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# Linear Realisability over nets: multiplicatives

Adrien Ragot<sup>\* †</sup>, Thomas Seiller<sup>‡ §</sup>, Lorenzo Tortora de Falco<sup>¶</sup>

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## Abstract

We provide a new realisability model based on orthogonality for the multiplicative fragment of linear logic, both in presence of generalised axioms (MLL<sup>⊗</sup>) and in the standard case (MLL). The novelty is the definition of cut elimination for generalised axioms. We prove that our model is adequate and complete both for MLL<sup>⊗</sup> and MLL.

## Introduction

Since the inception of Linear Logic (LL), proofs are represented as graphs that naturally live in a wider space of agents called proof structures (*nets* in this paper) that can freely interact. These nets were introduced by J.Y. Girard in [7], together with the *desequentialisation*: a simple process transforming proof trees from the sequent calculus of LL into nets. However, not every net is the desequentialisation of a proof: it is *impossible* to extract a proof tree from a net that “contains” cycles or disconnections [5]. Nets can therefore present forms of (what we call) *geometrical incorrectness*, and geometrically correct nets are (representants of) proof trees of LL. More recently, J.Y. Girard proposed *Ludics*, an interpretation of LL given in terms of “desseins”: proof trees of the LL sequent calculus with the addition of the daimon ( $\boxtimes$ ) rule, a generalised axiom allowing to prove any sequent. Ludics introduces a new kind of incorrectness that we call *provability incorrectness*: desseins are geometrically correct (they are proof trees) but can be provably incorrect. In the standard theory of proof nets geometrical and provability correctness coincide; it is the presence of daimons that allows to distinguish between provability correctness and geometrical correctness.

Understanding the relationship between correctness and computational behavior is (one of) the goal(s) of *realisability*, which, restricted to LL, will be our focus in this paper. We briefly sum up the existing works on linear realisability<sup>1</sup> by positioning them with respect to Table 1. We also recall if these models enjoy completeness or not. Two lines of research on realisability of LL can be identified.

One was initiated by V. De Paiva *Dialectica Interpretation* [6] and led to P. Oliva’s adequate and complete realisability model of first order LL [14] where realisers are proof trees (with standard axioms) from a decorated sequent calculus of LL. As a consequence realisers are typed and are “by construction” *geometrically and provably correct* (placing this model in the top left corner of Table 1).

The other originates in the work of J.Y. Girard: *Ludics* [10], whose “desseins” are geometrically correct but can be provably incorrect (top right corner of Table 1), which enjoys *completeness*. E. Beffara proposed adequate models in a concurrent  $\pi$ -calculus [2] and conjunctive structure [3]. T. Seiller’s *interaction graphs* (inspired by Girard’s Geometry of Interaction [8]) model various LL fragments adequately [15–19]. Beffara’s and Seiller’s approaches exhibit both geometrical and provability incorrectness (bottom-right corner of Table 1), but contain no completeness result.

We give the first complete realisability model of the multiplicative fragment of linear logic in terms of nets, essentially the well-known untyped proof-structures of LL [9] with *daimons*, as in the work of P.L. Curien [4]: this places us in the bottom-right corner of Table 1. The main tool we use in our approach to realisability is LL cut elimination: we interpret formulas as types, sets of nets closed under bi-orthogonality, where the notion of orthogonality is defined via the rewriting rules of nets induced by cut elimination. We prove completeness for MLL<sup>⊗</sup>, multiplicative LL with generalised axioms, meaning our model can capture *geometrical correctness*. As a

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<sup>1</sup>We use the expression linear realisability in the sense of [20] i.e. realisability models for LL.

	MLL	MLL <sup>✕</sup>
Proof Nets	no incorrectness	provability incorrectness
Nets	geometrical incorrectness	geometrical and provability incorrectness

Table 1: Presence of incorrectness, restricted to multiplicative linear logic, for realisability models.

byproduct we obtain completeness for the standard multiplicative fragment of linear logic (MLL), thus capturing *provability correctness*.

Although not expressed in the terms of realisability, a completeness result for MLL<sup>✕</sup> (in the atomic case) using a notion of orthogonality is already apparent in the work of P.L. Curien [4], where the partitions involved in the Danos Regnier criterion [5] are encoded using daimons. More precisely, one can test the geometrical correctness of a net by confronting it against carefully chosen *opponents* (which as in the work of B  chet [1] are geometrically correct nets). However the method in [4] does not allow to derive a completeness result for MLL. By contrast, we use *geometrically incorrect* opponents to prove completeness for MLL (Remark 87).

The novelty is the *cut elimination* of non-homogenous cuts (a generalised axiom against a connective – say a tensor): unlike in Ludics<sup>2</sup> [10] our daimon is the “perfect” opponent/evaluation context; it never stops responding during computation and never prevents proof search to go on (Figure 4 and Remark 24). These new cut elimination steps are key to interactively identify provability correctness and so to obtain our completeness result for MLL (Remark 86). The computational behavior of the daimon also differs from Krivine’s continuations involved in *classical realisability* [12]: they restore a previously stored context while our daimon rather behaves like an adaptive evaluation context.

The general aim is to understand the computational content of proofs and of (incorrect) nets, following a “purely interactive approach to logic” (to quote [10]). We follow the approach initiated with Ludics, we present a framework in which proofs and refutations are objects of the same nature that can freely interact: a proof–object proves a formula  $A$  whenever it “defeats” all the refutations of  $A$ . The correctness of an object is evaluated using a dynamic criterion (we make an object interact with each of its refutations) rather than a static one (such as a typing discipline).

**Outline.** In section 1, we give a detailed introduction of nets that we define as ordered hypergraphs. In section 2, we recall the elementary notions of multiplicative linear logic, we introduce the ✕-links and we formulate the criterion of Danos Regnier [5] in our setting. In section 3, we define orthogonality between two nets as “successful interaction” through cut elimination (Definition 42); this leads to the notion of type: a set of nets closed under bi-orthogonality. We then show how to perform the usual multiplicative constructions in the framework of types. In section 4, we define our realisability model interpreting formulas as types and we prove its adequacy: a net representing a proof of  $A$  is a realiser of  $A$  (Theorem 64). In section 5, we relate correctness criteria with orthogonality. The Danos-Regnier criterion applied to a cut-free net with conclusion  $A$  yields a set of nets called tests (Definition 74). We prove that the tests of  $A$  are proofs of  $A^\perp$  (Theorem 77) and that the interaction between a net  $\pi$  with conclusion  $A$  and its tests allows to determine whether or not  $\pi$  is indeed a proof: we thus extend to our framework a result of B  chet [1]. In section 6, we prove the completeness of our realisability model: if a net  $S$  realises  $A$  (in every basis), then  $S$  is a proof of  $A$  in MLL<sup>✕</sup> (Theorem 85). Finally we show that completeness of MLL<sup>✕</sup> implies that of MLL (Theorem 88).

## 1 Untyped nets

We introduce the framework of *nets* in which our construction takes place. Nets are a special kind of *directed hypergraphs* together with an order of *some* of their vertices which will come in play later on to define the notion of orthogonality. These hypergraphs enjoy a natural notion of sum (Definition 6). In subsection 1.2, we define our “realisers” that we call nets and their computational rules, the cut elimination procedure as known for multiplicative proof structures [9] but with a novelty: the generalised axiom or daimon–link (✕) which behave like an adaptive evaluation context.

### 1.1 Directed hypergraphs

Given a set  $X$  we will let  $\mathcal{P}_{\leq}(X)$  denote the set of totally ordered finite subsets of  $X$ . An element of  $\mathcal{P}_{\leq}(X)$  is equivalently a finite sequence of elements of  $X$  but, *without repetitions*.

<sup>2</sup>In Ludics, the daimon means the end of the game, or the end of the proof search.

**Definition 1.** Suppose given a set  $L$  of labels. A directed ( $L$ -labelled) hypergraph is a tuple  $(V, E, s, t, \ell)$  where  $V$  is a finite set of positions and  $E$  is a finite set of links,  $s : E \rightarrow \mathcal{P}_{\leq}(V)$  is the source map,  $t : E \rightarrow \mathcal{P}_{\leq}(V)$  is the target map and  $\ell : E \rightarrow L$  is the labelling map.

Given a link  $e \in E$ , since the finite sets  $t(e)$  and  $s(e)$  are totally ordered, to support readability we will represent them as sequences: they are respectively called the *target* and the *source* sets of  $e$ . A *source* (resp. *target*) of  $e$  is an element of its source (resp. target) set  $s(e)$  (resp.  $t(e)$ ). The set of targets and sources of  $e$  is the *domain* of the link  $e$ . We will use superscripts to denote sequences of positions  $(\bar{p}, \bar{q}, \bar{u}, \dots)$ . A link is a *loop* when its target set and source set are not disjoint.

**Convention.** Along this work we assume all the hypergraphs to be loop-free i.e. containing only links which are not loops.

Given an hypergraph  $\mathcal{H}$  with  $E$  as its set of links, we denote  $s(\mathcal{H})$  (resp.  $t(\mathcal{H})$ ) the set of all positions which are source (resp. target) of at least one link:

$$s(\mathcal{H}) = \bigcup_{e \in E} s(e), \quad t(\mathcal{H}) = \bigcup_{e \in E} t(e).$$

A *conclusion/output* (resp. a *premise/input*) of a directed hypergraph  $\mathcal{H}$  is a position which is the source (resp. target) of no link in  $\mathcal{H}$ , i.e. an element of  $V \setminus s(\mathcal{H})$  (resp. of  $V \setminus t(\mathcal{H})$ ). The set of conclusions (resp. premises) of an hypergraph  $\mathcal{H}$  is denoted  $\text{out}(\mathcal{H})$  (resp.  $\text{in}(\mathcal{H})$ ). A position  $p$  is *isolated* in an hypergraph  $\mathcal{H}$  if  $p$  is both an output and an input of  $\mathcal{H}$ , i.e.  $p \notin s(\mathcal{H}) \cup t(\mathcal{H})$ . The *size* of a directed hypergraph is the number of its links. There is a unique empty hypergraph  $\mathcal{H} = (V, E, s, t, \ell)$  with  $V = E = s = t = \emptyset$ .

An *isomorphism* of hypergraphs  $f : (V_1, E_1, s_1, t_1, \ell_1) \rightarrow (V_2, E_2, s_2, t_2, \ell_2)$  is a pair of functions  $(f_V, f_E)$  such that  $f_V : V_1 \rightarrow V_2$  and  $f_E : E_1 \rightarrow E_2$  are bijections,  $f_E$  preserve labels i.e.  $\ell(f_E(e)) = \ell(e)$ , and  $f_E$  preserves the target and source of a link, i.e.  $s_2(f_E(e)) = f_V^*(s_1(e))$  and  $t_2(f_E(e)) = f_V^*(t_1(e))$ , where  $f_V^*$  is the natural extension of  $f_V$  to sequences of positions. Along this work we work with hypergraphs up to isomorphism.

*Notation 2.* We denote  $\langle \bar{u} \triangleright_l \bar{v} \rangle$  the hypergraph  $(V, E, s, t, \ell)$  such that  $E = \{e\}$ ,  $V = s(e) \cup t(e)$ ,  $s(e) = \bar{u}$ ,  $t(e) = \bar{v}$  and  $\ell(e) = l$  (an example of such a single-link hypergraph is found in [Figure 1a](#)). In the sequel  $\langle \bar{u} \triangleright_l \bar{v} \rangle$  will denote both the described hypergraph and its unique link.

*Notation 3.* We write  $u \cdot v$  the concatenation of sequences. Given  $u = (u_1, \dots, u_n)$  a sequence of elements of a set  $X$  and an integer  $i \in \{1, \dots, n\}$ , we denote by  $u_{<i}$  (resp.  $u_{>i}$ ) the sequence  $(u_1, \dots, u_{i-1})$  (resp.  $(u_{i+1}, \dots, u_n)$ ). Moreover, given two – potentially empty – sequences  $u$  and  $v$  we denote by  $u[i \leftarrow v]$  the sequence  $u_{<i} \cdot v \cdot u_{>i}$ .

A link is *initial* (resp. *final*) when it has no input (resp. no output). A position is *initial* (resp. *final*) when it is an output (resp. input) of an initial (resp. final) link. In an hypergraph  $\mathcal{H}$ , a link  $e$  is *terminal* when every target of  $e$  is a conclusion of  $\mathcal{H}$  – thus a final link is a terminal link.

**Example 4.** For instance a link  $\langle \triangleright_\ell a, b, c \rangle$  is an initial link and the positions  $a, b$  and  $c$  are initial, on the other hand a link  $\langle a, b \triangleright_\ell c \rangle$  is not initial and neither are the positions  $a, b$  or  $c$ .

Hypergraphs enjoy a natural notion of sum based on the disjoint union of the set of links.

*Notation 5.* Given two sets  $X_0$  and  $X_1$  we denote  $X_0 \uplus X_1$  the set  $X_0 \cup X_1$  whenever  $X_0$  and  $X_1$  are disjoint. Given two functions  $f : X_0 \rightarrow E$  and  $g : X_1 \rightarrow E$  with disjoint domains we denote  $f \uplus g$  the function which takes an element  $x$  of  $X_0 \uplus X_1$ , and returns  $f(x)$  if  $x \in X_0$  and  $g(x)$  if  $x \in X_1$ .

**Definition 6.** Given two hypergraphs  $\mathcal{H}_1 = (V_1, E_1, t_1, s_1, \ell_1)$  and  $\mathcal{H}_2 = (V_2, E_2, t_2, s_2, \ell_2)$  such that  $E_1 \cap E_2 = \emptyset$ . The sum of  $\mathcal{H}_1$  and  $\mathcal{H}_2$  is defined as:

$$\mathcal{H}_1 + \mathcal{H}_2 = (V_1 \cup V_2, E_1 \uplus E_2, t_1 \uplus t_2, s_1 \uplus s_2, \ell_1 \uplus \ell_2).$$

**Remark 7.** Whenever  $\mathcal{H}_1 = (V_1, E_1, t_1, s_1, \ell_1)$  and  $\mathcal{H}_2 = (V_2, E_2, t_2, s_2, \ell_2)$  are such that  $E_1 \cap E_2 \neq \emptyset$ , we will abusively write their sum as  $\mathcal{H}_1 + \mathcal{H}_2 = (V_1 \cup V_2, E_1 \uplus E_2, t_1 \uplus t_2, s_1 \uplus s_2, \ell_1 \uplus \ell_2)$ , since up to renaming the sets of links of two hypergraphs can always be considered disjoint.

**Remark 8.** Vertices may overlap in a sum (as we take the union of vertex sets rather than the disjoint union). As a consequence, a position may be input (or output) of several distinct links ([Figure 1b](#)). We can describe hypergraphs as sums of simple hypergraphs; namely those that contain only one link. Indeed using [Notation 2](#), an hypergraph consisting of two links  $\langle \bar{a} \triangleright_\ell \bar{b} \rangle$  and  $\langle \bar{c} \triangleright_{\ell'} \bar{d} \rangle$  is in fact equal to the sum of the single-link hypergraphs  $\langle \bar{a} \triangleright_\ell \bar{b} \rangle$  and  $\langle \bar{c} \triangleright_{\ell'} \bar{d} \rangle$ . By induction on the number of links, this shows that any hypergraph  $\mathcal{H}$  without isolated positions can be written as  $\mathcal{H} = \sum_{e \in E} \langle s(e) \triangleright_{\ell(e)} t(e) \rangle$ .

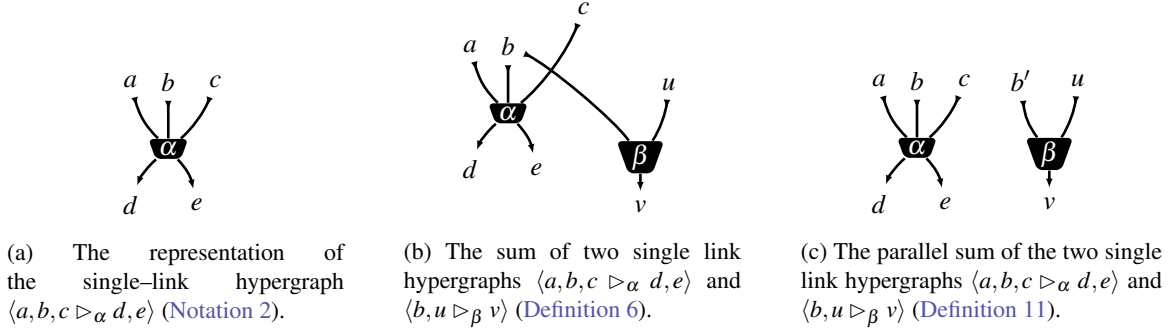


Figure 1: Hypergraphs can naturally be represented in a graphical way, we illustrate the notation of a hypergraph containing a single link, the sum of hypergraphs and the parallel sum of hypergraphs. In Figure 1c The position  $b$  is present in both hypergraphs therefore we rename it in one of the two hypergraphs: thus  $\langle a, b, c \triangleright_{\alpha} d, e \rangle \parallel \langle b, u \triangleright_{\beta} v \rangle$  equals  $\langle a, b, c \triangleright_{\alpha} d, e \rangle \parallel \langle b', u \triangleright_{\beta} v \rangle$  (that is, upto isomorphism).

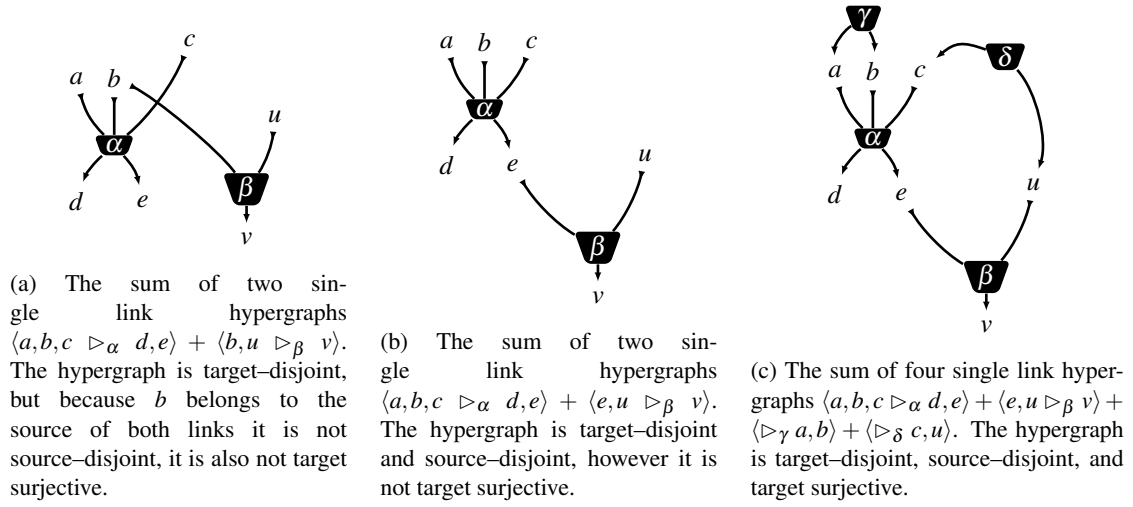


Figure 2: Properties of hypergraphs: source-disjoint, target-disjoint and target-surjective hypergraphs.

**Example 9.** In the hypergraph  $\langle \triangleright_{\ell_1} a, b, c \rangle + \langle a \triangleright_{\ell_2} d \rangle + \langle \triangleright_{\ell_3} e \rangle + \langle e \triangleright_{\ell_4} \rangle$  the set of initial positions is  $\{a, b, c, e\}$ , while  $e$  is the only final position of the hypergraph, and it belongs to the domain of the unique final link  $\langle e \triangleright_{\ell_4} \rangle$ .

**Remark 10.** The sum of hypergraphs enjoys the properties of an abelian monoid; associativity, commutativity, and a neutral element which is the empty hypergraph.

We will also use extensively the notion of *parallel composition* or *parallel sum* of hypergraphs, an analogue of the *union-graph* of two simple graphs.

**Definition 11.** Given  $\mathcal{H}_1 = (V_1, E_1, t_1, s_1, \ell_1)$  and  $\mathcal{H}_2 = (V_2, E_2, t_2, s_2, \ell_2)$  two hypergraphs such that  $V_1 \cap V_2 = E_1 \cap E_2 = \emptyset$ , we define their *parallel sum* as:  $\mathcal{H}_1 \parallel \mathcal{H}_2 = (V_1 \uplus V_2, E_1 \uplus E_2, t_1 \uplus t_2, s_1 \uplus s_2, \ell_1 \uplus \ell_2)$ .

**Remark 12.** The parallel sum of two hypergraphs  $\mathcal{H}_1$  and  $\mathcal{H}_2$  corresponds to a regular sum whenever the sets of vertices are disjoint. Just like the sum, parallel composition can always be performed between two hypergraphs (up to a renaming, see Figure 1c).

A hypergraph  $\mathcal{H} = (V, E, t, s, \ell)$  is: (1) *target-surjective* whenever  $t(\mathcal{H}) = V$ , (2) *source-disjoint* if the sets  $s(e)$  for  $e \in E$  are pairwise disjoint, (3) *target-disjoint* if the sets  $t(e)$  for  $e \in E$  are pairwise disjoint (Figure 2). A *module* is an hypergraph which is target-disjoint and source-disjoint, which means that for each position  $p$  there exists *at most* one link  $e$  such that  $s(e)$  (resp.  $t(e)$ ) contains  $p$ . Any single-link hypergraph is a module. Uncarefully summing two modules does not necessarily result in a module; for instance the single link hypergraphs  $e = \langle \triangleright_{\ell} a \rangle$  and  $e' = \langle \triangleright_{\ell'} a \rangle$  are both modules but their sum isn't as  $a$  is the target of the two links  $e$  and  $e'$ .

An *arrangement* of a directed hypergraph  $\mathcal{H}$  is a total order  $<_{\mathbf{a}}$  on its set of conclusions; equivalently the order may be identified as a bijection  $\mathbf{a} : \{1, \dots, \text{card}(\text{out}(\mathcal{H}))\} \rightarrow \text{out}(\mathcal{H})$ . An *ordered hypergraph* is a pair  $(\mathcal{H}, \mathbf{a})$  of an hypergraph  $\mathcal{H}$  together with an arrangement  $\mathbf{a}$  of  $\mathcal{H}$ . Given an ordered hypergraph  $(\mathcal{H}, \mathbf{a})$  with  $n$

conclusions for an integer  $1 \leq i \leq n$ , we denote  $\mathbf{a}(i)$  by  $\mathcal{H}(i)$  whenever there is no ambiguity. The arrangement  $\mathbf{a}$  is denoted  $\mathbf{a}(\mathcal{H})$ , and we might refer to  $\mathcal{H}$  as the *unordered hypergraph underlying*  $(\mathcal{H}, \mathbf{a})$ .

For  $n, m \in \mathbb{N}$  we denote by  $[n; m]$  the set of integers  $i$  such that  $n \leq i \leq m$ . Given two functions  $f : [1; n] \rightarrow E$  and  $g : [1; m] \rightarrow E$  we denote  $f \boxplus g : [1; n+m] \rightarrow E$  the function such that  $f \boxplus g(i) = f(i)$  when  $1 \leq i \leq n$  and  $f \boxplus g(i) = g(i-n)$  when  $n+1 \leq i \leq n+m$ . This operation is not commutative. The parallel sum of two ordered hypergraph  $(\mathcal{H}_1, \mathbf{a}_1)$  and  $(\mathcal{H}_2, \mathbf{a}_2)$  naturally yields an ordered hypergraph as  $(\mathcal{H}_1 \parallel \mathcal{H}_2, \mathbf{a}_1 \boxplus \mathbf{a}_2)$  (note that however this is not a commutative operation).

## 1.2 Multiplicative nets

Up to this point we have allowed any kind of link to occur in a hypergraph. We now consider untyped multiplicative nets in which only some specific kinds of links occur. We fix the set of labels as the set made of the *daimon* ( $\boxtimes$ ) the tensor ( $\otimes$ ) the par ( $\wp$ ) and the cut (cut) symbol. Furthermore we fix a family of links, namely  $\boxtimes$ -labelled links that have no inputs (they are initial links), cut-labelled links that have exactly two inputs and no outputs (they are final links),  $\otimes$ - and  $\wp$ -labelled links that have exactly two inputs and one output. As a consequence, the hypergraphs considered will closely resemble to multiplicative linear logic proof structures, with two important points of divergence: the absence of typing and the presence of generalised axioms, a standard MLL axiom link can be seen as daimon link with two conclusions <sup>3</sup>.

Formally we fix a countable set  $\text{Pos}$  of positions and a family of links  $\mathcal{L}$  defined as:

$$\mathcal{L} \triangleq \{ \langle p_1, p_2 \triangleright_{\otimes} p \rangle, \langle p_1, p_2 \triangleright_{\wp} p \rangle, \langle p_1, p_2 \triangleright_{\text{cut}} \rangle \mid p_1, p_2, p \in \text{Pos} \} \cup \{ \langle \triangleright_{\boxtimes} p_1, \dots, p_n \rangle \mid n \in \mathbb{N}, p_1, \dots, p_n \in \text{Pos} \}.$$

**Definition 13.** A multiplicative module is an ordered hypergraph  $M = (|M|, \mathbf{a}(M))$  where  $|M|$  is a sum of links of  $\mathcal{L}$  which is a module.

A multiplicative net is a multiplicative module  $S = (|S|, \mathbf{a}(S))$  where  $|S|$  is target-surjective.

From now on we will omit the word *multiplicative* but a module (resp. net) will always be a multiplicative module (resp. net). For a module  $M$  (resp. a net  $S$ ) we refer to  $|M|$  (resp.  $|S|$ ) as the unordered hypergraph underlying  $M$  (resp.  $S$ ). An *unordered* module (resp. net) is the unordered hypergraph underlying a module (resp. net).

**Remark 14.** For two nets  $S_1 = (V_1, E_1, s_1, t_1, \ell_1)$  and  $S_2 = (V_2, E_2, s_2, t_2, \ell_2)$ , if  $S_1 + S_2$  remains a net then  $S_1 + S_2 = S_1 \parallel S_2$ . Indeed, by [Definition 6](#),  $E_1 \cap E_2 = \emptyset$ . Then, by target-disjointness  $t(S_1) \cap t(S_2) = \emptyset$ ; and finally because  $S_1$  and  $S_2$  are target surjective we have  $V_1 \cap V_2 = t(S_1) \cap t(S_2) = \emptyset$ , so that [Definition 11](#) applies.

*Notation 15.* Given an integer  $n$  we denote by  $\boxtimes_n$  any multiplicative net consisting of a single daimon link with  $n$  outputs, i.e. isomorphic to  $\langle \triangleright_{\boxtimes} p_1, \dots, p_n \rangle$ .

**Definition 16.** Given a multiplicative net  $S$  the *type* of a cut link  $c = \langle p, q \triangleright_{\text{cut}} \rangle$  occurring in  $S$  is the multiset of the two labels of the links of output  $p$  and  $q$ ; for readability we write these multisets as ordered pairs. Thus there are six *types* of cuts (up to symmetry). More precisely, we distinguish: *multiplicative* cuts, of type  $(\otimes/\wp)$ ; *clash* cuts, of type  $(\otimes/\otimes)$  or  $(\wp/\wp)$ ; *glueing* cuts, of type  $(\boxtimes/\boxtimes)$ ; *non-homogeneous* cuts, of type  $(\otimes/\boxtimes)$  or  $(\wp/\boxtimes)$ , which are respectively called *reversible* and *irreversible* cuts. In a net  $S$ , a cut  $\langle p, q \triangleright_{\text{cut}} \rangle$  is *cyclic* whenever  $p$  and  $q$  are targets of the same link.

**Remark 17.** Each cut link occurring in a net  $S$  has a type since a net is target-surjective. However in a module this isn't true: for instance in the module  $\langle p, q \triangleright_{\text{cut}} \rangle$  consisting of a single cut link, the type of the cut link is not defined.

**Remark 18.** The inputs of a cut link  $\langle p, q \triangleright_{\text{cut}} \rangle$  are ordered, making the two links  $\langle p, q \triangleright_{\text{cut}} \rangle$  and  $\langle q, p \triangleright_{\text{cut}} \rangle$  distinct. However (up to isomorphism) this plays no role during cut elimination.

Multiplicative nets comes with their notion of computation called *cut elimination*: it is a rewriting on nets and more precisely it rewrites a redex (that is a sub-net made of a single cut link and two non-cut links) into redexes or daimons (in the very specific case of glueing cuts). Up to isomorphism, how a redex is rewritten depends solely on its type ([Definition 16](#)).

**Definition 19.** The relation of *homogeneous cut elimination* on unordered nets is denoted by  $\rightarrow_h$  and it is the rewriting relation defined as the contextual closure (with respect to the sum) of the relation defined in [Figure 3](#).

**Remark 20.** The (homogeneous) cut elimination procedure on unordered nets leave the conclusions unchanged. As a consequence the homogeneous cut elimination can be lifted from unordered nets to nets: whenever two unordered nets are such that  $S \rightarrow S'$ , for any arrangement  $\mathbf{a}$  of  $S$  we have  $(S, \mathbf{a}) \rightarrow (S', \mathbf{a})$ .

<sup>3</sup>To be precise one should say that an *atomic* standard MLL axiom link is a daimon link with two conclusions ([Remark 29](#)).

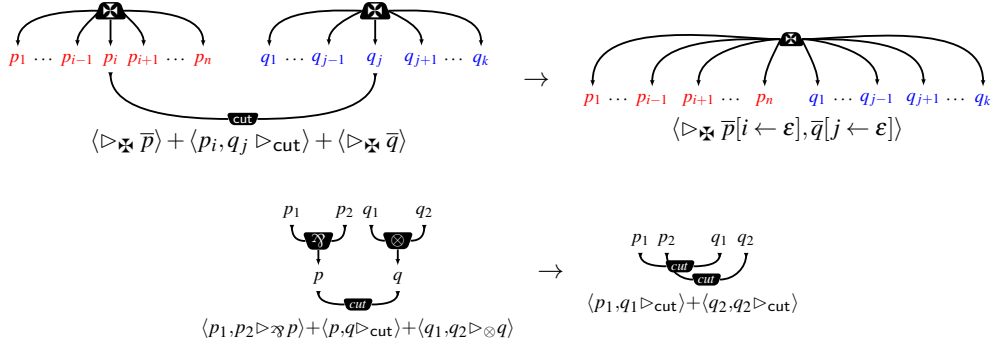


Figure 3: Rewriting defining the homogeneous cut elimination. We provide a representation of each hypergraph involved above its expression. In the step of the glueing cut we assume the two daimons to be distinct i.e. the cut is acyclic. In this figure  $\bar{p} = p_1, \dots, p_n$  while  $\bar{q} = q_1, \dots, q_k$ .

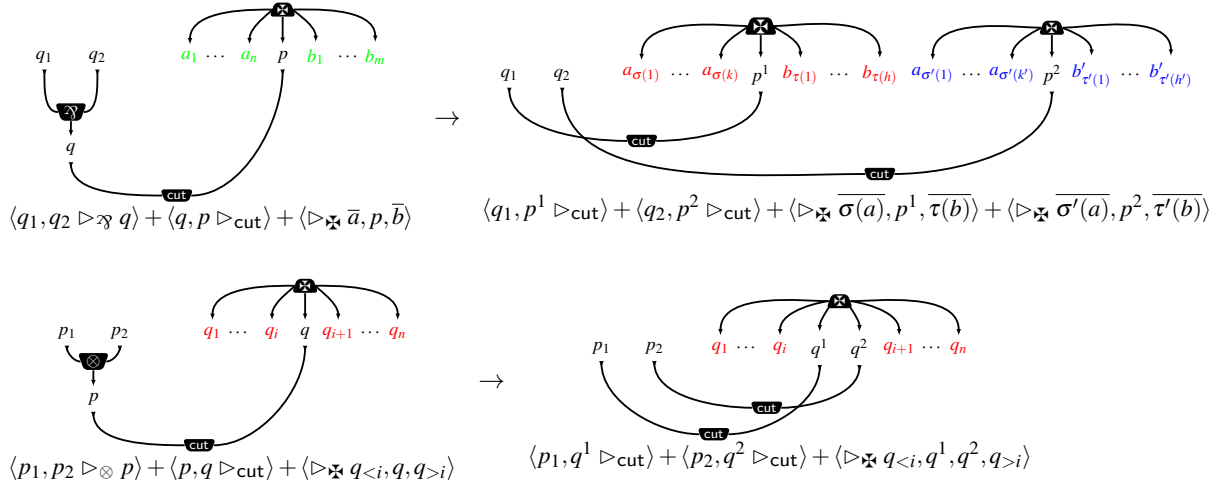


Figure 4: Rules defining the non-homogeneous cut elimination. In the elimination of the  $(\boxtimes/\boxtimes)$  cut - first row -  $\bar{a} = (a_1, \dots, a_n)$  and  $\bar{b} = (b_1, \dots, b_m)$  while  $\bar{\sigma}(\bar{a}) = (a_{\sigma(1)}, \dots, a_{\sigma(k)})$ ,  $\bar{\sigma}'(\bar{a}) = (a_{\sigma'(1)}, \dots, a_{\sigma'(k')})$ ,  $\tau(\bar{b}) = (b_{\tau(1)}, \dots, b_{\tau(h)})$ ,  $\tau'(\bar{b}) = (b_{\tau'(1)}, \dots, b_{\tau'(h')})$  (with  $n = k + k'$  and  $m = h + h'$ ) are sequences that define a partition of  $\{a_1, \dots, a_n, b_1, \dots, b_m\}$  more precisely  $\{a_1, \dots, a_n\} = \{a_{\sigma(1)}, \dots, a_{\sigma(k)}, a_{\sigma'(1)}, \dots, a_{\sigma'(k')}\}$  and  $\{b_1, \dots, b_m\} = \{b_{\tau(1)}, \dots, b_{\tau(h)}, b_{\tau'(1)}, \dots, b_{\tau'(h')}\}$ , and  $\bar{\sigma}, \bar{\sigma}', \tau, \tau'$  are permutations. Furthermore  $p^1, p^2, q^1, q^2$  are fresh positions. The figure is slightly misleading:  $q_1$  and  $q_2$  may be elements of  $\bar{a}$  or  $\bar{b}$  (in the first row) while  $p_1$  and  $p_2$  may be elements of  $q_1, \dots, q_{i-1}, q_{i+1}, \dots, q_n$  (in the second row), these cases are illustrated in Figure 13. This has an important consequence: a cut can belong to a cycle and still be reducible (Remark 28).

The following result is easily established, in particular since the number of links strictly decreases during homogeneous cut elimination.

**Proposition 21.** *Homogeneous cut elimination is confluent and strongly normalizing.*

**Definition 22.** The non homogeneous reduction is denoted  $\rightarrow_{nh}$  and it is defined on unordered nets as the contextual closure of the relation given in Figure 4.

**Remark 23.** The non-homogeneous reduction preserves the conclusion of the nets, hence it can be lifted to ordered nets – as in remark 20.

**Remark 24.** In the framework of Multiplicative Linear Logic (section 2, Figure 6c), non homogeneous cut elimination simulates proof search in the sequent calculus:

$$\frac{\frac{\frac{A^+, A \quad B^+, B}{\Gamma, A \boxtimes B} \quad \frac{A^+ \otimes B^+, A, B}{A^+ \otimes B^+, A \boxtimes B} \otimes}{\Gamma, A \boxtimes B} \quad \frac{A^+, A \quad B^+, B}{\Gamma, A \boxtimes B} \quad \frac{A^+, B^+, A \otimes B}{A^+ \boxtimes B^+, A \otimes B} \boxtimes}{\Gamma, A \otimes B} \text{cut} \rightarrow^* \frac{\frac{A^+, A \quad B^+, B}{\Gamma, A, B} \quad \frac{A^+, B^+, A \otimes B}{\Gamma, A \otimes B} \otimes}{\Gamma, A \otimes B} \quad \frac{A^+, A \quad B^+, B}{\Gamma, A \otimes B} \quad \frac{A^+, B^+, A \otimes B}{\Gamma, A \otimes B} \boxtimes}{\Gamma, A \otimes B} \text{cut} \rightarrow^* \frac{\frac{A^+, A \quad B^+, B}{\Gamma, A, B} \quad \frac{A^+, B^+, A \otimes B}{\Gamma, A \otimes B} \otimes}{\Gamma, A \otimes B} \quad \frac{A^+, A \quad B^+, B}{\Gamma, A \otimes B} \quad \frac{A^+, B^+, A \otimes B}{\Gamma, A \otimes B} \boxtimes}{\Gamma, A \otimes B} \text{cut}$$

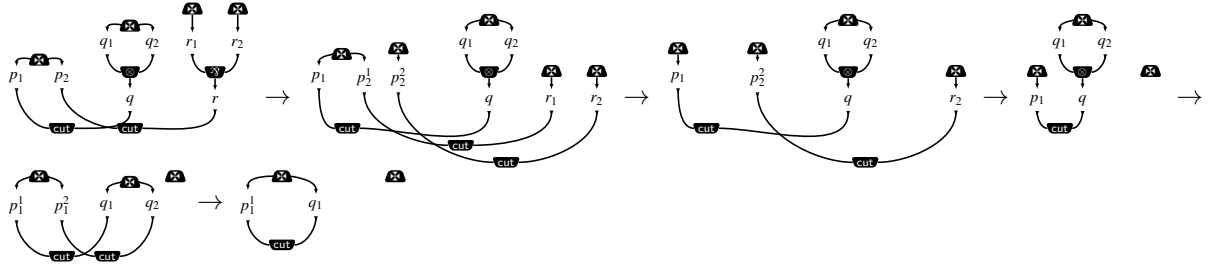
This also illustrates the non determinism of the  $(\boxtimes/\boxtimes)$  reduction step which corresponds to proof search on a formula of the form  $A \otimes B$ : going from bottom to top the  $\otimes$ -introduction rule splits the context  $\Gamma$ , which is a non deterministic process. A consequence of non determinism is the loss of confluence for cut elimination (but not of strong normalisation, Proposition 30); since splitting the context is irreversible, a net can have different normal forms, like the second net of figure 5b (from left to right) which coincides with the second net of figure 5c: this same net reduces, following the two figures, to two different normal forms.

**Remark 25.** A cyclic cut is a glueing cut. Indeed, given a cyclic cut link  $\langle p, q \triangleright_{\text{cut}} \rangle$  in a net, because  $p$  and  $q$  belong to the target of a same link  $e$  and the only links which may have several targets are daimon links it follows that  $e$  is a daimon link.

**Remark 26.** The side condition of Figure 3 entails that a cyclic cut is not reducible: for example the net  $\langle \triangleright_{\boxtimes} p, q \rangle + \langle p, q \triangleright_{\text{cut}} \rangle$  is a net in normal form.

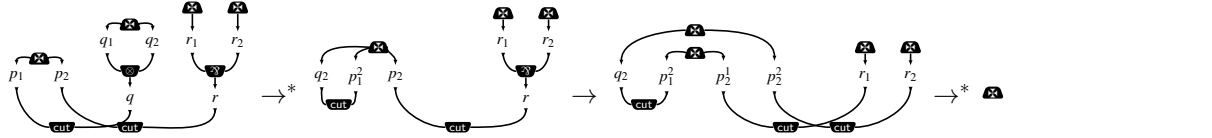
**Remark 27.** A cut link which is not reducible is either a clashing cut or a cyclic glueing cut. Notice, however, that while clashing cuts never disappear during cut elimination, cyclic cuts may disappear (see Figure 10b).

**Remark 28.** In the standard framework of MLL proof structures the cut elimination of an axiom against a cut is defined as the identification of the two extreme positions, therefore eliminating such a cut may create *loops* (section 1). To avoid loops from occurring during cut elimination an ad hoc condition is usually added (see for example [13]). In our framework, this condition is the rather natural side condition of Figure 3.

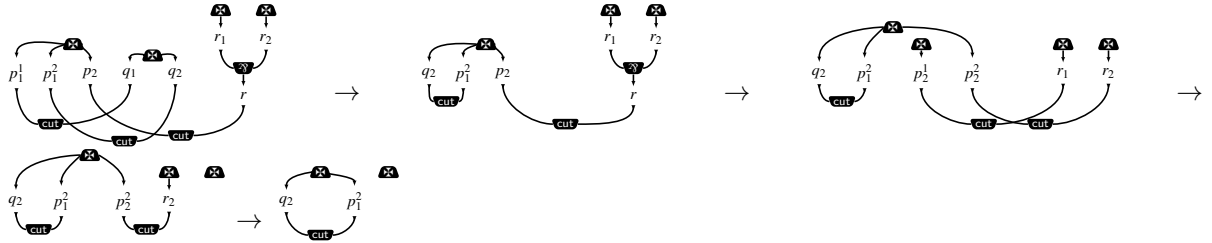


(a) Eliminating first the *irreversible cut*  $(\boxtimes/\boxtimes)$  produces a net<sup>a</sup> which cannot normalize in  $\boxtimes_0$ .

<sup>a</sup>The  $(\boxtimes/\boxtimes)$  reduction step is not deterministic but in this very special case any choice yields the same net.



(b) Eliminating the reversible cut  $(\boxtimes/\otimes)$  produces a cycle which can be eliminated by the elimination of the  $(\boxtimes/\boxtimes)$  cut remaining, hence that net can normalize in  $\boxtimes_0$ .



(c) Non determinism also comes from the choice of how we reduce  $(\boxtimes/\boxtimes)$  cuts, different choices leading to different normal forms: the “wrong” choice results in a net which cannot normalize to  $\boxtimes_0$ .

Figure 5: Non homogeneous cut eliminations contains two sources of non-determinism.

**Remark 29.** Notice that whenever daimons are binary and typed by dual atomic formulas the cut elimination procedure for  $\text{MLL}^{\boxtimes}$  defined in Definition 19 is exactly the standard cut elimination procedure for MLL [7], [13].

The rewriting rule, denoted  $\rightarrow$ , associated with cut elimination is the union of the homogeneous and non-homogeneous cut elimination i.e.  $\rightarrow_h \cup \rightarrow_{nh}$ . We write  $S \xrightarrow{c} S'$ , when  $S'$  is obtained from  $S$  by eliminating the cut  $c$ . We write by  $S \rightarrow_{\text{mult}} S'$  (resp.  $S \rightarrow_{\text{not mult}} S'$ ) whenever  $S \xrightarrow{c} S'$  and  $c$  is multiplicative (resp. not multiplicative). Given two binary relations  $R_1$  and  $R_2$  on a set  $X$  we denote by  $R_1 \cdot R_2$  their composition, i.e. for two  $x, y \in X$   $xR_1 \cdot R_2y$  if and only if there exists  $z$  such that  $xR_1z$  and  $zR_2y$ .



**Proposition 30.** *Cut elimination is strongly normalising, furthermore:*

1.  $\rightarrow^*$  can be factorised as  $\rightarrow_{\text{mult}}^* \cdot \rightarrow_{\neg\text{mult}}^*$ .
2. If  $c$  is a  $(\wp/\boxtimes)$  cut in  $S$ ; if  $S \xrightarrow{c} \rightarrow^* S'$  then  $S \rightarrow^* \cdot \xrightarrow{c} S'$ .
3. If  $c$  is not a  $(\wp/\boxtimes)$  cut in  $S$ ; if  $S \rightarrow^* \cdot \xrightarrow{c} S'$  then  $S \xrightarrow{c} \rightarrow^* S'$ .

$$\begin{array}{l}
A, B \triangleq X \in \text{Var} \\
\quad | A \wp B \quad | A \otimes B \\
\mathcal{H}_1, \mathcal{H}_2 \triangleq A \in \text{Form} \\
\quad | \mathcal{H}_1, \mathcal{H}_2 \quad | \mathcal{H}_1 \parallel \mathcal{H}_2
\end{array}
\quad (b) \text{ De Morgan laws lifting the involution } (\cdot)^\perp \text{ from Var to Form.}$$

$$\frac{}{\Gamma \boxtimes} \quad \frac{\Gamma, A, B \quad \Gamma, A \quad \Delta, B}{\Gamma, A \wp B} \wp \quad \frac{\Gamma, A \quad \Delta, B}{\Gamma, \Delta, A \otimes B} \otimes \quad \frac{\Gamma, A \quad \Delta, A^\perp}{\Gamma, \Delta} \text{cut} \quad \frac{\Gamma, A, B, \Delta}{\Gamma, B, A, \Delta} \text{ex} \quad \frac{}{A, A^\perp} \text{ax}$$

(a) Grammar defining Form (first two rows), and grammar defining Hseq (last two rows). (c) Rules used for constructing the proof trees. The rules  $(\boxtimes, \wp, \otimes, \text{cut}, \text{ex})$  define the  $\text{MLL}^{\boxtimes}$  fragment. Substituting the  $(\boxtimes)$ -daimon rule with the  $(\text{ax})$ -axiom rule results in the fragment MLL, that is  $(\text{ax}, \wp, \otimes, \text{cut}, \text{ex})$ .

Figure 6: Grammar of formulas and (hyper)sequent, de Morgan laws and inference rules.

$$\langle \bar{c} \triangleright_\ell \bar{a}, p_1, \bar{b} \rangle + \langle p_1, p_2 \triangleright_\wp p \rangle \rightarrow_{\wp}^l \langle \bar{c} \triangleright_\ell \bar{a}, p, \bar{b} \rangle \quad \langle \bar{c} \triangleright_\ell \bar{a}, p_2, \bar{b} \rangle + \langle p_1, p_2 \triangleright_\wp p \rangle \rightarrow_{\wp}^r \langle \bar{c} \triangleright_\ell \bar{a}, p, \bar{b} \rangle$$

Figure 7: The two cases (left and right) defining the switching rewriting. The left reduction  $\rightarrow_{\wp}^l$  destroys  $p_1$  and makes  $p_2$  a conclusion; while the right reduction  $\rightarrow_{\wp}^r$  destroys  $p_2$  and makes  $p_1$  a conclusion.

$$\frac{}{\langle \triangleright_{\boxtimes} p_1, \dots, p_n \rangle} \boxtimes \quad \frac{\frac{\frac{}{A, \Gamma} \pi_1 \quad \frac{}{A^\perp, \Delta} \pi_2}{\Gamma, \Delta} \text{cut}}{S_1 + S_2 +}}{\langle S_1(1), S_2(1) \triangleright_{\text{cut}} \rangle} \quad \frac{\frac{\frac{}{A, \Gamma} \pi_1 \quad \frac{}{B, \Delta} \pi_2}{\Gamma, \Delta, A \otimes B} \otimes}{S_1 + S_2 +}}{\langle S_1(1), S_2(1) \triangleright_{\otimes} p \rangle} \quad \frac{\frac{\frac{}{A, B, \Gamma} \pi_0}{A \wp B, \Gamma} \wp}{S_0 +}}{\langle S_0(1), S_0(2) \triangleright_{\wp} p \rangle} \quad \frac{\frac{}{\Gamma, B, A, \Delta} \pi_0}{\Gamma, A, B, \Delta} \text{ex}}{(S_0, a)}$$

Figure 8: Induction defining the relation  $\equiv_{\mathcal{D}}$ . The proof in the first row is represented by a net below it in the second row. The position  $p$  is always supposed fresh. In each case and for each  $0 \leq i \leq 2$ ,  $S_i$  is a net which represent  $\pi_i$  i.e.  $S_i \equiv_{\mathcal{D}} \pi_i$ . In the case of the exchange rule we explicitly mention the arrangement i.e. the order of the conclusion and assume  $(S_0, a') \equiv_{\mathcal{D}} \pi_0$  and  $a(i) = a'(i)$  whenever  $i \leq |\Gamma|$  or  $|\Gamma| + 2 < i$ . On the other hand,  $a'(|\Gamma| + 1) = a(|\Gamma| + 2)$  and  $a'(|\Gamma| + 2) = a(|\Gamma| + 1)$ .

## 2 Multiplicative Linear Logic and proof nets

We define the well-known notion of proof net [7] in our setting: in the presence of the generalised axiom  $(\boxtimes)$ , proof nets are similar to the *paraproof nets* of Curien [4] (which come from Girard Ludics [10]). We then formulate the Danos–Regnier criterion [5]: testing the acyclicity and connectedness of (several) graphs allows to determine whether a net is a (para)proof net or not [4].

We fix a countable set  $\text{Var}$  of *propositional variables*. The set  $\text{Var}$  comes with an (explicit) involution  $(\cdot)^\perp$ ; for each atomic variable  $X$  there exists its *dual* atomic variable  $X^\perp$  in  $\text{Var}$ . The set  $\text{Form}$  of *formulas* of multiplicative linear logic is defined by the grammar in Figure 6a. The involution  $(\cdot)^\perp$  is lifted from  $\text{Var}$  to  $\text{Form}$  as in Figure 6b. The set  $\text{Hseq}$  of *hypersequents* is defined by the grammar in Figure 6a, a *sequent* is an hypersequent without the *parallel* ‘ $\parallel$ ’ constructor. The introduction of hypersequents is naturally suggested by the constructions on types (section 3): indeed as the interpretation of the  $\wp$ -connective is based on the interpretation of the ‘ $,$ ’-connective, the interpretation of the  $\otimes$ -connective relies on that of the ‘ $\parallel$ ’-connective (Definition 49 and Definition 58). Technically hypersequents are necessary in *our* proof of the completeness theorem (Theorem 88).

A *proof* of MLL (resp.  $\text{MLL}^{\boxtimes}$ ) is a tree constructed using the rules  $(\text{ax}, \wp, \otimes, \text{cut}, \text{ex})$  (resp.  $(\boxtimes, \wp, \otimes, \text{cut}, \text{ex})$ ) of Figure 6c.

**Definition 31.** A net  $S$  represents<sup>4</sup> a proof  $\pi$  of  $\text{MLL}^{\boxtimes}$ , denoted  $\pi \equiv_{\mathcal{D}} S$  or  $S \equiv_{\mathcal{D}} \pi$ , whenever the relation defined in Figure 8 holds. A net represents a proof of MLL whenever it represents a proof of  $\text{MLL}^{\boxtimes}$  where every sequent conclusion of a  $(\boxtimes)$ -rule has shape  $A, A^\perp$  for  $A \in \text{Form}$ . A representation of a proof  $\pi$  is a net  $S$  which represents

<sup>4</sup>In the standard Linear Logic terminology  $\pi$  is a sequentialisation of the proof net  $S$ .

$\pi$ . A *proof net* of  $\text{MLL}^{\boxtimes}$  (resp.  $\text{MLL}$ ) is a net which represents a proof of  $\text{MLL}^{\boxtimes}$  (resp.  $\text{MLL}$ ): we say that  $S$  is *correct*. A net  $S$  is *correctly typeable*<sup>5</sup> by a sequent  $\Gamma$  whenever it represents a proof of  $\Gamma$  in  $\text{MLL}^{\boxtimes}$ .

*Notation 32.* Let  $\mathfrak{P}$  denote  $\text{MLL}$  or  $\text{MLL}^{\boxtimes}$  and let  $S$  be a net. We write  $S \vdash_{\mathfrak{P}} \Gamma$  whenever there exists a proof  $\pi$  in  $\mathfrak{P}$  such that  $S$  is the representation of  $\pi$ . Furthermore we denote  $\{\Gamma : \mathfrak{P}\}$  the set of all the nets  $S$  such that  $S \vdash_{\mathfrak{P}} \Gamma$ .

A *substitution* is a map  $\theta : \text{Var} \rightarrow \text{Form}$  such that  $\theta(X^\perp) = \theta(X)^\perp$  for each  $X \in \text{Var}$ . A substitution can be lifted to formulas and hypersequents by induction:  $\theta(A \otimes B) = \theta(A) \otimes \theta(B)$ ;  $\theta(A \wp B) = \theta(A) \wp \theta(B)$ ;  $\theta(A \parallel B) = \theta(A) \parallel \theta(B)$ ;  $\theta(A, B) = \theta(A), \theta(B)$ . Given two hypersequents, we denote  $\Delta \leq \Gamma$  whenever there exists a substitution  $\theta$  such that  $\theta\Delta = \Gamma$ .

**Proposition 33.** *Let  $\Gamma$  and  $\Delta$  be two sequents and suppose  $\Delta \leq \Gamma$ . For any net  $S$ : (1) if  $S \vdash_{\text{MLL}^{\boxtimes}} \Delta$  then  $S \vdash_{\text{MLL}^{\boxtimes}} \Gamma$  and (2) if  $S \vdash_{\text{MLL}} \Delta$  then  $S \vdash_{\text{MLL}} \Gamma$ .*

**Definition 34.** The *switching* rewriting is defined on unordered nets as the contextual closure of the rules in [Figure 7](#). A *switching* of a net  $S$  is a normal form of  $S$  for the switching rewriting: we often denote it  $\sigma S$ .

**Remark 35.** The switching rewriting strongly normalizes since every step reduces the number of links of the net. The rewriting is also non-deterministic and non-confluent, every normal form is a par-free net. The switching rewriting can be lifted to (ordered) nets; with the notations of [Figure 7](#) whenever an unordered net  $|S|$  with  $n$  conclusions is such that  $|S| \rightarrow_{\wp}^l |S'|$  we define  $(|S|, \mathbf{a}) \rightarrow_{\wp}^l (|S'|, \mathbf{a}')$  where  $\mathbf{a}'(i) = \mathbf{a}(i)$  for each  $1 \leq i \leq n$  and  $\mathbf{a}'(n+1) = p_2$  i.e. the new conclusion is made last conclusion (similarly we can define it for the case  $\rightarrow_{\wp}^r$ ).

**Definition 36.** The *undirected multigraph*<sup>6</sup> induced by two partitions  $P$  and  $Q$  of a set  $X$  is  $(V, E, \text{brd})$  denoted  $G(P, Q)$  where: (1)  $V = \{1\} \times P \cup \{2\} \times Q$  the vertices are the classes of  $P$  and  $Q$  (as a disjoint union); (2)  $E = X$ ; (3) For any edge  $x$  in  $X$ ;  $\text{brd}(x) = \{(1, P_x), (2, Q_x)\}$  where  $P_x \in P$  is such that  $x \in P_x$  and  $Q_x \in Q$  is such  $x \in Q_x$ .

Two partitions  $P$  and  $Q$  of a set  $X$  are *orthogonal* if the multigraph  $G(P, Q)$  is acyclic and connected.

**Definition 37.** In a net  $S$  denote  $p \geq_S q$  the relation which holds whenever there exists a link  $e$  such that  $p \in s(e)$  and  $q \in t(e)$ . Denote  $\geq_S^*$  its reflexive and transitive closure; a position  $p$  is *above* a position  $q$  whenever  $p \geq_S^* q$ . Given a position  $q$  we denote  $q \uparrow^i S$  the set of *initial* positions which are above  $q$  in  $S$ .

**Remark 38.** Given a *cut-free* net  $S$  with conclusions  $p_1, \dots, p_n$  the sets  $p_1 \uparrow^i S, \dots, p_n \uparrow^i S$  form a partition of the initial positions of  $S$ . We denote this partition  $\uparrow^i S$ .

*Notation 39.* Let  $S$  be a net and let  $\{d_1, \dots, d_n\}$  be the set of daimon links of  $S$ . The partition  $\{t(d_1), \dots, t(d_n)\}$  on the set of initial positions of  $S$  is denoted by  $P_{\boxtimes}(S)$ .

Reformulated in the context of hypergraphs we get the following theorem from [5].

**Theorem 40** ([4], [5]). *Given a cut-free net  $S$ , the following assertions are equivalent:*

1.  $S$  is a proof net of  $\text{MLL}^{\boxtimes}$ ;
2. For every switching  $\sigma S$  of  $S$ , the partitions  $P_{\boxtimes}(S)$  and  $\uparrow^i \sigma S$  of the set of initial positions of  $S$  are orthogonal;
3. Every switching  $\sigma S$  of  $S$  is acyclic and connected<sup>7</sup>.

### 3 Interaction of nets, orthogonality, and types

We define how nets can *interact* and if the interaction of two nets leads to the  $\boxtimes$ -link with no outputs ( $\boxtimes_0$ ) we say they are *orthogonal*. This recalls classical realisability proposed by J.-L. Krivine [12], where (the closure by antireduction of) the set  $\{\boxtimes_0\}$  will play the role of the *pole*. Notice, however, that our setting is fully symmetrical: both the elements of truth values and falsity values are nets.

The notion of ordered hypergraph and *arrangement* introduced in [section 1](#) will now explicitly come into play as it is necessary for defining the interactions of nets (see [Figure 9a](#)). We will denote by  $\#S$  the number of outputs of a net  $S$ . Given a partial function  $f : \mathbb{N} \rightarrow E$  with a finite domain of cardinality  $n$  and ordered as  $i_1 < i_2 < \dots < i_n$ , the *collapse* of  $f$ , denoted  $f_\downarrow$ , is the total function with domain  $[1; n]$  such that  $f_\downarrow(m) = f(i_m)$  for any integer  $1 \leq m \leq n$ .

<sup>5</sup>Notice that with the expressions “correctly typeable” we mean here that the net is both correct (it represents a proof) and that we can label its conclusions with the formulas of  $\Gamma$ .

<sup>6</sup>Recall that a multigraph is a graph where two vertices may be connected by several edges (not to be confused with the notion of hypergraph of [Definition 1](#)). The function  $\text{brd}$  maps each edge to its endpoints.

<sup>7</sup>We refer to the graph naturally induced by the net  $\sigma S$ .

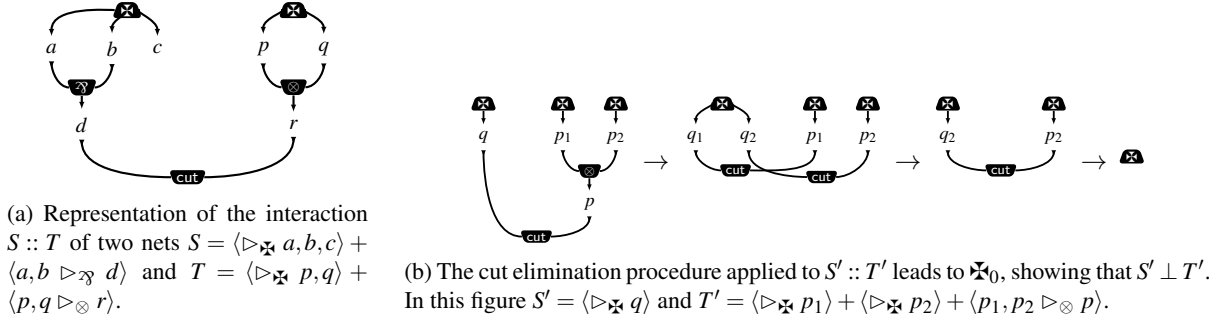


Figure 9: The interaction of two nets (Definition 41) and two orthogonal nets (Definition 42).

**Definition 41.** Let  $S = (|S|, \mathbf{a}(S))$  and  $T = (|T|, \mathbf{a}(T))$  be two nets and  $k = \min(\#S, \#T)$ , we define their *interaction*  $S :: T = (|S :: T|, \mathbf{a}(S :: T))$  as:

$$|S :: T| \triangleq |S| + |T| + \sum_{1 \leq i \leq \min(\#S, \#T)} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle \quad \mathbf{a}(S :: T) \triangleq \begin{cases} \emptyset & \text{when } \#S = \#T \\ \mathbf{a}(S) \upharpoonright_{[k+1, \#S]} \downarrow & \text{when } \#S > \#T \\ \mathbf{a}(T) \upharpoonright_{[k+1, \#T]} \downarrow & \text{when } \#S < \#T \end{cases}$$

**Definition 42.** Two nets  $S_1$  and  $S_2$  are *orthogonal* if  $S_1 :: S_2 \rightarrow^* \boxtimes_0$ <sup>8</sup>: when this holds we write  $S_1 \perp S_2$ . For a net  $S$  and a set of nets  $\Lambda$ , if for every  $\lambda \in \Lambda$  we have  $S \perp \lambda$  we write  $S \perp \Lambda$ .

**Remark 43.** Since cut links are asymmetric, namely  $\langle p, q \triangleright_{\text{cut}} \rangle$  and  $\langle q, p \triangleright_{\text{cut}} \rangle$  are distinct nets, the interactions  $S :: T$  and  $T :: S$  are not the same net. However, this has no consequence on cut elimination because the reduction steps do not depend on the order of the inputs of a cut link. Thus  $S :: T$  reduces to  $\boxtimes_0$  if and only if  $T :: S$  does, and as expected the relation of orthogonality is symmetric.

**Definition 44.** Given a set  $A$  of multiplicative nets, we define the *orthogonal* of  $A$  as  $A^\perp = \{P \mid \forall R \in A, P \perp R\}$ . A *type*  $\mathbf{A}$  is a set of multiplicative nets such that  $\mathbf{A}^{\perp\perp} = \mathbf{A}$ .<sup>9</sup>

**Remark 45.** Since cut elimination preserves the conclusions of a net and  $\boxtimes_0$  has no output, two orthogonal nets have the same number of conclusions. Thus, for every type  $\mathbf{A}$ , for every  $R \in \mathbf{A}$  and for every  $S \in \mathbf{A}^\perp$ , the nets  $R$  and  $S$  have the same number of conclusions: we denote by  $\#\mathbf{A}$  the number of conclusions of the nets in  $\mathbf{A}$ . Obviously  $\#\mathbf{A} = \#\mathbf{A}^\perp$ .

**Remark 46.** Clash cuts are preserved during cut elimination, thus a net containing such a cut cannot reduce to  $\boxtimes_0$ . Hence, there cannot be two nets  $S$  and  $S'$  respectively in  $\mathbf{A}$  and  $\mathbf{A}^\perp$  such that their  $i$ th conclusions  $S(i)$  and  $S'(i)$  are both outputs of a  $\boxtimes$ -link (or  $\otimes$ -link): their interaction  $S :: S'$  contains a clash cut and thus the nets cannot be orthogonal.

**Remark 47.** A net  $S$  which is orthogonal to the daimon link with a single output (i.e.  $\boxtimes_1$ ) has a single conclusion which can be the output of a daimon link, a tensor link or a par link. For instance the three cut-free nets  $\langle \triangleright_{\boxtimes} p \rangle$ ,  $\langle \triangleright_{\boxtimes} p_1 \rangle + \langle \triangleright_{\boxtimes} p_2 \rangle + \langle p_1, p_2 \triangleright_{\otimes} p \rangle$  and  $\langle \triangleright_{\boxtimes} p_1, p_2 \rangle + \langle p_1, p_2 \triangleright_{\boxtimes} p \rangle$  are all orthogonal to  $\boxtimes_1$  (one case is proved in Figure 9b).

The following proposition is a key step for proving propositions 51 and 54.

**Proposition 48.** *Given three net  $S$  and  $T$  and  $R$  such that  $\#S \geq \#T + \#R$ : the interaction  $S :: (T \parallel R)$  is equal to  $(S :: T) :: R$ .*

In the following definition 49 the side condition  $\#S \geq \#\mathbf{A}$  ensures that whenever a net  $S$  in  $\mathbf{A} \succ \mathbf{B}$  interacts with a net of  $T \in \mathbf{A}^\perp$  the remaining conclusions of  $S :: T$  are conclusions of  $S$ , this will allow to activate Proposition 48.

**Definition 49.** Given two sets of nets  $\mathbf{A}$  and  $\mathbf{B}$  their *functional composition* denoted  $\mathbf{A} \succ \mathbf{B}$ , and their *parallel composition* denoted  $\mathbf{A} \parallel \mathbf{B}$  are defined as follows:

$$\mathbf{A} \succ \mathbf{B} \triangleq \{S \mid \text{for any } T \in \mathbf{A}^\perp, S :: T \in \mathbf{B} \text{ and } \#S \geq \#\mathbf{A}\} \quad \mathbf{A} \parallel \mathbf{B} \triangleq \{S \parallel T \mid S \in \mathbf{A}, T \in \mathbf{B}\}^{\perp\perp}$$

**Remark 50** (Density of the parallel composition). For any two types  $\mathbf{A}$  and  $\mathbf{B}$  we have  $(\mathbf{A} \parallel \mathbf{B})^\perp = (\mathbf{A} \parallel \mathbf{B})^\perp$ , where  $\mathbf{A} \parallel \mathbf{B} = \{S \parallel T \mid S \in \mathbf{A}, T \in \mathbf{B}\}$ .

<sup>8</sup>Note that we require the *existence* of such a reduction, not all reductions need to behave this way.

<sup>9</sup>Equivalently, a type is a set  $\mathbf{A}$  such that  $\mathbf{A} = \mathbf{B}^\perp$  for some set  $\mathbf{B}$ , see, for instance, [11].

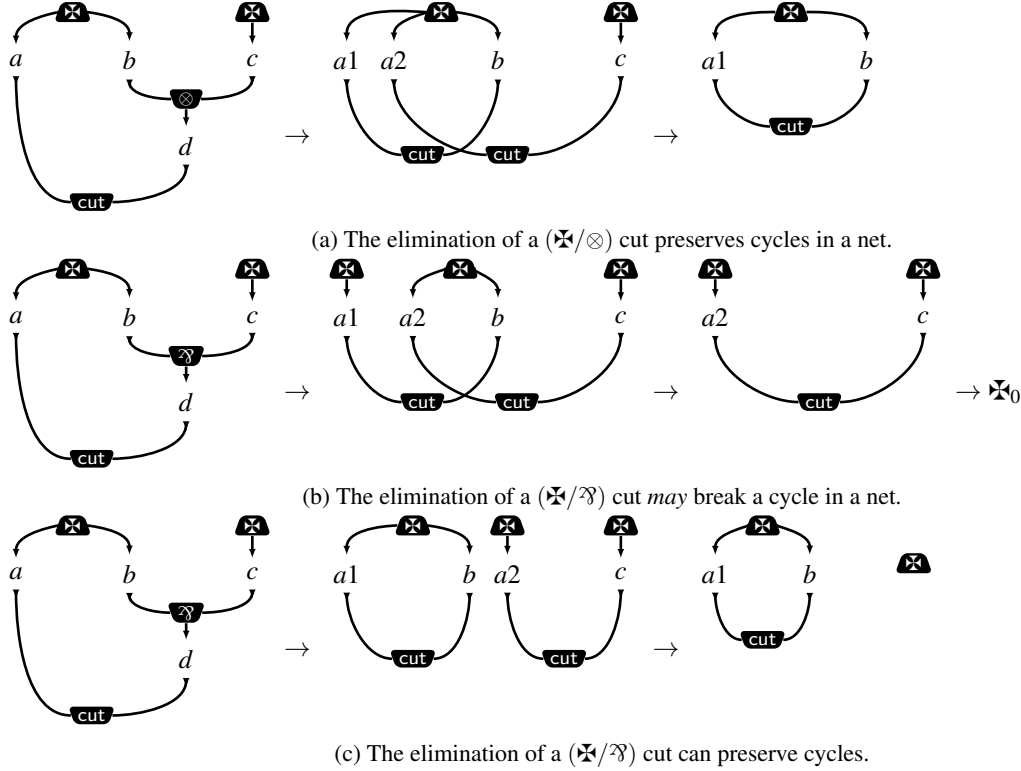


Figure 10: The evolution of (switching) cycles and (switching) disconnections during non homogeneous cut elimination.

**Proposition 51 (Duality).** *Given two types  $\mathbf{A}$  and  $\mathbf{B}$ :  $(\mathbf{A} \parallel \mathbf{B})^\perp = \mathbf{A}^\perp \succ \mathbf{B}^\perp$  and  $(\mathbf{A} \succ \mathbf{B})^\perp = \mathbf{A}^\perp \parallel \mathbf{B}^\perp$ .*

**Remark 52.** The duality of the constructions (Proposition 51) ensures that the set of types is closed under the  $\parallel$  and  $\succ$  operations. Moreover, the intersection of two types is still a type. This is not the case for the union which needs to be closed under bi-orthogonal.

**Remark 53.** For two types  $\mathbf{A}$  and  $\mathbf{B}$  the unordered nets of  $\mathbf{A} \parallel \mathbf{B}$  and of  $\mathbf{B} \parallel \mathbf{A}$  are the same, so as the unordered nets of  $\mathbf{A} \succ \mathbf{B}$  and  $\mathbf{B} \succ \mathbf{A}$ .

**Proposition 54.** *Given  $\mathbf{A}, \mathbf{B}$  and  $\mathbf{C}$  three types;  $(\mathbf{A} \succ \mathbf{B}) \succ \mathbf{C} = \mathbf{A} \succ (\mathbf{B} \succ \mathbf{C})$  and  $(\mathbf{A} \parallel \mathbf{B}) \parallel \mathbf{C} = \mathbf{A} \parallel (\mathbf{B} \parallel \mathbf{C})$ .*

**Definition 55.** Given  $\mathbf{A}$  and  $\mathbf{B}$  two types with one conclusion, we define their *tensor product* (denoted  $\otimes$ ) and their *compositional product* (denoted  $\wp$ ):

$$\mathbf{A} \otimes \mathbf{B} \triangleq \{S + \langle S(1), S(2) \triangleright_{\otimes} p \rangle \mid S \in \mathbf{A} \parallel \mathbf{B}\}^{\perp\perp} \quad \mathbf{A} \wp \mathbf{B} \triangleq \{S + \langle S(1), S(2) \triangleright_{\wp} p \rangle \mid S \in \mathbf{A} \succ \mathbf{B}\}^{\perp\perp}$$

where  $p$  denotes a fresh position.

**Proposition 56 (Duality).** *Given  $\mathbf{A}$  and  $\mathbf{B}$  two types with one conclusion,  $(\mathbf{A} \otimes \mathbf{B})^\perp = \mathbf{A}^\perp \wp \mathbf{B}^\perp$  and  $(\mathbf{A} \wp \mathbf{B})^\perp = \mathbf{A}^\perp \otimes \mathbf{B}^\perp$ .*

## 4 Realisability Model: Adequacy

We introduce our realisability model on untyped nets and prove it is adequate. We identify a sufficient property of interpretation bases to prove adequacy (Theorem 64): for any basis  $\mathcal{B}$  satisfying the property, a net  $S$  representing an  $\text{MLL}^{\otimes}$  proof of a sequent  $\Gamma$  is a realiser of  $\Gamma$  i.e. it belongs to  $\llbracket \Gamma \rrbracket_{\mathcal{B}}$ . This adequacy result immediately applies to  $\text{MLL}$ , since a net representing a proof of  $\text{MLL}$  represents, in particular, a proof of  $\text{MLL}^{\otimes}$ .

We start by giving an interpretation of formulas and hypersequents of multiplicative linear logic. We provide an interpretation of hypersequents instead of sequents as it turns out that handling hypersequents is more convenient and proving a result on hypersequents proves it on sequents too. However, do keep in mind that the proof trees we defined using Figure 6c are constructed with sequents.

**Definition 57.** An *interpretation basis*  $\mathcal{B}$  is a function that associates with each atomic proposition  $X$  a type  $\llbracket X \rrbracket_{\mathcal{B}}$ , the *interpretation* of  $X$ , such that:

- Each net in  $\llbracket X \rrbracket_{\mathcal{B}}$  has one conclusion.
- For any atomic proposition  $X$ , we have  $\llbracket X^\perp \rrbracket_{\mathcal{B}} \subseteq \llbracket X \rrbracket_{\mathcal{B}}^\perp$ .

**Definition 58.** Given an interpretation basis  $\mathcal{B}$ , the *interpretation* of MLL formulas and of hypersequents of MLL is defined by induction:

$$\begin{aligned} \llbracket A \otimes B \rrbracket_{\mathcal{B}} &\triangleq \llbracket A \rrbracket_{\mathcal{B}} \otimes \llbracket B \rrbracket_{\mathcal{B}}. & \llbracket \mathcal{H}_1, \mathcal{H}_2 \rrbracket_{\mathcal{B}} &\triangleq \llbracket \mathcal{H}_1 \rrbracket_{\mathcal{B}} \succ \llbracket \mathcal{H}_2 \rrbracket_{\mathcal{B}}. \\ \llbracket A \wp B \rrbracket_{\mathcal{B}} &\triangleq \llbracket A \rrbracket_{\mathcal{B}} \wp \llbracket B \rrbracket_{\mathcal{B}}. & \llbracket \mathcal{H}_1 \parallel \mathcal{H}_2 \rrbracket_{\mathcal{B}} &\triangleq \llbracket \mathcal{H}_1 \rrbracket_{\mathcal{B}} \parallel \llbracket \mathcal{H}_2 \rrbracket_{\mathcal{B}}. \end{aligned}$$

**Remark 59.** Using duality of types (Proposition 56) and the properties of orthogonality one proves that for an interpretation basis  $\mathcal{B}$  and an MLL formula  $A$  we have  $\llbracket A^\perp \rrbracket_{\mathcal{B}} \subseteq \llbracket A \rrbracket_{\mathcal{B}}^\perp$ .

**Definition 60.** A multiplicative net *realises* – with respect to an interpretation basis  $\mathcal{B}$  – an hypersequent  $\mathcal{H}$  of MLL formulas whenever it belongs to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}$ .

*Notation 61.* For a hypersequent  $\mathcal{H}$ , we will often write  $S \Vdash_{\mathcal{B}} \mathcal{H}$  instead of  $S \in \llbracket \mathcal{H} \rrbracket_{\mathcal{B}}$ , and sometimes  $S \Vdash \mathcal{H}$  or  $S \in \llbracket \mathcal{H} \rrbracket$  when there is no ambiguity on the basis  $\mathcal{B}$ .

From the point of view of cut elimination, a daimon link with  $n$  outputs may be thought as the approximation of a proof net with  $n$  outputs. More precisely, by iterating the process we have seen in Remark 24, every cut-free proof  $\pi$  of a formula  $C$  can be obtained by applying the cut elimination procedure to the daimon link  $\mathfrak{X}_1$  (of conclusion  $C$ ) cut against the appropriate identities of  $C, C^\perp$  (this generalises to a sequent  $\Gamma$  and  $\mathfrak{X}_n$ ). Furthermore daimon links and proof nets (with the same number of conclusions) are interchangeable with respect to geometrical correctness (Table 1): in a correct (resp. incorrect) net  $S$ , substituting a daimon link with  $n$  outputs by a proof net with  $n$  outputs produces a correct (resp. incorrect) net. However, proof nets and daimons (with the same number of conclusions) differ on realisability: for instance a proof net ending with a tensor link can never realise a formula of the form  $A \wp B$  whereas a daimon link can (Theorem 64). We will thus say that a daimon link “approximates” a sequent: this suggests Definition 62.

**Definition 62.** A type  $\mathbf{A}$  is *approximable* if and only if  $\mathfrak{X}_1 \in \mathbf{A}$ . A basis  $\mathcal{B}$  is *approximable* if for each  $X \in \text{Var}$ , the type  $\llbracket X \rrbracket_{\mathcal{B}}$  is approximable.

**Remark 63.** Because inclusion is preserved by bi-orthogonal closure, a type  $\mathbf{A}$  is approximable if and only if  $\{\mathfrak{X}_1\}^{\perp\perp} \subseteq \mathbf{A}$  which is equivalent to the inclusion  $\mathbf{A}^\perp \subseteq \{\mathfrak{X}_1\}^\perp$ .

**Theorem 64 (Adequacy).** Let  $\mathcal{B}$  be an approximable basis. For any net  $S$  and sequent  $\Gamma S \vdash_{\text{MLL}^\mathfrak{X}} \Gamma \Rightarrow S \Vdash_{\mathcal{B}} \Gamma$ .

*Proof.* The technique is standard in the works on realisability (see [12] or [14]): one proceeds by induction on the size of a proof  $\pi$  represented by  $S$ . For the base case one must show that  $\mathfrak{X}_n$  realises any sequent  $\Gamma$  with  $n$  formulas. To do so one first checks that, for any formula  $A$ ,  $\llbracket A \rrbracket_{\mathcal{B}}$  is approximable ( $\mathfrak{X}_1 \in \llbracket A \rrbracket_{\mathcal{B}}$ ).  $\square$

**Remark 65.** An approximable basis yields adequacy, in particular, for MLL. Notice, however, that there exist bases yielding an interpretation that is adequate for MLL but not for  $\text{MLL}^\mathfrak{X}$ .

## 5 Testability and tests

The partitions involved in the Danos Regnier criterion (Theorem 76) and their orthogonality with the daimons of a net can be translated as *tests*; so that for a formula  $A$ , a net  $S$  testable by  $A$  (definition 66 below) and orthogonal to tests( $A$ ) is a correct net (Theorem 76). We will show that these tests are proofs of  $\text{MLL}^\mathfrak{X}$  (Theorem 77). This means that for realisers in an approximable basis, testability (Definition 66) and correct typeability (Definition 31) coincide: this is Proposition 82.

**Definition 66** ((Atomic) testable cut-free nets). A *formula labelling* of a cut-free net  $S$  is a function  $\tau : V_S \rightarrow \text{Form}$  such that:

- **(Par)** When  $\langle p_1, p_2 \triangleright_{\wp} p \rangle$  occurs in  $S$ : if  $\tau(p_1) = A$  and  $\tau(p_2) = B$  then  $\tau(p) = A \wp B$ .
- **(Tens)** When  $\langle p_1, p_2 \triangleright_{\otimes} p \rangle$  occurs in  $S$ : if  $\tau(p_1) = A$  and  $\tau(p_2) = B$  then  $\tau(p) = A \otimes B$ .

A formula labelling of a cut-free net  $S$  is *atomic* when for each daimon link  $\langle \triangleright_{\mathfrak{X}} p_1, \dots, p_n \rangle$  in  $S$  the formula  $\tau(p_i)$  is a propositional variable.

A cut-free net  $S$  with  $n$  conclusions is *testable* (resp. *atomic testable*) by a sequent  $\Gamma = A_1, \dots, A_n$ , which we denote  $S \Vdash \Gamma$  (resp.  $S \Vdash^{\text{at}} \Gamma$ ), if there exists a formula (resp. an atomic formula) labelling  $\tau$  of  $S$  such that  $\tau(S(i)) = A_i$  for each  $1 \leq i \leq n$ .

**Remark 67.**  $S \vdash \Gamma$  iff  $S \stackrel{\text{af}}{\vdash} \Delta$  and  $\Gamma = \theta \Delta$  for some substitution  $\theta$  and sequent  $\Delta$ .

**Remark 68.**  $S \stackrel{\text{af}}{\vdash} \Gamma$  iff  $S$  without its  $\bowtie$ -links is the syntactic forest of (the formulas of)  $\Gamma$ .

**Remark 69.** A cut-free proof net  $S \vdash_{\text{MLL}}^{\bowtie} \Gamma$  is in particular testable by that sequent i.e.  $S \vdash \Gamma$ . However, a net  $S \vdash \Gamma$  which is testable by  $\Gamma$  may not be a proof net because it could contain cycles or disconnections: the testability condition only provides information on the multiplicative links constituting the net  $S$ . When is  $S$  atomic testable by  $A$ , orthogonality with the tests of  $A$  coincides with correctness ([Proposition 75](#)).

**Remark 70.** Let  $S \stackrel{\text{af}}{\vdash} A_1, \dots, A_n$  be a cut-free net. For any nets  $T_1, \dots, T_n$  cut-free and atomically testable respectively by  $A_1^\perp, \dots, A_n^\perp$  denoting  $S_0$  the normal form of  $S :: T_1 \parallel \dots \parallel T_n$ ,  $S_0$  is obtained by homogeneous cut-elimination, and we have (1)  $S_0$  equals  $\bowtie_0$  (2)  $S_0$  is equal to the sum of  $k \geq 2$  daimon without conclusions ( $S_0 = \sum_{1 \leq i \leq k} \bowtie_0$ ) or (3)  $S_0$  contains a cyclic cut ( $S_0 = R + \langle \triangleright_{\bowtie} \vec{q}, a, \vec{r}, b, \vec{p} \rangle + \langle a, b \triangleright_{\text{cut}} \rangle$ ).

**Remark 71.** Given a net  $S = (|S|, \mathbf{a}(S))$  we denote  $S^{\bowtie} = (|S^{\bowtie}|, \mathbf{a}(S^{\bowtie}))$  the net such that  $|S^{\bowtie}|$  is the hypergraph consisting of the daimon links occurring in  $S$ . The arrangement  $\mathbf{a}(S^{\bowtie})$  is induced by  $\mathbf{a}(S)$  because above every conclusion of  $S$  there is binary tree: each initial position  $p$  can be associated with a sequence  $\xi = \text{adr}(p)$  of  $\{\ell, r\}^*$  and an integer  $i = \text{root}(p)$  so that going up from  $S(i)$  following the left/right instruction of  $\xi$  one reaches the initial position  $p$ . The initial positions of  $S$  are then ordered by the lexicographical order of  $(\text{root}(p), \text{adr}(p))$  fixing  $\ell \leq r$ .

*Notation 72.* Given a net  $S$  with  $n$  initial positions, and  $P = \{C_1, \dots, C_k\}$  a partition of the initial positions of  $S$  we denote by  $\mathbf{Nat}_S(P)$  the partition  $\{\mathbf{a}(S^{\bowtie})^{-1}(C_1), \dots, \mathbf{a}(S^{\bowtie})^{-1}(C_k)\}$  of  $\{1, \dots, n\}$ . We might abusively write  $\mathbf{Nat}(P)$  for  $\mathbf{Nat}_S(P)$ .

**Proposition 73.** Let  $A$  be a formula, given two cut free nets  $S \stackrel{\text{af}}{\vdash} A$  and  $T \stackrel{\text{af}}{\vdash} A^\perp$  the assertions are equivalent:

1. The nets  $S$  and  $T$  are orthogonal.
2. The nets  $S^{\bowtie}$  and  $T^{\bowtie}$  are orthogonal.
3. The partition  $\mathbf{Nat}_S(\mathbf{P}_{\bowtie}(S))$  and  $\mathbf{Nat}_T(\mathbf{P}_{\bowtie}(T))$  are orthogonal.

**Definition 74.** A cut-free net  $T$  is a *test* of a formula  $A$  if  $T \stackrel{\text{af}}{\vdash} A^\perp$  and there exists a net  $S \stackrel{\text{af}}{\vdash} A$  and a switching  $\sigma S$  such that  $\mathbf{Nat}_T(\mathbf{P}_{\bowtie}(T)) = \mathbf{Nat}_S(\uparrow^i \sigma S)$ . We denote by  $\text{tests}(A)$  the set  $\{S \mid S \text{ is a test of } A\}$ .

**Proposition 75.** For  $S$  cut-free,  $S \stackrel{\text{af}}{\vdash} A$ , we have:  $S \vdash_{\text{MLL}}^{\bowtie} A \Leftrightarrow S \perp \text{tests}(A)$ .

A net  $S$  with  $n$  conclusion can always be transformed in a net with 1 conclusion by putting a bunch of par-links below its conclusions; this allows to generalise the previous proposition.

**Theorem 76 (Danos–Regnier Tests).** Given a cut-free net  $S \stackrel{\text{af}}{\vdash} A_1, \dots, A_n$ ;  $S \vdash_{\text{MLL}}^{\bowtie} A_1, \dots, A_n$  if and only if  $S$  is orthogonal to  $\text{tests}(A_1) \parallel \dots \parallel \text{tests}(A_n)$ .

**Theorem 77.** Any test  $T$  of a formula  $A$  is correctly typeable by  $A^\perp$ ,  $T \vdash_{\text{MLL}}^{\bowtie} A^\perp$ .

*Proof.* Consider a test  $T$  of  $A$  then by [Theorem 76](#) any net  $S \vdash_{\text{MLL}}^{\bowtie} A$  is orthogonal to  $T$ . By the counter-proof criterion [[4](#)] a net  $N \stackrel{\text{af}}{\vdash} A^\perp$  orthogonal to each proof of  $A$  is a proof; therefore it follows that  $T$  is a proof of  $A^\perp$ .  $\square$

**Remark 78.** [Theorem 76](#) is a refinement of the counter-proof criterion of P.L. Curien [[4](#)]: if  $S \stackrel{\text{af}}{\vdash} A$  and  $S \perp \text{tests}(A)$  then  $S \vdash_{\text{MLL}}^{\bowtie} A$  – and every element of  $\text{tests}(A)$  are proofs of  $A^\perp$  ([Theorem 77](#)), but the converse does not hold.

From [Theorem 76](#) and [Theorem 77](#) one obtains an “interactive” criterion for the nets of multiplicative linear logic (MLL). One takes a net of  $S$  of MLL (i.e. a net with binary daimons) and confronts it with the tests of the according formulas ([Definition 74](#)). A straightforward consequence of the [Theorem 76](#) is the reformulation of B echet’s theorem in our framework.

**Corollary 79.** Let  $S \stackrel{\text{af}}{\vdash} A_1, \dots, A_n$  be a cut-free net. If  $S$  is not correct then there exists nets  $T_1 \in \text{tests}(A_1), \dots, T_n \in \text{tests}(A_n)$  such that the normal form of  $S :: T_1 \parallel \dots \parallel T_n$  is not correct: we are in case (2) or (3) of [Remark 70](#).

**Remark 80.** The [Corollary 79](#) obviously applies to MLL nets, the main difference with B echet’s original result is that his opponents are MLL proof nets (in our framework they are  $\text{MLL}^{\bowtie}$  proof nets). However it is not difficult to adapt our techniques to obtain B echet’s result.

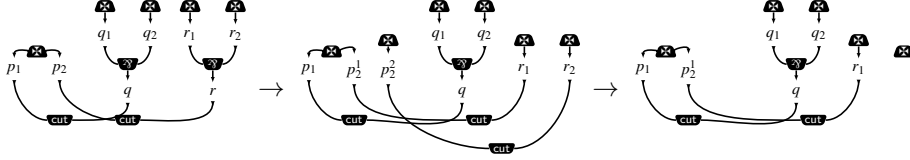


Figure 11: The daimon link  $\mathfrak{X}_2$  is not orthogonal to  $\mathfrak{X}_{\mathfrak{X}} \parallel \mathfrak{X}_{\mathfrak{X}}$ : a disconnected net never reduces to a connected one (and  $\mathfrak{X}_0$  is connected).

**Remark 81.** Consider an approximable basis  $\mathcal{B}$  and a sequent  $\Gamma = A_1, \dots, A_n$  we have  $\llbracket \Gamma \rrbracket_{\mathcal{B}} = (\llbracket A_1 \rrbracket_{\mathcal{B}}^\perp \parallel \dots \parallel \llbracket A_n \rrbracket_{\mathcal{B}}^\perp)^\perp$ . By [Theorem 64](#), for any  $A_i^\perp$  we have  $\llbracket A_i^\perp : \text{MLL}^{\mathfrak{X}} \rrbracket \subseteq \llbracket A_i^\perp \rrbracket_{\mathcal{B}}$  while  $\text{tests}(A_i) \subseteq \llbracket A_i^\perp : \text{MLL}^{\mathfrak{X}} \rrbracket$  ([Theorem 77](#)) thus  $\text{tests}(A_i) \subseteq \llbracket A_i^\perp \rrbracket_{\mathcal{B}} \subseteq \llbracket A_i \rrbracket_{\mathcal{B}}^\perp$  ([Remark 59](#)). Because the  $\parallel$ -construction preserves inclusions and orthogonality inverts inclusions we derive that  $\llbracket \Gamma \rrbracket_{\mathcal{B}} \subseteq (\text{tests}(A_1) \parallel \dots \parallel \text{tests}(A_n))^\perp$ .

[Remark 81](#) combined with the previous theorem ([Theorem 76](#)) means that for realisers in an approximable basis, testability and (correct) typeability collapse.

**Proposition 82.** Given  $\mathcal{B}$  an approximable basis<sup>10</sup> and a sequent  $\Gamma$  for any cut-free net  $S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$  the assertions are equivalent:

1.  $S \simeq \Gamma$  i.e.  $S \stackrel{\text{ad}}{\vdash} \Delta$  for some sequent  $\Delta \leq \Gamma$ .
2.  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Gamma$ .

## 6 Completeness

Using [Proposition 82](#) we provide a completeness result; we exhibit an approximable basis for which a net  $S$  realising a sequent  $\Gamma$  is testable, and so equivalently  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Gamma$ . This basis, denoted  $\mathbf{1}$ , maps each atomic formula to  $\{\mathfrak{X}_1\}^{\perp\perp}$ .

**Proposition 83.** For any sequent  $\Gamma$  and any cut-free net  $S$ ; if  $S \in \llbracket \Gamma \rrbracket_{\mathbf{1}}$  then  $S \simeq \Gamma$ .

**Remark 84.** By the [Proposition 83](#) and the [Theorem 64](#) we have that  $S \in \llbracket \Gamma \rrbracket_{\mathbf{1}}$  iff  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Gamma$ .

Since the base  $\mathbf{1}$  is approximable, [Proposition 82](#) allows to prove:

**Theorem 85** (MLL <sup>$\mathfrak{X}$</sup>  completeness). Given a cut-free net  $S$  and a sequent  $\Gamma$ ;

- If for all basis  $\mathcal{B}$  we have  $S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$ , then  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Gamma$ .
- $S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$  for any approximable basis  $\mathcal{B}$  iff  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Gamma$ .

**Remark 86.** The non homogeneous cut elimination allows to distinguish the types  $\llbracket X, X^\perp \rrbracket_{\mathcal{B}}$  and  $\llbracket X, Y \rrbracket_{\mathcal{B}}$  for a well chosen basis: for instance for the basis, that we will denote  $\mathcal{B}^{(\mathfrak{X})}$ , which maps positive propositional variables to  $\{\mathfrak{X}_{\mathfrak{X}}\}^\perp$  and negative propositional variables to  $\{\mathfrak{X}_{\mathfrak{X}}\}^{\perp\perp}$ , where  $\mathfrak{X}_{\mathfrak{X}}$  denotes the geometrically incorrect net  $\langle \triangleright_{\mathfrak{X}} a \rangle + \langle a, b \triangleright_{\mathfrak{X}} c \rangle$ .

In that case, (1) because  $\mathfrak{X}_2$  is not orthogonal to  $\mathfrak{X}_{\mathfrak{X}} \parallel \mathfrak{X}_{\mathfrak{X}}$  ([Figure 11](#)) it follows that  $\mathfrak{X}_2 \notin \llbracket X, X \rrbracket_{\mathcal{B}^{(\mathfrak{X})}}$  and more generally  $\mathfrak{X}_2 \notin \llbracket X, Y \rrbracket_{\mathcal{B}^{(\mathfrak{X})}}$ ; (2) by the property expressed in [Remark 90](#) (and illustrated in [Figure 12](#)),  $\mathfrak{X}_2 \in \llbracket X, X^\perp \rrbracket_{\mathcal{B}^{(\mathfrak{X})}}$ ; (3) point (1) above is not in contradiction with the theorem of adequacy ([Theorem 64](#)) because, even though  $\mathfrak{X}_2 \vdash_{\text{MLL}^{\mathfrak{X}}} X, Y$ , the basis  $\mathcal{B}^{(\mathfrak{X})}$  is not approximable.

**Remark 87.** The ability to distinguish realisers of the sequents  $X, X^\perp$  and  $X, Y$  ([Remark 86](#)) allows us to derive the completeness result for MLL ([Theorem 88](#)) from the completeness result for MLL <sup>$\mathfrak{X}$</sup>  ([Theorem 85](#)). In [Remark 86](#), to show that  $\mathfrak{X}_2 \notin \llbracket X, Y \rrbracket_{\mathcal{B}^{(\mathfrak{X})}}$  we have used incorrect nets (specifically  $\mathfrak{X}_{\mathfrak{X}}$ ), which explains that the completeness theorem for MLL ([Theorem 88](#)) refers to *any* basis  $\mathcal{B}$  (and not only to approximable basis). In the terms of [Table 1](#), we retrieve provability correctness by using interactions with geometrically incorrect nets.

**Theorem 88** (MLL completeness). Let  $S$  be a cut-free net such that each of its daimon link has exactly two outputs,  $\Gamma$  be a sequent such that  $S \stackrel{\text{ad}}{\vdash} \Gamma$ ; if  $S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$  for any basis  $\mathcal{B}$  then,  $S \vdash_{\text{MLL}} \Gamma$ .

**Remark 89.** A result of adequacy for MLL can also be stated: given an interpretation basis  $\mathcal{B}$  (not necessarily approximable) such that for each propositional variable  $X$  we have  $\llbracket X^\perp \rrbracket_{\mathcal{B}} = \llbracket X \rrbracket_{\mathcal{B}}^\perp$ , for any net  $S$ , if  $S \vdash_{\text{MLL}} \Gamma$  then  $S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$ .

<sup>10</sup>The [Proposition 82](#) actually holds for any “adequate” basis  $\mathcal{B}$ .

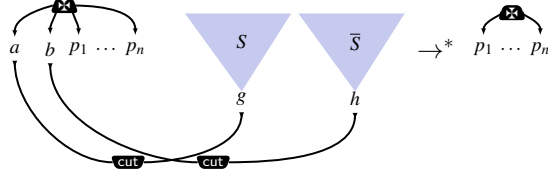


Figure 12: The interaction of two orthogonal nets  $S$  and  $\bar{S}$  with a daimon reduces to a daimon (with two less outputs).

**Remark 90.** The completeness result for MLL (Theorem 88) only identifies cut-free and *atomic* proofs (i.e. where axioms introduce sequents of the form  $X, X^\perp$ ). This is because for any atomic formulas  $X$  and  $Y$ , and for any basis  $\mathcal{B}$  such that  $\llbracket X^\perp \rrbracket_{\mathcal{B}} = \llbracket X \rrbracket_{\mathcal{B}}^\perp$ ,  $\mathfrak{X}_2 \in \llbracket X \wp X^\perp, Y \wp Y^\perp \rrbracket_{\mathcal{B}}$  while  $X \wp X^\perp$  and  $Y \wp Y^\perp$  are not dual formulas: contrary to the atomic case we cannot use  $\mathfrak{X}_2$  to distinguish  $\llbracket X \wp X^\perp, Y \wp Y^\perp \rrbracket_{\mathcal{B}(\wp)}$  from  $\llbracket X \wp X^\perp, X^\perp \otimes X \rrbracket_{\mathcal{B}(\wp)}$ .

The fact that  $\mathfrak{X}_2 \in \llbracket X \wp X^\perp, Y \wp Y^\perp \rrbracket_{\mathcal{B}(\wp)}$  (and more generally for any basis  $\mathcal{B}$  such that  $\llbracket X^\perp \rrbracket_{\mathcal{B}} = \llbracket X \rrbracket_{\mathcal{B}}^\perp$ ) is derived from the fact that, for any integer  $k$  and for any two orthogonal nets  $S_1$  and  $S_2$  with one conclusion, the interaction  $\mathfrak{X}_{k+2} :: (S_1 \parallel S_2)$  has *at least one* reduction to  $\mathfrak{X}_k$  by cut elimination (Figure 12). We use this property for  $k = 2$  and  $k = 4$  to show that  $\mathfrak{X}_2 \in \llbracket X \wp X^\perp, Y \wp Y^\perp \rrbracket_{\mathcal{B}}$ . More precisely, we prove that,  $\mathfrak{X}_2 \perp \llbracket X \wp X^\perp \rrbracket_{\mathcal{B}}^\perp \parallel \llbracket Y \wp Y^\perp \rrbracket_{\mathcal{B}}^\perp$ : given  $S, \bar{S}$  and  $R, \bar{R}$  two pairs of orthogonal nets (with one conclusion), when all nets  $S, \bar{S}, R, \bar{R}$  have disjoint sets of vertices, we can derive the following:

$$\begin{aligned}
& \langle \triangleright_{\mathfrak{X}} a, b \rangle :: S + \bar{S} + \langle S(1), \bar{S}(1) \triangleright_{\otimes} q \rangle + R + \bar{R} + \langle R(1), \bar{R}(1) \triangleright_{\otimes} r \rangle \\
\rightarrow \cdot \rightarrow & \langle \triangleright_{\mathfrak{X}} a_1, a_2, b_1, b_2 \rangle :: S + \bar{S} + R + \bar{R} \\
\rightarrow^* & \langle \triangleright_{\mathfrak{X}} b_1, b_2 \rangle :: R + \bar{R} \\
\rightarrow^* & \mathfrak{X}_0
\end{aligned}$$



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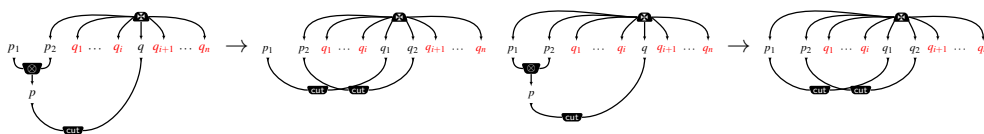
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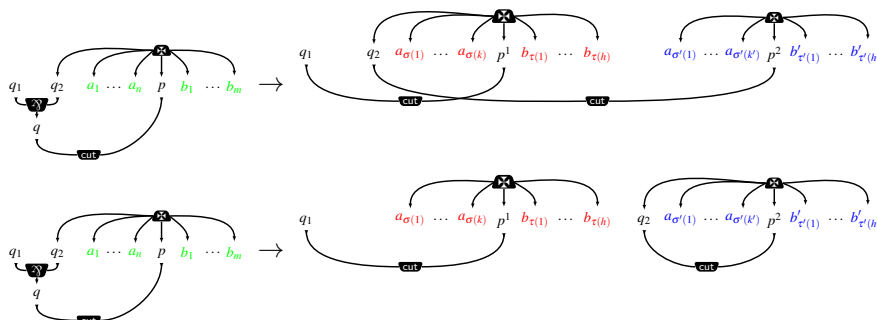
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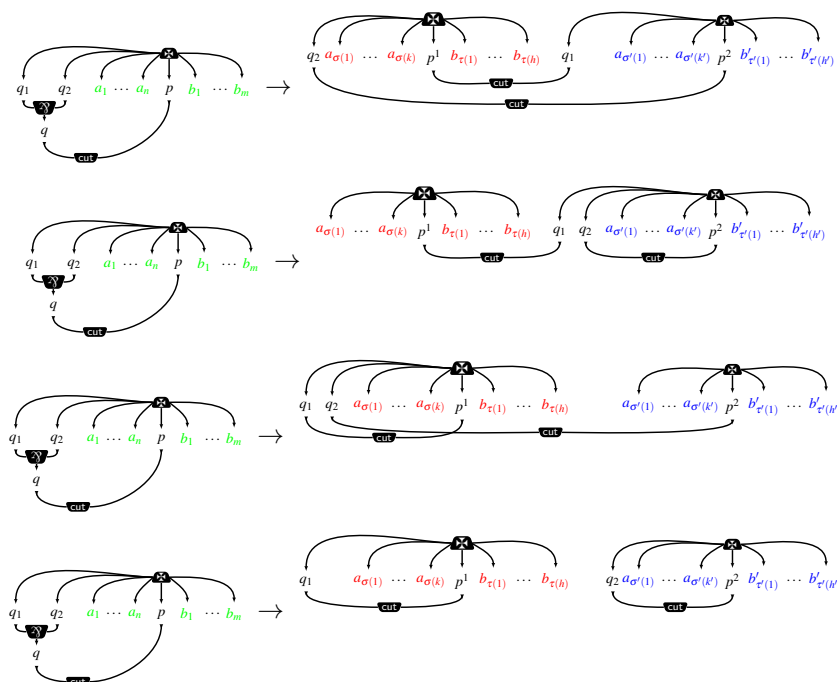
## A Additional Figures



(a) Extra cases for the elimination of  $(\boxtimes/\otimes)$  cuts, on the left the elimination step when one of the inputs belongs to the daimon above the cut, on the right the elimination step when both inputs belong to the daimon above the cut.



(b) Extra cases for the elimination of  $(\boxtimes/\otimes)$  cuts: when one of the inputs belongs to the daimon above the cut.



(c) Extra cases for the elimination of  $(\boxtimes/\otimes)$  cuts: when both inputs belong to the daimon above the cut.

Figure 13: Complements to Figure 4 for defining non homogeneous cut elimination (Definition 22).

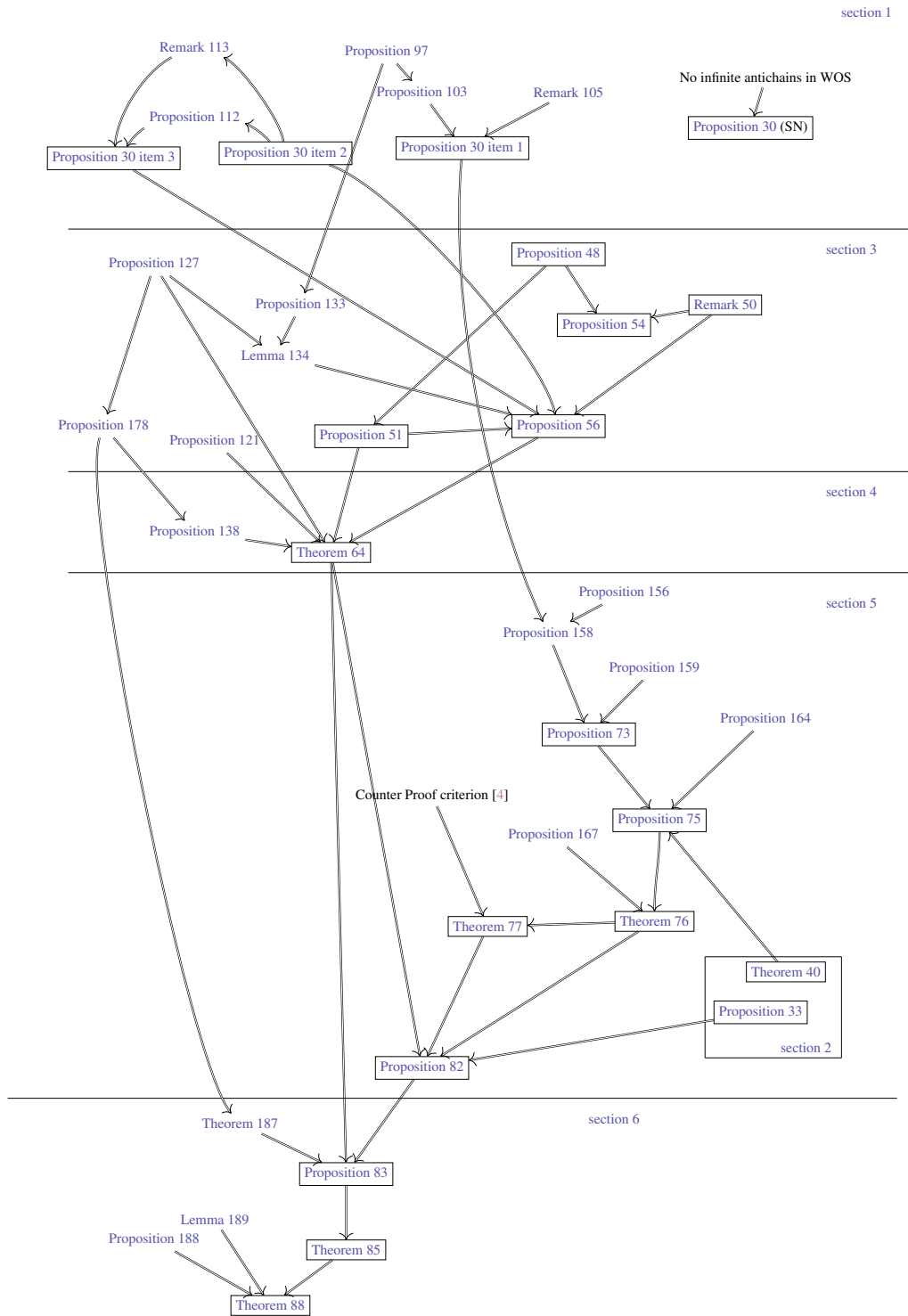


Figure 14: Relation of “dependency” between propositions in this document (one can observe that this is an acyclic graph!). The nodes of the graph are (some of) the propositions in the document, while an edge from a proposition  $P$  to proposition  $Q$  means that in order to prove  $Q$  we have used the proposition  $P$ . Nodes inside a rectangle are propositions which occur in the main text of the paper, while nodes without rectangle are proposition occurring only in the appendix.

## B Complements to section 1

We recall the notion of homomorphism and isomorphism between hypergraphs.

**Definition 91** (Homomorphism and isomorphisms of hypergraphs). An *homomorphism* between two (labelled) hypergraphs  $\mathcal{H}_1 = (V_1, E_1, s_1, t_1, \ell_1)$  and  $\mathcal{H}_2 = (V_2, E_2, s_2, t_2, \ell_2)$  is a pair of bijective maps  $\langle f^V, f^E \rangle$ , such that

- (COMMUTATION) For any edge  $e_1$  of  $\mathcal{H}_1$ ,  $s_2(f^E(e_1)) = f^V(s_1(e_1))$ , i.e. the following diagram commutes:

$$\begin{array}{ccc} e_1 & \xrightarrow{s_1} & s_1(e_1) \\ f^E \downarrow & & \downarrow f^V \\ e_2 & \xrightarrow{s_2} & s_2(e_2) \end{array}$$

- (LABEL-PRESERVING) The function  $f^E$  preserves the labels of the links, i.e. for any link  $e$  in  $\mathcal{H}_1$ ;  $\ell(e) = \ell(f^E(e))$

The homomorphism is an isomorphism if both  $f^V$  and  $f^E$  are bijective.

### B.1 Rewriting properties of nets: Strong Normalisation

**Proposition 92** (Strong normalisation [Proposition 30](#)). *The cut elimination rewriting  $\rightarrow$  is strongly normalising.*

*Proof.* This is obtained by observing that an appropriate measure on nets always decrease with cut elimination, i.e. whenever  $S \rightarrow S'$  then  $m(S) > m(S')$ . Such a measure  $m$  may be the pairs  $(\text{connective}(S), \text{cut}(S))$  where  $\text{connective}(S)$  is the number of  $\otimes$ - and  $\wp$ - links in  $S$ , while  $\text{cut}(S)$  is the number of cut links in  $S$ . The order involved is then the lexicographical order.

Indeed we have in that case that  $S \rightarrow S'$  implies  $m(S) > m(S')$ : the  $(\otimes/\wp)$ ,  $(\wp/\otimes)$  and multiplicative steps always decrease the number  $\text{connective}(S)$ , finally the glueing steps do not change  $\text{connective}(S)$  however they make the number of cuts  $\text{cut}(S)$  decrease.  $\square$

### B.2 Rewriting properties of nets: unrelated cuts

*Notation 93.* Given two binary relations  $R_1$  and  $R_2$  on a set  $X$  we denote by  $R_1 \cdot R_2$  the composition of the two relation, i.e. for two  $x, y \in X$   $xR_1 \cdot R_2 y$  if and only if there exists  $z$  such that  $xR_1 z$  and  $zR_2 y$ .

*Notation 94.* Given a net  $S$  and cut link  $c$  we denote by  $S \xrightarrow{c} S'$ , whenever  $S \rightarrow S'$  by eliminating the cut  $c$ . Note that whenever  $S \rightarrow S'$  there exists a cut  $c$  such that  $S \rightarrow cS'$ , namely the unique cut which belong to  $S$  but not  $S'$ .

*Notation 95.* We denote by  $\rightarrow_{\text{mult}}$  the relation between nets such that  $S \rightarrow_{\text{mult}} S'$  whenever  $S \xrightarrow{c} S'$  and  $c$  is multiplicative, on the other hand we denote  $S \rightarrow_{\neg \text{mult}} S'$  whenever  $S \xrightarrow{c} S'$  and  $c$  is not a multiplicative cut. Similarly one defined  $\rightarrow_{\text{glue}}$ ,  $\rightarrow_{\neg \text{glue}}$ ,  $\rightarrow_{(\wp/\otimes)}$ ,  $\rightarrow_{\neg(\wp/\otimes)}$ ,  $\rightarrow_{(\otimes/\wp)}$  and  $\rightarrow_{\neg(\otimes/\wp)}$ .

**Definition 96.** Given a net  $S$  two cuts  $c_1$  and  $c_2$  in  $S$  are *unrelated* whenever:

- $S$  can be written as  $C + R_1 + R_2$  where  $R_1$  is the redex of the cut  $c_1$  and  $R_2$  is the redex of the cut  $c_2$ .
- By eliminating  $c_1$ ,  $S$  reduces to  $C + R'_1 + R_2$ .
- By eliminating  $c_2$ ,  $S$  reduces to  $C + R_1 + R'_2$ .

**Proposition 97** (Strong Confluence of unrelated cuts). *Given  $S$  some net containing two reducible and unrelated cuts  $c_1$  and  $c_2$ . The diagram below commutes:*

$$\begin{array}{ccc} S & \xrightarrow{c_1} & S_1 \\ \downarrow c_2 & & \downarrow c_1 \\ S'_1 & \xrightarrow{c_2} & S_2 \end{array}$$

Where the dotted arrows are the existence of a reduction.

*Proof.* Let us fix a net  $S$  containing (at least) two unrelated cuts that we denote  $c_1$  and  $c_2$ . This means that one can write  $S$  as  $C + R_1 + R_2$  where the  $R_i$  are the corresponding redexes. Assuming (without loss of generality) that  $R_i \rightarrow R'_i$ : by contextual closure, the elimination of  $c_1$  rewrites  $S$  in  $S_1 = C + R'_1 + R_2$  and the elimination of  $c_2$  rewrites  $S$  in  $S_2 = C + R_1 + R'_2$

Using contextual closure once more we conclude that  $S_1$  reduces to  $C + R'_1 + R'_2$  by eliminating  $c_2$  while  $S_2$  reduces to  $C + R'_1 + R'_2$  by eliminating  $c_1$ .  $\square$

**Remark 98.** Consider a cut  $c$  which is not multiplicative in a net  $S$  if  $c$  is not related with a cut  $c'$  in  $S$  then if  $S \xrightarrow{c} S'$  the cut produced by  $c$  are also unrelated with  $c'$ .

**Definition 99** (Strongly unrelated cuts). We say that two cuts  $c_1 = \langle p_1, q_1 \triangleright_{\text{cut}} \rangle$  and  $c_2 = \langle p_2, q_2 \triangleright_{\text{cut}} \rangle$  are *strongly unrelated* whenever they are distinct and the set  $D_1$  of daimon links above  $p_1$  or  $q_1$  does not intersect with the set  $D_2$  of daimon links above  $p_2$  or  $q_2$ .

**Remark 100.** The notion of strongly unrelated cuts is to obtain the stability of unrelated cuts (as in [Remark 98](#)) also for multiplicative cuts. Given a multiplicative cut  $c$  of a net  $S$  and a cut  $c'$  of  $S$  unrelated with  $c$ , if  $S \xrightarrow{c} S'$  then the cut produced by the elimination of  $c$  remain unrelated with  $c$  in  $S'$ .

**Proposition 101.** Given a net  $S$  and a partition of its set of cut links  $\text{cut}(S) = C_1 \uplus C_2$  such that each cuts of  $C_1$  and  $C_2$  are strongly unrelated then the assertions are equivalent:

1.  $S \rightarrow^* T$
2.  $S \xrightarrow{*}_1 S' \xrightarrow{*}_2 T$ , where  $\rightarrow_1$  eliminates cut from  $C_1$  or the cut it produces and  $\rightarrow_2$  eliminates cuts from  $C_2$  or the cut it produces.

*Proof.*  $1 \Rightarrow 2$ . If two cut  $c$  and  $d$  are unrelated: if  $c$  is not multiplicative, any cut created by the elimination of  $c$  will still be unrelated with  $d$  ([Remark 98](#)). If  $c$  is multiplicative since  $C_1$  and  $C_2$  are strongly unrelated and thus  $c$  and  $d$  are strongly unrelated cuts, the non multiplicative cuts say  $c_1$  and  $c_2$  created by the elimination of  $c$  involve a different daimon link than that of  $d$  thus  $c_1$  and  $c_2$  are still unrelated with  $d$  ([Remark 100](#)). Therefore, it follows that for each step of cut elimination  $\xrightarrow{c}_1$  and  $\xrightarrow{c}_2$  the cuts  $c$  and  $c'$  are unrelated thus using the property of confluence of unrelated cuts ([Proposition 97](#)) one concludes.

$2 \Rightarrow 1$ . indeed because  $\rightarrow_1$  and  $\rightarrow_2$  are special case of the cut elimination  $\rightarrow$ . □

### B.3 Multiplicative Factorization of Cut Elimination – Proof of [Proposition 30 item 1](#)

**Remark 102.** Given a cut  $c$  in a net  $S$  if  $c$  is a multiplicative cut then for any other cut  $c'$  of  $S$ :  $c$  and  $c'$  are unrelated, because the redexes of cuts only involve three links made of the cut link and the two links above the input of that cut. Therefore two cuts of two distinct multiplicative redexes cannot involve a same connective link since otherwise it means that these two cuts have for input a same position i.e. the target of the connective link (this violates the property of *target-disjointness* of nets).

**Proposition 103** (General commutation of the Multiplicative cuts). For any net  $S$  containing a multiplicative cut  $c$ , for any sequence of cuts  $\alpha^*$  in  $S$  then the following diagrams commutes;

$$\begin{array}{ccc} S & \xrightarrow{c} & S_1 \\ \downarrow \alpha^* & & \downarrow \alpha^* \\ S'_1 & \xrightarrow{c} & S_2 \end{array}$$

Where the dotted arrows are the existence of a reduction.

*Proof.* Because the multiplicative cut  $c$  is unrelated with any cut  $d$  of  $S$  ([Remark 102](#)) we conclude using the confluence of unrelated cuts ([Proposition 97](#)). □

**Remark 104.** If  $S$  reduces to  $S'$  by eliminating a non multiplicative cut then any multiplicative cut  $c$  occurring in  $S'$  is a multiplicative cut occurring in  $S$ .

**Remark 105.** Given a net  $S$  containing a cut  $c$  and one of its reduction  $S \rightarrow_c S'$ , if a multiplicative or clash cut  $c'$  occurs in  $S'$  and not  $S$ : then  $c$  was a multiplicative cut.

This is because (1) glueing cuts make cuts disappear (2)  $(\otimes/\boxtimes)$  and  $(\boxtimes/\boxtimes)$  cuts create two new cuts which can be of types  $(\otimes/\boxtimes)$ ,  $(\boxtimes/\boxtimes)$  or  $(\boxtimes/\boxtimes)$  but these new cuts are never multiplicative cuts.

**Proposition 106** (Cut Elimination Factorisation). The relation induced by the cut-elimination rewriting  $\rightarrow^*$  of [Definition 22](#) and [Definition 19](#) is equal to  $\rightarrow_{\text{mult}}^* \cdot \rightarrow_{\text{-mult}}^*$ .

*Proof.* The inclusion of  $\rightarrow_{\text{mult}}^* \cdot \rightarrow_{\text{-mult}}^*$  in  $\rightarrow^*$  is obvious.

Let us now show the converse inclusion. We do so by induction on the length of the reduction sequence. If there is only one reduction this is trivial.

Let us treat the general case when  $S \rightarrow^* S'$  doing a sequence of  $n > 1$  steps. Then this can be decomposed as  $S \rightarrow^* S_0 \xrightarrow{c} S'$ ; apply the induction hypothesis yielding that there exists a net  $S_1$  such that  $S \rightarrow_{\text{mult}}^* S_1 \rightarrow_{\text{-mult}}^* S_0$ . Two case may occur depending on the type of the cut  $c$ .



- If  $c$  is not multiplicative; then  $S_0 \rightarrow_{\neg\text{mult}} S'$  and from  $S \rightarrow_{\text{mult}}^* \cdot \rightarrow_{\neg\text{mult}}^* S_0$ . we derive  $S \rightarrow_{\text{mult}}^* \cdot \rightarrow_{\neg\text{mult}}^* S'$ .
  - If  $c$  is multiplicative; note that whenever  $T \rightarrow_{\neg\text{mult}}^* T'$  a multiplicative cut  $c$  occurring in  $T'$  will also occur in  $T$  and be also a multiplicative cut (**Remark 104**). To conclude however it will be necessary to do an induction on the reduction sequence  $S_1 \rightarrow_{\neg\text{mult}}^* S_0$ :
    - If it contains only one step of reduction; then the situation is the following  $S_1 \xrightarrow{c'} S_0 \xrightarrow{c} S'$  where  $c'$  is not multiplicative and  $c$  is a multiplicative cut. Applying **Remark 104** non multiplicative cuts cannot create multiplicative cuts, therefore the cut  $c$  also appears in  $S_1$  and thus both cuts  $c$  and  $c'$  occur in  $S_1$ . As a consequence since multiplicative commute (**Proposition 103**) it follows that  $S_1 \xrightarrow{c} S_0 \xrightarrow{c'} S'$ . Therefore  $S_1 \rightarrow_{\text{mult}}^* \rightarrow_{\neg\text{mult}}^* S'$ .
    - If the sequence of non multiplicative cut–elimination contains  $n$  element. Note that it may be factorized as  $S_1 \rightarrow_{\neg\text{mult}}^* S_2 \rightarrow_{\neg\text{mult}} S_0$  while  $S_0 \rightarrow_{\text{mult}} S'$ . Applying the previous result (the case when the  $\rightarrow_{\neg\text{mult}}$  sequence is of length 1) we derive that  $S_1 \rightarrow_{\neg\text{mult}}^* S_2 \rightarrow_{\text{mult}} S'_0 \rightarrow_{\neg\text{mult}} S'$ . Since the sequence  $S_1 \rightarrow_{\neg\text{mult}}^* S_2$  has a smaller length than that of  $S_1 \rightarrow_{\neg\text{mult}}^* S_2 \rightarrow_{\neg\text{mult}} S_0$  we call the induction hypothesis and ensure  $S_1 \rightarrow_{\text{mult}}^* \cdot \rightarrow_{\neg\text{mult}}^* S'_0$ . Finally since  $S'_0 \rightarrow_{\neg\text{mult}} S'$  we derive  $S_1 \rightarrow_{\text{mult}}^* \cdot \rightarrow_{\neg\text{mult}}^* S'$ .
- Now we can conclude; the situation was  $S \rightarrow_{\text{mult}}^* S_1 \rightarrow_{\neg\text{mult}}^* S_0 \rightarrow_{\text{mult}} S'$ . But we have shown that  $S_1 \rightarrow_{\neg\text{mult}}^* S_0 \rightarrow_{\text{mult}} S'$  yields  $S_1 \rightarrow_{\text{mult}}^* \cdot \rightarrow_{\neg\text{mult}}^* S'$ . This yields  $S \rightarrow_{\text{mult}}^* S_1 \rightarrow_{\text{mult}}^* \cdot \rightarrow_{\neg\text{mult}}^* S'$  which means  $S \rightarrow_{\text{mult}}^* \cdot \rightarrow_{\neg\text{mult}}^* S'$ . □

**Remark 107.** The previous proposition is a novelty of the cut elimination with generalised axiom we have introduced. The standard cut–elimination procedure of MLL proof structure ([7]) does not enjoy such a factorisation: mainly because the **Remark 105** fails, a non multiplicative cut (that is in that case an (ax/cut) cut) can be eliminated creating a single cut that may be a multiplicative cut.

**Remark 108.** If  $S \rightarrow_{\text{mult}}^* S_0$  i.e.  $S$  reduces to  $S_0$  by eliminating multiplicative cuts only, then  $S$  and  $S_0$  have the same orthogonal.

## B.4 Delaying Irreversible cuts elimination – proof of **Proposition 30 item 2**

**Proposition 109** (Non–Deterministic non homogeneous cut elimination commutes to the right). *Given  $S$  some net containing a non homogeneous cut link  $c$  of a daimon link against a  $\bowtie$ –link. The diagram below commutes, for any cut kind of cut link  $c'$ ;*

$$\begin{array}{ccc} S & \xrightarrow{c} & S_1 \\ \downarrow c' & & \downarrow c' \\ S'_1 & \dashrightarrow^{c'} & S_2 \end{array}$$

Where the dotted arrows are the existence of a reduction.

*Proof.* If  $c'$  is a multiplicative cut the commutation holds (**Proposition 103**). And more generally if  $c'$  and  $c$  are unrelated then the commutation also hold (**Proposition 97**). Thus  $c'$  is a glueing or non homogeneous cut involving the same daimon link as  $c$ .

Let us treat each cases depending on the type of the cut  $c'$ ;

- Assuming that  $c'$  is a glueing cut, and assuming (without loss of generality) that the two first conclusions of the daimon link are the premisses of the cut links  $c$  and  $c'$ , the reduction will be of the form:

$$\begin{aligned} & \langle \triangleright_{\bowtie} p_1, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\bowtie} r \rangle + \langle \triangleright_{\bowtie} q_1, \dots, q_k \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2, q_1 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\bowtie} p_1, p_2, A \rangle + \langle \triangleright_{\bowtie} p_1^2, B \rangle + \langle \triangleright_{\bowtie} q_1, \dots, q_k \rangle + \langle p_1^1, r_2 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle + \langle p_2, q_1 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\bowtie} p_1^1, q_2, \dots, q_k, A \rangle + \langle \triangleright_{\bowtie} p_1^2, B \rangle + \langle p_1^1, r_2 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle \end{aligned}$$

By consistently choosing the partition during the elimination of the cut  $c$  this can be matched by first starting with the elimination of the glueing cut  $c'$ .

$$\begin{aligned} & \langle \triangleright_{\bowtie} p_1, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\bowtie} r \rangle + \langle \triangleright_{\bowtie} q_1, \dots, q_k \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2, q_1 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\bowtie} p_1, q_2, \dots, q_k, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\bowtie} r \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\bowtie} p_1^1, q_2, \dots, q_k, A \rangle + \langle \triangleright_{\bowtie} p_1^2, B \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle \end{aligned}$$

- In the first non homogeneous case the redex is of the following form

$$\langle \triangleright_{\bowtie} p_1, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\bowtie} r \rangle + \langle q_1, q_2 \triangleright_{\otimes} q \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2, q \triangleright_{\text{cut}} \rangle.$$

After one step of cut elimination this becomes in all generality, given that  $A = \{a_1, \dots, a_k\}$  and  $B = \{b_1, \dots, b_l\}$  partition  $p_2, \dots, p_n$ ,

$$\langle \triangleright_{\boxtimes} p_1^1, p_2, A' \rangle + \langle \triangleright_{\boxtimes} p_1^2, B \rangle + \langle q_1, q_2 \triangleright_{\otimes} q \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle + \langle p_2, q \triangleright_{\text{cut}} \rangle.$$

Without loss of generality assume that  $p_2$  occurs in the class  $A = \{p_2\} \cup A'$ , then after one step of cut elimination this becomes;

$$\langle \triangleright_{\boxtimes} p_1^1, p_2^1, p_2^2, A' \rangle + \langle \triangleright_{\boxtimes} p_1^2, B \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle.$$

Indeed one can obtain the same redex by first eliminating  $c'$ , when eliminating the  $\boxtimes$ -link we need to make a consistent choice e.g. the partition of  $p_2^1, p_2^2, \dots, p_n$  made of the two classes  $B$  and  $A' \cup \{p_2^1, p_2^2\}$ :

$$\begin{aligned} & \langle \triangleright_{\boxtimes} p_1, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\boxtimes} r \rangle + \langle q_1, q_2 \triangleright_{\otimes} q \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2, q \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\boxtimes} p_1, p_2^1, p_2^2, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\boxtimes} r \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\boxtimes} p_1, p_2^1, p_2^2, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\boxtimes} r \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\boxtimes} p_1^1, p_2^1, p_2^2, A' \rangle + \langle \triangleright_{\boxtimes} p_1^2, B \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle. \end{aligned}$$

- In the non homogeneous second case both cuts are made of  $\boxtimes$  links against a daimon thus in all generality a reduction is of the following form form;

$$\begin{aligned} & \langle \triangleright_{\boxtimes} p_1, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\boxtimes} r \rangle + \langle q_1, q_2 \triangleright_{\boxtimes} q \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2, q \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\boxtimes} p_1^1, p_2, A \rangle + \langle \triangleright_{\boxtimes} p_1^2, B \rangle + \langle q_1, q_2 \triangleright_{\boxtimes} q \rangle + \langle p_2, q \triangleright_{\text{cut}} \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\boxtimes} p_1^1, p_2^1, A_1 \rangle + \langle \triangleright_{\boxtimes} p_2^2, A_2 \rangle + \langle \triangleright_{\boxtimes} p_1^2, B \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle \end{aligned}$$

Indeed starting by eliminating the cut  $c'$  we can obtain the same redex by making consistent choices in the partitions:

$$\begin{aligned} & \langle \triangleright_{\boxtimes} p_1, \dots, p_n \rangle + \langle r_1, r_2 \triangleright_{\boxtimes} r \rangle + \langle q_1, q_2 \triangleright_{\boxtimes} q \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2, q \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\boxtimes} p_2^1, p_1, A_1 \cup B \rangle + \langle \triangleright_{\boxtimes} p_2^2, A_2 \rangle + \langle r_1, r_2 \triangleright_{\boxtimes} r \rangle + \langle p_1, r \triangleright_{\text{cut}} \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle \\ \rightarrow & \langle \triangleright_{\boxtimes} p_1^1, p_2^1, A_1 \rangle + \langle \triangleright_{\boxtimes} p_2^2, A_2 \rangle + \langle \triangleright_{\boxtimes} p_1^2, B \rangle + \langle p_2^1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2^2, q_2 \triangleright_{\text{cut}} \rangle + \langle p_1^1, r_1 \triangleright_{\text{cut}} \rangle + \langle p_1^2, r_2 \triangleright_{\text{cut}} \rangle \end{aligned}$$

□

**Proposition 110** (General right commutation, [Proposition 30item 2](#)). *For any net  $S$  containing an irreversible cut  $c$ , and given  $\alpha^*$  a series of cut elimination of cuts occurring in  $S$  then the following diagram commutes;*

$$\begin{array}{ccc} S & \xrightarrow{c} & S_1 \\ \downarrow \alpha^* & & \downarrow \alpha^* \\ S'_1 & \xrightarrow{c'} & S_2 \end{array}$$

*Proof.* This follows from a simple induction on  $\alpha^*$ . If  $\alpha^*$  is made of only one cut the proposition 109 gets us the conclusion. Otherwise we decompose the sequence of reductions as  $\alpha^* = \beta \cdot \beta^*$  and by applying the proposition 109 we obtain the following diagram.

$$\begin{array}{ccc} S & \xrightarrow{c} & S_1 \\ \downarrow \beta & & \downarrow \beta \\ S'_1 & \xrightarrow{c'} & S_2 \\ & & \downarrow \beta^* \\ & & S_3 \end{array}$$

Calling the induction hypothesis on  $\beta^*$  we can complete the diagram as follows and conclude.

$$\begin{array}{ccc} S & \xrightarrow{c} & S_1 \\ \downarrow \beta & & \downarrow \beta \\ S'_1 & \xrightarrow{c'} & S_2 \\ \downarrow \beta^* & & \downarrow \beta^* \\ S'_2 & \xrightarrow{c'} & S_3 \end{array}$$

□

## B.5 Reversible cuts can be anticipated – Proof of Proposition 30 item 3

**Proposition 111** (Deterministic non homogeneous cut elimination commutes to the left). *Given  $S$  some net containing a non homogeneous cut link  $c$  of a daimon link against a  $\otimes$ -link. The diagram below commutes, for any cut kind of cut link  $c'$ ;*

$$\begin{array}{ccc} S & \xrightarrow{c'} & S_1 \\ \downarrow c & & \downarrow c \\ S'_1 & \xrightarrow{c'} & S_2 \end{array}$$

Where the dotted arrows are the existence of a reduction.

*Proof.* We reason as in the proof for proposition 109, if the two cuts involved different link of the net we conclude. Furthermore if  $c'$  is an homogeneous cut the proposition hold. Thus we assume that  $c$  and  $c'$  are both non-homogeneous and involve the same daimon link.

- If  $c'$  is also a tensor link its clear that the elimination of the two cuts commute.
- If  $c'$  is a  $\wp$  link we can call the previous proposition 109 on the cut  $c'$ , claiming that its elimination commutes on the right with any step of cut elimination. In particular any elimination of  $c'$  followed by any elimination of  $c$  can be matched by an elimination of  $c$  followed by an elimination of  $c'$ . Thus we conclude.  $\square$

**Proposition 112** (General left commutation). *For any net  $S$  containing an reversible cut  $c$ , and given  $\alpha^*$  a series of cut elimination of cuts occuring in  $S$  then the following diagram commutes;*

$$\begin{array}{ccc} S & \xrightarrow{\alpha^*} & S_1 \\ \downarrow c & & \downarrow c \\ S'_1 & \xrightarrow{\alpha^*} & S_2 \end{array}$$

*Proof.* As for the previous proposition it follows from a simple induction on  $\alpha^*$ . If  $\alpha^*$  is made of only one cut the proposition 111 gets us to conclude. Otherwise we decompose the sequence of reductions as  $\alpha^* = \beta^* \cdot \beta$  and by applying the proposition 111 together with the induction hypothesis we obtain the following diagram.

$$\begin{array}{ccccc} S & \xrightarrow{\beta^*} & S_1 & \xrightarrow{\beta} & S_2 \\ \downarrow c & & \downarrow c & & \downarrow c \\ S'_1 & \xrightarrow{\beta^*} & S_2 & \xrightarrow{\beta} & S_3 \end{array}$$

$\square$

**Remark 113.** Moreover, following a similar method, we can easily establish that glueing cuts commute to the left with any kind of cut. The case with a  $(\wp/\boxtimes)$  cut is taken care of since we know these cuts can be delayed (Proposition 30 item 2). For the other cases note that two related glueing cuts indeed commute, furthermore by some simple calculations, one can prove that a glueing cut and  $(\otimes/\boxtimes)$  cut will also commute. Indeed the result of commutation of two cuts may then be generalised to sequences of cut elimination as in Proposition 112.

**Proposition 114** (Proposition 30 item 3). *Given a net  $S$  containing a cut  $c$  that is not a  $(\wp/\boxtimes)$  cut. If  $S \xrightarrow{*} \xrightarrow{c} S'$  then  $S \xrightarrow{c} \xrightarrow{*} S'$ .*

*Proof.* This is a consequence of Proposition 112 and Remark 113.  $\square$

## C Complements to section 2

### C.1 Multigraph

The notion of multigraph is used several times: (1) in order to define orthogonal partitions one must compute a multigraph and (2) in all generality the graph underlying an hypergraph should also be a multigraph – this will be defined in the next section of this appendix (subsection C.2).

**Definition 115.** An (undirected) multigraph is a tuple  $G = (V_G, E_G, \text{brd}_G)$  such that:

- $V_G$  and  $E_G$  are two disjoint sets, whose elements are respectively called *nodes* and *edges*.
- The border function  $\text{brd}_G : E_G \rightarrow \mathcal{P}(V_G)$  maps edges to sets of nodes of cardinal 1 or 2. The intuition is that whenever  $\text{brd}_G(e) = \{x, y\}$  then  $x$  and  $y$  are the two endpoints of the edge  $e$ , while if  $\text{brd}_G(e) = \{x\}$  the edge  $e$  forms a *loop*.

## C.2 Graph underlying an hypergraph

In [section 2](#) we informally refer to the notion of graph underlying an hypergraph. This induced graph can be formally defined as in the following [Definition 116](#)

**Definition 116** (Underlying graph). Given an hypergraph  $\mathcal{H} = (V_{\mathcal{H}}, E_{\mathcal{H}}, \text{t}_{\mathcal{H}}, \text{s}_{\mathcal{H}})$  its *underlying directed graph* is a directed graph denoted  $G(\mathcal{H}) = (V_G, E_G, \text{t}_G, \text{s}_G)$  obtained from  $\mathcal{H}$  as follow;

- A vertex of  $G(\mathcal{H})$  is either a position of the hypergraph  $\mathcal{H}$ , or a link of  $\mathcal{H}$ , i.e.  $V_G = V_{\mathcal{H}} \uplus E_{\mathcal{H}}$ .
- An edge of  $G(\mathcal{H})$  is a tuple  $(e, p, \text{in})$  of a link  $e \in E_{\mathcal{H}}$  with  $p \in \text{s}(e)$  or a tuple  $(e, p, \text{out})$  with  $p \in \text{t}(e)$ .
- For an edge  $(e, p, \text{in})$  we have  $\text{t}(e, p, \text{in}) = e$  and  $\text{s}(e, p, \text{in}) = p$ . For an edge  $(e, p, \text{out})$  we have  $\text{t}(e, p, \text{out}) = p$  and  $\text{s}(e, p, \text{out}) = e$ .

**Definition 117** (Induced undirected graph). Given a directed graph  $G = (V_G, E_G, \text{t}_G, \text{s}_G)$  its induced undirected graph is the graph  $G^* = (V_G^*, E_G^*, \text{brd}_{G^*})$  such that:

- The set of vertices  $V_G$  and  $V_G^*$  are the same;
- The set of edges  $E_G$  and  $E_G^*$  are the same;
- For any edge  $e$  its border in  $G^*$  is given by  $\text{brd}_{G^*}(e) = \{\text{s}_G(e), \text{t}_G(e)\}$ .

**Remark 118.** The paths of the induced undirected graph  $G^*$  of a directed (multi)graph  $G = (V_G, E_G, \text{t}_G, \text{s}_G)$  are exactly the paths in  $G$  where we allow edges to be travelled through in both direction (source–target or target–source).

**Definition 119.** Given an hypergraph  $\mathcal{H}$  its *underlying graph*  $G(\mathcal{H})^*$  is the induced undirected graph of  $G(\mathcal{H})$ .

The previous definition is the one involved in in the theorem of Danos Regnier ([Theorem 40](#)): given a switching  $\sigma S$  of a net  $S$  the acyclicity and connectedness of  $\sigma S$  is that of its underlying undirected graph  $G(\sigma S)^*$ .

## C.3 On substitutions – proof of [Proposition 33](#)

**Proposition 33.** Let  $\Gamma$  and  $\Delta$  be two sequents and suppose  $\Delta \leq \Gamma$ . For any net  $S$ : (1) if  $S \vdash_{\text{MLL}^{\boxtimes}} \Delta$  then  $S \vdash_{\text{MLL}^{\boxtimes}} \Gamma$  and (2) if  $S \vdash_{\text{MLL}} \Delta$  then  $S \vdash_{\text{MLL}} \Gamma$ .

*Proof.* This is because if a sequent  $\Delta$  can be introduced by a daimon rule (resp. an axiom rule for MLL) then for any substitution  $\theta$  the sequent  $\theta\Delta$  can be introduced by a daimon rule (resp. an axiom rule for MLL). Thus a proof by induction on the represented proof tree allows us to conclude.  $\square$

**Remark 120.** The [Proposition 33](#) implies that whenever  $A \leq B$  the set  $\{\{A : \text{MLL}^{\boxtimes}\}\}$  is contained in  $\{\{B : \text{MLL}^{\boxtimes}\}\}$  but also that  $\{\{A : \text{MLL}\}\}$  is contained in  $\{\{B : \text{MLL}\}\}$

# D Complements to [section 3](#)

## D.1 On the notion of Interaction

### D.1.1 Proof of [Proposition 48](#)

We have presented a notion of interaction  $S :: T$  between two net  $S$  and  $T$  in [Definition 41](#), this definition makes direct use of the order on the conclusions of the nets (the arrangement of [section 1](#)). An alternative can be to define the interaction between two nets  $S$  and  $T$  using a notion of *interface*  $\sigma$ , a functional and injective relation between the conclusions of  $S$  and  $T$ , then  $S ::_{\sigma} T$  is equal to  $S + T + \sum_{(p,q) \in \sigma} \langle S(p), T(p) \triangleright_{\text{cut}} \rangle$ . This makes orthogonality more subtle to define so we chose to work with ordered conclusions which determines the interaction  $S :: T$ , in some sense the order on the conclusions is used to define a canonical interface.

**Proposition 48.** Given three net  $S$  and  $T$  and  $R$  such that  $\#S \geq \#T + \#R$ : the interaction  $S :: (T \parallel R)$  is equal to  $(S :: T) :: R$ .

*Proof.* Consider three net  $S, T$  and  $R$  such that  $\#S \geq \#T + \#R$ . Note that  $\#T \parallel \#R = \#T + \#R$  thus  $\#S \geq \#T \parallel \#R$  and so  $\min(\#S, \#T \parallel \#R) = \#T \parallel \#R$ . As a consequence  $S :: (T \parallel R)$  is equal to

$$S :: (T \parallel R) = S + T + R + \sum_{1 \leq i \leq \#T \parallel \#R} \langle S(i), (T \parallel R)(i) \triangleright_{\text{cut}} \rangle$$

Note that for any  $1 \leq i \leq \#T$  we have  $(T \parallel R)(i) = T(i)$  thus:

$$S :: (T \parallel R) = \left( S + T + \sum_{1 \leq i \leq \#T} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle \right) + R + \sum_{\#T+1 \leq i \leq \#T \parallel \#R} \langle S(i), (T \parallel R)(i) \triangleright_{\text{cut}} \rangle$$

Furthermore note that for each  $\#T + 1 \leq i \leq \#S$  we have that the conclusion  $S(i)$  is the conclusion  $(S :: T)(i - \#T)$  of  $S$ . Also for any  $\#T + 1 \leq i \leq \#T + \#R$  we have  $(T \parallel R)(i)$  equal to  $R(i - \#T)$ . Thus we can rewrite the interaction  $S :: (T \parallel R)$  as follows:

$$\begin{aligned} & S :: (T \parallel R) \\ &= \left( S + T + \sum_{1 \leq i \leq \#T} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle \right) + R + \sum_{\#T+1 \leq i \leq \#T \parallel \#R} \langle S(i), (T \parallel R)(i) \triangleright_{\text{cut}} \rangle \\ &= \left( S + T + \sum_{1 \leq i \leq \#T} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle \right) + R + \sum_{\#T+1 \leq i \leq \#T \parallel \#R} \langle (S :: T)(i - \#T), R(i - \#T) \triangleright_{\text{cut}} \rangle \\ &= \left( S + T + \sum_{1 \leq i \leq \#T} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle \right) + R + \sum_{\#T+1 - \#T \leq i \leq \#T \parallel \#R - \#T} \langle (S :: T)(i), R(i) \triangleright_{\text{cut}} \rangle \\ &= \left( S + T + \sum_{1 \leq i \leq \#T} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle \right) + R + \sum_{1 \leq i \leq \#R} \langle (S :: T)(i), R(i) \triangleright_{\text{cut}} \rangle \\ &= S :: T + R + \sum_{1 \leq i \leq \#R} \langle (S :: T)(i), R(i) \triangleright_{\text{cut}} \rangle \\ &= (S :: T) :: R \end{aligned}$$

□

### D.1.2 Interaction of nets and unrelated connective links

The following proposition will be used in the proof of the result of adequacy ([Theorem 64](#)) in the inductive cases. In particular when one interprets  $S$  as the image of a proof  $\pi_0$  of  $\Gamma, A, B$ , the link  $e$  represents a  $\mathfrak{A}$ -link (say a link  $\langle p_1, p_2 \triangleright_{\mathfrak{A}} p \rangle$ ) and  $T$  is an opponent from  $\llbracket \Gamma \rrbracket_{\mathfrak{A}}^{\perp}$ .

**Proposition 121.** *Given two nets  $S$  and  $T$  with  $\#S \geq \#T$  and a link  $e$  such that the positions of  $s(e)$  are included in the outputs of  $S :: T$  and are the only conclusions of  $S$ . Then we have:*

$$(S :: T) + e = (S + e) :: T.$$

*Proof.* Recall the definition of the interaction

$$S :: T = S + \sum_{1 \leq i \leq \#T} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle + T$$

Thus we have that

$$(S :: T) + e = \left( S + \sum_{1 \leq i \leq \#T} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle + T \right) + e$$

Because each input of  $e$  is a conclusion of  $S$  in  $S :: T$  then each input of  $e$  is a conclusions  $S(j)$  of  $S$  with  $j \geq \#T$ . Furthermore say  $S = (|S|, \mathbf{a}(S))$  then  $S + e = (|S| + e, (\mathbf{a}(S) \setminus s(e)) \cdot t(e))$  and for each  $1 \leq i \leq \#T$  we have  $S(i) = (S + e)(i)$ . It follows that:

$$\left( S + \sum_{1 \leq i \leq \#T} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle + T \right) + e = \left( S + \sum_{1 \leq i \leq \#T} \langle (S + e)(i), T(i) \triangleright_{\text{cut}} \rangle + T \right) + e$$

Which itself is equal to

$$(S + e) + \sum_{1 \leq i \leq \#T} \langle (S + e)(i), T(i) \triangleright_{\text{cut}} \rangle + T = (S + e) :: T$$

□

## D.2 On types, proving duality results and associativity

### D.2.1 The Merge Operator

**Definition 122** (Merge of two nets). The *merge* of two daimon links  $d = \langle \triangleright_{\boxtimes} \bar{p} \rangle$  and  $d' = \langle \triangleright_{\boxtimes} \bar{q} \rangle$  is the daimon link  $d \boxtimes d' = \langle \triangleright_{\boxtimes} \bar{p}, \bar{q} \rangle$ . Given a net  $S$  containing a daimon link  $d$  i.e.  $S = d + S_0$  and a net  $S'$  containing a daimon link  $d'$  i.e.  $S' = d' + S'_0$ . We denote by  $S \boxtimes^{(d,d')} S'$  the net  $S_0 + S'_0 + (d \boxtimes d')$ .

**Proposition 123.** *Given two nets  $S$  and  $S'$  respectively containing daimons  $d$  and  $d'$  so that  $S = S_0 + d$  and  $S' = S'_0 + d'$ :  $S \boxtimes^{(d,d')} S'$  is equal to  $S \boxtimes^{(d,d')} d' + S'_0$ .*

*Proof.* One merely needs to write the nets  $S \boxtimes^{(d,d')} S'$  and  $(S \boxtimes^{(d,d')} d') + S'_0$  as sum of links to conclude.  $\square$

*Notation 124.* We fix a convention to denote the daimon links resulting from cut elimination steps involving daimons (glueing and non homogeneous):

- In a glueing step say  $d = \langle \triangleright_{\boxtimes} \bar{p}, a \rangle$  and  $d' = \langle \triangleright_{\boxtimes} \bar{q}, b \rangle$  the glueing redex  $d + \langle a, b \triangleright_{\text{cut}} \rangle + d'$  reducing to  $\langle \triangleright_{\boxtimes} \bar{p}, \bar{q} \rangle$  we denote that resulting daimon link by  $g_{a,b}(d, d')$ .
- In a  $(\otimes/\boxtimes)$  step say  $d = \langle \triangleright_{\boxtimes} \bar{p}, a \rangle$  the redex  $d + \langle a, b \triangleright_{\text{cut}} \rangle + \langle b_1, b_2 \triangleright_{\otimes} b \rangle$  reducing to  $\langle \triangleright_{\boxtimes} \bar{p}, a_1, a_2 \rangle + \langle a_1, b_1 \triangleright_{\text{cut}} \rangle + \langle a_2, b_2 \triangleright_{\text{cut}} \rangle$  we denote that resulting daimon link by  $p_a(d)$ .
- In a  $(\boxtimes/\boxtimes)$  step say  $d = \langle \triangleright_{\boxtimes} \bar{p}, a \rangle$  the redex  $d + \langle a, b \triangleright_{\text{cut}} \rangle + \langle b_1, b_2 \triangleright_{\boxtimes} b \rangle$  reducing to  $\langle \triangleright_{\boxtimes} \bar{p}^A, a_1 \rangle + \langle \triangleright_{\boxtimes} \bar{p}^B, a_2 \rangle + \langle a_1, b_1 \triangleright_{\text{cut}} \rangle + \langle a_2, b_2 \triangleright_{\text{cut}} \rangle$  for a partition  $\bar{p} = \bar{p}^A \uplus \bar{p}^B$  we denote that resulting daimon links by  $t_{a,A}^1(d)$  and  $t_{a,B}^2(d)$ .

Given a reduction step  $S \rightarrow S'$  eliminating a cut  $c$  we denote  $S \rightarrow^c S'$ . Given an elimination step  $S \rightarrow^c S'$  that isn't multiplicative, we denote by  $d_c$  the daimon involve in the redex if it is unique, otherwise we denote the unordered pair of daimons  $g(d_c)$  in the case of a glueing cut.

**Proposition 125.** *Given a net  $S$  containing a cut  $c = \langle a, b \triangleright_{\text{cut}} \rangle$  and daimon  $d$  of  $S$  if  $S \rightarrow^c S'$ :*

- If  $c$  is multiplicative  $S \boxtimes^d \boxtimes_1 \rightarrow^c S' \boxtimes^d \boxtimes_1$ .
- If  $d$  isn't  $d_c$  or contained in  $g(d_c)$  then  $S \boxtimes^d \boxtimes_1 \rightarrow^c S' \boxtimes^d \boxtimes_1$ .
- If  $c$  is a glueing cut and  $d$  belongs to  $g(d_c)$  then  $S \boxtimes^d \boxtimes_1 \rightarrow S' \boxtimes^{g_s(c)(g(d_c))} \boxtimes_1$
- If  $c$  is a  $(\otimes/\boxtimes)$  cut and  $d = d_c$  then  $S \boxtimes^d \boxtimes_1 \rightarrow S' \boxtimes^{p_a(d)} \boxtimes_1$
- If  $c$  is a  $(\boxtimes/\boxtimes)$  cut and  $d = d_c$  and  $S \rightarrow^c S'$  by choosing a partition  $A \uplus B$  of the daimon  $d = \langle \triangleright_{\boxtimes} \bar{p} \rangle$  then  $S \boxtimes^d \boxtimes_1 \rightarrow S' \boxtimes^{t_{a,A}^1(d)} \boxtimes_1$  and  $S \boxtimes^d \boxtimes_1 \rightarrow S' \boxtimes^{t_{a,B}^2(d)} \boxtimes_1$ .

*Proof.* If the reduction  $S \rightarrow S'$  is a multiplicative step, then it does not affects the daimon of  $S$  and therefore we conclude. In the other cases one (or two for the glueing cut) daimon are involved in the cut-elimination, if these daimons don't contain  $d$  then by contextual closure one can ensure the proposition.

Now let us treat the three cases where the daimon  $d$  is involved.

- Say the reduction in  $S$  is that of a glueing cut:

$$\langle \triangleright_{\boxtimes} \bar{p}, a \rangle + \langle a, b \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\boxtimes} \bar{q}, b \rangle \rightarrow \langle \triangleright_{\boxtimes} \bar{p}, \bar{q} \rangle.$$

Say (without loss of generality) that  $d = \langle \triangleright_{\boxtimes} \bar{p}, a \rangle$  then in  $S \boxtimes^d \boxtimes_1$  the daimon  $d$  is isomorphic to  $\langle \triangleright_{\boxtimes} \bar{p}, a, u \rangle$  and the redex becomes:

$$\langle \triangleright_{\boxtimes} \bar{p}, a, u \rangle + \langle a, b \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\boxtimes} \bar{q}, b \rangle \rightarrow \langle \triangleright_{\boxtimes} \bar{p}, u, \bar{q} \rangle.$$

Indeed the resulting daimon is  $g_{a,b}(d_1, d_2) \boxtimes \boxtimes_1$  and therefore the redex of  $S \boxtimes^d \boxtimes_1$  is  $S' \boxtimes^{g_{a,b}(d_1, d_2)} \boxtimes_1$ .

- Say the reduction is that of a  $(\otimes/\boxtimes)$  cut:

$$\langle \triangleright_{\boxtimes} \bar{p}, a \rangle + \langle a, b \triangleright_{\text{cut}} \rangle + \langle b_1, b_2 \triangleright_{\otimes} b \rangle \rightarrow \langle \triangleright_{\boxtimes} \bar{p}, a_1, a_2 \rangle + \langle a_1, b_1 \triangleright_{\text{cut}} \rangle + \langle a_2, b_2 \triangleright_{\text{cut}} \rangle.$$

It follows that:

$$\begin{aligned} & \langle \triangleright_{\boxtimes} \bar{p}, a \rangle + \langle a, b \triangleright_{\text{cut}} \rangle + \langle b_1, b_2 \triangleright_{\otimes} b \rangle \boxtimes \boxtimes_1 \\ &= \langle \triangleright_{\boxtimes} \bar{p}, a, u \rangle + \langle a, b \triangleright_{\text{cut}} \rangle + \langle b_1, b_2 \triangleright_{\otimes} b \rangle \\ &\rightarrow \langle \triangleright_{\boxtimes} \bar{p}, a_1, a_2, u \rangle + \langle a_1, b_1 \triangleright_{\text{cut}} \rangle + \langle a_2, b_2 \triangleright_{\text{cut}} \rangle \\ &= \langle \triangleright_{\boxtimes} \bar{p}, a_1, a_2 \rangle + \langle a_1, b_1 \triangleright_{\text{cut}} \rangle + \langle a_2, b_2 \triangleright_{\text{cut}} \rangle \boxtimes \boxtimes_1 \end{aligned}$$

Thus we have shown that in  $S \rightarrow S'$  the daimon  $d$  became  $p_a(d)$  and we have that  $S \boxtimes^d \boxtimes_1$  rewrites to  $S' \boxtimes^{p_a(d)} \boxtimes_1$ .

- Say the reduction is that of a  $(\mathfrak{F}, \mathfrak{X})$  cut:

$$\langle \triangleright_{\mathfrak{X}} \bar{p}, a \rangle + \langle a, b \triangleright_{\text{cut}} \rangle + \langle b_1, b_2 \triangleright_{\mathfrak{F}} b \rangle \rightarrow \langle \triangleright_{\mathfrak{X}} \bar{p}_A, a_1 \rangle + \langle \triangleright_{\mathfrak{X}} \bar{p}_B, a_2 \rangle + \langle a_1, b_1 \triangleright_{\text{cut}} \rangle + \langle a_2, b_2 \triangleright_{\text{cut}} \rangle.$$

It follows that:

$$\begin{aligned} & \langle \triangleright_{\mathfrak{X}} \bar{p}, a \rangle + \langle a, b \triangleright_{\text{cut}} \rangle + \langle b_1, b_2 \triangleright_{\mathfrak{F}} b \rangle \bowtie \mathfrak{X}_1 \\ &= \langle \triangleright_{\mathfrak{X}} \bar{p}, a, u \rangle + \langle a, b \triangleright_{\text{cut}} \rangle + \langle b_1, b_2 \triangleright_{\mathfrak{F}} b \rangle \\ &\rightarrow \langle \triangleright_{\mathfrak{X}} \bar{p}_A, a_1, u \rangle + \langle \triangleright_{\mathfrak{X}} \bar{p}_B, a_2 \rangle + \langle a_1, b_1 \triangleright_{\text{cut}} \rangle + \langle a_2, b_2 \triangleright_{\text{cut}} \rangle \\ &= \langle \triangleright_{\mathfrak{X}} \bar{p}_A, a_1 \rangle + \langle \triangleright_{\mathfrak{X}} \bar{p}_B, a_2 \rangle + \langle a_1, b_1 \triangleright_{\text{cut}} \rangle + \langle a_2, b_2 \triangleright_{\text{cut}} \rangle \bowtie \mathfrak{X}_1 \end{aligned}$$

Thus we have shown that in  $S \rightarrow S'$  the daimon  $d$  can be rewritten in two daimons  $t_{a,A}^1(d)$  or  $t_{b,B}^2(d)$  for any choice of partition  $A$  and  $B$  of  $\bar{p}$ . As a consequence  $S \bowtie^d \mathfrak{X}_1$  rewrites to  $S' \bowtie_{t_{a,A}^1(d)} \mathfrak{X}_1$  or to  $S' \bowtie_{t_{b,B}^2(d)} \mathfrak{X}_1$  still for any choice of partition.  $\square$

**Remark 126.** The previous proposition ensures the (less precise) statement: given a net  $S$  with a daimon link  $d$  if  $S \rightarrow S'$  then there exists a daimon  $d'$  in  $S'$  so that  $S \bowtie^d \mathfrak{X}_1 \rightarrow S' \bowtie^{d'} \mathfrak{X}_1$ . By a simple induction, this can be generalised to multiple steps of cut elimination.

**Proposition 127.** Given  $S$  and  $T$  two nets such that  $S :: T \rightarrow \mathfrak{X}_k$ ,  $d$  a daimon of  $S$ , and  $d_0 = \mathfrak{X}_n$  some daimon link: the net  $(S \bowtie^d \mathfrak{X}_n) :: T$  reduces to  $\mathfrak{X}_{n+k}$ .

*Proof.* Assume that  $S$  and  $T$  are orthogonal, then  $S :: T \rightarrow \mathfrak{X}_k$ . Consider  $d$  some daimon of  $S$  then in particular it is a daimon of  $S :: T$  and  $S \bowtie^d d_0 :: T$  is equal to  $S :: T \bowtie^d d_0$ . Because  $S :: T \rightarrow \mathfrak{X}_k$  it follows that  $S :: T \bowtie^d \mathfrak{X}_n \rightarrow \mathfrak{X}_k \bowtie^{d'} d_0$  where  $d'$  is a daimon link of  $\mathfrak{X}_k$  (Remark 126). Since  $\mathfrak{X}_k$  is a net made of a single daimon link, necessarily  $d' = \mathfrak{X}_k$  and thus since  $d_0 = \mathfrak{X}_n \mathfrak{X}_k \bowtie^{d'} d_0 = \mathfrak{X}_{n+k}$ . Therefore we conclude  $S :: T \bowtie^d d_0 \rightarrow \mathfrak{X}_{n+k}$ .  $\square$

**Proposition 128.** Given two nets  $S$  and  $T$  are orthogonal.  $S'$  a net obtained from  $S$  by adding  $n$  conclusions to some (eventually distinct) daimons of  $S$  then  $S' :: T$  reduces to  $\mathfrak{X}_n$ .

*Proof.* One inductively uses the previous proposition (Proposition 127).  $\square$

## D.2.2 Behavior of Identity cut–nets

**Definition 129.** We call a cut  $c$  occurring in a net  $S$  *irreversible* whenever it is of type  $(\mathfrak{X}/\mathfrak{F})$  (Definition 16). An *identity cut–net* whenever it is of the following form

$$\langle \triangleright_{\mathfrak{X}} \vec{a}, p_1, \vec{b}, p_2, \vec{c} \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{F}} p \rangle + \langle p, q \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\mathfrak{X}} \vec{d}, q, \vec{e} \rangle$$

Namely, the  $\mathfrak{F}$ -involved above the input  $p$  of the cut link is such that both its inputs are outputs of the same daimon.

**Proposition 130** (Normal Form of Identity cut–nets). *An identity cut–net  $S$  with  $n$  conclusions has a unique normal form which is (up to isomorphism)  $\mathfrak{X}_n$ .*

*Proof.* Note that the conclusions of an identity cut–net are all outputs of (one of the two) daimon links. Writing  $S$  as the following:

$$S ::= \langle \triangleright_{\mathfrak{X}} \vec{a}, p_1, \vec{b}, p_2, \vec{c} \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{F}} p \rangle + \langle p, q \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\mathfrak{X}} \vec{d}, q, \vec{e} \rangle$$

The  $n$  conclusions of  $S$  correspond to the set obtained from the vector  $\vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e}$  (note that this vector contains no repetition).

Fixing a choice for the partition of the context  $\vec{d}, \vec{e}$  of the daimon containing the position  $q$  we obtain a reduction:

$$\begin{aligned} & \langle \triangleright_{\mathfrak{X}} \vec{a}, p_1, \vec{b}, p_2, \vec{c} \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{F}} p \rangle + \langle p, q \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\mathfrak{X}} \vec{d}, q, \vec{e} \rangle \\ \rightarrow & \langle \triangleright_{\mathfrak{X}} \vec{a}, p_1, \vec{b}, p_2, \vec{c} \rangle + \langle p_1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2, q_2 \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\mathfrak{X}} \vec{d}_\ell, q_1, \vec{e}_\ell \rangle + \langle \triangleright_{\mathfrak{X}} \vec{d}_r, q_2, \vec{e}_r \rangle \end{aligned}$$

At this point two cuts occur thus two reductions are possible, both lead to the same net. Eliminating first  $\langle p_1, q_1 \triangleright_{\text{cut}} \rangle$ :

$$\begin{aligned}
& \langle \triangleright_{\boxtimes} \vec{a}, p_1, \vec{b}, p_2, \vec{c} \rangle + \langle p_1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2, q_2 \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\boxtimes} \vec{d}_\ell, q_1, \vec{e}_\ell \rangle + \langle \triangleright_{\boxtimes} \vec{d}_r, q_2, \vec{e}_r \rangle \\
\rightarrow & \langle \triangleright_{\boxtimes} \vec{a}, \vec{b}, p_2, \vec{c}, \vec{d}_\ell, \vec{e}_\ell \rangle + \langle p_2, q_2 \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\boxtimes} \vec{d}_r, q_2, \vec{e}_r \rangle \\
\rightarrow & \langle \triangleright_{\boxtimes} \vec{a}, \vec{b}, \vec{d}_r, \vec{e}_r, \vec{c}, \vec{d}_\ell, \vec{e}_\ell \rangle \\
\equiv & \langle \triangleright_{\boxtimes} \vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e} \rangle \\
\equiv & \langle \triangleright_{\boxtimes} S(1), \dots, S(n) \rangle
\end{aligned}$$

Eliminating first  $\langle p_2, q_2 \triangleright_{\text{cut}} \rangle$ :

$$\begin{aligned}
& \langle \triangleright_{\boxtimes} \vec{a}, p_1, \vec{b}, p_2, \vec{c} \rangle + \langle p_1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2, q_2 \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\boxtimes} \vec{d}_\ell, q_1, \vec{e}_\ell \rangle + \langle \triangleright_{\boxtimes} \vec{d}_r, q_2, \vec{e}_r \rangle \\
\rightarrow & \langle \triangleright_{\boxtimes} \vec{a}, p_1, \vec{b}, \vec{d}_r, \vec{e}_r, \vec{c} \rangle + \langle p_1, q_1 \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\boxtimes} \vec{d}_\ell, q_1, \vec{e}_\ell \rangle \\
\rightarrow & \langle \triangleright_{\boxtimes} \vec{a}, \vec{d}_\ell, \vec{e}_\ell, \vec{b}, \vec{d}_r, \vec{e}_r, \vec{c} \rangle \\
\equiv & \langle \triangleright_{\boxtimes} \vec{a}, \vec{b}, \vec{c}, \vec{d}, \vec{e} \rangle \\
\equiv & \langle \triangleright_{\boxtimes} S(1), \dots, S(n) \rangle
\end{aligned}$$

This is true for any choice of partition during the first  $(\boxtimes/\boxtimes)$  step, therefore  $S$  has a unique normal which is (upto iso)  $\langle \triangleright_{\boxtimes} S(1), \dots, S(n) \rangle$ .  $\square$

**Remark 131.** From [Proposition 130](#) one derive that an identity cut-net without conclusions always reduces to  $\boxtimes_0$ .

**Remark 132.** A net which is cut-free and has no conclusions is a sum of daimon links without conclusions: such a net cannot contain connective otherwise to have no conclusion a cut needs to be added, similarly its daimon links cannot have outputs otherwise to have no conclusions a cut must be present in the net.

**Proposition 133** (Identity cut-net behavior). *Consider an identity cut  $S$  without conclusions and denote its two daimons  $d_1$  and  $d_2$ . Let  $T_1$  and  $T_2$  be two nets without conclusions, and  $d_1'$  a daimon of  $T_1$  while  $d_2'$  denotes a daimon of  $T_2$ , the assertions are equivalent:*

1.  $T_1 \rightarrow^* \boxtimes_0$  and  $T_2 \rightarrow^* \boxtimes_0$ .
2.  $S \boxtimes^{d_1, d_1'} T_1 \boxtimes^{d_2, d_2'} T_2$  reduces to  $\boxtimes_0$

*Proof.*  $1 \Rightarrow 2$ . This is the easy implication.  $T_2 \rightarrow^* \boxtimes_0$  thus by applying [Proposition 127](#)  $S \boxtimes^{d_1, d_1'} T_1 \boxtimes^{d_2, d_2'} T_2$  reduces to  $S \boxtimes^{d_1, d_1'} T_1 \boxtimes^{d_2, f(d_2')} \boxtimes_0$  that is  $S \boxtimes^{d_1, d_1'} T_1$  again applying [Proposition 127](#) with  $T_1 \rightarrow^* \boxtimes_0$  yields that  $S \boxtimes^{d_1, d_1'} T_1$  reduces to  $S \boxtimes^{d_1} \boxtimes_0$  that is  $S$ . We conclude since  $S \rightarrow^* \boxtimes_0$  ([Remark 131](#)).

$2 \Rightarrow 1$ . The other direction requires the use of [Proposition 30](#) (item 2) and [Proposition 30](#) (item 3). The cut that we will denote  $c$  occurring in  $S$  is a  $(\boxtimes/\boxtimes)$  cut thus it can be performed last ([Proposition 30](#) item 2). Thus if  $S \boxtimes^{d_1, d_1'} T_1 \boxtimes^{d_2, d_2'} T_2$  reduces to  $\boxtimes_0$  one can factorize the reduction as  $S \boxtimes^{d_1, d_1'} T_1 \boxtimes^{d_2, d_2'} T_2 \rightarrow^* U$  with  $U \rightarrow_c \boxtimes_0$ , one can verify that this implies that  $U$  is (isomorphic to) the following net

$$\langle \triangleright_{\boxtimes} p_1, p_2 \rangle p + \langle p, q \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\boxtimes} q \rangle$$

That is,  $U$  and  $S$  are isomorphic nets.

In that case note that the cuts of  $T_1$  and  $T_2$  in the net  $S \boxtimes^{d_1, d_1'} T_1 \boxtimes^{d_2, d_2'} T_2$  are unrelated and thus applying [Proposition 97](#), one can factorize the reduction leading to  $U$  as follows:

$$S \boxtimes^{d_1, d_1'} T_1 \boxtimes^{d_2, d_2'} T_2 \rightarrow_2^* S_1 \rightarrow_1^* S \rightarrow_c \rightarrow^* \boxtimes_0 \quad (1)$$

where  $\rightarrow_1$  eliminates cuts occurring in  $T_1$  or its reducts only and  $\rightarrow_2$  eliminates cuts occurring in  $T_2$  or its reducts only. As a consequence using [Proposition 127](#) one may rewrite the first step of the reduction:  $(S \boxtimes^{d_1, d_1'} T_1) \boxtimes^{d_2, d_2'} T_2 \rightarrow_2^* (S \boxtimes^{d_1, d_1'} T_1) \boxtimes^{d_2, f(d_2')} T_2'$

Necessarily because  $c$  (and the cut it produces) is a cut that is not in  $T_2$  or any of its redexes, and because the cuts of  $T_1$  and  $T_2$  also are disjoint the reduction [Equation 1](#) implies that  $T_2'$  must be cut-free and thus in particular a normal form of  $T_2$ .

Similarly because  $(S \boxtimes^{d_1, d_1'} T_1) \boxtimes^{d_2, f(d_2')} T_2'$  reduces to  $S$  by  $\rightarrow_1^*$  applying [Proposition 127](#) one obtains that  $S_1 = (S \boxtimes^{d_1, d_1'} T_1) \boxtimes^{d_2, f(d_2')} T_2'$  reduces to  $S$  in the following way: there is a reduction  $T_1 \rightarrow T_1'$  and  $S$  equals  $(S \boxtimes^{d_1, d_1'} T_1') \boxtimes^{d_2, f(d_2')} T_2'$ . Again because  $c$  (and the cut it produces) is a cut that is not in  $T_1$  or any of its redexes,



and because the cuts of  $T_1$  and  $T_2$  also are disjoint the reduction Equation 1 implies that  $T'_1$  must be cut-free and thus in particular a normal form of  $T_1$ .

From this argument one shows that

$$S = (S \bowtie^{d_1, d'_1} T'_1) \bowtie^{d_2, f(d'_2)} T'_2. \quad (2)$$

Furthermore  $T'_1$  and  $T'_2$  are both cut-free and without conclusions they are therefore sums of  $\mathfrak{X}_0$  links Remark 132. If  $T'_1$  (resp  $T'_2$ ) is  $\sum_{1 \leq i \leq n} \mathfrak{X}_0$  (resp.  $\sum_{1 \leq i \leq k} \mathfrak{X}_0$ ) then  $S = (S \bowtie^{d_1, d'_1} \sum_{1 \leq i \leq n} \mathfrak{X}_0) \bowtie^{d_2, f(d'_2)} T'_2$  equals  $(S + \sum_{1 \leq i \leq n-1} \mathfrak{X}_0) \bowtie^{d_2, f(d'_2)} T'_2$  that will be  $(S + \sum_{1 \leq i \leq n-1} \mathfrak{X}_0) \bowtie^{d_2, f(d'_2)} \sum_{1 \leq i \leq k} \mathfrak{X}_0$  that is  $S + \sum_{1 \leq i \leq n-1} \mathfrak{X}_0 + \sum_{1 \leq i \leq k-1} \mathfrak{X}_0$ .

From Equation 2 this means that necessarily  $n-1 = 0$  and  $k-1 = 0$  i.e.  $T'_1$  and  $T'_2$  equal  $\mathfrak{X}_0$ . Therefore we conclude that  $T_1 \rightarrow^* \mathfrak{X}_0$  and  $T_2 \rightarrow^* \mathfrak{X}_0$  □

### D.2.3 Proofs of section section 3 – Proposition 51 , Proposition 56 and Proposition 54

**Proposition 51 (Duality).** *Given two types  $\mathbf{A}$  and  $\mathbf{B}$ :  $(\mathbf{A} \parallel \mathbf{B})^\perp = \mathbf{A}^\perp \succ \mathbf{B}^\perp$  and  $(\mathbf{A} \succ \mathbf{B})^\perp = \mathbf{A}^\perp \parallel \mathbf{B}^\perp$ .*

*Proof.* Consider a net  $S$  orthogonal to  $\mathbf{A} \parallel \mathbf{B}$ . For any  $a \in \mathbf{A}$  and  $b \in \mathbf{B}$  the net  $S :: (a \parallel b)$  reduces to  $\mathfrak{X}_0$ . Since  $S :: (a \parallel b) = S :: a :: b$  (Proposition 48) and since the orthogonality holds for any pair  $(a, b)$ , in particular for any net  $b \in \mathbf{B}$  the net  $(S :: a) :: b$  reduces to  $\mathfrak{X}_0$ . This means that, for any  $a \in \mathbf{A}$ ,  $S :: a$  is orthogonal to  $\mathbf{B}$  i.e.  $S :: a \in \mathbf{B}^\perp$ . Since  $\mathbf{A} = (\mathbf{A}^\perp)^\perp$ , this means that  $S \in \mathbf{A}^\perp \succ \mathbf{B}^\perp$ .

On the other hand, consider a net  $S \in \mathbf{A}^\perp \succ \mathbf{B}^\perp$ : for any  $a \in \mathbf{A} = (\mathbf{A}^\perp)^\perp$  the net  $S :: a$  belongs to  $\mathbf{B}^\perp$  and so for any  $b \in \mathbf{B}$  the net  $S :: a :: b$  reduces to  $\mathfrak{X}_0$ . Since  $S :: a :: b = S :: (a \parallel b)$  (Proposition 48), we have that  $S$  is orthogonal to  $\mathbf{A} \parallel \mathbf{B}$  (using Remark 50).

Hence we showed  $(\mathbf{A} \parallel \mathbf{B})^\perp = \mathbf{A}^\perp \succ \mathbf{B}^\perp$ . As a consequence  $(\mathbf{A} \succ \mathbf{B})^\perp = (\mathbf{A}^\perp \succ \mathbf{B}^\perp)^\perp = (\mathbf{A}^\perp \parallel \mathbf{B}^\perp)^\perp = \mathbf{A}^\perp \parallel \mathbf{B}^\perp$ . □

**Lemma 134.** *Given a net with no conclusions  $R$  and two types with one conclusion  $\mathbf{A}$  and  $\mathbf{B}$ : If  $R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle$  belongs to  $\mathbf{A}^\perp \mathfrak{X} \mathbf{B}^\perp$  then  $R \bowtie^d \langle \triangleright_{\mathfrak{X}} p \rangle$  belongs to  $\mathbf{A}^\perp \mathfrak{X} \mathbf{B}^\perp$*

*Proof.* To do so one shows that  $R \bowtie^d \langle \triangleright_{\mathfrak{X}} p \rangle$  is orthogonal to any net orthogonal to  $\mathbf{A}^\perp \mathfrak{X} \mathbf{B}^\perp$ . Consider therefore an opponent  $U$  orthogonal to  $\mathbf{A}^\perp \mathfrak{X} \mathbf{B}^\perp$ . We distinguish two cases:

- If  $U = U' + \langle U'(1), U'(2) \triangleright_{\otimes} q \rangle$  has a terminal tensor link. Note that, by assumption:

$$R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle :: U \rightarrow^* \mathfrak{X}_0$$

Since multiplicative cuts can be performed first and  $R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle :: U$  reduces to  $R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle :: U'$  by eliminating a multiplicative cut hence we derive the following (Proposition 106)

$$R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle :: U' \rightarrow^* \mathfrak{X}_0.$$

In the meanwhile we have the following reduction:

$$\begin{aligned} & R \bowtie^d \langle \triangleright_{\mathfrak{X}} p \rangle :: U \\ &= R \bowtie^d \langle \triangleright_{\mathfrak{X}} p \rangle :: U' + \langle U'(1), U'(2) \triangleright_{\otimes} q \rangle \\ &\rightarrow R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle :: U' \\ &\rightarrow \mathfrak{X}_0(\text{previous argument}) \end{aligned}$$

- If  $U$  has a terminal daimon link outputting its only conclusion  $q$ . Then  $U$  may written as as  $\langle \triangleright_{\mathfrak{X}} q \rangle \bowtie^{d'} R'$ . Therefore one can identify an identity cut-net  $V$  in the interaction:

$$\begin{aligned} & R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle :: U \\ &= R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle :: \langle \triangleright_{\mathfrak{X}} q \rangle \bowtie^{d'} R' \\ &= (R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle) + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle + \langle p, q \triangleright_{\text{cut}} \rangle + (\langle \triangleright_{\mathfrak{X}} q \rangle \bowtie^{d'} R') \\ &= [\langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle + \langle p, q \triangleright_{\text{cut}} \rangle + (\langle \triangleright_{\mathfrak{X}} q \rangle \bowtie^{d'} R')] \bowtie^d R \\ &= [\langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle + \langle p, q \triangleright_{\text{cut}} \rangle + \langle \triangleright_{\mathfrak{X}} q \rangle] \bowtie^{d'} R' \bowtie^d R \\ &= [V] \bowtie^{d'} R' \bowtie^d R \end{aligned}$$

Now since  $V$  is a cut net without conclusions while  $R'$  and  $R$  both have no conclusion while  $[V] \bowtie^{d'} R' \bowtie^d R \rightarrow^* \mathfrak{X}_0$  (because by assumption  $R \bowtie^d \langle \triangleright_{\mathfrak{X}} p \rangle :: U \rightarrow \mathfrak{X}_0$ ) we derive that  $R \rightarrow^* \mathfrak{X}_0$  and  $R' \rightarrow \mathfrak{X}_0$  (Proposition 133).

We then derive the following

$$\begin{aligned}
& R \bowtie^d \langle \triangleright_{\mathfrak{X}} p \rangle :: U \\
&= R \bowtie^d \langle \triangleright_{\mathfrak{X}} p \rangle :: \langle \triangleright_{\mathfrak{X}} q \rangle \bowtie^{d'} R' \\
\rightarrow^* & \mathfrak{X}_0 \bowtie^d \langle \triangleright_{\mathfrak{X}} p \rangle :: \langle \triangleright_{\mathfrak{X}} q \rangle \bowtie^{d'} R' \\
&= \langle \triangleright_{\mathfrak{X}} p \rangle :: \langle \triangleright_{\mathfrak{X}} q \rangle \bowtie^{d'} R' \\
\rightarrow^* & \langle \triangleright_{\mathfrak{X}} p \rangle :: \langle \triangleright_{\mathfrak{X}} q \rangle \bowtie^{d'} \mathfrak{X}_0 \\
&= \langle \triangleright_{\mathfrak{X}} p \rangle :: \langle \triangleright_{\mathfrak{X}} q \rangle \\
\rightarrow & \mathfrak{X}_0
\end{aligned}$$

in the end we have shown that  $R \bowtie^d \langle \triangleright_{\mathfrak{X}} p \rangle$  belongs to  $(\mathbf{A}^\perp \wp \mathbf{B}^\perp)^{\perp\perp}$  which is  $\mathbf{A}^\perp \wp \mathbf{B}^\perp$  and conclude.  $\square$

**Proposition 56 (Duality).** *Given  $\mathbf{A}$  and  $\mathbf{B}$  two types with one conclusion,  $(\mathbf{A} \otimes \mathbf{B})^\perp = \mathbf{A}^\perp \wp \mathbf{B}^\perp$  and  $(\mathbf{A} \wp \mathbf{B})^\perp = \mathbf{A}^\perp \otimes \mathbf{B}^\perp$ .*

*Proof.* One equality implies the other: say that  $(\mathbf{A} \otimes \mathbf{B})^\perp = \mathbf{A}^\perp \wp \mathbf{B}^\perp$  holds for any pair of types. Then  $\mathbf{A}^\perp \otimes \mathbf{B}^\perp = (\mathbf{A}^\perp \otimes \mathbf{B}^\perp)^{\perp\perp} = (\mathbf{A}^{\perp\perp} \wp \mathbf{B}^{\perp\perp})^\perp = (\mathbf{A} \wp \mathbf{B})^\perp$ .

Let us now prove  $\mathbf{A}^\perp \wp \mathbf{B}^\perp = (\mathbf{A} \otimes \mathbf{B})^\perp$ . As for  $\mathbf{A}^\perp \wp \mathbf{B}^\perp \subseteq (\mathbf{A} \otimes \mathbf{B})^\perp$ , we prove

$$\{S + \langle S(1), S(2) \triangleright_{\otimes} p \rangle \mid S \in \mathbf{A} \parallel \mathbf{B}\} \subseteq \{T + \langle T(1), T(2) \triangleright_{\wp} p \rangle \mid T \in \mathbf{A}^\perp \wp \mathbf{B}^\perp\}^\perp$$

which is enough to conclude because the previous inclusion yields (since bi-orthogonality preserves inclusion)

$$\mathbf{A} \otimes \mathbf{B} \subseteq \{T + \langle T(1), T(2) \triangleright_{\wp} p \rangle \mid T \in \mathbf{A}^\perp \wp \mathbf{B}^\perp\}^\perp$$

that is (since the tri-orthogonal of a set is the orthogonal of this set):

$$\mathbf{A} \otimes \mathbf{B} \subseteq (\mathbf{A}^\perp \wp \mathbf{B}^\perp)^\perp.$$

Which implies  $(\mathbf{A} \otimes \mathbf{B})^\perp \supseteq (\mathbf{A}^\perp \wp \mathbf{B}^\perp)$  (Orthogonality inverts inclusion).

So let  $S_0 \in \{S + \langle S(1), S(2) \triangleright_{\otimes} p \rangle \mid S \in \mathbf{A} \parallel \mathbf{B}\}$  and  $T_0 \in \{T + \langle T(1), T(2) \triangleright_{\wp} p \rangle \mid T \in \mathbf{A}^\perp \wp \mathbf{B}^\perp\}$ . One easily sees that  $S_0 \perp T_0$ : after eliminating the multiplicative cut we obtain a net  $S :: T$ , where  $S \in \mathbf{A}^\perp \wp \mathbf{B}^\perp$  and  $T \in \mathbf{A} \parallel \mathbf{B}$ : these constructions are orthogonal (Proposition 51).

As for  $(\mathbf{A} \otimes \mathbf{B})^\perp \subseteq \mathbf{A}^\perp \wp \mathbf{B}^\perp$ , let  $T \in (\mathbf{A} \otimes \mathbf{B})^\perp$  i.e.  $T \perp \{S + \langle S(1), S(2) \triangleright_{\otimes} p \rangle \mid S \in \mathbf{A} \parallel \mathbf{B}\}^{\perp\perp}$ ; we can deduce that the unique terminal link of  $T$  is either a  $\wp$ -link or a  $\mathfrak{X}$ -link:

- if  $T = T_0 + l$  has a terminal  $\wp$ -link  $l$  then, for every  $S_0 \in \mathbf{A} \parallel \mathbf{B}$  the net  $S ::= S_0 + \langle S(1), S(2) \triangleright_{\otimes} p \rangle$  belongs  $\mathbf{A} \otimes \mathbf{B}$ . In a single multiplicative step of cut elimination  $T :: S$  reduces to  $T_0 :: S_0$  and since  $S :: T \rightarrow^* \mathfrak{X}_0$  it follows that  $T_0 :: S_0 \rightarrow^* \mathfrak{X}_0$ , using Proposition 30 item 3. As a consequence  $T_0 \perp \mathbf{A} \parallel \mathbf{B}$  i.e.  $T_0 \in \mathbf{A}^\perp \wp \mathbf{B}^\perp$  and  $T \in \mathbf{A}^\perp \wp \mathbf{B}^\perp$ ;
- if  $T$  has for terminal link a daimon link then  $T$  may be written as  $R \bowtie^d \langle \triangleright_{\mathfrak{X}} p \rangle$  where  $R$  is a net without

conclusions. Let us start an argument from the assumption that  $T$  belongs to  $(\mathbf{A} \otimes \mathbf{B})^\perp$ :

$$\begin{aligned}
& T \perp \mathbf{A} \otimes \mathbf{B} \\
\Leftrightarrow & T \perp \{S_a \parallel S_b \mid S_a \in \mathbf{A}, S_b \in \mathbf{B}\}^{\perp\perp} \\
\Leftrightarrow & T \perp \{S_a \parallel S_b + \langle S_a(1), S_b(1) \triangleright_\otimes p \rangle \mid S_a \in \mathbf{A}, S_b \in \mathbf{B}\} \\
\Leftrightarrow & T :: S_a \parallel S_b + \langle S_a(1), S_b(1) \triangleright_\otimes p \rangle \rightarrow^* \mathfrak{X}_0 \quad (\text{for any } S_a \in \mathbf{A}, S_b \in \mathbf{B}) \\
\Leftrightarrow & R \bowtie^d \langle \triangleright_{\mathfrak{X}} p \rangle :: S_a \parallel S_b + \langle S_a(1), S_b(1) \triangleright_\otimes p \rangle \rightarrow^* \mathfrak{X}_0 \quad (\text{for any } S_a \in \mathbf{A}, S_b \in \mathbf{B}) \\
\Leftrightarrow & R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle :: S_a \parallel S_b \rightarrow^* \mathfrak{X}_0 \quad (\text{Proposition 30item 3}) \\
\Leftrightarrow & R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle \perp \{S_a \parallel S_b \mid S_a \in \mathbf{A}, S_b \in \mathbf{B}\} \\
\Leftrightarrow & R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle \perp \mathbf{A} \parallel \mathbf{B} \\
\Leftrightarrow & R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle \in \mathbf{A}^\perp \succ \mathbf{B}^\perp \\
\Rightarrow & R \bowtie^d \langle \triangleright_{\mathfrak{X}} p_1, p_2 \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{X}} p \rangle \in \mathbf{A}^\perp \mathfrak{X} \mathbf{B}^\perp \\
\Rightarrow & R \bowtie^d \langle \triangleright_{\mathfrak{X}} p \rangle \in \mathbf{A}^\perp \mathfrak{X} \mathbf{B}^\perp \quad (\text{Lemma 134}) \\
\Leftrightarrow & T \in \mathbf{A}^\perp \mathfrak{X} \mathbf{B}^\perp
\end{aligned}$$

□

**Proposition 54.** Given  $\mathbf{A}, \mathbf{B}$  and  $\mathbf{C}$  three types;  $(\mathbf{A} \succ \mathbf{B}) \succ \mathbf{C} = \mathbf{A} \succ (\mathbf{B} \succ \mathbf{C})$  and  $(\mathbf{A} \parallel \mathbf{B}) \parallel \mathbf{C} = \mathbf{A} \parallel (\mathbf{B} \parallel \mathbf{C})$ .

*Proof.* The result is a consequence of the two following properties:

1. the density of the parallel composition (Remark 50);
2. Proposition 48: more precisely, for any nets  $S_1, S_2, S_3, S_4$  such that  $\#S_1 \geq \#S_2 + \#S_3 + \#S_4$ , we have

$$(S_1 :: (S_2 \parallel S_3)) :: S_4 = ((S_1 :: S_2) :: S_3) :: S_4 = (S_1 :: S_2) :: (S_3 \parallel S_4).$$

By Proposition 51 it is enough to show that one of the two constructions is associative. Let us do it for the parallel composition: more precisely, using property 1 we prove that  $(A \parallel (B \parallel C))^\perp = (A \parallel^- (B^- \parallel C^-))^\perp$  and  $((A \parallel B) \parallel C)^\perp = ((A \parallel^- B) \parallel^- C)^\perp$ , and using property 2 that  $(A \parallel^- (B^- \parallel C^-))^\perp = ((A \parallel^- B) \parallel^- C)^\perp$ . From the previous equalities one deduces  $A \parallel (B \parallel C) = (A \parallel (B \parallel C))^{\perp\perp} = ((A \parallel B) \parallel C)^{\perp\perp} = (A \parallel B) \parallel C$ .

To prove the equality  $(A \parallel (B \parallel C))^\perp = (A \parallel^- (B^- \parallel C^-))^\perp$  one proves that, for every net  $S$ , the following equivalence holds:  $S \perp A \parallel (B \parallel C) \iff S \perp A \parallel^- (B^- \parallel C^-)$ . Indeed:

$$\begin{aligned}
& S \perp A \parallel (B \parallel C) \\
\Leftrightarrow & S \perp A \parallel^- (B \parallel C) \quad (\text{by property 1}) \\
\Leftrightarrow & \forall a \in A, \forall x \in B \parallel C, S :: (a \parallel x) \rightarrow \mathfrak{X}_0 \quad (\text{by definition}) \\
\Leftrightarrow & \forall a \in A, \forall x \in B \parallel C, (S :: a) :: x \rightarrow \mathfrak{X}_0 \quad (\text{by Proposition 48}) \\
\Leftrightarrow & \forall a \in A, S :: a \perp (B \parallel C) \quad (\text{by definition}) \\
\Leftrightarrow & \forall a \in A, S :: a \perp (B \parallel^- C) \quad (\text{by property 1}) \\
\Leftrightarrow & \forall a \in A, \forall x \in (B \parallel^- C), (S :: a) :: x \rightarrow \mathfrak{X}_0 \quad (\text{by definition}) \\
\Leftrightarrow & \forall a \in A, \forall x \in (B \parallel^- C), S :: (a \parallel x) \rightarrow \mathfrak{X}_0 \quad (\text{by Proposition 48}) \\
\Leftrightarrow & S \perp A \parallel^- (B \parallel^- C) \quad (\text{by definition}).
\end{aligned}$$

To prove the equality  $((A \parallel B) \parallel C)^\perp = ((A \parallel^- B) \parallel^- C)^\perp$ , we need to express the (obvious) fact that, given three nets  $S, T, U$  such that  $\#S = \#T + \#U$ , once the list of  $S$ 's conclusions that one decides to cut with the list of conclusions of  $T$  (resp.  $U$ ) is chosen, the interaction between  $S$  on the one hand and  $T$  and  $U$  on the other hand is uniquely determined. With the conventions of definition 41, when we write  $(S :: T) :: U$  we cut the first part ("the head") of the sequence  $a(S)$  with  $a(T)$  and the last part ("the tail") of the sequence  $a(S)$  with  $a(U)$ . We denote by  $ex_U(S)$  the net  $S$  where we have swapped the tail and the head of  $a(S)$ : the previous fact can then be expressed by the equality  $(S :: T) :: U = (ex_U(S) :: U) :: T$ .

Like for the previous equality, we prove that, for every net  $S$ , the following equivalence holds:  $S \perp (A \parallel B) \parallel C \iff S \perp (A \parallel^- B) \parallel^- C$ . Indeed:

$$\begin{aligned}
& S \perp (A \parallel B) \parallel C \\
\Leftrightarrow & S \perp (A \parallel B) \parallel^- C && \text{(by property 1)} \\
\Leftrightarrow & \forall x \in A \parallel B, \forall c \in C, S :: (x \parallel c) \rightarrow \mathfrak{X}_0 && \text{(by definition)} \\
\Leftrightarrow & \forall x \in A \parallel B, \forall c \in C, (S :: x) :: c \rightarrow \mathfrak{X}_0 && \text{(by Proposition 48)} \\
\Leftrightarrow & \forall x \in A \parallel B, \forall c \in C, (ex_c(S) :: c) :: x \rightarrow \mathfrak{X}_0 && \text{(by the obvious fact above)} \\
\Leftrightarrow & \forall c \in C, ex_c(S) :: c \in (A \parallel B)^\perp && \text{(by definition)} \\
\Leftrightarrow & \forall c \in C, ex_c(S) :: c \in (A \parallel^- B)^\perp && \text{(by property 1)} \\
\Leftrightarrow & \forall a \in A, \forall b \in B, \forall c \in C, (ex_c(S) :: c) :: (a \parallel b) \rightarrow \mathfrak{X}_0 && \text{(by definition)} \\
\Leftrightarrow & \forall a \in A, \forall b \in B, \forall c \in C, ((ex_c(S) :: c) :: a) :: b \rightarrow \mathfrak{X}_0 && \text{(by Proposition 48)} \\
\Leftrightarrow & \forall a \in A, \forall b \in B, \forall c \in C, ((S :: a) :: b) :: c \rightarrow \mathfrak{X}_0 && \text{(by the obvious fact above)} \\
\Leftrightarrow & \forall a \in A, \forall b \in B, \forall c \in C, (S :: (a \parallel b)) :: c \rightarrow \mathfrak{X}_0 && \text{(by Proposition 48)} \\
\Leftrightarrow & \forall a \in A, \forall b \in B, \forall c \in C, S :: ((a \parallel b) \parallel c) \rightarrow \mathfrak{X}_0 && \text{(by Proposition 48)} \\
\Leftrightarrow & S \perp (A \parallel^- B) \parallel^- C && \text{(by definition).}
\end{aligned}$$

To prove the last equality  $(A \parallel^- (B^- \parallel C^-))^\perp = ((A \parallel^- B) \parallel^- C)^\perp$ , we proceed like before and we show that, for every net  $S$ , the following equivalence holds:  $S \perp A \parallel^- (B^- \parallel C^-) \iff S \perp (A \parallel^- B) \parallel^- C$ . Indeed:

$$\begin{aligned}
& S \perp A \parallel^- (B^- \parallel C^-) \\
\Leftrightarrow & \forall a \in A, \forall b \in B, \forall c \in C, S :: (a \parallel (b \parallel c)) \rightarrow \mathfrak{X}_0 && \text{(by definition)} \\
\Leftrightarrow & \forall a \in A, \forall b \in B, \forall c \in C, (S :: a) :: (b \parallel c) \rightarrow \mathfrak{X}_0 && \text{(by Proposition 48)} \\
\Leftrightarrow & \forall a \in A, \forall b \in B, \forall c \in C, ((S :: a) :: b) :: c \rightarrow \mathfrak{X}_0 && \text{(by Proposition 48)} \\
\Leftrightarrow & \forall a \in A, \forall b \in B, \forall c \in C, (S :: (a \parallel b)) :: c \rightarrow \mathfrak{X}_0 && \text{(by property 2)} \\
\Leftrightarrow & \forall a \in A, \forall b \in B, \forall c \in C, S :: ((a \parallel b) \parallel c) \rightarrow \mathfrak{X}_0 && \text{(by Proposition 48)} \\
\Leftrightarrow & S \perp (A \parallel^- B) \parallel^- C && \text{(by definition).}
\end{aligned}$$

□

## E Complements to section 4

### E.1 Proving Remark 59

**Proposition 135** (Constructions preserve inclusion). *Consider four types  $\mathbf{A}_0 \subseteq \mathbf{A}$  and  $\mathbf{B}_0 \subseteq \mathbf{B}$  then:*

1.  $\mathbf{A}_0 \parallel \mathbf{B}_0 \subseteq \mathbf{A} \parallel \mathbf{B}$
2.  $\mathbf{A}_0 \succ \mathbf{B}_0 \subseteq \mathbf{A} \succ \mathbf{B}$
3.  $\mathbf{A}_0 \otimes \mathbf{B}_0 \subseteq \mathbf{A} \otimes \mathbf{B}$
4.  $\mathbf{A}_0 \wp \mathbf{B}_0 \subseteq \mathbf{A} \wp \mathbf{B}$

*Proof.* We treat each point independently.

1. Consider  $x$  an element of  $\mathbf{A}_0 \parallel^- \mathbf{B}_0$  then  $x$  is of the form  $a_0 \parallel b_0$  with  $a_0 \in \mathbf{A}_0$  and  $b_0 \in \mathbf{B}_0$ . Because we have the inclusion  $\mathbf{A}_0 \subseteq \mathbf{A}$  and  $\mathbf{B}_0 \subseteq \mathbf{B}$  it follows then that  $x = a_0 \parallel b_0$  belongs to  $\mathbf{A} \parallel^- \mathbf{B}$  and thus to  $\mathbf{A} \parallel \mathbf{B}$ . As a consequence  $\mathbf{A}_0 \parallel^- \mathbf{B}_0 \subseteq \mathbf{A} \parallel^- \mathbf{B}$  thus, because bi orthogonality preserves inclusion, it follows that  $(\mathbf{A}_0 \parallel^- \mathbf{B}_0)^{\perp\perp} \subseteq (\mathbf{A} \parallel^- \mathbf{B})^{\perp\perp}$  therefore  $\mathbf{A}_0 \parallel \mathbf{B}_0 \subseteq \mathbf{A} \parallel \mathbf{B}$  by density (Remark 50)
2. If  $\mathbf{A}_0 \subseteq \mathbf{A}$  and  $\mathbf{B}_0 \subseteq \mathbf{B}$  we equivalently have the inclusions  $\mathbf{A}_0^\perp \supseteq \mathbf{A}^\perp$  and  $\mathbf{B}_0^\perp \supseteq \mathbf{B}^\perp$ . Using the previous demonstrated fact it follows that  $\mathbf{A}_0^\perp \parallel \mathbf{B}_0^\perp$  contains  $\mathbf{A}^\perp \parallel \mathbf{B}^\perp$ . Again using the fact that orthogonality invert inclusions we then derive that  $(\mathbf{A}_0^\perp \parallel \mathbf{B}_0^\perp)^\perp$  is included in  $(\mathbf{A}^\perp \parallel \mathbf{B}^\perp)^\perp$ . By duality this means  $\mathbf{A}_0 \succ \mathbf{B}_0 \subseteq \mathbf{A} \succ \mathbf{B}$  (Proposition 51).
3. In the  $\otimes$ -case we reason similarly to the  $\parallel$  case.
4. For the  $\wp$ -case we can reason by duality using the  $\otimes$ -case.

□

**Proposition 136 (Remark 59).** Given a formula  $A$  and a basis  $\mathcal{B}$ , we have  $\llbracket A^\perp \rrbracket_{\mathcal{B}} \subseteq \llbracket A \rrbracket_{\mathcal{B}}^\perp$ .

*Proof.* By induction on the formula. If  $A = X$  is an atomic formula this is trivial. If  $A = B \otimes C$  then:

$$\begin{aligned}
\llbracket (B \otimes C)^\perp \rrbracket_{\mathcal{B}} &= \llbracket B^\perp \wp C^\perp \rrbracket_{\mathcal{B}} \\
&= \llbracket B^\perp \rrbracket_{\mathcal{B}} \wp \llbracket C^\perp \rrbracket_{\mathcal{B}} \\
&\subseteq \llbracket B \rrbracket_{\mathcal{B}}^\perp \wp \llbracket C \rrbracket_{\mathcal{B}}^\perp && \text{(Proposition 135 and induction hypothesis)} \\
&= (\llbracket B \rrbracket_{\mathcal{B}} \otimes \llbracket C \rrbracket_{\mathcal{B}})^\perp && \text{(Proposition 56)} \\
&= \llbracket B \otimes C \rrbracket_{\mathcal{B}}^\perp && \text{(Proposition 56)}
\end{aligned}$$

Similarly, if  $A = B \wp C$ :

$$\begin{aligned}
\llbracket (B \wp C)^\perp \rrbracket_{\mathcal{B}} &= \llbracket B^\perp \otimes C^\perp \rrbracket_{\mathcal{B}} \\
&= \llbracket B^\perp \rrbracket_{\mathcal{B}} \otimes \llbracket C^\perp \rrbracket_{\mathcal{B}} \\
&\subseteq \llbracket B \rrbracket_{\mathcal{B}}^\perp \otimes \llbracket C \rrbracket_{\mathcal{B}}^\perp && \text{(Proposition 135 and induction hypothesis)} \\
&= (\llbracket B \rrbracket_{\mathcal{B}} \wp \llbracket C \rrbracket_{\mathcal{B}})^\perp && \text{(Proposition 56)} \\
&= \llbracket B \wp C \rrbracket_{\mathcal{B}}^\perp && \text{(Proposition 56)}
\end{aligned}$$

□

## E.2 On approximable basis

**Remark 137.** Given two nets  $S$  and  $T$  the net  $S :: T$  is equal to the net  $T :: S$ , that is, up to isomorphism of the cut links mapping each  $\langle S(i), T(i) \triangleright_{\text{cut}} \rangle$  to a cut link  $\langle T(i), S(i) \triangleright_{\text{cut}} \rangle$ .

**Proposition 138.** Given  $\mathcal{B}$  an approximable basis for any formula  $A$  of MLL the type  $\llbracket A \rrbracket_{\mathcal{B}}$  is approximable.

*Proof.* By induction on  $A$ . In the case where  $A$  is atomic this follows from the definition.

- For a formula  $A \otimes B$ . Let us show  $\mathfrak{X}_1$  belongs to  $\llbracket A \otimes B \rrbracket_{\mathcal{B}}$  by showing that  $\mathfrak{X}_1$  is orthogonal to  $\llbracket A \otimes B \rrbracket_{\mathcal{B}}^\perp$ . The interpretation  $\llbracket A \otimes B \rrbracket_{\mathcal{B}}^\perp$  is equal to  $(\llbracket A \rrbracket_{\mathcal{B}} \otimes \llbracket B \rrbracket_{\mathcal{B}})^\perp$  and so to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \wp \llbracket B \rrbracket_{\mathcal{B}}^\perp$ . To be orthogonal to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \wp \llbracket B \rrbracket_{\mathcal{B}}^\perp$  is to be orthogonal to the nets  $S + \langle p_1, p_2 \triangleright_{\wp} p \rangle$  where  $S$  belongs to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \succ \llbracket B \rrbracket_{\mathcal{B}}^\perp$ . Consider  $S$  such a net, then:

$$\begin{aligned}
&\mathfrak{X}_1 :: S + \langle p_1, p_2 \triangleright_{\wp} p \rangle \\
\rightarrow &\mathfrak{X}_1 \parallel \mathfrak{X}_1 :: S && \text{(Definition 22)} \\
= &S :: (\mathfrak{X}_1 \parallel \mathfrak{X}_1) && \text{(Remark 137)} \\
= &(S :: \mathfrak{X}_1) :: \mathfrak{X}_1 && \text{(Proposition 48)}
\end{aligned}$$

By induction  $\mathfrak{X}_1$  belongs to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp$  and  $S$  belongs to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \succ \llbracket B \rrbracket_{\mathcal{B}}^\perp$  we conclude that  $S :: \mathfrak{X}_1$  belongs to  $\llbracket B^\perp \rrbracket_{\mathcal{B}}$ . Again by induction  $\mathfrak{X}_1$  belongs to  $\llbracket B \rrbracket_{\mathcal{B}}^\perp$  and so in particular it belongs to  $\llbracket B^\perp \rrbracket_{\mathcal{B}}$ . To conclude: we have shown that  $S :: \mathfrak{X}_1$  belongs to  $\llbracket B^\perp \rrbracket_{\mathcal{B}}$ , hence it is orthogonal to  $\llbracket B^\perp \rrbracket_{\mathcal{B}}^\perp$  so in particular it is orthogonal to  $\mathfrak{X}_1$ . As a consequence  $S :: \mathfrak{X}_1 :: \mathfrak{X}_1$  reduces to  $\mathfrak{X}_0$ .

- Now consider a formula of the form  $A \wp B$ . Again let us show that  $\mathfrak{X}_1$  is orthogonal to  $\llbracket A \wp B \rrbracket_{\mathcal{B}}^\perp$  i.e. to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \otimes \llbracket B \rrbracket_{\mathcal{B}}^\perp$ . To be orthogonal to that type is to be orthogonal to the nets  $\bar{a} \parallel \bar{b} + \langle p_1, p_2 \triangleright_{\otimes} p \rangle$ . Consider such a net, then:

$$\begin{aligned}
&\mathfrak{X}_1 :: \bar{a} \parallel \bar{b} + \langle p_1, p_2 \triangleright_{\otimes} p \rangle \\
\rightarrow &\mathfrak{X}_2 :: \bar{a} \parallel \bar{b} && \text{(Definition 22)} \\
= &\mathfrak{X}_1 \bowtie \mathfrak{X}_1 :: \bar{a} \parallel \bar{b} && \text{(Definition 122)}
\end{aligned}$$

By induction both  $\llbracket A \rrbracket_{\mathcal{B}}$  and  $\llbracket B \rrbracket_{\mathcal{B}}$  contains  $\mathfrak{X}_1$  thus  $\llbracket A \rrbracket_{\mathcal{B}} \succ \llbracket B \rrbracket_{\mathcal{B}}$  contains  $\mathfrak{X}_2$  (Proposition 178). Equivalently this means that  $\mathfrak{X}_2$  is orthogonal to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \parallel \llbracket B \rrbracket_{\mathcal{B}}^\perp$ , hence  $\mathfrak{X}_2 \perp \bar{a} \parallel \bar{b}$ . This concludes to show that  $\mathfrak{X}_1$  belongs to  $\llbracket A \wp B \rrbracket_{\mathcal{B}}$ .

□

### E.3 Adequacy – Proof of Theorem 64

**Theorem 64** (Adequacy). *Let  $\mathcal{B}$  be an approximable basis. For any net  $S$  and sequent  $\Gamma \vdash_{\text{MLL}^\times} \Gamma \Rightarrow S \Vdash_{\mathcal{B}} \Gamma$ .*

*Proof.* We proceed by induction on the proof  $\pi$  which is represented by the net  $S$ .

*Base case.* In the base the proof  $\pi$  consists of a proof tree using a single  $\boxtimes$ -rule introducing any sequent  $\Gamma = A_1, \dots, A_n$  of length  $n$ . Then any net  $S$  representing  $\pi$  is isomorphic to  $\boxtimes_n$ , we must therefore show that  $\boxtimes_n$  belongs to  $\llbracket \Gamma \rrbracket_{\mathcal{B}}$ .

Because  $\mathcal{B}$  is approximable any interpretation  $\llbracket A \rrbracket_{\mathcal{B}}$  is approximable (Proposition 138): for every  $1 \leq i \leq n$  we have  $\boxtimes_1 \in \llbracket A_i \rrbracket_{\mathcal{B}}$  i.e.  $u \perp \boxtimes_1$  for every  $u \in \llbracket A_i \rrbracket_{\mathcal{B}}^\perp$ . We want to show that  $\boxtimes_n \in \llbracket A_1, \dots, A_n \rrbracket_{\mathcal{B}}$  or equivalently that  $\boxtimes_n$  is orthogonal to  $\llbracket A_1, \dots, A_n \rrbracket_{\mathcal{B}}^\perp = (\llbracket A_1 \rrbracket_{\mathcal{B}} \triangleright \dots \triangleright \llbracket A_n \rrbracket_{\mathcal{B}})^\perp = \llbracket A_1 \rrbracket_{\mathcal{B}}^\perp \parallel \dots \parallel \llbracket A_n \rrbracket_{\mathcal{B}}^\perp$ . By density (Remark 50) to prove  $\boxtimes_n \perp \llbracket A_1 \rrbracket_{\mathcal{B}}^\perp \parallel \dots \parallel \llbracket A_n \rrbracket_{\mathcal{B}}^\perp$  it is enough to prove that  $u_1 \parallel \dots \parallel u_n \perp \boxtimes_n$  for every  $u_i \in \llbracket A_i \rrbracket_{\mathcal{B}}^\perp$ . We proceed by induction on  $n$ :

$$\begin{aligned}
& \boxtimes_n :: (u_1 \parallel \dots \parallel u_n) \\
&= (\dots (\boxtimes_n :: u_1) :: \dots) :: u_n && \text{(Proposition 48)} \\
\rightarrow^* & (\dots (\boxtimes_{n-1} :: u_2) :: \dots) :: u_n && \text{(Proposition 128 and } u_1 \perp \boxtimes_1) \\
&= \boxtimes_{n-1} :: (u_2 \parallel \dots \parallel u_n) && \text{(Proposition 48)} \\
\rightarrow^* & \boxtimes_0 && \text{(Induction hypothesis)}
\end{aligned}$$

As a consequence  $\boxtimes_n$  is contained in the type  $\llbracket A_1, \dots, A_n \rrbracket_{\mathcal{B}}$ , i.e.  $\boxtimes_n$  realises the sequent  $\Gamma$ .

To conclude observe that any proof  $\pi$  of conclusion  $\Gamma = A_1, \dots, A_n$  which uses only one inference rule in  $\text{MLL}^\times$ , must use a daimon inference rule. Then any such proof  $\pi$  is mapped by  $\llbracket \pi \rrbracket$  to the daimon link with  $n$  outputs  $\boxtimes_n$ . Hence we have showed the base case of the induction, where the induction is performed on the size (its number of inference rules) of the proof  $\pi$ .

*Inductive cases.* We look at the last rule of  $\pi$  which may be a  $\otimes, \wp$  or cut-rule:

IND-1 Assume that the last rule in the represented proof is a  $\wp$ -rule with main conclusion  $A \wp B$ . Thus the sequent is of the form  $\Gamma, A \wp B$ , and by assumption  $S \vdash_{\text{MLL}^\times} \Gamma, A \wp B$ . Say the conclusions of  $S$  are ordered as  $q_1, \dots, q_n, p$  then since  $p$  is given the type  $A \wp B$  the net  $S$  is of the form  $S' + \langle p_1, p_2 \triangleright_{\wp} p \rangle$  where  $S' \vdash_{\text{MLL}^\times} \Gamma, A, B$ .

Calling the induction hypothesis we can deduce  $S' \Vdash_{\mathcal{B}} \Gamma, A, B$ . By definition, this means that for any  $\gamma$  in  $\llbracket \Gamma \rrbracket_{\mathcal{B}}^\perp$ , we have  $S' :: \gamma \Vdash_{\mathcal{B}} A, B$ , and thus  $S' :: \gamma + \langle p_1, p_2 \triangleright_{\wp} p \rangle \Vdash_{\mathcal{B}} A \wp B$ .

Furthermore interaction and sum commute (Proposition 121) hence,  $S' :: \gamma + \langle p_1, p_2 \triangleright_{\wp} p \rangle = S' + \langle p_1, p_2 \triangleright_{\wp} p \rangle :: \gamma$  so that  $S :: \gamma \Vdash_{\mathcal{B}} A \wp B$  for any  $\gamma \in \llbracket \Gamma \rrbracket_{\mathcal{B}}^\perp$ : this allows us to conclude that  $S \Vdash_{\mathcal{B}} \Gamma, A \wp B$ .

IND-2 Assume that the last rule in the represented proof is a  $\otimes$ -rule with main conclusion  $A \otimes B$ . Thus the sequent is of the form  $\Gamma, \Delta, A \otimes B$ , and  $S$  is of the form  $S_1 \otimes S_2$  where  $S_1 \vdash_{\text{MLL}^\times} \Gamma, A$  and  $S_2 \vdash_{\text{MLL}^\times} \Delta, B$ .

Calling the induction hypothesis we obtain  $S_1 \Vdash_{\mathcal{B}} \Gamma, A$  and  $S_2 \Vdash_{\mathcal{B}} \Delta, B$ . Thus for any  $\gamma \in \llbracket \Gamma \rrbracket_{\mathcal{B}}^\perp$  and  $\delta \in \llbracket \Delta \rrbracket_{\mathcal{B}}^\perp$   $S_1 :: \gamma \Vdash_{\mathcal{B}} A$  and  $S_2 :: \delta \Vdash_{\mathcal{B}} B$ . In particular this means that the tensor union of the two nets  $S_1 :: \gamma + S_2 :: \delta + \langle p_1, p_2 \triangleright_{\otimes} p \rangle$  is in  $\llbracket A \rrbracket_{\mathcal{B}} \otimes \llbracket B \rrbracket_{\mathcal{B}} = \llbracket A \otimes B \rrbracket_{\mathcal{B}}$ .

Note that  $S_1 :: \gamma + S_2 :: \delta + \langle p_1, p_2 \triangleright_{\otimes} p \rangle = S_1 + S_2 + \langle p_1, p_2 \triangleright_{\otimes} p \rangle :: \gamma \parallel \delta$  (Proposition 48 and Proposition 121). Thus  $S_1 + S_2 + \langle p_1, p_2 \triangleright_{\otimes} p \rangle :: \gamma \parallel \delta \Vdash_{\mathcal{B}} A \otimes B$ . Since this hold for any  $\gamma \in \llbracket \Gamma \rrbracket_{\mathcal{B}}^\perp$  and  $\delta \in \llbracket \Delta \rrbracket_{\mathcal{B}}^\perp$  we conclude that  $S_1 + S_2 + \langle p_1, p_2 \triangleright_{\otimes} p \rangle \Vdash_{\mathcal{B}} \Gamma, \Delta, A \otimes B$  i.e.  $S \Vdash_{\mathcal{B}} \Gamma, \Delta, A \otimes B$ .

IND-3 Assume that the last rule in  $\pi$  is a cut rule between two subproofs  $\pi_1$  and  $\pi_2$  of respective conclusions  $\Gamma, A$  and  $\Delta, A^\perp$ . Then a net  $S$  which represents  $\pi$  is of the form  $S = S_1 + S_2 + \langle S_1(\#S_1), S_2(\#S_2) \triangleright_{\text{cut}} \rangle$  where  $S_1$  (resp.  $S_2$ ) is a representation of the proof  $\pi_1$  (resp.  $\pi_2$ ).

Applying the induction hypothesis we derive  $S_1$  belongs to  $\llbracket \Gamma, A \rrbracket_{\mathcal{B}}$  and  $S_2$  belongs to  $\llbracket \Delta \rrbracket_{\mathcal{B}}$ . Hence for any  $u \in \llbracket \Gamma \rrbracket_{\mathcal{B}}^\perp$  and  $v \in \llbracket \Delta \rrbracket_{\mathcal{B}}^\perp$  we have  $S_1 :: u \in \llbracket A \rrbracket_{\mathcal{B}}$  and  $S_2 :: v \in \llbracket A^\perp \rrbracket_{\mathcal{B}}$ . Because  $\llbracket A^\perp \rrbracket_{\mathcal{B}} \subset \llbracket A \rrbracket_{\mathcal{B}}^\perp$  (Remark 59) it follows that  $(S_1 :: u) \perp (S_2 :: v)$ . Let us rewrite this net to conclude:

$$\begin{aligned}
& S_1 :: u \quad :: \quad S_2 :: v \\
&= S_1 :: u \quad + \langle (S_1 :: u)(1), (S_2 :: v)(1) \triangleright_{\text{cut}} \rangle + \quad S_2 :: v && \text{(Definition 41)} \\
&= S_1 :: u \quad + \langle S_1(\#S_1), S_2(\#S_2) \triangleright_{\text{cut}} \rangle + \quad S_2 :: v && \text{(Identity)} \\
&= (S_1 + \langle S_1(\#S_1), S_2(\#S_2) \triangleright_{\text{cut}} \rangle) :: u + S_2 :: v && \text{(Proposition 121)} \\
&= (S_1 + \langle S_1(\#S_1), S_2(\#S_2) \triangleright_{\text{cut}} \rangle + S_2) :: u :: v && \text{(Proposition 121)} \\
&= S :: u :: v && \text{(Identify } S) \\
&= S :: (u \parallel v) && \text{(Proposition 48)}
\end{aligned}$$

This shows that  $S$  is orthogonal to any  $u \parallel v$  when  $u \in \llbracket \Gamma \rrbracket_{\mathcal{B}}^{\perp}$  and  $v \in \llbracket \Delta \rrbracket_{\mathcal{B}}^{\perp}$ . In other words  $S$  is orthogonal to  $\llbracket \Gamma \rrbracket_{\mathcal{B}}^{\perp} \parallel \llbracket \Delta \rrbracket_{\mathcal{B}}^{\perp}$  and therefore to  $\llbracket \Gamma \rrbracket_{\mathcal{B}}^{\perp} \parallel \llbracket \Delta \rrbracket_{\mathcal{B}}^{\perp}$  (Remark 50) thus by duality it belongs to  $\llbracket \Gamma \rrbracket_{\mathcal{B}} \succ \llbracket \Delta \rrbracket_{\mathcal{B}}$  (Proposition 51) i.e. by definition to  $\llbracket \Gamma, \Delta \rrbracket_{\mathcal{B}}$ . □

## F Complements to section 5

### F.1 On orthogonality of Partitions

#### F.1.1 Orthogonal Partitions and bijections

**Definition 139** (image of a partition). Given a partition  $P = \{C_1, \dots, C_n\}$  of a set  $X$  and a function  $f : X \rightarrow Y$  the image of  $P$  by  $f$  is the set  $\{f(C_1), \dots, f(C_n)\}$ , it is denoted  $f(P)$ .

**Proposition 140** (Bijections preserve partitions). Given  $f : X \rightarrow Y$  a bijection between two sets. For any partition  $P$  of  $X$  the image  $f(P)$  is a partition of  $Y$ .

**Definition 141.** An isomorphism  $f : G \rightarrow H$  between two undirected multigraph  $G = (V_G, E_G, \text{brd}_G)$  and  $H = (V_H, E_H, \text{brd}_H)$  is a pair of functions  $(f_V, f_E)$  such that:

- $f_V : V_G \rightarrow V_H$  is a bijection.
- $f_E : E_G \rightarrow E_H$  is a bijection.
- For any edge  $e$  of  $E_G$  we have that  $f_V(\text{brd}_{G_e})$  equals  $\text{brd}_{H_{f(e)}}$ .

**Proposition 142.** Given  $f : X \rightarrow Y$  a bijection  $P$  and  $Q$  two partitions of  $X$ . The assertions are equivalent:

- The partitions  $P$  and  $Q$  are orthogonal.
- The partitions  $f(P)$  and  $f(Q)$  are orthogonal.

*Proof.* We show that the induced graph by  $P$  and  $Q$  and by  $f(P)$  and  $f(Q)$  are isomorphic, i.e. that  $G_1 ::= G(P, Q)$  and  $G_2 ::= G(f(P), f(Q))$  are isomorphic. The bijection between the set of edges is given by  $f : X \rightarrow Y$  while the bijection between the vertex-sets  $V_{G_1}$  and  $V_{G_2}$  is given by the map associating with a subset of  $X$  its image under  $f$  i.e.  $F : \{x_1, \dots, x_n\} \rightarrow \{f(x_1), \dots, f(x_n)\}$ .

Finally, one can check that the border function are coherent: for  $A \in P$  the image  $f(\text{brd}_{G(P,Q)}(A))$  is  $\text{brd}_{G(f(P), f(Q))}(f(A))$ . □

#### F.1.2 Orthogonal Partitions and their representations as nets

**Definition 143.** A natural partition is a partition of a subset of  $\mathbb{N}$ . A natural partition of size  $n$  is a partition of the set of integers  $\{1, \dots, n\}$ .

**Remark 144.** Given a natural partition  $P$  of size  $n$ , any partition to  $Q$  which is orthogonal to  $P$  is also a natural partition of size  $n$ .

**Definition 145.** A net  $S = (|S|, \mathbf{a}(S))$  containing only daimon links represents a natural partition  $P$  of size  $n$  (which we denote  $S \equiv P$ ) whenever:

- For each class  $\{i_1, \dots, i_k\}$  in  $P$  there exists a daimon link  $\langle \triangleright_{\mathbf{x}} S(i_1), \dots, S(i_k) \rangle$  in  $S$ .
- For each daimon link  $\langle \triangleright_{\mathbf{x}} S(a_1), \dots, S(a_m) \rangle$  contained in  $S$  the set of integers  $\{a_1, \dots, a_m\}$  is a class of the partition  $P$ .

**Remark 146.** Given a natural partition  $P$  of size  $n$  there exists (infinitely) many nets  $S$  which represent  $P$ , however all these nets are isomorphic.

*Notation 147.* Given a natural partition  $P$  of size  $n$  we abusively denote by  $P^{\mathbf{x}}$  a representation of  $P$ .

**Proposition 148.** Given  $P$  and  $Q$  two natural partition of size  $n$ , and two nets  $S$  and  $T$  such that  $S \equiv P$  while  $T \equiv Q$ , the assertions are equivalent:

- The partitions  $P$  and  $Q$  are orthogonal.
- The nets  $S$  and  $T$  are orthogonal.

*Proof.* Observe that  $G(P, Q)$  is acyclic and connected if and only if  $S :: T$  is acyclic and connected. Then we can prove each implication:

- 1  $\Rightarrow$  2. Assuming that  $G(P, Q)$  then  $S :: T$  is acyclic and connected. Because  $S$  and  $T$  contain only daimon links,  $S :: T$  contains only glueing cuts. Furthermore the elimination of glueing cuts preserve acyclicity and connectedness while  $S :: T$  is a net without conclusion (and cut-elimination preserve the conclusion of a net). The only cut free, acyclic-connected net with no conclusions is  $\mathfrak{X}_0$  thus we conclude  $S :: T \rightarrow \mathfrak{X}_0$  meaning that  $S$  and  $T$  are orthogonal.
- 2  $\Rightarrow$  2. Assume that  $S :: T \rightarrow \mathfrak{X}_0$  note that the reduction step of glueing cuts preserve acyclicity and connect- edness *in both directions*, as a consequence since  $S :: T$  reduces to  $\mathfrak{X}_0$  by eliminating only glueing cuts and since  $\mathfrak{X}_0$  is acyclic and connected we conclude that  $S :: T$  is also acyclic and connected. This means equivalently that  $G(P, Q)$  is acyclic and connected. □

## F.2 Proving Proposition 75

### F.2.1 Elimination of Multiplicative cuts and addresses

**Definition 149.** An *address* is a sequence of elements of  $\{\ell, \varkappa\}$ . The element at position  $\ell$  (resp.  $\varkappa$ ) relatively to  $p$  in a net  $S$  is as follows:

$$\text{Find}(p; \ell) \triangleq \begin{cases} p_1 & \text{if } p \in \langle p_1, p_2 \triangleright_{\square} p \rangle \text{ belongs to } S \\ & \text{undefined otherwise.} \end{cases}$$

$$\text{Find}(p; \varkappa) \triangleq \begin{cases} p_2 & \text{if } p \in \langle p_1, p_2 \triangleright_{\square} p \rangle \text{ belongs to } S \\ & \text{undefined otherwise.} \end{cases}$$

The element at address  $\xi$  relatively to  $p$  in a net  $S$  is defined inductively:

$$\text{Find}(p; \varepsilon) = p \quad ; \quad \text{Find}(p; \ell \cdot \xi) = \text{Find}(\text{Find}(p; \ell); \xi) \quad ; \quad \text{Find}(p; \varkappa \cdot \xi) = \text{Find}(\text{Find}(p; \varkappa); \xi)$$

**Remark 150.** For each position  $p$  of a net  $S = (|S|, \mathbf{a}(S))$  there exists an address  $\xi$  and an integer  $i$  so that  $p = \text{Find}(S(i); \xi)$  (This can be shown by induction on the number of links in  $S$ ). For a position  $p$  we denote that index  $i$  by  $\text{root}(p)$  and  $\xi$  by  $\text{adr}(p)$ .

**Remark 151.** Addresses are ordered in the following way: we set  $\ell \leq \varkappa$  and lift to addresses using the lexicographical order.

**Definition 152** (cut-free initial order). Given a cut-free net  $S = (|S|, \mathbf{a}(S))$  we can order the initial positions of  $S$ : A position  $p$  is smaller than  $q$  if  $\text{root}(p) < \text{root}(q)$  or if  $\text{root}(p) = \text{root}(q)$  while  $\text{adr}(p) \leq \text{adr}(q)$  (for the lexicographical order). This induced order is denoted  $\leq^{\mathbf{a}(S)}$ .

Given a conclusion  $p$  of a net  $S$  an address  $\xi$  is *defined* for  $p$  if  $\text{Find}(p; \xi)$  is defined in  $S$ . An address is *maximal* for a position  $p$  if  $\xi$  is defined for  $p$  and no address that is defined for  $p$  has  $\xi$  for strict prefix. We denote  $\text{max}(p)$  the maximal addresses of a position  $p$ .

**Proposition 153.** Given a net  $S$ ,  $p$  a conclusion of  $S$  and  $\xi$  an address. The assertions are equivalent:

- $\xi$  is maximal for  $p$ .
- $\text{Find}(p; \xi)$  is an initial position of  $S$ .

**Corollary 154.** Given a net  $S = (|S|, \mathbf{a}(S))$  the set of initial positions of  $S$  is equal to

$$\bigcup_{1 \leq i \leq \#S} \text{Find}(S(i); \text{max}(S(i))).$$

*Notation 155.* Given a position  $p$  in a net  $S$  we denote  $\text{link}_S(p)$  the set of connective links above  $p$  in  $S$ .

Given an (unordered) net  $S = (V, E, s, t)$  and a set of link  $E_0$  of  $S$  we denote by  $S \setminus E_0$  the net  $(V \cap \bigcup_{e \in E \setminus E_0} t(e) \cup s(e), E \setminus E_0, s, t)$ .

**Proposition 156.** Given  $S$  and  $T$  two nets such that  $S \stackrel{\text{dual}}{\vdash} A_1, \dots, A_n$  while  $T \stackrel{\text{dual}}{\vdash} A_1^\perp, \dots, A_n^\perp$  then by eliminating only multiplicative cuts the interaction  $S :: T$  reduces to

$$S^{\mathfrak{X}} + T^{\mathfrak{X}} + \sum_{1 \leq i \leq n} \sum_{\xi \in \{\ell, \varkappa\}^{N_i}} \langle \text{Find}(S(i); \xi)_S, \text{Find}(T(i); \xi)_T \triangleright_{\text{cut}} \rangle$$

*Proof.* Assume that  $S$  and  $T$  contain connective links (i.e.  $\otimes$ - and  $\wp$ -links) and more precisely, without loss of generality assume that their first conclusion is the output of a connective link. As a consequence since  $S \stackrel{\text{dual}}{\vdash} A_1, \dots, A_n$  and  $T \stackrel{\text{dual}}{\vdash} A_1^\perp, \dots, A_n^\perp$  witness dual sequents it follows that  $A_1 = B_1 \otimes B_2$  while  $A_1^\perp = B_1 \wp B_2$ .



First note that  $S :: T$  is equal to

$$S + T + \sum_{1 \leq i \leq n} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle = S + T + \langle S(1), T(1) \triangleright_{\text{cut}} \rangle + \sum_{2 \leq i \leq n} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle$$

Furthermore since  $A_1 = B_1 \otimes B_2$  while  $A_1^\perp = B_1 \wp B_2$  we can ensure that a link  $\langle p_1, p_2 \triangleright_{\otimes} S(1) \rangle$  occurs in  $S$  and is terminal while a link  $\langle q_1, q_2 \triangleright_{\wp} T(1) \rangle$  occurs in  $T$  and is terminal.

As a consequence  $\langle S(1), T(1) \triangleright_{\text{cut}} \rangle$  is a multiplicative cut in  $S :: T$  and its elimination produces the following reduction:

$$\begin{aligned} & S + T + \langle S(1), T(1) \triangleright_{\text{cut}} \rangle \\ \rightarrow & S \setminus \langle p_1, p_2 \triangleright_{\otimes} S(1) \rangle + T \setminus \langle q_1, q_2 \triangleright_{\wp} T(1) \rangle + \langle p_1, q_1 \triangleright_{\text{cut}} \rangle + \langle p_2, q_2 \triangleright_{\text{cut}} \rangle \\ = & S \setminus \langle p_1, p_2 \triangleright_{\otimes} S(1) \rangle + T \setminus \langle q_1, q_2 \triangleright_{\wp} T(1) \rangle \\ & + \langle \text{Find}(S(1); \ell), \text{Find}(T(1); \ell) \triangleright_{\text{cut}} \rangle + \langle \text{Find}(S(1); \mathcal{r}), \text{Find}(T(1); \mathcal{r}) \triangleright_{\text{cut}} \rangle \end{aligned}$$

By contextual closure of the cut elimination procedure we derive the following reduction:

$$\begin{aligned} S :: T &= S + T + \sum_{1 \leq i \leq n} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle \\ &= S + T + \langle S(1), T(1) \triangleright_{\text{cut}} \rangle + \sum_{2 \leq i \leq n} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle \\ \rightarrow & S \setminus \langle p_1, p_2 \triangleright_{\otimes} S(1) \rangle + T \setminus \langle q_1, q_2 \triangleright_{\wp} T(1) \rangle \\ & + \langle \text{Find}(S(1); \ell), \text{Find}(T(1); \ell) \triangleright_{\text{cut}} \rangle + \langle \text{Find}(S(1); \mathcal{r}), \text{Find}(T(1); \mathcal{r}) \triangleright_{\text{cut}} \rangle + \sum_{2 \leq i \leq n} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle \end{aligned}$$

Now consider the two nets  $S_0 = (S \setminus \langle p_1, p_2 \triangleright_{\otimes} S(1) \rangle, \text{Find}(S(1); \ell) \cdot \text{Find}(S(1); \mathcal{r}) \cdot \mathbf{a}(S))$  and  $T_0 = (T \setminus \langle q_1, q_2 \triangleright_{\wp} T(1) \rangle, \text{Find}(T(1); \ell) \cdot \text{Find}(T(1); \mathcal{r}) \cdot \mathbf{a}(T))$ . Observe that because  $S \stackrel{\text{at}}{\cong} A_1, \dots, A_n$  it follows that  $S_0 \stackrel{\text{at}}{\cong} B_1, B_2, A_2, \dots, A_n$  while because  $T \stackrel{\text{at}}{\cong} A_1^\perp, \dots, A_n^\perp$  it follows that  $T_0 \stackrel{\text{at}}{\cong} B_1^\perp, B_2^\perp, A_2^\perp, \dots, A_n^\perp$ . Because  $S_0$  and  $T_0$  have less connective links than  $S$  and  $T$  we can apply the induction hypothesis. Furthermore we observe that

$$\begin{aligned} & S \setminus \langle p_1, p_2 \triangleright_{\otimes} S(1) \rangle + T \setminus \langle q_1, q_2 \triangleright_{\wp} T(1) \rangle \\ & + \langle \text{Find}(S(1); \ell), \text{Find}(T(1); \ell) \triangleright_{\text{cut}} \rangle + \langle \text{Find}(S(1); \mathcal{r}), \text{Find}(T(1); \mathcal{r}) \triangleright_{\text{cut}} \rangle + \sum_{2 \leq i \leq n} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle \\ = & S_0 + T_0 + \langle \text{Find}(S(1); \ell), \text{Find}(T(1); \ell) \triangleright_{\text{cut}} \rangle + \langle \text{Find}(S(1); \mathcal{r}), \text{Find}(T(1); \mathcal{r}) \triangleright_{\text{cut}} \rangle + \sum_{2 \leq i \leq n} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle \\ = & S_0 + T_0 + \langle S_0(1), T_0(1) \triangleright_{\text{cut}} \rangle + \langle S_0(2), T_0(2) \triangleright_{\text{cut}} \rangle + \sum_{2 \leq i \leq n} \langle S(i), T(i) \triangleright_{\text{cut}} \rangle \\ = & S_0 + T_0 + \langle S_0(1), T_0(1) \triangleright_{\text{cut}} \rangle + \langle S_0(2), T_0(2) \triangleright_{\text{cut}} \rangle + \sum_{3 \leq i \leq n+1} \langle S_0(i), T_0(i) \triangleright_{\text{cut}} \rangle \\ = & S_0 + T_0 + \sum_{1 \leq i \leq n+1} \langle S_0(i), T_0(i) \triangleright_{\text{cut}} \rangle \\ = & S_0 :: T_0 \end{aligned}$$

By calling the induction hypothesis on  $S_0 :: T_0$  we derive the reduction:

$$S_0 :: T_0 \rightarrow^* S_0^{\mathfrak{X}} + T_0^{\mathfrak{X}} + \sum_{1 \leq i \leq \#S_0} \sum_{\xi \in \max(S_0(i))} \langle \text{Find}(S_0(i); \xi)_S, \text{Find}(T_0(i); \xi)_T \triangleright_{\text{cut}} \rangle$$

First observe  $S_0^{\mathfrak{X}} = S^{\mathfrak{X}}$  and  $T_0^{\mathfrak{X}} = T^{\mathfrak{X}}$ . Now note that any for any  $3 \leq i \leq \#S_0$  we have  $S_0(i) = S(i-1)$  while  $T_0(i) = T(i-1)$  thus the sum of cut links can be rewritten:

$$\begin{aligned} & \sum_{1 \leq i \leq \#S_0} \sum_{\xi \in \max(S_0(i))} \langle \text{Find}(S_0(i); \xi)_S, \text{Find}(T_0(i); \xi)_T \triangleright_{\text{cut}} \rangle \\ = & \sum_{1 \leq i \leq 2} \sum_{\xi \in \max(S_0(i))} \langle \text{Find}(S_0(i); \xi)_S, \text{Find}(T_0(i); \xi)_T \triangleright_{\text{cut}} \rangle + \sum_{3 \leq i \leq \#S_0} \sum_{\xi \in \max(S_0(i))} \langle \text{Find}(S_0(i); \xi)_S, \text{Find}(T_0(i); \xi)_T \triangleright_{\text{cut}} \rangle \\ = & \sum_{1 \leq i \leq 2} \sum_{\xi \in \max(S_0(i))} \langle \text{Find}(S_0(i); \xi)_S, \text{Find}(T_0(i); \xi)_T \triangleright_{\text{cut}} \rangle + \sum_{3 \leq i \leq \#S_0} \sum_{\xi \in \max(S(i-1))} \langle \text{Find}(S(i-1); \xi)_S, \text{Find}(T(i-1); \xi)_T \triangleright_{\text{cut}} \rangle \\ = & \sum_{1 \leq i \leq 2} \sum_{\xi \in \max(S_0(i))} \langle \text{Find}(S_0(i); \xi)_S, \text{Find}(T_0(i); \xi)_T \triangleright_{\text{cut}} \rangle + \sum_{2 \leq i \leq \#S} \sum_{\xi \in \max(S(i))} \langle \text{Find}(S(i); \xi)_S, \text{Find}(T(i); \xi)_T \triangleright_{\text{cut}} \rangle \end{aligned}$$

To conclude observe that  $\max(S_0(i))$  is equal to  $\{s \mid \exists i \in \{\ell, \mathcal{r}\} \cdot s \in \max(S(i))\}$  Furthermore  $\text{Find}(T(1); \ell)$

is equal to  $T_0(1)$ . Thus we derive the following:

$$\begin{aligned}
& \sum_{1 \leq i \leq 2} \sum_{\xi \in \max(S_0(i))} \langle \text{Find}(S_0(i); \xi)_{S_0}, \text{Find}(T_0(i); \xi)_{T_0} \triangleright_{\text{cut}} \rangle \\
&= \sum_{i \in \{\ell, r\}} \sum_{\xi \in \max(\text{Find}(S(1); i)_S)} \langle \text{Find}(X; \xi)_{S_0}, \text{Find}(X; \xi)_{T_0} \triangleright_{\text{cut}} \rangle \\
&= \sum_{i \in \{\ell, r\}} \sum_{\xi \in \max(\text{Find}(S(1); i)_S)} \langle \text{Find}(S(1); i\xi)_S, \text{Find}(T(1); i\xi)_T \triangleright_{\text{cut}} \rangle \\
&= \sum_{\xi \in \max(S(1))} \langle \text{Find}(S(1); \xi)_S, \text{Find}(T(1); \xi)_T \triangleright_{\text{cut}} \rangle
\end{aligned}$$

Putting all together the sum of cut links can be written as:

$$\begin{aligned}
& \sum_{\xi \in \max(S(1))} \langle \text{Find}(S(1); \xi)_S, \text{Find}(T(1); \xi)_T \triangleright_{\text{cut}} \rangle + \sum_{2 \leq i \leq \#S} \sum_{\xi \in \max(S(i))} \langle \text{Find}(S(i); \xi)_S, \text{Find}(T(i); \xi)_T \triangleright_{\text{cut}} \rangle \\
&= \sum_{1 \leq i \leq \#S} \sum_{\xi \in \max(S(i))} \langle \text{Find}(S(i); \xi)_S, \text{Find}(T(i); \xi)_T \triangleright_{\text{cut}} \rangle
\end{aligned}$$

As a consequence we conclude that  $S_0 :: T_0$  reduces to

$$S^{\mathfrak{X}} + T^{\mathfrak{X}} + \sum_{1 \leq i \leq \#S} \sum_{\xi \in \max(S(i))} \langle \text{Find}(S(i); \xi)_S, \text{Find}(T(i); \xi)_T \triangleright_{\text{cut}} \rangle$$

Finally, because  $S :: T \rightarrow S_0 :: T_0$  we conclude.  $\square$

## F.2.2 Proof of Proposition 75

**Remark 157.** Given a net  $S = (|S|, \mathbf{a}(S))$  the net induced by its daimon  $S^{\mathfrak{X}}$  comes with the natural order induced by  $\mathbf{a}(S)$  and the lexicographical order on address. More specifically given  $p$  and  $q$  two points of  $S^{\mathfrak{X}}$  both can be written as  $\text{Find}(a; \xi)$  where  $a = \text{root}(p)_S$  is a conclusion of  $S$  while  $\xi = \text{adr}(p)_S$ . Then  $p \leq q$  if and only if  $(\text{root}(p), \text{adr}(p)) \leq (\text{root}(q), \text{adr}(q))$  for the lexicographical order.

**Proposition 158.** Given two cut free nets  $S$  and  $T$  atomically testable respectively  $A_1, \dots, A_n$  and  $A_1^\perp, \dots, A_n^\perp$ . The assertions are equivalent;

- $S$  and  $T$  are orthogonal.
- $S^{\mathfrak{X}}$  and  $T^{\mathfrak{X}}$  are orthogonal.

*Proof.* Consider  $S$  and  $T$  two nets (atomically) testable by dual sequents so that  $\#S = \#T = n$ . The following reduction can be derived (Proposition 156):

$$S :: T \rightarrow_{\text{mult}} S^{\mathfrak{X}} + T^{\mathfrak{X}} + \sum_{1 \leq i \leq n} \sum_{\xi \in \max(S(i))} \langle \text{Find}(S(i); \xi), \text{Find}(T(i); \xi) \triangleright_{\text{cut}} \rangle$$

But observe that the outputs of  $S^{\mathfrak{X}}$  and  $T^{\mathfrak{X}}$  are ordered according to the lexicographical order on the pairs  $(\text{root}(p), \text{adr}(p))$ . As a consequence two conclusion have the same index if and only if their respective roots are  $S(i)$  and  $T(i)$  for some  $i$  and their address is the same. We derive the following

$$S^{\mathfrak{X}} + T^{\mathfrak{X}} + \sum_{1 \leq i \leq n} \sum_{\xi \in \max(S(i))} \langle \text{Find}(S(i); \xi), \text{Find}(T(i); \xi) \triangleright_{\text{cut}} \rangle = S^{\mathfrak{X}} + T^{\mathfrak{X}} + \sum_{1 \leq i \leq N} \langle S^{\mathfrak{X}}(i), T^{\mathfrak{X}}(i) \triangleright_{\text{cut}} \rangle$$

1  $\Rightarrow$  2. By assumption  $S :: T \rightarrow \mathfrak{X}_0$ , because the multiplicative cuts get eliminated and can always be performed first due to the commutation result (Proposition 103), this ensures that  $S :: T \rightarrow S^{\mathfrak{X}} :: T^{\mathfrak{X}}$  and since  $S :: T \rightarrow \mathfrak{X}_0$  it follows that  $S^{\mathfrak{X}} :: T^{\mathfrak{X}} \rightarrow \mathfrak{X}_0$ .

2  $\Rightarrow$  1. We have shown with the previous equality that  $S :: T \rightarrow S^{\mathfrak{X}} :: T^{\mathfrak{X}}$  thus we immediately conclude.  $\square$

**Proposition 159** (Orthogonality of nets and of natural partitions). Given  $S^{\mathfrak{X}}$  and  $T^{\mathfrak{X}}$  two nets made only of daimon links, the assertions are equivalent:

1.  $S^{\mathfrak{X}}$  and  $T^{\mathfrak{X}}$  are orthogonal.
2.  $\text{Nat}(P_{\mathfrak{X}}(S))$  and  $\text{Nat}(P_{\mathfrak{X}}(T))$  are orthogonal.

*Proof.* Assume that  $S^{\mathfrak{X}} :: T^{\mathfrak{X}}$  reduces to  $\mathfrak{X}_0$ . In other words the following net reduces to  $\mathfrak{X}_0$ :

$$S^{\mathfrak{X}} + T^{\mathfrak{X}} + \sum_{1 \leq i \leq n} \langle S^{\mathfrak{X}}(i), T^{\mathfrak{X}}(i) \triangleright_{\text{cut}} \rangle.$$

Equivalently this means that  $S^{\mathfrak{X}} :: T^{\mathfrak{X}}$  is an acyclic and connected graph. In particular if we contract each cut link  $\langle S^{\mathfrak{X}}(i), T^{\mathfrak{X}}(i) \triangleright_{\text{cut}} \rangle$  in a single point that we call  $i$  the acyclicity and connectedness is preserved. And in that case the resulting graph is  $G(\text{Nat}(\text{P}_{\mathfrak{X}}(S)), \text{Nat}(\text{P}_{\mathfrak{X}}(T)))$  showing that these two partitions are orthogonal. The same argument in the other direction hold.  $\square$

**Proposition 73.** *Let  $A$  be a formula, given two cut free nets  $S \stackrel{\text{at}}{\simeq} A$  and  $T \stackrel{\text{at}}{\simeq} A^\perp$  the assertions are equivalent:*

1. *The nets  $S$  and  $T$  are orthogonal.*
2. *The nets  $S^{\mathfrak{X}}$  and  $T^{\mathfrak{X}}$  are orthogonal.*
3. *The partition  $\text{Nat}_S(\text{P}_{\mathfrak{X}}(S))$  and  $\text{Nat}_T(\text{P}_{\mathfrak{X}}(T))$  are orthogonal.*

*Proof.* This follows from the two previous proposition, [Proposition 159](#) and [Proposition 158](#).

**1  $\Leftrightarrow$  2.** As for  $1 \Rightarrow 2$ , we need to establish a property of the rewriting of nets: if  $N \rightarrow_{\text{mult}} N'$  and  $N \rightarrow^* \mathfrak{X}_0$  then  $N' \rightarrow^* \mathfrak{X}_0$ . As for  $2 \Rightarrow 1$ , it follows from  $S :: T \rightarrow^* S^{\mathfrak{X}} :: T^{\mathfrak{X}}$ .

**2  $\Leftrightarrow$  3.** One first proves that  $S^{\mathfrak{X}} :: T^{\mathfrak{X}}$  is acyclic and connected (ACC) if and only if  $G(\text{Nat}(\text{P}_{\mathfrak{X}}(S)), \text{Nat}(\text{P}_{\mathfrak{X}}(T)))$  is.

As for  $3 \Rightarrow 2$ , since no cycle occurs in  $S^{\mathfrak{X}} :: T^{\mathfrak{X}}$  all (glueing) cuts are acyclic and thus all cuts can be eliminated, furthermore the elimination of a glueing cut preserves ACC. The net  $S^{\mathfrak{X}} :: T^{\mathfrak{X}}$  contains no connective link, thus its normal form does not contain clash cuts. Hence because the only net without conclusion that is normal, ACC and contains no clash cuts is  $\mathfrak{X}_0$  we conclude  $S^{\mathfrak{X}} :: T^{\mathfrak{X}} \rightarrow \mathfrak{X}_0$ . As for  $2 \Rightarrow 3$ , note that anti-steps of cut elimination for glueing cuts also preserve the ACC property hence if  $S^{\mathfrak{X}} :: T^{\mathfrak{X}} \rightarrow \mathfrak{X}_0$  it must be that  $S^{\mathfrak{X}} :: T^{\mathfrak{X}}$  is ACC.  $\square$

### F.3 Description of tests

The object of this section is to show that the *tests* of a formula  $A$  ([Definition 74](#)) may be defined relatively to a single cut free net  $S \stackrel{\text{at}}{\simeq} A$ : this is because the partitions  $\uparrow^i \sigma S$  for some switching of  $S$ , only depend on the  $\otimes$ - and  $\wp$ -link of  $S$ , which is the same for any cut free net  $U \stackrel{\text{at}}{\simeq} A$ .

We now fix a special propositional variable that we denote  $\mathbf{V}$ . A formula which contains only the  $\mathbf{V}$  propositional variable is called a *formula pattern*.

**Definition 160.** An hypergraph  $M$  which contains only  $\otimes$ - or  $\wp$ -links represents a formula pattern  $F$ , denoted  $M \equiv_{\mathcal{F}} F$  whenever:

- If  $F = \mathbf{V}$  is the propositional variable  $\mathbf{V}$  then  $M$  consists of  $(\{p\}, \emptyset, \emptyset, \emptyset)$  for some position  $p$ .
- If  $F = A \otimes B$  then  $M$  is equal to  $M_A + M_B + \langle M_A(1), M_B(1) \triangleright_{\otimes} p \rangle$  where  $p$  is a fresh position and  $M_A \equiv_{\mathcal{F}} A$  while  $M_B \equiv_{\mathcal{F}} B$ .
- If  $F = A \wp B$  then  $M$  is equal to  $M_A + M_B + \langle M_A(1), M_B(1) \triangleright_{\wp} p \rangle$  where  $p$  is a fresh position and  $M_A \equiv_{\mathcal{F}} A$  while  $M_B \equiv_{\mathcal{F}} B$ .

**Proposition 161.** *Given an hypergraph  $M$  and a formula pattern  $A$ . If  $M \equiv_{\mathcal{F}} A$  then  $M$  is target-surjective and source surjective, and  $M$  has a single conclusion.*

*Proof.* By doing a simple induction on  $A$ .  $\square$

*Notation 162.* Given a formula pattern  $A$  and a position  $p$  we denote by  $\underline{A}(p)$  a hypergraph which represents the formula  $A$  and has for conclusion  $p$ . Furthermore we denote by  $A \langle p_1, \dots, p_n \triangleright p \rangle$  a hypergraph  $M$  of conclusion  $p$  representing  $A$  and such that  $p_1, \dots, p_n$  are the input positions<sup>11</sup> of the hypergraph ordered by lexicographical order on  $\{\ell, \nu\}^*$  for the addresses of the positions i.e. their addresses  $\text{adr}(p_i)$  in  $M$ .

**Lemma 163** (Decomposition Lemma). *Given a cut-free net  $S$  with  $n$  conclusion there exists a unique sequence of  $n$  formula patterns  $A_1, \dots, A_n$  such that:*

$$S = S^{\mathfrak{X}} + \sum_{1 \leq i \leq n} \underline{A}_i(S(i)).$$

*Proof.* One proceeds by induction on the number of connective links of  $S$ .  $\square$

We prove that test of  $A$  can be defined relatively to any single net  $S \stackrel{\text{at}}{\simeq} A$ : the next proposition ensures that the test of a formula may be described using any single net  $S \stackrel{\text{at}}{\simeq} A$ : this is because the switchings only depend on the  $\wp$ - and  $\otimes$ -links of  $S$  which are the same for all the nets  $U \stackrel{\text{at}}{\simeq} A$ .

<sup>11</sup>that is positions which are the target of no link in  $M$ .

**Proposition 164.** *Given a formula  $A$  and a net  $S \stackrel{a}{\models} A$ . For any net  $T$ ,  $T$  is a test of  $A$  iff  $T \stackrel{a}{\models} A^\perp$  and for some switching  $\sigma S$  of  $S$  we have  $\mathbf{Nat}_T(\mathbf{P}_{\mathfrak{X}}(T)) = \mathbf{Nat}_S(\uparrow^i \sigma S)$ .*

*Proof.*  $2 \Rightarrow 1$ . It is the obvious direction.

$1 \Rightarrow 2$ .  $S$  is a cut-free net atomically testable by  $A$ , consider any  $U$  which is also cut-free and atomically testable by  $A$ : we show that for any switching  $\sigma S$  of  $S$  there exists a switching  $\tau U$  of  $U$  such that the partitions  $\mathbf{Nat}_S(\uparrow^i \sigma S)$  and  $\mathbf{Nat}_U(\uparrow^i \tau U)$  are the same.

$S$  and  $U$  can be both written as  $S^{\mathfrak{X}} + A\langle \bar{u} \triangleright p \rangle$  and  $U^{\mathfrak{X}} + A\langle \bar{v} \triangleright q \rangle$ . The nets  $S^{\mathfrak{X}}$  and  $U^{\mathfrak{X}}$  are normal for the switching rewriting. Observe that the sum of two normal nets for the switching rewriting is still normal for the switching rewriting; therefore one can show that  $S^{\mathfrak{X}} + A\langle \bar{u} \triangleright p \rangle \rightarrow_{\mathfrak{X}} S'$  implies  $A\langle \bar{u} \triangleright p \rangle \rightarrow_{\mathfrak{X}} R$  (by showing the contraposition). Indeed this will also be true for the net  $U$ .

Therefore we derive the following:

- A switching of  $S$  is a net of the form  $S^{\mathfrak{X}} + R_S$  where  $R_S$  is a normal form of  $A\langle \bar{v} \triangleright q \rangle$ .
- Similarly, a switching of  $U$  is a net of the form  $U^{\mathfrak{X}} + R_U$  where  $R_U$  is a normal form of  $A\langle \bar{v} \triangleright q \rangle$ .
- Finally observe that two cut-free nets  $U^{\mathfrak{X}} + R = V^{\mathfrak{X}} + R$  where  $R$  is made of connective links only then  $\mathbf{Nat}_U(\uparrow^i U) = \mathbf{Nat}_V(\uparrow^i V)$ .

Consider then any two switchings of  $S$  and  $U$  with the same reduct  $R$  and denote them respectively  $\sigma S$  and  $\tau U$ : it follows that  $\mathbf{Nat}_S(\uparrow^i \sigma S)$  equals  $\mathbf{Nat}_T(\uparrow^i \sigma T)$ .

To conclude, fix the net  $S \stackrel{a}{\models} A$ . A test  $T$  of  $A$  is such that  $T \stackrel{a}{\models} A^\perp$  and for some  $U \stackrel{a}{\models} A$  and some of its switching  $\tau U$   $\mathbf{Nat}_T(\mathbf{P}_{\mathfrak{X}}(T)) = \mathbf{Nat}_U(\uparrow^i \tau U)$ . By the previous considerations there exists a switching  $\sigma S$  of  $S$  such that  $\mathbf{Nat}_S(\uparrow^i \sigma S)$  equals  $\mathbf{Nat}_U(\uparrow^i \tau U)$  therefore  $\mathbf{Nat}_T(\mathbf{P}_{\mathfrak{X}}(T)) = \mathbf{Nat}_S(\uparrow^i \sigma S)$ : this allows us to conclude.  $\square$

## F.4 Orthogonality with tests – proof of Proposition 75 and Theorem 76

Using the previous proposition and the well-known result of Danos Regnier (Theorem 40) we obtain the following proposition of section 5.

**Proposition 75.** *For  $S$  cut-free,  $S \stackrel{a}{\models} A$ , we have:  $S \vdash_{\text{MLL}\mathfrak{X}} A \Leftrightarrow S \perp \text{tests}(A)$ .*

*Proof.* Consider a test  $T$  of  $\text{tests}(A)$  then  $T \stackrel{a}{\models} A^\perp$  and  $\mathbf{Nat}_T(\mathbf{P}_{\mathfrak{X}}(T)) = \mathbf{Nat}_S(\uparrow^i \sigma S)$  for some switching  $\sigma S$  of  $S$  (Proposition 164). By assumption  $S$  and  $T$  are orthogonal thus equivalently  $\mathbf{Nat}_S(\mathbf{P}_{\mathfrak{X}}(S))$  and  $\mathbf{Nat}_T(\mathbf{P}_{\mathfrak{X}}(T))$  are orthogonal (Proposition 73). Then  $\mathbf{Nat}_S(\mathbf{P}_{\mathfrak{X}}(S))$  is orthogonal to each  $\mathbf{Nat}_S(\uparrow^i \sigma S)$ . we then conclude that  $\mathbf{P}_{\mathfrak{X}}(S)$  and  $\uparrow^i \sigma S$  are orthogonal. Since this holds for any switching  $\sigma S$  thus calling theorem 40 we conclude that  $S$  is a proof net. The other direction of the equivalence is obtained using the same argument in the opposite direction.  $\square$

We can indeed generalise this result the case where  $S$  has multiple conclusions, as usual we do this by transform the net with multiple conclusions in a net with one conclusion by adding a bunch of  $\mathfrak{X}$ -links.

**Definition 165** (General connectives). A generalised  $\mathfrak{X}$ -link on the positions  $p_1, \dots, p_n$  is a module denoted  $\langle p_0, \dots, p_n \triangleright_{\mathfrak{X}^n} p \rangle$  and defined by the following induction;

- $\langle p_0, p_1 \triangleright_{\mathfrak{X}^1} p \rangle = \langle p_0, p_1 \triangleright_{\mathfrak{X}} p^1 \rangle$ .
- For any  $n > 0$  we defined  $\langle p_1, \dots, p_{n+2} \triangleright_{\mathfrak{X}^{n+1}} p \rangle = \langle p^{n-1}, p_n \triangleright_{\mathfrak{X}^n} p \rangle + \langle p^n, p_{n+1} \triangleright_{\mathfrak{X}} p^{n+1} \rangle$ .

Similarly we defined the generalised tensor links  $\langle p_0, \dots, p_n \triangleright_{\otimes^n} p \rangle$ .

**Proposition 166.** *Let  $S$  be a net with conclusions  $p_0, \dots, p_n, q_1, \dots, q_k$ . Let  $S_1, \dots, S_n, T_1, \dots, T_k$  be  $n$  nets with one conclusion the assertions are equivalent;*

1.  $S \perp S_1 \parallel \dots \parallel S_n \parallel T_1 \parallel \dots \parallel T_k$ .
2.  $S + \langle p_0, \dots, p_n \triangleright_{\mathfrak{X}^n} p \rangle \perp S_1 + \dots + S_n + \langle S_1(1), \dots, S_n(1) \triangleright_{\otimes^n} q \rangle \parallel T_1 \parallel \dots \parallel T_k$

*Proof.* By a simple induction of the size of the generalised  $\mathfrak{X}$  connective.  $\square$

To derive the generalised theorem one must observe that the tests of  $\mathfrak{X}$ -formulas are tensors of the tests of the subformulas.

**Proposition 167.** *A test  $T$  of  $A \mathfrak{X} B$  is of the form  $T_A \parallel T_B + \langle T_A(1), T_B(1) \triangleright_{\otimes} p \rangle$  where  $p$  a fresh position,  $T_A$  is a test of  $A$  and  $T_B$  is a test of  $B$ .*

*Proof.* This comes down to analysing the partitions generated by a the switchings of a net  $S \stackrel{a}{\models} A \mathfrak{X} B$  indeed this these are partitions  $P_A \cup P_B$  where  $P_A$  is a partition of the representation of  $A$  and  $P_B$  is a partition of the representation of  $B$ .  $\square$

This easily leads to a proof of the Danos Regnier theorem as stated in section 5.

**Theorem 76** (Danos–Regnier Tests). *Given a cut-free net  $S \Vdash^{\text{at}} A_1, \dots, A_n$ ;  $S \vdash_{\text{MLL}^{\times}} A_1, \dots, A_n$  if and only if  $S$  is orthogonal to  $\text{tests}(A_1) \parallel \dots \parallel \text{tests}(A_n)$ .*

## F.5 Correctness of Tests and counter proofs – proof of Theorem 77

**Theorem 77.** *Any test  $T$  of a formula  $A$  is correctly typeable by  $A^\perp$ ,  $T \vdash_{\text{MLL}^{\times}} A^\perp$ .*

We have established the previous theorem using the counter proof criterion of P.L. Curien found in [4]. Let us clearly state the theorem of counter proofs.

**Theorem 168** (Counter Proof Criterion). *Given a cut-free net  $S \Vdash^{\text{at}} A$  the assertions are equivalent:*

- $S$  is orthogonal to each net  $T \Vdash^{\text{at}} A^\perp$  representing a proof of  $A^\perp$ .
- $S \vdash_{\text{MLL}^{\times}} A$ .

**Remark 169.** The counter proof criterion is formulated in [4] where non-homogeneous cut elimination does not exist. Because this is not the case in our work we add the hypothesis that  $S \Vdash^{\text{at}} A$  and the proof-opponents are of the form  $T \Vdash^{\text{at}} A^\perp$  so that only homogeneous cut will appear and be eliminated. Also observe that a proof of  $A^\perp$  such that  $T \Vdash^{\text{at}} A^\perp$  is an atomic proof of  $\text{MLL}^{\times}$  that is, a proof such that its daimon rules introduce sequents which contains only propositional variables.

One can establish that the fact that tests of  $A$  are proofs of  $A^\perp$  is equivalent to an implication of the counter proof criterion.

**Proposition 170.** *Given a formula  $A$  the assertions are equivalent:*

1. Each test of  $A$  is the representation of a proof of  $A^\perp$ .
2. For any cut-free net  $S \Vdash^{\text{at}} A$  if  $S$  is orthogonal to each (cut-free)  $T \Vdash^{\text{at}} A^\perp$  representing a proof of  $A^\perp$  then  $S \vdash_{\text{MLL}^{\times}} A$ .

*Proof.* The fact that 2  $\Rightarrow$  1 is the proof of Theorem 77 (it uses the counter proof criterion [4], Theorem 168). To show 1  $\Rightarrow$  2, consider a net  $S \Vdash^{\text{at}} A$  and assume that each test  $t$  of  $A$  represents a proof of  $A^\perp$  and is such that  $t \Vdash^{\text{at}} A^\perp$ . If the net  $S$  is orthogonal to each net  $T \vdash_{\text{MLL}^{\times}} A^\perp$  with  $T \Vdash^{\text{at}} A^\perp$  then in particular it is orthogonal to the set of tests  $\text{tests}(A)$ : it follows that  $S \vdash_{\text{MLL}^{\times}} A$  (Theorem 76).  $\square$

In fact the result of adequacy we have obtained for  $\text{MLL}^{\times}$  (Theorem 64) allows to generalise the counter proof criterion of P.L. Curien [4] in presence of non homogeneous cuts. More precisely, having an adequate basis implies a “soundness” result of a counter proof criterion, i.e. any proof of  $A$  is orthogonal to any proof of  $A^\perp$ .

**Proposition 171.** *The first assertion implies the second:*

1. There exists an adequate basis.
2. Any  $S \vdash_{\text{MLL}^{\times}} A$  is orthogonal to any  $T \vdash_{\text{MLL}^{\times}} A^\perp$ .

*Proof.* 1  $\Rightarrow$  2. Say  $\mathcal{B}$  is an adequate basis then for any formula  $A$  we have  $\llbracket A : \text{MLL}^{\times} \rrbracket \subseteq \llbracket A \rrbracket_{\mathcal{B}}$ . Now since  $\llbracket A \rrbracket_{\mathcal{B}}$  equals  $\llbracket A^\perp \rrbracket_{\mathcal{B}}$  we derive  $\llbracket A \rrbracket_{\mathcal{B}} \subseteq \llbracket A^\perp \rrbracket_{\mathcal{B}}$  (Definition 58). This means that any realiser of  $A$  is a orthogonal to any realiser of  $A^\perp$ , since by adequacy a net  $S \vdash_{\text{MLL}^{\times}} A$  realises  $A$  and a net  $T \vdash_{\text{MLL}^{\times}} A^\perp$  realises  $A^\perp$  we conclude.  $\square$

Then we can obtain a general form of counter proof criterion namely when we don’t have the restriction that the proofs are atomic i.e.  $S \Vdash^{\text{at}} A$  and  $T \Vdash^{\text{at}} A^\perp$ .

**Proposition 172** (Generalised Counter Proof criterion). *Given a formula  $A$  and  $S$  a cut-free net the assertions are equivalent:*

1.  $S$  is orthogonal to each net  $T$  representing a proof of  $A^\perp$ .
2.  $S \vdash_{\text{MLL}^{\times}} A$ .

*Proof.* 1  $\Rightarrow$  2. Assume that  $S \Vdash^{\text{at}} A_0$  with  $A_0$  such that there exists  $\theta$  with  $\theta A_0 = A$  and that  $S$  is orthogonal to each proof nets  $T \vdash_{\text{MLL}^{\times}} A^\perp$ . In particular any net  $T \vdash_{\text{MLL}^{\times}} A_0^\perp$  with  $T \Vdash^{\text{at}} A_0^\perp$  is such that  $T \vdash_{\text{MLL}^{\times}} A^\perp$  (Proposition 33).

As a consequence:  $S$  is orthogonal to each  $T \vdash_{\text{MLL}^{\times}} A_0^\perp$  with  $T \Vdash^{\text{at}} A_0^\perp$ . by the counter proof criterion [4] we derive  $S \vdash_{\text{MLL}^{\times}} A_0$  and thus  $S \vdash_{\text{MLL}^{\times}} A$  (again using Proposition 33).

2  $\Rightarrow$  1. We aim at showing that any proof  $S \vdash_{\text{MLL}^{\times}} A$  is orthogonal to any proof  $T \vdash_{\text{MLL}^{\times}} A^\perp$  which may have a different syntax tree (and here lies the novelty with respect to P-L. Curien’s theorem [4]). Showing this implication by analyzing the rewriting rules of cut-elimination can be delicate, however in the language of realisability this becomes a trivial consequence of the adequacy theorem (Proposition 171). Hence because adequacy hold (Theorem 64), we conclude.  $\square$

## F.6 On Substitutions and Testability – Proof of Proposition 82

Recall that a *substitution* is a map  $\theta$  which maps propositional variables to formulas. Naturally substitutions can be lifted by induction to formulas and to sequents;  $\theta(A \square B) \triangleq \theta A \square \theta B$  and  $\theta(\Gamma, A) = \theta(\Gamma), \theta A$ . A sequent  $\Gamma$  is an *instance* of a sequent  $\Delta$  whenever there exists a substitution  $\theta$  such that  $\theta\Delta = \Gamma$ . In that case we denote  $\Delta \leq \Gamma$ .

**Remark 173.** Given two representation of sequents  $\Gamma = A_1, \dots, A_n$  and  $\Delta = B_1, \dots, B_n$  are such that  $\Delta \leq \Gamma$  this implies that for each index  $1 \leq i \leq n$  we have  $B_i \leq A_i$  specifically  $A_i[X_1 \mapsto F_1, \dots, X_k \mapsto F_k] = B_i$ .

**Remark 174.** Whenever a cut-free net  $S$  is such that  $S \simeq \Gamma$  then there exists a substitution  $\theta$  and a sequent  $\Delta$  such that  $S \stackrel{\omega}{\simeq} \Delta$  and  $\theta\Delta$ .

Let us provide a proof of the previous remark.

**Proposition 175.** *Given a cut-free net  $S$  and a sequent  $\Gamma$ . If  $S \simeq \Gamma$  then there exists a sequent  $\Delta$  and a substitution  $\theta$  such that  $\theta\Delta = \Gamma$  and  $S \stackrel{\omega}{\simeq} \Delta$ .*

*Proof.* If  $S \stackrel{\omega}{\simeq} \Gamma$  we can already conclude. Otherwise  $S \simeq \Gamma$  and let us denote  $p_1, \dots, p_n$  the initial positions of  $S$ , and  $\tau$  the formula labelling witnessing  $S \simeq \Gamma$ . Because formula labellings are total functions each  $p_i$  as a formula  $\tau(p_i) = F_i$  associated with it consider then  $X_1, \dots, X_n$  a family of distinct propositional variables and the substitution  $\theta[X_1 \mapsto F_1; \dots; X_n \mapsto F_n]$  then consider the (unique) atomic formula labelling such that  $\tau'(p_i) = X_i$  then  $S \stackrel{\omega}{\simeq} \tau'(S(1)), \dots, \tau'(S(n))$ . Indeed then applying  $\theta$  to the sequent  $\tau'(S(1)), \dots, \tau'(S(n))$  will result in  $\tau(S(1)), \dots, \tau(S(n))$  which is  $\Gamma$ .  $\square$

**Proposition 82.** *Given  $\mathcal{B}$  an approximable basis<sup>12</sup> and a sequent  $\Gamma$  for any cut-free net  $S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$  the assertions are equivalent:*

1.  $S \simeq \Gamma$  i.e.  $S \stackrel{\omega}{\simeq} \Delta$  for some sequent  $\Delta \leq \Gamma$ .
2.  $S \vdash_{\text{MLL}^*} \Gamma$ .

*Proof.* Using remark 81 and the fact that the tests of a formula  $B$  in  $\Delta$  are proofs of  $B^\perp$  and by proposition 33 are proofs of  $A^\perp$ .  $\square$

## G Complements to section 6

### G.1 Decomposition

**Proposition 176** (Decomposition). *Let  $\mathcal{B}$  be an interpretation basis,  $\mathcal{H}$  be an hypersequent,  $A, B$  two formulas and  $S$  be a multiplicative net.*

- If  $S = S_0 + l$  has a terminal  $\wp$  link  $l$  above its last conclusion and  $S \in \llbracket \mathcal{H}, A \wp B \rrbracket_{\mathcal{B}}$ ; then  $S_0$  belongs to  $\llbracket \mathcal{H}, A, B \rrbracket_{\mathcal{B}}$
- If  $S = S_0 + l$  has a terminal  $\wp$  link  $l$  above its last conclusion and  $S \in \llbracket \mathcal{H} \parallel A \wp B \rrbracket_{\mathcal{B}}$ ; then  $S_0$  belongs to  $\llbracket \mathcal{H} \parallel (A, B) \rrbracket_{\mathcal{B}}$
- If  $S = S_0 + l$  has a terminal  $\otimes$  link  $l$  above its last conclusion and  $S \in \llbracket \mathcal{H}, A \otimes B \rrbracket_{\mathcal{B}}$ ; then  $S_0$  belongs to  $\llbracket \mathcal{H}, (A \parallel B) \rrbracket_{\mathcal{B}}$
- If  $S = S_0 + l$  has a terminal  $\otimes$  link  $l$  above its last conclusion and  $S \in \llbracket \mathcal{H} \parallel A \otimes B \rrbracket_{\mathcal{B}}$ ; then  $S_0$  belongs to  $\llbracket \mathcal{H} \parallel A \parallel B \rrbracket_{\mathcal{B}}$

*Proof.*

- Consider  $S$  a net with a terminal  $\wp$ -link  $l$  decomposing  $S = S_0 + l$  such that  $l$  outputs the only conclusion of  $S$ . Say  $S$  belongs to  $\llbracket A \wp B \rrbracket_{\mathcal{B}}$  equivalently  $S$  is orthogonal to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \otimes \llbracket B \rrbracket_{\mathcal{B}}^\perp$ , since a multiplicative cut can always be performed first this implies that  $S_0$  is orthogonal to  $\llbracket A \rrbracket_{\mathcal{B}}^\perp \parallel \llbracket B \rrbracket_{\mathcal{B}}^\perp$  thus  $S_0$  belongs to  $\llbracket A \rrbracket_{\mathcal{B}} \wp \llbracket B \rrbracket_{\mathcal{B}}$ . By allowing types to contain cuts, this arguments easily adapts to  $\llbracket \mathcal{H}, A, B \rrbracket_{\mathcal{B}}$ .
- The reasonment for a net in  $\llbracket A \otimes B \rrbracket_{\mathcal{B}}$  is similar and also easily adapts to the case  $\llbracket \mathcal{H}, A \otimes B \rrbracket_{\mathcal{B}}$ .
- Consider on the other hand a net  $S$  in  $\llbracket \mathcal{H} \parallel A \otimes B \rrbracket_{\mathcal{B}}$ , such that  $S = S_0 + l$  where  $l$  is a tensor link and outputs the last conclusion of  $S$ . Equivalently  $S$  is a net orthogonal to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^\perp \wp \llbracket A \otimes B \rrbracket_{\mathcal{B}}^\perp$  i.e.  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^\perp \wp \llbracket A \rrbracket_{\mathcal{B}}^\perp \wp \llbracket B \rrbracket_{\mathcal{B}}^\perp$ . Since the multiplicative cut commute to the left (Proposition 103) we derive that  $S_0$  is orthogonal to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^\perp \wp \llbracket A \rrbracket_{\mathcal{B}}^\perp \wp \llbracket B \rrbracket_{\mathcal{B}}^\perp$ . Equivalently (by duality)  $S_0$  belongs to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}} \parallel \llbracket A \rrbracket_{\mathcal{B}} \parallel \llbracket B \rrbracket_{\mathcal{B}}$ .
- Similarly we treat the case  $\llbracket \mathcal{H} \parallel A \wp B \rrbracket_{\mathcal{B}}$ . Consider on the other hand a net  $S$  in  $\llbracket \mathcal{H} \parallel A \wp B \rrbracket_{\mathcal{B}}$ , such that  $S = S_0 + l$  where  $l$  is a parr link and outputs the last conclusion of  $S$ . Equivalently  $S$  is a net orthogonal to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^\perp \wp \llbracket A \wp B \rrbracket_{\mathcal{B}}^\perp$  i.e.  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^\perp \wp \llbracket A \rrbracket_{\mathcal{B}}^\perp \otimes \llbracket B \rrbracket_{\mathcal{B}}^\perp$ . Because multiplicative cuts can always be performed first in a reduction sequence (Proposition 103) it follows that  $S_0$  is orthogonal to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^\perp \wp \llbracket A \rrbracket_{\mathcal{B}}^\perp \parallel \llbracket B \rrbracket_{\mathcal{B}}^\perp$ . Equivalently  $S_0$  belongs to  $\llbracket \mathcal{H}, A \parallel B \rrbracket_{\mathcal{B}}$ .  $\square$

<sup>12</sup>The Proposition 82 actually holds for any “adequate” basis  $\mathcal{B}$ .

## G.2 The merge construction belongs to the composition

From [Proposition 127](#) we can derive easily that an explicitly defined type (called merge and denoted  $\mathbf{A} \bowtie \mathbf{B}$ ) belongs to the composition of two types  $\mathbf{A} \succ \mathbf{B}$ . This will be used to show that the functional composition ( $\succ$ ) and the parallel composition ( $\parallel$ ) of full types remains full (that is the [Proposition 181](#) of [Appendix G](#)) therefore by duality this will imply that the parallel and functional composition of daimon types is a daimon type.

**Definition 177.** Given  $A$  and  $B$  two sets of nets we denote by  $A \bowtie B$  the following set:

$$\{a \bowtie^{(d,d')} b \mid a \in A, b \in B, \quad d \in E_a, \ell_a(d) = \bowtie, \quad d \in E_b, \ell_b(d) = \bowtie\}.$$

**Proposition 178** (Merges belongs to the compositional construction). *Given  $\mathbf{A}$  and  $\mathbf{B}$  two types;*

$$(\mathbf{A} \bowtie \mathbf{B})^{\perp\perp} \subseteq \mathbf{A} \succ \mathbf{B}.$$

*Proof.* Consider  $S$  a net in  $A \bowtie B$ .  $S$  may therefore be written as  $a \bowtie^{(d_a, d_b)} b$  for two elements  $a \in \mathbf{A}$  and  $b \in \mathbf{B}$  with  $d_a$  a daimon of  $a$  and  $d_b$  a daimon of  $b$ . We can thus decompose  $b$  as  $d_b + b_0$  and  $a \bowtie^{(d_a, d_b)} b$  is equal to  $(a \bowtie^{(d_a, d_b)} d_b) + b_0$  ([Proposition 123](#)).

Now consider an element  $\bar{a}$  of  $\mathbf{A}$ , then  $a$  and  $\bar{a}$  are orthogonal, therefore  $(a \bowtie^{(d_a, d_b)} d_b) :: \bar{a}$  reduces to  $d_b$  ([Proposition 127](#)). Furthermore  $(a \bowtie^{(d_a, d_b)} d_b) + b_0 :: \bar{a}$  is equal to  $((a \bowtie^{(d_a, d_b)} d_b) :: \bar{a}) + b_0$  because  $\bar{a}$  only interacts with the conclusions of  $a$  thus that net reduces to  $d_b + b_0$  which is exactly  $b$  and so belongs to  $\mathbf{B}$ . Thus it follows that the net  $((a \bowtie^{(d_a, d_b)} d_b) + b_0) :: \bar{a}$  that is  $a \bowtie^{(d_a, d_b)} b_0 :: \bar{a}$  belongs to  $\mathbf{B}$  for any  $\bar{a} \in \mathbf{A}$  thus that it belongs to  $\mathbf{A} \succ \mathbf{B}$ .

We have shown  $\mathbf{A} \bowtie \mathbf{B} \subseteq \mathbf{A} \succ \mathbf{B}$  and thus since inclusion is stable under bi orthogonal,  $(\mathbf{A} \bowtie \mathbf{B})^{\perp\perp} \subseteq \mathbf{A} \succ \mathbf{B}$   $\square$

## G.3 Full Types and daimon types

We introduce some terminology regarding types in order to identify types which contain only nets whose conclusions are outputs of daimon links. Such basis are for instance daimon basis, and, one can easily see that  $\mathbf{1}$  is a daimon basis.

**Definition 179.** A type  $\mathbf{A}$  is *full* when for each  $1 \leq i \leq \#\mathbf{A}$  there exists a net  $S$  (resp.  $T$ ) such that its conclusion  $S(i)$  (resp.  $T(i)$ ) is the output of a  $\bowtie$ -link (resp.  $\otimes$ -link).

**Remark 180.** Observe that in a daimon type  $\mathbf{A}$  each conclusion of a net  $S \in \mathbf{A}$  is the output of a daimon link ([Remark 185](#)): as a consequence there exists an atomic (that is, containing only propositional variable) sequent  $\Gamma$  made of  $\#\mathbf{A}$  formulas such that any cut free net  $S$  of  $\mathbf{A}$  is testable by  $\Gamma$  i.e.  $S \stackrel{\text{at}}{\vdash} \Gamma$

**Proposition 181.** *Given two full types  $\mathbf{A}$  and  $\mathbf{B}$ ;*

- *Their functional composition  $\mathbf{A} \succ \mathbf{B}$  is still full.*
- *Their parallel composition  $\mathbf{A} \parallel \mathbf{B}$  is still full.*

*Given two daimon types  $\mathbf{A}$  and  $\mathbf{B}$ ;*

- *Their functional composition  $\mathbf{A} \succ \mathbf{B}$  is still a daimon type.*
- *Their parallel composition  $\mathbf{A} \parallel \mathbf{B}$  is still a daimon type.*

*Proof.* Indeed the parallel  $\mathbf{A} \parallel \mathbf{B}$  remains full since it contains in particular the nets of the form  $a \parallel b$  with  $a \in \mathbf{A}$  and  $b \in \mathbf{B}$ . Similarly this remains true for  $\mathbf{A} \succ \mathbf{B}$  since it contains the merge  $\mathbf{A} \bowtie \mathbf{B}$  ([Proposition 178](#)).

For the daimon types we conclude using the duality results ([Proposition 51](#)).  $\square$

**Proposition 182.** *Let  $\mathbf{A}$  and  $\mathbf{A}^\perp$  be two orthogonal and non-empty types, the assertions are equivalent:*

- *$\mathbf{A}$  is a daimon type.*
- *Each conclusion of a net  $S$  of  $\mathbf{A}$  is the output of a daimon link.*

*Proof.*  $\mathbf{1} \Rightarrow \mathbf{2}$ . This is already discussed in [Remark 185](#). A net  $S$  in  $\mathbf{A}$  is orthogonal to any nets in  $\mathbf{A}^\perp$ , and because  $\mathbf{A}^\perp$  is full the conclusions of  $S$  cannot be the outputs of connectives otherwise a clashing cut will occur in some interaction  $S :: T$  for some  $T$  in  $\mathbf{A}^\perp$ .

$\mathbf{2} \Rightarrow \mathbf{1}$ . This requires a much more detailed analysis of the rewriting rules of cut elimination. We will not give the detail here because this implication is not used in this work and the proof requires the addition of a lot of content in order to be obtained.  $\square$

## G.4 Daimon Basis are Compact – proof of Proposition 83

The object of this section is to provide a proof of Theorem 187. Let us define *compact basis* as basis which satisfy the first item of Proposition 82; such a basis if also enjoys adequacy, will obviously be complete for MLL<sup>⊗</sup> (using Proposition 82).

**Definition 183.** A basis  $\mathcal{B}$  is *compact* if for any  $\Gamma$  and any net  $S$ ;  $S \Vdash_{\mathcal{B}} \Gamma \Rightarrow S \Vdash^{\text{all}} \Gamma$ .

To obtain MLL<sup>⊗</sup>-completeness (for cut-free nets) Proposition 82 ensure that it is enough to find a basis that is approximable and compact: such kind of basis are *daimon basis* (Theorem 187), for instance the basis mapping each propositional variable to  $\{\boxtimes_1\}^{\perp\perp}$ . Technically this is where it is useful to interpret hypersequents; a basis is compact is equivalently a basis such that for any hypersequent  $\mathcal{H}$  which is  $\wp$ - and  $\otimes$ -free the interpretation  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}$  contain only nets of whose conclusions are targets of daimon links.

**Definition 184** (Daimon type). A type  $\mathbf{A}$  is a *daimon type* whenever for each  $1 \leq i \leq \#\mathbf{A}$  its dual  $\mathbf{A}^{\perp}$  contains a net  $S$  with a terminal  $\wp$ -link which outputs  $S(i)$  and a net  $T$  with a terminal  $\otimes$ -link which outputs  $T(i)$ . A basis  $\mathcal{B}$  is a *daimon basis* when for each propositional variable  $X$  its interpretation  $\llbracket X \rrbracket_{\mathcal{B}}$  is a daimon type.

**Remark 185.** The conclusions of a net which belong to a daimon type  $\mathbf{A}$  must be the outputs of daimon links (Remark 46).

A daimon basis is indeed a compact basis; the elements of a (non-empty) daimon type can only have conclusions which are outputs of a daimon link otherwise clashing cuts occur (Remark 46), then an induction on the formulas show compactedness of the basis. First we must show a more general result which holds on hypersequents.

There is a simple inductive process to associate a sequent  $\downarrow \mathcal{H}$  to an hypersequent  $\mathcal{H}$ ;  $\downarrow A \triangleq A$  while  $\downarrow \mathcal{H}_1, \mathcal{H}_2 \triangleq \downarrow \mathcal{H}_1, \downarrow \mathcal{H}_2$  and  $\downarrow \mathcal{H}_1 \parallel \mathcal{H}_2 \triangleq \downarrow \mathcal{H}_1, \downarrow \mathcal{H}_2$ .

**Lemma 186** (Daimon Basis Property). *Let  $\mathcal{B}$  be a daimon basis. For any cut-free net  $S$ ;  $S \in \llbracket \mathcal{H} \rrbracket_{\mathcal{B}} \Rightarrow S \Vdash^{\text{all}} \downarrow \mathcal{H}$ .*

*Proof.* of lemma 186. By induction on the hypersequent using the measure  $(c, n)$  where  $c$  is the number of connectives in the hypersequent and  $n$  is the size of the hypersequent.

- If the hypersequent is made of one atomic formula  $X$  then  $S \in \llbracket X \rrbracket_{\mathcal{B}}$  implies that  $S$  belongs to a daimon type, thus the outputs of  $S$  are outputs of a daimon (Remark 185). This by definition means  $S \Vdash^{\text{all}} X$  and therefore  $S \Vdash^{\text{all}} \downarrow X$ .
- Say the hypersequent is made only of atomic formulas. The hypersequent may be of the form  $\mathcal{H}_1, \mathcal{H}_2$  then  $\llbracket \mathcal{H}_1, \mathcal{H}_2 \rrbracket_{\mathcal{B}} = (\llbracket \mathcal{H}_1 \rrbracket_{\mathcal{B}}^{\perp} \parallel \llbracket \mathcal{H}_2 \rrbracket_{\mathcal{B}}^{\perp})^{\perp}$  since,  $\llbracket \mathcal{H}_1 \rrbracket_{\mathcal{B}}^{\perp}$  and  $\llbracket \mathcal{H}_2 \rrbracket_{\mathcal{B}}^{\perp}$  are open types their parallel composition remains an open type.  $\llbracket \mathcal{H}_1, \mathcal{H}_2 \rrbracket_{\mathcal{B}}$  is the orthogonal of an full type hence it is a daimon type. Thus any sequent in that type as outputs which comes from a daimon link and thus is the approximation of any sequent of size  $n$  in particular  $S \Vdash^{\text{all}} \downarrow \mathcal{H}$ . We do a similar reasonment when  $\mathcal{H} = \mathcal{H}_1 \parallel \mathcal{H}_2$ .
- Case of non-atomic hypersequent with a virgula as main connective. Assume that  $S$  has a terminal connective link, say the hypersequent is of the form  $\mathcal{H}, A \wp B$  such that  $S = S_0 + l$  where  $l$  is a  $\wp$ -link and is the last conclusion of  $S$ . By proposition 176 it follows that  $S_0$  belongs to  $\llbracket \mathcal{H}, A, B \rrbracket_{\mathcal{B}}$  the measure of that hypersequent as decreased and so we apply the induction hypothesis;  $S_0 \Vdash^{\text{all}} \downarrow (\mathcal{H}, A, B)$  indeed it follows that  $S \Vdash^{\text{all}} \downarrow (\mathcal{H}, A \wp B)$ . A similar argument works for an hypersequent of the form  $\mathcal{H}, A \otimes B$ .
- Case of non-atomic hypersequent with a parallel as main connective. On the other hand say  $S$  belongs to the interpretation  $\llbracket \mathcal{H} \parallel A \wp B \rrbracket_{\mathcal{B}}$  then it is orthogonal to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^{\perp} \triangleright \llbracket A \wp B \rrbracket_{\mathcal{B}}^{\perp}$  and thus  $S_0$  is orthogonal to  $\llbracket \mathcal{H} \rrbracket_{\mathcal{B}}^{\perp} \triangleright \llbracket A, B \rrbracket_{\mathcal{B}}^{\perp}$ . Equivalently using proposition 176  $S_0$  belongs to  $\llbracket \mathcal{H} \parallel A, B \rrbracket_{\mathcal{B}}$  the size of the hypersequent as decreased and so we can apply the induction hypothesis;  $S_0 \Vdash^{\text{all}} \downarrow (\mathcal{H}, A, B)$  and thus  $S \Vdash^{\text{all}} \downarrow (\mathcal{H}, A \wp B)$ . A similar argument works for an hypersequent of the form  $\mathcal{H} \parallel A \otimes B$ . Similarly we treat the case of  $\mathcal{H}, A \otimes B$ .  $\square$

Indeed the previous lemma is a more general form of Theorem 187 as stated in this subsection. One can now easily derive it:

**Theorem 187.** *Let  $\mathcal{B}$  be an approximable daimon basis,  $\Gamma$  a sequent, and  $S$  a cut-free net;  $S \Vdash_{\mathcal{B}} \Gamma \Rightarrow S \Vdash_{\text{MLL}}^{\boxtimes} \Gamma$ .*

From this theorem the fact that the base  $\mathbf{1}$  is compact easily follows, because one can show that  $\mathbf{1}$  is a daimon basis.

**Proposition 83.** *For any sequent  $\Gamma$  and any cut-free net  $S$ ; if  $S \in \llbracket \Gamma \rrbracket_{\mathbf{1}}$  then  $S \Vdash^{\text{all}} \Gamma$ .*

*Proof.* Observe that the type  $\{\boxtimes_1\}^{\perp\perp}$  is indeed a daimon type (Remark 47), therefore the basis mapping each propositional variable  $X$  to  $\{\boxtimes_1\}^{\perp\perp}$  is a daimon basis. We conclude using Theorem 187.  $\square$



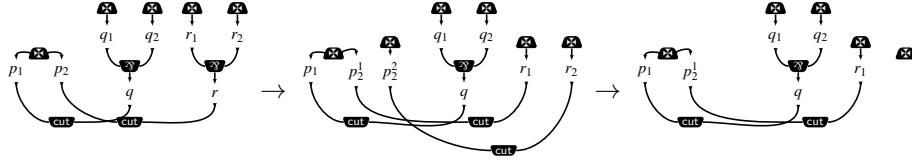


Figure 15: The interaction of  $\mathfrak{X}_2$  with the net  $\mathfrak{X}_{\mathfrak{A}} \parallel \mathfrak{X}_{\mathfrak{B}}$ , this cannot reduce to  $\mathfrak{X}_0$  since disconnection of the net is preserved by cut-elimination and  $\mathfrak{X}_0$  is connected.

## G.5 Proof of Theorem 85

**Theorem 85** (MLL $^{\mathfrak{X}}$  completeness). *Given a cut-free net  $S$  and a sequent  $\Gamma$ ;*

- *If for all basis  $\mathcal{B}$  we have  $S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$ , then  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Gamma$ .*
- *$S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$  for any approximable basis  $\mathcal{B}$  iff  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Gamma$ .*

*Proof.* As for (1) note that if  $S$  realises  $\Gamma$  for any basis  $\mathcal{B}$  then in particular it realises  $\Gamma$  for  $\mathbf{1}$ . Thus applying Proposition 83  $S$  must be testable by  $\Gamma$ , furthermore  $\mathbf{1}$  is approximable hence applying Proposition 82 we conclude  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Gamma$ .

As for (2) to have the implication that  $S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$  yields  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Gamma$  simply use the same argument as for point (1) merely noting that  $\mathbf{1}$  is an approximable basis. To show that  $S \vdash_{\text{MLL}^{\mathfrak{X}}} \Gamma$  implies  $S \in \llbracket \Gamma \rrbracket_{\mathcal{B}}$  is exactly the theorem of adequacy (Theorem 64).  $\square$

## G.6 Proof of Theorem 88

The *ground form* of an MLL formula  $A$  is the atomic hypersequent obtain when each  $\otimes$  is replaced by a  $\parallel$  and each  $\mathfrak{A}$  is replaced by a  $\succ$ .

**Proposition 188.** *Given a cut-free net  $a \parallel b$  with binary daimon links and two formulas  $A$  and  $B$  which have no variable in common if  $a \parallel b$  belongs to  $\llbracket A \rrbracket_{\mathcal{B}} \parallel \llbracket B \rrbracket_{\mathcal{B}}$  for any base  $\mathcal{B}$  then  $a \in \llbracket A \rrbracket_{\mathcal{B}}$  and  $b \in \llbracket B \rrbracket_{\mathcal{B}}$  for any basis  $\mathcal{B}$ .*

*Proof.* We rely on the following fact,  $a \parallel b$  belongs to  $\llbracket A \rrbracket_{\mathcal{B}} \parallel \llbracket B \rrbracket_{\mathcal{B}}$  so in particular  $a \parallel b$  is orthogonal to  $\llbracket A \rrbracket_{\mathcal{B}}^{\perp} \bowtie \llbracket B \rrbracket_{\mathcal{B}}^{\perp}$  (Proposition 178).

By induction on the formula  $A$  and  $B$  if both are equal to  $X$  and  $Y$  some propositional variable, then  $a \parallel b$  is orthogonal to  $\llbracket X \rrbracket_{\mathcal{B}} \bowtie \llbracket Y \rrbracket_{\mathcal{B}}$  for any choice of basis therefore the conclusion of  $a \parallel b$  may only be daimon links. Since furthermore  $\bigcap_{\mathcal{B}} \llbracket X \rrbracket_{\mathcal{B}}$  is empty it follows that  $\llbracket X \rrbracket_{\mathcal{B}} \parallel \llbracket Y \rrbracket_{\mathcal{B}}$  is also empty thus the proposition hold.

Note that because  $a \parallel b \in \llbracket A \parallel B \rrbracket_{\mathcal{B}}$  for any basis and in particular for  $\mathbf{1}$  which is daimon basis it follows that  $a \stackrel{\text{at}}{\preceq} A$  and  $b \stackrel{\text{at}}{\preceq} B$  (Lemma 186). Since  $a \parallel b \perp \llbracket A \rrbracket_{\mathcal{B}}^{\perp} \bowtie \llbracket B \rrbracket_{\mathcal{B}}^{\perp}$  from Proposition 106 one can show using Proposition 176 that this equivalently means that  $a^{\mathfrak{X}} \parallel b^{\mathfrak{X}}$  is orthogonal to  $\llbracket g(A) \rrbracket_{\mathcal{B}}^{\perp} \bowtie \llbracket g(B) \rrbracket_{\mathcal{B}}^{\perp}$ .  $a^{\mathfrak{X}} \in \llbracket g(A) \rrbracket_{\mathcal{B}}$  iff  $a \in \llbracket A \rrbracket_{\mathcal{B}}$  we assume that  $a^{\mathfrak{X}} \in \llbracket g(A) \rrbracket_{\mathcal{B}}$  for some basis. Consider a basis  $\mathcal{B}$  such that  $a^{\mathfrak{X}} :: s$  does not reduce to  $\mathfrak{X}_0$ , although for this to be possible the outputs of a same daimon link in  $a$  must be realising  $X, Y$ . one can easily check that  $a^{\mathfrak{X}} \parallel b^{\mathfrak{X}}$  is not orthogonal to  $s \bowtie s'$  for all  $s' \in \llbracket g(B) \rrbracket_{\mathcal{B}}$  (Because  $A$  and  $B$  don't share any propositional variable the interpretation of  $g(B)$  can be chosen so that orthogonality fails) and thus that it does not belong to  $\llbracket A \rrbracket_{\mathcal{B}} \parallel \llbracket B \rrbracket_{\mathcal{B}}$  for all basis.  $\square$

**Lemma 189** (Splitting a Parallel Interaction). *Given a sequent  $\Gamma$  with  $n$  formulas and two formulas  $A$  and  $B$  a net  $S ::= S_1 \parallel S_2$  with  $n+2$  conclusions such that  $S(n+1)$  is a conclusion of  $S_1$  and  $S(n+2)$  is a conclusion of  $S_2$ . There exists two unique sequents  $\Gamma_1$  and  $\Gamma_2$  such that for any basis:*

$$S \in \llbracket \Gamma, A \parallel B \rrbracket_{\mathcal{B}} \Rightarrow S \in \llbracket \Gamma_1, A \rrbracket_{\mathcal{B}} \parallel \llbracket \Gamma_2, B \rrbracket_{\mathcal{B}}$$

*Proof.* The inclusions of  $\llbracket \Gamma_1, A \rrbracket_{\mathcal{B}} \parallel \llbracket \Gamma_2, B \rrbracket_{\mathcal{B}}$  in  $\llbracket \Gamma, A \parallel B \rrbracket_{\mathcal{B}}$  is obtained by standard calculations.

on the other hand consider a net  $S$  in  $\llbracket \Gamma, A \parallel B \rrbracket_{\mathcal{B}}$  and is of the form  $S_1 \parallel S_2$ .  $S_1 \parallel S_2 :: \bar{\gamma}$  belongs to  $A \parallel B$ . And such that  $S_1 \parallel S_2 :: \bar{\gamma}$  belongs to  $A \parallel B$ . The opponent  $\bar{\gamma}$  of  $\llbracket \Gamma \rrbracket_{\mathcal{B}}^{\perp}$  is an element of  $\llbracket A_1 \rrbracket_{\mathcal{B}}^{\perp} \parallel \dots \parallel \llbracket A_n \rrbracket_{\mathcal{B}}^{\perp}$ , thus it can range over  $\llbracket A_1 \rrbracket_{\mathcal{B}}^{\perp} \parallel \dots \parallel \llbracket A_n \rrbracket_{\mathcal{B}}^{\perp}$  and we can set  $\bar{\gamma} = U_1 \parallel \dots \parallel U_n$ . In  $S_1 \parallel S_2 :: (U_1 \parallel \dots \parallel U_n)$  each  $U_i$  is cut with a conclusion of  $S_1 \parallel S_2$  i.e. a conclusion of  $S_1$  or a conclusion of  $S_2$ . We can therefore split the  $U_i$ 's according to those that are cut with  $S_1$  and those that are cut with  $S_2$  let us denote the nets made of the parallel of these families

respectively  $U^1$  and  $U^2$ . This can be rewritten using [Proposition 121](#) to obtain:

$$\begin{aligned}
& S_1 \parallel S_2 :: (U_1 \parallel \cdots \parallel U_n) \\
&= S_1 \parallel S_2 :: (U^1 \parallel U^2) \\
&= (S_1 :: U^1 \parallel S_2) :: U^2 && \text{(Proposition 121)} \\
&= (S_1 :: U^1) \parallel (S_2 :: U^2) && \text{(Proposition 121)}
\end{aligned}$$

Now  $U^1$  range in  $[\Gamma_1]_{\mathcal{B}}$  while  $U^2$  range in  $[\Gamma_2]_{\mathcal{B}}$ . Since  $S_1 \parallel S_2 :: (U_1 \parallel \cdots \parallel U_n)$  belongs to  $[[A]]_{\mathcal{B}} \parallel [[B]]_{\mathcal{B}}$  it follows that  $S_1 :: U^1 \parallel S_2 :: U^2$  belongs to  $[[A]]_{\mathcal{B}} \parallel [[B]]_{\mathcal{B}}$ . By [Proposition 188](#) it follows that  $S_1 :: U^1$  belongs to  $[[A]]_{\mathcal{B}}$  and  $S_2 :: U^2$  belongs to  $[[B]]_{\mathcal{B}}$ .  $\square$

**Theorem 88** (MLL completeness). *Let  $S$  be a cut-free net such that each of its daimon link has exactly two outputs,  $\Gamma$  be a sequent such that  $S \stackrel{at}{\models} \Gamma$ ; if  $S \in [\Gamma]_{\mathcal{B}}$  for any basis  $\mathcal{B}$  then,  $S \vdash_{\text{MLL}} \Gamma$ .*

*Proof.* Assume that  $S$  realise the sequent  $\Gamma$  for any basis. In particular then  $S$  realises  $\Gamma$  for the basis  $\mathbf{1}$ . By [Proposition 83](#) and [Proposition 82](#) this implies that  $S \vdash_{\text{MLL}^*} \Gamma$ . To conclude one must show that the proof tree  $\pi$  represented by  $S$  is indeed from MLL: this means that (1) its daimon rules should introduce only sequents of size 2 i.e. of the form  $X, Y$  and (2) these sequents must be such that  $Y = X^\perp$ . The point (1) is guaranteed by the assumption that the daimons of  $S$  are binary <sup>13</sup>.

To prove (2), we reason by induction on the proof tree  $\pi$  represented by  $S$  however we will need to show in the inductive cases that the assumption of the theorem hold.

*Base case.* If  $\pi$  is made of a single daimon rule, then since the daimons are binary it must introduce a sequent  $X, Y$ , now we show that it cannot be that  $X \neq Y$ , we assume that  $S \in [[X, Y]]_{\mathcal{B}}$  for any base  $\mathcal{B}$  meaning  $S \in [[X]]_{\mathcal{B}} \succ [[Y]]_{\mathcal{B}}$  or  $S \perp [[X]]_{\mathcal{B}}^\perp \parallel [[Y]]_{\mathcal{B}}^\perp$  ([Proposition 56](#)). Since we range over all basis we may map  $X$  and  $Y$  to the same type in particular we can map both atomic proposition to the type  $\{\mathfrak{X}_{\mathfrak{A}}\}^\perp$  where  $\mathfrak{X}_{\mathfrak{A}} ::= \langle \triangleright_{\mathfrak{A}} p_1 \rangle + \langle \triangleright_{\mathfrak{A}} p_2 \rangle + \langle p_1, p_2 \triangleright_{\mathfrak{A}} p \rangle$ . In that case;

$$\begin{aligned}
& \mathfrak{X}_2 \perp [[X]]_{\mathcal{B}}^\perp \parallel [[Y]]_{\mathcal{B}}^\perp \\
\Rightarrow & \mathfrak{X}_2 \perp (\{\mathfrak{X}_{\mathfrak{A}}\}^\perp)^\perp \parallel (\{\mathfrak{X}_{\mathfrak{A}}\}^\perp)^\perp \\
\Leftrightarrow & \mathfrak{X}_2 \perp \{\mathfrak{X}_{\mathfrak{A}}\}^{\perp\perp} \parallel \{\mathfrak{X}_{\mathfrak{A}}\}^{\perp\perp} \\
\Leftrightarrow & \mathfrak{X}_2 \perp \{\mathfrak{X}_{\mathfrak{A}}\}^{\perp\perp} \parallel \{\mathfrak{X}_{\mathfrak{A}}\}^{\perp\perp} \\
\Leftrightarrow & \mathfrak{X}_2 \perp \{\mathfrak{X}_{\mathfrak{A}}\}^{\perp\perp} \parallel \neg \{\mathfrak{X}_{\mathfrak{A}}\}^{\perp\perp} \\
\Rightarrow & \mathfrak{X}_2 \perp \{\mathfrak{X}_{\mathfrak{A}}\} \parallel \neg \{\mathfrak{X}_{\mathfrak{A}}\} \\
\Rightarrow & \mathfrak{X}_2 \perp \mathfrak{X}_{\mathfrak{A}} \parallel \mathfrak{X}_{\mathfrak{A}}
\end{aligned}$$

Indeed one can check that the last assertion is false by computing the reduction of  $\mathfrak{X}_2 :: \mathfrak{X}_{\mathfrak{A}} \parallel \mathfrak{X}_{\mathfrak{A}}$ , this is illustrated in [Figure 15](#). This shows that  $\mathfrak{X}_2$  can realise in all basis only sequents of the form  $X, X^\perp$ .

*Inductive cases.* Assume that  $S$  represents a proof terminating with a tensor then it is of the form  $S ::= S_1 + S_2 + \langle S_1(1), S_2(1) \triangleright_{\otimes} p \rangle$  where  $p$  is a fresh position (without loss of generality assume the tensor is made on the first formula of the subproofs). Let us show we can call again the hypothesis on the subproof  $S_1$  and  $S_2$ . Since  $S$  represents a proof terminating with a tensor it proves a sequent  $\Gamma, A \otimes B$ . by [Proposition 176](#) it follows that  $S_1 \parallel S_2$  belongs to  $\Gamma, A \parallel B$  and thus using [Lemma 189](#)  $S_1 \parallel S_2$  belongs to  $[\Gamma_1, A]_{\mathcal{B}} \parallel [\Gamma_2, B]_{\mathcal{B}}$ . Since  $S$  is the tensor of the two proofs  $\pi_1$  and  $\pi_2$  represented by  $S_1$  and  $S_2$  respectively we may assume that  $\pi_1$  and  $\pi_2$  prove sequents which have no propositional variable in common. We apply now [Proposition 188](#) and conclude that  $S_1$  belongs to  $[\Gamma_1, A]_{\mathcal{B}}$  while  $S_2$  belongs to  $[\Gamma_2, B]_{\mathcal{B}}$  for any basis  $\mathcal{B}$ . Calling the induction hypothesis this yields  $S_1 \vdash_{\text{MLL}} \Gamma_1, A$  and  $S_2 \vdash_{\text{MLL}} \Gamma_2, B$  therefore,  $S \vdash_{\text{MLL}} \Gamma, A \otimes B$ .

Assume that  $S$  represents a proof terminating with a par-rule then it is of the form  $S ::= S_0 + \langle S_0(1), S_0(1) \triangleright_{\wp} p \rangle$  where  $p$  is a fresh position. Let us show we can call again the hypothesis on the subproof  $S_0$ . Since  $S$  represents a proof terminating with a  $\wp$  and belongs to  $[\Gamma, A \wp B]_{\mathcal{B}}$ . by [Proposition 176](#) it follows that  $S_0$  belongs to  $[\Gamma, A, B]_{\mathcal{B}}$  for any basis, applying the induction hypothesis it follows that  $S_0$  is a proof of MLL of  $\Gamma, A, B$  and thus  $S \vdash_{\text{MLL}} \Gamma, A \wp B$ .  $\square$

<sup>13</sup>in terms of classical realisability one can call the nets with binary daimon *proof like*.