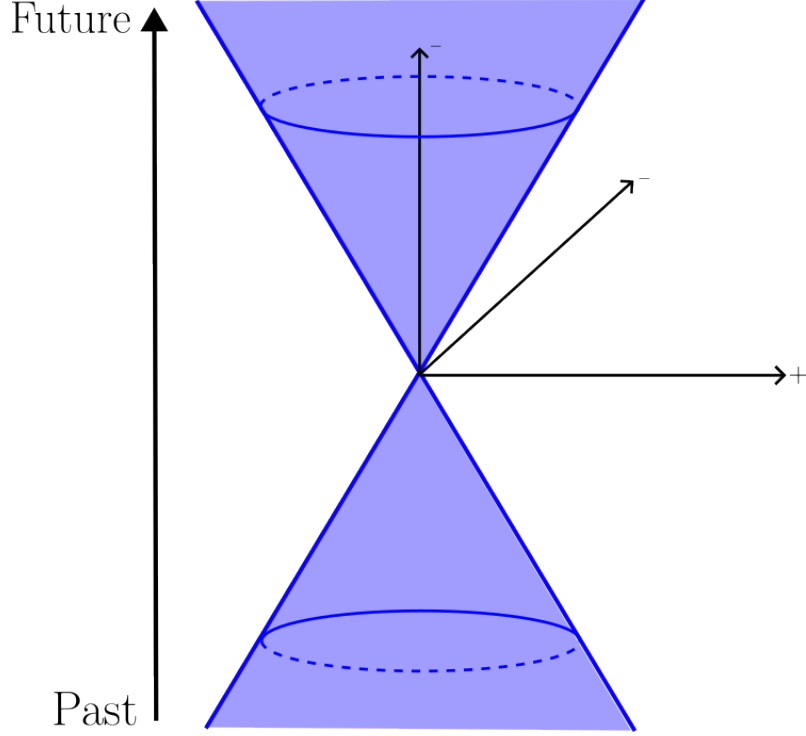


Figure 1: The causal vectors in the Lorentzian space $\mathbb{R}^{2,1}$

Definition 2.3. The choice of one of those components gives a *local time orientation* of M in x . The chosen component will be denoted by $\mathbb{S}^{<0}(T_x M)^+$ whereas the opposite component will be denoted by $\mathbb{S}^{<0}(T_x M)^-$. A tangent timelike ray in x will be said to be *future-oriented* if it is contained in $\mathbb{S}^{<0}(T_x M)^+$. Otherwise, it will be *past-oriented*. A tangent timelike vector $v \in T_x M$ will be future (resp. past) oriented if $[v] \in \mathbb{S}^{<0}(T_x M)^+$ (resp. $[v] \in \mathbb{S}^{<0}(T_x M)^-$).

Remark 2.4. This notion can be extended to causal tangent rays ; for such a ray $[v] \in \mathbb{S}(T_x M)$, we'll say that $[v]$ is future (resp. past) oriented if it is in the closure of $\mathbb{S}^{<0}(T_x M)^+$ (resp. $\mathbb{S}^{<0}(T_x M)^-$) in $\mathbb{S}(T_x M)$. This also applies to non-zero tangent causal vectors.

This time orientation can be extended to the whole manifold M , provided that the local orientation in each point stays coherent on the whole manifold.

Definition 2.5. When the fiber bundle $\mathbb{S}^{<0}(TM)$ over M is trivial, M is said to be *time orientable*. The choice of one of its two connected components gives a *time orientation* of M . We'll denote by $\mathbb{S}^{<0}(TM)^+$ (resp. $\mathbb{S}^{<0}(TM)^-$) the chosen component, called the bundle of *future-oriented* (resp. *past-oriented*) tangent rays.

Definition 2.6. A causal path c in a time oriented manifold M is said to be future (resp. past) oriented if for all t , $\dot{c}(t)$ is future (resp. past) oriented.

A time orientation on a lorentzian manifold is sometimes defined by a nowhere vanishing causal vector field X on M . Those two definitions are in fact equivalent.

Proposition 2.7. *Let M be a time oriented lorentzian manifold. There exists a causal future-oriented vector field X on M .*

Proof. Each fiber of the bundle $\mathbb{S}^{<0}(TM)^+$ is contractible, meaning there must exist a smooth section $X : M \rightarrow \mathbb{S}^{<0}(TM)^+$ (see [9]), hence the result. \square

Remark 2.8. Let M be a smooth manifold admitting a smooth nowhere-vanishing vector field X . Let \bar{g} be a Riemannian metric on M and for each $x \in M$ and $v \in T_xM$, let

$$g(v) = \bar{g}(\pi_{X(x)^\perp}(v)) - \bar{g}(\pi_{X(x)}(v)),$$

where π_E is the orthogonal projection with image E . This defines a Lorentzian metric g on M for which X is timelike, thus giving us the following result : The smooth manifolds admitting a Lorentzian metric are exactly the manifolds admitting a smooth non-vanishing vector field.

Corollary 2.9. *Since each smooth vector field is integrable, every time oriented Lorentzian manifold admits a smooth future-oriented one dimensional timelike foliation.*

Mathematicians and physicists alike often assume that M is both orientable and time-orientable, as it gives both a reasonable model for our spacetime when $p = 3$ and a convenient context in which to do Lorentzian geometry.

Definition 2.10. A Lorentzian manifold M which is both orientable and time orientable is called a *spacetime*.

Unless specified otherwise, every Lorentzian manifold we will consider in this paper will be a Lorentzian spacetime.

2.2 Causality in a Lorentzian spacetime

Now that we have defined what it means for a path to be causal, for $x \in M$, we can consider the set of points which can be causally related to x .

Definition 2.11. The set $I^+(x)$ (resp. $J^+(x)$) is the set of points $y \in M$ such that there exists a timelike (resp causal) future-oriented path c coming from x to y .

It is possible to show that the notion of causality is locally linked to the nature of the geodesics coming from a point x of M .

Definition 2.12. Let $x \in M$. A normal neighborhood of x in M is an open subset U of T_xM containing 0 on which $\exp : T_xM \rightarrow M$ is a diffeomorphism. The sets U and $\exp(U)$ will sometimes be identified.

Proposition 2.13. *Let $U \subset T_xM$ be a normal subset of M centered on x and let $c : I \rightarrow U$ be a timelike future-oriented path such that $c(0) = 0$. Then for all $t > 0$ (resp. $t < 0$), we have $g_x(c(t)) < 0$ and $c(t)$ is in the future (resp. past) time cone $g_x < 0$ of T_xM .*

Remark 2.14. The same result can be stated for causal paths ; the points $y \in U$ such that $g_x(y) = 0$ are exactly those linked to 0 by a lightlike geodesic.

This describes the local structure of a smooth future-oriented causal path and allows us to weaken the regularity condition on c . We are thus able to consider paths which are only continuous but locally verify a property analogous to 2.13.

Definition 2.15. A continuous path $c : I \rightarrow M$ is said to be timelike (resp. causal) if for each $t_0 \in I$, there exists a normal neighborhood U of $c(t_0)$ such that for each $t < t_0$ (resp. $t > t_0$) with $c(t) \in U$, the geodesic $[c(t_0), c(t)]$ is causal past-oriented (resp. future-oriented).

It is possible to show that the set of smooth causal paths is dense in the set of continuous causal paths for the uniform topology.

Proposition 2.16. *Let $c : I \rightarrow M$ be a continuous causal path and d a Riemannian distance on M . For each positive continuous map $\delta : I \rightarrow \mathbb{R}$, there exists a smooth causal path $\bar{c} : I \rightarrow M$ such that for all $t \in I$, $d(c(t), \bar{c}(t)) \leq \delta(t)$.*

In particular, piecewise-smooth causal paths are continuous causal paths in this sense, as long as every smooth piece keeps the same time orientation.

Corollary 2.17. *Let $c_1 : (a, b) \rightarrow M$ and $c_2 : (b, c) \rightarrow M$ be two smooth timelike (resp. causal) paths in M such that $c_1(b) = c_2(b)$ and $\dot{c}_1(b)$ and $\dot{c}_2(b)$ are both in the same connected component of $\{g_{c_1(b)} < 0\}$. Then the concatenation c of c_1 and c_2 is a continuous timelike (resp. causal) path.*

Proof. By using proposition 2.13, this tells us that for ε small enough and for each $b - \varepsilon < t < b$, $c_1(t)$ is in the past of $c_1(b)$ and for each $b + \varepsilon > t > b$, $c_2(t)$ is in the future of $c_2(b)$, hence the result. \square

In what follows, unless specified otherwise, a future-oriented causal path will only be assumed to be continuous. As Lorentzian geometry was primarily developed as a tool to modelize general relativity, we would like to give conditions on spacetimes to get a physically relevant notion of causality. One of those obvious conditions is that we do not want a point in a spacetime M to be causally related to itself.

Definition 2.18. A spacetime M is said to be *non-causal* if there exists a point x and a non-trivial causal path $c : [a, b] \rightarrow M$ such that $c(a) = c(b) = x$. Alternatively, M is non-causal if there exists a future-oriented causal embedding $c : \mathbb{S}^1 \rightarrow M$. When M is non-non-causal, it is said to be *causal*.

Here are some examples of non-causal spacetimes.

Example 2.19. • The Einstein space, $\text{Ein}^{n,1} \simeq (\mathbb{S}^n \times \mathbb{S}^1, ds_n^2 - ds_1^2)$ where ds_n^2 is the round metric on \mathbb{S}^n . The map $t \in \mathbb{S}^1 \mapsto (p, t)$ is a causal loop.

- Any space of the form $(M \times \mathbb{S}^1, ds^2 - ds_1^2)$ where ds^2 is a Riemannian metric on M , for the same reason.
- It is possible to show that any compact spacetime is non-causal (see [10]).

One can reinforce this notion of causality with the existence of a time function.

Definition 2.20. A continuous map $T : M \rightarrow \mathbb{R}$ is said to be a *time function* if for each future causal curve $c : I \rightarrow M$, $T \circ c : I \rightarrow \mathbb{R}$ is increasing.

Such a map gives us a way to parametrize all timelike curves by intervals of \mathbb{R} . In particular, the existence of a time function on M is a stronger causality condition than that of causality.

Corollary 2.21. *If M admits a time function, M must be causal.*

Proof. Assume there exists a future-oriented causal path $c : (a, b) \rightarrow M$ such that $c(a) = c(b)$. Then we must have $T(c(a)) = T(c(b))$, which is impossible since T is assumed to be increasing on every causal curve. \square

Definition 2.22. A path $c : (a, b) \rightarrow M$ is said to be *extendible* if c converges in M near a or b . Otherwise, c is *inextendible*.

As we've seen, it is quite easy to construct smooth future oriented paths in a space-time M , thus it is also possible to extend an extendible future-oriented path into an inextendible future path.

Proposition 2.23. *Let $c : (a, b) \rightarrow M$ be a future path which is extendible in b . There exists an inextendible future path $\tilde{c} : I \rightarrow M$ such that $(a, b) \subset I$ and for all $t \in (a, b)$, $\tilde{c}(t) = c(t)$.*

Proof. The path c can be extended indefinitely by taking a normal neighborhood centered on its endpoint, hence the result. When c is smooth, it is also possible to extend $\dot{c}(t)$, $t \in \bar{I}$ into a causal vector field on M and to take the flow line containing c . \square

2.3 Cauchy surfaces and global hyperbolicity

We will now give a stronger notion of causality on a spacetime M . This notion detects both the fact that M is causal, and the absence of causal paths which go to the boundary of the spacetime M in finite time.

Definition 2.24. Let $x, y \in M$. We will denote by $\mathcal{C}(x, y)$ the set of future-oriented causal paths coming from x to y , which we will endow with the uniform C^0 topology, meaning that the open set of $\mathcal{C}(x, y)$ will be the future-oriented paths from x to y entirely contained in an open set U of M .

Definition 2.25. A spacetime M is said to be *globally hyperbolic* if for each $x, y \in M$, $\mathcal{C}(x, y)$ is compact.

This is actually a very strong condition, which as we will see, implies both the causality of M and the existence of a time function.

Definition 2.26. A subset $S \subset M$ is a *Cauchy surface* of M if for each inextendible future causal curve $c : I \rightarrow M$, there exists a unique $t \in I$ such that $c(t) \in S$.

Definition 2.27. A map $T : M \rightarrow \mathbb{R}$ is said to be a *Cauchy time function* if for each $t \in \mathbb{R}$, $T^{-1}(t)$ is a Cauchy surface of M . Alternatively, this means that for each inextendible future causal curve $c : I \rightarrow M$, $T \circ c : I \rightarrow \mathbb{R}$ is bijective and increasing.

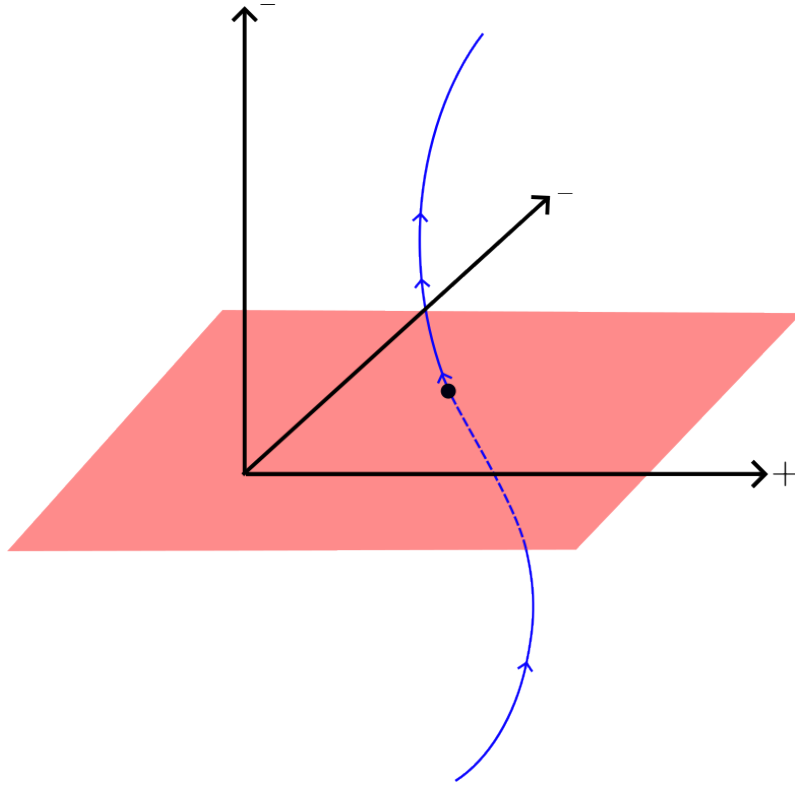


Figure 2: A Cauchy surface of $\mathbb{R}^{2,1}$ which an inextendible timelike curve intersects in a unique point.

Remark 2.28. In particular, a Cauchy time function is a time function.

The fact that a Cauchy time function T is injective gives us the causality of M , and the fact that it must be surjective on each inextendible curve tells us that we cannot find a future causal curve which approaches the boundary of the spacetime M without passing by every "time".

Proposition 2.29. *The three following properties are equivalent :*

- M is globally hyperbolic,
- M admits a Cauchy surface,
- M admits a Cauchy time function.

This condition actually implies a strong topological rigidity on M .

Proposition 2.30. *Let M be a globally hyperbolic space with a Cauchy surface S . Then M is homeomorphic to $S \times \mathbb{R}$ where each slice $S \times \{*\}$ is a Cauchy surface and each line $\{*\} \times \mathbb{R}$ is a timelike curve.*

Proof. Let T be a Cauchy time function with $S = T^{-1}(0)$ and let X be a smooth timelike vector field on M , whose existence is guaranteed by 2.7. For each $x \in M$, let c_x be the flow line containing x . It is an inextendible timelike path, meaning it must encounter S in a unique point which we will denote by $\varphi(x)$. Finally, let us define the map

$$\psi : x \in M \longmapsto (\varphi(x), T(x)) \in S \times \mathbb{R}.$$

By definition of a Cauchy time function, ψ is a homeomorphism for which the slices $S \times \{t\}$ correspond to the Cauchy surfaces $T^{-1}(t)$ and the slices $\{p\} \times \mathbb{R}$ to the timelike flow lines, hence the result. \square

For the rest of this paper, our goal will be to generalize the preceding notions to pseudo-Riemannian spaces of higher signature.

3 Causality in higher signature

3.1 Time orientation in higher signature

Let M be a smooth manifold and $(g_x)_{x \in M}$ a smooth family of quadratic forms of signature (p, q) .

Definition 3.1. Under these conditions, the couple (M, g) is said to be a *pseudo-Riemannian manifold* of signature (p, q) .

Remark 3.2. Taking $q = 1$ gives us a Lorentzian geometry on M .

As this is a generalization of Lorentzian geometry, we will use the same terminology.

Definition 3.3. A vector $v \in T_x M$ is said to be

- *spacelike* when $g(v) > 0$,
- *lightlike* when $g(v) = 0$,
- *timelike* when $g(v) < 0$.

When $g(v)$ is lightlike or timelike, v is said to be *causal*.

Since p and q can both be greater than 2, it is also possible to discuss the nature of the 2-planes in $T_x M$, or more generally, of the k -planes for $k \leq q$.

Definition 3.4. A tangent vector space $Q \subset T_x M$ of dimension k is said to be

- *spacelike* if $g|_Q > 0$,
- *lightlike* if $g|_Q = 0$,
- *causal* if $g|_Q \leq 0$,
- *timelike* if $g|_Q < 0$.

It is possible to extend the notion of time orientability where, instead of taking a coherent orientation of timelike vectors, we take an orientation of timelike tangent vector spaces of dimension q . Let x be a point in M . The set $Gr_q^{<0}(T_x M)$ of q -planes of $T_x M$ on which g_x is definite negative can be viewed as the Riemannian symmetric space $SO_0(p, q)/SO(p) \times SO(q)$. In particular, it is contractible. Let $\widetilde{Gr}_q^{<0}(T_x M)$ be the set of oriented timelike q -planes of $T_x M$. It is a double covering of $Gr_q^{<0}(T_x M)$ which is simply connected, therefore it is the disjoint union of two copies of $Gr_q^{<0}(T_x M)$.

Definition 3.5. The choice of one of those components gives us a *local time orientation* of M in x .

In the same manner as in the Lorentzian case, we are going to require that this local time orientation on the timelike q -vector spaces in $T_x M$ extends to the whole manifold M in a coherent fashion.

Definition 3.6. When the fiber bundle $Gr_q^{<0}(TM)$ is trivial, M is said to be *time orientable*. The choice of a connected component gives a *time orientation* of M . The chosen component $Gr_q^{<0}(TM)^+$ is the bundle of *future-oriented* tangent q -spaces.

From now on, we will assume that M is time-oriented.

Definition 3.7. An oriented causal tangent q -vector space $Q \subset T_x M$ is futur oriented if it is in the closure of $Gr_q^{<0}(T_x M)^+$ in the Grassmanian of q -vector spaces of $T_x M$.

Taking a time orientation on M is equivalent to taking a distribution of oriented timelike q -vector spaces of M .

Proposition 3.8. *Let M be a time oriented pseudo-Riemannian manifold M of signature (p, q) . There exists a smooth distribution of oriented timelike tangent vector spaces of M of dimension q .*

Proof. The bundle $Gr_q^{<0}(TM)^+$ has contractible fibers, meaning it must admit a smooth section (see [9]), hence the result. \square

The existence of a smooth distribution of dimension q on a smooth manifold M^{p+q} characterises the existence of a pseudo-Riemannian metric of signature (p, q) on M .

Proposition 3.9. *Let M be a smooth manifold of dimension $p + q$. M admits a smooth distribution of dimension q if and only if M admits a smooth pseudo-Riemannian metric.*

Proof. Assume M admits a pseudo-Riemannian metric g . The fiber bundle $Gr_q^{<0}(TM)$ has contractible fibers, thus it must admit a smooth section, hence the result. Conversely, assume that M admits a smooth distribution D of dimension q . Let \bar{g} be a Riemannian metric on M and let $x \in M, v \in T_x M$. We'll write

$$g(v) = \bar{g}(\pi_{D(x)^\perp}(v)) - \bar{g}(\pi_{D(x)}(v)),$$

where π_E is the orthogonal projection of image E . This gives us a metric of signature (p, q) on M for which D is a smooth distribution of timelike tangent spaces, hence the result. \square

Remark 3.10. Contrary to the Lorentzian case, since not every q -dimensional distribution of vector spaces is integrable, this does not give a timelike foliation of M . It was however proven by Thurston in [11], [12] that every q -dimensional distribution on a manifold M is always homotopic to an integrable distribution. A natural question one could ask would be whether it is always possible, given a pseudo-Riemannian metric, to find an integrable distribution of dimension q which is timelike at every point. As we will see later, the answer to this question is negative.

Definition 3.11. A pseudo-Riemannian manifold of signature (p, q) is called a (p, q) -spacetime if it is both orientable and time orientable.

Examples of non time-orientable pseudo-Riemannian manifolds include but are not limited to the metric products $(P \times N, ds_P^2 - ds_N^2)$ where N is a non-orientable manifold of dimension q .

In the rest of this paper, it will be assumed that M is a (p, q) spacetime. Let us now generalize the notion of causal curve in higher signature. As we have given a way to distinguish different k -vector spaces based on their induced signature, it is natural to extend the notion of causal curve to the whole family of immersions of dimension k for $k \leq q$.

Definition 3.12. An immersion of dimension $k \leq q$, $f : N_0^k \rightarrow M$ is said to be *timelike* (resp. *causal*, *lightlike*, *spacelike*) if for each $x \in N_0$, the image of $d_x f$ in $T_{f(x)}M$ is timelike (resp. causal, lightlike, spacelike).

Alternatively, one could define $f : N_0 \rightarrow M$ to be timelike (resp. causal) if the pullback metric f^*g on N_0 is negative (resp. non-positive). Since we have endowed M with a coherent orientation for its tangent causal vector spaces of dimension q , it is possible to ask for an immersion of dimension q to be future-oriented or past-oriented.

Definition 3.13. Let N_0 be an oriented space of dimension q . A causal immersion $f : N_0 \rightarrow M$ is future-oriented if for each $x \in N_0$, $Im(df)$ with the orientation induced by N_0 is future-oriented.

In fact, the existence of a time orientation on M insures us that any causal immersion of dimension q is orientable in a way that makes it future-oriented.

Proposition 3.14. *Let $f : N_0 \rightarrow M$ be a causal immersion of dimension q in a spacetime M . Then the manifold N_0 is orientable and one of its orientations makes f future-oriented.*

Proof. It suffices to take the pullback orientation of the time-orientation of M on N_0 via f . This also makes f future-oriented. \square

3.2 Causality in the pseudo-Euclidian space

In this section, we will assume that M is the pseudo-Euclidian space $\mathbb{R}^{p,q}$, i.e the flat space \mathbb{R}^{p+q} endowed with the constant pseudo-Riemannian metric $g = dx_1^2 + \dots + dx_p^2 - dy_1^2 - \dots - dy_q^2$. When $q = 1$, M is the Minkowski space $\mathbb{R}^{p,1}$ and is the simplest Lorentzian space in which we can study causality.

In $\mathbb{R}^{p,1}$, a vector v verifies $g(v) < 0$ (resp. $g(v) \leq 0$) if and only if v is in the future (resp. causal future) of the origin. When $q \geq 2$, we can also differentiate a vector v depending on $g(v)$. Since we have already generalized the notion of time orientation, this gives us a good motivation for the elaboration of a non-trivial notion of causality in $\mathbb{R}^{p,q}$.

A first natural attempt would be to say that a vector v is causally related to the origin if and only if there exists a causal path $c : I \rightarrow \mathbb{R}^{p,q}$ which starts at the origin and ends at the vector v , however, it is easy to see that this definition is trivial.

Proposition 3.15. *Let $v \in \mathbb{R}^{p,q}$, $q \geq 2$. There exists a causal path $c : I \rightarrow \mathbb{R}^{p,q}$ which goes from the origin to v .*

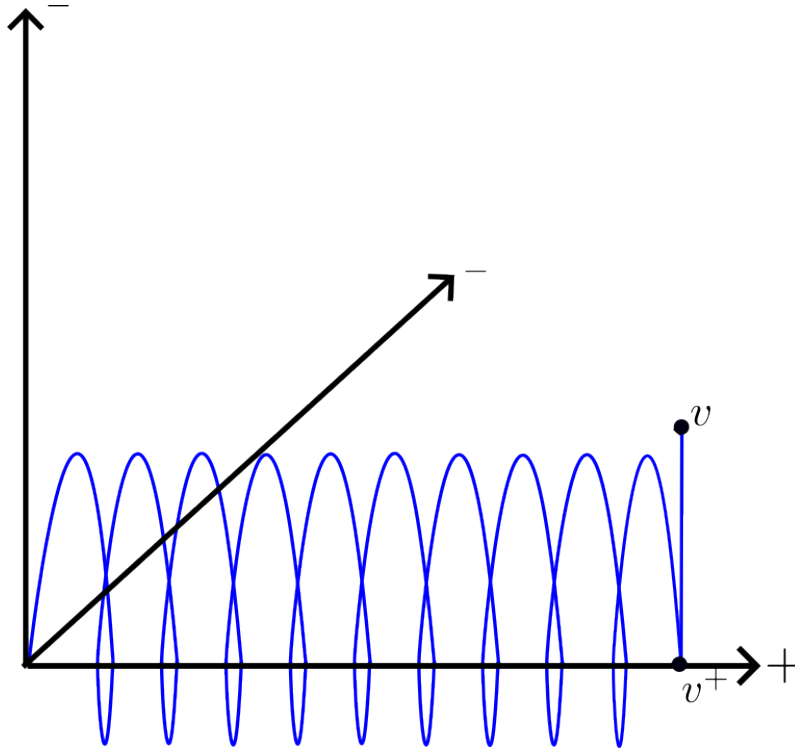


Figure 3: The timelike curve goes from the origin to the vector v .

Proof. Let $v^+ = \pi_{\mathbb{R}^{p,0}}(v)$. Let us define the path

$$c_1 : t \in [0, 2g(v^+)] \mapsto t \frac{v^+}{2g(v^+)} + (\cos(t) - 1)e_{q+1} + \sin(t)e_{q+2},$$

for which $\forall t < 2g(v^+)$, $g(\dot{c}_1(t)) = -\frac{1}{2}$, $c_1(0) = 0$ and $\pi_{\mathbb{R}^{p,0}}(c_1(2g(v^+))) = v^+$. Since $\pi_{\mathbb{R}^{p,0}}(c_2(2g(v^+))) = v^+$, it is possible to take a smooth path c_2 which goes from $c_1(2g(v^+))$ to v while staying in an affine copy of $\mathbb{R}^{0,q}$. The concatenation of c_1 and c_2 (up to a smoothing) gives us a smooth timelike path which goes from the origin to v . \square

This naïve approach to causality in higher signature seems to be doomed. Another natural generalization would be to consider smooth timelike immersions of dimension q instead of smooth paths, since the notion of time orientation we have introduced concerns timelike q -vector spaces. One would then be tempted to define that v is causally related to the origin if and only if there exists a timelike immersion $f : N_0 \rightarrow \mathbb{R}^{p,q}$ of dimension q containing in its image both v and the origin. However, this definition also fails to yield an interesting relation.

Proposition 3.16. *Let $v \in \mathbb{R}^{p,q}$, $q \geq 2$. There exists a timelike immersion $f : N_0 \rightarrow \mathbb{R}^{p,q}$ of dimension q containing both v and the origin in its image.*

Proof. Let $c : I \rightarrow \mathbb{R}^{p,q}$ be a smooth timelike path from a segment I to $\mathbb{R}^{p,q}$ containing both v and the origin in its interior. This path can be constructed in the same manner as before. Since I is compact, it is possible to consider a tubular neighborhood $\bar{c} : \Omega \rightarrow \mathbb{R}^{p,q}$, with $\Omega = \overset{\circ}{I} \times D^{p+q-1}(0, r)$, $\bar{c}|_{\overset{\circ}{I}} = c|_{\overset{\circ}{I}}$ and \bar{c} an immersion. The space Ω can be endowed with a pseudo-Riemannian metric of signature (p, q) via the pullback of g by \bar{c} . We now only have to take $N_0 = \overset{\circ}{I} \times D^{q-1}(0, r)$. We have thickened c into a timelike immersion \bar{c} of dimension q whose image contains the image of c , and thus both the vector v and the origin. \square

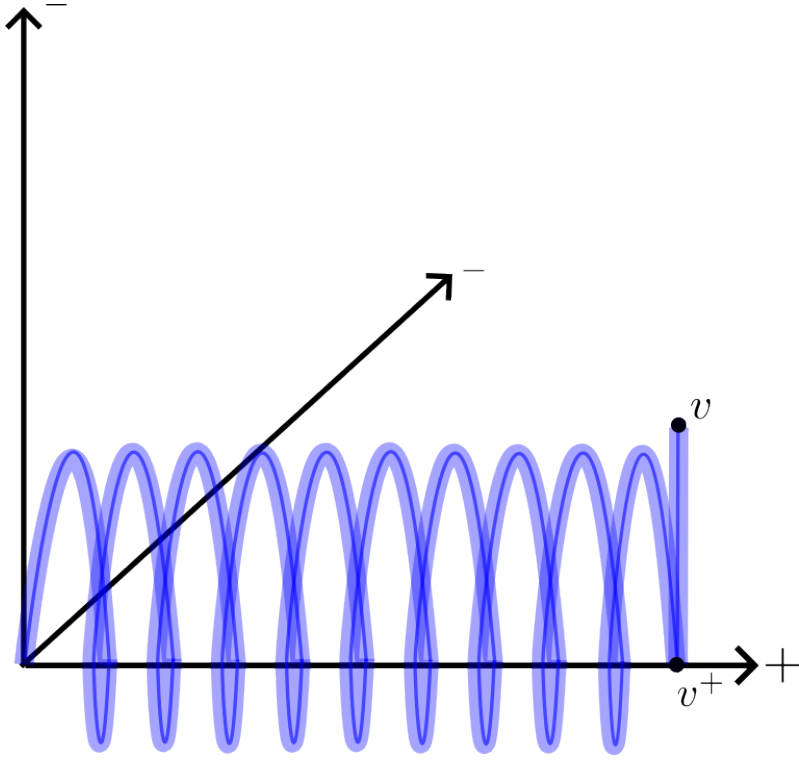


Figure 4: The same curve, thickened into a timelike immersion of dimension 2.

The property we need in order to get an interesting notion of causality in $\mathbb{R}^{p,q}$ is that of *inextendability*. We want our notion of inextendability to generalize the one for paths. Intuitively, we want for the immersion f to be "without boundary" within M . Being proper does not satisfy our needs as we would like the path $\mathbb{R} \rightarrow \mathbb{S}^1$ to be inextendible. Only taking the boundary of the image of f also does not work ; One would like for the endpoint b to be accounted for in the boundary of f .

Definition 3.17. Let M be a (p, q) -spacetime. A continuous map $f : N_0 \rightarrow M$ from a k -dimensional smooth manifold N_0 is said to be *inextendible* if for each continuous path $c : I \rightarrow M$ which is inextendible in N_0 , $f \circ c$ is inextendible in M .

Note that any path which is not inextendible can always be extended into an inextendible path. That is not the case of upper-dimensional immersions.

Definition 3.18. Let $x \in \mathbb{R}^{p,q}$. We will denote by $I(x)$ (resp. $J(x)$) the set of points $y \in \mathbb{R}^{p,q}$ for which there exists an inextendible timelike (resp. causal) immersion of dimension q containing both x and y . When $y \in I(x)$ (resp. $y \in J(x)$), the set $\{x, y\}$

will be said to be in *timelike position* (resp. *causal position*). More generally, when a set $E \subset \mathbb{R}^{p,q}$ is included in the image of an inextendible timelike (resp. causal) immersion of dimension q , the set E will be said to be in timelike position (resp. causal position).

In order to show that this notion of causality is non-trivial, let us study the inextendible timelike immersions of dimension q in $\mathbb{R}^{p,q}$. Let $T : \mathbb{R}^{p,q} \rightarrow \mathbb{R}^q$ be the projection on the negative coordinates of the pseudo-Euclidian space.

Proposition 3.19. *Let $f : N_0 \rightarrow \mathbb{R}^{p,q}$ be an inextendible timelike immersion. Then the map $T \circ f : N_0 \rightarrow \mathbb{R}^q$ is a diffeomorphism. In particular, f can be seen as the graph of a 1-contracting map from \mathbb{R}^q to \mathbb{R}^p .*

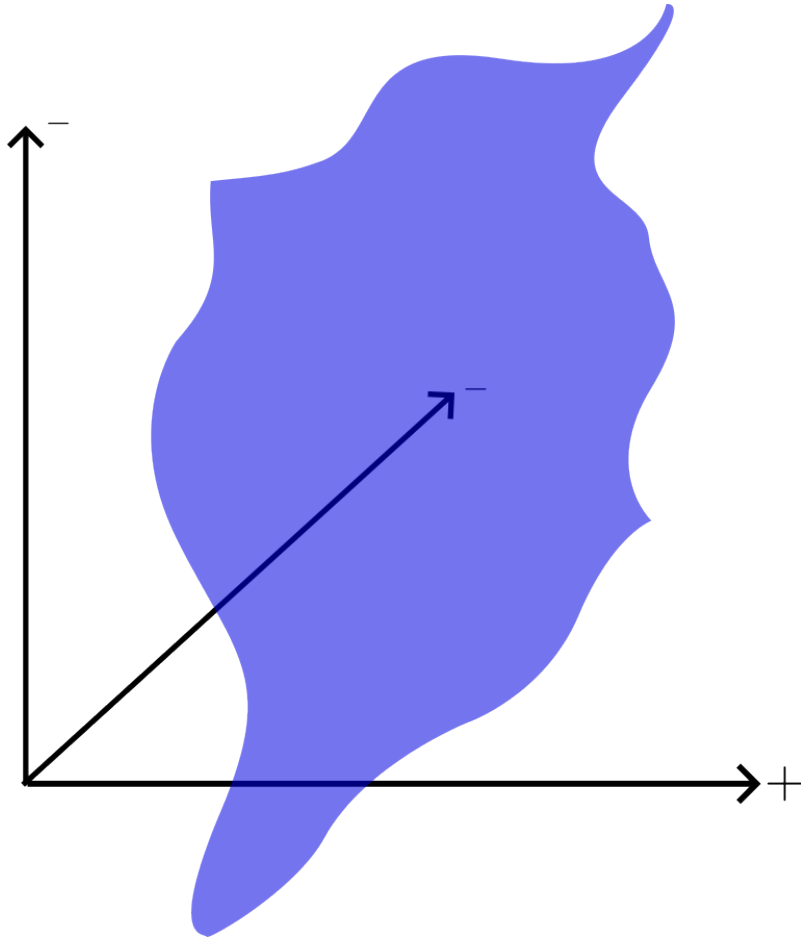


Figure 5: An inextendible timelike immersion of dimension 2 in $\mathbb{R}^{1,2}$.

Proof. Let us first show that the map $T \circ f$ is a local diffeomorphism. Since the projection T removes the negative coordinates, the map $T \circ f$ increases the distances between the riemannian manifolds $(N_0, -f^*g_{\mathbb{R}^{p,q}})$ and $(\mathbb{R}^q, g_{\mathbb{R}^q})$, i.e for each $x \in N_0, v \in T_x N_0$,

$$g_{\mathbb{R}^q}(d(T \circ f)(v)) \geq -f^*g_{\mathbb{R}^{p,q}}(v).$$

This tells us in particular that $T \circ f$ is a local diffeomorphism ; in particular, its image $Im(T \circ f)$ is open in \mathbb{R}^q .

Let's assume that $T \circ f$ is not surjective and let $y \in \partial Im(T \circ f)$. Let $c : (a, b) \rightarrow \mathbb{R}^q$ be a smooth path of speed 1 in $Im(T \circ f)$ converging to y in b .

We'll denote by $\bar{g} = g_{\mathbb{R}^p} + g_{\mathbb{R}^q}$ the flat Riemannian metric on $\mathbb{R}^{p,q} = \mathbb{R}^{p+q}$. For each $x \in \mathbb{R}^{p,q}$ and $v \in T_x \mathbb{R}^{p,q}$ timelike, we have that

$$\begin{aligned}\bar{g}(v) &= g_{\mathbb{R}^p}(v) + g_{\mathbb{R}^q}(v) \\ &\leq 2g_{\mathbb{R}^q}(v),\end{aligned}$$

since $g_{\mathbb{R}^{p,q}}(v) \leq 0$. In particular, if \tilde{c} is a path in N_0 such that $T \circ f \circ \tilde{c} = c$, we have that the speed of $f(\tilde{c})$ for \bar{g} is lower than 2. Since $\mathbb{R}^{p,q}$ is complete, $f(\tilde{c})$ must converge, and since f is inextendible so must \tilde{c} . However we have assumed that $y \notin Im(T \circ f)$, hence the contradiction ; the map $T \circ f$ must be surjective.

Let us now show that $T \circ f$ is a covering on \mathbb{R}^q . Let $y \in \mathbb{R}^q$ and let $c : (0, 1) \rightarrow \mathbb{R}^q$ be a smooth path such that $c(0) = y$. Let $x \in N_0$ such that $T \circ f(x) = y$ and $\tilde{c} : I = (0, b) \rightarrow N_0$ such that $T \circ f(\tilde{c}) = c$. Assume that $b < 1$. By the same reasoning as before, the smooth path $f(\tilde{c})$ must converge in M . Since f is inextendible, the path \tilde{c} must also converge in N_0 . However since $T \circ f$ is a local diffeomorphism, \tilde{c} can then be defined further than b , meaning that $b = 1$. The map $T \circ f$ must be a covering from N_0 to N .

Finally, since \mathbb{R}^q is simply connected, the map $T \circ f$ must be a diffeomorphism from N_0 to \mathbb{R}^q , hence the result.

This result tells us that $N_0 \simeq \mathbb{R}^q$ and that f can be seen as the graph of a map $g : \mathbb{R}^q \rightarrow \mathbb{R}^q$ such that for each $x \in \mathbb{R}^q$, $f(x) = (g(x), x)$. The map f is timelike if and only if $\|dg\| < 1$, i.e if and only if g is 1-contracting. \square

We can extend this result to the case where f is a causal immersion.

Definition 3.20. Let g_1 and g_2 be two metrics of signature (p, q) on a manifold M . We say that $g_1 \leq g_2$ (resp. $g_1 < g_2$) if every causal tangent vector for g_2 is causal for g_1 (resp. every causal tangent vector for g_2 is timelike for g_1).

Remark 3.21. This actually defines a relation on the conformal classes of g_1 and g_2 .

Proposition 3.22. *Let $f : N_0 \rightarrow \mathbb{R}^{p,q}$ be an inextendible causal immersion. Then the map $T \circ f : N_0 \rightarrow \mathbb{R}^q$ is a diffeomorphism. In particular, f can be seen as the graph of a 1-Lipschitz map from \mathbb{R}^q to \mathbb{R}^p which is 1-contracting if and only if f is timelike.*

Proof. Let g_- be the constant (p, q) metric on \mathbb{R}^{p+q} defined by

$$g_- = dx_1^2 + \dots dx_p^2 - 2dy_1^2 - \dots - 2dy_q^2.$$

It is clear that $g_- < g$. In particular, every causal immersion for g is a timelike immersion for g_- . Thus $f : N_0 \rightarrow \mathbb{R}^{p+q}$ is an inextendible timelike immersion for (\mathbb{R}^{p+q}, g_-) . With the same reasoning as before, we can show that the projection $T : \mathbb{R}^{p,q} \rightarrow \mathbb{R}^q$ verifies that $T \circ f$ is a diffeomorphism between N_0 and \mathbb{R}^q , hence the

first part of the result. The map f can then be seen as the graph of a smooth map from \mathbb{R}^q to \mathbb{R}^p , and it is clear that this map is 1-Lipschitz if and only if f is causal and 1-contracting if and only if f is timelike. \square

In particular, this makes it easy to realize that the causality relation between two points coincides with the nature of the segment joining them.

Proposition 3.23. *Let x, y be two points in $\mathbb{R}^{p,q}$. The set $\{x, y\}$ is in timelike position (resp. causal position) if and only if the segment $[x, y]$ is timelike (resp. causal).*

Proof. Assume that $[x, y]$ is timelike (resp. causal) and include the vector $y - x$ into a vector space V of dimension q such that $q|_V < 0$ (resp. $q|_V \leq 0$). The space $x + V$ is an inextendible timelike (resp. causal) immersion.

Inversely, assume that there exists an inextendible causal immersion f containing both x and y . This immersion can be seen as the graph of a 1-Lipschitz map g from \mathbb{R}^q to \mathbb{R}^p . In particular, if $x = (g(x_0), x_0)$ and $y = (g(y_0), y_0)$, we have that

$$d(g(x_0), g(y_0)) \leq d(x_0, y_0),$$

thus $[x, y]$ is a causal segment. When the immersion is timelike, the map g is 1-contracting and $[x, y]$ is timelike, hence the result. \square

We can actually characterise the subsets of $\mathbb{R}^{p,q}$ which are in causal position.

Proposition 3.24. *Let E be a subset of $\mathbb{R}^{p,q}$. The set E is in causal position if and only if for each $x, y \in E$, the segment $[x, y]$ is in causal position.*

Proof. Assume that E is in causal position. Then for each x, y in E , the set $\{x, y\}$ is in causal position, meaning that the segment $[x, y]$ is causal according to proposition 3.23.

Inversely, if E verifies that for each $x, y \in E$, $[x, y]$ is causal, then E is the graph of a 1-Lipschitz map from a subset $T(E)$ of \mathbb{R}^q to \mathbb{R}^p . This map can then be extended into a 1-Lipschitz map defined on the whole of \mathbb{R}^q by Kirszbraun Theorem (see [8], p. 21). Then, by definition, E is in causal position. \square

Remark 3.25. We may wish to also characterise the sets E of $\mathbb{R}^{p,q}$ which are in timelike position ; such a set must be the graph of a strictly 1-Lipschitz map $g : T(E) \rightarrow \mathbb{R}^p$, i.e such that $d(g(x), g(y)) < d(x, y)$. However Kirszbraun Theorem does not guarantee the existence of a strictly 1-Lipschitz extension. The proof only works if E is compact or more generally if E is the graph of a k -Lipschitz map with $k < 1$.

We will come back for a more in-depth study of the pseudo-Euclidian space later on. Since we will almost exclusively be interested in immersions of dimension q , unless specified otherwise, every timelike or causal immersion we consider will be of dimension q .

3.3 Local structure of causal immersions

We can extend the notion of causality we introduced on $\mathbb{R}^{p,q}$ to any (p, q) -spacetime with the same definitions.

Definition 3.26. A set E of M is said to be in timelike position (resp. causal position) when there exists an inextendible timelike (resp. causal) immersion containing E in its image.

A first fact we can observe is that depending on the spacetime M , it is entirely possible that there does not exist any timelike or causal immersion which is inextendible in the sense we defined as soon as $q \geq 2$. We will give examples of such spacetimes later on.

Definition 3.27. A (p, q) -spacetime M is said to exhibit property (A) if for each point x in M , the set $\{x\}$ is in timelike position, meaning there always exists an inextendible timelike immersion of dimension q containing x .

Unless specified otherwise, every spacetime we consider will be assumed to verify property (A). Please note that in the Lorentzian case, every spacetime verifies (A). In Lorentzian geometry, it is a well known fact that the local causality in a normal neighborhood of a point x is entirely determined by the sign of the geodesics originating from x . We are going to show a similar result in higher signatures.

Lemma 3.28. *Let $x \in M$. There exist a normal neighborhood $U \subset T_x M$ and two pseudo-Riemannian metrics g_- and g_+ constant on U such that $g_- < g < g_+$.*

Proof. Let $V \subset T_x M$ be a normal neighborhood of x in which the metric at the origin g_0 is equal to $x_1^2 + \dots + x_p^2 - y_1^2 - \dots - y_q^2$. Let g_- and g_+ be the constant metrics on V such that

$$g_- = x_1^2 + \dots + x_p^2 - 2y_1^2 - \dots - 2y_q^2,$$

$$g_+ = x_1^2 + \dots + x_p^2 - \frac{1}{2}y_1^2 - \dots - \frac{1}{2}y_q^2.$$

Since g is smooth on V , we know that there exists a neighborhood U of 0 in V such that in U , we have

$$g_- < g < g_+,$$

hence the result. □

Definition 3.29. We will take U to be of the shape $C(a, b) = (-a, a)^p \times \mathbb{D}^q(0, b)$ with $a \gg b$. We will call such a neighborhood a *cylindrical neighborhood*.

Lemma 3.30. *Let $U = C(a, b) \subset T_x M$ be a cylindrical neighborhood of x with constant metrics g_- and g_+ and let $v \in U$ such that $g_-(v) < 0$. Let F be a p -dimensional vector space in U which is positive for g_- and orthogonal to v for g_- . Let $L = F \oplus v$. The vector space L is a lorentzian manifold in (U, g) .*

Proof. Since $g_-(v) < 0$, g_- is of signature $(p, 1)$ on L . Furthermore $\dim(L) = p + 1$, thus g_+ cannot be definite positive on L , meaning that there exists a non-zero w such that $g_+(w) \leq 0$. Let $y \in L$. Since $g_- < g$, $g_L = g|_L$ must be positive on $F \subset T_y L$, meaning that $g_L|_y$ is either of signature $(p, 1)$ or degenerate. However, since $g < g_+$ we must have $g_L(w) < 0$, and g_L must be of signature $(p, 1)$. \square

Proposition 3.31. *Let x be a point in a spacetime M . There exists a normal neighborhood $U \subset T_x M$ such that if there exists a timelike immersion $f : N_0 \rightarrow U$ inextendible in U containing both 0 and $v \in V$, then $[0, v]$ is timelike.*

Proof. Let $U = C(a, b)$ be a cylindrical neighborhood around x defined as before and let $f : N_0 \rightarrow U$ be a timelike immersion inextendible in U . Since f is timelike for g , it must also be timelike for g_- . We then know with proposition 3.23 that the composition of f with the projection on $\mathbb{D}^q(0, b)$ must be a diffeomorphism. In other words, f can also be regarded as a map $\bar{f} : \mathbb{D}^q(0, b) \rightarrow (-a, a)^p$ whose graph is the image of the inextendible immersion f .

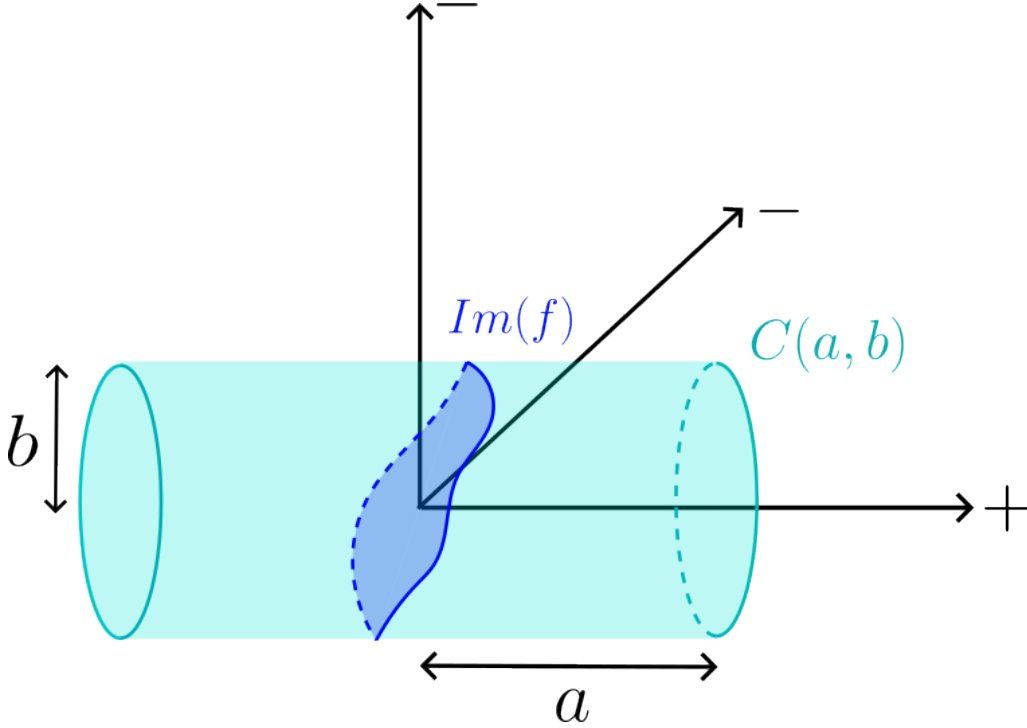


Figure 6: An inextendible timelike immersion f inside a cylindrical neighborhood $C(a, b)$.

Let v be in the image of f . We are going to show that the line $[0, v]$ is timelike. What we have done before shows that $g_-(v) < 0$. Let F be a p -dimensional vector space in U which is positive for g_- (and thus for g_+) and orthogonal to v for g_- . Let $L = F \oplus v$. By lemma 3.30, we know that L is a Lorentzian manifold for g .

Since f is the graph of \bar{f} , the intersection of L and $Im(f)$ gives a smooth path from the origin to v . Since it is contained in $Im(f)$, this path must be timelike. However, since it is also contained in the Lorentzian space L and since $L \cap U$ is a normal neighborhood for L , results of Lorentzian geometry tell us that the geodesic

from the origin to v in L must be timelike. Since L is a vector space in the normal neighborhood U , the geodesic $[0, v]$ in M is contained in L , meaning that $[0, v]$ must also be the geodesic in L , hence the result. \square

We can prove the same result when the immersion is causal.

Proposition 3.32. *Let x be a point in a spacetime M . There exists a normal neighborhood $U \subset T_x M$ such that if there exists an inextendible causal immersion $f : N_0 \rightarrow U$ containing both 0 and $v \in U$, then $[0, v]$ is causal. When $[0, v]$ is lightlike, the image of the immersion must contain the whole segment $[0, v]$.*

Proof. We proceed in the same fashion. Since f is causal for g , it must be timelike for g_- , meaning that it must be the graph of a map $\bar{f} : \mathbb{D}^q(0, b) \rightarrow (-a, a)^p$. Let F be a p -dimensional vector space in U which is positive for g_- and orthogonal to v for g_- and let $L = F \oplus v$. By lemma 3.30, L must be Lorentzian for g . The intersection $Im(f) \cap L$ is a path between the origin and v which is causal in L . This means that the line $[0, v]$ is either timelike or lightlike. In the case where it is lightlike, results of Lorentzian geometry (see [10], proposition 4.5.1) tell us that the path $Im(f) \cap L$ must be the lightlike geodesic $[0, v]$, hence the result. \square

It also gives us a characterisation of smooth maps which are causal immersions, meaning that $f : N_0 \rightarrow M$ is a timelike (resp. causal) immersion if and only if for each $y \in N_0$, there exists a cylindrical neighborhood $U = C(a, b)$ centered on $x = f(y)$ such that the restriction of f to $f^{-1}(U)$ is the graph of a map from $\mathbb{D}^q(0, b)$ to $(-a, a)^p$ which takes values in the area $\{g|_0 < 0\} \subset U$ (resp. $\{g|_0 \leq 0\}$). This characterisation is interesting, as it can easily be generalized when f is only a continuous map, which gives us a bigger class of maps that we can call causal.

Definition 3.33. A continuous map $f : N_0 \rightarrow M$ from a smooth manifold N_0 to M will be said to be timelike (resp. causal) if for each $y \in N_0$, there exists a cylindrical neighborhood $U = C(a, b)$ of $x = f(y)$ such that the restriction of f to $f^{-1}(U)$ is the graph of a map from $\mathbb{D}^q(0, b)$ to $(-a, a)^p$ taking values in the area $\{g|_x < 0\}$ (resp. $\{g|_x \leq 0\}$).

This generalization will become useful later on when we will construct limit maps which are not necessarily smooth.

Proposition 3.34. *Let $f : N_0 \rightarrow M$ be a continuous causal map. Then f is differentiable almost everywhere and for each x in which f is differentiable, $Im(df_x)$ is causal.*

Proof. Let $x \in N_0$ and $C(a, b)$ a cylindrical neighborhood around $f(x)$ such that the image of f in $C(a, b)$ is the graph of a map $\hat{f} : \mathbb{D}^q(0, b) \rightarrow (-a, a)^p$ with image in $\{g|_0 \leq 0\}$. Since $g_- < g$, the map f is also timelike for g_- as in each point, the lightcone of g_- contains the lightcone of g . Thus the map \hat{f} must be 1-lipschitz for g_- , which guarantees that \hat{f} is differentiable almost everywhere in $\mathbb{D}^q(0, b)$ by Rademacher's theorem. Assume now that f is differentiable in x and that $Im(df_x)$ is not causal. There must exist a direction on which df is positive, meaning that f cannot take values in $\{g|_0 \leq 0\}$, hence the result. \square

Proposition 3.35. *Let d be a Riemannian distance on M and $f : N_0 \rightarrow M$ a continuous timelike map. For each positive continuous map $\delta : N_0 \rightarrow \mathbb{R}$, there exists a smooth timelike immersion $g : N_0 \rightarrow M$ such that for each $x \in N_0$,*

$$d(f(x), g(x)) \leq \delta(x).$$

Proof. Let us consider a locally finite covering of compacts (U_i) of N_0 . Let $x \in U_i$. Since f is timelike, one can find a cylindrical neighborhood $C(a, b)$ of $f(x)$ such that f can be seen as the graph of a map $\hat{f} : \mathbb{D}^q(0, b) \rightarrow (-a, a)^p$. For each $\varepsilon > 0$, let us define a metric

$$g_\varepsilon = x_1^2 + \dots + x_p^2 - (1 - \varepsilon)(y_1^2 + \dots + y_q^2)$$

which is constant on $C(a, b)$ and which verifies $g_\varepsilon|_0 > g|_0$. Since f is timelike and by taking a small enough neighborhood $V_x = C(a', b') \subset C(a, b)$, one can find a ε such that f is timelike for g_ε and $g_\varepsilon > g$ on $C(a', b')$. In some cases, this may require a continuous deformation of \hat{f} near zero, for instance when f is a timelike path in a lorentzien manifold for which $g(\dot{f}(x)) = 0$.

The map f is timelike for g_ε , meaning that \hat{f} must be a 1-contracting map for g_ε from $\mathbb{D}^q(0, b)$ to $(-a, a)^p$. Since $g_\varepsilon > g$, it is enough to find a smooth approximation of \hat{f} which is 1-contracting for g_ε ; one may find such a uniform approximation by taking a convolution of \hat{f} . Furthermore, it is possible to ask that this approximation be equal to \hat{f} on $\mathbb{D}^q(0, b)$.

Since the covering (U_i) is locally finite and each $\overline{U_i}$ is compact, it is possible to extract a locally finite covering of the (V_x) . We can then build the smooth timelike immersion g on the entirety of N_0 , hence the result. \square

Remark 3.36. With this same proof, it is also possible to approximate f with a timelike immersion g such that $g = f$ on a finite set of points.

The proof for the approximation of causal, non necessarily timelike curves is slightly more involved.

Proposition 3.37. *Let d be a Riemannian distance on M and $f : N_0 \rightarrow M$ a continuous causal map. For each positive continuous map $\delta : N_0 \rightarrow \mathbb{R}$, there exists a smooth causal immersion $g : N_0 \rightarrow M$ such that for each $x \in N_0$,*

$$d(f(x), g(x)) \leq \delta(x).$$

Proof. Let (U_i) , $x \in U_i$, $C(a, b)$ be as before. By proposition 3.34, we know that f is differentiable almost everywhere. Let $x \in U_i$ such that f is differentiable in x . The sub-vector space $Im(df_x)$ in $C(a, b)$ must be causal, meaning it must be the orthogonal sum of an isotropic vector space F of dimension k and a negative vector space Q of dimension $q - k$. Let us decompose $T_{f(x)}M = \mathbb{R}^{k,k} \oplus \mathbb{R}^{p-k} \oplus Q$ where $F \subset \mathbb{R}^{k,k}$. Finally, let us define the constant metric on $C(a, b)$

$$g_\varepsilon = g|_{\mathbb{R}^{k,k}} + g|_{\mathbb{R}^{p-k}} + (1 - \varepsilon)g|_Q$$

for each $\varepsilon > 0$. For a small enough neighborhood $V_x = C(a', b')$ and a small enough epsilon, we have that $g_\varepsilon \geq g$ and that f is causal for g_ε . The map \hat{f} must then be 1-Lipschitz for g_ε and we can uniformly approximate \hat{f} by a smooth 1-Lipschitz map for g_ε . Since $g_\varepsilon \geq g$, this map is still causal for g , hence the result. \square

Since most of our results concern smooth causal immersions which are inextendible, let's show an even stronger result when the continuous map f is assumed to be inextendible.

Lemma 3.38. *Let $f : N_0 \rightarrow M$ be an inextendible continuous causal map from a smooth manifold N_0 to M and let $\delta : N_0 \rightarrow \mathbb{R}$ be a positive continuous map. There exists a smooth inextendible causal immersion $g : N_0 \rightarrow M$ such that for each $x \in N_0$, $d(f(x), g(x)) < \delta(x)$.*

Proof. Let φ be a coercive map greater than 1 on N_0 and let $\bar{\delta} = \frac{1}{\varphi}\delta$. The map $\bar{\delta}$ verifies that $\bar{\delta} < \delta$ and that for each $\varepsilon > 0$, there exists a compact K of N_0 such that $\bar{\delta} < \varepsilon$ outside of K . Proposition 3.37 tells us that there exists a causal immersion $g : N_0 \rightarrow M$ which verifies that $d(f, g) \leq \bar{\delta} \leq \delta$. Let us show that g must be inextendible.

Let c be an inextendible path in N_0 and assume that $g \circ c$ converges in M to a point ℓ . Let $\varepsilon > 0$. Let $T_1 \in \mathbb{R}$ such that for all $t > T_1$, $d(g(c(t)), \ell) < \varepsilon$. Let K be a compact of N_0 such that $\bar{\delta} < \varepsilon$ outside of K . Since c is inextendible in N_0 , it exits every compact of N_0 ; in particular, there exists a $T_2 > T_1$ such that for all $t > T_2$, $c(t) \notin K$. We must then have for all $t > T_2$,

$$\begin{aligned} d(f(c(t)), \ell) &\leq d(f(c(t)), g(c(t))) + d(g(c(t)), \ell) \\ &\leq 2\varepsilon. \end{aligned}$$

Hence the path $f \circ c$ must converge to ℓ which contradicts the assumption that f is inextendible, hence the result. \square

Remark 3.39. This shows that property (A) can be weakened to its continuous counterpart, as the existence of an inextendible timelike map containing $x \in M$ implies the existence of an inextendible timelike immersion containing x .

From now on, unless specified otherwise, timelike (resp. causal) maps from N_0 to M will be assumed to only be continuous

3.4 Future and past of oriented causal spaces of dimension $q - 1$

One defect of the notion of causality we defined is that contrary to its Lorentzian counterpart, it lacks the existence of a future and a past. In $\mathbb{R}^{p,q}$, the set of points which are in timelike position with the origin is connected as soon as $q \geq 2$. In order to obtain a nice notion of past and future, one must make a few more assumptions on the sets we consider.

Let $f : N_1 \rightarrow M$ be a causal map of dimension $q - 1$ where N_1 is an oriented closed manifold. Assume that f can be extended into an inextendible causal map of dimension q , $\bar{f} : N_0 \rightarrow M$ where N_1 is a closed submanifold of N_0 of co-dimension one. Since the orientation of N_1 has been fixed and the orientation of N_0 stems from the time orientation of M , there is a canonical way to choose between the two connected components of $N_0 \setminus N_1$. One will be the past of f in \bar{f} and the other will be its future.

Definition 3.40. Let $f : N_1 \rightarrow M$ be a causal map of dimension $q - 1$ where N_1 is an oriented closed manifold. A point $x \in M$ will be said to be in the *future* of f (resp. its *past*) when there exists an inextendible map $\bar{f} : N_0 \rightarrow M$ of dimension q which extends f and which contains x in the future of f in \bar{f} . The future (resp. past) of f will be noted $J_q^+(f)$ (resp. $J_q^-(f)$). When there is no possible confusion, we will not make the distinction between f and its image.

This definition can actually be extended to a larger class of maps $f : N_1 \rightarrow M$ of dimension $q - 1$. The only thing we need is that for each inextendible map $\bar{f} : N_0 \rightarrow M$ of f of dimension q , $N_1 \subset N_0$ separates N_0 into two connected components.

Definition 3.41. A causal map $f : N_1 \rightarrow M$ of dimension $q - 1$ will be said to be *splitting* if for each inextendible causal map $\bar{f} : N_0 \rightarrow M$ of f , $N_0 \setminus N_1$ is the union of two connected components.

We define the past and the future of f in the same manner when f is splitting.

Remark 3.42. The notion of future and past of f depends on the orientation of N_1 . When taking the opposite orientation, one exchanges the past and future of f .

3.5 Causal diamonds in the pseudo-Euclidian space

Let us continue the study of the pseudo-Euclidian space from the standpoint of causality that we began in section 2. As shown in proposition 3.23, any inextendible causal immersion is the graph of a smooth map from \mathbb{R}^q to \mathbb{R}^p which is 1-Lipschitz.

Remark 3.43. Since inextendible smooth causal maps can be approximated by inextendible continuous causal maps, this result extends to the continuous case. Meaning that any inextendible causal map in $\mathbb{R}^{p,q}$ is the graph of a continuous map from \mathbb{R}^q to \mathbb{R}^p which is 1-Lipschitz, or 1-contracting when it is a timelike map.

This, in particular, allows us to determine the causal maps of dimensions $q - 1$ which are splitting in the sense previously defined.

Proposition 3.44. *An oriented causal map of dimension $q - 1$ is extendible into an inextendible causal map of dimension q if and only if it is the graph of a map f defined on a submanifold of co-dimension 1 of \mathbb{R}^q which is 1-Lipschitz for the Riemannian distances of \mathbb{R}^q and \mathbb{R}^p . Furthermore, this immersion is splitting if and only if N_1 separates \mathbb{R}^q into two connected components*

Proof. The first part of the result comes from 3.24. The second part comes immediately from the fact that any inextendible causal map extending f is a graph over \mathbb{R}^q . \square

In particular, we are going to study two examples. Let $S^- = \{q = -1\} \cap \mathbb{R}^{0,q}$ be the sphere of radius 1 in $\mathbb{R}^{0,q}$ with the standard orientation and $L = \mathbb{R}^{0,q-1} \subset \mathbb{R}^{0,q}$ be a hyperspace of $\mathbb{R}^{0,q}$ bounding the half-space $L \times \mathbb{R}^{>0}$ in $\mathbb{R}^{0,q}$ with the induced orientation. By the previous result, both are timelike splitting immersions of dimension $q - 1$. Let us first study the future of S^- . Let $S^+ = \{q = 1\} \cap \mathbb{R}^{p,0}$ be the positive sphere of radius 1 in $\mathbb{R}^{p,0}$. The topological joint of S^- and S^+ is topologically a sphere of dimension $p + q - 1$ which is the boundary of a bounded area in $\mathbb{R}^{p,q}$. We will write this open space $Diam_{p,q}$.

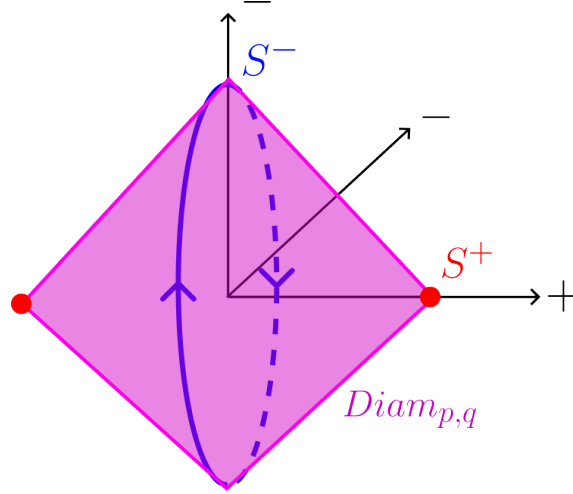


Figure 7: The 1-sphere S_- and the 0-sphere S_+ .

Proposition 3.45. *The open set $Diam_{p,q}$ is the 0-future of S^- in $\mathbb{R}^{p,q}$.*

Proof. We first wish to determine the set of points x such that $\{x\} \cup S^-$ is in timelike position and $T(x)$ is in the disk bounded by S^- in $\mathbb{R}^{0,q}$. Using the result proven in 3.24, those points are exactly those in $T^{-1}(D(0,1))$ such that for each $y \in S^-$, $[x,y]$ is timelike. This area is clearly bounded by the joint of the spheres S^- and S^+ , hence the result. \square

Remark 3.46. The space $Diam_{p,q}$ can be interpreted as a generalization of the causal diamond in Lorentzian geometry, where the two points defining the diamond are replaced by an oriented timelike mobius sphere of dimension $q - 1$.

Remark 3.47. When S is a topological sphere in timelike position of dimension $q - 1$ in $\mathbb{R}^{p,q}$, the open future of S will be called a causal diamond. When, like S_- , S is a mobius sphere, the causal diamond will be said to be *flat*.

Let's now move on to the space L .

Proposition 3.48. *The future $J_0^+(L)$ of L is the set of points $x \in \mathbb{R}^{p,q}$ such that $T(x)$ is in $L \times \mathbb{R}^{>0}$ and there exists $y \in L$ such that the segment $[x,y]$ is timelike and orthogonal to L .*

Proof. Assume that $T(x) \in L \times \mathbb{R}^{>0}$ and that there exists $y \in L$ such that $[x,y]$ is timelike and orthogonal to L . The first condition tells us that x is going to be in the future of L rather than in its past. Since $[x,y]$ is orthogonal to L , the vector space $L \oplus x$ is negative of dimension q , meaning that $L \cup \{x\}$ is in timelike position, hence $x \in J_0^+(L)$.

Inversely, if x is in the future of L , then for each $y \in L$, $[x,y]$ is timelike. In particular, there must exist a y in L such that $[x,y]$ is orthogonal to L , hence the result. \square

Remark 3.49. This second example is to be compared with the open future cone of a point in the Lorentzian Minkowski space. In Lorentzian geometry, both this

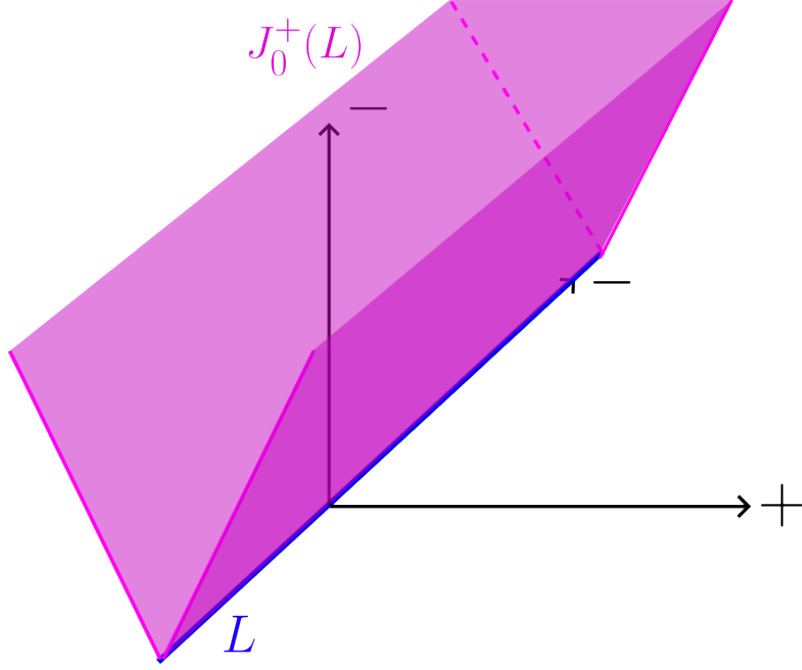


Figure 8: The timelike line L and its future $J_0^+(L)$.

space and the Lorentzian diamond are actually equivalent, in the sense that there exists a conformal transformation taking the first one to the second. The same is true in higher signature.

Proposition 3.50. *The space $Diam_{p,q}$ is conformally equivalent to $J_0^+(L)$.*

Proof. We are going to use the map

$$\varphi : x \in Diam_{p,q} \mapsto -\frac{x - e_1}{q(x - e_1)}.$$

The map φ is the restriction of an inversion in $\text{Ein}^{p,q}$, in particular it is a conformal transformation. We now only need to show that φ takes $Diam_{p,q}$ to $J_0^+(L)$. Since the notion of causality only depends on the conformal class of the metric, time-orientation preserving conformal maps preserve the futures. Thus it is enough to show that φ takes S^- to L , which is clear as its restriction to $\mathbb{R}^{0,q}$ is the usual map taking the unit disk to the half-space, hence the result. \square

This equivalence is particularly interesting, as there is a nice conformal model for $J_0^+(L)$.

Proposition 3.51. *$J_0^+(L)$ is conformally equivalent to $\mathbb{H}^p \times \mathbb{H}^q$.*

Proof. The map we are going to consider is the following,

$$\psi : (x, y, t) \in \mathbb{H}^p \times \mathbb{H}^q \mapsto tx + y,$$

where we take the half-plane model of $\mathbb{H}^q \simeq \mathbb{R}^{q-1} \times \mathbb{R}^{>0}$, \mathbb{R}^{q-1} is identified to L and $\mathbb{H}^p \subset \mathbb{R}^{p,q}$ is taken to be the space $\{q = -1\} \cap (\mathbb{R}^{p,0} \oplus \mathbb{R}^{0,1})$ where $\mathbb{R}^{0,1}$ is the orthogonal of L in $\mathbb{R}^{0,q}$. It is clear that ψ takes $\mathbb{H}^p \times \mathbb{H}^q$ to $J_0^+(L)$, we now have to show that it is a conformal transformation.

Let us first show that the image by ψ of the spaces $\mathbb{H}^p \times \{*\}$ and $\{*\} \times \mathbb{H}^q$ are orthogonal. Let $(x, y, t) \in \mathbb{H}^p \times \mathbb{H}^q$. The image by ψ of the set $\mathbb{H}^p \times \{(y, t)\}$ is one of the two hyperbolic map of the orthogonal of L in y , which is an affine copy of the Minkowski Lorentzian space. The image of $\{x\} \times \mathbb{H}^q$ is the vector space $L \oplus x$ on which q is definite negative. We see that those two spaces are orthogonal in $tx + y$. In order to show that ψ is a conformal transformation, we now only have to show that $\psi|_{\mathbb{H}^p}$ and $\psi|_{\mathbb{H}^q}$ are both conformal transformations on their image with the same conformal factor. It is then easy to see that

$$\psi_* g_{\mathbb{H}^p \times \mathbb{H}^q} = \frac{1}{t} g_{\mathbb{R}^{p,q}},$$

hence the result. \square

From this result, we can deduce two corollaries.

Corollary 3.52. *The pseudo-Riemannian space $\mathbb{H}^p \times \mathbb{H}^q$ is conformally flat*

Corollary 3.53. *The diamond $Diam_{p,q}$ is conformally equivalent to $\mathbb{H}^p \times \mathbb{H}^q$. In particular, $Diam_{p,q}$ gives an example of a proper open subspace of $\text{Ein}^{p,q}$ which is conformally homogenous.*

Remark 3.54. Conjectures on the proper open spaces of flag manifolds predicted that such spaces in $\text{Ein}^{p,q}$ should be classifiable (see [6]); In [3], Chalumeau and Galiay proved that the flat causal diamonds were the only proper homogenous subspaces of $\text{Ein}^{p,q}$.

Via the conformal transformations of $\text{Ein}^{p,q}$, the sphere S^- can be taken to any timelike mobius sphere of $\text{Ein}^{p,q}$ of dimension $q - 1$. Thus, any two causal diamonds can be linked by a conformal transformation of $\text{Ein}^{p,q}$. In particular, the past of S_- is also a causal diamond.

As we have seen before, one of the way higher signature differs from Lorentzian signature is that not every timelike map can be extended into an inextendible timelike map. One consequence is that when taking an open set Ω of a spacetime M (for instance $M = \mathbb{R}^{p,q}$), it is possible that some timelike maps could be inextendible in Ω but not in M , and furthermore that this map could not be extended into an inextendible timelike map in M . In other words, for x, y in Ω , it is possible that $\{x, y\}$ is in timelike position in Ω but not in M .

Example 3.55. Let $T : \mathbb{R}^{p,q} \rightarrow \mathbb{R}^q$ be the projection on the negative coordinates and let $U = D(0, 2) \setminus \overline{D(0, 1)}$ be an open ring in \mathbb{R}^q . Let $\Omega = T^{-1}(U)$. For each $x, y \in \Omega$, $\{x, y\}$ is in timelike position in Ω .

Proof. Let $f : \tilde{U} \rightarrow \Omega$ be an upward spiral of Ω as in figure 4, where \tilde{U} is the universal covering of U . Up to modifying the vertical speed of the spiral and translating it, one can assume that f contains both x and y in its image. The immersion f is inextendible in Ω , hence the result. \square

Notice that in the previous proof, f is not inextendible in $\mathbb{R}^{p,q}$ and cannot be extended into an inextendible timelike map, since it is not even the graph of a map from a subset of \mathbb{R}^q into \mathbb{R}^p . To end this subsection, we will give a property on open subspaces Ω of $\mathbb{R}^{p,q}$ which guarantees that Ω and $\mathbb{R}^{p,q}$ share the same notion of causality.

Proposition 3.56. *Let U be an open subset of $\mathbb{R}^{p,q}$. The set $\Omega = T^{-1}(U)$ has the same causality as $\mathbb{R}^{p,q}$ if and only if U is convex in \mathbb{R}^q .*

Proof. Assume U is convex and let $\tilde{f} : N_0 \rightarrow \Omega$ be an inextendible timelike immersion. For the same reasons as before, $T \circ \tilde{f}$ must be both surjective and a covering on U . However since U is convex it must be simply connected, meaning that \tilde{f} can be seen as the graph of a map $f : U \rightarrow \mathbb{R}^p$ which verifies that for all $x \in U$, $\|df_x\| < 1$. This means that for each smooth path c of U ,

$$L(f(c)) < L(c).$$

Since U is convex, for each $x, y \in U$, the segment $[x, y]$ is in U and minimises the length of the curves c from x to y , meaning that $L([x, y]) = d(x, y)$. This means that we must have

$$d(f(x), f(y)) < d(x, y)$$

for each $x, y \in U$, meaning that f is a 1-contracting map defined on $U \subset \mathbb{R}^q$. By 3.24, this means that f can be extended into a 1-contracting map defined on the whole of \mathbb{R}^q , hence the result.

Inversely, assume that U is not convex. It will always be possible to build a map $f : U \rightarrow \mathbb{R}^p$ which verifies $\forall x \in U, \|df_x\| < 1$ but is not 1-contracting. \square

4 Global hyperbolicity in higher signature

4.1 Cauchy surfaces

The third section of this article will be dedicated to a generalization of the notion of global hyperbolicity in the context of pseudo-Riemannian spacetimes of higher signature. In Lorentzian geometry, being globally hyperbolic can be characterised by each one of those three equivalent properties :

1. The spacetime M admits a Cauchy surface,
2. M admits a Cauchy time function $T : M \rightarrow \mathbb{R}$,
3. For each $x, y \in M$, the space $\mathcal{C}(x, y)$ of future-oriented causal curves from x to y is compact for the uniform topology.

We are going to start by generalizing the notion of Cauchy surface.

Definition 4.1. Let M be a spacetime of signature (p, q) . A subset S of M is said to be a *Cauchy surface* if for each inextendible causal map $f : N_0 \rightarrow M$ of dimension q , there exists a unique $x \in N_0$ such that $f(x) \in S$.

Remark 4.2. In Lorentzian geometry, the usual definition for a Cauchy surface only requires that every inextendible timelike curve meets the set S exactly once, thus allowing the existence of achronal Cauchy surfaces which are not acausal. However, it is possible to show that every globally hyperbolic spacetime admits an acausal Cauchy surface. For our generalization, we will restrict ourselves to acausal Cauchy surfaces, however there are probably results to be established regarding non-acausal Cauchy surfaces in higher signature.

We have already seen an example of Cauchy surfaces in the pseudo-Euclidian case in 3.19.

Proposition 4.3. *Let $S = \mathbb{R}^{p,0} \subset \mathbb{R}^{p,q}$. S is a Cauchy surface of $\mathbb{R}^{p,q}$.*

This result is comforting as the Lorentzian Minkowski space is the basic example of a globally hyperbolic Lorentzian space. We can actually exhibit more Cauchy surfaces in $\mathbb{R}^{p,q}$.

Proposition 4.4. *Let $g : P_0 \rightarrow \mathbb{R}^{p,q}$ be an inextendible spacelike map, i.e an inextendible timelike map of the space $-\mathbb{R}^{p,q}$. Then $S = \text{Im}(g)$ is a Cauchy surface in $\mathbb{R}^{p,q}$.*

Proof. Since g is spacelike in $(\mathbb{R}^{p,q}, g_{\mathbb{R}^{p,q}})$, it must be timelike in $(\mathbb{R}^{q,p}, -g_{\mathbb{R}^{p,q}})$ which means that g is the graph of a 1-contracting map \bar{g} from \mathbb{R}^p to \mathbb{R}^q . Let $f : N_0 \rightarrow \mathbb{R}^{p,q}$ be an inextendible timelike map of $\mathbb{R}^{p,q}$ which by 3.19 is the graph of a 1-contracting map \bar{f} from \mathbb{R}^q to \mathbb{R}^p . We have that $\bar{g} \circ \bar{f}$ is a 1-contracting map from \mathbb{R}^q to \mathbb{R}^q , meaning that it must admit a unique fixed point $x \in \mathbb{R}^q$, hence the result. \square

A first observation arises regarding the uniqueness of x in N_0 such that $f(x) \in S$. In Lorentzian geometry, an immediate consequence of the existence of a Cauchy surface is the causality of the spacetime M .

Every non-causal spacetime automatically fails to be globally hyperbolic, as a timelike curve $c : \mathbb{S}^1 \rightarrow M$ can be seen as an inextendible periodic timelike path $\hat{c} : \mathbb{R} \rightarrow M$. If it encounters a Cauchy surface, it must do so an infinite number of times, hence the result. The same thing happens in higher signature when timelike curves fail to be simply connected.

Definition 4.5. A spacetime of signature (p, q) is said to be *causal* if there does not exist an inextendible causal map $f : N_0 \rightarrow M$ of dimension q where N_0 is not simply connected.

Proposition 4.6. *If M admits a Cauchy surface S , it must be causal.*

Proof. Assume there exists an inextendible causal map $f : N_0 \rightarrow M$ where N_0 is not simply connected and let $\tilde{f} : \tilde{N}_0 \rightarrow M$ be the map induced by f on the universal covering \tilde{N}_0 of N_0 . Since \tilde{f} is also an inextendible causal map, there must exist a point x_1 in \tilde{N}_0 such that $\tilde{f}(x_1) = f(\pi(x_1)) \in S$. However since N_0 is not simply connected, there exists a point $x_2 \neq x_1$ such that $\pi(x_1) = \pi(x_2)$, meaning that $\tilde{f}(x_1) = \tilde{f}(x_2)$. The point $x \in \tilde{N}_0$ such that $\tilde{f}(x) \in S$ is supposed to be unique, hence the result. \square

In most cases, it will be easier to check that a subset $S \subset M$ is a Cauchy subset by verifying the property on smooth inextendible causal immersion. Let us prove that this is enough to get the property on continuous inextendible causal maps.

Lemma 4.7. *Let $S \subset M$ be a subset verifying the Cauchy property for smooth maps. Then S is closed in M .*

Proof. Assume that S is not closed and let $y \in \bar{S} \setminus S$. Let $f : N_0 \rightarrow M$ be a smooth inextendible timelike map such that there exists $x \in N_0$ with $f(x) = y$. There must exist a unique $x_1 \in N_0$ such that $f(x_1) \in S$. However, by taking a

normal neighborhood of y , one can slightly deform f into an inextendible timelike immersion \tilde{f} such that $\tilde{f}(x) \in S$ and $\tilde{f} = f$ outside of a small neighborhood around x . The map \tilde{f} thus verifies that $\tilde{f}(x_1)$ and $\tilde{f}(x)$ are both in S which is impossible, hence the result. \square

Proposition 4.8. *Let $S \subset M$ be a subset verifying the Cauchy property for smooth maps. Then S also verifies the property for continuous maps, meaning that S is a Cauchy surface.*

Proof. Let $f : N_0 \rightarrow M$ be an inextendible continuous causal map and let's assume the image of f does not intersect S . Since $M \setminus S$ is open, one can uniformly approximate f by a smooth inextendible causal immersion f_1 which also does not meet S , which is impossible. Hence, there must exist a x in N_0 such that $f(x) \in S$.

Assume now that there exists two distinct points x_1 and x_2 in N_0 such that $f(x_1)$ and $f(x_2)$ are both in S . Then one can uniformly approximate f by a smooth inextendible causal immersion f_2 such that $f_2(x_1) = f(x_1)$ and $f_2(x_2) = f(x_2)$ which are both in S . This is impossible as S verifies the Cauchy property for smooth maps, hence the result. \square

4.2 Causal foliations and Cauchy surfaces

One interesting property about Cauchy hypersurfaces in Lorentzian geometry is that their topology is determined up to homeomorphism (or diffeomorphism when both surfaces are smooth). The proof of this result relies on the existence of timelike vector fields on Lorentzian spaces, which are always integrable. In signature (p, q) , there always exists timelike distributions of q -vector spaces, however those distributions sometimes fail to be integrable. Examples of spacetimes which do not admit a timelike foliation of dimension q will be given later. We will prove a few results under the assumption of the existence of such a foliation.

Proposition 4.9. *Let \mathcal{F} be a timelike (resp. causal) foliation of dimension q on a spacetime M . Every leaf of \mathcal{F} is an inextendible timelike (resp. causal) immersion of dimension q .*

Proof. Let F be a leaf of \mathcal{F} . The inclusion $i : F \rightarrow M$ is a timelike (resp. causal) embedding of F in M . Let D be the distribution associated with \mathcal{F} and assume there exists a path $c : (a, b) \rightarrow F$ and a point x in $M \setminus F$ such that $c(t) \xrightarrow{t \rightarrow b} x$. The distribution D is locally integrable around x which is in the closure of F , meaning that x must also be in the leaf F , hence the result. \square

Definition 4.10. A foliation is said to be *regular* if there exists a submersion $\varphi : M \rightarrow P$ such that the leaves of F are exactly the level sets of φ .

Proposition 4.11. *Let M be a (p, q) -spacetime admitting a smooth Cauchy surface S and a causal foliation F . Then F is a regular foliation.*

Proof. Let $x \in M$ and let F_x be the leaf of F containing x . Since F_x is an inextendible causal immersion of dimension q , there exists a unique point $\varphi(x)$ both in F_x and in S . The map $x \in M \mapsto \varphi(x) \in S$ is a smooth submersion for which the level sets are exactly the leaves of F , hence the result. \square

Proposition 4.12. *Let M be a (p, q) -spacetime with a timelike foliation F of dimension q and two Cauchy surfaces S_1 and S_2 . Then S_1 and S_2 are homeomorphic.*

Proof. Let $x \in S_1$ and let F_x be the leaf of F containing x . Let $\varphi(x)$ be the only point both in F_x and S_2 . Similarly, for $y \in S_2$, let $\psi(y)$ be the unique point in both F_y and S_1 . Since ψ is the inverse of φ , φ gives us a homeomorphism between S_1 and S_2 , hence the result. \square

Remark 4.13. It is not known to the author whether one can prove that every two Cauchy surfaces are homeomorphic without the existence of a timelike foliation.

4.3 Time functions

Let us now generalize the notion of time function to the spacetimes of signature (p, q) .

Definition 4.14. Let M be a (p, q) spacetime and N a manifold of dimension q . A continuous map $T : M \rightarrow N$ between two smooth manifolds is a *time function* if for every inextendible causal map $f : N_0 \rightarrow M$, the map $T \circ f : N_0 \rightarrow N$ is injective.

Remark 4.15. In particular, this implies that every inextendible causal map $f : N_0 \rightarrow M$ is an embedding.

Proposition 4.16. *If M admits a time function, it is causal.*

Proof. Assume M is not causal ; there must then exist an inextendible causal map $f : N_0 \rightarrow M$ in M where $\pi_1(N_0)$ is non-trivial. Let $\tilde{f} : \tilde{N}_0 \rightarrow M$ be the map induced by f on the universal covering \tilde{N}_0 of N_0 . If there existed a time function $T : M \rightarrow N$, both the maps $T \circ f$ and $T \circ \tilde{f}$ would be injective which is not possible since $\#\pi_1(N_0) \geq 2$, hence the result. \square

Proposition 4.17. *Let M be a (p, q) spacetime with property (A) with a smooth time function $T : M \rightarrow N$. Then T is a submersion.*

Proof. Let $x \in M$. M has property (A), so there exists an inextendible timelike immersion $f : N_0 \rightarrow M$ and $y \in N_0$ such that $f(y) = x$. The map $T \circ f$ is injective between two manifolds of dimension q , meaning its differential $dT \circ df$ must be an isomorphism. This implies that dT is surjective onto $T_{T(x)}N$, hence the result. \square

In the same manner as for Cauchy surfaces, it is enough to check that T is a time function for smooth maps.

Proposition 4.18. *Let $T : M \rightarrow N$ be a continuous map such that for every smooth inextendible causal immersion $f : N_0 \rightarrow M$, the map $T \circ f$ is injective. Then T is a time function.*

Proof. Assume that there exists a continuous inextendible causal map $f : N_0 \rightarrow M$ and two distinct points x_1, x_2 in N_0 such that $T(f(x_1)) = T(f(x_2))$. Let $\tilde{f} : \tilde{N}_0 \rightarrow M$ be a smooth inextendible causal immersion such that $\tilde{f}(x_1) = f(x_1)$ and $\tilde{f}(x_2) = f(x_2)$, the existence of which is given by the construction in 3.37. The map $T \circ \tilde{f}$ is not injective, hence the result. \square

4.4 Cauchy time functions and globally hyperbolic spaces.

We will generalize the notion of Cauchy time functions in the same manner.

Definition 4.19. Let M be a spacetime of signature (p, q) . A *Cauchy time function* is a map $T : M \rightarrow N$ where N is a q -dimensional manifold such that for each $y \in N$, $T^{-1}(y)$ is a Cauchy surface.

Remark 4.20. An equivalent formulation is that for each inextendible causal map $f : N_0 \rightarrow M$, the map $T \circ f : N_0 \rightarrow N$ is a homeomorphism.

Once again, the Cauchy time function property only has to be checked for smooth maps.

Proposition 4.21. *Let $T : M \rightarrow N$ be a map such that for each smooth inextendible causal immersion $f : N_0 \rightarrow M$, the map $T \circ f$ is a homeomorphism. Then T is a Cauchy time function.*

Proof. This is a consequence of the same result about Cauchy surfaces. \square

Definition 4.22. A (p, q) -spacetime M is said to be *globally hyperbolic* if and only if it admits a Cauchy time function $T : M \rightarrow N$.

Corollary 4.23. *Let $T : M \rightarrow N$ be a globally hyperbolic space. Then every inextendible causal map is homeomorphic to N . More precisely, every inextendible causal map can be seen as a section of the map T .*

This gives us an easy way to verify that a spacetime is not globally hyperbolic. For instance, $\mathbb{R}^{p,q} \setminus \{0\}$ is not globally hyperbolic since some of the inextendible timelike maps are homeomorphic to \mathbb{R}^q while others are homeomorphic to $\mathbb{R}^q \setminus \{0\}$.

In 3.19, we have shown that $T : \mathbb{R}^{p,q} \rightarrow \mathbb{R}^q$ is a Cauchy time function. By using the same arguments, we can exhibit a very large class of examples of globally hyperbolic spaces.

Proposition 4.24. *Let $T : (M^{p+q}, g_M) \rightarrow (N^q, g_N)$ be a complete Riemannian bundle where N is simply connected and let g be the (p, q) metric defined on M by the splitting $\text{Ker}(dT) \oplus (\text{Ker}(dT))^\perp$ where each fiber is positive and its orthogonal is negative. Then the map $T : M \rightarrow N$ is a Cauchy time function.*

Proof. Let us first show that the map $T \circ f$ is a local diffeomorphism. Since the projection T removes the negative coordinates, the map $T \circ f$ increases the distances between the riemannian manifolds $(N_0, -f^*g)$ and (N, g_N) , i.e for each $x \in N_0, v \in T_x N_0$,

$$g_N(d(T \circ f)(v)) \geq -f^*g(v).$$

This tells us in particular that $T \circ f$ is a local diffeomorphism ; in particular, its image $\text{Im}(T \circ f)$ is open in N .

Let's assume that $T \circ f$ is not surjective and let $y \in \partial \text{Im}(T \circ f)$. Let $c : (a, b) \rightarrow N$ be a smooth path of speed 1 in $\text{Im}(T \circ f)$ converging to y in b .

For each $x \in \mathbb{R}^{p,q}$ and $v \in T_x \mathbb{R}^{p,q}$ timelike, we have that $g_M(v) \leq 2g_N(v)$, since $g_{\mathbb{R}^{p,q}}(v) \leq 0$. In particular, if \tilde{c} is a path in N_0 such that $T \circ f \circ \tilde{c} = c$, we have that the speed of $f(\tilde{c})$ for \bar{g} is lower than 2. Since (M, g_M) is complete, $f(\tilde{c})$ must

converge, and since f is inextendible so must \tilde{c} . However we have assumed that $y \notin \text{Im}(T \circ f)$, hence the contradiction ; the map $T \circ f$ must be surjective.

Let us now show that $T \circ f$ is a covering on N . Let $y \in N$ and let $c : (0, 1) \rightarrow N$ be a smooth path such that $c(0) = y$. Let $x \in N_0$ such that $T \circ f(x) = y$ and $\tilde{c} : I = (0, b) \rightarrow N_0$ such that $T \circ f(\tilde{c}) = c$. Assume that $b < 1$. By the same reasoning as before, the smooth path $f(\tilde{c})$ must converge in M . Since f is inextendible, the path \tilde{c} must also converge in N_0 . However since $T \circ f$ is a local diffeomorphism, \tilde{c} can then be defined further than b , meaning that $b = 1$. The map $T \circ f$ must be a covering from N_0 to N .

Finally, since N is simply connected, the map $T \circ f$ must be a diffeomorphism from N_0 to \mathbb{R}^q , hence the result.

Assume now that f is an inextendible causal immersion and for each $x \in M, v \in T_x M$, let

$$g'_M(v) = g_M(T_{\text{Ker}(d\pi)}(v)) + 2g_M(T_{\text{Ker}(dT)^\perp}(v)),$$

$$g'(v) = g_M(T_{\text{Ker}(dT)}(v)) - 2g_M(T_{\text{Ker}(dT)^\perp}(v)).$$

The projection T is still an isometry on the orthogonal of its fibers for the metric $g'_N = 2g_N$ on N and g'_M on M . Since $g'_M \geq g_M$, the Riemannian manifold (M, g'_M) is still complete. Furthermore, every causal immersion for g is timelike for g' , thus by the same reasoning as before, the map $T \circ f$ must be a diffeomorphism, hence the result on causal immersions.

Since it is enough to prove the Cauchy property on smooth maps, we have shown that $T : M \rightarrow N$ is a Cauchy time function. \square

Remark 4.25. Since the notion of inextendible causal map does not change when taking another metric in the same conformal class, the previous result applies to spaces conformally equivalent to complete pseudo-Riemannian bundles. In particular, when P is a complete Riemannian manifold and N a complete simply connected Riemannian manifold, the projection T on the second factor of the product $P \times N$ is a Cauchy time function and the product $P \times N$ is globally hyperbolic. For instance, this tells us that when $q \geq 2$, the space $\text{Ein}^{p,q}$ is globally hyperbolic and that for each decomposition $\text{Ein}^{p,q} \simeq \mathbb{S}^p \times \mathbb{S}^q$, the projection on \mathbb{S}^q is a time function.

Remark 4.26. When $T : M \rightarrow N$ is a Cauchy time function, the inextendible causal maps of M are all global sections $s : N \rightarrow M$ of the map T . In particular, taking a complete pseudo-riemannian bundle which does not admit a section gives us an example of a spacetime which does not verify property (A). The Hopff bundle $\mathbb{S}^3 \rightarrow \mathbb{S}^2$ is such a space. Non-trivial complete pseudo-riemannian bundles which verify property (A) include but are not limited to tangent bundles on a simply connected Riemannian space endowed with the corresponding Sasaki metric. It is not known whether the topological obstruction to the existence of a section is the only obstruction to property (A) on a complete pseudo-riemannian bundle.

We have shown in 3.51 that the future of a timelike mobius sphere of dimension $q - 1$ in $\mathbb{R}^{p,q}$ is conformally equivalent to $\mathbb{H}^p \times \mathbb{H}^q$. This means in particular that those spaces are globally hyperbolic with the projection unto \mathbb{H}^q as a time function. From this we can deduce two results.

Corollary 4.27. *Let P be a complete Riemannian manifold and N a simply connected Riemannian manifold. The projection on the second coordinates on the pseudo-Riemannian product $M = P \times N$ is a Cauchy time function.*

Proof. In order to apply proposition 4.24, we need to show that in the case of a product, we only need the completeness of the fibers to conclude. Since N is Riemannian, there exists a smooth map $f : N \rightarrow \mathbb{R}$ such that $f \geq 1$ and (N, fg_N) is complete. The manifold $(M, fg_P - fg_N)$ is conformally equivalent to $(M, g_P - g_N)$ and is a Riemannian submersion for $M \rightarrow N$. Furthermore, we have that

$$g_P + fg_N \leq fg_P + fg_N,$$

and the Riemannian manifold $(M, g_P + fg_N)$ is complete as the product of two complete manifolds, thus $(M, fg_P + fg_N)$ is also complete and we can apply proposition 4.24 which gives us the result. \square

It is a well known result that every Riemannian submersion $T : M \rightarrow N$ where M is complete is actually a fiber bundle. Inversely, for every fiber bundle $T : M \rightarrow N$, it is possible to find Riemannian metrics on M and N such that M is complete and T is a Riemannian submersion. In our case, this tells us that the topology of any smooth fiber bundle with simply connected base is possible for a globally hyperbolic space M .

5 A few results on globally hyperbolic spaces

5.1 Global hyperbolicity and causality

A famous result of the theory of globally hyperbolic spaces is the equivalence between the existence of a Cauchy time function and the compactness of the space $\mathcal{C}(x, y)$ of future-oriented causal paths coming from x to y . In higher signature, there are many possibilities for the manifold N onto which a time function maps M , meaning that constructing a Cauchy time function seems too hard a task. However, given a Cauchy time function $T : M \rightarrow N$, it is possible to prove a result of compactness similar to the result in Lorentzian geometry.

Before proving this result, we must first show a set of technical lemmas regarding the construction of limits of sequence of causal maps.

Lemma 5.1. *Let M be a globally hyperbolic space with a Cauchy time function $T : M \rightarrow N$ and let $s_n : N \rightarrow M$ be a sequence of causal sections of T . Assume there exists a compact set E of M such that for each $x \in E$, the sequence $s_n(T(x))$ converges to x up to a subsequence. Then there exists a subsequence of (s_n) and a causal section s_∞ which converges locally uniformly to s_∞ .*

Proof. Let $x \in E$ and let $C(a, b) \subset T_x M$ be a cylindrical neighborhood centered on x . Let K be a compact of $\overline{C(a, b)}$ containing an open neighborhood of $\{q|_x \leq 0\}$. Since the $s_n(T(x))$ converges to x up to a subsequence and every s_n is causal, we know that up to a subsequence the images of s_n in $C(a, b)$ are in K . Let g_- be a constant metric on $C(a, b)$ such that $g_- \leq g$. Since every s_n is causal for g , every s_n is also causal for g_- ; in particular every s_n can be seen as the graph of a 1-Lipschitz map for g_- from $\mathbb{D}^q(0, b)$ to $(-a, a)$, each of whom have values in a

compact set. Arzela-Ascoli theorem then tells us that there exists a 1-Lipschitz map $s_\infty : \mathbb{D}^q(0, b) \rightarrow (-a, a)$ to which every s_n converge uniformly. By doing this process on a finite open cover of cylindrical neighborhoods of E (which is compact), we define a map $s_\infty : U \subset N \rightarrow M$ on an open neighborhood U of $T(E)$. The converges is uniform on U , thus the limit map must be continuous and a causal continuous map from U to M .

Let's assume that U is a maximal open set on which s_∞ can be defined. Assume that s_∞ is not inextendible. Then there must exist an inextendible path $c : \mathbb{R} \rightarrow U$ such that $s_\infty \circ c$ converges to $x \in M$. Since the s_n converge uniformly to s_∞ , the sequence $s_n(T(x))$ must converge to x , and by repeating the previous construction, it becomes possible to extend U by adding an open set containing $T(x)$. Thus the map s_∞ must be an inextendible causal map, and since T is a Cauchy time function, s_∞ must be defined on the whole of N . Hence the result. \square

Now equipped with this lemma, we can prove the following generalization.

Theorem 5.2. *Let $T : M \rightarrow N$ be a Cauchy time function, let $B \subset N$ be a connected q -dimensional compact submanifold with connected boundary and let $s : N \rightarrow M$ be a causal section in M . Let $C(B, s)$ be the set of causal sections $\hat{s} : B \rightarrow M$ such that $\hat{s}|_{\partial B} = s|_{\partial B}$ endowed with the uniform topology. Then $C(B, s)$ is compact.*

Proof. Let s_n be a sequence of causal sections $s_n : B \rightarrow M$ with $s_n|_{\partial B} = s$. By attaching s_n inside B and s outside of B , one can consider that each s_n is an inextendible causal limit map in M which is constant on $\overline{N \setminus B}$. By using the lemma 5.1, there must exist an inextendible causal map $s_\infty : N \rightarrow M$ equal to s outside of B to which the s_n converge locally uniformly. Since B is compact, the s_n converge uniformly to s_∞ , hence the result. \square

5.2 Plateau problem in globally hyperbolic spaces

In the Lorentzian setting, the compactness of the space of causal paths from one point to another allows us to prove a central results in globally hyperbolic spaces, the existence of causal geodesics between any two causally related points (see [10], proposition 6.7.1). We may prove an analogous result in higher signatures, by generalising the existence of a length-maximising geodesic to the solution of a Plateau problem. Let $M, T : M \rightarrow N, B, s : N \rightarrow M$ be as defined in the previous section. As seen in proposition 3.34, s must be differentiable everywhere, which allows us to define an area fonctionnal.

Definition 5.3. Let $\hat{s} \in C(B, s)$ and let

$$\mathcal{A}(\hat{s}) = \text{Vol}(B^\circ, -\hat{s}^*g_M).$$

The function \mathcal{A} is the area fonctionnal on the set $C(B, s)$.

Theorem 5.4. *There exists a \hat{s} in $C(B, s)$ for which $\mathcal{A}(\hat{s})$ is maximal.*

Definition 5.5. Let $C^1(B, s)$ be the set of C^1 sections in $C(B, s)$. The set $C^1(B, s)$ is dense for the uniform topology in $C(B, s)$.

Lemma 5.6. *Let $\hat{s}_0 \in C^1(B, s)$. There exists a neighborhood U of $Im(\hat{s}_0)$ and a map $f : U \rightarrow B$ such that $f = T$ on $Im(\hat{s}_0)$, the set $(f = cste)$ are spacelike and orthogonal to $Im(\hat{s}_0)$.*

Proof. Let $x \in Im(\hat{s}_0)$. Let U_x be a normal neighborhood centered on x . For each y in the orthogonal of $Im(d\hat{s}_0)$, we will define $f(y) = T(x)$. Since B is compact, we may only do this on a finite number of U_x which gives us a neighborhood $U \bigcup_x U_x$ on which f is defined. \square

Lemma 5.7. *The functionnal \mathcal{A} is upper semi-continuous on $C^1(B, s)$.*

Proof. Let $\hat{s}_0 \in C^1(B, s)$ and f as in the previous lemma. Each section $\hat{s} \in C^1(B, s)$ with image in U can be parametrised by f . Let \hat{s} be such a section. For each x in U , let π_1 be the orthogonal projection on $\text{Ker}(df)$ and π_2 be the orthogonal projection on $\text{Ker}(df)^\perp$. For each vector v tangent to $Im(\hat{s})$, we have

$$\begin{aligned} g(v) &= g(\pi_1(v)) + g(\pi_2(v)), \\ &\geq g(\pi_2(v)), \end{aligned}$$

since $\text{Ker}(df)$ is spacelike. Let v_0 be a tangent vector to \hat{s}_0 in the same level set $(f = cste)$ as v . There exists a small neighborhood $U' \subset U$ containing $Im(\hat{s}_0)$ such that $g(\pi_2(v)) > g(v_0) + \varepsilon$. For each \hat{s} in U' , we must then have

$$\mathcal{A}(\hat{s}) \leq \mathcal{A}(\hat{s}_0)\sqrt{1 + \varepsilon},$$

hence the result. \square

Proof of theorem 5.2. Since the functionnal \mathcal{A} is upper semi-continuous on $C^1(B, s)$ which is dense in $C(B, s)$, it must also be upper continuous on $C(B, s)$. However the set $C(B, s)$ is compact, meaning that there must exist a \hat{s} in $C(B, s)$ such that \mathcal{A} is maximal, hence the result. \square

5.3 More examples of globally hyperbolic spaces

The examples of globally hyperbolic spaces we have exhibited in 4.24 have the downside of being rather rigid, whereas global hyperbolicity in the lorentzian setting could appear in many spaces which do not have a simple conformal model, for instance any causally convex open set of $\mathbb{R}^{p,1}$. This subsection will be dedicated to exhibiting such examples, first by considering spacetimes which are comparable to the examples given by proposition 4.24 and then by considering some open spaces in the product $M = P \times N$ where P and N are complete simply connected manifolds.

Proposition 5.8. *Let (M, g) be a pseudo-Riemannian manifold and let \bar{g} be another pseudo-Riemannian metric such that $\bar{g} \geq g$. If $T : M \rightarrow N$ is a Cauchy time function for (M, \bar{g}) , then it is a Cauchy time function for (M, g) .*

Proof. Consider an inextendible map $f : N_0 \rightarrow M$ which is causal for g . Since $\bar{g} \geq g$, it is also causal for \bar{g} . Since T is a Cauchy time function for \bar{g} , $T \circ f$ is a diffeomorphism from N_0 to N , hence the result. \square

Corollary 5.9. *Let x be a point in a spacetime M . There exists a globally hyperbolic neighborhood U of x in M .*

Proof. Let $C(a, b)$ be a cylindrical neighborhood of x in M , let U be flat causal diamond in $C(a, b) \subset \mathbb{R}^{p,q}$ for the flat metric g_- and let $T : U \rightarrow \mathbb{H}^q$ be a Cauchy time function on U for g_- . Since $g_- < g$, T is also a Cauchy time function for (U, g) , hence the result. \square

Proposition 5.8 allows us to give a bigger class of example of globally hyperbolic spaces by comparing them to the more rigid examples given by proposition 4.24. Here are some of those spacetimes.

Proposition 5.10. *Let (M, g) be a pseudo-Riemannian spacetime which is topologically a product $M = P \times N$ where the $P \times \{*\}$ are spacelike and the $\{*\} \times N$ are timelike. For each point, we have the decomposition $g = g_P - g_N$ where g_P is definite positive on TP and g_N is definite positive on TN . Assume that :*

- *there exists a complete Riemannian metric \bar{g}_P on P such that on each fiber of $M \rightarrow N$, we have $\bar{g}_P \geq g_P$,*
- *there exists a Riemannian metric \bar{g}_N on N such that on each fiber of $M \rightarrow P$, we have $\bar{g}_N \geq g_N$,*
- *the manifold N is simply connected.*

Then the projection $T : M \rightarrow N$ is a Cauchy time function for (M, g) .

Proof. Let $\bar{g} = \bar{g}_P - \bar{g}_N$. The manifold (M, \bar{g}) is the pseudo-Riemannian product of (P, \bar{g}_P) and (N, \bar{g}_N) . Corollary 4.27 tells us that $T : M \rightarrow N$ is a Cauchy time function for (M, \bar{g}) and the assumptions that $\bar{g}_P \leq g_P$ and $\bar{g}_N \geq g_N$ tell us that $\bar{g} \leq g$, thus T is a Cauchy time function for (M, g) , hence the result. \square

Let us now take $B \subset P$ a compact oriented submanifold with boundary of dimension p which we will assume is geodesically convex. Let $s : P \rightarrow N$ be an inextendible timelike map and let Ω be the causal diamond of the graph of s in M .

Proposition 5.11. *The graph of $s|_B$ is a Cauchy surface in Ω .*

Lemma 5.12. *Let $f : N_0 \rightarrow \Omega$ be a smooth inextendible timelike map of Ω . Then f is the graph of a map $\hat{f} : A \subset \mathbb{R}^q \rightarrow B$.*

5.4 Topological structure of globally hyperbolic spaces

As we have seen in proposition 2.30, being globally hyperbolic in the Lorentzian case implies strong topological characteristics on the manifold M . In order to obtain the same kinds of results in higher signature, one has to require the existence of a causal foliation of dimension q .

Proposition 5.13. *Let $T : M \rightarrow N$ be a globally hyperbolic space with a timelike (resp. causal) foliation F . Then the bundle $T : M \rightarrow N$ is homeomorphic to the product $S \times N$ where S is one of the fibers. In particular, all the fibers are homeomorphic. Furthermore, the sets $\{*\} \times N$ form a timelike (resp. causal) foliation of M .*

Proof. Let $x \in M$, F_x the leaf of F containing x and $\varphi(x)$ the only point both in F_x and S . The map

$$x \in M \longmapsto (\varphi(x), T(x)) \in S \times N$$

is a fiber bundle isomorphism, hence the result. \square

The examples given all have the topological structure of a smooth fiber bundle $T : M \rightarrow N$. We are going to show that this is actually implied by the definition of Cauchy time function when we assume that T is smooth and that at least one of the level sets is compact.

Definition 5.14. Let $T : M \rightarrow N$ be a smooth Cauchy time function on a globally hyperbolic spacetime and let $\gamma : I = (a, b) \rightarrow N$ be a smooth path on N . We define the pullback γ^*M as the space

$$\bigsqcup_{t \in I} T^{-1}(\gamma(t)),$$

along with the projection $\gamma^*T : \gamma^*M \rightarrow I$. This defines a smooth manifold for which the tangent space in $x \in \gamma^*M$ is $T_x\gamma^*M = (\text{Ker}(dT)) \oplus dT^{-1}(\dot{\gamma}(t))$. This also gives us a Lorentzian metric on γ^*M induced by the metric on M .

Those spaces we built using the structure on M keep some of its properties.

Proposition 5.15. *Let $\gamma : I \rightarrow N$ be a smooth path on N . The map $\gamma^*T : \gamma^*M \rightarrow I$ is a time function on the Lorentzian space γ^*M .*

Proof. Let $c : J \rightarrow \gamma^*M$ be a timelike path. We are going to show that the map $\gamma^*T \circ c : J \rightarrow I$ is injective. Let $t \in J$. Since the vector $\dot{c}(t) \in T_{c(t)}M$ is timelike, there exists an inextendible timelike immersion $f : N \rightarrow M$ such that f contains $c(t')$ for $t' \in]t - \varepsilon, t + \varepsilon[$. To build this immersion, one can for instance take an inextendible timelike immersion going through $c(t)$ and modify it in the neighborhood of $c(t)$. Since $T \circ f$ is a diffeomorphism, this implies that the map $\gamma^*T \circ c$ is injective on $]t - \varepsilon, t + \varepsilon[$, meaning that $\gamma^*T \circ c$ is locally injective. Since it is a map between two intervals, it must be globally injective, hence the result. \square

We would like to show that those projections are not only time functions, but also Cauchy time functions in the Lorentzian sense. However, we are unable to do so without the key assumption that T admits at least one level sets which is compact. It is not known whether we can get rid of this hypothesis.

Proposition 5.16. *Let $T : M \rightarrow N$ be a smooth Cauchy time function on M such that there exists $x \in N$ such that $T^{-1}(x)$ is compact and let $\gamma : I = (a, b) \rightarrow N$ be a smooth path. Then the map $\gamma^*T : \gamma^*M \rightarrow I$ is a Cauchy time function on the Lorentzian space γ^*M .*

Proof. Let $c : (T_1, T_2) \subset I \rightarrow \gamma^*M$ be an inextendible timelike path such that $\gamma^*T \circ c = \gamma|_{(T_1, T_2)}$. Assume that $T_2 < b$ and let (t_n) be an increasing sequence of (T_1, T_2) converging to b . For each n , let $f_n : N \rightarrow M$ be an inextendible timelike immersion containing $c(t_n) = f_n(\gamma(t_n))$ and let p_n be its unique intersection with $T^{-1}(x)$. Since $T^{-1}(x)$ is compact, we can assume that (p_n) converges to $p_\infty \in T^{-1}(x)$.

Since p_∞ is an accumulation point of the f_n , we can build a causal inextendible map $f_\infty : N \rightarrow M$ containing p_∞ . The map f_∞ is defined on the whole of N since T is a Cauchy time function. Let $c_\infty = f_\infty(T_2)$. Since t_n converges to T_2 , $c(t_n)$ is contained in the image of f_n and c_∞ is an accumulation point of the f_n , we must have that $c(t_n)$ converges to c_∞ which is a contradiction as c was assumed to be inextendible, hence the result. \square

This result has a few immediate consequences.

Corollary 5.17. *When T admits at least one compact level set, the map $T : M \rightarrow N$ is a smooth fiber bundle on N . In particular, all of the level sets are diffeomorphic, and thus compact.*

Proof. It is enough to show that for each smooth path $\gamma : I \rightarrow M$, the map $\gamma^*T : \gamma^*M \rightarrow I$ is a fiber bundle. However it is a Cauchy time function, meaning that it must be diffeomorphic to the trivial bundle $S \times I$ where S is a level set of T , hence the result. \square

It is possible to improve proposition 5.16 by taking an immersion in N instead of a smooth path.

Definition 5.18. Let $f : N' \rightarrow N$ be a submersion from a smooth manifold N' of dimension $q' \leq q$ to N . In the same manner as definition 5.14, one can define the space f^*M along with a projection $f^*T : f^*M \rightarrow N'$. The metric on M endows f^*M with a pseudo-Riemannian metric of signature (p, q') .

Proposition 5.19. *Let $T : M \rightarrow N$ be a smooth Cauchy time function on M with at least one compact fiber and let $f : N' \rightarrow N$ be an immersion in N with N' simply connected. Then $f^*T : f^*M \rightarrow N'$ is a Cauchy time function.*

Proof. Let $g : N_0 \rightarrow f^*M$ be an inextendible smooth causal map in f^*M . Let us first show that $f^*T \circ g : N_0 \rightarrow N'$ is surjective. Assume it is not and let (y_n) be a sequence in $Im(f^*T \circ g)$ which converges to $y_\infty \in N' \setminus Im(f^*T \circ g)$. Let (x_n) in N_0 such that $f^*T \circ g(x_n) = y_n$. For each n , there exists an inextendible timelike map $f_n : N \rightarrow M$ which goes through $g(x_n)$. Since T is a Cauchy time function, this map also goes through $T^{-1}(x)$ which is compact, therefore one may extract a limit map f_∞ to the f_n which must be defined on the whole of N . In particular it is defined on $f(y_\infty)$, which gives an inconsistency.

Let us now show that $f^*T \circ g$ is a covering on N' . Since it is a local diffeomorphism, we only have to show that every smooth path in N' can be uniquely lifted to a

smooth path from any point in a fiber. Let $\gamma : [a, b] \rightarrow N'$ be a smooth path in N' . By proposition 5.16, we know that $\gamma^*T : \gamma^*M \rightarrow (a, b)$ is a Cauchy time function. Let $x \in (f^*T \circ g)^{-1}(\gamma(a))$. Since g is a local diffeomorphism, we know that near $g(x)$, the intersection of $Im(g)$ with γ^*M gives a unique timelike path c . Since γ^*T is a Cauchy time function, $\gamma^*T \circ c$ must be a diffeomorphism, which means that there is a unique lifting of γ in N_0 starting from x and defined for all $t \in (a, b)$. Thus $f^*T \circ g$ must be a covering.

Since N' is assumed to be simply connected, we know that $f^*T \circ g$ must be a diffeomorphism, hence the result. \square

5.5 Closed spacetimes and time functions

In Lorentzian geometry, it is possible to show that any compact spacetime must admit a timelike immersion $f : \mathbb{S}^1 \rightarrow M$, meaning that it cannot be globally hyperbolic. This stems from the fact that the only compact one dimensional manifold is not simply connected. However, in higher signature, there exists globally hyperbolic spacetimes which are topologically closed, for instance the einstein space $\text{Ein}^{p,q}$ for $q \geq 2$. This section is dedicated to results on such spacetimes.

Lemma 5.20. *Let $T : M \rightarrow N$ be a time function on a spacetime M and let $f : N_0 \rightarrow M$ be an inextendible causal map. Then the image of f is closed in M .*

Proof. Assume this is not the case and take x in the boundary of $Im(f)$ in M . The points $y = T(x)$ is in the boundary of $T(Im(f))$ in N . Let $c :]a, +\infty[\rightarrow N$ be a smooth injective path in $T(Im(f))$ such that $c(+\infty) = y$. Since $T \circ f$ is injective, there exists a unique path $\tilde{c} :]a, +\infty[\rightarrow N_0$ such that $T \circ f \circ \tilde{c} = c$. The path \tilde{c} is right-inextendible in N_0 but $f \circ \tilde{c}$ converges to x in infinity, which is not possible as f was taken to be inextendible, hence the result. \square

Proposition 5.21. *Let M be a closed (p, q) -spacetime with a time function $T : M \rightarrow N$. Then T is a Cauchy time function.*

Proof. Let $f : N_0 \rightarrow M$ be an inextendible causal map. Using the previous lemma, we can deduce that the image of f is closed in M . Since M is compact, $Im(f)$ must then also be compact. This means that the map $T : Im(f) \rightarrow N$ is an embedding of a closed manifold into a manifold of the same dimension; it must then be a homeomorphism, hence the result. \square

Remark 5.22. In this case, the fact that $T : M \rightarrow N$ is a fiber bundle comes immediately since any smooth submersion from a compact space is a bundle.

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