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# Asymptotic probability for connectedness

Thierry Monteil\*

Khaydar Nurligareev†

## Abstract

We study the structure of the asymptotic expansion of the probability that a combinatorial object is connected. We show that the coefficients appearing in those asymptotics are integers and can be interpreted as the counting sequences of other derivative combinatorial classes. The general result applies to rapidly growing combinatorial structures, which we call gargantuan, that also admit a sequence decomposition. The result is then applied to several models of graphs, of surfaces (square-tiled surfaces, combinatorial maps), and to geometric models of higher dimension (constellations, graph encoded manifolds). The corresponding derivative combinatorial classes are irreducible (multi)tournaments, indecomposable (multi)permutations and indecomposable perfect (multi)matchings.

## 1 Introduction

Many combinatorial structures with a topological flavour admit a decomposition into connected components. We are interested in the probability that an object of such a class is connected, when its size goes to infinity. More precisely, for a given size  $n \geq 0$ , we endow the subset of objects of size  $n$  in the class with the uniform probability, and then study the behavior of the probability that an object is connected, when  $n$  goes to infinity.

For example, the probability that a labeled graph with  $n$  vertices is connected tends to 1 as  $n$  goes to infinity [26]. The probability that a permutation of size  $n$  consists of a single cycle is equal to  $1/n$ , which goes to 0 as  $n$  goes to infinity (the expression of a permutation into the commutative product of cycles with disjoint support can be understood as a decomposition of the underlying graph into its connected components). In between, the probability that a labeled forest of size  $n$  is connected goes to  $1/\sqrt{e}$  as  $n$  goes to infinity [43].

The question of how to distinguish between these three cases was raised by Wright [48].

The main tool to deal with that question is the use of generating functions. To the counting sequence  $(\mathbf{a}_n)$  of a labeled combinatorial class  $\mathcal{A}$ , one can associate the *generating*

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function

$$A(z) = \sum_{n \geq 0} \mathbf{a}_n \frac{z^n}{n!}. \quad (1)$$

The fact that any element of  $\mathcal{A}$  can be uniquely decomposed into the disjoint union of its connected components translates into the following *exponential formula* (see e.g. [45], chapter 5):

$$A(z) = \exp(C(z)), \quad (2)$$

where  $C(z) = \sum_{n=0}^{\infty} \mathbf{c}_n \frac{z^n}{n!}$  is the generating function associated with the class  $\mathcal{C}$  of elements of  $\mathcal{A}$  that are connected. Equivalently,

$$C(z) = \log(A(z)), \quad (3)$$

so that the number  $\mathbf{c}_n$  of connected objects of size  $n$  only depends on the numbers  $\mathbf{a}_k$  of all objects of sizes  $k \leq n$ . Hence, case-by-case topological considerations can be avoided in estimating the probability  $\mathbf{c}_n/\mathbf{a}_n$ .

Wright showed [48, 49] the following equivalence:

$$\lim_{n \rightarrow \infty} (\mathbf{c}_n/\mathbf{a}_n) = 1 \quad \Leftrightarrow \quad \sum_{k=1}^{n-1} \binom{n}{k} \mathbf{a}_k \mathbf{a}_{n-k} = o(\mathbf{a}_n) \quad \Leftrightarrow \quad \sum_{k=1}^{n-1} \binom{n}{k} \mathbf{c}_k \mathbf{c}_{n-k} = o(\mathbf{c}_n)$$

Wright's approach was developed further [14, 10, 4, 1] and culminated in the paper of Bell, Bender, Cameron and Richmond [2] who proved the following characterization. Assuming that the limit  $\rho = \lim_{n \rightarrow \infty} (\mathbf{c}_n/\mathbf{a}_n)$  exists, its value is in the following correspondence with the radius of convergence  $R$  of the generating function  $C(z)$ :

$$\begin{aligned} \rho = 1 &\Leftrightarrow R = 0 \\ \rho = 0 &\Leftrightarrow R > 0 \text{ and } C(R) \text{ diverges} \\ 0 < \rho < 1 &\Leftrightarrow R > 0 \text{ and } C(R) \text{ converges} \end{aligned}$$

Moreover, they studied the case when the limit does not exist, and described the behavior of  $\limsup(\mathbf{c}_n/\mathbf{a}_n)$  and  $\liminf(\mathbf{c}_n/\mathbf{a}_n)$  depending on the parameter  $R$ .

The present paper focuses on the first case: the sequences  $(\mathbf{a}_n)$  and  $(\mathbf{c}_n)$  grow rapidly and the generating functions  $A(z)$  and  $C(z)$  have a radius of convergence  $R = 0$ . Our interest is not restricted to the limiting probability. One of our aims is to establish the full asymptotic expansion for the probability that a random object is irreducible. In other words, we wish to have an expression of the type

$$\frac{\mathbf{c}_n}{\mathbf{a}_n} = \rho \left( 1 + f_1(n) + f_2(n) + \dots + f_r(n) + o(f_r(n)) \right), \quad (4)$$

where  $f_{k+1}(n) = o(f_k(n))$  for every  $k \in \mathbb{N}$ , as  $n \rightarrow \infty$ .

If we look for example at labeled graphs, in 1959, Gilbert [26] provided the first nontrivial term of the asymptotic expansion:

$$p_n = 1 - \frac{2n}{2^n} + O\left(\frac{n^2}{2^{3n/2}}\right).$$

Eleven years later, Wright [50] provided the first three nontrivial terms:

$$p_n = 1 - \binom{n}{1} \frac{1}{2^{n-1}} - 2 \binom{n}{3} \frac{1}{2^{3n-6}} - 24 \binom{n}{4} \frac{1}{2^{4n-10}} + O\left(\frac{n^5}{2^{5n}}\right).$$

While Wright's method allows to compute more terms recursively, it does not provide a way to grasp the structure of the whole asymptotic expansion, if only because there is no interpretation to the coefficients  $1, 2, 24, \dots$ .

The goal of this paper is to provide the structure of the whole asymptotic expansion. As we shall see, when the sequence  $(\mathbf{a}_n)$  grows fast enough (see Definition 2.2 of a gargantuan class), the above-mentioned coefficients are integers that can be interpreted as the counting sequence  $(\mathbf{d}_n)$  of a derivative class  $\mathcal{D}$  that depends on the decomposition of the elements of  $\mathcal{A}$  into a sequence of elements of a class  $\mathcal{D}$ .

Our main result is:

**Theorem 4.1** (SET asymptotics). *Let  $\mathcal{A}$  be a gargantuan labeled combinatorial class with positive counting sequence, such that  $\mathcal{A} = \text{SET}(\mathcal{C}) = \text{SEQ}(\mathcal{D})$  for some labeled combinatorial classes  $\mathcal{C}$  and  $\mathcal{D}$ . Suppose that  $a \in \mathcal{A}$  is a random object of size  $n$ . Then*

$$\mathbb{P}(a \text{ is SET-irreducible}) \approx 1 - \sum_{k \geq 1} \mathbf{d}_k \cdot \binom{n}{k} \cdot \frac{\mathbf{a}_{n-k}}{\mathbf{a}_n}.$$

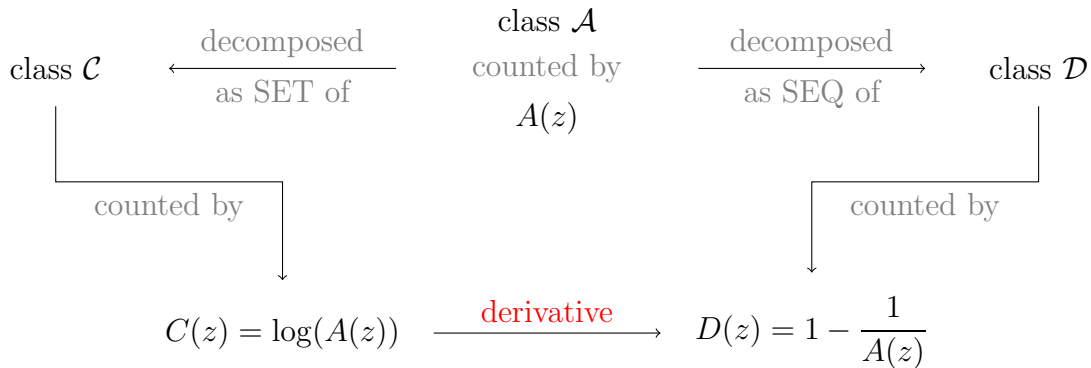
As an example of application, we get the following asymptotic for connected labeled graphs:

**Corollary 5.3.** *The asymptotic probability that a random labeled simple graph  $g$  with  $n$  vertices is connected satisfies*

$$\mathbb{P}(g \text{ is connected}) \approx 1 - \sum_{k \geq 1} \mathbf{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{kn}},$$

where  $\mathbf{it}_k$  denotes the number of irreducible tournaments of size  $k$ .

In this example, the number of irreducible tournaments plays the role of the derivative sequence. Irreducible tournaments come from the decomposition of a tournament into a sequence of irreducible ones. While the class of tournaments and the class of graphs are not isomorphic (e.g. when we swap the labels, the two tournaments of size 2 are exchanged, while the two graphs of size two are fixed), they share the same counting sequence, hence they can be identified from an enumerative combinatorics perspective. We could summarize the double decomposition as follows:



We will also list several other applications that include the connectedness for other models of graphs (Propositions 5.2, 5.6, 5.7), square-tiled surfaces (Proposition 5.10), combinatorial maps (5.15), as well as higher dimensional models of graph encoded manifolds (Proposition 5.17) and constellations (Proposition 5.21).

## 2 Tools

### 2.1 Asymptotic expansion

**Notation 2.1.** For a sequence  $(a_n)$  of numbers and an integer  $m$ , we write

$$a_n \approx \sum_{k \geq m} f_k(n) \tag{5}$$

if

$$\forall r \geq m, \quad a_n = \sum_{k=m}^r f_k(n) + O(f_{r+1}(n))$$

and

$$\forall k \geq m, \quad f_{k+1}(n) = o(f_k(n)).$$

The expression (5) is called an *asymptotic expansion* of the sequence  $(a_n)$ .

### 2.2 Combinatorial classes and decompositions

We use the standard notion of *labeled combinatorial structure (or class)*, as used in the textbooks [45, 23, 6]. Briefly, and to fix the notations, a combinatorial structure  $\mathcal{A}$  is a collection of objects of finite size such that for each integer  $n \geq 0$ , the number  $\mathbf{a}_n$  of objects of size  $n$  is finite. The class  $\mathcal{A}$  is *labeled* if the objects of size  $n$  are defined on a set of labels of cardinality  $n$ , whose elements are distinguishable (hence, the elements of  $\mathcal{A}$  have only trivial automorphisms). It is common to fix the ground set to  $[n] = \{1, \dots, n\}$ , but we should keep in mind that the nature of the ground set is irrelevant. The notion of combinatorial species [31, 5] makes this point precise. In this paper, every combinatorial structure is assumed labeled. To such a structure  $\mathcal{A}$ , we associate its generating function

$$A(z) = \sum_{n \geq 0} \mathbf{a}_n \frac{z^n}{n!}.$$

Combinatorial classes can be combined to create new classes: for example, we can construct the disjoint union or the labeled product of finitely many combinatorial classes. The corresponding generating functions are the sum and the product of the original ones, respectively. A combinatorial class can be combined with itself in various ways. We can form the *sequence* of a single class  $\mathcal{A}$  by considering the labeled products of  $\mathcal{A}$  by itself, involving arbitrary many terms. We denote  $\text{SEQ}(\mathcal{A})$  the corresponding class. Its generating function, denoted by  $\text{SEQ}(\mathcal{A})(z)$ , satisfies:

$$\text{SEQ}(\mathcal{A})(z) = \sum_{k \geq 0} A(z)^k = \frac{1}{1 - A(z)}.$$

The *set* of a single class  $\mathcal{A}$  is obtained from the sequence  $\text{SEQ}(\mathcal{A})$  by forgetting the order of the components. We denote  $\text{SET}(\mathcal{A})$  the corresponding class. Its generating function, denoted by  $\text{SET}(\mathcal{A})(z)$ , satisfies:

$$\text{SET}(\mathcal{A})(z) = \sum_{k \geq 0} \frac{A(z)^k}{k!} = \exp(A(z)).$$

The SET and SEQ constructions are only possible when  $A(0) = \mathbf{a}_0 = 0$ , in which case we have  $\text{SET}(\mathcal{A})(0) = 1$ .

To a combinatorial class  $\mathcal{A}$  and an integer  $n$  such that  $\mathbf{a}_n > 0$ , we endow the nonempty finite set  $\mathcal{A}_n$  of objects of size  $n$  with the uniform probability  $\mathbb{P}_n$ : each object of size  $n$  has probability  $1/\mathbf{a}_n$ . If  $Q$  is some property about objects of  $\mathcal{A}$ , we denote by

$$\mathbb{P}(a \text{ satisfies } Q)$$

the sequence of probabilities

$$(\mathbb{P}_n\{a \in \mathcal{A}_n \mid a \text{ satisfies } Q\})_{\mathbf{a}_n > 0}.$$

When the objects of a class  $\mathcal{A}$  have a topological flavour so that connectedness makes sense, we can consider the subclass  $\mathcal{C}$  of connected objects from  $\mathcal{A}$ . In such a case, when the connected components of objects from  $\mathcal{A}$  belong to  $\mathcal{C}$  and when the knowledge of the connected components of an object from  $\mathcal{A}$  is sufficient to reconstruct it, we have  $\mathcal{A} = \text{SET}(\mathcal{C})$ . In this case, we have  $C(z) = \log(A(z))$ , so that the probability

$$\mathbb{P}_n(a \text{ is connected}) = \mathbf{c}_n/\mathbf{a}_n$$

only depends on the counting sequence  $(\mathbf{a}_n)$ . Hence, we will identify labeled combinatorial classes if they share the same counting sequence (such classes are called “combinatorially isomorphic” in [23]). This identification is adapted to our objective and offers a lot of flexibility.

Note that the fact that objects of a combinatorial structure admit a topological model is not sufficient to state that it is the SET of its connected objects. For example, chord diagrams [46] and meandric systems [32] have a topological representation, but the knowledge of their connected components is not enough to reconstruct the original objects, since connected objects can be intertwined in several ways.

## 2.3 Gargantuan classes and Bender theorem

**Definition 2.2.** A sequence  $(a_n)$  is *gargantuan* if, for any integer  $r$ , as  $n \rightarrow \infty$ , the following two conditions hold:

$$(i) \quad \frac{a_{n-1}}{a_n} \rightarrow 0; \quad (ii) \quad \sum_{k=r}^{n-r} |a_k a_{n-k}| = O(a_{n-r}).$$

A labeled combinatorial class  $\mathcal{A}$  is *gargantuan* if its counting sequence  $(\mathbf{a}_n)$  is such that  $(\mathbf{a}_n/n!)$  is gargantuan.

**Theorem 2.3** (Bender [3], in the simplified form of [37]). *Consider the formal power series*

$$U(z) = \sum_{n=1}^{\infty} u_n z^n$$

and a function  $F(x)$  analytic in a neighborhood of the origin. Define

$$V(z) = \sum_{n=0}^{\infty} v_n z^n = F(U(z)) \quad \text{and} \quad W(z) = \sum_{n=0}^{\infty} w_n z^n = \left[ \frac{\partial}{\partial x} F(x) \right]_{x=U(z)}.$$

Assume that  $u_n \neq 0$  for any  $n > 0$  and the sequence  $(u_n)$  is gargantuan. Then

$$v_n \approx \sum_{k \geq 0} w_k u_{n-k}$$

and the sequence  $(v_n)$  is gargantuan.

In order to verify the gargantuan property in some applications, we will rely on the following sufficient condition.

**Lemma 2.4.** *If a sequence  $(a_n)$  satisfies the following two conditions*

- (i)'  $na_{n-1} = O(a_n)$ , as  $n \rightarrow \infty$ ;
- (ii)'  $x_k = |a_k a_{n-k}|$  is decreasing for  $k < n/2$  and for all but finitely many  $n$ ,

then  $(a_n)$  is gargantuan.

*Proof.* The first condition of Definition 2.2 immediately follows from (i)'. To check the second condition, note that the value of  $x_k$  is symmetric with respect to  $k = n/2$  and it is increasing for  $k < n/2$ . Hence, conditions (i)' and (ii)' imply that

$$\sum_{k=r}^{n-r} |a_k a_{n-k}| = \sum_{k=r}^{n-r} x_k = 2x_r + \sum_{k=r+1}^{n-r-1} x_k \leq 2x_r + (n-2r-1)x_{r+1} = O(a_{n-r}).$$

□

**Lemma 2.5.** *If  $(a_n)$  and  $(b_n)$  are two non-negative gargantuan sequences, then the sequence  $(a_n b_n)$  is non-negative gargantuan as well.*

*Proof.* Let us check the conditions of Definition 2.2. The first condition trivially holds. The second condition reads

$$\sum_{k=r}^{n-r} (a_k b_k)(a_{n-k} b_{n-r}) \leq \left( \sum_{k=r}^{n-r} a_k a_{n-k} \right) \left( \sum_{k=r}^{n-r} b_k b_{n-k} \right) = O(a_{n-r})O(b_{n-r}) = O(a_{n-r} b_{n-r}).$$

The condition  $a_n b_n \geq 0$  is trivial. □

### 3 Double SET/SEQ decomposition

The goal of this section is to see how the existence of a SEQ-decomposition is a natural hypothesis for the computation of an asymptotic expansion for the probability that an object is SET-irreducible. To this end, we will try to estimate this probability by a direct computation, under the hypothesis that the counting sequence of the studied class grows rapidly.

Let us start with two labeled combinatorial classes  $\mathcal{A}$  and  $\mathcal{C}$ , such that  $\mathcal{A} = \text{SET}(\mathcal{C})$ . Each object  $a \in \mathcal{A}$  of size  $n$  consists of several connected components that are considered as objects from  $\mathcal{C}$ . The sizes of these components determine a partition  $\lambda \vdash n$ , which can be expressed as an  $n$ -tuple  $p$  of non-negative integers counting the number of components for each size:  $p \in P_n = \{(p_1, \dots, p_n) \mid p_1 + 2p_2 + \dots + np_n = n\}$ . The exponential formula (2) can be expanded to obtain the following expression for  $\mathbf{a}_n$ :

$$\mathbf{a}_n = \sum_{p \in P_n} \frac{n!}{(1!)^{p_1} \dots (n!)^{p_n}} \cdot \frac{1}{p_1! \dots p_n!} \cdot (\mathbf{c}_1^{p_1} \dots \mathbf{c}_n^{p_n}). \quad (6)$$

If the sequence  $(\mathbf{a}_n)$  grows fast enough, then the radius of convergence of  $A(z)$  is zero, hence  $\mathbf{a}_n \sim \mathbf{c}_n$  [2]. Let us look at the asymptotic expansion of the difference  $(\mathbf{a}_n - \mathbf{c}_n)$ . For gargantuan sequences, this difference is decomposed into a sum of  $\mathbf{c}_{n-k}$ ,  $k = 1, 2, 3, \dots$ , taken with certain coefficients that can be deduced from (6). For instance, the first three terms of the asymptotics are

$$\mathbf{a}_n - \mathbf{c}_n = \binom{n}{1} \mathbf{c}_1 \mathbf{c}_{n-1} + \binom{n}{2} (\mathbf{c}_2 + \mathbf{c}_1^2) \mathbf{c}_{n-2} + \binom{n}{3} (\mathbf{c}_3 + 3\mathbf{c}_2 \mathbf{c}_1 + \mathbf{c}_1^3) \mathbf{c}_{n-3} + \dots$$

or, even simpler,

$$\mathbf{a}_n - \mathbf{c}_n = \binom{n}{1} \mathbf{a}_1 \mathbf{c}_{n-1} + \binom{n}{2} \mathbf{a}_2 \mathbf{c}_{n-2} + \binom{n}{3} \mathbf{a}_3 \mathbf{c}_{n-3} + O(n^4 \mathbf{c}_{n-4}).$$

Thus, a finite number of terms of the asymptotic expansion are determined by objects possessing a large connected component.

In practice, we suppose that  $(\mathbf{a}_n)$  is known, while  $(\mathbf{c}_n)$  is not. That is the reason why we would like to have an asymptotic expression in terms of  $\mathbf{a}_{n-k}$  rather than  $\mathbf{c}_{n-k}$ . In order to get one, let us apply the inclusion-exclusion principle.

The main idea is to count objects with respect to large components, replacing  $\mathbf{c}_{n-k}$  by  $\mathbf{a}_{n-k}$  that is asymptotically the same. For instance, if we are interested only in the leading term of  $(\mathbf{a}_n - \mathbf{c}_n)$ , as  $n \rightarrow \infty$ , then it is sufficient to replace  $\mathbf{c}_{n-1}$  by  $\mathbf{a}_{n-1}$ :

$$\mathbf{a}_n - \mathbf{c}_n = \binom{n}{1} \mathbf{c}_1 \mathbf{a}_{n-1} + O(n^2 \mathbf{a}_{n-2}).$$

Combinatorially, the term  $\binom{n}{1} \mathbf{c}_1 \mathbf{a}_{n-1}$  counts objects of size  $n$  with at least one isolated vertex. Any object with  $k$  isolated vertices is counted  $k$  times, since we mark one of them. However, the contribution of that object is significantly smaller, and hence, can be ignored.



When the second term is needed, we must accurately count objects possessing connected components of size at least  $(n - 2)$ . Therefore, we need to deduce objects with connected components of sizes  $(n - 2)$ , 1 and 1, which are counted twice in the first approximation:

$$\mathbf{a}_n - \mathbf{c}_n = \binom{n}{1} \mathbf{c}_1 \mathbf{a}_{n-1} + \binom{n}{2} (\mathbf{c}_2 - \mathbf{c}_1^2) \mathbf{a}_{n-2} + O(n^3 \mathbf{a}_{n-3}).$$

Again, combinatorially, objects with several components of size 1 or 2 are counted several times within these two terms, but their contribution is  $O(n^3 \mathbf{a}_{n-3})$ .

The same approach works in searching for more terms of the asymptotic expansion. Indeed, given a positive integer  $n$ , for any partition  $\lambda = (\lambda_1, \dots, \lambda_{l(\lambda)})$  of an integer  $k \leq n$ , define  $B_{\lambda,n}$  to be the number of objects with marked connected components of sizes  $\lambda_1, \lambda_2, \dots, \lambda_{l(\lambda)}$ . For example, if  $n = 3$ ,  $k = 2$  and  $\lambda = (1, 1)$ , then  $B_{(1,1),3}$  is three times the number of objects of size 3 with 3 isolated vertices:  $B_{(1,1),3} = 3\mathbf{c}_1^3$ . In the general case,  $B_{\lambda,n}$  satisfies

$$B_{\lambda,n} = \binom{n}{k} \frac{k!}{(1!)^{p_1} \dots (k!)^{p_k}} \frac{\mathbf{c}_1^{p_1} \dots \mathbf{c}_k^{p_k}}{p_1! \dots p_k!} \mathbf{a}_{n-k}$$

and, for a given number  $r$  of terms of the asymptotic expansion, the inclusion-exclusion principle provides

$$\mathbf{a}_n - \mathbf{c}_n = \sum_{\lambda: |\lambda| < r} (-1)^{l(\lambda)-1} B_{\lambda,n} + O(n^r \mathbf{a}_{n-r}). \quad (7)$$

Let us factor out  $\binom{n}{k} \mathbf{a}_{n-k}$  in the relation (7). The result is of the form

$$\mathbf{a}_n - \mathbf{c}_n \approx \sum_{k \geq 1} \binom{n}{k} \mathfrak{d}_k \mathbf{a}_{n-k},$$

where

$$\mathfrak{d}_k = \sum_{P_k} (-1)^{(p_1 + \dots + p_k) - 1} \cdot \frac{k!}{(1!)^{p_1} \dots (k!)^{p_k}} \cdot \frac{\mathbf{c}_1^{p_1} \dots \mathbf{c}_k^{p_k}}{p_1! \dots p_k!}. \quad (8)$$

It turns out that these coefficients might have a combinatorial meaning on their own. In order to make this fact clear, let us apply the inclusion-exclusion principle, so that we obtain an exact formula for  $\mathbf{a}_n$ :

$$\mathbf{a}_n = \sum_{\lambda: |\lambda| \leq n} (-1)^{l(\lambda)-1} B_{\lambda,n}$$

or

$$\mathbf{a}_n = \sum_{k=1}^n \mathfrak{d}_k \binom{n}{k} \mathbf{a}_{n-k}.$$

If  $D(z)$  denotes the power series  $D(z) = \sum_{n \geq 1} \mathfrak{d}_n z^n / n!$ , the last equation can be rewritten as

$$A(z) = 1 + A(z)D(z).$$

Equivalently,

$$A(z) = \frac{1}{1 - D(z)},$$

so that if we could find a labeled combinatorial class  $\mathcal{D}$  satisfying  $\mathcal{A} = \text{SEQ}(\mathcal{D})$ , then  $(\mathfrak{d}_n)$  would be the counting sequence of  $\mathcal{D}$ .

This leads us to the following definition:

**Definition 3.1.** A combinatorial class  $\mathcal{A}$  admits a double SET/SEQ decomposition if there exist two classes  $\mathcal{C}$  and  $\mathcal{D}$  such that  $\mathcal{A} = \text{SET}(\mathcal{C}) = \text{SEQ}(\mathcal{D})$ . The class  $\mathcal{D}$  is called the *derivative class* of  $\mathcal{A}$  and the counting sequence  $(\mathfrak{d}_n)$  is called the *derivative sequence* of the sequence  $(\mathfrak{a}_n)$ .

## 4 Main results

**Theorem 4.1** (SET asymptotics). *Let  $\mathcal{A}$  be a gargantuan labeled combinatorial class with positive counting sequence, such that  $\mathcal{A} = \text{SET}(\mathcal{C}) = \text{SEQ}(\mathcal{D})$  for some labeled combinatorial classes  $\mathcal{C}$  and  $\mathcal{D}$ . Suppose that  $a \in \mathcal{A}$  is a random object of size  $n$ . Then*

$$\mathbb{P}(a \text{ is SET-irreducible}) \approx 1 - \sum_{k \geq 1} \mathfrak{d}_k \cdot \binom{n}{k} \cdot \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}. \quad (9)$$

*Proof.* Let us consider the labeled combinatorial class  $\mathcal{U} = \mathcal{A} - \{\epsilon\}$  with generating function  $U(z)$ , and the function

$$F(y) = \log(1 + y).$$

Since  $\mathcal{U}$  is gargantuan and  $F$  is analytic at the origin, we can apply Theorem 2.3. We have

$$V(z) = \log(A(z)) = C(z)$$

and

$$W(z) = \left[ \frac{\partial}{\partial x} F(x) \right]_{x=A(z)-1} = \frac{1}{A(z)} = 1 - D(z).$$

Hence,

$$v_n \approx \sum_{k \geq 0} w_k u_{n-k}.$$

By substituting with the original counting sequences, we have

$$\frac{\mathfrak{c}_n}{n!} \approx \frac{\mathfrak{a}_n}{n!} - \sum_{k \geq 1} \frac{\mathfrak{d}_k}{k!} \frac{\mathfrak{a}_{n-k}}{(n-k)!}.$$

Dividing by  $\frac{\mathfrak{a}_n}{n!}$ , we obtain

$$\frac{\mathfrak{c}_n}{\mathfrak{a}_n} \approx 1 - \sum_{k \geq 1} \binom{n}{k} \mathfrak{d}_k \frac{\mathfrak{a}_{n-k}}{\mathfrak{a}_n}.$$

□

**Remark 4.2.** Theorem 4.1 provides, for each  $k$ , a convergence result when  $n$  goes to infinity. If we fix  $n$  in the right-hand side of (9), the sum will be 0.

**Remark 4.3.** In the proof of Theorem 4.1, the relationship between the SET and the SEQ decompositions is witnessed by the fact that

$$\frac{\partial \log(z)}{\partial z} = \frac{1}{z}.$$

This fact also supports the name “derivative sequence”.

## 4.1 Asymptotics for $p$ -periodic sequences

Counting sequences of certain combinatorial classes possess zeroes. For example, perfect matchings only exist when the number of vertices is even. Similarly, some combinatorial structures admit elements only when the number of vertices is a multiple of  $p$  for some integer  $p > 1$ .

**Definition 4.4.** Let  $p \geq 1$  be an integer. A sequence  $(\mathbf{a}_n)$  is  $p$ -periodic if:

$$\mathbf{a}_n \neq 0 \quad \Leftrightarrow \quad \exists k \in \mathbb{N}, \quad n = pk.$$

This situation is quite general for a combinatorial class  $\mathcal{A}$  that admits a SET decomposition  $\mathcal{A} = \text{SET}(\mathcal{C})$ : if its counting sequence  $(\mathbf{a}_n)$  vanishes for infinitely many  $n$ , then the sequence  $(\mathbf{c}_n)$  is  $p$ -periodic for some  $p > 1$  [48, 49]. In particular, both sequences  $(\mathbf{a}_{pn})$  and  $(\mathbf{c}_{pn})$  are eventually positive.

Let us adapt Theorem 4.1 for labeled combinatorial classes whose counting sequences are  $p$ -periodic. This will be useful in Sections 5.2.2 and 5.3.1.

**Proposition 4.5.** Let  $\mathcal{A}$  be a labeled combinatorial class, such that  $\mathcal{A} = \text{SET}(\mathcal{C}) = \text{SEQ}(\mathcal{D})$  for some labeled combinatorial classes  $\mathcal{C}$  and  $\mathcal{D}$ . Suppose that the counting sequence  $(\mathbf{a}_n)$  is  $p$ -periodic for some  $p \geq 1$ , and that the sequence  $(\mathbf{a}_{pn}/(pn)!)$  is gargantuan. Suppose also that  $a \in \mathcal{A}$  is a random object of size  $pn$ . Then

$$\mathbb{P}(a \text{ is SET-irreducible}) \approx 1 - \sum_{k \geq 1} \mathfrak{d}_{pk} \cdot \binom{pn}{pk} \cdot \frac{\mathbf{a}_{p(n-k)}}{\mathbf{a}_{pn}}. \quad (10)$$

*Proof.* Conditions of the proposition ensure that we can apply Theorem 2.3 to the formal power series  $U(z) = A(z^{1/p}) - 1$  and the function  $F(y) = \log(1 + y)$ . Repeating the reasoning of the proof of Theorem 4.1 leads to (10).  $\square$

# 5 Applications

## 5.1 Graphs

Graphs are versatile objects, the asymptotic probability for the connectedness of several variants can be approached with our methods.

**Definition 5.1.** For  $d \geq 1$ , a (labeled)  $d$ -multigraph of size  $n$  is a graph defined on the set of vertices  $[n]$ , such that any pair of distinct vertices are joined by at most  $d$  indistinguishable edges. A (labeled)  $d$ -multitournament of size  $n$  is a directed graph defined on the set of vertices  $[n]$ , such that any ordered pair of distinct vertices  $i \neq j$  is joined by  $d$  directed edges (which consist of  $l$  indistinguishable directed edges from  $i$  to  $j$  and  $d - l$  indistinguishable directed edges from  $j$  to  $i$  for some  $0 \leq l \leq d$  that varies from pair to pair). A *simple graph* is a 1-multigraph, a *tournament* is a 1-multitournament. A  $d$ -multitournament is *reducible* if there exists a partition of its vertices into two nonempty parts  $A$  and  $B$  such that any pair of vertices  $(a, b) \in A \times B$  are joined by  $d$  oriented edges that all go from  $a$  to  $b$ . A  $d$ -multitournament is *irreducible* if it is not reducible. We denote by  $\mathcal{G}(d)$ ,  $\mathcal{CG}(d)$ ,  $\mathcal{T}(d)$ , and  $\mathcal{IT}(d)$  the combinatorial classes of  $d$ -multigraphs, connected  $d$ -multigraphs,  $d$ -multitournaments, and irreducible  $d$ -multitournaments, respectively.

**Proposition 5.2.** *The asymptotic probability that a random labeled  $d$ -multigraph  $g$  with  $n$  vertices is connected satisfies*

$$\mathbb{P}(g \text{ is connected}) \approx 1 - \sum_{k \geq 1} \mathbf{it}_k(d) \cdot \binom{n}{k} \cdot \frac{(d+1)^{k(k+1)/2}}{(d+1)^{kn}}, \quad (11)$$

where  $\mathbf{it}_k(d)$  denotes the number of irreducible  $d$ -multitournaments of size  $k$ .

*Proof.* First, we have  $\mathcal{G}(d) = \mathcal{T}(d)$  since both classes have  $(d+1)^{\binom{n}{2}}$  objects of size  $n$  for any  $n$ . The decomposition  $\mathcal{G}(d) = \text{SET}(\mathcal{CG}(d))$  follows from the decomposition of a graph into its connected components. To prove that  $\mathcal{T}(d) = \text{SEQ}(\mathcal{IT}(d))$ , let us consider a  $d$ -multitournament  $t$ . We decompose  $t$  into its irreducible components (note that the irreducible components of  $t$  are also its strongly connected components [42, 44]). If  $A$  and  $B$  are distinct components, we write  $A < B$  if for every pair of vertices  $a \in A$  and  $b \in B$ , every edge between  $a$  and  $b$  is oriented from  $a$  to  $b$ . We check that  $<$  is a linear order on the set of irreducible components of  $t$ , so that we can enumerate them as  $t_1 < t_2 < \dots < t_l$  and construct the sequence of irreducible  $d$ -multitournaments  $(t_1, \dots, t_l)$  out of  $t$ . Conversely, if  $(t_1, \dots, t_l)$  is a (labeled) sequence of irreducible  $d$ -multitournaments, we can construct a  $d$ -multitournament  $t$  as the disjoint union of the  $t_i$  and then adding  $d$  oriented edges from any vertex of  $t_i$  to any vertex of  $t_j$  if  $i < j$ .

Let us show that the sequence

$$a_n = \frac{(d+1)^{\binom{n}{2}}}{n!}$$

is gargantuan by applying Lemma 2.4. Condition (i)' follows from

$$\frac{a_{n-1}}{a_n} = \frac{(d+1)^{\binom{n-1}{2}}}{(n-1)!} \cdot \frac{n!}{(d+1)^{\binom{n}{2}}} = \frac{n}{(d+1)^n} \rightarrow 0.$$

Regarding condition (ii)', let us check that the sequence  $x_k = a_k a_{n-k}$  is decreasing for  $k < n/2$ . To this end, let us consider

$$\frac{x_{k+1}}{x_k} = \frac{a_{k+1} a_{n-k-1}}{a_k a_{n-k}} = \frac{(n-k)}{(k+1)} \cdot \frac{(d+1)^{k+1}}{(d+1)^{n-k}}.$$

Since the function

$$f(x) = \frac{(d+1)^x}{x}$$

is increasing for large  $x$ , we have the following equivalences for  $n$  large enough:

$$\frac{x_{k+1}}{x_k} \leq 1 \quad \Leftrightarrow \quad \frac{(d+1)^{k+1}}{(k+1)} \leq \frac{(d+1)^{n-k}}{(n-k)} \quad \Leftrightarrow \quad k+1 \leq n-k.$$

Hence,  $(x_k)$  is decreasing for  $k < n/2$ . According to Lemma 2.4,  $(a_n)$  is gargantuan.

Therefore, we can apply Theorem 4.1:

$$\begin{aligned} \mathbb{P}(g \text{ is SET-irreducible}) &\approx 1 - \sum_{k \geq 1} \mathbf{it}_k(d) \cdot \binom{n}{k} \cdot \frac{\mathfrak{g}_{n-k}(d)}{\mathfrak{g}_n(d)} \\ &\approx 1 - \sum_{k \geq 1} \mathbf{it}_k(d) \cdot \binom{n}{k} \cdot \frac{(d+1)^{k(k+1)/2}}{(d+1)^{kn}}. \end{aligned}$$

□

**Corollary 5.3.** *The asymptotic probability that a random labeled simple graph  $g$  with  $n$  vertices is connected satisfies*

$$\mathbb{P}(g \text{ is connected}) \approx 1 - \sum_{k \geq 1} \mathbf{it}_k \cdot \binom{n}{k} \cdot \frac{2^{k(k+1)/2}}{2^{kn}}, \quad (12)$$

where  $\mathbf{it}_k$  denotes the number of irreducible tournaments of size  $k$ .

*Proof.* Apply Proposition 5.2 with  $d = 1$ . □

**Remark 5.4.** The counting sequence of irreducible tournaments (A054946 in the OEIS [38]) is

$$(\mathbf{it}_k) = 1, 0, 2, 24, 544, 22\,320, 1\,677\,488, 236\,522\,496, 64\,026\,088\,576, \dots$$

Hence, Corollary 5.3 is consistent with the result of Wright [50]:

$$\mathbb{P}(g \text{ is connected}) = 1 - \binom{n}{1} \cdot 2^{1-n} - 2 \cdot \binom{n}{3} \cdot 2^{6-3n} - 24 \cdot \binom{n}{4} \cdot 2^{10-4n} + O(n^5 \cdot 2^{-5n}).$$

The fact that there are no irreducible tournaments of size 2 is witnessed by the fact that the second nontrivial term (in  $2^{-2n}$ ) is hidden.

**Remark 5.5.** Theorem 4.1 can be applied to several other models of graphs. For example [36, Statement 10.1.18] deals with multigraphs with distinguished edges (which is essentially Proposition 5.2 with  $d = 2^k - 1$ ). We can also apply it to estimate the probability that a directed graph is weakly connected. Let us recall that a *digraph* of size  $n$  is a directed graph defined on the set of vertices  $[n]$ , such that any ordered pair of distinct vertices  $i \neq j$  is joined by at most one directed edge in each direction. An *oriented graph* is a digraph with at most one oriented edge between two distinct vertices. We have:

**Corollary 5.6.** *The asymptotic probability that a random labeled oriented graph  $g$  with  $n$  vertices is weakly connected satisfies*

$$\mathbb{P}(g \text{ is weakly connected}) \approx 1 - \sum_{k \geq 1} \mathbf{it}_k(2) \cdot \binom{n}{k} \cdot \frac{3^{k(k+1)/2}}{3^{kn}}, \quad (13)$$

where  $\mathbf{it}_k(2)$  denotes the number of irreducible 2-multitournaments of size  $k$ .

*Proof.* Apply Proposition 5.2 with  $d = 2$ . The class of oriented graphs can be identified with the class of 2-multigraphs, since in both cases, there are 3 ways to link two distinct vertices.  $\square$

**Corollary 5.7.** *The asymptotic probability that a random digraph  $g$  with  $n$  vertices is weakly connected satisfies*

$$\mathbb{P}(g \text{ is weakly connected}) \approx 1 - \sum_{k \geq 1} \mathbf{it}_k(3) \cdot \binom{n}{k} \cdot \frac{4^{k(k+1)/2}}{4^{kn}}, \quad (14)$$

where  $\mathbf{it}_k(3)$  denotes the number of irreducible 3-multitournaments of size  $k$ .

*Proof.* Apply Proposition 5.2 with  $d = 3$ .  $\square$

Note however that Theorem 4.1 is not adapted to estimating the probability that a directed graph is strongly connected. Indeed, a digraph can be decomposed as a directed acyclic graph (DAG) of its strongly connected components, not a SET. The ideas presented here can however be adapted in such a more complex context [19].

## 5.2 Surfaces

### 5.2.1 Square-tiled surfaces

Square-tiled surfaces play a key role in the study of abelian and quadratic differentials on Riemann surfaces. They first appeared in disguise in the proof by Douady and Hubbard that among quadratic differentials, the ones that can be decomposed into a disjoint union of vertical cylinders (Jenkins–Strebel forms) is dense [18]. The dense subset they used is the set of quadratic differentials whose (relative) periods have rational coordinates, which turn out to be square-tiled surfaces. Beyond density, Eskin and Okounkov used the uniform distribution of such surfaces to estimate the volume of the strata of abelian [20] and quadratic [21] differentials (the computation of explicit values was implemented by Goujard [27]). Square-tiled surfaces also provide examples of differentials with exceptional behavior, such as the “Eierlegende Wollmilchsau” [30] or the “Ornithorynque” [24].

**Definition 5.8.** A *quadratic square-tiled surface* is obtained from finitely many squares by identifying pairs of sides of the squares by isometries in such way that horizontal sides are glued to horizontal sides and vertical sides to vertical. An *abelian square tiled surface* (also known as *origami*) is obtained by gluing finitely many squares such that left edges are glued to right edges and top edges are glued to bottom edges. Equivalently, an origami of size  $n$  is a pair of permutations  $(\sigma, \tau)$  on the ground set  $[n]$ , where  $\sigma(i) = j$  if the right side of the square labeled  $i$  is glued to the left side of the square labeled  $j$ ,

and  $\tau(i) = k$  if the top side of the square labeled  $i$  is glued to the bottom side of the side labeled  $k$  [51]. We denote by  $\mathcal{O}$  and  $\mathcal{CO}$  the combinatorial classes of origamis and connected origamis, respectively.

In particular, there are  $((2n - 1)!)^2$  labeled quadratic square-tiled surface of size  $n$ , and  $(n!)^2$  origamis of size  $n$ .

Note that, in contrast with the historical references that focus on a fixed stratum, we consider a model with unconstrained genus: every origami involving  $n$  labeled squares appears with the same probability (generically, the genus grows in  $\Theta(n)$ ).

**Definition 5.9.** A permutation  $\sigma$  on  $[n]$  is *indecomposable* if there is no  $k < n$  such that  $\sigma([k]) = [k]$ . The class of indecomposable permutations is denoted by  $\mathcal{IP}$ . For an integer  $d \geq 1$ , a  $d$ -multiple linear order of size  $n$  is a  $d$ -tuple of linear orders of the set  $[n]$ . A  $d$ -multiple linear order  $L = (\langle_1, \dots, \langle_d)$  is *reducible*, if there is a non-trivial partition  $A \sqcup B$  of the ground set  $[n]$  such that  $a \langle_k b$  for every pair  $(a, b) \in A \times B$  and any order  $\langle_k$ . Otherwise, we call  $L$  *irreducible*. We denote by  $\mathcal{L}(d)$  and  $\mathcal{IL}(d)$  the combinatorial classes of  $d$ -multiple linear orders and irreducible  $d$ -multiple linear order, respectively.

**Proposition 5.10.** *The asymptotic probability that an origami  $o$  made up of  $n$  unit squares is connected satisfies*

$$\mathbb{P}(o \text{ is connected}) \approx 1 - \sum_{k \geq 1} \frac{\mathbf{il}_k(2)}{k! \cdot (n)_k} \quad (15)$$

$$\approx 1 - \sum_{k \geq 1} \frac{\mathbf{ip}_k}{(n)_k}, \quad (16)$$

where  $\mathbf{il}_k(2)$  is the number of irreducible pairs of linear orders of size  $k$ , and  $\mathbf{ip}_k$  is the number of indecomposable permutations of size  $k$ .

*Proof.* Since an origami can be decomposed into the disjoint union of its connected components, we have  $\mathcal{O} = \text{SET}(\mathcal{CO})$ . Since each origami is determined by a pair of permutations, its counting sequence is  $\mathbf{o}_n = (n!)^2$ . Hence,  $\mathcal{O} = \mathcal{L}(2)$ , since the number of pairs of linear orders of size  $n$  is also  $(n!)^2$ . This change of perspective, from permutations to linear orders, allows to decompose with respect to the SEQ construction: we have  $\mathcal{L}(2) = \text{SEQ}(\mathcal{IL}(2))$ .

Let us show that the sequence  $a_n = (n!)^2/n! = n!$  is gargantuan. Indeed, the first condition of Definition 2.2 trivially holds. To check the second condition, note that  $a_k a_{n-k} \geq a_{k+1} a_{n-k-1}$  as long as  $k \leq (n-1)/2$ . Note also, that  $n a_{n-r-1} = O(a_{n-r})$ . Hence, replacing  $a_k a_{n-k}$  by  $a_{r+1} a_{n-r-1}$  for all  $k$  such that  $r < k < n-r$ , we have

$$\sum_{k=r}^{n-r} a_k a_{n-k} \leq 2a_r a_{n-r} + (n-2r+1)a_{r+1} a_{n-r-1} = O(a_{n-r}).$$

Thus, Theorem 4.1 is applicable and gives the following asymptotics:

$$\mathbb{P}(o \text{ is SET-irreducible}) \approx 1 - \sum_{k \geq 1} \mathbf{il}_k(2) \cdot \binom{n}{k} \cdot \frac{\mathbf{o}_{n-k}(d)}{\mathbf{o}_n(d)} = 1 - \sum_{k \geq 1} \frac{\mathbf{il}_k(2)}{k! \cdot (n)_k}.$$

The second asymptotics follows from (see Remark 5.11)

$$\mathfrak{ip}_n = \frac{\mathfrak{il}_n(2)}{n!}. \quad (17)$$

□

**Remark 5.11.** The nature of the elements of the ground set  $[n]$  plays a role in the definition of indecomposable permutations, as those labels need to be compared with each other. The ground set both serves as a label set and as a way to compare elements: the notion of indecomposable permutation is not stable by relabeling. In particular, it is not possible to construct the product or the sequence of indecomposable permutations, since this operation involves relabelling (how to define the sequence of the three irreducible permutations 321, 21, 3142?). The class  $\mathcal{IP}$  can therefore not be considered as a combinatorial class as its definition is not functorial (we can not transport its structure to any label set). We can see the class  $\mathcal{IL}(2)$  as a *lift* of the class  $\mathcal{IP}$  that consists in decoupling the role of the ground set  $[n]$  as a label set on the one hand, and as a linear order on the other. The first order  $<_1$  can be interpreted as the linear order of the ground set  $1 < 2 < \dots < n$ , and the second as the linear order of the image by the permutation  $\sigma(1) < \sigma(2) < \dots < \sigma(n)$ . The  $n!$  available relabelings explain the relation (17).

**Remark 5.12.** Proposition 5.10 can be seen as a topological version of results of Comtet, Dixon and Cori. Indeed, the connectedness of an origami  $o$  of size  $n$  is equivalent to the fact that the two permutations in the pair  $(\sigma, \tau)$  determining  $o$  generate a transitive subgroup of  $S_n$ . In 2005, Dixon [17] showed that the asymptotic probability  $t_n$  that two permutations of size  $n$  form a transitive subgroup satisfies

$$t_n \sim 1 - \frac{1}{(n)_1} - \frac{1}{(n)_2} - \frac{3}{(n)_3} - \frac{13}{(n)_4} - \frac{71}{(n)_5} - \frac{461}{(n)_6} - \dots,$$

where the numerators are coefficients of the multiplicative inverse of the formal power series comprised by factorials. The latter, as it was showed by Comtet [15] in 1972, count indecomposable permutations. Alternatively, this fact was established by Cori [16] in 2009.

### 5.2.2 $p$ -angulations and combinatorial maps

There are several models of random discrete oriented surfaces. The simplest model consists of oriented triangles glued together along their edges. It was introduced in 2004 by Brooks and Makover [8] in order to study the “typical” Riemann surfaces with high genus, and independently by Pippenger and Schleich [41] who were motivated by the needs of quantum gravity. Pippenger and Schleich showed that a random gluing of  $2n$  triangles form a connected surface with probability

$$1 - \frac{5}{36n} + O\left(\frac{1}{n^2}\right) \quad (18)$$

and studied their topological characteristics. Gamburd [25] generalized this model by replacing triangles by  $p$ -gons for a fixed integer  $p \geq 3$ . Then, Chmutov and Pittel [13]



extended the model to polygons whose set of allowed perimeters is a given nonempty subset  $J$  of  $\mathbb{N}_{\geq 3}$ , and showed that the probability to get a connected surface always satisfies

$$1 - O\left(\frac{1}{n}\right).$$

Finally, in 2019, Budzinski, Curien and Petri [9] considered a model without any constraint on the perimeters of the polygons (1-gons and 2-gons are permitted), and found that the probability to obtain a connected surface is

$$1 - \frac{1}{n} + O\left(\frac{1}{n^2}\right). \quad (19)$$

The main goal of the above authors was to study topological characteristics of the proposed models, such as their Euler characteristic, genus, diameter, etc. In all the mentioned models, glued polygons are assumed to be oriented and the identified sides are oriented opposite-wise. Let us focus on this last model studied by Budzinski, Curien and Petri.

**Definition 5.13.** A *perfect matching* is a permutation whose orbits have length 2. We denote by  $\mathcal{IM}$  the class of indecomposable perfect matchings. A *combinatorial map* of size  $n$  is obtained from a collection of polygons whose set of sides is labeled by integers from 1 to  $n$ , by identifying in pairs the sides of these polygons. Equivalently, a combinatorial map of size  $n$  is a pair of permutations  $(\sigma, \alpha)$  on  $[n]$  such that  $\alpha$  is a perfect matching. The orbits of  $\sigma$  correspond to the enumeration of the sides of the polygons in the cyclic order. The orbits of  $\alpha$  correspond to the identification of the edges. We denote by  $\mathcal{CM}$  and  $\mathcal{CCM}$  the classes of combinatorial maps and connected combinatorial maps, respectively.

There are  $n!(n-1)!!$  combinatorial maps of size  $n$  when  $n$  is even. There is no combinatorial map of odd size because the sides of the polygons they are made of are identified in pairs. The elements of the ground set  $[n]$  are sometimes called “darts” or “half-edges”.

**Definition 5.14.** A *linear matching* is a pair of linear orders  $(\langle_1, \langle_2) \in \mathcal{L}(2)$  such that the transform

$$(\langle_1, \langle_2) \mapsto (\langle_2, \langle_1)$$

coincides with some relabeling  $(1, \dots, n) \mapsto (1, \dots, n)$  that is an involution without any fixed point. We denote by  $\mathcal{LM}(2)$  and  $\mathcal{ILM}(2)$  the classes of linear matchings and irreducible linear matchings, respectively.

**Proposition 5.15.** *The asymptotic probability that a random combinatorial map  $m \in \mathcal{CM}$  of size  $2n$  is connected satisfies*

$$\mathbb{P}(m \text{ is connected}) \approx 1 - \sum_{k \geq 1} \frac{\mathbf{ilm}_{2k}}{(2k)!} \cdot \frac{(2(n-k)-1)!!}{(2n-1)!!} \quad (20)$$

$$\approx 1 - \sum_{k \geq 1} \mathbf{im}_{2k} \cdot \frac{(2(n-k)-1)!!}{(2n-1)!!}, \quad (21)$$

where  $\mathbf{ilm}_k$  is the number of irreducible linear matchings, and  $\mathbf{im}_k$  is the number of indecomposable perfect matchings of size  $k$ .

*Proof.* We have  $\mathcal{CM} = \mathcal{LM}(2)$ , since the 2-periodic counting sequence of both classes is  $n! \cdot (n-1)!! \cdot \mathbb{I}_{\{n \text{ is even}\}}$ . As in the proof of Proposition 5.10, we have  $\mathcal{CM} = \text{SET}(\mathcal{CCM})$  and  $\mathcal{LM}(2) = \text{SEQ}(\mathcal{ILM}(2))$ .

Let us show that the sequence

$$a_n = \frac{(2n)! \cdot (2n-1)!!}{(2n)!} = (2n-1)!!$$

is gargantuan. To this end, we apply Lemma 2.4. The first condition holds, since

$$na_{n-1} = n(2n-3)!! = O((2n-1)!!) = O(a_n).$$

Regarding the second condition, for  $k \leq (n-1)/2$  we have

$$\frac{x_{k+1}}{x_k} = \frac{(2k+1)!!(2n-2k-3)!!}{(2k-1)!!(2n-2k-1)!!} = \frac{2k+1}{2n-2k-1} \leq 1.$$

Therefore,  $(x_k)$  is decreasing for  $k < n/2$ , and the sequence  $(a_n)$  is gargantuan.

Hence, we can apply Proposition 4.5 to obtain

$$\mathbb{P}(s \text{ is connected}) \approx 1 - \sum_{k \geq 1} \frac{\mathbf{ilm}_{2k}}{(2k)!} \cdot \frac{(2(n-k)-1)!!}{(2n-1)!!}.$$

The second asymptotics follows from  $\mathbf{ilm}_n(2) = n! \cdot \mathbf{im}_n$ , as explained in Remark 5.11 for (17).  $\square$

## 5.3 Higher dimensions

### 5.3.1 Graph Encoded Manifolds

GEMs, for Graph Encoded Manifolds, were introduced in the framework of “crystallization theory” as a way to encode compact PL-manifolds [40, 39, 22]. Those objects recently attracted attention of theoretical physicists, where GEMs encode colored tensor models, seen as quantum gravity theories [28, 7, 29, 47].

**Definition 5.16.** Let us fix a dimension  $D \geq 2$  and a size  $n \geq 1$ . A simplex of dimension  $D$  has its vertices colored from 1 to  $D+1$ . A *GEM of dimension  $D$*  of size  $n$  is a simplicial complex obtained from  $n$  copies of the simplex glued according to the following rule: for each  $1 \leq k \leq D+1$ , consider a perfect matching  $\alpha_k$  of  $[n]$  and glue the hyperface opposite to the  $k$ th vertex of the  $i$ th simplex to the hyperface that opposite to the  $k$ th vertex of the  $\alpha_k(i)$ th simplex. We denote by  $\mathcal{GEM}(D)$  the class of GEMs of dimension  $D$ . Equivalently, a *GEM of dimension  $D$*  and size  $n$  is a graph on  $[n]$  vertices, where each vertex has a degree  $(D+1)$  and each edge is colored into one of  $D+1$  colors such that every vertex is incident to exactly one edge of each color.

A GEM has an even number of simplices. Every compact PL-manifold can be represented as a GEM. A GEM is orientable if, and only if, the associated graph is bipartite [12]: there exists a partition of the simplices into two sets  $A$  and  $B$  of size  $n/2$  such that the

simplices in  $A$  the are glued to simplices of  $B$ . We denote  $\mathcal{OGEM}(D)$  and  $\mathcal{COGEM}(D)$  the class of orientable GEMs and connected orientable GEMs of dimension  $D$ , respectively. In the colored tensor model, orientable GEMs of dimension  $D$  are called *closed  $(D+1)$ -colored graphs*. There are  $((2n-1)!!)^{D+1}$  GEMs and  $\binom{2n}{n}(n!)^{D+1} = (2n)!(n!)^{D-1}$  orientable GEMs of dimension  $D$  and size  $2n$ .

In 2019, within the framework of the colored tensor model, Carrance [11] showed that a random orientable GEM of dimension  $D$  and size  $n$  is connected with probability

$$1 - \frac{1}{n^{D-1}} + O\left(\frac{1}{n^{2(D-1)}}\right). \quad (22)$$

The whole asymptotic expansion is provided by the following proposition.

**Proposition 5.17.** *Let  $D \geq 2$ . The asymptotic probability that a random orientable GEM  $g$  of dimension  $D$  of size  $2n$  is connected satisfies*

$$\mathbb{P}(s \text{ is connected}) \approx 1 - \sum_{k \geq 1} \frac{\mathbf{if}_k(D)}{k! \cdot \binom{(n)_k}{D-1}}. \quad (23)$$

*Proof.* The counting sequence of the class  $\mathcal{OGEM}(D)$  is 2-periodic with

$$\mathbf{ogem}_n(D+1) = \binom{2n}{n}(n!)^{D+1} = (2n)!(n!)^{D-1}.$$

Hence, the exponential generating function of this class coincides with the one of the class  $\mathcal{L}(D)$  of  $D$ -multiple linear orders taken at  $z^2$ :

$$\sum_{n=0}^{\infty} (n!)^{D-1} z^{2n}.$$

We have the following decompositions:  $\mathcal{OGEM}(D) = \text{SET}(\mathcal{OGEM}(D))$  and  $\mathcal{L}(D) = \text{SEQ}(\mathcal{IL}(D))$ . We have seen in the proof of Proposition 5.10 that the sequence  $(n!)$  is gargantuan. Hence, according to Lemma 2.5, the sequence

$$\frac{\mathbf{ogem}_{2n}(D)}{(2n)!} = (n!)^{D-1}$$

is gargantuan too.

This allows us to apply Proposition 4.5. Note that we need to formally rewrite the exponential generating function of the class  $\mathcal{IL}(D)$  taken at  $z^2$  as if it corresponds to a combinatorial class with 2-periodic counting sequence:

$$\sum_{n=1}^{\infty} \frac{(2n)! \cdot \mathbf{if}_k(D)}{n!} \cdot \frac{z^{2n}}{(2n)!}.$$

Doing so, we get the asymptotics in the form

$$\begin{aligned} \mathbb{P}(g \text{ is SET-irreducible}) &\approx 1 - \sum_{k \geq 1} \frac{(2k)! \cdot \mathbf{if}_k(D)}{k!} \cdot \binom{2n}{2k} \cdot \frac{\mathbf{ogem}_{n-k}(D)}{\mathbf{ogem}_n(D)} \\ &\approx 1 - \sum_{k \geq 1} \frac{\mathbf{if}_k(D)}{k!} \cdot \frac{1}{\binom{(n)_k}{D-1}}. \end{aligned}$$

□

**Remark 5.18.** We can extend the notion of indecomposable permutation in higher dimension. Let us define a  $d$ -multipermutation of size  $n$  as a  $d$ -tuple of permutations

$$(\sigma_1, \dots, \sigma_d) \in S_n^d.$$

A  $d$ -multipermutation is said *indecomposable*, if there is no  $k < n$  such that  $\sigma_i([k]) = [k]$  for all  $i \in [d]$ . The same way as for relation (17), the counting sequence of indecomposable  $d$ -multipermutations satisfies

$$\mathbf{imp}_n(d) = \frac{\mathbf{il}(d+1)}{n!} \quad (24)$$

This fact allows us to reformulate Proposition 5.17 in the following way.

**Corollary 5.19.** *Let  $D \geq 2$ . The asymptotic probability that a random orientable GEM  $g$  of dimension  $D$  of size  $2n$  is connected satisfies*

$$\mathbb{P}(s \text{ is connected}) \approx 1 - \sum_{k \geq 1} \frac{\mathbf{imp}_k(D-1)}{\binom{(n)_k}{D-1}}. \quad (25)$$

where  $\mathbf{imp}_k(D-1)$  denotes the number of indecomposable  $(D-1)$ -multipermutations of size  $k$ .

### 5.3.2 Constellations

**Definition 5.20.** Given a positive integer  $d$ , a  $d$ -constellation of size  $n$  is a  $d$ -multipermutation  $(\sigma_1, \dots, \sigma_d) \in S_n^d$  such that:

- the group  $\langle \sigma_1, \dots, \sigma_d \rangle$  acts transitively on the set  $[n]$ ,
- the product of  $\sigma_k$  is the identity permutation, i.e.  $\sigma_1 \dots \sigma_d = \text{id}$ .

We denote the class of  $d$ -constellations by  $\mathcal{CN}(d)$ .

While constellation have been introduced to study unramified covering of the punctured sphere [33], they can be interpreted as a higher dimensional generalization of origami studied in Section 5.2.1. Indeed, the second condition is equivalent to  $\sigma_d = (\sigma_1 \dots \sigma_{d-1})^{-1}$  so that the first  $d-1$  permutations can be chosen independently, and the last one is determined by those. The permutations  $\sigma_1, \dots, \sigma_{d-1}$  of  $[n]$  describe how  $n$   $(d-1)$ -hypercubes are glued in each direction to form a generalized origami. In this interpretation, the first condition is equivalent to the connectedness of the generalized origami.

**Proposition 5.21.** *Let  $d \geq 3$ . The number  $c_n(d)$  of  $d$ -constellations of size  $n$  satisfies:*

$$c_n(d) \approx (n!)^{d-1} \left( 1 - \sum_{k \geq 1} \frac{\mathbf{imp}_k(d-2)}{\binom{(n)_k}{d-2}} \right) \quad (26)$$

$$\approx (n!)^{d-1} \left( 1 - \sum_{k \geq 1} \frac{\mathbf{il}_k(d-1)}{k! \cdot \binom{(n)_k}{d-2}} \right). \quad (27)$$

*Proof.* The estimation of the number of constellations follows from the computation of the probability that a uniformly chosen  $(d-1)$ -multipermutation determines  $d$ -constellation (after adding the last permutation). This result is a generalisation of Proposition 5.10. If  $\mathcal{S}(d-1)$  denotes the class of  $(d-1)$ -multipermutations, we have  $\mathcal{S}(d-1) = \text{SET}(\mathcal{CN}(d))$ . The corresponding counting sequence is  $(n!)^{d-1}$ , which coincides with the one of the class  $\mathcal{L}(d-1) = \text{SEQ}(\mathcal{IL}(d-1))$ . As we have seen in the proof of Proposition 5.17, the sequence  $(n!)^{d-2}$  is gargantuan for  $d > 2$ . Therefore, we can apply Theorem 4.1 that, together with relation (24) and multiplication of both sides by  $(n!)^{d-1}$ , gives the asymptotic expansions (26) and (27).  $\square$

## 6 Conclusion

We have seen how to provide the whole asymptotic expansion of the probability that a combinatorial class admitting a double SET/SEQ decomposition is connected. The relationship between those two decompositions corresponds, symbolically, to the relation

$$\frac{\partial \log(z)}{\partial z} = \frac{1}{z}.$$

It is tempting to iterate this idea to get the asymptotic probability that a combinatorial structure is SEQ-irreducible from the equation

$$\frac{\partial}{\partial z} \left( \frac{1}{z} \right) = -\frac{1}{z^2}.$$

The derivative sequence should therefore be related to structures admitting a decomposition into two SEQ-irreducible components. This will be the subject of the forthcoming paper [34].

In the applications, we relied on the existence of a double SET/SEQ decomposition for some combinatorial classes: graphs *vs* tournaments, permutations *vs* linear orders, and combinatorial maps *vs* linear matchings. However, some models ( $p$ -angulated surfaces, quadratic square-tiled surface, not necessarily orientable GEMs) have been described without an explicit SEQ decomposition. First, note that it is still possible to apply (the proof of) Theorem 4.1, by replacing the combinatorial sequence  $\mathfrak{d}_k$  with the  $k$ th coefficient of the formal power series  $D(z) := 1 - 1/A(z)$ . In particular, we have:

**Proposition 6.1.**

$$\mathbb{P}(\text{a triangulated surface is connected}) \approx 1 - \frac{5}{36n} - \frac{695}{2592n^2} - \frac{216305}{279936n^3} - \dots$$

$$\mathbb{P}(\text{a quadrangulated surface is connected}) \approx 1 - \frac{3}{16n} - \frac{183}{512n^2} - \frac{8313}{8192n^3} - \dots$$

$$\mathbb{P}(\text{a quadratic square-tiled surface is connected}) \approx 1 - \frac{1}{4n} - \frac{15}{32n^2} - \frac{167}{128n^3} - \frac{11845}{2048n^4} - \dots$$

$$\mathbb{P}(\text{a GEM of dimension 3 is connected}) \approx 1 - \frac{1}{8n^2} - \frac{3}{16n^3} - \frac{49}{128n^4} - \frac{145}{128n^5} - \dots$$

*Proof.* Apply the proof of Theorem 4.1, where  $\mathfrak{d}_k/k!$  is the  $k$ th coefficient the of

$$1 - \left( \sum_{n \geq 0} (6n - 1)!! \frac{z^{2n}}{(2n)!} \right)^{-1},$$

$$1 - \left( \sum_{n \geq 0} (4n - 1)!! \frac{z^n}{n!} \right)^{-1}$$

$$1 - \left( \sum_{n \geq 0} ((2n - 1)!!)^2 \frac{z^n}{n!} \right)^{-1},$$

and

$$1 - \left( \sum_{n \geq 0} ((2n - 1)!!)^4 \frac{z^{2n}}{(2n)!} \right)^{-1},$$

respectively. □

However, a combinatorial interpretation for the coefficients is still missing. In the second forthcoming paper [35], we will construct an “anti-SEQ” operator, which provides a combinatorial interpretation of such classes.

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