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ST and TS as Product and Sum

Quentin Blomet^{1,2}  · Paul Égré^{1,3}

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Abstract

The set of ST-valid inferences is neither the intersection, nor the union of the sets of K_3 -valid and LP-valid inferences, but despite the proximity to both systems, an extensional characterization of ST in terms of a natural set-theoretic operation on the sets of K_3 -valid and LP-valid inferences is still wanting. In this paper, we show that it is their *relational product*. Similarly, we prove that the set of TS-valid inferences can be identified using a dual notion, namely as the *relational sum* of the sets of LP-valid and K_3 -valid inferences. We discuss links between these results and the interpolation property of classical logic. We also use those results to revisit the duality between ST and TS. We present a notion of duality on which ST and TS are dual in exactly the same sense in which LP and K_3 are dual to each other.

Keywords Strict-Tolerant logics · Non-transitive logic · Non-reflexive logic · Substructural logics · Strong Kleene Logic · Logic of Paradox · Interpolation · Duality

1 Introduction

This paper deals with the substructural logics ST and TS, first introduced under those names by [8], and also known as p -consequence and q -consequence logics [15, 20]. These logics, which admit a canonical three-valued characterization [7, 9, 26], are closely related to the well-known systems LP and K_3 , namely Priest’s Logic of Paradox [2, 23] and Kleene’s Strong three-valued logic [17]. However, whereas LP-validity and K_3 -validity are defined in terms of the preservation of a designated set of truth-values (strict truth for K_3 , tolerant truth for LP), ST and TS are systems of “mixed consequence”,

✉ Quentin Blomet
quentin.blomet@ens.psl.eu

✉ Paul Égré
paul.egre@ens.psl.eu

¹ Institut Jean-Nicod (CNRS, ENS-PSL, EHESS), PSL University, 29 rue d’Ulm, Paris 75005, France

² Department of Philosophy, University of Greifswald, Baderstraße 6, Greifswald 17489, Germany

³ IRL CROSSING, CNRS, Adelaide, Australia

in which validity means that the strict truth of the premises entails the tolerant truth of the conclusions, and conversely for TS.

The question this paper answers is the following: is there a natural characterization of ST-validity and TS-validity, directly in terms of the valid inferences of LP and the valid inferences of K_3 ? Some results exist that relate the *metainferences* of ST to the inferences of LP [5, 14], or indeed the metainferences of TS to the inferences of K_3 [10]. Other results show that classical logic is the least extension of Dunn-Belnap's four-valued logic (FDE) that contains both K_3 and LP [1, 25], and so is ST in that regard, since it coincides with classical logic when the language is classical [8]. However, in this paper we are interested in whether ST-valid and TS-valid inferences can be defined in terms of some more direct set-theoretic operations on the sets of valid inferences of K_3 and LP.

It is known that the set of ST-valid inferences is neither the intersection, nor the union of the sets of K_3 -valid and LP-valid inferences (viz. [6]), but which operation relates them exactly remained to be determined. The first result we establish in this paper — originally a conjecture formulated in 2011 by the second author, but long left sitting in a drawer — is that ST is the *relational product* of K_3 and LP (in the sense of [19]): this means that Γ ST-entails Δ if and only if there exists a formula ϕ such that Γ K_3 -entails ϕ , and ϕ LP-entails Δ . Similarly, we prove that the set of TS-valid inferences can be identified using a dual notion, namely as the *relational sum* of the sets of LP-valid and K_3 -valid inferences: this means that Γ TS-entails Δ if and only if every formula ϕ is such that either Γ LP-entails ϕ , or ϕ K_3 -entails Δ . These results give us a more precise articulation of the intuition that ST and TS are hybrids of K_3 and LP (viz. [10, 32]).

To prove our results, we start by reviewing the definitions of validity in K_3 , LP, ST and TS in Section 2. In Section 3, we briefly review why intersection and union fail to adequately characterize ST and TS, and we introduce the operations of relational product and relational sum that underlie our main result. Sections 4 and 5 give the proofs of the main claims. Section 6 shows that order matters in these identities, that is ST and TS differ from the logics obtained when swapping K_3 and LP as the operands of sum and product. Section 7 defines a notion of duality between logics that relates ST to TS in exactly the sense in which LP relates to K_3 , this notion is then used to characterize extensionally ST and TS in terms of either LP or K_3 . In Section 8, finally, we close with some comparisons: first we examine the way in which the relational characterization of ST can be related to a refinement of the interpolation theorem proved in [21] for classical logic in a trivalent setting. Then we compare our characterization of the notion of duality of logic with those found in [11] and [12], and show that our notion gives us an invariant characterization that was missing. Finally, we conclude with some open problems in order to indicate the fruitfulness of our approach.

2 LP, K_3 , ST and TS

This section provides some technical preliminaries needed for our results. We define the basic propositional logics of interest in this paper, namely K_3 , LP, TS, and ST, and the accompanying concepts of satisfiability and validity.

Definition 2.1 (Language) The propositional language \mathcal{L} is built from a denumerably infinite set $Var = \{p, p', \dots\}$ of propositional variables, using the logical constants *verum* (\top), *falsum* (\perp), *lambda* (λ), negation (\neg), disjunction (\vee) and conjunction (\wedge).

Elements of \mathcal{L} will be denoted by lower case Greek letters. Subsets of \mathcal{L} will be denoted by upper case Greek letters.

Definition 2.2 (Inference) An inference on \mathcal{L} is an ordered pair $\langle \Gamma, \Delta \rangle$ (hereafter symbolized by $\Gamma \Rightarrow \Delta$) where $\Gamma, \Delta \subseteq \mathcal{L}$ are finite sets. We denote by $INF(\mathcal{L})$ the set of all inferences on \mathcal{L} .

Definition 2.3 (SK-valuation) A Strong-Kleene valuation v (or SK-valuation for short) is a function from propositional variables to truth-values:

$$v : Var \longrightarrow \{0, 1/2, 1\}$$

that is extended to the whole language by letting

$$\begin{aligned} v^*(p) &= v(p), \text{ if } p \in Var, \\ v^*(\top) &= 1, \\ v^*(\perp) &= 0, \\ v^*(\lambda) &= 1/2, \\ v^*(\neg\phi) &= 1 - v^*(\phi), \\ v^*(\phi \vee \psi) &= \max(v^*(\phi), v^*(\psi)), \\ v^*(\phi \wedge \psi) &= \min(v^*(\phi), v^*(\psi)). \end{aligned}$$

Strong Kleene valuations permit the definition of two standards of truth for a propositional formula, namely strict truth and tolerant truth [9]: a formula ϕ is said to be *strictly true* if $v(\phi) \in \{1\}$, and *tolerantly true* if $v(\phi) \in \{1/2, 1\}$. By extension, we will say that a formula is strictly satisfiable (tolerantly satisfiable), if and only if there is a valuation that makes it strictly true (tolerantly true).

We get two well-known infra-classical logics from those definitions, depending on whether logical consequence is defined as the preservation of strict truth or of tolerant truth, corresponding to Strong Kleene logic K_3 [17], and to the Logic of Paradox LP [2, 23], respectively. Each preserves a different set of designated values from the premises to the conclusions. While the logical consequence of K_3 preserves the value in $S = \{1\}$, the one of LP preserves the values in $T = \{1/2, 1\}$.

Definition 2.4 (LP-satisfaction, LP-validity) An SK-valuation v satisfies an inference $\Gamma \Rightarrow \Delta$ in LP (symbolized by $v \models_{LP} \Gamma \Rightarrow \Delta$) if and only if $v(\gamma) \neq 0$ for all $\gamma \in \Gamma$ only if $v(\delta) \neq 0$ for some $\delta \in \Delta$. An inference $\Gamma \Rightarrow \Delta$ is valid in LP (symbolized by $\models_{LP} \Gamma \Rightarrow \Delta$) if and only if $v \models_{LP} \Gamma \Rightarrow \Delta$ for all SK-valuations v .

Definition 2.5 (K_3 -satisfaction, K_3 -validity) An SK-valuation v satisfies an inference $\Gamma \Rightarrow \Delta$ in K_3 (symbolized by $v \models_{K_3} \Gamma \Rightarrow \Delta$) if and only if $v(\gamma) = 1$ for all $\gamma \in \Gamma$

only if $v(\delta) = 1$ for some $\delta \in \Delta$. An inference $\Gamma \Rightarrow \Delta$ is valid in K_3 (symbolized by $\models_{K_3} \Gamma \Rightarrow \Delta$) if and only if $v \models_{K_3} \Gamma \Rightarrow \Delta$ for all SK-valuations v .

It can easily be checked that K_3 is paracomplete in the sense that $\not\models_{K_3} \emptyset \Rightarrow \phi \vee \neg\phi$, and that LP is paraconsistent in the sense that $\not\models_{LP} \phi \wedge \neg\phi \Rightarrow \emptyset$.

The two definitions of validity for K_3 and LP rely on the standard definition of logical consequence as necessary preservation of the same set of designated values from premises to conclusions. Following [31], [8, 9] consider a more liberal definition of validity, also known as mixed. Rather than moving from premises to conclusions from one set of designated values to the same set, they consider the idea of moving from one set of designated values to another set, different from the first. This idea gives rise to two other logics: ST and TS. As indicated by the names, ST validity involves moving from strict to tolerant truth, while TS validity involves moving from tolerant truth to strict truth.¹

Definition 2.6 (ST-satisfaction, ST-validity) An SK-valuation v satisfies an inference $\Gamma \Rightarrow \Delta$ in ST (symbolized by $v \models_{ST} \Gamma \Rightarrow \Delta$) if and only if $v(\gamma) = 1$ for all $\gamma \in \Gamma$ only if $v(\delta) \neq 0$ for some $\delta \in \Delta$. An inference $\Gamma \Rightarrow \Delta$ is valid in ST (symbolized by $\models_{ST} \Gamma \Rightarrow \Delta$) if and only if $v \models_{ST} \Gamma \Rightarrow \Delta$ for all SK-valuations v .

Definition 2.7 (TS-satisfaction, TS-validity) An SK-valuation v satisfies an inference $\Gamma \Rightarrow \Delta$ in TS (symbolized by $v \models_{TS} \Gamma \Rightarrow \Delta$) if and only if $v(\gamma) \neq 0$ for all $\gamma \in \Gamma$ only if $v(\delta) = 1$ for some $\delta \in \Delta$. An inference $\Gamma \Rightarrow \Delta$ is valid in TS (symbolized by $\models_{TS} \Gamma \Rightarrow \Delta$) if and only if $v \models_{TS} \Gamma \Rightarrow \Delta$ for all SK-valuations v .

It follows from the definition of ST-validity that the set of valid inferences of the λ -free fragment of ST coincides exactly with the set of inferences valid in classical propositional logic [8]: an inference is valid in the λ -free fragment of ST if and only if it is classically valid. The λ -including fragment of ST is non-classical, however, in particular it fails structural transitivity, because although $\top \Rightarrow \lambda$ and $\lambda \Rightarrow \perp$ are both ST-valid, $\top \Rightarrow \perp$ is not. TS on the other hand fails structural reflexivity, already on the λ -free fragment: $p \Rightarrow p$ is not valid. Moreover, TS is empty of validities in the fragment not involving \top and \perp , and its validities in the language including both constants are very limited (more on this below).

Hereinafter, we will distinguish between a logic L , its set of valid inferences \mathbb{L}^+ , and its set of antivalid inferences \mathbb{L}^- . The reason for this distinction is twofold. Firstly, as exemplified by the difference between classical logic and ST regarding transitivity, two logics can share the same set of valid inferences but still differ at the metainferential level.² Secondly, as shown by [27], two logics can have the same valid inferences and

¹ Both notions of validity made earlier appearances, for TS in [20] (under the name of q -consequence, and restricted to single conclusions), and for ST in [16] (on the cut-free sequent calculus for LK) and in [15] (under the name p -consequence, also restricted to single conclusions). Even earlier incarnations of ST-consequence can be found in Strawson and in Belnap (see [3] for a historical overview).

² At least on the local definition of metainferential validity, though not on the global one. For the distinction between these two notions, which lies beyond the scope of this paper, see [14] and [4].

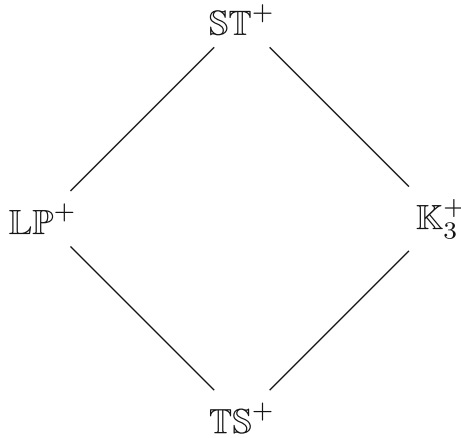


Fig. 1 Relations between TS^+ , K_3^+ , LP^+ and ST^+

metainferences while having different antivalidities.³ Hence, we will reserve the notation L — for what we call a logic — to refer to what fixes the satisfaction (respectively antisatisfaction) and validity (respectively antivalidity) conditions of an argument over a language, and we will use the notations L^+ and L^- to denote a set of ordered pairs of sentences of the language, determined by either the validity or the antivalidity conditions of L , respectively. Informally, however, we will allow ourselves to say that the logic ST is the product of K_3 and LP , to mean the more accurate claim that ST^+ is the product of K_3^+ and LP^+ .

We won't need to introduce the notion of antivalidity until Section 7. First, however, we will need the notion of a set of valid inferences:

Definition 2.8 (Set of valid inferences)

$$L^+ = \{\Gamma \Rightarrow \Delta : \models_L \Gamma \Rightarrow \Delta\}.$$

Inclusion relations between the set of validities of the above four logics are well-known and depicted in Figure 1 [6, 8]: TS^+ is properly included in LP^+ and in K_3^+ , and both of those sets are incomparable with each other but are included in ST^+ . What is yet to be found concerns the way in which ST^+ and TS^+ can be characterized in terms of K_3^+ and LP^+ .

3 Relational sum and product

In order to introduce the operations that allow us to define ST^+ and TS^+ in terms of K_3^+ and LP^+ , we start by first reviewing why neither of those systems coincides with

³ Several notions of antivalidities have been put forward in the literature on ST . [27] and [12] say that an inference $\Gamma \Rightarrow \Delta$ is antivalid when for all v , $v \not\models \Gamma \Rightarrow \Delta$, while [11] say that an inference $\Gamma \Rightarrow \Delta$ is antivalid when for all v , if $v(\gamma)$ is not designated for all $\gamma \in \Gamma$, then $v(\delta)$ is not designated for some $\delta \in \Delta$. It can easily be checked that given any of these notions, classical logic and ST fail to have the same antivalidities. In Section 7, we will make use of the second notion.

the intersection or the union of both logics. While it is well-known that ST^+ is not the intersection of K_3^+ and LP^+ , it is tempting to think that ST^+ might be the union of K_3^+ and LP^+ , but in fact it is not (see the references given in [30] regarding the misidentification of classical logic with $K_3^+ \cup LP^+$). It is useful, therefore, to review counterexamples to both identities.

Fact 3.1

$$TS^+ \neq K_3^+ \cap LP^+ \neq ST^+.$$

$$TS^+ \neq K_3^+ \cup LP^+ \neq ST^+.$$

Proof For the intersection: $p \Rightarrow p$ is not in TS^+ , but it is both in LP^+ and in K_3^+ . Conversely, $\emptyset \Rightarrow \neg p \vee p$ is in ST^+ , but is not in K_3^+ , so it is not in $K_3^+ \cap LP^+$ either. Hence, $K_3^+ \cap LP^+$ is equal neither to TS^+ nor to ST^+ .

Regarding the union: since both K_3^+ and LP^+ are reflexive, it follows that TS^+ , which is not reflexive, cannot be the union. For ST^+ , the following inference — first brought to our notice in 2011 by David Ripley, and featuring in e.g. [30, p. 511] —, gives a counterexample:

$$p \vee (q \wedge \neg q) \Rightarrow p \wedge (q \vee \neg q).$$

This inference is valid in ST , but it is neither valid in K_3 nor valid in LP . Let v and v' be two SK-valuations such that $v(p) = 1, v'(p) = 0$ and $v(q) = v'(q) = 1/2$. Then $v(p \vee (q \wedge \neg q)) = 1, v(p \wedge (q \vee \neg q)) = 1/2, v'(p \vee (q \wedge \neg q)) = 1/2$ and $v'(p \wedge (q \vee \neg q)) = 0$. So, $v \not\models_{K_3} p \vee (q \wedge \neg q) \Rightarrow p \wedge (q \vee \neg q)$ and $v' \not\models_{LP} p \vee (q \wedge \neg q) \Rightarrow p \wedge (q \vee \neg q)$. \square

One may nonetheless observe that $\models_{K_3} p \vee (q \wedge \neg q) \Rightarrow p$ and $\models_{LP} p \Rightarrow p \wedge (q \vee \neg q)$. This suggests that p acts as a connecting formula. More generally, therefore, let the relational composition of K_3^+ and LP^+ be defined as follows:

Definition 3.1 (Relational composition of K_3^+ and LP^+)

$$K_3^+ \circ LP^+ = \{ \langle \Gamma, \Delta \rangle : \langle \Gamma, \Theta \rangle \in K_3^+ \text{ and } \langle \Theta, \Delta \rangle \in LP^+ \text{ for some } \Theta \}.$$

Relational composition is strictly more inclusive than the union:

Fact 3.2 $K_3^+ \cup LP^+ \subsetneq K_3^+ \circ LP^+$.

Proof Assume first that $\Gamma, \Delta \neq \emptyset$. If $\Gamma \Rightarrow \Delta \in K_3^+$, given that $\Delta \Rightarrow \Delta \in LP^+$, then $\Gamma \Rightarrow \Delta \in K_3^+ \circ LP^+$. If $\Gamma \Rightarrow \Delta \in LP^+$, given that $\Gamma \Rightarrow \Gamma \in K_3^+$, then $\Gamma \Rightarrow \Delta \in K_3^+ \circ LP^+$. Assume now that $\Gamma = \emptyset$. If $\emptyset \Rightarrow \Delta \in K_3^+$, then $\Delta \neq \emptyset$, so $\Delta \Rightarrow \Delta \in LP^+$, and hence $\emptyset \Rightarrow \Delta \in K_3^+ \circ LP^+$. Now, if $\emptyset \Rightarrow \Delta \in LP^+$, $\top \Rightarrow \Delta \in LP^+$, and given that $\emptyset \Rightarrow \top \in K_3^+$, we obtain $\emptyset \Rightarrow \Delta \in K_3^+ \circ LP^+$. Finally, assume $\Delta = \emptyset$. If $\Gamma \Rightarrow \emptyset \in K_3^+$, $\Gamma \Rightarrow \perp \in K_3^+$, and given that $\perp \Rightarrow \emptyset \in LP^+$, we obtain $\Gamma \Rightarrow \emptyset \in K_3^+ \circ LP^+$. If $\Gamma \Rightarrow \emptyset \in LP^+$, then $\Gamma \neq \emptyset$, so $\Gamma \Rightarrow \Gamma \in K_3^+$, and hence $\Gamma \Rightarrow \emptyset \in K_3^+ \circ LP^+$.

To prove proper inclusion, note that the previous inference $p \vee (q \wedge \neg q) \Rightarrow p \wedge (q \vee \neg q)$ belongs to $K_3^+ \circ LP^+$. \square

However, as defined, relational composition is still inadequate to exactly match ST^+ , an observation originally due to Pablo Cobrerros:

Fact 3.3 $ST^+ \neq \mathbb{K}_3^+ \circ LP^+$.

Proof $\models_{\mathbb{K}_3} p \vee \neg p \Rightarrow p, \neg p$ and $\models_{LP} p, \neg p \Rightarrow p \wedge \neg p$. Hence, $p \vee \neg p \Rightarrow p \wedge \neg p \in \mathbb{K}_3^+ \circ LP^+$ but $p \vee \neg p \Rightarrow p \wedge \neg p \notin ST^+$.

The problematic case arises because composition is defined over sets of formulae, and on our definitions of validity, the meaning of a comma on the left-hand side of the sequent is different from its meaning on its right-hand side. While on the left-hand side it must be read as a conjunction, on the right-hand side it must be read as a disjunction. In order to rule out this problematic case, we need to restrict the middle-term to a singleton set, that is to a single formula.

To that end, we introduce the following enhanced definition of *relative product* and *relative sum* (see [19] for an exposition, and [22] for the original definitions):

Definition 3.2 (Relative product) Let R and S be two arbitrary binary relations on a set U and $V \subseteq U$, then

$$R \underset{V}{|} S = \{ \langle x, z \rangle : \langle x, y \rangle \in R \text{ and } \langle y, z \rangle \in S \text{ for some } y \in V \}.$$

Definition 3.3 (Relative sum) Let R and S be two arbitrary binary relations on a set U and $V \subseteq U$, then

$$R \underset{V}{\dagger} S = \{ \langle x, z \rangle : \langle x, y \rangle \in R \text{ or } \langle y, z \rangle \in S \text{ for all } y \in V \}.$$

The duality between these two notions is expressed by the following identities:

Fact 3.4 (*Duality of product and sum*)

$$\begin{aligned} R \underset{V}{|} S &= \overline{\overline{R \underset{V}{\dagger} S}} \\ R \underset{V}{\dagger} S &= \overline{\overline{R \underset{V}{|} S}} \end{aligned}$$

Proof

$\langle x, z \rangle \in R \underset{V}{ } S$ iff there is $y \in V$ s.t. $\langle x, y \rangle \in R$ and $\langle y, z \rangle \in S$ iff there is $y \in V$ s.t. $\langle x, y \rangle \notin \overline{R}$ and $\langle y, z \rangle \notin \overline{S}$ iff not for all $y \in V, \langle x, y \rangle \in \overline{R}$ or $\langle y, z \rangle \in \overline{S}$ iff $\langle x, z \rangle \notin \overline{\overline{R \underset{V}{\dagger} S}}$ iff $\langle x, z \rangle \in \overline{\overline{R \underset{V}{\dagger} S}}$	$\langle x, z \rangle \in R \underset{V}{\dagger} S$ iff for all $y \in V, \langle x, y \rangle \in R$ or $\langle y, z \rangle \in S$ iff for all $y \in V, \langle x, y \rangle \notin \overline{R}$ or $\langle y, z \rangle \notin \overline{S}$ iff for no $y \in V, \langle x, y \rangle \in \overline{R}$ and $\langle y, z \rangle \in \overline{S}$ iff $\langle x, z \rangle \notin \overline{\overline{R \underset{V}{ } S}}$ iff $\langle x, z \rangle \in \overline{\overline{R \underset{V}{ } S}}$
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□

With these definitions at our disposal, we can provide an exact characterization of ST^+ and TS^+ as product and sum. We will use $\mathcal{P}(\mathcal{L})$ as U and $\{\{\phi\} : \phi \in \mathcal{L}\}$ as V (omitting from now on its mention under the symbols for relative multiplication and relative sum). The main theorems we shall prove are:

Theorem 4.1 $ST^+ = \mathbb{K}_3^+ \mid LP^+$.

Theorem 5.1 $TS^+ = LP^+ \dagger \mathbb{K}_3^+$.

We proceed to prove Theorem 4.1 in Section 4 and Theorem 5.1 in Section 5. Before doing so, however, we pause with an important caveat, regarding the fact that these results concern the connection between the validities of \mathbb{K}_3 and LP each understood *globally*. In the case of ST, for example, Theorem 4.1 implies that if *every* valuation ST-satisfies an argument $\Gamma \Rightarrow \Delta$, then *there is* a formula ϕ such that *every* valuation \mathbb{K}_3 -satisfies $\Gamma \Rightarrow \phi$ and *every* valuation LP-satisfies $\phi \Rightarrow \Delta$. This is stronger than saying that if every valuation ST-satisfies an argument $\Gamma \Rightarrow \Delta$, then for *every* valuation v , *there is* a formula ϕ such that v \mathbb{K}_3 -satisfies $\Gamma \Rightarrow \phi$ and v LP-satisfies $\phi \Rightarrow \Delta$. The latter formulation is weaker than the former, in that it permits the connecting formula to vary depending on the valuation considered. Instead, our approach constructs a uniform connecting formula, one that, given the ST-validity of an argument, is meant to hold across all valuations.

4 $ST^+ = \mathbb{K}_3^+ \mid LP^+$

In order to prove the equality expressed by Theorem 4.1, we proceed in two steps, proving first the left inclusion and then the right inclusion. For the left inclusion, we show how to construct a connecting formula ϕ such that if $\models_{ST} \Gamma \Rightarrow \Delta$, then $\models_{\mathbb{K}_3} \Gamma \Rightarrow \phi$ and $\models_{LP} \phi \Rightarrow \Delta$. For the right inclusion, we will see that under the assumption that $\models_{\mathbb{K}_3} \Gamma \Rightarrow \phi$ and $\models_{LP} \phi \Rightarrow \Delta$ for some $\phi \in \mathcal{L}$, the ST-validity of $\Gamma \Rightarrow \Delta$ follows almost immediately from the definition of \mathbb{K}_3 -validity and LP-validity.

We start by defining the function At that allows us to isolate the atoms and the constants of a formula:

Definition 4.1 (Set of atoms of a formula) The function $At : \mathcal{L} \longrightarrow \mathcal{P}(Var \cup \{\top, \perp, \lambda\})$ is a recursively defined function such that:

$$\begin{aligned} At(p) &= \{p\}, \text{ if } p \in Var, \\ At(\top) &= \{\top\}, \\ At(\perp) &= \{\perp\}, \\ At(\lambda) &= \{\lambda\}, \\ At(\neg\phi) &= At(\phi), \\ At(\phi \vee \psi) &= At(\phi \wedge \psi) = At(\phi) \cup At(\psi). \end{aligned}$$

If $\Gamma \subseteq \mathcal{L}$, $At(\Gamma)$ abbreviates $\bigcup\{At(\gamma) : \gamma \in \Gamma\}$.

We then introduce the notion of *partial sharpening* adapted from the notion of *sharpening*, found in [27]. Intuitively, given an SK-valuation v , an SK-valuation v^*

qualifies as a sharpening of v if and only if it agrees with v on all the propositional variables that take a classical value, and possibly disagrees with v on the others. The notion of partial sharpening of a valuation with respect to a set of atoms extends the definition of a sharpening to any subset of the set of atoms of the language.

Definition 4.2 (Partial sharpening) Let v be an SK-valuation. An SK-valuation v^* is said to be a partial sharpening of v with respect to a set of atoms $\Sigma \subseteq \text{Var} \cup \{\top, \perp, \lambda\}$ if and only if for all $\alpha \in \Sigma$, if $v(\alpha) \neq 1/2$, then $v^*(\alpha) = v(\alpha)$.

The next two corollaries describe useful properties of the notion of partial sharpening.

Corollary 4.1 *If v^* is a partial sharpening of v with respect to Σ , then v^* is a partial sharpening of v with respect to any $\Theta \subseteq \Sigma$.*

Proof Note that if the condition holds for all $\alpha_1 \in \Sigma$, it also holds for all $\alpha_2 \in \Theta \subseteq \Sigma$. \square

Corollary 4.2 *If v^* is a partial sharpening of v with respect to Σ and Θ , then v^* is a partial sharpening of v with respect to $\Sigma \cup \Theta$.*

Proof Note that if the condition holds for all $\alpha_1 \in \Sigma$ and for all $\alpha_2 \in \Theta$, it also holds for all $\alpha \in \Sigma \cup \Theta$. \square

We then turn to the first lemma of importance, which illustrates the monotonicity of the SK-scheme (see Da Ré et al. [13]).⁴ Given a formula ϕ , if ϕ is assigned a classical value by a valuation v , any valuation v^* that agrees with v on the classical values assigned to the atoms of ϕ will also agree with v on the value of ϕ .

Lemma 4.1 *For all SK-valuations v , if $v(\phi) \neq 1/2$, then for all partial sharpenings v^* of v with respect to $\text{At}(\phi)$, it holds that $v^*(\phi) = v(\phi)$.*

Proof The proof is by simple induction on the complexity of ϕ . To prove the inductive step we rely on Corollary 4.1 to show that if v^* is a partial sharpening of v with respect to $\text{At}(\psi \vee \chi)$ or $\text{At}(\psi \wedge \chi)$, it is also a partial sharpening of v with respect to $\text{At}(\psi)$ and $\text{At}(\chi)$, allowing thus the use of the inductive hypothesis. \square

For any ST-valid inference $\Gamma \Rightarrow \Delta$ such that all the $\gamma \in \Gamma$ jointly take the value 1 for some valuation v , we show how to construct a term ϕ such that $\Gamma \Rightarrow \phi$ is K_3 -valid and $\phi \Rightarrow \Delta$ is LP-valid. This term ϕ will be the K_3 disjunctive normal form of $\bigwedge \Gamma$. The next two definitions show how to construct such a term.

Definition 4.3 (Γ, v -conjunction) Let $\Gamma \subseteq \mathcal{L}$ be a finite nonempty set of formulae. If v is such that $v(\gamma) = 1$ for every $\gamma \in \Gamma$, let $C_{\Gamma, v} := \bigwedge_{v(\alpha) \neq 1/2} \alpha^{\sim}$, with $\alpha \in \text{At}(\Gamma)$ and $\alpha^{\sim} = \alpha$ if $v(\alpha) = 1$ and $\alpha^{\sim} = \neg\alpha$ if $v(\alpha) = 0$.

⁴ Given the order \leq_I defined on the set $\{0, 1/2, 1\}$ so that $1/2 <_I 0$ and $1/2 <_I 1$, a scheme is said to be *monotonic* if each of its n -ary operations is order-preserving with respect to the Cartesian order \leq_I^n defined by $\langle x_1, \dots, x_n \rangle \leq_I^n \langle y_1, \dots, y_n \rangle \iff (i = 1, \dots, n) x_i \leq_I y_i$.

In other words, if there is a valuation v that strictly satisfies each $\gamma \in \Gamma$, we construct the conjunction of all the literals $\alpha \sim$ — built from the atoms of Γ — that take value 1 according to v . That $C_{\Gamma,v}$ is well-defined when v is such that $v(\gamma) = 1$ for every $\gamma \in \Gamma$ follows from the fact that necessarily some atom α of the formulae in Γ is such that $v(\alpha) = 1$ or $v(\alpha) = 0$, for otherwise we would have $v(\gamma) = 1/2$ for every $\gamma \in \Gamma$.

Definition 4.4 Given a finite nonempty set of formulae Γ , let $D_\Gamma := \bigvee\{C_{\Gamma,v} : v(\bigwedge \Gamma) = 1\}$ when there is v such that $v(\bigwedge \Gamma) = 1$, and let $D_\Gamma := \perp$ otherwise.

The first clause of the definition tells us how to construct the K_3 disjunctive normal form of $\bigwedge \Gamma$ from each $C_{\Gamma,v}$, while the last clause enables us to deal with the case where no such $C_{\Gamma,v}$ exists, namely when there is no v such that $v(\bigwedge \Gamma) = 1$.

If the clause of Definition 4.3 is satisfied, $C_{\Gamma,v}$ necessarily takes the value 1. This fact is illustrated by the following lemma:

Lemma 4.2 *Let Γ be a nonempty set of formulae. For all v , if $v(\bigwedge \Gamma) = 1$, then $v(C_{\Gamma,v}) = 1$.*

Proof If $v(\bigwedge \Gamma) = 1$, then v is such that $v(\gamma) = 1$ for every $\gamma \in \Gamma$. So $C_{\Gamma,v}$ is defined. By construction, and by the SK evaluation of a conjunction, $v(C_{\Gamma,v}) = 1$. \square

The next lemma shows that for any nonempty set of formulae Γ , D_Γ is K_3 -equivalent to $\bigwedge \Gamma$.

Lemma 4.3 *Let Γ be a nonempty set of formulae. Then*

$$\models_{K_3} \Gamma \Rightarrow D_\Gamma \text{ and } \models_{K_3} D_\Gamma \Rightarrow \bigwedge \Gamma.$$

Proof The case for which there is no valuation making the formulae in Γ strictly true together is obvious, since then $D_\Gamma = \perp$.

So consider the case in which the formulae in Γ can be made strictly true jointly. If $v(\bigwedge \Gamma) = 1$ then $v(C_{\Gamma,v}) = 1$ by the previous lemma, and $v(D_\Gamma) = 1$. So $\models_{K_3} \Gamma \Rightarrow D_\Gamma$.

If there is v' such that $v'(D_\Gamma) = 1$, then $v'(\bigvee\{C_{\Gamma,v} : v(\bigwedge \Gamma) = 1\}) = 1$. So for some v such that $v(\bigwedge \Gamma) = 1$, we have $v'(C_{\Gamma,v}) = 1$. Then v' must agree on the atoms of Γ to which v gives a classical value, and possibly differs on the atoms of Γ to which v gives the value $1/2$. Thus, v' is a partial sharpening of v with respect to $\text{At}(\bigwedge \Gamma)$, and therefore since $v(\bigwedge \Gamma) = 1$, we also have $v'(\bigwedge \Gamma) = 1$ by Lemma 4.1. \square

We are now in a position to prove the main theorem of the section. Namely that $\models_{ST} \Gamma \Rightarrow \Delta$ if and only if there is $\phi \in \mathcal{L}$ such that $\models_{K_3} \Gamma \Rightarrow \phi$ and $\models_{LP} \phi \Rightarrow \Delta$.

Theorem 4.1 $ST^+ = K_3^+ \mid LP^+$.

Proof (\subseteq) Assume that $\Gamma \Rightarrow \Delta \in \mathbb{ST}^+$, then we prove that there is a connecting formula ϕ such that $\Gamma \Rightarrow \phi \in \mathbb{K}_3^+$ and $\phi \Rightarrow \Delta \in \mathbb{LP}^+$.

When Γ is empty, we let $\phi := \top$. Clearly, $\models_{\mathbb{K}_3} \Gamma \Rightarrow \top$, and since $\models_{\mathbb{ST}} \emptyset \Rightarrow \Delta$ by assumption, for all v , $v(\delta) \neq 0$ for some $\delta \in \Delta$, and therefore $\models_{\mathbb{LP}} \top \Rightarrow \Delta$.

When Γ is nonempty, we let $\phi := D_\Gamma$. $\models_{\mathbb{K}_3} \Gamma \Rightarrow D_\Gamma$ follows from Lemma 4.3. So let us prove $\models_{\mathbb{LP}} D_\Gamma \Rightarrow \Delta$.

Assume $\not\models_{\mathbb{LP}} D_\Gamma \Rightarrow \Delta$ for the sake of contradiction. It follows that there is v such that $v(D_\Gamma) \neq 0$ and $v(\delta) = 0$ for all $\delta \in \Delta$. If $v(D_\Gamma) = 1/2$, there is by Definition 4.4 v' such that $v(C_{\Gamma,v'}) = 1/2$. So, for all conjuncts $\alpha \sim$ of $C_{\Gamma,v'}$, $v(\alpha \sim) \neq 0$. Consider v^* which is exactly like v except that for all the conjuncts $\alpha \sim$ of $C_{\Gamma,v'}$, if $v(\alpha \sim) = 1/2$, $v^*(\alpha \sim) = 1$. Then $v^*(C_{\Gamma,v'}) = 1$, so $v^*(D_\Gamma) = 1$. Moreover, v^* is a partial sharpening of v with respect to $\text{At}(C_{\Gamma,v'})$ by construction, and is exactly like v for all $\beta \in \text{At}(\Delta) - \text{At}(C_{\Gamma,v'})$, so trivially it is also a partial sharpening of v with respect to $\text{At}(\Delta) - \text{At}(C_{\Gamma,v'})$. Therefore, by Corollary 4.2, v^* is a partial sharpening of v with respect to $\text{At}(C_{\Gamma,v'}) \cup (\text{At}(\Delta) - \text{At}(C_{\Gamma,v'}))$, and by Corollary 4.1 to $\text{At}(\Delta) \subseteq \text{At}(C_{\Gamma,v'}) \cup (\text{At}(\Delta) - \text{At}(C_{\Gamma,v'}))$. Now, given that $v(\delta) = 0$ for all $\delta \in \Delta$, by Lemma 4.1, $v^*(\delta) = 0$ for all $\delta \in \Delta$. But by Lemma 4.3, $\models_{\mathbb{K}_3} D_\Gamma \Rightarrow \bigwedge \Gamma$, so $v^*(\bigwedge \Gamma) = 1$ since $v^*(D_\Gamma) = 1$, and thus $v^* \not\models_{\mathbb{ST}} \Gamma \Rightarrow \Delta$, which is impossible. If $v(D_\Gamma) = 1$, then $v(\bigwedge \Gamma) = 1$ by the same lemma, and $v(\delta) = 0$ for all $\delta \in \Delta$ by assumption, which again contradicts $\models_{\mathbb{ST}} \Gamma \Rightarrow \Delta$.

(\supseteq) Assume that $\Gamma \Rightarrow \Delta \in \mathbb{K}_3^+ \mid \mathbb{LP}^+$. Then there is $\phi \in \mathcal{L}$ such that $\models_{\mathbb{K}_3} \Gamma \Rightarrow \phi$ and $\models_{\mathbb{LP}} \phi \Rightarrow \Delta$. So, for all v if $v(\gamma) = 1$ for all $\gamma \in \Gamma$, then $v(\phi) = 1$, and if $v(\phi) \neq 0$, then $v(\delta) \neq 0$ for some $\delta \in \Delta$. Hence, for all v if $v(\gamma) = 1$ for all $\gamma \in \Gamma$, then $v(\delta) \neq 0$ for some $\delta \in \Delta$, thus $\models_{\mathbb{ST}} \Gamma \Rightarrow \Delta$, and we have $\Gamma \Rightarrow \Delta \in \mathbb{ST}^+$, as desired. \square

To illustrate the theorem, let us go back to the case of the ST-valid argument $p \vee (q \wedge \neg q) \Rightarrow p \wedge (q \vee \neg q)$. The theorem implies that there is a formula D such that $p \vee (q \wedge \neg q) \Rightarrow D \in \mathbb{K}_3^+$ and $D \Rightarrow p \wedge (q \vee \neg q) \in \mathbb{LP}^+$. In this case, the method described produces $D = (p \wedge q) \vee (p \wedge \neg q) \vee p$, and one can check that $p \vee (q \wedge \neg q) \Rightarrow (p \wedge q) \vee (p \wedge \neg q) \vee p \in \mathbb{K}_3^+$ and $(p \wedge q) \vee (p \wedge \neg q) \vee p \Rightarrow p \wedge (q \vee \neg q) \in \mathbb{LP}^+$. This formula is more complex than the atomic formula p which we saw above would suffice, but we note that it too may be described as an interpolant in the sense of Craig’s interpolation theorem, since it only contains atoms common to Γ and Δ . We return to this connection to interpolation in Section 8.

5 $\mathbb{TS}^+ = \mathbb{LP}^+ \dagger \mathbb{K}_3^+$

Turning now to the proof of the second theorem, we start by proving a simple but useful lemma showing that for any formula ϕ , if ϕ is tolerantly satisfiable, then the SK-valuation v that maps each propositional variable to $1/2$ also makes ϕ tolerantly satisfiable, and if ϕ is strictly falsifiable, then the same v makes ϕ strictly falsifiable too.

Lemma 5.1 *Let v' be such that for all $p \in \text{Var}$, $v'(p) = 1/2$. Then for all $\phi \in \mathcal{L}$,*

- (i) *If there is v such that $v(\phi) \neq 0$, then $v'(\phi) \neq 0$.*

(ii) If there is v such that $v(\phi) \neq 1$, then $v'(\phi) \neq 1$.

Proof (i) and (ii) are proved simultaneously by induction on the complexity of ϕ . \square

The next lemma shows a sense in which TS-validity requires either that one of its premise be tolerantly unsatisfiable or that one of its conclusion be strictly unfalsifiable:

Lemma 5.2 *If $\models_{\text{TS}} \Gamma \Rightarrow \Delta$, then*

- (i) *For some $\gamma \in \Gamma$, it holds for all v that $v(\gamma) = 0$*
or
- (ii) *for some $\delta \in \Delta$, it holds for all v that $v(\delta) = 1$.*

Proof We prove the contrapositive. Assume that for all $\gamma \in \Gamma$ there is v such that $v(\gamma) \neq 0$ and for all $\delta \in \Delta$, there is v such that $v(\delta) \neq 1$. Let v' be such that $v'(p) = 1/2$ for all $p \in \text{Var}$. By Lemma 5.1, for all $\gamma \in \Gamma$, $v'(\gamma) \neq 0$ and for all $\delta \in \Delta$, $v'(\delta) \neq 1$. Hence, $\not\models_{\text{TS}} \Gamma \Rightarrow \Delta$. \square

We then turn to the proof of the main theorem:

Theorem 5.1 $\text{TS}^+ = \text{LP}^+ \dagger \mathbb{K}_3^+$.

Proof (\subseteq) Assume $\Gamma \Rightarrow \Delta \in \text{TS}^+$. Then $\models_{\text{TS}} \Gamma \Rightarrow \Delta$, and by Lemma 5.2, for some $\gamma \in \Gamma$, it holds for all v that $v(\gamma) = 0$ or for some $\delta \in \Delta$, it holds for all v that $v(\delta) = 1$. So, for all $\phi \in \mathcal{L}$, $\models_{\text{LP}} \Gamma \Rightarrow \phi$ or $\models_{\mathbb{K}_3} \phi \Rightarrow \Delta$, and therefore $\Gamma \Rightarrow \Delta \in \text{LP}^+ \dagger \mathbb{K}_3^+$.

(\supseteq) We prove the contrapositive. Assume $\Gamma \Rightarrow \Delta \notin \text{TS}^+$. Then $\not\models_{\text{TS}} \Gamma \Rightarrow \Delta$. So, there is a valuation v such that $v(\gamma) \neq 0$ for all $\gamma \in \Gamma$ and $v(\delta) \neq 1$ for all $\delta \in \Delta$. Let $p \in \text{Var} - (\text{At}(\Gamma) \cup \text{At}(\Delta))$. Since Var is an infinite set and both $\text{At}(\Gamma)$ and $\text{At}(\Delta)$ are finite sets, there is such a p . Let v' be exactly like v except that if $v(p) \neq 0$, $v'(p) = 0$. Then $v'(\gamma) \neq 0$ for all $\gamma \in \Gamma$ since p does not appear in Γ , and $v'(p) = 0$, so $v' \not\models_{\text{LP}} \Gamma \Rightarrow p$. Now, let v'' be exactly like v except that if $v(p) \neq 1$, $v''(p) = 1$. Then $v''(p) = 1$ and $v''(\delta) \neq 1$ for all $\delta \in \Delta$ since p does not appear in Δ , so $v'' \not\models_{\mathbb{K}_3} p \Rightarrow \Delta$. Hence, there is $\phi := p$ such that $\not\models_{\text{LP}} \Gamma \Rightarrow \phi$ and $\not\models_{\mathbb{K}_3} \phi \Rightarrow \Delta$, meaning that $\Gamma \Rightarrow \Delta \notin \text{LP}^+ \dagger \mathbb{K}_3^+$. \square

As we shall see in the next section, this characterization shows that TS can be conceptualized in terms of the theorems of \mathbb{K}_3 and the antitheorems of LP. The theorems of \mathbb{K}_3 indeed consist only of the trivial inferences involving \top in the consequent (e.g. $\Gamma \Rightarrow \psi \vee \top$) and the antitheorems of LP only of the trivial inferences involving \perp in the antecedent (e.g. $\phi \wedge \perp \Rightarrow \Delta$).

6 The case of $\text{LP}^+ \mid \mathbb{K}_3^+$ and $\mathbb{K}_3^+ \dagger \text{LP}^+$

How tight are the previous characterizations of ST^+ and TS^+ ? Could LP^+ and \mathbb{K}_3^+ be swapped in the statement of either theorem? We show that the answer is negative.

Let us consider TS^+ first. The relative sum of \mathbb{K}_3^+ and LP^+ is not equal to TS^+ , but to the set of antitheorems and theorems of ST. In order to prove this claim, we first

define the antitheorems and the theorems of a logic, and then show that the operation of relative sum extracts the antitheorems of the first operand and the theorems of the second. After reviewing a well-known fact about K_3 and LP, namely that the first has the same antitheorems as ST and the second the same theorems as ST, we prove our main claim.

Definition 6.1 (Theorem and antitheorem) Given a logic L , a set of formulae Γ is said to be a theorem of L when $\models_L \emptyset \Rightarrow \Gamma$, and an antitheorem of L when $\models_L \Gamma \Rightarrow \emptyset$.

Definition 6.2 (Trivial theorem and antitheorem) An antitheorem Γ is trivial if there is $\gamma \in \Gamma$ such that for all $v, v': v(\gamma) = v'(\gamma) \neq 1$. A theorem Δ is trivial if there is $\delta \in \Delta$ such that for all $v, v': v(\delta) = v'(\delta) \neq 0$.

We now focus on two well-known facts about antitheorems and theorems.⁵

Fact 6.1 Let L be a finitary logic with a consequence relation defined from the pair of sets of designated values $\langle D_1, D_2 \rangle$. The following are equivalent:

- (i) Γ is an antitheorem of L .
- (ii) $\models_L \Gamma \Rightarrow \Delta$ for all $\Delta \subseteq \mathcal{L}$.
- (iii) $\models_L \Gamma \Rightarrow \phi$ for all $\phi \in \mathcal{L}$.
- (iv) $\models_L \Gamma \Rightarrow p$ for some $p \notin \text{At}(\Gamma)$.

Proof That (ii) follows from (i), (iii) from (ii), and (iv) from (iii) is straightforward to prove, so we just show that (i) follows from (iv). Assume that $\not\models_L \Gamma \Rightarrow \emptyset$. Then there is v such that $v(\gamma) \in D_1$ for all $\gamma \in \Gamma$. Let $p \notin \text{At}(\Gamma)$, since L is finitary and Γ finite, such a p exists. Assume further that v' is exactly like v except that if $v(p) \in D_2$, $v'(p) \notin D_2$. Then $v'(\gamma) \in D_1$ for all $\gamma \in \Gamma$ since $p \notin \text{At}(\Gamma)$ and hence $\not\models_L \Gamma \Rightarrow p$ for some $p \notin \text{At}(\Gamma)$. \square

Fact 6.2 Let L be a finitary logic with a consequence relation defined from the pair of sets of designated values $\langle D_1, D_2 \rangle$. The following are equivalent:

- (i) Δ is a theorem of L .
- (ii) $\models_L \Gamma \Rightarrow \Delta$ for all $\Gamma \subseteq \mathcal{L}$.
- (iii) $\models_L \phi \Rightarrow \Delta$ for all $\phi \in \mathcal{L}$.
- (iv) $\models_L p \Rightarrow \Delta$ for some $p \notin \text{At}(\Delta)$.

Proof The proof is symmetric to the proof of Fact 6.1. \square

Given Fact 6.1, we identify the set of antitheorems of a logic with the set of all the valid inferences of the form $\Gamma \Rightarrow \Delta$ with Γ an antitheorem, and similarly for the set of theorems given Fact 6.2.

⁵ When L is an infinitary logic, (i) of Fact 6.1 does not necessarily entail (iv). To see this, note that for any $p \in \text{Var}$, $p \in \text{At}(q \wedge \neg q \wedge \bigwedge \text{Var})$, so (iv) is false when $\Gamma = \{q \wedge \neg q \wedge \bigwedge \text{Var}\}$, yet $q \wedge \neg q \wedge \bigwedge \text{Var}$ is an antitheorem of the classical infinitary logic \mathcal{L}_{ω_1} permitting conjunctions and disjunctions of length $< \omega_1$ and with $|\text{Var}| < \omega_1$. To remedy this, one can restate (iv) as $\models_L \sigma_p(\Gamma) \Rightarrow p$ with σ_p a substitution that renames the variables in such a way that p does not appear in the antecedent anymore (see e.g. [25]). Since $p \notin \sigma_p(q \wedge \neg q \wedge \bigwedge \text{Var})$ and $\sigma_p(q \wedge \neg q \wedge \bigwedge \text{Var}) \Rightarrow p$ is valid in \mathcal{L}_{ω_1} , the preceding case does not constitute a counterexample to the entailment of (iv) by (i) anymore.

Definition 6.3 (Sets of theorems and antitheorems) The sets of theorems and antitheorems of a logic L are defined respectively as follows:

$$\mathbb{L}^t := \{\Gamma \Rightarrow \Delta \in INF(\mathcal{L}) : \models_L \emptyset \Rightarrow \Delta\},$$

$$\mathbb{L}^a := \{\Gamma \Rightarrow \Delta \in INF(\mathcal{L}) : \models_L \Gamma \Rightarrow \emptyset\}.$$

The next fact shows that the operation of relative sum extracts the antitheorems of the first operand and the theorems of the second.

Fact 6.3 Given L_1 and L_2 two finitary logics with each a consequence relation defined from a pair of sets of designated values:

$$\mathbb{L}_1^+ \dagger \mathbb{L}_2^+ = \mathbb{L}_1^a \cup \mathbb{L}_2^t.$$

Proof (\subseteq) If $\Gamma \Rightarrow \Delta \in \mathbb{L}_1^+ \dagger \mathbb{L}_2^+$, then for all $\phi \in \mathcal{L}$, $\models_{L_1} \Gamma \Rightarrow \phi$ or $\models_{L_2} \phi \Rightarrow \Delta$. Since L_1, L_2 are finitary and Γ, Δ are finite, there is $p \in Var$ such that $p \notin At(\Gamma) \cup At(\Delta)$ and $\models_{L_1} \Gamma \Rightarrow p$ or $\models_{L_2} p \Rightarrow \Delta$. So $\models_{L_1} \Gamma \Rightarrow \Delta$ or $\models_{L_2} \Gamma \Rightarrow \Delta$ by Facts 6.1 and 6.2, and therefore $\Gamma \Rightarrow \Delta \in \mathbb{L}_1^a \cup \mathbb{L}_2^t$.

(\supseteq) Let $\Gamma \Rightarrow \Delta \in \mathbb{L}_1^a \cup \mathbb{L}_2^t$. If $\Gamma \Rightarrow \Delta \in \mathbb{L}_1^a$, then $\models_{L_1} \Gamma \Rightarrow \phi$ for all $\phi \in \mathcal{L}$ by Fact 6.1. If $\Gamma \Rightarrow \Delta \in \mathbb{L}_2^t$, then $\models_{L_2} \psi \Rightarrow \Delta$ for all $\psi \in \mathcal{L}$ by Fact 6.2. In both cases we obtain $\models_{L_1} \Gamma \Rightarrow \phi$ for all $\phi \in \mathcal{L}$ or $\models_{L_2} \psi \Rightarrow \Delta$ for all $\psi \in \mathcal{L}$, which entails in turn $\models_{L_1} \Gamma \Rightarrow \phi$ or $\models_{L_2} \phi \Rightarrow \Delta$ for all $\phi \in \mathcal{L}$. And therefore $\Gamma \Rightarrow \Delta \in \mathbb{L}_1^+ \dagger \mathbb{L}_2^+$. \square

We then go over a well-known fact about K_3 and LP. Namely that K_3 has the same antitheorems as ST, and LP the same theorems.

Fact 6.4 $\mathbb{K}_3^a = ST^a$ and $LP^t = ST^t$.

Proof Note on the one hand that $\models_{K_3} \Gamma \Rightarrow \emptyset$ iff for all $v, v(\gamma) \neq 1$ for some $\gamma \in \Gamma$ iff $\models_{ST} \Gamma \Rightarrow \emptyset$, and on the other that $\models_{LP} \emptyset \Rightarrow \Delta$ iff for all $v, v(\delta) \neq 0$ for some $\delta \in \Delta$ iff $\models_{ST} \emptyset \Rightarrow \Delta$. \square

This fact enables us to characterize the relative sum of \mathbb{K}_3^+ and LP^+ as the set of antitheorems and theorems of ST.

Theorem 6.1 $\mathbb{K}_3^+ \dagger LP^+ = \mathbb{K}_3^a \cup LP^t = ST^a \cup ST^t$.

Proof From Facts 6.3 and 6.4. \square

It follows that the relative sum of \mathbb{K}_3^+ and LP^+ is not equal to TS^+ since for instance $\emptyset \Rightarrow p \vee \neg p$ is an element of $ST^a \cup ST^t$ but not of TS^+ , and similarly for $p \wedge \neg p \Rightarrow \emptyset$.

This result provides a better understanding of the role played by the relative sum in the characterization of TS^+ . Just as $\mathbb{K}_3^+ \dagger LP^+$ is the set of antitheorems of K_3 and of theorems of LP — that is the set of antitheorems and theorems of ST, as shown by the previous fact —, $TS^+ = LP^+ \dagger \mathbb{K}_3^+$ is the set consisting of the antitheorems of LP^+ and the theorems of \mathbb{K}_3^+ . Since it is known that LP has no non-trivial antitheorems and K_3 no non-trivial theorems, TS^+ is therefore the set of trivial antitheorems of LP and trivial theorems of K_3 .

Turning to the case of ST, we show that in the presence of λ , the relative product of the sets of LP-valid and K_3 -valid inferences is the set of valid inferences of the trivial, universal logic, in which anything follows from anything.

Fact 6.5 $ST^+ \neq LP^+ | K_3^+ = \mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$.

Proof Note that for any $\Gamma, \Delta \subseteq \mathcal{L}$, $\models_{LP} \Gamma \Rightarrow \lambda$ and $\models_{K_3} \lambda \Rightarrow \Delta$. Hence, for any $\Gamma, \Delta, \Gamma \Rightarrow \Delta \in LP^+ | K_3^+$. \square

However, we can prove that $LP^+ | K_3^+ = ST^+$ when the constant λ is removed from the language. The connecting formula will be constructed by taking the conjunction of the premises with the classical tautologies built from each atom of the conclusions. Doing so forces one of the conclusions to take a classical value whenever the connecting formula is K_3 -satisfied.

Theorem 6.2 Let \mathcal{L}^- be the language defined in Definition 2.1 without the constant λ , then

$$ST^+ = LP^+ | K_3^+.$$

Proof (\subseteq) Assume $\Gamma \Rightarrow \Delta \in ST^+$, that is $\models_{ST} \Gamma \Rightarrow \Delta$. We start with the limit cases. When $\Gamma = \emptyset$, we let $D := \bigvee \Delta$. Since $\models_{ST} \emptyset \Rightarrow \Delta$, $\models_{LP} \emptyset \Rightarrow D$ by Fact 6.4, and clearly $\models_{K_3} D \Rightarrow \Delta$. When $\Delta = \emptyset$, we let $D := \bigwedge \Gamma$. Given that $\models_{ST} \Gamma \Rightarrow \emptyset$, we have $\models_{K_3} D \Rightarrow \emptyset$ by Fact 6.4, and clearly $\models_{LP} \Gamma \Rightarrow D$. In both cases $\Gamma \Rightarrow \Delta \in LP^+ | K_3^+$. For the main case, we let

$$C := \bigwedge \{(\alpha \vee \neg\alpha) : \alpha \in At(\Delta)\}$$

and $D := \bigwedge \Gamma \wedge C$.

We first show that $\models_{LP} \Gamma \Rightarrow D$. Assume $v(\gamma) \in \{1/2, 1\}$ for all $\gamma \in \Gamma$. For all $p \in Var$, $v(p \vee \neg p) \neq 0$, so $v(C) \neq 0$, and since $v(\gamma) \neq 0$ for all $\gamma \in \Gamma$ by assumption, $v(D) \neq 0$. Hence, $\models_{LP} \Gamma \Rightarrow D$.

Assume now that $\not\models_{K_3} D \Rightarrow \Delta$. Then there is v such that $v(D) = 1$ and $v(\delta) \neq 1$ for all $\delta \in \Delta$. Given that $v(D) = 1$, $v(\gamma) = 1$ for all $\gamma \in \Gamma$ and $v(C) = 1$. From the latter, it follows that for all $\alpha \in At(\Delta)$, $v(\alpha \vee \neg\alpha) = 1$, therefore for all $\alpha \in At(\Delta)$, $v(\alpha) \neq 1/2$. So, $v(\delta) \neq 1/2$ for all $\delta \in \Delta$, meaning that $v(\delta) = 0$ for all $\delta \in \Delta$. Hence, $v \not\models_{ST} \Gamma \Rightarrow \Delta$, which contradicts our first assumption. Therefore, $\models_{K_3} D \Rightarrow \Delta$. Finally, since there is $\phi := D$ such that $\models_{LP} \Gamma \Rightarrow \phi$ and $\models_{K_3} \phi \Rightarrow \Delta$, $\Gamma \Rightarrow \Delta \in LP^+ | K_3^+$.

(\supseteq) Assume $\Gamma \Rightarrow \Delta \in LP^+ | K_3^+$ but $\Gamma \Rightarrow \Delta \notin ST^+$. From the latter, it follows that $\not\models_{ST} \Gamma \Rightarrow \Delta$. Hence, there is v such that $v(\gamma) = 1$ for all $\gamma \in \Gamma$ and $v(\delta) = 0$ for all $\delta \in \Delta$. From the former, it follows that there is ϕ such that $\models_{LP} \Gamma \Rightarrow \phi$ and $\models_{K_3} \phi \Rightarrow \Delta$. If $v(\phi) = 0$, then $\not\models_{LP} \Gamma \Rightarrow \phi$ and if $v(\phi) = 1$, then $\not\models_{K_3} \phi \Rightarrow \Delta$, therefore $v(\phi) = 1/2$. Now, let v' be exactly like v except that for all $p \in Var$, if $v(p) = 1/2$, then $v'(p) \neq 1/2$. Then $v'(\gamma) = 1$ for all $\gamma \in \Gamma$ and $v'(\delta) = 0$ for all $\delta \in \Delta$ by Lemma 4.1, and $v'(\phi) = 1$ or $v'(\phi) = 0$ since all its atoms get a classical value. And again, $\not\models_{LP} \Gamma \Rightarrow \phi$ or $\not\models_{K_3} \phi \Rightarrow \Delta$, which is inconsistent with our first assumption. So, $\Gamma \Rightarrow \Delta \in ST^+$, as desired. \square

The upshot is that \mathbb{LP}^+ and \mathbb{K}_3^+ are not interchangeable in the definitions of \mathbb{ST}^+ and \mathbb{TS}^+ using relational product and sum. The symmetry found for \mathbb{ST}^+ in the λ -free fragment is special, and does not function for \mathbb{TS}^+ . This may reflect the fact that \mathbb{TS}^+ is already non-reflexive in the λ -free fragment, whereas \mathbb{ST}^+ is only nontransitive in the λ -including fragment (under the notion of global notion of metainferential validity, see fn. 2).

7 On the duality between ST and TS

The results of the previous section tell us that \mathbb{ST}^+ and \mathbb{TS}^+ are both definable in terms of \mathbb{K}_3^+ and \mathbb{LP}^+ , from the dual operations of sum and product. This suggests that \mathbb{ST}^+ and \mathbb{TS}^+ are dual logics. Whether \mathbb{ST}^+ and \mathbb{TS}^+ can be seen as such is moot, however, and depends on how duality between logics is defined. Cobreros, La Rosa and Tranchini in [11] distinguish two notions of duality, which they call *operational* duality and *structural* duality, following Gentzen's distinction between operational rules and structural rules for a logic, and they show that \mathbb{ST}^+ and \mathbb{TS}^+ are structural duals, but not operational duals, and conversely for \mathbb{K}_3^+ and \mathbb{LP}^+ . In this section, we propose a revision of the notion of operational duality allowing us to establish that by combining the two notions of operational and structural duality, \mathbb{ST}^+ and \mathbb{TS}^+ stand in a relation that is exactly the same as the relation to which \mathbb{K}_3^+ and \mathbb{LP}^+ stand to each other, and similarly for their counterparts defined in terms of antivalidity. We proceed in three steps: we first introduce our modified notion of operational duality, then Cobreros et al.'s notion of structural duality, finally we use the apparatus of relative sum and product to provide general interdefinability results for \mathbb{ST}^+ , \mathbb{ST}^- , \mathbb{TS}^+ and \mathbb{TS}^- in terms of the validities and antivalidities of either LP or \mathbb{K}_3 .

7.1 Operational duality, revised

The notion of operational duality used by [11] originates in [8], and for the sake of clarity we propose to call it negation duality. It has the following definition:

Definition 7.1 (Negation duality) A logic L_1 is *negation dual* to a logic L_2 when

$$\models_{L_1} \Gamma \Rightarrow \Delta \text{ iff } \models_{L_2} \neg(\Delta) \Rightarrow \neg(\Gamma)$$

with $\neg(\Gamma) = \{\neg\gamma : \gamma \in \Gamma\}$ for all $\Gamma \subseteq \mathcal{L}$.

The logics \mathbb{K}_3 and LP are *negation* duals of each other. Concomitantly, it is known that ST and TS are *negation* self-dual (see e.g. [8, 12]).

Despite its intrinsic interest, such a notion does not license a straightforward extensional interdefinability of \mathbb{K}_3 and LP. This can easily be noticed by letting $n : \mathcal{P}(\text{INF}(\mathcal{L})) \rightarrow \mathcal{P}(\text{INF}(\mathcal{L}))$ be a mapping such that $n(\mathbb{X}) = \{\neg(\Delta) \Rightarrow \neg(\Gamma) : \Gamma \Rightarrow \Delta \in \mathbb{X}\}$. Clearly $\mathbb{K}_3^+ \neq n(\mathbb{LP}^+)$, since not all \mathbb{K}_3 -valid inferences are of the form $\neg(\Delta) \Rightarrow \neg(\Gamma)$.

Because of that, we propose a distinct notion of operational duality, which will enable us to prove the extensional interdefinability of \mathbb{K}_3 and LP. This notion of duality

can be found in Kleene’s [17, 18] (and goes back to Schröder in [28]), where it is proven that for any two formulae ϕ, ψ , if the inference $\phi \Rightarrow \psi$ is classically valid, then so is the inference $\psi' \Rightarrow \phi'$, with ϕ' and ψ' obtained from ϕ and ψ by interchanging \wedge with \vee . In our case, we define a mapping \sim that will substitute \top and \perp for each other in all formulae, and similarly \vee and \wedge .

Definition 7.2 Let $\sim : \mathcal{L} \rightarrow \mathcal{L}$ be a mapping such that:

$$\begin{aligned} \sim p &= p, \text{ if } p \in \text{Var}, \\ \sim \top &= \perp, \\ \sim \perp &= \top, \\ \sim \lambda &= \lambda, \\ \sim \neg \phi &= \neg \sim \phi, \\ \sim(\phi \wedge \psi) &= \sim \phi \vee \sim \psi, \\ \sim(\phi \vee \psi) &= \sim \phi \wedge \sim \psi. \end{aligned}$$

The function \sim is extended to all $\Gamma \subseteq \mathcal{L}$ by $\sim(\Gamma) := \{\sim \gamma : \gamma \in \Gamma\}$, and to all $\mathbb{X} \subseteq \text{INF}(\mathcal{L})$ by $\sim(\mathbb{X}) := \{\sim(\Gamma) \Rightarrow \sim(\Delta) : \Gamma \Rightarrow \Delta \in \mathbb{X}\}$.

For instance, this function applied to the formula $p \wedge q$ will produce the formula $p \vee q$, and applied to the formula $p \wedge (q \vee \neg q)$ will produce the formula $p \vee (q \wedge \neg q)$. We use the mapping \sim for the following revision of the notion of operational duality considered by Cobreros et al.:

Definition 7.3 (Operational duality for formulae, and inferences) For any two formulae ϕ and ψ , ψ is called the (revised) operational dual of ϕ when $\sim \phi = \psi$. For any two inferences $\Gamma \Rightarrow \Delta$ and $\Delta' \Rightarrow \Gamma'$, $\Delta' \Rightarrow \Gamma'$ is called the operational dual of $\Gamma \Rightarrow \Delta$ when $\sim \Gamma = \Gamma'$ and $\sim \Delta = \Delta'$.

The next two lemmas express useful properties of \sim , namely that it is involutive and commutes with the relational inverse function.

Lemma 7.1 For all $\phi \in \mathcal{L}$, $\sim \sim \phi = \phi$.

Proof The proof is by induction on the length of ϕ . □

Lemma 7.2 For all $\mathbb{X} \subseteq \text{INF}(\mathcal{L})$, $(\sim \mathbb{X})^{-1} = \sim(\mathbb{X}^{-1})$.

Proof

$$\begin{aligned} \Gamma \Rightarrow \Delta \in (\sim \mathbb{X})^{-1} &\text{ iff } \Delta \Rightarrow \Gamma \in \sim \mathbb{X} \\ &\text{ iff } \sim \Delta \Rightarrow \sim \Gamma \in \sim \sim \mathbb{X} = \mathbb{X} \text{ by Lemma 7.1} \\ &\text{ iff } \sim \Gamma \Rightarrow \sim \Delta \in \mathbb{X}^{-1} \\ &\text{ iff } \sim \sim \Gamma \Rightarrow \sim \sim \Delta \in \sim(\mathbb{X}^{-1}) \\ &\text{ iff } \Gamma \Rightarrow \Delta \in \sim(\mathbb{X}^{-1}) \text{ by Lemma 7.1.} \end{aligned}$$

□

To show that K_3 and LP are extensionally interdefinable, we will have to prove first that they are operationally dual. The next lemma describes the operational duality between two formulae in terms of dual valuations. Two SK valuations v and v' are said to be dual when for all $p \in \text{Var}$, $v'(p) = 1 - v(p)$. The dual of a valuation v will be noted v^* .

Lemma 7.3 *For all valuations v and all $\phi \in \mathcal{L}$, $v(\phi) = 0$ if and only if $v^*(\sim\phi) = 1$ and $v(\phi) = 1$ if and only if $v^*(\sim\phi) = 0$.*

Proof The proof is by induction on the length of ϕ . □

As highlighted earlier, K_3 and LP are negation dual to each other, and ST and TS are negation self-dual. We prove here that in a similar way K_3 and LP are operationally dual, and ST and TS operationally self-dual.

Proposition 7.1 $\models_{K_3} \Gamma \Rightarrow \Delta$ if and only if $\models_{LP} \sim(\Delta) \Rightarrow \sim(\Gamma)$ and $\models_{LP} \Gamma \Rightarrow \Delta$ if and only if $\models_{K_3} \sim(\Delta) \Rightarrow \sim(\Gamma)$.

Proof Assume $\models_{K_3} \Gamma \Rightarrow \Delta$ and $v^*(\sim\delta) \neq 0$ for all $\delta \in \Delta$. By the preceding lemma, $v(\delta) \neq 1$ for all $\delta \in \Delta$, so $v(\gamma) \neq 1$ for some $\gamma \in \Gamma$. Again, by the preceding lemma, $v^*(\sim\gamma) \neq 0$ for some $\gamma \in \Gamma$, and therefore $\models_{LP} \sim(\Delta) \Rightarrow \sim(\Gamma)$. Assume now $\models_{LP} \sim(\Delta) \Rightarrow \sim(\Gamma)$ and $v(\gamma) = 1$ for all $\gamma \in \Gamma$. By the preceding lemma, $v^*(\sim\gamma) = 0$ for all $\gamma \in \Gamma$, so $v^*(\sim\delta) = 0$ for some $\delta \in \Delta$. Again, by the preceding lemma, $v(\delta) = 1$ for some $\delta \in \Delta$, and therefore $\models_{K_3} \Gamma \Rightarrow \Delta$. The second equivalence can be proved symmetrically. □

Proposition 7.2 $\models_{ST} \Gamma \Rightarrow \Delta$ if and only if $\models_{ST} \sim(\Delta) \Rightarrow \sim(\Gamma)$ and $\models_{TS} \Gamma \Rightarrow \Delta$ if and only if $\models_{TS} \sim(\Delta) \Rightarrow \sim(\Gamma)$.

Proof Assume $\models_{ST} \Gamma \Rightarrow \Delta$ and $v^*(\sim\delta) = 1$ for all $\delta \in \Delta$. By Lemma 7.3, $v(\delta) = 0$ for all $\delta \in \Delta$, so $v(\gamma) \neq 1$ for some $\gamma \in \Gamma$. Again, by Lemma 7.3, $v^*(\sim\gamma) \neq 0$ for some $\gamma \in \Gamma$, and therefore $\models_{ST} \sim(\Delta) \Rightarrow \sim(\Gamma)$. Assume now $\models_{ST} \sim(\Delta) \Rightarrow \sim(\Gamma)$ and $v(\gamma) = 1$ for all $\gamma \in \Gamma$. By Lemma 7.3, $v^*(\sim\gamma) = 0$ for all $\gamma \in \Gamma$, so $v^*(\sim\delta) \neq 1$ for some $\delta \in \Delta$. Again, by Lemma 7.3, $v(\delta) \neq 0$ for some $\delta \in \Delta$, and therefore $\models_{ST} \Gamma \Rightarrow \Delta$. The case of TS can be proved similarly. □

Definition 7.4 (Operational duality for logics) For any two logics L_1 and L_2 based on the language of Definition 2.1, we will say that L_1 is operationally dual to L_2 when $\models_{L_1} \Gamma \Rightarrow \Delta$ if and only if $\models_{L_2} \sim(\Delta) \Rightarrow \sim(\Gamma)$ for all $\Gamma, \Delta \subseteq \mathcal{L}$.

Given this terminology, K_3 and LP are therefore operationally dual, while ST and TS are both operationally self-dual.

It remains to show that this notion of duality licenses an extensional interdefinability between a logic and its operational dual. This is proven in the next lemma.

Lemma 7.4 *Given two logics L_1 and L_2 defined on the language of Definition 2.1, L_1 is operationally dual to L_2 if and only if $\mathbb{L}_1^+ = \sim(\mathbb{L}_2^+)^{-1}$.*

Proof Assume that a logic L_1 is operationally dual to a logic L_2 . Then $\Gamma \Rightarrow \Delta \in \mathbb{L}_1^+$ if and only if $\sim(\Delta) \Rightarrow \sim(\Gamma) \in \mathbb{L}_2^+$. Which is equivalent to $\sim(\Gamma) \Rightarrow \sim(\Delta) \in (\mathbb{L}_2^+)^{-1}$, and in turn to $\sim\sim(\Gamma) \Rightarrow \sim\sim(\Delta) \in \sim(\mathbb{L}_2^+)^{-1}$. Given that \sim is involutive by Lemma 7.1, the previous statement amounts to $\Gamma \Rightarrow \Delta \in \sim(\mathbb{L}_2^+)^{-1}$.

Assume now that $\mathbb{L}_1^+ = \sim(\mathbb{L}_2^+)^{-1}$. We have that $\Gamma \Rightarrow \Delta \in \mathbb{L}_1^+$ if and only if $\Gamma \Rightarrow \Delta \in \sim(\mathbb{L}_2^+)^{-1}$. But this is equivalent to $\sim\sim\Gamma \Rightarrow \sim\sim\Delta \in \sim(\mathbb{L}_2^+)^{-1}$ since \sim is involutive. Which is to say that $\sim\Gamma \Rightarrow \sim\Delta \in (\mathbb{L}_2^+)^{-1}$, and in turn that $\sim\Delta \Rightarrow \sim\Gamma \in \mathbb{L}_2^+$. \square

With this lemma at hand, we are in a position to characterize extensionally each of the logics discussed in terms of their operational dual.

Theorem 7.1

$$\begin{aligned} \mathbb{K}_3^+ &= \sim(\mathbb{LP}^+)^{-1} & \mathbb{LP}^+ &= \sim(\mathbb{K}_3^+)^{-1} \\ \mathbb{ST}^+ &= \sim(\mathbb{ST}^+)^{-1} & \mathbb{TS}^+ &= \sim(\mathbb{TS}^+)^{-1}. \end{aligned}$$

Proof This follows directly from Lemma 7.4, Proposition 7.1 and Proposition 7.2. \square

The composition $\sim \circ^{-1}$ is therefore a function mapping a set of valid inferences of a logic to the set of valid inferences of its operational dual. \mathbb{LP}^+ is mapped to \mathbb{K}_3^+ , \mathbb{K}_3^+ to \mathbb{LP}^+ , while both \mathbb{ST}^+ and \mathbb{TS}^+ are fixed points of the composition.

7.2 Structural duality

In the second half of this section, we combine our proposed notion of operational duality with the notion of structural duality to show that ST and TS are each other’s dual under this combined notion, in the same way in which \mathbb{K}_3 and LP are each other’s dual. In addition, the results of Sections 4 and 5 will allow for an extensional characterization of the sets of ST- and TS-valid inferences solely from the set of \mathbb{K}_3 - or LP-valid inferences. Along the way, we will prove a similar duality result for the sets of \mathbb{K}_3 - and LP- *antivalid* inferences, and see that the operations of relative product and relative sum can be used to characterize the sets of ST and TS antivalid inferences from the sets of LP and \mathbb{K}_3 antivalid inferences.

The notion of antivalidity used throughout this section was first introduced by Cobreros, La Rosa and Tranchini in [11] to account for the notion of structural duality. It is defined as follows:

Definition 7.5 (L-antisatisfaction, L-antivalidity) Let L be a logic based on a consequence relation defined on the standard D_1 for the premises and D_2 for the conclusions.

- An SK-valuation v *antisatisfies* an inference $\Gamma \Rightarrow \Delta$ in L (symbolized by $v \models_L \Gamma \Rightarrow \Delta$) if and only if $v(\gamma) \notin D_1$ for all $\gamma \in \Gamma$ only if $v(\delta) \notin D_2$ for some $\delta \in \Delta$.
- An inference $\Gamma \Rightarrow \Delta$ is *antivalid* in L (symbolized by $\models_L \Gamma \Rightarrow \Delta$) if and only if $v \models_L \Gamma \Rightarrow \Delta$ for all SK-valuations v .

For instance, $\models_{LP} p \Rightarrow p \wedge q$, since for all v , if $v(p) \notin \{1/2, 1\}$, then $v(p \wedge q) \notin \{1/2, 1\}$; while $\not\models_{ST} p \Rightarrow p \wedge q$, because there is v such that $v(p) = v(q) = v(p \wedge q) = 1/2$, and $v(p) \neq 1$ but $v(p \wedge q) \in \{1/2, 1\}$.

Cobrerros et al. [11] say that a logic is structurally dual to another whenever, if an argument is valid in the first logic, the converse (or inverse) argument is antivalid in the second logic. Formally:

Definition 7.6 (Structural duality) A logic L_1 is structurally dual to a logic L_2 when

$$\models_{L_1} \Gamma \Rightarrow \Delta \text{ iff } \models_{L_2} \Delta \Rightarrow \Gamma.$$

Structural duality implicitly involves a form of negation in the metalanguage, more precisely, a form of metalinguistic contraposition. Validity is expressed through a universally closed metalinguistic conditional statement connecting the satisfaction of an inference’s premises to the satisfaction of its conclusions. In turn, antivalidity is obtained by contraposing this metalinguistic conditional while keeping fixed the sets of designated values. Structural duality between two logics L_1 and L_2 requires therefore an equivalence between two universally closed conditional statements: the first one connecting the L_1 -satisfaction of an inference’s premises to the L_1 -satisfaction of its conclusions, and the second one connecting the non- L_2 -satisfaction of its conclusions to the non- L_2 -satisfaction of its premises.

The following proposition is proven by Cobrerros, La Rosa, and Tranchini in [11]:

Proposition 7.3 (Structural duals) Let $\Gamma, \Delta \subseteq \mathcal{L}$,

$$\begin{aligned} \models_{LP} \Gamma \Rightarrow \Delta \text{ if and only if } \models_{LP} \Delta \Rightarrow \Gamma. \\ \models_{K_3} \Gamma \Rightarrow \Delta \text{ if and only if } \models_{K_3} \Delta \Rightarrow \Gamma. \\ \models_{ST} \Gamma \Rightarrow \Delta \text{ if and only if } \models_{TS} \Delta \Rightarrow \Gamma. \\ \models_{TS} \Gamma \Rightarrow \Delta \text{ if and only if } \models_{ST} \Delta \Rightarrow \Gamma. \end{aligned}$$

Proof Let XY and YX be two logics defined on the same scheme for the connectives and XY be a logic based on a mixed consequence relation defined on the standard X for the premises and Y for the conclusions (and conversely for YX). Then $\models_{XY} \Gamma \Rightarrow \Delta$ if and only if $v(\gamma) \in X$ for all $\gamma \in \Gamma$ only if $v(\delta) \in Y$ for some $\delta \in \Delta$. Which is equivalent to $v(\delta) \notin Y$ for all $\delta \in \Delta$ only if $v(\gamma) \notin X$ for some $\gamma \in \Gamma$, that is $\models_{YX} \Delta \Rightarrow \Gamma$. \square

The set of antivalidities of a logic is defined in a similar way as the set of validities of a logic:

Definition 7.7 (Set of antivalidities)

$$\mathbb{L}^- = \{\Gamma \Rightarrow \Delta : \models_L \Gamma \Rightarrow \Delta\}.$$

Proposition 7.3 can thus be rephrased in terms of inverse: the sets of K_3 - and LP -antivalid inferences are respectively the inverses of the sets of K_3 - and LP -valid inferences; while the sets of ST - and TS -antivalid inferences are respectively the inverses of the sets of TS - and ST -valid inferences.

Corollary 7.1 (*Inverse of \mathbb{K}_3^+ , LP^+ , ST^+ and TS^+*)

$$\begin{aligned} \text{LP}^- &= (\text{LP}^+)^{-1}. \\ \mathbb{K}_3^- &= (\mathbb{K}_3^+)^{-1}. \\ \text{TS}^- &= (\text{ST}^+)^{-1}. \\ \text{ST}^- &= (\text{TS}^+)^{-1}. \end{aligned}$$

Proof This follows from Proposition 7.3. □

The results of Theorem 7.1 can now be restated in light of these equalities.

Theorem 7.2

$$\begin{aligned} \mathbb{K}_3^+ &= \sim\text{LP}^- & \text{LP}^+ &= \sim\mathbb{K}_3^- \\ \text{ST}^+ &= \sim\text{TS}^- & \text{TS}^+ &= \sim\text{ST}^-. \end{aligned}$$

Proof From Corollary 7.1 and Theorem 7.1. □

A dual result can be given by interchanging the validities and the antivalidities:

Theorem 7.3

$$\begin{aligned} \mathbb{K}_3^- &= \sim\text{LP}^+ & \text{LP}^- &= \sim\mathbb{K}_3^+ \\ \text{ST}^- &= \sim\text{TS}^+ & \text{TS}^- &= \sim\text{ST}^+. \end{aligned}$$

Proof We only prove the first equivalence, the proof of the others being similar. By Corollary 7.1, $\mathbb{K}_3^- = (\mathbb{K}_3^+)^{-1}$, and by the previous theorem $(\mathbb{K}_3^+)^{-1} = (\sim\text{LP}^-)^{-1} = (\sim((\text{LP}^+)^{-1}))^{-1}$. Finally, by Lemma 7.2, the last expression is equal to $\sim(((\text{LP}^+)^{-1})^{-1}) = \sim(\text{LP}^+)$. □

Although neither the revised notion of operational duality nor the notion of structural duality makes \mathbb{K}_3 and LP dual simultaneously with ST and TS, the combination of the two leads to a unified extensional definition of \mathbb{K}_3 in terms of LP and of ST in terms of TS (and conversely). In the case of \mathbb{K}_3 and LP, this combined notion of duality collapses with operational duality, since both are structurally self-dual; in the case of ST and TS, the notion collapses with structural duality since both are operationally self-dual.

7.3 Putting it all together

We now turn to the characterization of the sets of ST and TS antivalid inferences out of the sets of LP and \mathbb{K}_3 antivalid inferences. It is known that the inverse of the relational product of two relations R and S is equal to the relational product of the inverse of S and the inverse of R , and similarly for the relational sum [19]. The next fact attests that this remains true for the modified versions of relative product and relative sum introduced in Definition 3.2 and Definition 3.3.

Fact 7.1 (Inverse product and inverse sum)

$$\begin{aligned} (R \mid_v S)^{-1} &= S^{-1} \mid_v R^{-1} \\ (R \dagger_v S)^{-1} &= S^{-1} \dagger_v R^{-1}. \end{aligned}$$

Proof

$$\begin{array}{ll} \langle x, z \rangle \in (R \mid_v S)^{-1} & \langle x, z \rangle \in (R \dagger_v S)^{-1} \\ \text{iff } \langle z, x \rangle \in R \mid_v S & \text{iff } \langle z, x \rangle \in R \dagger_v S \\ \text{iff there is } y \in V \text{ s.t. } \langle z, y \rangle \in R & \text{iff for all } y \in V, \langle z, y \rangle \in R \\ \text{and } \langle y, x \rangle \in S & \text{or } \langle y, x \rangle \in S \\ \text{iff there is } y \in V \text{ s.t. } \langle y, z \rangle \in R^{-1} & \text{iff for all } y \in V, \langle y, z \rangle \in R^{-1} \\ \text{and } \langle x, y \rangle \in S^{-1} & \text{or } \langle x, y \rangle \in S^{-1} \\ \text{iff } \langle x, z \rangle \in S^{-1} \mid_v R^{-1}. & \text{iff } \langle x, z \rangle \in S^{-1} \dagger_v R^{-1}. \end{array}$$

□

We are now in a position to identify the set of ST-antivalid inferences with the relative sum of the sets of K_3 -antivalid inferences and LP-antivalid inferences.

Theorem 7.4 $ST^- = K_3^- \dagger LP^-$.

Proof By Corollary 7.1, $ST^- = (TS^+)^{-1}$, moreover $TS^+ = LP^+ \dagger K_3^+$ by Theorem 5.1, so $(TS^+)^{-1} = (LP^+ \dagger K_3^+)^{-1}$. Finally, by Fact 7.1 and Corollary 7.1 again, $(LP^+ \dagger K_3^+)^{-1} = (K_3^+)^{-1} \dagger (LP^+)^{-1} = K_3^- \dagger LP^-$. □

The set of TS-antivalid inferences can in turn be characterized as the relative product of the set of LP-antivalid inferences and the set of K_3 -antivalid inferences.

Theorem 7.5 $TS^- = LP^- \mid K_3^-$.

Proof By Corollary 7.1, $TS^- = (ST^+)^{-1}$, moreover $ST^+ = K_3^+ \mid LP^+$ by Theorem 4.1, so $(ST^+)^{-1} = (K_3^+ \mid LP^+)^{-1}$. Finally, by Fact 7.1 and Corollary 7.1 again, $(K_3^+ \mid LP^+)^{-1} = (LP^+)^{-1} \mid (K_3^+)^{-1} = LP^- \mid K_3^-$. □

Combining Theorem 4.1 and Theorem 5.1 with Theorem 7.2, and Theorem 7.4 and 7.5 with Theorem 7.3 leads to the following proposition, which shows that ST and TS can be extensionally characterized by either LP or K_3 .

Theorem 7.6

$$\begin{aligned} ST^+ &= K_3^+ \mid \sim K_3^- = \sim LP^- \mid LP^+ & TS^+ &= \sim K_3^- \dagger K_3^+ = LP^+ \dagger \sim LP^- \\ ST^- &= K_3^- \dagger \sim K_3^+ = \sim LP^+ \dagger LP^- & TS^- &= \sim K_3^+ \mid K_3^- = LP^- \mid \sim LP^+. \end{aligned}$$

Proof This follows directly from Theorem 4.1, Theorem 5.1, Theorem 7.4, Theorem 7.5, Theorem 7.2 and Theorem 7.3. \square

Those identities lend support to the view that ST isn't more fundamentally tied to LP than it is to K_3 (see Cobreros et al. [10]): either system can be used as a primitive in order to define either ST or TS at the inferential level.

8 Comparisons

We have shown that ST and TS can be extensionally defined from either K_3 or LP. In this section, we comment on the significance of these findings by comparing them to two sets of results found in the literature. We start by discussing the link between the characterization proposed of ST and a generalization of Craig's interpolation theorem proven by Peter Milne in [21] for classical logic. Then, we return to the significance of the duality results presented in the previous section, in particular in comparison to those of [11] and [12].

8.1 Interpolation

Given an ST-valid inference $\Gamma \Rightarrow \Delta$, Theorem 4.1 gives us a constructive method of finding a formula ϕ such that $\Gamma \Rightarrow \phi$ is K_3 -valid, and $\phi \Rightarrow \Delta$ is LP-valid, namely the K_3 disjunctive normal form of $\bigwedge \Gamma$. We called such a formula a connecting formula, but in the example we produced, it has the features of an interpolant in the sense of Craig's theorem for propositional logic. Craig's interpolation theorem for classical logic states that when $\phi \Rightarrow \psi$ is classically valid, provided that ϕ is not a contradiction and ψ not a tautology, there exists an interpolant formula χ such that every propositional variable in χ occurs in both in ϕ and ψ , and $\phi \Rightarrow \chi$ and $\chi \Rightarrow \psi$ are classically valid.⁶ Hence, it is natural to wonder about the connection between our result and Craig's interpolation theorem.

As it turns out, Milne in [21] proved a refinement of Craig's theorem, which is that when ϕ classically entails ψ and neither ϕ is a contradiction nor ψ a tautology, then there is an interpolant formula χ such that $\phi \Rightarrow \chi$ is K_3 -valid, and $\chi \Rightarrow \psi$ is LP-valid (see also [24] for a statement of other interpolation properties for three-valued logics in the vicinity of LP and K_3). An examination of Milne's proof shows that he used the same method we used in the first half of the proof of Theorem 4.1, except that he restricts it to atoms common to the antecedent and succedent of an argument. In the example we gave of $p \vee (q \wedge \neg q) \Rightarrow p \wedge (q \vee \neg q)$, Milne's method would produce as interpolant formula the same as ours, namely $(p \wedge q) \vee (p \wedge \neg q) \wedge p$. But in the variant $p \vee (q \wedge \neg q) \Rightarrow p \wedge (r \vee \neg r)$, Milne's method would produce p as the

⁶ This is sometimes called the *deductive* interpolation theorem, since it states the existence of a deductive interpolant. In classical logic it is equivalent to the *conditional* interpolation theorem, which states that when $\phi \rightarrow \psi$ is a theorem and ϕ and ψ have at least one propositional variable in common, there is an interpolant formula χ such that every propositional variable in χ occurs both in ϕ and ψ , and $\phi \rightarrow \chi$ and $\chi \rightarrow \psi$ are theorems.

interpolant given Craig's constraint on atom-sharing. We do not have that constraint and our method would still produce $(p \wedge q) \vee (p \wedge \neg q) \wedge p$ as the connecting formula, since it is based solely on producing a K_3 DNF.

Milne's result is not concerned with ST, it is also limited to the case of inferences with a satisfiable single premise and a falsifiable single conclusion, and it is proven for the language of propositional logic without constants. However, it bears a direct connection to our characterization of ST, since it can be used to show that classical logic is the product of K_3 and LP. This raises a natural question, namely whether we may strengthen the characterization of ST in terms of relational product by making systematic reference to the interpolation property.

This question also holds broader interest. As discussed in [13], and as originally proved in [29], the logic ST can also be obtained in terms of mixed consequence on the basis of the Weak Kleene logic K_3^w and its dual system Paraconsistent Weak Kleene PWK: both systems rest on the Weak Kleene scheme, unlike K_3 and LP. So does ST equal the relational product of K_3^w and PWK, and if so, is the connecting formula an interpolant? More generally, [13] show that ST can be characterized in terms of mixed consequence over even more schemes intermediate between the Weak Kleene and the Strong Kleene scheme: for those logics, does the product characterization still hold, and if so, is the connecting formula always an interpolant in the sense of Craig? We reserve the answer to these questions for further work.

8.2 Duality

Several duality results have been offered in the literature on Strong Kleene logics. As reviewed above, [8] have shown that K_3 and LP are negation dual, and ST and TS negation self-dual; [11] that ST and TS are structurally dual and K_3 and LP structurally self-dual; and [12] that ST and TS are metainferentially dual.⁷ Yet, none of these notions of duality makes ST the dual of TS in the same sense in which K_3 is the dual of LP.

The notion of operational duality put forward in this paper differs from each of these notions. The difference with structural duality and metainferential duality is readily noticeable, since none of them involves connectives and are thus not operational. The difference with negation duality is less obvious, both notions involve operators and make K_3 dual to LP and ST and TS self-dual. But as shown earlier, no function can be straightforwardly defined from negation duality with the intent of characterizing the set of K_3 -valid inferences from the set of LP-valid inferences, contrary to operational duality, where such a function was easily specified. Moreover, negation duality only involves a single operator, negation, whereas operational duality involves all the operators of the language. Given a formula, all the operators occurring in it are interchanged by the function \sim with their dual: \top with \perp , λ with λ , \neg with \neg , and \wedge with \vee . To see why for all the logics discussed above this duality between operators obtains, it is enough to notice that if 1 and 0 are renamed 0 and 1, then the truth-conditions of \perp and \top , and \wedge and \vee are swapped, while the truth-conditions of λ and \neg are

⁷ A metainference is an inference between inferences. [12] show, with the help of a notion of antisatisfaction different from the one used here, that every metainference can be dualized. They prove that for every ST-valid metainference, its dual is TS-valid (and conversely).

left unchanged.⁸ Despite these differences, the two notions are connected through the following fact:

Fact 8.1 *For any truth-functional logic L based on the language of Definition 2.1 and equipped with an involutive negation satisfying the De Morgan laws:*

$$\models_L \sim(\Gamma) \Rightarrow \sim(\Delta) \quad \text{if and only if} \quad \models_L \neg(\Gamma) \Rightarrow \neg(\Delta).$$

Proof Let σ be a substitution of every atom by its negation. By a simple but tedious induction one can show that for all v , $v(\sigma(\sim\phi)) = v(\neg\phi)$ and $v(\sigma(\neg\phi)) = v(\sim\phi)$. Hence, if $\models_L \sim(\Gamma) \Rightarrow \sim(\Delta)$, then $\models_L \sigma(\sim(\Gamma)) \Rightarrow \sigma(\sim(\Delta))$, and therefore $\models_L \neg(\Gamma) \Rightarrow \neg(\Delta)$. The other direction is similar. \square

This explains why both negation duality and operational duality make K_3 dual to LP and ST and TS self-dual. In the context of truth-functional logics based on the language of Definition 2.1 and equipped with an involutive negation satisfying the De Morgan laws, the two notions are equivalent.

Still, our notion of operational duality, like the others mentioned, does not by itself offer an account that simultaneously makes ST and TS dual, and K_3 and LP dual. It is only when combined with structural duality, that such a unified account can be obtained. Theorems 7.2 and 7.3 thus provide a uniform characterization of K_3 in terms of LP and of ST in terms of TS. Given one of these logics, one can obtain the set of (antivalid) valid inferences of its dual by first taking the set of (valid) antivalid inferences of this logic and then applying the mapping \sim to this set. ST is dual to TS under this combined notion inasmuch as ST is operationally self-dual but structurally dual to TS; K_3 is dual to LP under this same notion inasmuch as K_3 is structurally self-dual but operationally dual to LP.

As a concluding remark, let us note that since its inception, ST has been presented as being of type K_3 on the side of premises and of type LP on the side of conclusions (viz. Cobreros et al. [9, 10]), based on the very definition of ST in terms of mixed standards of truth. For example, Zardini in [32] calls ST “ K_3LP ”, precisely on account of this hybrid behavior. One might therefore be tempted to call TS “ LPK_3 ” on the same grounds. The results of this paper may be used to vindicate those denominations. But what matters is to see that the operation that connects K_3 and LP extensionally is not the same depending on the case. In particular, TS^+ is not the relational product of LP^+ and K_3^+ , which corresponds to ST^+ in the absence of constants, and otherwise to $\mathcal{P}(\mathcal{L}) \times \mathcal{P}(\mathcal{L})$. Conversely, ST^+ is not the relational sum of K_3^+ and LP^+ , which corresponds to the set of theorems and antitheorems of ST^+ . What is more, the results proven in Theorem 7.6 from the duality of K_3 and LP might equally license alternative denominations of ST and TS as “ K_3K_3 ” or “ $LPLP$ ”, which blurs the distinction between the two logics. Pending a conclusive argument favoring the characterization of a logic in terms of validity rather than antivalidity, we do not see why ST and TS should deserve the denominations “ K_3LP ” and “ LPK_3 ”, more than “ K_3K_3 ” or “ $LPLP$ ”.

⁸ See Kleene [18, p. 23-24] for an illustration of this duality between operators that can easily be adapted to the logics discussed here.

9 Conclusion

In this paper we have proved that ST and TS each have a natural set-theoretic characterization based on the inferences of the dual logics K_3 and LP: ST is the relational product of the inferences of K_3 and LP, while TS is their relational sum.

We reckon that those results are of importance philosophically and proof-theoretically. Philosophically, because whereas ST and TS are semantically defined using a notion of mixed consequence, the results given here provide a characterization of each logic directly in terms of the inferences of K_3 and LP, without explicitly invoking shifting standards of truth between premises and conclusions. Proof-theoretically, because ST and TS are obtained directly via some operation on the logics K_3 and LP. For ST, in particular, one can determine whether Γ ST-entails Δ by constructing the K_3 disjunctive normal form Γ' of Γ and by checking if Γ' LP-entails Δ .

The present characterization is of interest more generally, in particular when considering the metainferential hierarchy discussed by [4]. Barrio and associates have shown that ST and TS are both only the first level of a more extended hierarchy of systems of mixed consequence. The system TS/ST, for example, is defined by letting $\Gamma \Rightarrow \Delta$ be in TS/ST provided every SK-valuation v either fails to TS-satisfy γ for some γ in Γ or ST-satisfies δ for some δ in Δ (with γ, δ inferences). This raises the question of whether TS/ST can also be defined as the relational product of TS and ST, and whether dual results can be obtained for the TS-hierarchy. As it turns out, the answer to this question is positive, but it would lie beyond the scope of this paper to examine the ST-hierarchy, and a presentation of these results and of their proof is postponed to another occasion.

Several questions remain open. The most pressing one for us concerns the possibility, discussed by [13], of defining the logic ST using other schemes than the Strong-Kleene scheme. Over those more extended schemes, does the characterization of ST in terms of relational product remain adequate? And can it be strengthened using the interpolation property in all cases? We have obtained some preliminary results on this matter, but more work needs to be done before we can answer these questions in full generality.

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