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From Microscopic Driver Models to Macroscopic PDEs in Ring Road Traffic Dynamics

S. Fueyo and C. Canudas-de-Wit

Abstract—In this paper, using a continuation method [22], we provide a simple procedure to derive macroscopic models from second order ordinary differential equations modeling individual vehicles moving on a ring. In particular, the result is applied to classical traffic models such as the optimal velocity follow the leader (OV-FTL) models, the intelligent driver model (IDM) or delayed driver models and we are able to modelize the macroscopic behavior of the drivers through a 2×2 partial differential equations whose variables are the density of the traffic and the speed of the vehicles. The paper concludes with open questions regarding stability and microscopic control design.

Keywords: Continuation method, Microscopic and macroscopic driver models, Partial differential equations (PDEs)

I. INTRODUCTION

Road traffic consists of a large number of interconnected vehicles, making the direct analysis of such complex systems nearly impossible. Typically, the dynamics of vehicles on a traffic road are described using microscopic models, which are often expressed through high-dimensional first or second-order ordinary differential equations (ODEs). Examples of such models include the optimal velocity follow-the-leader (OV-FTL) model [4], the intelligent driver model (IDM) [25] and delayed driver model [7]. To simplify the analysis of complex networks, it is common in the literature to employ a continuous representation, typically in the form of hyperbolic partial differential equations (PDEs); various macroscopic models have been proposed, such as the Lighthill-Whitham-Richards (LWR) model [17] and the Aw-Rascle-Zhang (ARZ) model [3].

The dynamics of vehicular traffic represent a system where large-scale patterns arise from the combined actions of individual drivers. One notable example is the *phantom traffic jam*, a phenomenon in which, under specific density conditions, steady traffic flow becomes unstable. Minor disruptions grow into significant nonlinear oscillations, leading to *stop-and-go phenomenon*, often referred to as *jamitons*, see [11] for further references.

With the advancement of computing, the discretization of PDEs into ODEs has seen significant development. In opposition, the reverse problem—transforming ODEs into PDEs—has received less attention. This process is known as the continuation method. As with discretization, different continuation methods are possible, depending on the specific

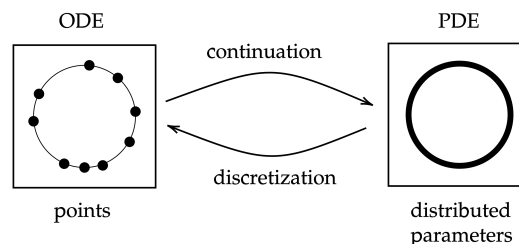


Fig. 1. Continuation and discretization methods for a ring.

ODEs and the objectives we aim to achieve by considering the models.

Several studies [1], [14], [24], [16], [6], [18], [19] and [21] employ continuation methods to derive PDEs from first-order differential equations. Specifically, references [6], [18], [16] utilized a convolution operator for representation, while [1], [14], [24], [19], [21] applied Taylor expansion to formulate continuous models. Notably, in the context of modeling the dynamics of an infinite number of vehicles on a real line, several studies in focus on deriving LWR models, [1], [14] deriving from the OV-FTL microscopic model, and [21] from a different first-order ODE.

In traffic flow analysis, it is crucial to consider the more complex behaviors produced by second-order microscopic models, as first-order models often oversimplify the dynamics of congested traffic. Second-order models are particularly effective in capturing phenomena such as stop-and-go waves and phantom traffic jams, which are common under congested conditions. While [21], [22] explored specific second-order ODEs, the application of continuation methods, particularly regarding Taylor expansions, to second-order ODEs remains an open question.

On one hand, [15] proposes a general method known as coarse-graining, which enables the derivation of macroscopic models from general second-order OV-FTL models. However, this method is intricate and significantly more complex than the approach based on Taylor expansions [8]. On the other hand, the studies [2], [8] converted a specific second-order OV-FTL model defined on a real line into an ARZ macroscopic model using Taylor expansion. Nonetheless, they did not address the continuation of general models, nor did they consider vehicles moving on a ring instead of a straight road.

This paper marks an initial attempt to apply the continu-

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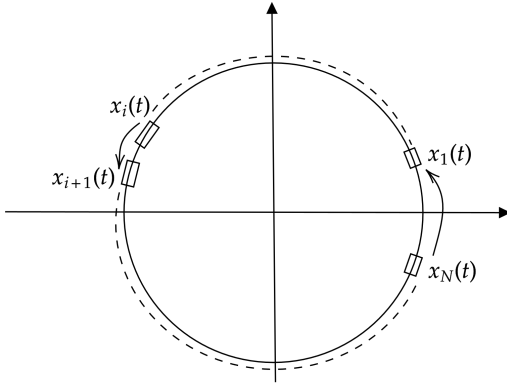


Fig. 2. N vehicles moving on a ring.

ation method framework introduced in [21] to second-order models that characterize the behavior of vehicles on a ring. By utilizing a straightforward continuation method based on Taylor expansion, we describe the evolution of vehicle density and speed through a 2×2 partial differential equation. Specifically, we derive hyperbolic balanced law systems from both the OV-FTL models and delayed driver models, some of which align with ARZ models.

This approach allows for a reciprocal analysis between microscopic and macroscopic models. In opposition, the IDM model [25], when approached through our method, results in a PDE that does not conform to a balanced law framework, thereby limiting the insights that can be drawn about these systems.

Section II introduces the general continuation method for inferring macroscopic models from second-order microscopic traffic models on a circular road. The approach is then illustrated in the sections III, IV and Section V using various vehicle models, such as the OV-FTL, IDM, and delay driver models, allowing for the generalization of existing results in the literature. In particular, we derive macroscopic models from classical microscopic drivers models while, conversely we deduce microscopic models from the macroscopic ARZ model agreeing with our method. The paper is concluded by Section VI, discussing how the continuation method will eventually lead to control strategies for microscopic models.

II. CONTINUATION METHOD FOR SECOND ORDER MODELS OF N VEHICLES MOVING ON A RING

For N an integer, consider the variable set $\{x_i(t), i = 1, \dots, N\}$ representing the position of N vehicles at time $t \geq 0$, moving on a ring and defined modulo 2π , satisfying the equations

$$\ddot{x}_i(t) = f(x_{i+1}(t) - x_i(t), \dot{x}_{i+1}(t) - \dot{x}_i(t), \dot{x}_i(t)), \quad (1)$$

where $t \geq 0$, $x_{i+1}(t) \geq x_i(t)$ for $i = 1, \dots, N$, $x_{N+1}(t) = x_1(t)$. Here, f is a nonlinear function mapping \mathbb{R}^3 to \mathbb{R} . Equation (1) represents the behavior of the vehicle i where its acceleration depends on the spacing and the difference of speed with the vehicle $i + 1$ ahead ($x_{i+1}(t) - x_i(t)$ and $\dot{x}_{i+1}(t) - \dot{x}_i(t)$ respectively), and its own velocity \dot{x}_i .

Following [21], we can apply a continuation method, with respect to Taylor expansions, to Equation (1). We summarize the continuation procedure as follows

- 1) see the discrete set of vehicles $(x_i)_{i=1, \dots, N}$ as a continuum of vehicles described by a function x ;
- 2) introduce the function $M(t, x)$ corresponding to the number of vehicles passing through the point x at time t . This function is called the Moscowitz function and it translate the idea that the density of the system of vehicles (mass in unit length), is a mass which is transferred through the point x up to the time t , see [20] for further details;
- 3) define the density ρ and flux φ of the vehicles in function of M and write the conservation law satisfied by the Moscowitz function;
- 4) define the speed variable v and do Taylor approximations in (1) to obtain PDEs.

Let us follow the procedure that we described. We consider a continuous function x defined on $\mathbb{R}_+ \times [0, N]$ with value in $[0, 2\pi)$ such that $x(t, i) = x_i(t)$ (modulo 2π). The function x is a continuation of the discrete set of the vehicles $(x_i(t))_{i=1, \dots, N}$. Since $x_i(t)$ is defined modulo 2π , we define the Moscowitz function M from $\mathbb{R}_+ \times [0, 2\pi)$ into $[0, N]$. Specifically, we have $M(t, x_i(t)) = i$, which can be seen as a sensor mounted on the agent $x_i(t)$. In contrast note that $M(t, x)$ can be interpreted as a count sensor at the position x , counting all vehicles passing at a fix point x . The Moskowitz function, $M(t, x)$, naturally allows to define the vehicles density and flow, ρ and φ respectively, as

$$\rho := \frac{\partial M}{\partial x} \quad \text{and} \quad \varphi = -\frac{\partial M}{\partial t}, \quad (2)$$

where density and flow are related by the conservation law

$$\frac{\partial \rho}{\partial t} + \frac{\partial \varphi}{\partial x} = 0. \quad (3)$$

This equally implies that M satisfies:

$$\frac{\partial^2 M}{\partial t \partial x} = \frac{\partial^2 M}{\partial x \partial t}.$$

On the other side, the quantities x_i , $x_{i+1} - x_i$ and $\dot{x}_{i+1} - \dot{x}_i$ can be approximated by Taylor expansions as follows:

$$\begin{aligned} x_i &\approx x, \\ x_{i+1} - x_i &\approx \frac{\partial x}{\partial M} \\ \dot{x}_{i+1} - \dot{x}_i &\approx \frac{\partial}{\partial M} \frac{\partial x}{\partial t}. \end{aligned}$$

and we get from Equation (1) that

$$\frac{\partial^2 x}{\partial t^2} = f\left(\frac{\partial x}{\partial M}, \frac{\partial^2 x}{\partial M \partial t}, \frac{\partial x}{\partial t}\right). \quad (4)$$

By definition, we have $x(t, M(t, x)) = x$ for all $x \in [0, 2\pi)$ so that if we take the derivative with respect to t and x , we get

$$\frac{\partial x}{\partial M} \frac{\partial M}{\partial x} = 1 \quad \text{and} \quad \frac{\partial x}{\partial t} + \frac{\partial x}{\partial M} \frac{\partial M}{\partial t} = 0. \quad (5)$$

Thus defining the speed v as the partial derivative in time of x , namely $v := \partial x / \partial t$, we get from (5) that

$$\varphi = v\rho, \quad \frac{\partial x}{\partial M} = \frac{1}{\rho} \quad \text{and} \quad \frac{\partial^2 x}{\partial M \partial t} = \frac{\partial v}{\partial x}. \quad (6)$$

Putting (6) into the equations (3)-(4) yields to the following one dimensional PDEs defined on $\mathbb{R}_+ \times (0, 2\pi)$

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial(v\rho)}{\partial x} = 0, \\ \frac{\partial v}{\partial t} = f\left(\frac{1}{\rho}, \frac{\partial v}{\partial x}, v\right). \end{cases} \quad (7)$$

As boundary conditions, since the vehicles move on a ring, we assume that $\rho(t, 0) = \rho(t, 2\pi)$ and $v(t, 0) = v(t, 2\pi)$ for all $t \geq 0$. Thus System (1) can be construed as the following PDE with periodic boundary conditions, where the equations are given by:

$$\begin{cases} \frac{\partial}{\partial t} \begin{bmatrix} \rho \\ v \end{bmatrix} + \begin{bmatrix} v & \rho \\ 0 & 0 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \rho \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ f\left(\frac{1}{\rho}, \frac{\partial v}{\partial x}, v\right) \end{bmatrix}, \end{cases} \quad (8a)$$

$$\begin{cases} \rho(t, 0) = \rho(t, 2\pi), & \forall t \geq 0, \\ v(t, 0) = v(t, 2\pi), & \forall t \geq 0, \end{cases} \quad (8b)$$

and the initial data of the system are

$$\rho(t, x) = \rho^0(x), \quad v(t, x) = v^0(x), \quad \forall x \in [0, 2\pi], \quad (9)$$

for some scalar functions $\rho^0(\cdot)$ and $v^0(\cdot)$ defined on $[0, 2\pi]$. As desired, integrating the first equation of (8a) and utilizing the periodic boundaries (8b) demonstrates that the integral of the density remains constant as it evolves over time.

We remark that the macroscopic model (8a) has the partial derivative of the speed depending nonlinearly on the spacial derivative of v . However, for some particular f , it is possible to arrive to one dimensional hyperbolic PDE with balanced laws property. In fact, assuming that there exist two functions f_1 and f_2 such that f has the particular form

$$f\left(\frac{1}{\rho}, \frac{\partial v}{\partial x}, v\right) = f_1\left(\frac{1}{\rho}, v\right) \frac{\partial v}{\partial x} + f_2\left(\frac{1}{\rho}, v\right), \quad (10)$$

it allows to rewrite (8a) as

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ v \end{bmatrix} + \begin{bmatrix} v & \rho \\ 0 & -f_1\left(\frac{1}{\rho}, v\right) \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \rho \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ f_2\left(\frac{1}{\rho}, v\right) \end{bmatrix}. \quad (11)$$

with boundary conditions given in (8b). System (11) is of hyperbolic type and it allows one to use all theoretical results in the literature such that the existence of solutions, the stability analysis of equilibrium points and the boundary stabilization process that we can find in the references [29], [12], [5], [10].

We shall illustrate in the next sections the macroscopic models that can be derived from microscopic traffic models such as the OV-FTL model.

III. OPTIMAL VELOCITY FOLLOW THE LEADER (OV-FTL) MODELS

In this section, we introduce one type of microscopic traffic models called OV-FTL and its macroscopic derivation using the continuation method outlined in Section II.

A. General optimal velocity follow the leader (OV-FTL) models

A OV-FTL model is given by the equations

$$\begin{aligned} \ddot{x}_i &= f_1(x_{i+1} - x_i, \dot{x}_i) (\dot{x}_{i+1} - \dot{x}_i) \\ &\quad + f_2(x_{i+1} - x_i, \dot{x}_i) (V(x_{i+1} - x_i) - \dot{x}_i), \end{aligned} \quad (12)$$

for $i = 1, \dots, N$. We assume that f_1 and f_2 are two strictly positive functions representing gains that depend on whether drivers exhibit cautious or aggressive behaviors. The map $V(\cdot)$ is an increasing and uniformly bounded function translating the idea that if the vehicle is on a free road then the vehicle goes at maximal speed and if the road is congested then the speed is reduced. The function $V(\cdot)$ is called *fundamental diagram*. Several forms have been proposed in the literature, see for instance [12] for a non-exhaustive list.

Interpreting the model given by Equation (12), the second term in the right-hand side of Equation (12), namely $f_2(x_{i+1} - x_i, \dot{x}_i) (V(x_{i+1} - x_i) - \dot{x}_i)$, indicates that when the leading vehicle $i+1$ is far away, the vehicle i accelerates without exceeding a certain threshold. Conversely, when the distance to the leading vehicle $i+1$ is short, the acceleration of the vehicle i decreases as the spacing reduces. The first term in the right-hand side of Equation (12), *i.e.* $f_1(x_{i+1} - x_i, \dot{x}_i) (\dot{x}_{i+1} - \dot{x}_i)$, demonstrates that the vehicle decelerate (resp. accelerate) if the relative speed $\dot{x}_{i+1} - \dot{x}_i$ is negative (resp. positive). A pair of real positive numbers (d^*, v^*) is called an equilibrium point of Equation (12) if, for all $i = 1, \dots, N$,

$$x_{i+1} - x_i := d^*, \quad \dot{x}_{i+1} - \dot{x}_i = 0 \quad \text{and} \quad \dot{x}_i = v^*$$

implies that the acceleration \ddot{x}_i is zero. Thus at an equilibrium point, the accelerations of the vehicles are zero, and the positions are uniformly spaced at distance d^* and each vehicle maintains a constant velocity v^* . System (12) has the advantageous property of possessing equilibrium points because, by definition of the function V , there exists a unique pair (d^*, v^*) of strictly positive numbers such that $V(d^*) = v^*$. Unfortunately equilibrium points of System (1) may be unstable (depending on the functions f_1 and f_2) resulting in a stop and go, or phantom traffic jam, phenomena, see the papers [26], [23] for empirical proofs and the reference [9] for a theoretical argument.

We presented the OV-FTL model in its full generality in Equation (12). However, such models might be too general to prevent the collision between the vehicles. In the next section, we address now the macroscopic derivation of a specific OV-FTL model that is widely used in the literature preventing the colliding vehicles situation.

B. Instability analysis of a particular OV-FTL model and its continuation model

More precisely, we consider the OV-FTL model introduced in [9], where $f_1 = a/(x_{i+1} - x_i)^2$ and $f_2 = b$ are equal to, for some strictly positive constants a and b . This model

generalizes the one proposed in [4] and is described by the following equations

$$\ddot{x}_i = a \frac{\dot{x}_{i+1} - \dot{x}_i}{(x_{i+1} - x_i)^2} + b(V(x_{i+1} - x_i) - \dot{x}_i), \quad (13)$$

where the first term in the right-hand side of (13) implies that the relative velocity decreases with increased spacing, thereby preventing collisions between vehicles. Following the procedure described in Section II, we arrive to the following hyperbolic PDEs system

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ v \end{bmatrix} + \begin{bmatrix} v & \rho \\ 0 & -a\rho^2 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \rho \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ b \left(V\left(\frac{1}{\rho}\right) - v \right) \end{bmatrix} \quad (14)$$

with periodic boundaries given in (8b). Of course $\rho = 1/d^*$ and $v = v^*$ are equilibrium points of (14). We denote by $(\rho^*, v^*) := (1/d^*, v^*)$ the equilibrium point of (14). Since the equilibrium point of Equation (13) is known to be unstable (see [9]). More precisely, we have the following proposition.

Proposition III.1 ([9]). *The equilibrium point of the OV-FTL model (13) is locally unstable if the following inequality holds*

$$a - \frac{2b}{(d^*)^2} - 2V'(d^*) < 0. \quad (15)$$

Proof. Let us define $f(h_j, \dot{h}_j, v_j) = a(V(h_j) - v_j) + b\frac{\dot{h}_j}{h_j^2}$. We have $f(d^*, 0, v^*) = 0$ so that we get

$$\begin{aligned} \alpha_1 &:= \frac{\partial f}{\partial h_j}(d^*, 0, v^*) = aV'(d^*), \\ \alpha_2 &:= \frac{\partial f}{\partial \dot{h}_j}(d^*, 0, v^*) - \frac{\partial f}{\partial v_j}(d^*, 0, v^*) = \frac{b}{(d^*)^2} - a, \\ \alpha_3 &:= \frac{\partial f}{\partial \dot{h}_j}(d^*, 0, v^*) = \frac{b}{(d^*)^2}. \end{aligned} \quad (16)$$

From Equation (10) in [9], the equilibrium point of the OV-FTL model (13) is locally unstable if the condition $\alpha_2^2 - \alpha_3^2 - 2\alpha_1 \geq 0$ is violated. Thus we deduce from the previous inequality the result when replacing α_1 , α_2 and α_3 by the values given in (16). \square

Proposition III.1 provided conditions through an inequality for the local instability of the equilibrium point of System (13). Is it correct to assume that, under the same conditions, the equilibrium point of the PDE is also locally unstable? We give an insight by adopting the strategy of the paper [28], we are able to provide the following result on the local instability of equilibrium point for the macroscopic model of the OV-FTL model.

Proposition III.2. *The equilibrium point of the hyperbolic system (18) is locally unstable if*

$$a + v^*(d^*)^2 < (d^*)^3 V'(d^*). \quad (17)$$

Proof. To study the local stability of the equilibrium point for (14), we linearize the system around the equilibrium point and we get the following linear hyperbolic system

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ v \end{bmatrix} + \begin{bmatrix} v^* & \rho^* \\ 0 & -a(\rho^*)^2 \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \rho \\ v \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ ba_1 & -bv^* \end{bmatrix} \begin{bmatrix} \rho \\ v \end{bmatrix}, \quad (18)$$

where

$$a_1 = \frac{\partial}{\partial \rho} \left(V\left(\frac{1}{\rho}\right) \right)_{|\rho=\rho^*}, \quad (19)$$

with boundary conditions given in (8b). Interpreting this system as an ARZ model so that we by defining $p'(\rho^*) := (a(\rho^*)^2 + v^*)/\rho^*$, we get from the spectral study of [28, Section III] that the equilibrium point is locally unstable if the inequality

$$\frac{a(\rho^*)^2 + v^*}{\rho^*} < -a_1. \quad (20)$$

Using the fact that $\rho^* = 1/d^*$ and $a_1 = -V'(1/\rho^*)/(\rho^*)^2$, we get from (20) that the inequality (17) holds. \square

The conditions for the local instability coming from Proposition III.1 and Proposition III.2 are complicated to compare because they are different in their essence. One depends on the constant b or v^* while it is not the case of the other. However, if we assume that $d^*V'(d^*) > v^*$ the two inequality holds for a small enough, meaning that if the drivers do not adopt a careful driving, *i.e.* if the constant a is small, then the systems tends to be unstable around the equilibrium point in both the microscopic and the macroscopic models.

IV. OTHER INTERESTING MACROSCOPIC MODELS DERIVED FROM MICROSCOPIC VEHICULAR TRAFFIC MODELS

The OV-FTL models are a bit unrealistic because the accelerations and decelerations are symmetric with respect to the deviation of the actual velocity from the equilibrium velocity. To overcome this issue, two classes of systems has been developed, namely the IMD driver models and a OV-FTL model including delays. We infer in this section new macroscopic models which can be useful to describe traffic road from the macroscopic model perspective.

A. Intelligent driver model (IDM)

The following model has been proposed in [25] to introduce acceleration asymmetries:

$$\ddot{x}_i = a \left(1 - \left(\frac{\dot{x}_i}{v_0} \right)^\delta - \left(\frac{s^*(\dot{x}_i, \dot{x}_{i+1} - \dot{x}_i)}{x_{i+1} - x_i} \right)^2 \right), \quad (21)$$

with

$$s^*(y, z) = s_0 + \tau y - \frac{yz}{2\sqrt{ab}},$$

for $y, z \in \mathbb{R}$.

In this context, v_0 signifies the target speed, typically aligned with the speed limit. The variable τ stands for the time headway, while a and b correspond to the maximum acceleration and deceleration capacities. The parameter δ

accounts for the acceleration exponent, and s_0 defines the minimum spacing between vehicles in congested traffic, also known as the jam distance.

Equation (21) can be interpreted as follows. The vehicle i accelerate according with $1 - \left(\frac{v}{v_0}\right)^\delta$ on free-flow whereas it tends to decelerate by $\left(\frac{s^*(v, \dot{x}_{i+1} - \dot{x}_i)}{x_{i+1} - x_i}\right)^2$ when the traffic is jammed. The deceleration coefficient depends on the ratio between the ‘‘desired minimum gap’’ s^* and the actual gap $x_{i+1} - x_i$. Applying the continuation method, we get the following PDE

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v)}{\partial x} = 0, \\ \frac{\partial v}{\partial t} = a \left(1 - \left(\frac{v}{v_0}\right)^\delta - (\rho s^*(v, \frac{\partial v}{\partial x}))^2 \right), \end{cases} \quad (22)$$

with the boundary conditions given by (8b). In the limit of our knowledge, the 2×2 PDE (22) given by the IMD driver model through the continuation method is not known in the literature. The analysis of this new model might lead to some interesting insight in futur works.

B. Second order traffic flow models with time delays

In addition to the IDM models, which were introduced to address the unrealistic assumption of the OV-FTL models that neglect the latency in drivers’ behaviors, other models have been developed to directly account for delays, see [7], [27]. The continuation method presented in Section II can also be adapted for simple second-order traffic models that include time delays. For instance, consider the simple system (13), where the response delay denoted by $T > 0$, implies that the acceleration of the vehicles depends on the delayed speed and position by $T > 0$:

$$\begin{aligned} \ddot{x}_i(t) = a \frac{\dot{x}_{i+1}(t-T) - \dot{x}_i(t-T)}{(x_{i+1}(t-T) - x_i(t-T))^2} \\ + b(V(x_{i+1}(t-T) - x_i(t-T)) - \dot{x}_i(t-T)). \end{aligned} \quad (23)$$

An immediate adaptation of Section II allows us to construe the second order delay systems (23) as the following PDE

$$\begin{cases} \frac{\partial \rho(t,x)}{\partial t} + \frac{\partial(v(t,x)\rho(t,x))}{\partial x} = 0, \\ \frac{\partial v(t,x)}{\partial t} = a\rho(t-T,x)^2 \frac{\partial}{\partial x} v(t-T,x) \\ + b \left(V \left(\frac{1}{\rho(t-T,x)} \right) - v(t-T,x) \right) \end{cases}, \quad (24)$$

with periodic boundaries given in (8b). System (24) is a hyperbolic delay partial differential equations.

Let us examine the right-hand side of Equation (24) more closely. Unlike the classical model, the temporal variation in speed now depends on both the density and the spatial derivative of speed from a previous time step. The derivation of (24) from (23) is a generalization of the results obtained in [8, Section 2.2]. In fact, [8] performed a similar continuation with $b = 0$. They remarked that the introduction of delays can change drastically the behavior of the systems so that it is a thing we might expect when comparing the difference of

behavior between (13) and (24) from the study they perform for PDE’s systems.

V. DERIVATION OF A AW-RASCLE-ZHANG (ARZ) MODEL

From Section III, our continuation method does not lead to a ARZ model but to a some alike model. As curiosity, we infer in this section the microscopic model leading to the ARZ model through the continuation procedure developed in Section II. To do so, we start by examining a particular fundamental diagram. Let $\gamma > 0$, and consider the Greenshield model, which characterizes the relationship between traffic density and speed:

$$V(\rho) = v_m - v_m \left(\frac{\rho}{\rho_m} \right)^\gamma, \quad (25)$$

where v_m is the maximal speed and ρ_m the maximal density. In the Greenshield model, if the speed is zero then the traffic is jammed ($\rho = \rho_m$), and conversely, if the speed is v_m then the road is free of traffic. Let us consider the following microscopic model defined for $i = 1, \dots, N$

$$\begin{aligned} \ddot{x}_i = \left[\gamma \left(\frac{1}{x_{i+1} - x_i} \right)^\gamma - \dot{x}_i \right] (\dot{x}_{i+1} - \dot{x}_i) \\ + \frac{V \left(\frac{1}{x_{i+1} - x_i} \right) - \dot{x}_i}{\tau(x_{i+1} - x_i)}, \end{aligned} \quad (26)$$

where $\tau > 0$ is a relaxation term. The microscopic model described in (26) does not seem to appear in the literature. The second equation in the right-hand side of (26) has the same interpretation than for the OV-FTL model (12). What differs from (12) is the first equation in the right-hand side of (26) where the meaning is subtle and does not have a simple microscopic interpretation. Using the continuation method of Section II yields

$$\frac{\partial}{\partial t} \begin{bmatrix} \rho \\ v \end{bmatrix} + \begin{bmatrix} v & \rho \\ 0 & v - \gamma \rho^\gamma \end{bmatrix} \frac{\partial}{\partial x} \begin{bmatrix} \rho \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{\rho}{\tau} (V(\rho) - v) \end{bmatrix} \quad (27)$$

with the periodic boundary given in (8b). On its equilibrium point ρ^* and v^* , System (27) is known to be unstable for some parameters γ , v_m and ρ_m , see for instance [29], [28] for ARZ models with different boundary conditions or [13] for some numerical simulations. As with the analysis conducted for the OV-FTL model in Section III, we may expect that the instability conditions for model (27) have a non-empty intersection with the instability conditions of System (26).

VI. CONCLUSION AND PERSPECTIVES

In this paper, we developed a continuation method that is consistent with [22] for second-order microscopic driver models on a ring. It leads to 2×2 PDEs systems depending on the density and the speed of the vehicles. We applied the result to the modeling of macroscopic models from classical microscopic driver models. The study has opened up several non-exhaustive questions:

- Initially, we presented a preliminary analysis in Section III on the OV-FTL model and its PDE derived through the continuation method; however, this analysis remains inconclusive in its current form. A definitive outcome would involve identifying specific conditions on the control gains that prevent the emergence of stop-and-go waves, thereby offering an initial tuning approach to avoid them. Alternatively, a second approach could involve a redesign of the feedback functions to ensure consistent damping behavior, effectively mitigating stop-and-go waves across all potential gain values.
- The second result infers partial differential equations (PDEs) from microscopic IMD and delayed OV-FTL models; however, we have yet to draw any substantial conclusions from these derivations. Despite the challenges posed by the structure of these PDEs—such as delays or nonlinearities in the spatial derivatives—could numerical methods help us identify stability conditions or parameter domains that yield stability?
- Finally, we presented the microscopic driver model, enabling our continuation method to derive the ARZ model in Section V. In conclusion, we hope this procedure will contribute to a clearer understanding of complex microscopic driver models by leveraging well-known properties on PDEs systems. As for the two points mentioned above, the analysis remains challenging, and numerical methods may prove useful in identifying stability regions for the parameters of the microscopic model.

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