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NEW RESULTS ON RELAXATION AND HOMOGENIZATION OF UNBOUNDED SINGULAR INTEGRALS

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Abstract. In this paper, we extend our study of the relaxation and homogenization of unbounded singular integral functionals, previously developed in [AHLM11, AHCM17] where we examined the case of integrands whose quasiconvexification has polynomial growth. Here, we focus on the more general case where the quasiconvexification has convex growth. The distinguishing feature of this study is that such a singularity on the integrands is compatible with the fundamental constraint of hyperelasticity which says that compressing a volume of matter to a point requires an infinite amount of energy. However, our framework is not consistent with the constraint of noninterpenetration of matter.

Contents

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1. Introduction

In this paper we consider unbounded singular integral functionals of the form:

$$\mathscr{F}(\phi) := \int_O F\left(x, \nabla\phi(x)\right) dx,$$

where $F : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ is a Borel measurable function representing the hyperelastic energy of a periodic composite material, $O \subset \mathbb{R}^d$ is a bounded open set representing the reference configuration of the material and $\phi : O \to \mathbb{R}^m$ denotes the deformation field. The distinguishing feature here is that F can take infinite values. This paper aims to study relaxation and homogenization, via Γ-convergence (see Definition 2.7), of such unbounded singular integrals. In the scalar case, i.e. when $m = 1$, unbounded relaxation and homogenization problems were intensively studied by Carbone and De Arcangelis (see [CDA02] and the reference therein). Here, we focus on the vectorial case with applications to hyperelasticity in mind.

In previous papers [AHLM11, AHCM17], we proved that under certain conditions, if the Dacorogna relaxation formula [Dac82] for F, i.e. the quasiconvexification $\mathcal{Z}F$ of F, see (1.5), has polynomial growth, then the relaxation $\overline{\mathscr{F}}$ of $\mathscr{F}$, see (2.1), has an integral representation:

$$\overline{\mathscr{F}}(\phi) = \int_O \overline{F}(x, \nabla\phi(x))dx \tag{1.1}$$

with $\overline{F} : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ given by $\overline{F} = \mathcal{Z}F$. In the present paper, we consider the more general case where $\mathcal{Z}F$ has convex growth. Under this condition, we generalize our previous relaxation result by proving that (1.1) holds with

$$\overline{F} = \widehat{\mathcal{Q}\mathcal{Z}F},$$

where $\widehat{\mathcal{Q}\mathcal{Z}F}$ is defined through the composition of the operations given by the formulas (1.4), (1.5), (1.6) (see Theorem 2.5). We also demonstrate (see Corollary 2.6) that our result is consistent with the fundamental constraint of hyperelasticity which states that compressing a volume of matter to a point requires an infinite amount of energy, i.e.

$$F(x,\xi) \to \infty \text{ as } \det\xi \to 0. \tag{1.2}$$

It turns out that we can also extend our work on homogenization developed in [AHLM11, AHCM17]. Considering unbounded singular integral functionals of the form:

$$\mathscr{F}_\varepsilon(\phi) := \int_O F\left(\frac{x}{\varepsilon}, \nabla\phi(x)\right) dx,$$

where $F : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ is 1-periodic and $\varepsilon > 0$ characterizes the periodicity scale, we proved in [AHLM11, AHCM17] that under certain conditions, if $\mathcal{Z}F$ has polynomial growth, then homogenization holds, i.e.

$$\Gamma\text{-}\lim_{\varepsilon\to 0} \mathscr{F}_\varepsilon(\phi) = \int_O F_{\text{hom}}\left(\nabla\phi(x)\right) dx \tag{1.3}$$

with $F_{\text{hom}} : \mathbb{M}^{m\times d} \to [0,\infty]$ given by the Braides-Müller formula [Bra85, Mül87], i.e. $F_{\text{hom}} = \mathcal{H}F$ with $\mathcal{H}F$ given by (1.7). By using this new relaxation theorem (Theorem 2.5), we can

generalize our previous homogenization result to the more general case where $\mathcal{Z}F$ has convex growth by proving that (1.3) holds with

$$F_{\text{hom}} = \widehat{\mathcal{H}\widehat{\mathcal{Q}\mathcal{Z}F}}$$

where $\widehat{\mathcal{H}\widehat{\mathcal{Q}\mathcal{Z}F}}$ is defined through the composition of the operations given by the formulas (1.4), (1.5), (1.6) and (1.7) (see Theorem 2.11). We also demonstrate (see Corollary 2.12) that our result is consistent with (1.2).

The plan of the paper is as follows. Section 2 presents the main results. We begin in §2.1 with the statement of a new relaxation theorem (see Theorem 2.5) for unbounded singular integral functionals whose the integrands have a quasiconvexification with convex growth. This theorem is applicable to a new class of integrands (see (2.7), (2.8) and (2.9)) which are compatible with the fundamental constraint of hyperelasticity which asserts that compressing a volume of matter to a point requires an infinite energy (see Corollary 2.6). To prove Theorem 2.5, we utilize a known relaxation theorem (see Theorem 2.1) for unbounded integrands with convex growth. As mentioned, this theorem is applicable to integrands of the type (2.3) (see Corollary 2.2). After relaxation, in §2.2 we present a new homogenization theorem (see Theorem 2.11) for unbounded singular integral functionals whose the integrands have a quasiconvexication with convex growth. The proof of this theorem relies on both the relaxation theorem, Theorem 2.5, and a known homogenization theorem (see Theorem 2.9) for unbounded integrands with convex growth. Theorem 2.11 can be applied to the new class of integrands introduced in §2.1. Finally, in Section 3, we prove our main results, Theorems 2.5 and 2.11, in §3.1 and §3.2 respectively. For the convenience of the reader, auxiliary known results needed for these proofs are compiled in the appendix.

Notation. Throughout the paper we will use the following notation.

- Given $d \in \mathbb{N}^*$, $O \subset \mathbb{R}^d$ denotes a bounded open set.
- The Lebesgue measure on $\mathbb{R}^d$ is denoted by $\mathscr{L}^d$ and for each Borel set $A \subset \mathbb{R}^d$, the measure of A with respect to $\mathscr{L}^d$ is denoted by $\mathscr{L}^d(A)$.
- For every $x \in \mathbb{R}^d$ and every $\rho > 0$, we set $Q_\rho(x) := \frac{\rho}{2}]-1,1[^d + x = \,]x - \frac{\rho}{2}, x + \frac{\rho}{2}[^d$, which is the open cube centered at x and of side ρ.
- Given $p > 1$ and $m \in \mathbb{N}^*$, the space of p-Lebesgue functions from O to $\mathbb{R}^m$ is denoted by $L^p(O;\mathbb{R}^m)$.
- The space of $(1,p)$-Sobolev functions from O to $\mathbb{R}^m$ is denoted by $W^{1,p}(O;\mathbb{R}^m)$ and we set $W^{1,p}_0(O;\mathbb{R}^m) := \{\phi \in W^{1,p}(O;\mathbb{R}^m) : \phi = 0 \text{ on } \partial O\}$.
- The space of continuous piecewise affine functions from O to $\mathbb{R}^m$ is denoted by $\text{Aff}(O;\mathbb{R}^m)$ and we set $\text{Aff}_0(O;\mathbb{R}^m) := \{\phi \in \text{Aff}(O;\mathbb{R}^m) : \phi = 0 \text{ on } \partial O\}$.
- The space of $m \times d$ matrices is denoted by $\mathbb{M}^{m\times d}$.
- Given any Borel measurable $F : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$, the radial extension of F is denoted by $\widehat{F} : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ and is defined by

$$\widehat{F}(x,\xi) := \underline{\lim}_{t\to 1^-} F(x,t\xi). \tag{1.4}$$

- The quasiconvexification of F is denoted by $\mathcal{Z}F : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ and is defined by

$$\mathcal{Z}F(x,\xi) := \inf\left\{\int_{]0,1[^d} F(x,\xi+\nabla\varphi(y))dy : \varphi \in W_0^{1,\infty}(]0,1[^d;\mathbb{R}^m)\right\}. \tag{1.5}$$

- The generalized quasiconvexification of F is denoted by $\mathcal{Q}F : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ and is defined by

$$\mathcal{Q}F(x,\xi) := \overline{\lim_{\rho\to 0}} \inf\left\{\fint_{Q_\rho(x)} F(y,\xi+\nabla\varphi(y))dy : \varphi \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)\right\}. \tag{1.6}$$

- The homogenization of F is denoted by $\mathcal{H}F : \mathbb{M}^{m\times d} \to [0,\infty]$ and is defined by

$$\mathcal{H}F(\xi) := \inf_{k\geqslant 1} \inf\left\{\fint_{]0,k[^d} F(x,\xi+\nabla\varphi(x))dx : \varphi \in W_0^{1,p}(]0,k[^d;\mathbb{R}^m)\right\}. \tag{1.7}$$

2. Main results

From now on, $m,d \geqslant 1$ are two integers, $p>1$ is a real number and $O \subset \mathbb{R}^d$ is a bounded open set with Lipschitz boundary.

2.1. **Relaxation of unbounded singular integrals.** Let $F : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ be a Borel measurable function and let $\mathscr{F} : W^{1,p}(O;\mathbb{R}^m) \to [0,\infty]$ be defined by

$$\mathscr{F}(\phi) := \int_O F(x,\nabla\phi(x))dx,$$

let $\overline{\mathscr{F}} : W^{1,p}(O;\mathbb{R}^m) \to [0,\infty]$ be the relaxed functional given by

$$\overline{\mathscr{F}}(\phi) := \inf\left\{\underline{\lim_{n\to\infty}} \mathscr{F}(\phi_n) : \phi_n \to \phi \text{ in } L^p(O;\mathbb{R}^m)\right\} \tag{2.1}$$

and let $G : \mathbb{M}^{m\times d} \to [0,\infty]$ be a Borel measurable function. In what follows, we consider the following hypotheses:

- (A_1) G is convex;
- (A_2) $0 \in \mathrm{int}(\mathbb{G})$ where $\mathbb{G} := \{\xi \in \mathbb{M}^{m\times d} : G(\xi) < \infty\}$ denotes the effective domain of G;
- (A_3) G is p-coercive, i.e. there exists $C>0$ such that for every $\xi \in \mathbb{M}^{m\times d}$,
$$G(\xi) \geqslant C|\xi|^p;$$
- (A_4) F has G-growth, i.e. there exist $\alpha,\beta>0$ such that for every $(x,\xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d}$,
$$\alpha G(\xi) \leqslant F(x,\xi) \leqslant \beta(1+G(\xi));$$
- (A_5) F is radially uniformly upper semicontinuous (ru-usc), i.e. there exists $a>0$ such that
$$\overline{\lim_{t\to 1^-}} \Delta_F^a(t) \leqslant 0$$
with $\Delta_F^a : [0,1] \to]-\infty,\infty]$ defined by
$$\Delta_F^a(t) := \sup_{x\in\mathbb{R}^d} \sup_{\xi\in\mathbb{F}_x} \frac{F(x,t\xi)-F(x,\xi)}{a+F(x,\xi)},$$

where $\mathbb{F}_x = \{\xi \in \mathbb{M}^{m\times d} : F(x,\xi) < \infty\}$ denotes the effective domain of $F(x,\cdot)$.

We begin with the following integral representation result. For a proof we refer to [AHM24, Corollary 5.1] (see also [AHM18, Theorem 2.7] and [AHM23, Corollary 4.9]).

Theorem 2.1. *Assume that $p > d$ and (A_1)–(A_5) hold. Then*

$$\overline{\mathscr{F}}(\phi) = \int_O \widehat{\mathcal{Q}F}(x, \nabla\phi(x))dx \text{ for all } \phi \in W^{1,p}(O;\mathbb{R}^m). \tag{2.2}$$

Moreover, we have

$$\widehat{\mathcal{Q}F}(x,\xi) = \begin{cases} \lim\limits_{t\to 1^-} \mathcal{Q}F(x,t\xi) & \text{if } (x,\xi) \in \mathbb{R}^d \times \overline{\mathbb{G}} \\ \infty & \text{otherwise.} \end{cases}$$

Classically, Theorem 2.1 can be applied to unbounded functions $F : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ of the form:

$$F(x,\xi) = \widetilde{F}(x,\xi) + \widetilde{G}(\xi) \tag{2.3}$$

with $\widetilde{F} : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty[$ and $\widetilde{G} : \mathbb{M}^{m\times d} \to [0,\infty]$ satisfying the following assumptions:

- $(\widetilde{A}_1)$ $\widetilde{G}$ is convex;
- $(\widetilde{A}_2)$ $0 \in \mathrm{int}(\widetilde{\mathbb{G}})$ where $\widetilde{\mathbb{G}} := \{\xi \in \mathbb{M}^{m\times d} : \widetilde{G}(\xi) < \infty\}$ denotes the effective domain of $\widetilde{G}$;
- $(\widetilde{A}_3)$ $\widetilde{F}$ has p-growth, i.e. there exist $\mu, \nu > 0$ such that for every $(x,\xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d}$,
$$\mu|\xi|^p \leqslant \widetilde{F}(x,\xi) \leqslant \nu(1+|\xi|^p);$$
- $(\widetilde{A}_4)$ there exists $L > 0$ such that for every $x \in \mathbb{R}^d$ and every $\xi_1, \xi_2 \in \mathbb{M}^{m\times d}$,
$$\left|\widetilde{F}(x,\xi_1) - \widetilde{F}(x,\xi_2)\right| \leqslant L\,|\xi_1 - \xi_2|\left(1 + |\xi_1|^{p-1} + |\xi_2|^{p-1}\right).$$

More precisely, as a consequence of Theorem 2.1, we have the following result.

Corollary 2.2. *Assume that $p > d$. Under $(\widetilde{A}_1)$–$(\widetilde{A}_4)$, if F is given by (2.3) then (2.2) holds.*

Proof of Corollary 2.2. It suffices to prove that F given by (2.3) satisfies (A_1)–(A_5) and to apply Theorem 2.1.

First of all, by $(\widetilde{A}_3)$ we see that (A_4) is verified with $\alpha = \min\{\mu, 1\}$, $\beta = \max\{\nu, 1\}$ and $G(\xi) = |\xi|^p + \widetilde{G}(\xi)$ where the effective domain $\mathbb{G}$ of G is equal to $\widetilde{\mathbb{G}}$. Hence (A_3) holds and (A_1) and (A_2) follow from $(\widetilde{A}_1)$ and $(\widetilde{A}_2)$ respectively.

So, it remains to establish (A_5), i.e. F is ru-usc. Fix any $t \in]0,1[$, any $x \in \mathbb{R}^d$ and any $\xi \in \widetilde{\mathbb{G}}$. Using $(\widetilde{A}_4)$ with $\xi_1 = t\xi$ and $\xi_2 = \xi$ and taking the left inequality in $(\widetilde{A}_3)$ into account, we obtain

$$\widetilde{F}(x,t\xi) - \widetilde{F}(x,\xi) \leqslant L'(1-t)(1+\widetilde{F}(x,\xi)) \tag{2.4}$$

with $L' := L\max\{1, \frac{3}{\mu}\}$. On the other hand, as, by $(\widetilde{A}_1)$, $\widetilde{G}$ is convex we have

$$\widetilde{G}(t\xi) - \widetilde{G}(\xi) \leqslant t\widetilde{G}(\xi) + (1-t)\widetilde{G}(0) - \widetilde{G}(\xi) \leqslant (1-t)\widetilde{G}(0),$$

and consequently

$$\widetilde{G}(t\xi) - \widetilde{G}(\xi) \leqslant (1-t)\widetilde{G}(0)(1+\widetilde{G}(\xi)). \tag{2.5}$$

From (2.4) and (2.5) we deduce that

$$F(x,t\xi)-F(x,\xi)\leqslant \max\{L',\widetilde{G}(0)\}(1-t)(2+F(x,\xi)).$$

Passing to the supremum on x and ξ we obtain

$$\sup_{x\in\mathbb{R}^d}\sup_{\xi\in\widetilde{\mathbb{G}}}\frac{F(x,t\xi)-F(x,\xi)}{2+F(x,\xi)}\leqslant \max\{L',\widetilde{G}(0)\}(1-t),$$

and, noticing that for any $x\in\mathbb{R}^d$ the effective domain of $F(x,\cdot)$, $\mathbb{F}_x$, is equal to $\widetilde{\mathbb{G}}$ and $\widetilde{G}(0)<\infty$ by $(\widetilde{\mathrm{A}}_2)$, we conclude that

$$\varlimsup_{t\to 1^-}\Delta_F^2(t):=\varlimsup_{t\to 1^-}\sup_{x\in\mathbb{R}^d}\sup_{\xi\in\mathbb{F}_x}\frac{F(x,t\xi)-F(x,\xi)}{2+F(x,\xi)}\leqslant 0,$$

which proves that F is ru-usc with $a=2$, and the proof is complete. ■

Theorem 2.1 can be improved to incorporate determinant-type constraints (see Theorem 2.5 and corollary 2.6). To do this, let Λ be the class of $\lambda\in L^\infty(\mathbb{R}^d;[0,\infty[)$ satisfying the following property:

(P) for every bounded open subset $U\subset\mathbb{R}^d$ with $\mathscr{L}^d(\partial U)=0$ and every $\delta\in]0,\delta_0]$ with $\delta_0>0$, there exists a compact $K_\delta\subset\overline{U}$ such that

$$\begin{cases}\mathscr{L}^d(\partial K_\delta)=0\\ \mathscr{L}^d(\overline{U}\backslash K_\delta)<\delta\\ \lambda|_{K_\delta}\text{ is continuous}\end{cases}$$

with $\overline{U}$ denoting the closure of U.

Remark 2.3. (i) If λ is continuous then (P) is verified with $K_\delta=\overline{U}$.

(ii) If λ is continuous $\mathscr{L}^d$-a.e., i.e. $\mathscr{L}^d(N:=\{x\in\mathbb{R}^N:\lambda\text{ is not continuous at }x\})=0$, and if $\lim_{\delta\to 0}\mathscr{L}^d(V_\delta):=\{x\in U:\operatorname{dist}(x,N)<\delta\})=0$ and $\mathscr{L}^d(\partial V_\delta)=0$, then (P) is verified with $K_\delta=\overline{U}\backslash V_\delta$.

(iii) If (P) holds then λ is continuous $\mathscr{L}^d$-a.e. (for a proof, see [AHCM17, Lemma 2.5]).

In what follows, we consider the following complementary hypotheses:

(B_0) O is is strongly star-shaped[1];

(B_1) $\mathbb{G}$ is open;

(B_2) there exists $\lambda\in\Lambda$ such that for every $(x_1,x_2)\in\mathbb{R}^d\times\mathbb{R}^d$ and every $\xi\in\mathbb{M}^{m\times d}$,

$$F(x_1,\xi)\leqslant|\lambda(x_1)-\lambda(x_2)|(1+F(x_2,\xi))+F(x_2,\xi);$$

(B_3) $\mathcal{Z}F$ has G-growth, i.e. there exist $\alpha,\beta>0$ such that for every $(x,\xi)\in\mathbb{R}^d\times\mathbb{M}^{m\times d}$,

$$\alpha G(\xi)\leqslant\mathcal{Z}F(x,\xi)\leqslant\beta(1+G(\xi)).$$

Remark 2.4. (i) If G is continuous then (B_1) holds.

(ii) If F satisfies (B_2) then $\mathbb{F}_{x_1}=\mathbb{F}_{x_2}$ for all $(x_1,x_2)\in\mathbb{R}^d\times\mathbb{R}^d$, where $\mathbb{F}_{x_1}$ and $\mathbb{F}_{x_2}$ denote the effective domain of $F(x_1,\cdot)$ and $F(x_2,\cdot)$ respectively.

[1]An open set $O\subset\mathbb{R}^d$ is said to be strongly star-shaped if there exists $x_0\in O$ such that $t(\overline{-x_0+O})\subset -x_0+O$ for all $t\in]0,1[$.

(iii) If F satisfies (B_2) and if λ is continuous at $x \in \mathbb{R}^d$ then $F(\cdot,\xi)$ is continuous at x for all $\xi \in \mathbb{M}^{m\times d}$. More generally, if (B_2) holds and if $\lambda|_K$ is continuous for $K \subset \mathbb{R}^d$ then $F(\cdot,\xi)|_K$ is continuous for all $\xi \in \mathbb{M}^{m\times d}$.

(iv) If F satisfies (B_2) then for every $(x_1,x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and every $\xi \in \mathbb{M}^{m\times d}$,

$$\mathcal{Z}F(x_1,\xi) \leqslant |\lambda(x_1)-\lambda(x_2)|(1+\mathcal{Z}F(x_2,\xi)) + \mathcal{Z}F(x_2,\xi).$$

Hence, if (B_2) holds and if $\lambda|_K$ is continuous for $K \subset \mathbb{R}^d$ then $\mathcal{Z}F(\cdot,\xi)|_K$ is continuous for all $\xi \in \mathbb{M}^{m\times d}$ and $\mathcal{Z}\mathbb{F}_{x_1} = \mathcal{Z}\mathbb{F}_{x_2}$ for all $(x_1,x_2) \in \mathbb{R}^d \times \mathbb{R}^d$, where, for any $x \in \mathbb{R}^d$, $\mathcal{Z}\mathbb{F}_x$ denotes the effective domain of $\mathcal{Z}F(x,\cdot)$, i.e. $\mathcal{Z}\mathbb{F}_x := \{\xi \in \mathbb{M}^{m\times d} : \mathcal{Z}F(x,\xi) < \infty\}$.

(v) As a consequence of (ii)–(iii), if F satisfies (B_2) then, for each $\xi \in \mathbb{M}^{m\times d}$, $F(\cdot,\xi)$ and $\mathcal{Z}F(\cdot,\xi)$ are continuous a.e. because λ is continuous a.e. (see Remark 2.3-(iii)).

Here is the first main result of the paper.

Theorem 2.5. *Assume that $p > d$ and (A_1)–(A_3), (A_5) and (B_0)–(B_3) hold. Then*

$$\overline{\mathscr{F}}(\phi) = \int_O \widehat{Q\mathcal{Z}F}(x,\nabla\phi(x))dx \text{ for all } \phi \in W^{1,p}(O;\mathbb{R}^m). \tag{2.6}$$

Moreover, we have

$$\widehat{Q\mathcal{Z}F}(x,\xi) = \begin{cases} \lim\limits_{t\to 1^-} Q\mathcal{Z}F(x,t\xi) & \text{if } (x,\xi) \in \mathbb{R}^d \times \overline{\mathbb{G}} \\ \infty & \text{otherwise.} \end{cases}$$

Theorem 2.5 can be applied to unbounded singular functions $F : \mathbb{R}^d \times \mathbb{M}^{d\times d} \to [0,\infty]$ of the form:

$$F(x,\xi) = \widetilde{F}(x,\xi) + \widetilde{G}(\xi) + a(x)H(\det\xi). \tag{2.7}$$

The functions $a : \mathbb{R}^d \to [0,\infty[$ and $H : \mathbb{R} \to [0,\infty]$ satisfy the following conditions:

(C_1) $a \in \Lambda$ and there exists $\eta > 0$ such that $a \geqslant \eta$;

(C_2) H is Borel measurable and there exist $\gamma,\delta > 0$ such that $H(s) \leqslant \delta$ for all $|s| \geqslant \gamma$;

(C_3) $\mathbb{R}\backslash\{0\} \subset \mathbb{H}$, where $\mathbb{H}$ denotes the effective domain of H;

(C_4) there exists $\kappa > 0$ such that for every $t \in]0,1]$ and every $s \in \mathbb{H}$, $H(t^d s) \leqslant \frac{1}{t^\kappa}H(s)$.

The fonction $\widetilde{G} : \mathbb{M}^{m\times d} \to [0,\infty[$ is of the form

$$\widetilde{G}(\xi) = \widetilde{g}\left(|\xi|^2\right) \tag{2.8}$$

with $\widetilde{g} : [0,\infty[\to [0,\infty[$ verifying the following conditions:

$(\widetilde{\mathrm{a}}_1)$ the function $\widetilde{g}$ is convex and nondecreasing;

$(\widetilde{\mathrm{a}}_2)$ there exist $\gamma_1,\gamma_2 \geqslant 0$ such that for every $(s_1,s_2) \in [0,\infty[\times[0,\infty[$,

$$\widetilde{g}(s_1+s_2) \leqslant \gamma_1\widetilde{g}(s_1)\widetilde{g}(s_2) + \gamma_2\big(\widetilde{g}(s_1)+\widetilde{g}(s_2)\big).$$

The function $\widetilde{F} : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty[$ satisfies $(\widetilde{\mathrm{A}}_3)$–$(\widetilde{\mathrm{A}}_4)$ and the following additional condition:

$(\widetilde{\mathrm{A}}_5)$ for every $(x_1,x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and every $\xi \in \mathbb{M}^{m\times d}$,

$$\widetilde{F}(x_1,\xi) \leqslant |a(x_1)-a(x_2)|(1+\widetilde{F}(x_2,\xi)) + \widetilde{F}(x_2,\xi).$$

For example, the conditions $(\widetilde{\mathrm{a}}_1)$–$(\widetilde{\mathrm{a}}_2)$ are satisfied with $\widetilde{g}(s) = \mathrm{e}^s$ by taking $\gamma_1 = 1$ and $\gamma_2 = 0$, and the conditions (C_2)–(C_4) are verified with

$$H(s) = \begin{cases} \dfrac{1}{|s|} & \text{if } s \neq 0 \\ \infty & \text{if } s = 0. \end{cases} \tag{2.9}$$

Note that F as in (2.7) with H given by (2.9) is compatible with the singular behavior $F(x,\xi) \to \infty$ as $\det\xi \to 0$. However, such a F is not consistent with the noninterpenetration of the matter.

As a consequence of Theorem 2.5, we have the following result.

Corollary 2.6. *Assume that $m = d$ and $p > d$. Under (B_0), $(\widetilde{\mathrm{A}}_3)$–$(\widetilde{\mathrm{A}}_5)$, $(\widetilde{\mathrm{a}}_1)$–$(\widetilde{\mathrm{a}}_2)$ and (C_1)–(C_4), if F is given by (2.7) then (2.6) holds.*

Proof of Corollary 2.6. It suffices to prove that F given by (2.7) satisfies (A_1)–(A_3), (A_5) and (B_1)–(B_3), and to apply Theorem 2.5.

Define $G : \mathbb{M}^{d\times d} \to [0,\infty[$ by

$$G(\xi) := |\xi|^p + \widetilde{G}(\xi). \tag{2.10}$$

Taking (2.8) and $(\widetilde{\mathrm{a}}_1)$ into account, as $\widetilde{G}$ is finite it is clear that (A_1)–(A_3) and (B_1) are verified.

From (C_1) and $(\widetilde{\mathrm{A}}_5)$ it is easily seen that (B_2) holds with $\lambda = \max\left\{1, \frac{1}{\eta}\right\}a$.

Noticing that by (C_1), $a \in L^\infty(\mathbb{R}^d;[0,\infty[)$, from $(\widetilde{\mathrm{A}}_3)$ and (C_2) we see that F satisfies the following condition:

$(\widetilde{\mathrm{A}}_{\mathrm{det}})$ for every $(x,\xi) \in \mathbb{R}^d \times \mathbb{M}^{d\times d}$,

$$\text{if } |\det\xi| \geqslant \gamma \text{ then } F(x,\xi) \leqslant \widetilde{\delta}(1 + G(\xi)).$$

where $\gamma > 0$ is given by (C_2) and $\widetilde{\delta} := \max\{\nu + \delta\|a\|_{L^\infty}, 1\}$.

Let us prove (B_3). Fix any $x \in \mathbb{R}^d$ and any $\xi \in \mathbb{M}^{d\times d}$. First of all, from the left inequality in $(\widetilde{\mathrm{A}}_3)$ we see that $\mathcal{Z}F(x,\xi) \geqslant \alpha G(\xi)$ with $\alpha = \min\{\mu, 1\}$. On the other hand, since $\widetilde{G}$ is finite, so is G, and so from $(\widetilde{\mathrm{A}}_{\mathrm{det}})$ we see that if $|\det\xi| \geqslant \gamma$ then $\mathcal{Z}F(x,\xi) \leqslant F(x,\xi) < \infty$. Suppose now that $|\det\xi| < \gamma$. Then, by Lemma A.11, there exists $\varphi \in W_0^{1,\infty}(]0,1[^d;\mathbb{R}^d)$ such that $\big|\det(\xi + \nabla\varphi(y))\big| = \gamma$ for $\mathscr{L}^d$-a.a. $y \in]0,1[^d$, and using $(\widetilde{\mathrm{A}}_{\mathrm{det}})$ and $(\widetilde{\mathrm{a}}_1)$–$(\widetilde{\mathrm{a}}_2)$ we have

$$\begin{aligned}
\mathcal{Z}F(x,\xi) &\leqslant \int_{]0,1[^d} F(x,\xi + \nabla\varphi(y))dy \\
&\leqslant \int_{]0,1[^d} \widetilde{\delta}(1 + G(\xi + \nabla\varphi(y)))dy \\
&\leqslant \widetilde{\delta}\Big[1 + 2^p(|\xi|^p + \|\nabla\varphi\|_{L^\infty}^p) + \gamma_2\widetilde{g}(2|\xi|^2) + (\gamma_1\widetilde{g}(2|\xi|^2) + \gamma_2)\widetilde{g}(2\|\nabla\varphi\|_{L^\infty}^2)\Big] < \infty.
\end{aligned}$$

Thus $\mathcal{Z}F(x,\xi) < \infty$ for all $\xi \in \mathbb{M}^{d\times d}$, i.e. $\mathcal{Z}F(x,\cdot)$ is finite. From Proposition A.2(b) we deduce that $\mathcal{Z}F(x,\cdot)$ is rank-one convex, and consequently $\mathcal{Z}F(x,\cdot) \leqslant \mathcal{R}F(x,\cdot)$. Taking (2.8), (2.10) and $(\widetilde{\mathrm{A}}_{\mathrm{det}})$ into account, from Lemma A.9 (and Remark A.10) we conclude that

for every $(x,\xi)\in\mathbb{R}^d\times\mathbb{M}^{d\times d}$, $\mathcal{Z}F(x,\xi)\leqslant\beta(1+G(\xi))$ with $\beta>0$ given by Lemma A.9, which proves (B_3).

Let us prove (A_5). Fix any $t\in]0,1]$, any $x\in\mathbb{R}^d$ and any $\xi\in\mathbb{M}^{d\times d}$ such that $\det\xi\in\mathbb{H}$. Using $(\widetilde{\mathrm{A}}_4)$ with $\xi_1=t\xi$ and $\xi_2=\xi$ and taking the left inequality in $(\widetilde{\mathrm{A}}_3)$ into account, we obtain

$$\widetilde{F}(x,t\xi)-\widetilde{F}(x,\xi)\leqslant L'(1-t)(1+\widetilde{F}(x,\xi)) \tag{2.11}$$

with $L':=L\max\{1,\frac{3}{\mu}\}$. On the other hand, as, by $(\widetilde{\mathrm{a}}_1)$, $\widetilde{G}$ is convex we have

$$\widetilde{G}(t\xi)-\widetilde{G}(\xi)\leqslant t\widetilde{G}(\xi)+(1-t)\widetilde{G}(0)-\widetilde{G}(\xi)\leqslant(1-t)\widetilde{G}(0),$$

and consequently

$$\widetilde{G}(t\xi)-\widetilde{G}(\xi)\leqslant(1-t)\widetilde{G}(0)(1+\widetilde{G}(\xi)). \tag{2.12}$$

Moreover, by (C_3)–(C_4) we have

$$h(\det(t\xi))-h(\det\xi)=h(t^d\det\xi)-h(\det\xi)\leqslant\left(\frac{1}{t^\kappa}-1\right)(1+h(\det\xi)) \tag{2.13}$$

with $\kappa>0$ given by (C_4). From (2.11), (2.12) and (2.13) we deduce that

$$F(x,t\xi)-F(x,\xi)\leqslant\max\left\{L'(1-t),\widetilde{G}(0)(1-t),\frac{1}{t^\kappa}-1\right\}(3+F(x,\xi)).$$

Passing to the supremum on x and ξ we obtain

$$\sup_{x\in\mathbb{R}^d}\sup_{\xi\in\det\mathbb{H}}\frac{F(x,t\xi)-F(x,\xi)}{3+F(x,\xi)}\leqslant\max\left\{L'(1-t),\widetilde{G}(0)(1-t),\frac{1}{t^\kappa}-1\right\},$$

Where $\det\mathbb{H}:=\left\{\xi\in\mathbb{M}^{d\times d}:\det\xi\in\mathbb{H}\right\}$. Noticing that for any $x\in\mathbb{R}^d$ the effective of $F(x,\cdot)$, $\mathbb{F}_x$, is equal to $\det\mathbb{H}$, we conclude that

$$\varlimsup_{t\to1^-}\Delta_F^3(t):=\varlimsup_{t\to1^-}\sup_{x\in\mathbb{R}^d}\sup_{\xi\in\mathbb{F}_x}\frac{F(x,t\xi)-F(x,\xi)}{3+F(x,\xi)}\leqslant0,$$

which proves that F is ru-usc with $a=3$, i.e. (A_5) holds, and the proof is complete. ■

2.2. **Homogenization of unbounded singular integrals.** Let $F:\mathbb{R}^d\times\mathbb{M}^{m\times d}\to[0,\infty]$ be a Borel measurable function satisfying the following assumption:

- (A_6) F is 1-periodic, i.e. $F(x+e_i,\xi)=F(x,\xi)$ for all $\xi\in\mathbb{M}^{m\times d}$ and all $i\in\{1,\cdots,d\}$, where $(e_1,\cdots,e_d)$ is the standard basis of $\mathbb{R}^d$,

and, for each $\varepsilon>0$, let $\mathscr{F}_\varepsilon:W^{1,p}(O;\mathbb{R}^m)\to[0,\infty]$ be defined by

$$\mathscr{F}_\varepsilon(\phi):=\int_O F\left(\frac{x}{\varepsilon},\nabla\phi(x)\right)dx.$$

We aim to compute the Γ-limit of $\{\mathscr{F}_\varepsilon\}_{\varepsilon>0}$ as $\varepsilon\to0$ with respect to the $L^p(O;\mathbb{R}^m)$-convergence. Here is the definition of the Γ-limit. (For more details on this concept we refer to [DM93, BD98, Bra02].)

Definition 2.7. Let $\Gamma(L^p)\text{-}\underline{\lim}_{\varepsilon\to 0}\mathscr{F}_\varepsilon$, $\Gamma(L^p)\text{-}\overline{\lim}_{\varepsilon\to 0}\mathscr{F}_\varepsilon : W^{1,p}(O;\mathbb{R}^m) \to [0,\infty]$ be respectively defined by:

$$\Gamma(L^p)\text{-}\underline{\lim}_{\varepsilon\to 0}\mathscr{F}_\varepsilon(\phi) := \inf\left\{\underline{\lim}_{\varepsilon\to 0}\mathscr{F}_\varepsilon(\phi_\varepsilon) : \phi_\varepsilon \overset{L^p}{\to} \phi\right\};$$
$$\Gamma(L^p)\text{-}\overline{\lim}_{\varepsilon\to 0}\mathscr{F}_\varepsilon(\phi) := \inf\left\{\overline{\lim}_{\varepsilon\to 0}\mathscr{F}_\varepsilon(\phi_\varepsilon) : \phi_\varepsilon \overset{L^p}{\to} \phi\right\}.$$

Let $\mathscr{F}_{\rm hom} : W^{1,p}(O;\mathbb{R}^m) \to [0,\infty]$. We say that $\{\mathscr{F}_\varepsilon\}_{\varepsilon>0}$ $\Gamma(L^p)$-converges to $\mathscr{F}_{\rm hom}$ as $\varepsilon\to 0$, and we write $\mathscr{F}_{\rm hom} = \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0}\mathscr{F}_\varepsilon$, if the following two inequalities hold:

$$\mathscr{F}_{\rm hom} \leqslant \Gamma(L^p)\text{-}\underline{\lim}_{\varepsilon\to 0}\mathscr{F}_\varepsilon;$$
$$\Gamma(L^p)\text{-}\overline{\lim}_{\varepsilon\to 0}\mathscr{F}_\varepsilon \leqslant \mathscr{F}_{\rm hom}.$$

Let us mention the following property of the Γ-liminf and Γ-limsup that we will use in the proof of Theorem 2.11.

Proposition 2.8. *The Γ-liminf and the Γ-limsup are stable under the substitution of $\mathscr{F}_\varepsilon$ with its relaxed functional $\overline{\mathscr{F}}_\varepsilon$, i.e.*

$$\Gamma(L^p)\text{-}\underline{\lim}_{\varepsilon\to 0}\mathscr{F}_\varepsilon = \Gamma(L^p)\text{-}\underline{\lim}_{\varepsilon\to 0}\overline{\mathscr{F}}_\varepsilon \text{ and } \Gamma(L^p)\text{-}\overline{\lim}_{\varepsilon\to 0}\mathscr{F}_\varepsilon = \Gamma(L^p)\text{-}\overline{\lim}_{\varepsilon\to 0}\overline{\mathscr{F}}_\varepsilon,$$

where, for each $\varepsilon>0$, $\overline{\mathscr{F}}_\varepsilon(\phi) := \inf\big\{\underline{\lim}_{n\to\infty}\mathscr{F}_\varepsilon(\phi_n) : \phi_n \to \phi \text{ in } L^p(O;\mathbb{R}^m)\big\}$.

We begin with the following homogenization result. For a proof we refer to [AHMZ15, Theorem 1.1] (see also [AHM11]).

Theorem 2.9. *Assume $p>d$ and (A_1)–(A_6) hold. Then, $\mathscr{F}_{\rm hom} = \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0}\mathscr{F}_\varepsilon$ with $\mathscr{F}_{\rm hom}$ given by*

$$\mathscr{F}_{\rm hom}(\phi) = \int_O \widehat{\mathcal{H}F}(\nabla\phi(x))dx \text{ for all } \phi\in W^{1,p}(O;\mathbb{R}^m). \tag{2.14}$$

Moreover, we have

$$\widehat{\mathcal{H}F}(\xi) = \begin{cases} \lim\limits_{t\to 1^-}\mathcal{H}F(t\xi) & \text{if } \xi\in\overline{\mathbb{G}} \\ \infty & \text{otherwise.}\end{cases}$$

According to the proof of Corollary 2.2, the following result is a straightforward consequence of Theorem 2.9.

Corollary 2.10. *Assume that $p>d$ and F given by (2.3) verifies $(\widetilde{\mathrm{A}}_1)$–$(\widetilde{\mathrm{A}}_4)$ and the following additional assumption*:

$(\widetilde{\mathrm{A}}_6)$ *$\widetilde{F}$ is 1-periodic.*

Then, $\mathscr{F}_{\rm hom} = \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0}\mathscr{F}_\varepsilon$ with $\mathscr{F}_{\rm hom}$ given by (2.14).

The following theorem, which is the second main result of the paper, improves Theorem 2.9 and allows to incorporate determinant-type constraints (see Corollary 2.12).

Theorem 2.11. *Assume $p > d$ and (A_1)–(A_3), (A_5)–(A_6) and (B_0)–(B_3) hold. Then, $\mathscr{F}_{\rm hom} = \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0}\mathscr{F}_\varepsilon$ with $\mathscr{F}_{\rm hom}$ given by*

$$\mathscr{F}_{\rm hom}(\phi) = \int_O \widehat{\mathcal{H}\widehat{QZF}}(\nabla\phi(x))dx \text{ for all } \phi \in W^{1,p}(O;\mathbb{R}^m). \tag{2.15}$$

Moreover, we have

$$\widehat{\mathcal{H}\widehat{QZF}}(\xi) = \begin{cases} \lim\limits_{t\to 1^-} \mathcal{H}\widehat{QZF}(t\xi) & \text{if } \xi \in \overline{\mathbb{G}} \\ \infty & \text{otherwise.} \end{cases}$$

As a consequence of Theorem 2.11 we obtain the following result that applies to integrands F of the form (2.7) which are compatible with the singular behavior $F(x,\xi) \to \infty$ as $\det\xi \to 0$.

Corollary 2.12. *Assume that $m = d$, $p > d$ and F given by (2.7) verifies (B_0), $(\widetilde{A}_3)$–$(\widetilde{A}_6)$, $(\widetilde{a}_1)$–$(\widetilde{a}_2)$ and (C_1)–(C_4). Then, $\mathscr{F}_{\rm hom} = \Gamma(L^p)\text{-}\lim_{\varepsilon\to 0}\mathscr{F}_\varepsilon$ with $\mathscr{F}_{\rm hom}$ given by (2.15).*

The proof of Corollary 2.12 follows the same lines as Corollary 2.6, applying Theorem 2.11 instead of Theorem 2.5.

3. Proof of the main results

3.1. **Proof of the unbounded singular relaxation theorem.** Let $\mathcal{Z}\mathscr{F} : W^{1,p}(O;\mathbb{R}^m) \to [0,\infty]$ be defined by

$$\mathcal{Z}\mathscr{F}(\phi) := \int_O \mathcal{Z}F(x,\nabla\phi(x))dx \tag{3.1}$$

and let $\overline{\mathcal{Z}\mathscr{F}} : W^{1,p}(O;\mathbb{R}^m) \to [0,\infty]$ be the relaxed functional given by

$$\overline{\mathcal{Z}\mathscr{F}}(\phi) := \inf\left\{ \underline{\lim_{n\to\infty}}\, \mathcal{Z}\mathscr{F}(\phi_n) : \phi_n \to \phi \text{ in } L^p(O;\mathbb{R}^m) \right\}. \tag{3.2}$$

We begin with the following result which asserts that (under some conditions) the relaxed functional in (2.1) is equal to the relaxed functional in (3.2).

Proposition 3.1. *If (A_1) and (B_0)–(B_3) hold then $\overline{\mathscr{F}} = \overline{\mathcal{Z}\mathscr{F}}$.*

Proof of Proposition 3.1. First of all, it is easy to see that $\overline{\mathcal{Z}\mathscr{F}} \leqslant \overline{\mathscr{F}}$. So, it remains to prove that $\overline{\mathscr{F}} \leqslant \overline{\mathcal{Z}\mathscr{F}}$ for which it is sufficient to show that for every $\phi \in W^{1,p}(O;\mathbb{R}^m)$,

$$\overline{\mathscr{F}}(\phi) \leqslant \int_O \mathcal{Z}F\left(x,\nabla\phi(x)\right)dx. \tag{3.3}$$

To do this, we need the following lemma whose proof can be found in [AHCM17, Lemma 3.9].

Lemma 3.2. *Under (B_2),*

$$\text{if } \phi \in \mathrm{Aff}(O;\mathbb{R}^m) \text{ then } \overline{\mathscr{F}}(\phi) \leqslant \int_O \mathcal{Z}F\left(x,\nabla\phi(x)\right)dx.$$

Let $\phi \in W^{1,p}(O;\mathbb{R}^m)$. Without loss of generality we can assume that $\int_O \mathcal{Z}F\left(x,\nabla\phi(x)\right)dx < \infty$. Then, by (B_3),

$$\int_O G(\nabla\phi(x))dx < \infty, \tag{3.4}$$

and so

$$\nabla\phi(x) \in \mathbb{G} \text{ for } \mathscr{L}^d\text{-a.a. } x \in O. \tag{3.5}$$

According to (A_1) and (B_0)–(B_1), by Lemma A.1 there exists $\{\phi_n\}_n \subset \mathrm{Aff}(O;\mathbb{R}^m)$ satisfying (D_1)–(D_3) and, up to a subsequence,

$$\lim_{n\to\infty} |\nabla\phi_n(x) - \nabla\phi(x)| = 0 \text{ for } \mathscr{L}^d\text{-a.a. } x \in O. \tag{3.6}$$

From (B_3) we see that for every $x \in O$, $\mathcal{Z}\mathbb{F}_x = \mathbb{G}$, where $\mathcal{Z}\mathbb{F}_x := \{\xi \in \mathbb{M}^{m\times d} : \mathcal{Z}F(x,\xi) < \infty\}$ denotes the effective domain of $\mathcal{Z}F(x,\cdot)$. By (B_1) and Proposition A.2-(c) it follows that for every $x \in O$, $\mathcal{Z}F(x,\cdot)$ in continuous on $\mathbb{G}$, and consequently, by using (3.5) and (D_3),

$$\lim_{n\to\infty} |\mathcal{Z}F(x,\nabla\phi_n(x)) - \mathcal{Z}F(x,\nabla\phi(x))| = 0 \text{ for } \mathscr{L}^d\text{-a.a. } x \in O. \tag{3.7}$$

Moreover, from the right inequality in (B_3), it is easy to see that for every $n \geqslant 1$ and every Borel sets $E \subset O$,

$$\int_E \mathcal{Z}F(x,\nabla\phi_n(x))dx \leqslant \beta\mathscr{L}^d(E) + \beta\int_E G(\nabla\phi(x))dx + \beta\|G(\nabla\phi_n) - G(\nabla\phi)\|_{L^1(O)},$$

which shows that $\{\mathcal{Z}F(\cdot,\nabla\phi_n)\}_n$ is uniformly integrable when combined with (3.4) and (D_2). Recalling (3.7), from Vitali's theorem we deduce that

$$\lim_{n\to\infty}\int_O \mathcal{Z}F(x,\nabla\phi_n(x))dx = \int_O \mathcal{Z}F(x,\nabla\phi(x))dx \tag{3.8}$$

On the other hand, by Lemma 3.2 we have

$$\overline{\mathscr{F}}(\phi_n) \leqslant \int_O \mathcal{Z}F\left(x,\nabla\phi_n(x)\right)dx \text{ for all } n \geqslant 1. \tag{3.9}$$

Letting $n \to \infty$ in (3.9) and using (3.8) we obtain

$$\varliminf_{n\to\infty} \overline{\mathscr{F}}(\phi_n) \leqslant \int_O \mathcal{Z}F\left(x,\nabla\phi(x)\right)dx. \tag{3.10}$$

By (D_1), $\phi_n \to \phi$ in $L^p(O;\mathbb{R}^m)$, hence

$$\overline{\mathscr{F}}(\phi) \leqslant \varliminf_{n\to\infty} \overline{\mathscr{F}}(\phi_n) \tag{3.11}$$

because $\overline{\mathscr{F}}$ is lower semicontinuous with respect to the $L^p(O;\mathbb{R}^m)$-convergence, and (3.3) follows by combining (3.10) with (3.11). ■

We can now prove Theorem 2.5.

Proof of Theorem 2.5. According to (B_3) we see that (A_4) is verified with $\mathcal{Z}F$ instead of F. By Lemma A.6 we can assert that (A_5) is satisfied with $\mathcal{Z}F$ instead of F. Since (A_1)–(A_3) are also fulfilled we can applied Theorem 2.1 to $\mathcal{Z}\mathscr{F}$ defined in (3.1). Consequently, we deduce that

$$\overline{\mathcal{Z}\mathscr{F}}(\phi) = \int_O \widehat{\mathcal{Q}\mathcal{Z}F}(x, \nabla\phi(x))dx \text{ for all } \phi \in W^{1,p}(O;\mathbb{R}^m).$$

Furthermore, by Proposition 3.1, we have $\overline{\mathscr{F}} = \overline{\mathcal{Z}\mathscr{F}}$, which implies (2.6). ■

3.2. **Proof of the unbounded singular homogenization theorem.** Here we prove Theorem 2.11. From now on, for each $\varepsilon > 0$, let $F_\varepsilon : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ be defined by

$$F_\varepsilon(x,\xi) := F\left(\frac{x}{\varepsilon}, \xi\right). \tag{3.12}$$

Proof of Theorem 2.11. The proof is divided into two steps: first, we apply the new relaxation theorem, Theorem 2.5, to every F_ε, and then we conclude via the known homogenization theorem, Theorem 2.9.

Step 1: applying Theorem 2.5 to every F_ε. Fix any $\varepsilon > 0$. Since (A_1)–(A_3) and (B_0)–(B_1) are verified and are independent of the integrand F, to apply Theorem 2.5 to F_ε, it is sufficient to prove that (A_5) and (B_2)–(B_3) hold with F_ε instead of F.

Fix any $t \in [0,1]$, any $x \in \mathbb{R}^d$ and any $\xi \in \mathbb{F}_{\varepsilon,x}$, where $\mathbb{F}_{\varepsilon,x}$ denotes the effective domain of $F_\varepsilon(x,\cdot)$. As $\mathbb{F}_{\varepsilon,x} = \mathbb{F}_{\frac{x}{\varepsilon}}$ by (3.12), and F is ru-usc with $a > 0$, we can assert that

$$\frac{F_\varepsilon(x,t\xi) - F_\varepsilon(x,\xi)}{a + F_\varepsilon(x,\xi)} = \frac{F\left(\frac{x}{\varepsilon}, t\xi\right) - F\left(\frac{x}{\varepsilon},\xi\right)}{a + F\left(\frac{x}{\varepsilon},\xi\right)} \leqslant \sup_{y\in\mathbb{R}^d}\sup_{\zeta\in\mathbb{F}_y} \frac{F(y,t\zeta) - F(y,\zeta)}{a + F(y,\zeta)} = \Delta_F^a(t).$$

Hence, for every $t \in [0,1]$,

$$\Delta_{F_\varepsilon}^a(t) = \sup_{x\in\mathbb{R}^d}\sup_{\xi\in\mathbb{F}_{\varepsilon,x}} \frac{F_\varepsilon(x,t\xi) - F_\varepsilon(x,\xi)}{a + F_\varepsilon(x,\xi)} \leqslant \Delta_F^a(t).$$

But, since F is ru-usc with $a > 0$, $\overline{\lim}_{t\to 1^-} \Delta_F^a(t) \leqslant 0$, and consequently $\overline{\lim}_{t\to 1^-} \Delta_{F_\varepsilon}^a(t) \leqslant 0$, which shows that F_ε is ru-usc with $a > 0$, i.e. (A_5) holds with F_ε instead of F.

By assumption, there exists $\lambda \in \Lambda$ such that

$$F(y_1,\xi) \leqslant |\lambda(y_1) - \lambda(y_2)|(1 + F(y_2,\xi)) + F(y_2,\xi) \text{ for all } (y_1,y_2,\xi) \in \mathbb{R}^d\times\mathbb{R}^d\times\mathbb{M}^{m\times d}. \tag{3.13}$$

Setting $\lambda_\varepsilon(\cdot) := \lambda\left(\frac{\cdot}{\varepsilon}\right)$ where, without loss of generality, $\varepsilon \in]0,1[$, and taking (3.12) into account, from (3.13) we see that for every $(x_1,x_2) \in \mathbb{R}^d \times \mathbb{R}^d$ and every $\xi \in \mathbb{M}^{m\times d}$,

$$F_\varepsilon(x_1,\xi) \leqslant |\lambda_\varepsilon(x_1) - \lambda_\varepsilon(x_2)|(1 + F_\varepsilon(x_2,\xi)) + F_\varepsilon(x_2,\xi).$$

We claim that $\lambda_\varepsilon \in \Lambda$. Indeed, first of all it is clear that $\lambda_\varepsilon \in L^\infty(\mathbb{R}^d;[0,\infty[)$. On the other hand, let $U \subset \mathbb{R}^d$ be a bounded open set with $\mathscr{L}^d(\partial U) = 0$ and let $\delta > 0$. As $\lambda \in \Lambda$, there exists a compact $\widetilde{K}_\delta \subset \overline{\frac{1}{\varepsilon}U}$ such that

$$\begin{cases} \mathscr{L}^d\left(\partial\widetilde{K}_\delta\right) = 0 \\ \mathscr{L}^d\left(\overline{\frac{1}{\varepsilon}U}\backslash\widetilde{K}_\delta\right) < \delta \\ \lambda|_{\widetilde{K}_\delta} \text{ is continuous.} \end{cases} \tag{3.14}$$

Set $K_\delta := \varepsilon\widetilde{K}_\delta$. Then, K_δ is compact, $K_\delta \subset \overline{U}$ and from (3.14) we have

$$\begin{cases} \mathscr{L}^d\left(\partial K_\delta\right) = \varepsilon^d \mathscr{L}^d\left(\partial \widetilde{K}_\delta\right) = 0 \\ \mathscr{L}^d\left(\overline{U}\backslash K_\delta\right) = \varepsilon^d \mathscr{L}^d\left(\overline{\frac{1}{\varepsilon}U}\backslash \widetilde{K}_\delta\right) < \varepsilon^d\delta < \delta \\ \lambda_\varepsilon|_{K_\delta} = \lambda\left(\frac{\cdot}{\varepsilon}\right)|_{\varepsilon\widetilde{K}_\delta} \text{ is continuous,} \end{cases}$$

which means that (P) is satisfied with λ_ε in place of λ, and proves the claim. Thus, (B_2) holds with F_ε instead of F.

By assumption there exist $\alpha, \beta > 0$ such that

$$\alpha G(\xi) \leqslant \mathcal{Z}F(y,\xi) \leqslant \beta(1+G(\xi)) \text{ for all } (y,\xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d}. \tag{3.15}$$

Fix any $(x,\xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d}$. By definition of $\mathcal{Z}F$ in (1.5) it is easily seen that

$$\mathcal{Z}F_\varepsilon(x,\xi) = \mathcal{Z}F\left(\frac{x}{\varepsilon},\xi\right). \tag{3.16}$$

From (3.15) and (3.16) we deduce that for every $(x,\xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d}$,

$$\alpha G(\xi) \leqslant \mathcal{Z}F_\varepsilon(x,\xi) \leqslant \beta(1+G(\xi)),$$

which shows that (B_3) holds with F_ε instead of F.

Consequently, by applying Theorem 2.5 to F_ε, we conclude that for every $\varepsilon > 0$,

$$\overline{\mathscr{F}_\varepsilon}(\phi) = \int_O \widehat{\mathcal{Q}\mathcal{Z}F_\varepsilon}(x,\nabla\phi(x))dx = \int_O \widehat{\mathcal{Q}\mathcal{Z}F}\left(\frac{x}{\varepsilon},\nabla\phi(x)\right)dx \text{ for all } \phi \in W^{1,p}(O;\mathbb{R}^m). \tag{3.17}$$

Step 2: end of the proof by applying Theorem 2.9. By Proposition 2.8, $\Gamma(L^p)\text{-}\underline{\lim}_{\varepsilon\to 0}\mathscr{F}_\varepsilon = \Gamma(L^p)\text{-}\underline{\lim}_{\varepsilon\to 0}\overline{\mathscr{F}}_\varepsilon$ and $\Gamma(L^p)\text{-}\overline{\lim}_{\varepsilon\to 0}\mathscr{F}_\varepsilon = \Gamma(L^p)\text{-}\overline{\lim}_{\varepsilon\to 0}\overline{\mathscr{F}}_\varepsilon$. So, the proof will be completed by showing that

$$\left(\Gamma(L^p)\text{-}\lim_{\varepsilon\to 0}\overline{\mathscr{F}}_\varepsilon\right)(\phi) = \int_O \widehat{\mathcal{H}\widehat{\mathcal{Q}\mathcal{Z}F}}(\nabla\phi(x))dx \text{ for all } \phi \in W^{1,p}(O;\mathbb{R}^m). \tag{3.18}$$

For this, we are going to apply Theorem 2.9 to $\widehat{\mathcal{Q}\mathcal{Z}F}$. Let $\overline{G} : \mathbb{M}^{m\times d} \to [0,\infty]$ denote the lower semicontinuous envelope of G. To apply Theorem 2.9 to $\widehat{\mathcal{Q}\mathcal{Z}F}$, it suffices to show that (A_1)–(A_3) hold with $\overline{G}$ instead of G, (A_4) holds with $(\overline{G}, \widehat{\mathcal{Q}\mathcal{Z}F})$ instead of (G,F) and (A_5)–(A_6) hold with $\widehat{\mathcal{Q}\mathcal{Z}F}$ instead of F.

First of all, since (A_1)–(A_3) hold for G, it is clear that (A_1)–(A_3) hold for $\overline{G}$ as well.

On the other hand, since F is ru-usc, by successively using the stability results for ru-usc from Lemmas A.6, A.7, and A.5(a), we can assert that $\widehat{\mathcal{Q}\mathcal{Z}F}$ is ru-usc, i.e. (A_5) holds with $\widehat{\mathcal{Q}\mathcal{Z}F}$ instead of F.

Moreover, as F is 1-periodic it is easily seen that $\widehat{\mathcal{Q}\mathcal{Z}F}$ is also 1-periodic, i.e. (A_6) holds with $\widehat{\mathcal{Q}\mathcal{Z}F}$ instead of F.

Finally, by assumption there exist $\alpha, \beta > 0$ such that

$$\alpha G(\xi) \leqslant \mathcal{Z}F(x,\xi) \leqslant \beta(1+G(\xi)) \text{ for all } (x,\xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d}.$$

Hence, by using Lemma A.5(d),

$$\alpha\overline{G}(\xi) \leqslant \widehat{\mathcal{QZ}F}(x,\xi) \leqslant \beta(1+\overline{G}(\xi)) \text{ for all } (x,\xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d},$$

which means that (A_4) holds with $(\overline{G}, \widehat{\mathcal{QZ}F})$ instead of (G,F).

Consequently, (3.18) follows by applying Theorem 2.9 to $\widehat{\mathcal{QZ}F}$, and the proof is complete. ■

Appendix. Auxiliary results

A.1. **Approximation by continuous piecewise affine functions.** The following lemma can be found in [Mül87, Lemma 3.6(b)] (see also [ET74, Chapitre X, §2.3, pp. 288-293] and [AHM11, §3.3]).

Lemma A.1. *Let $O \subset \mathbb{R}^d$ be a bounded open set with Lipschitz boundary and let $G : \mathbb{M}^{m\times d} \to [0,\infty]$. If (B_0) holds then for every $\phi \in W^{1,p}(O;\mathbb{R}^m)$ such that*

$$\int_O G(\nabla\phi(x))dx < \infty,$$

there exists $\{\phi_n\}_n \subset \mathrm{Aff}(O;\mathbb{R}^m)$ such that:

- (D_1) $\lim\limits_{n\to\infty} \|\phi_n - \phi\|_{W^{1,p}(O;\mathbb{R}^m)} = 0$;
- (D_2) $\lim\limits_{n\to\infty} \|G(\nabla\phi_n) - G(\nabla\phi)\|_{L^1(O)} = 0$;
- (D_3) $\nabla\phi_n(x) \in \mathbb{G}$ *for all* $n \geqslant 1$ *and* $\mathscr{L}^d$*-a.a.* $x \in O$.

A.2. **Properties of $\mathcal{Z}F$.** The following result is due to Fonseca (see [Fon88, lemma 2.16, Theorem 2.17 and Proposition 2.3]).

Proposition A.2. *Let $F : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ be a Borel measurable function. The function $\mathcal{Z}F$ satisfies the following properties.*

(a) *For every bounded open set $U \subset \mathbb{R}^d$ with $|\partial U| = 0$ and every $(x,\xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d}$,*

$$\mathcal{Z}F(x,\xi) = \inf\left\{ \fint_U F(x,\xi+\nabla\varphi(y))dy : \varphi \in W^{1,\infty}_0(U;\mathbb{R}^m)\right\} =: \mathcal{Z}_U F(x,\xi).$$

More precisely, $\mathcal{Z}_U F \leqslant \mathcal{Z}F$ for all bounded open set $U \subset \mathbb{R}^d$, and $\mathcal{Z}F \leqslant \mathcal{Z}_U F$ for all bounded open set $U \subset \mathbb{R}^d$ with $|\partial U| = 0$.

(b) *For every $x \in \mathbb{R}^d$, $\mathcal{Z}F(x,\cdot)$ is rank-one convex in $\mathrm{int}(\mathcal{Z}\mathbb{F}_x)$, where $\mathcal{Z}\mathbb{F}_x$ denotes the effective domain of $\mathcal{Z}F(x,\cdot)$, i.e. $\mathcal{Z}\mathbb{F}_x := \{\xi \in \mathbb{M}^{m\times d} : \mathcal{Z}F(x,\xi) < \infty\}$.*

(c) *For every $x \in \mathbb{R}^d$, $\mathcal{Z}F(x,\cdot)$ is continuous on $\mathrm{int}(\mathcal{Z}\mathbb{F}_x)$.*

(d) *For every bounded open set $U \subset \mathbb{R}^d$ with $|\partial U| = 0$, every $(x,\xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d}$ and every $\varphi \in \mathrm{Aff}_0(U;\mathbb{R}^m)$,*

$$\mathcal{Z}F(x,\xi) \leqslant \fint_U \mathcal{Z}F(x,\xi+\nabla\varphi(y))dy.$$

Remark A.3. Proposition A.2 is also valid with "$\widehat{\mathcal{Z}}F$" instead of "$\mathcal{Z}F$" (see [AHM09, Proposition 2.3]) where $\widehat{\mathcal{Z}}F : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ is given by

$$\widehat{\mathcal{Z}}F(x,\xi) := \inf\left\{ \int_Y F(x,\xi+\nabla\varphi(y))dy : \varphi \in \mathrm{Aff}_0(Y;\mathbb{R}^m)\right\}.$$

In particular, Proposition A.2(d) can be rewritten as $\widehat{\mathcal{Z}}[\mathcal{Z}F] = \mathcal{Z}F$.

A.3. **Stability properties of ru-usc functions.** We begin by recalling the definition of a radially uniformly upper semicontinuous (ru-usc) function. Let $F : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ be a Borel measurable function.

Definition A.4. We say that F is ru-usc if there exists $a > 0$ such that

$$\overline{\lim_{t\to 1^-}}\ \Delta_F^a(t) \leqslant 0$$

with $\Delta_F^a : [0,1] \to]-\infty,\infty]$ defined by

$$\Delta_F^a(t) := \sup_{x\in\mathbb{R}^d}\sup_{\xi\in\mathbb{F}_x} \frac{F(x,t\xi) - F(x,\xi)}{a + F(x,\xi)},$$

where $\mathbb{F}_x$ denotes the effective domain of $F(x,\cdot)$.

Let $\widehat{F} : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ be defined by

$$\widehat{F}(x,\xi) := \underline{\lim_{t\to 1^-}}\ F(x,t\xi).$$

The operation $F \mapsto \widehat{F}$ is stable under ru-usc. More precisely, we have the following result, which makes clear the properties of $\widehat{F}$ (see [AHM11, Theorem 3.5] for a proof).

Lemma A.5. *If F is ru-usc and if $t\overline{\mathbb{F}_x} \subset \operatorname{int}(\mathbb{F}_x)$ for all $x \in \mathbb{R}^d$ and all $t \in]0,1[$, then*:

(a) *$\widehat{F}$ is ru-usc*;

(b) *For every $(x,\xi) \in \mathbb{R}^d \times \mathbb{M}^{m\times d}$,*

$$\widehat{F}(x,\xi) = \begin{cases} \lim\limits_{t\to 1^-} F(x,t\xi) & \text{if } \xi \in \overline{\mathbb{F}_x} \\ \infty & \text{otherwise.} \end{cases}$$

If moreover, for every $x \in \mathbb{R}^d$, $F(x,\cdot)$ is lsc on $\operatorname{int}(\mathbb{F}_x)$ then:

(c) $\widehat{F}(x,\xi) = \begin{cases} F(x,\xi) & \text{if } \xi \in \operatorname{int}(\mathbb{F}_x) \\ \lim\limits_{t\to 1^-} F(x,t\xi) & \text{if } \xi \in \partial\mathbb{F}_x \\ \infty & \text{otherwise;} \end{cases}$

(d) *for every $x \in \mathbb{R}^d$, $\widehat{F}(x,\cdot)$ is the lsc envelope of $F(x,\cdot)$.*

Let $\mathcal{Z}F : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ be defined by

$$\mathcal{Z}F(x,\xi) := \inf\left\{\int_{]0,1[^d} F(x,\xi+\nabla\varphi(y))dy : \varphi \in W_0^{1,\infty}(]0,1[^d;\mathbb{R}^m)\right\}.$$

The following lemma states that the operation $F \mapsto \mathcal{Z}F$ is stable under ru-usc (see [AHM11, Proposition 3.6] for a proof).

Lemma A.6. *If F is ru-usc then $\mathcal{Z}F$ is ru-usc.*

Let $\mathcal{Q}F : \mathbb{R}^d \times \mathbb{M}^{m\times d} \to [0,\infty]$ by

$$\mathcal{Q}F(x,\xi) := \overline{\lim_{\rho\to 0}} \inf\left\{\fint_{Q_\rho(x)} F(y,\xi+\nabla\varphi(y))dy : \varphi \in W_0^{1,p}(Q_\rho(x);\mathbb{R}^m)\right\}.$$

The following lemma asserts the stability of the operation $F \mapsto \mathcal{Q}F$ under ru-usc (see [AHM23, Corollary 2.22] for a proof).

Lemma A.7. *If F is ru-usc then $\mathcal{Q}F$ is ru-usc.*

Let $\mathcal{H}F : \mathbb{M}^{m\times d} \to [0,\infty]$ be defined by

$$\mathcal{H}F(\xi) := \inf_{k\geqslant 1} \inf \left\{ \fint_{]0,k[^d} F(x,\xi+\nabla\varphi(x))dx : \varphi \in W^{1,p}_0(]0,k[^d;\mathbb{R}^m) \right\}.$$

As with the three operations mentioned above, the operation $F \mapsto \mathcal{H}F$ is stable under ru-usc (see [AHM11, Proposition 3.7] for a proof).

Lemma A.8. *If F is ru-usc then $\mathcal{H}F$ is ru-usc.*

A.4. **A variant of Ben Belgacem's lemma.** In what follows $d \leqslant m$ and given $\xi \in \mathbb{M}^{m\times d}$, $0 \leqslant v_1(\xi) \leqslant \cdots \leqslant v_d(\xi)$ denote the singular values of ξ. Set

$$v(\xi) := \prod_{i=1}^{d} v_i(\xi).$$

The following lemma is a variant of the result by Ben Belgacem [BB96] (see also [AHM12, Theorem 3.20]).

Lemma A.9. *Let $G : \mathbb{M}^{m\times d} \to [0,\infty[$ be defined by*

$$G(\xi) := |\xi|^p + \widetilde{g}\left(|\xi|^2\right) \tag{A.1}$$

with $p > 1$ and $\widetilde{g} : [0,\infty[\to [0,\infty[$ verifying $(\widetilde{a}_1)$–$(\widetilde{a}_2)$. Let $F : \mathbb{M}^{m\times d} \to [0,\infty]$ be a Borel measurable function. Assume that there exist $\gamma, \widetilde{\delta} > 0$ such that for every $\xi \in \mathbb{M}^{m\times d}$,

$$v(\xi) \geqslant \gamma \text{ implies } F(\xi) \leqslant \widetilde{\delta}(1+G(\xi)). \tag{A.2}$$

Then, there exists $\beta > 0$ such that for every $\xi \in \mathbb{M}^{m\times d}$,

$$\mathcal{R}F(\xi) \leqslant \beta(1+G(\xi)),$$

where $\mathcal{R}F$ denotes the rank-one convex envelope of F.

Proof of Lemma A.9. Without loss of generality we can assume that $\gamma \geqslant 1$, and it is sufficient to prove that $\mathcal{R}F(\xi) \leqslant \beta(1+|\xi|^p)$ for all $\xi \in \mathbb{M}^{m\times d}$ such that $v(\xi) < \gamma$.
Let $\xi \in \mathbb{M}^{m\times d}$ be such that $v(\xi) < \gamma$. Let $P \in \mathbb{O}(m)$ be such that

$$\xi = PJU,$$

where $U := \sqrt{\xi^{\mathrm{T}}\xi}$ and $J = (J_{ij}) \in \mathbb{M}^{m\times d}$ with $J_{ij} = 0$ if $i \neq j$ and $J_{ii} = 1$, where $\mathbb{O}(m)$ is the group of $m \times m$ orthogonal matrices. Let $Q \in \mathbb{SO}(d)$ be such that

$$U = Q^{\mathrm{T}}\mathrm{diag}(v_1(\xi),\cdots,v_d(\xi))Q,$$

where $\mathbb{SO}(d)$ is the group of $d \times d$ orthogonal matrices with determinant equal to 1. Then

$$\begin{cases} \xi = PJQ^{\mathrm{T}}\mathrm{diag}(v_1(\xi),\cdots,v_d(\xi))Q \\ |\xi|^2 = \displaystyle\sum_{i=1}^{d} v_i^2(\xi). \end{cases}$$

As $v(\xi) < \gamma$ there exists $1 \leqslant i_1 \leqslant \cdots \leqslant i_k \leqslant d$, with $k \in \{1, \cdots, d\}$, such that $v_{i_1}(\xi) < \gamma, \cdots, v_{i_k}(\xi) < \gamma$ (and $v_i(\xi) \geqslant \gamma$ for all $i \notin \{i_1, \cdots, i_k\}$). For every $j \in \{1, \cdots, k\}$, let $t_j \in]0,1[$ be such that $v_{i_j}(\xi) = (1-t_j)(-\gamma) + t_j\gamma$. Then

$$\begin{aligned}\mathrm{diag}(v_1(\xi), \cdots, v_{i_1}(\xi), \cdots, v_d(\xi)) = {} & (1-t_1)\mathrm{diag}(v_1(\xi), \cdots, -\alpha, \cdots, v_d(\xi)) \\ & + t_1\mathrm{diag}(v_1(\xi), \cdots, \alpha, \cdots, v_d(\xi)),\end{aligned}$$

and so $\xi = (1-t_1)\xi_1^- + t_1\xi_1^+$ with

$$\begin{cases} \xi_1^- := PJQ^{\mathrm{T}}\mathrm{diag}(v_1(\xi), \cdots, -\gamma, \cdots, v_d(\xi))Q \\ \xi_1^+ := PJQ^{\mathrm{T}}\mathrm{diag}(v_1(\xi), \cdots, \gamma, \cdots, v_d(\xi))Q \\ \mathrm{rank}\left(\xi_1^- - \xi_1^+\right) = 1. \end{cases}$$

Moreover, we have

$$\begin{aligned}\mathrm{diag}(v_1(\xi), \cdots, -\gamma, \cdots, v_{i_2}(\xi), \cdots, v_d(\xi)) = {} & (1-t_2)\mathrm{diag}(v_1(\xi), \cdots, -\gamma, \cdots, -\gamma, \cdots, v_d(\xi)) \\ & + t_2\mathrm{diag}(v_1(\xi), \cdots, -\gamma, \cdots, \gamma, \cdots, v_d(\xi)),\end{aligned}$$

hence $\xi_1^- = (1-t_2)\xi_2^{-,-} + t_2\xi_2^{-,+}$ with

$$\begin{cases} \xi_2^{-,-} := PJQ^{\mathrm{T}}\mathrm{diag}(v_1(\xi), \cdots, -\gamma, \cdots, -\gamma, \cdots, v_d(\xi))Q \\ \xi_2^{-,+} := PJQ^{\mathrm{T}}\mathrm{diag}(v_1(\xi), \cdots, -\gamma, \cdots, \gamma, \cdots, v_d(\xi))Q \\ \mathrm{rank}\left(\xi_2^{-,-} - \xi_2^{-,+}\right) = 1. \end{cases}$$

In the same manner, we obtain $\xi_1^+ = (1-t_2)\xi_2^{+,-} + t_2\xi_2^{+,+}$ with

$$\begin{cases} \xi_2^{+,-} := PJQ^{\mathrm{T}}\mathrm{diag}(v_1(\xi), \cdots, \gamma, \cdots, -\gamma, \cdots, v_d(\xi))Q \\ \xi_2^{+,+} := PJQ^{\mathrm{T}}\mathrm{diag}(v_1(\xi), \cdots, \gamma, \cdots, \gamma, \cdots, v_d(\xi))Q \\ \mathrm{rank}\left(\xi_2^{+,-} - \xi_2^{+,+}\right) = 1. \end{cases}$$

We continue in this fashion, obtaining a finite sequence $\{\xi_j^\sigma\}_{j\in\{1,\cdots,k\}}^{\sigma\in\mathfrak{S}_j} \subset \mathbb{M}^{m\times d}$, where $\mathfrak{S}_j$ denotes the set of all maps $\sigma : \{1, \cdots, j\} \to \{-,+\}$, with the following properties:

- $\xi_j^\sigma = PJQ^{\mathrm{T}}\mathrm{diag}(v_1(\xi), \cdots, \sigma(1)\gamma, \cdots, \sigma(j)\gamma, \cdots, v_d(\xi))$;
- if $\sigma(j) \neq \sigma'(j)$ and $\sigma(l) = \sigma'(l)$ for all $l \in \{1, \cdots, j-1\}$ then $\mathrm{rank}\left(\xi_j^\sigma - \xi_j^{\sigma'}\right) = 1$;
- if $\sigma(1) \neq \sigma'(1)$ then $\xi = (1-t_1)\xi_1^\sigma + t_1\xi_1^{\sigma'}$;
- if $\sigma'(j+1) \neq \sigma''(j+1)$ and $\sigma'(l) = \sigma''(l) = \sigma(l)$ for all $l \in \{1, \cdots, j\}$, then $\xi_j^\sigma = (1-t_{j+1})\xi_{j+1}^{\sigma'} + t_{j+1}\xi_{j+1}^{\sigma''}$.

It follows that:

- if $\sigma(1) \neq \sigma'(1)$ then $\mathcal{R}F(\xi) \leqslant \mathcal{R}F(\xi_1^\sigma) + \mathcal{R}F(\xi_1^{\sigma'})$;
- if $\sigma'(j+1) \neq \sigma''(j+1)$ and $\sigma'(l) = \sigma''(l) = \sigma(l)$ for all $l \in \{1, \cdots, j\}$, then $\mathcal{R}F(\xi_j^\sigma) \leqslant \mathcal{R}F(\xi_{j+1}^{\sigma'}) + \mathcal{R}W(\xi_{j+1}^{\sigma''})$.

From the above we conclude that

$$\mathcal{R}F(\xi) \leqslant \sum_{\sigma\in\mathfrak{S}_k} \mathcal{R}F(\xi_k^\sigma).$$

On the other hand, we have

$$\begin{aligned} v(\xi_k^\sigma) &= \left|\det(\mathrm{diag}(v_1(\xi),\cdots,\sigma(1)\gamma,\cdots,\sigma(k)\gamma,\cdots,v_d(\xi)))\right| \\ &= \gamma^k \prod_{i\notin\{i_1,\cdots i_k\}} v_i(\xi), \end{aligned}$$

hence $v(\xi_k^\sigma) \geqslant \gamma^d \geqslant \gamma$ for all $\sigma \in \mathfrak{S}_k$. Using (A.1)–(A.2) we deduce that

$$\mathcal{R}F(\xi) \leqslant \sum_{\sigma\in\mathfrak{S}_k} \widetilde{\delta}\left(1 + |\xi_k^\sigma|^p + \widetilde{g}\left(|\xi_k^\sigma|^2\right)\right).$$

Moreover, we have

$$\begin{aligned} |\xi_k^\sigma|^2 &= \left|\mathrm{diag}(v_1(\xi),\cdots,\sigma(1)\gamma,\cdots,\sigma(k)\gamma,\cdots,v_d(\xi))\right|^2 \\ &= k\gamma^2 + \sum_{i\notin\{i_1,\cdots,i_k\}} v_i^2(\xi) \\ &\leqslant d\gamma^2 + |\xi|^2, \end{aligned}$$

and so $|\xi_k^\sigma|^p \leqslant 2^{\frac{p}{2}} d^{\frac{p}{2}}\gamma^p + 2^{\frac{p}{2}}|\xi|^p$. Consequently, by using $(\widetilde{\mathrm{a}}_1)$–$(\widetilde{\mathrm{a}}_2)$,

$$\begin{aligned} \mathcal{R}F(\xi) &\leqslant \sum_{\sigma\in\mathfrak{S}_k} \widetilde{\delta}\left(1 + 2^{\frac{p}{2}} d^{\frac{p}{2}}\gamma^p + 2^{\frac{p}{2}}|\xi|^p + \gamma_2\widetilde{g}(d\gamma^2) + (\gamma_1\widetilde{g}(d\gamma^2) + \gamma_2)\widetilde{g}(|\xi|^2)\right) \\ &\leqslant \beta(1 + G(\xi)) \end{aligned}$$

with $\beta = 2^d\widetilde{\delta}\max\left\{1 + 2^{\frac{p}{2}} d^{\frac{p}{2}}\gamma^p + \gamma_2\widetilde{g}(d\gamma^2), 2^{\frac{p}{2}}, \gamma_1\widetilde{g}(d\gamma^2) + \gamma_2\right\}$, and the proof is complete. ■

Remark A.10. When $m = d$, it is easy to check that $v(\xi) = |\det\xi|$ for all $\xi \in \mathbb{M}^{d\times d}$. Consequently, if F satisifies $(\widetilde{\mathrm{A}}_{\det})$ then $\mathcal{R}F$ has convex growth.

A.5. **A useful result concerning the determinant.** The following lemma is a special case of a theorem due to Dacorogna and Ribeiro [DR04, Theorem 1.3] (see also [Dac08, Theorem 10.29, pp. 462]).

Lemma A.11. *Let $s_1 < s_2$ and $\xi \in \mathbb{M}^{d\times d}$ with $\det\xi \in]s_1, s_2[$. There exists $\varphi \in W_0^{1,\infty}(]0,1[^d;\mathbb{R}^d)$ such that $\det(\xi + \nabla\varphi(y)) \in \{s_1, s_2\}$ for $\mathscr{L}^d$-a.a. $y \in]0,1[^d$.*

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