



Observations on the hex clusters of the Spectre tilings

Arnaud Chéritat

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Observations on the hex clusters of the Spectre tilings

Arnaud Chéritat*

Decorating the Spectre tile with hexagons reveals triangular hexagonal clusters whose structure we study. In the process we reprove that the Spectre tilings exist and are uniquely hierarchical. The proof is not computer-assisted.

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1 Introduction

This article focuses on the second aperiodic monotile discovered by Smith *et alii*, the chiral one called Tile(1,1) a.k.a. the Spectre¹ and announced in [Smi+23a].

The tile has a shape that is (bounded by) a non-convex 13-gon, which, as the authors explain, is better viewed as a 14-gon whose edges all have the same length but which has one flat angle.

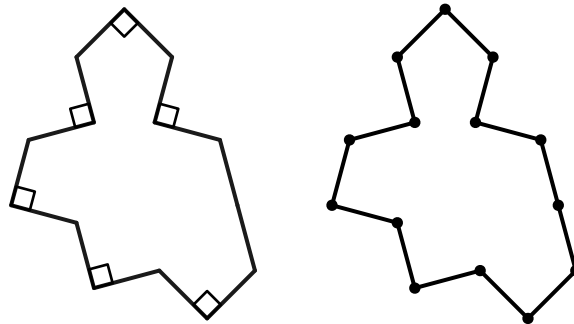


Figure 1 – The Spectre. The 7 non-right or flat angles measure 120° .

It was proved in [Smi+23a] that it can tile the plane without reflections, but only aperiodically. The key ingredient is that tilings with this shape are *uniquely*

¹We deviate slightly from the terminology of [Smi+23a], see Footnote 2.

hierarchical. We will not reprove this classical implication but among other things we give an independent proof of the unique hierarchical nature of tilings with the Spectre.

Here is a sample of a whole plane tiling:

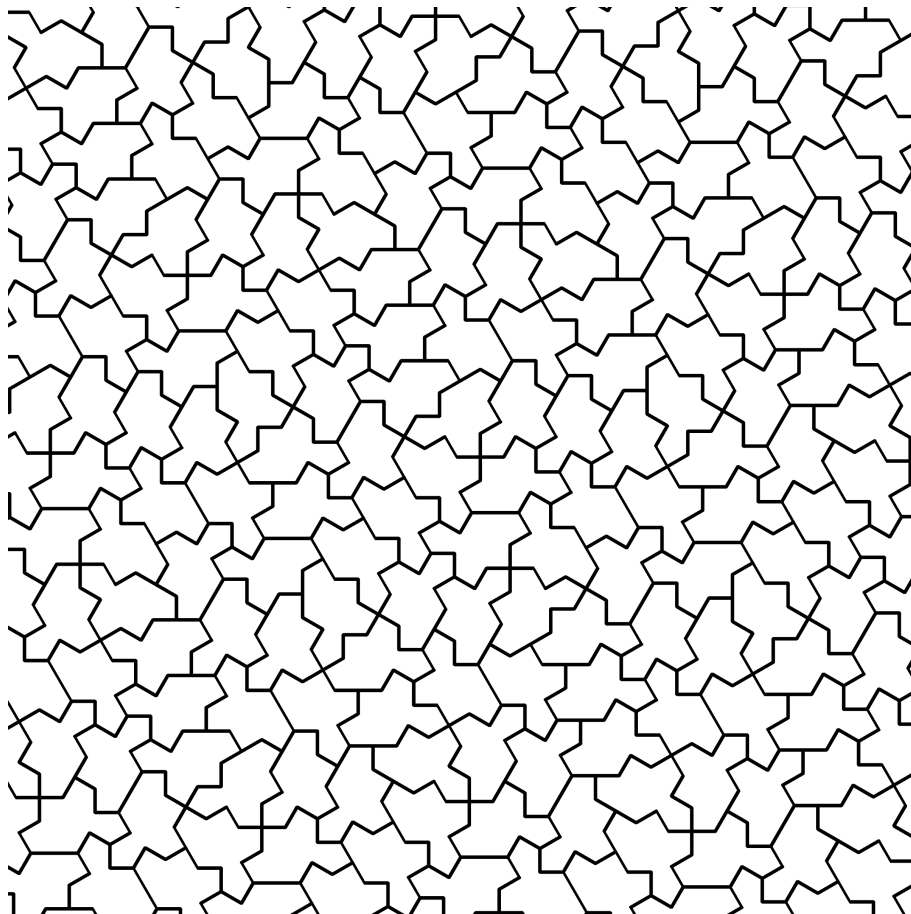


Figure 2

It was obtained by applying the construction algorithm explained in [Smi+23a], Figure A1, page 21, to construct a finite patch that is ensured to be extendable into a whole plane tiling.

In such tessellations, we do not allow ourselves to reflect the tile.² In [Smi+23a], it was proved that all tiles appear in 12 possible orientations differing of a turn/12, with a majority of them differing only by a turn/6. They are called the **even** tiles and the other ones are called the **odd** tiles, which is coherent with their bearing w.r.t. a reference even tile being written as $k \times 30^\circ$, with k even or odd. The asymptotic proportion of odd tiles is known (see [Smi+23a]) and is between $1/8$ and $1/9$.

²In [Smi+23a] the authors explain how to modify the tile outline so as to prevent reflected tiles to assemble with unreflected tiles. The result is actually what they call a Spectre. Here, since mixing the tile with its reflection is never considered, we also call Spectre the undeformed Tile(1,1).

The construction can be pushed to get a tiling as big as one wants, for instance:

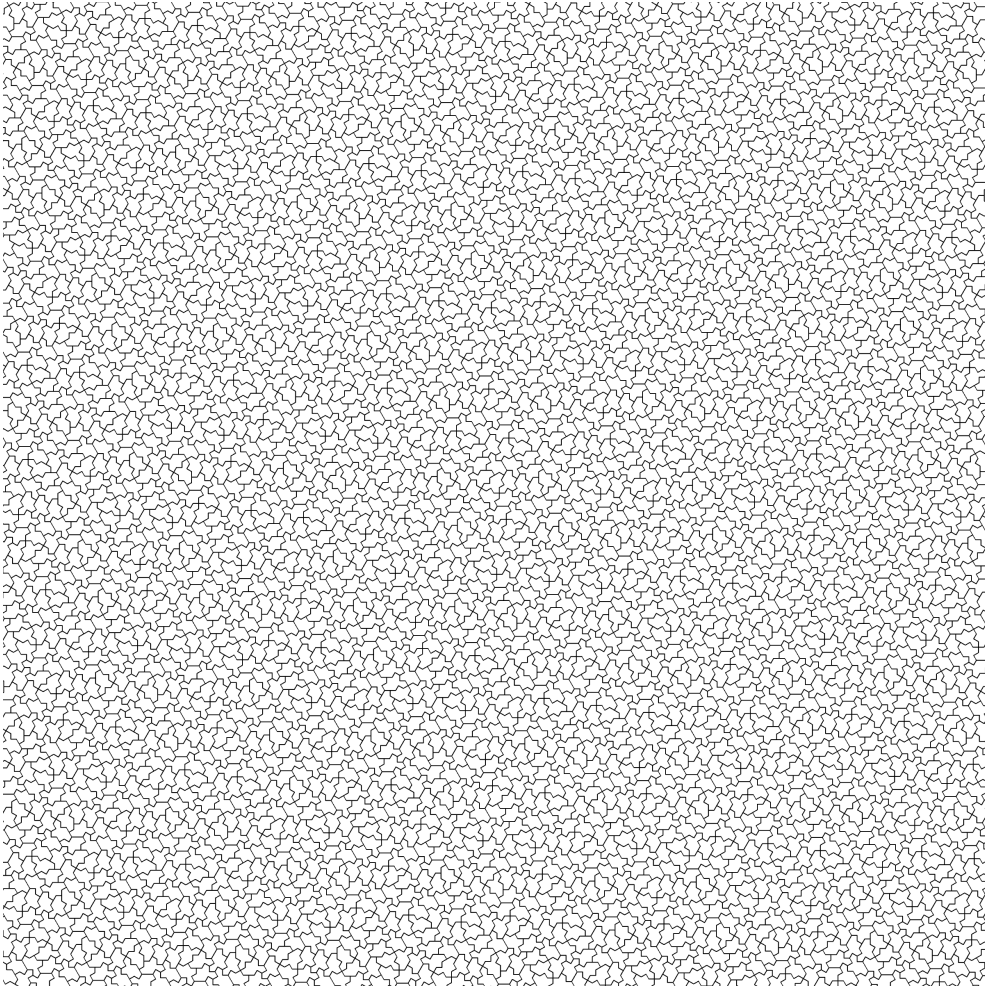


Figure 3

Without further markings, it is hard to distinguish any structure. In [Smi+23a] the authors investigate this structure to prove that the shape can tile the whole Euclidean plane, but also that no whole plane tiling can have a translation period.

Yoshiaki Araki³ realized one could decorate Tile(1,1) as follows:

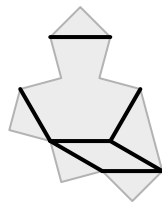


Figure 4

³Inspired by a remark Craig Kaplan made on another decoration of the tile by Dale Walton.

so as to reveal a beautiful subjacent tiling into hexagons, squares and rhombs:

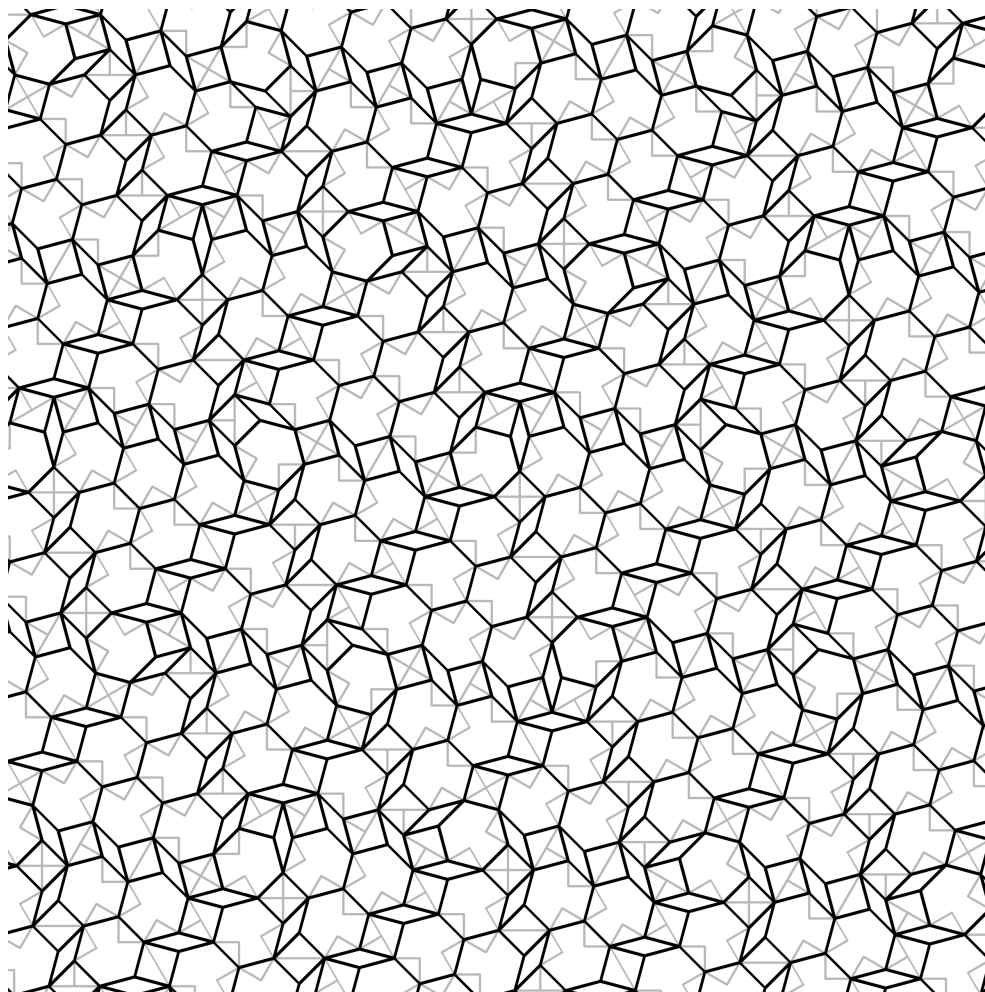


Figure 5

Here is the same picture but showing the hexagons only, using yellow for odd

tiles and blue for even ones:

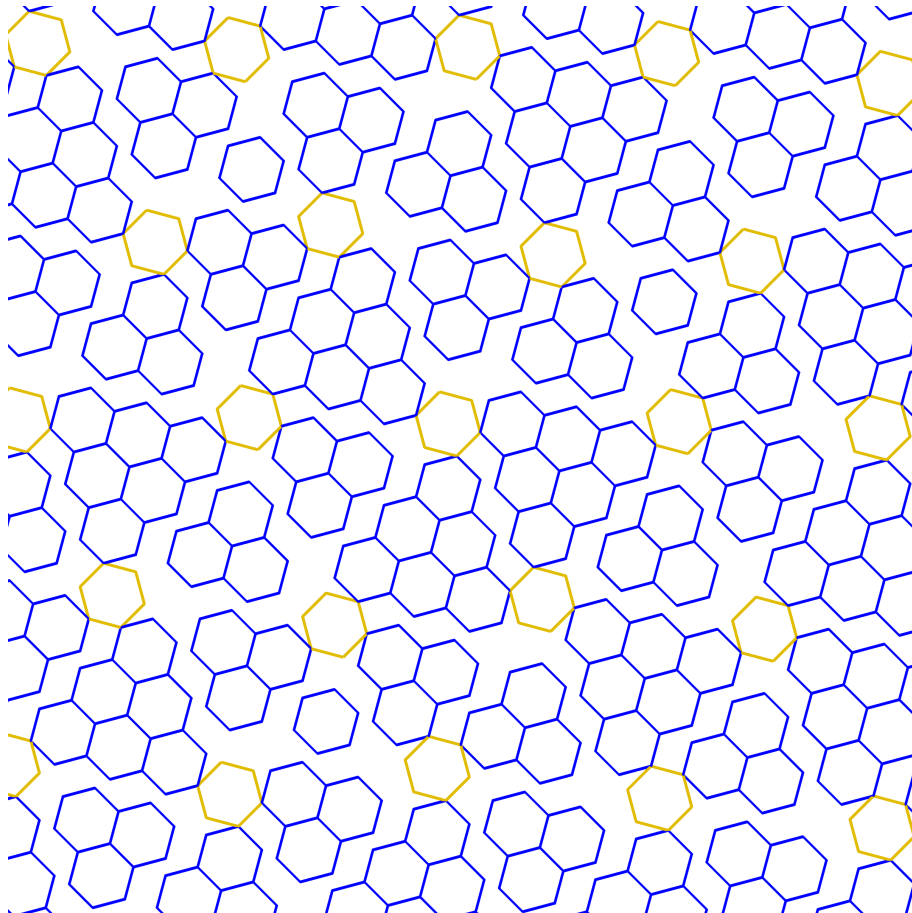


Figure 6

In the sequel we will often abbreviate hexagon as **hex** and hexagons as **hexes**. An interesting blue/yellow colouring of the whole hex/rhomb/squares graph is possible using the left decoration for even tiles and the right one for odd tiles:

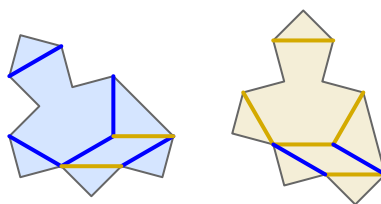


Figure 7

Varying the shade of blue of Spectres to distinguish better their boundaries, the

decorated Spectre tiling looks as follows:

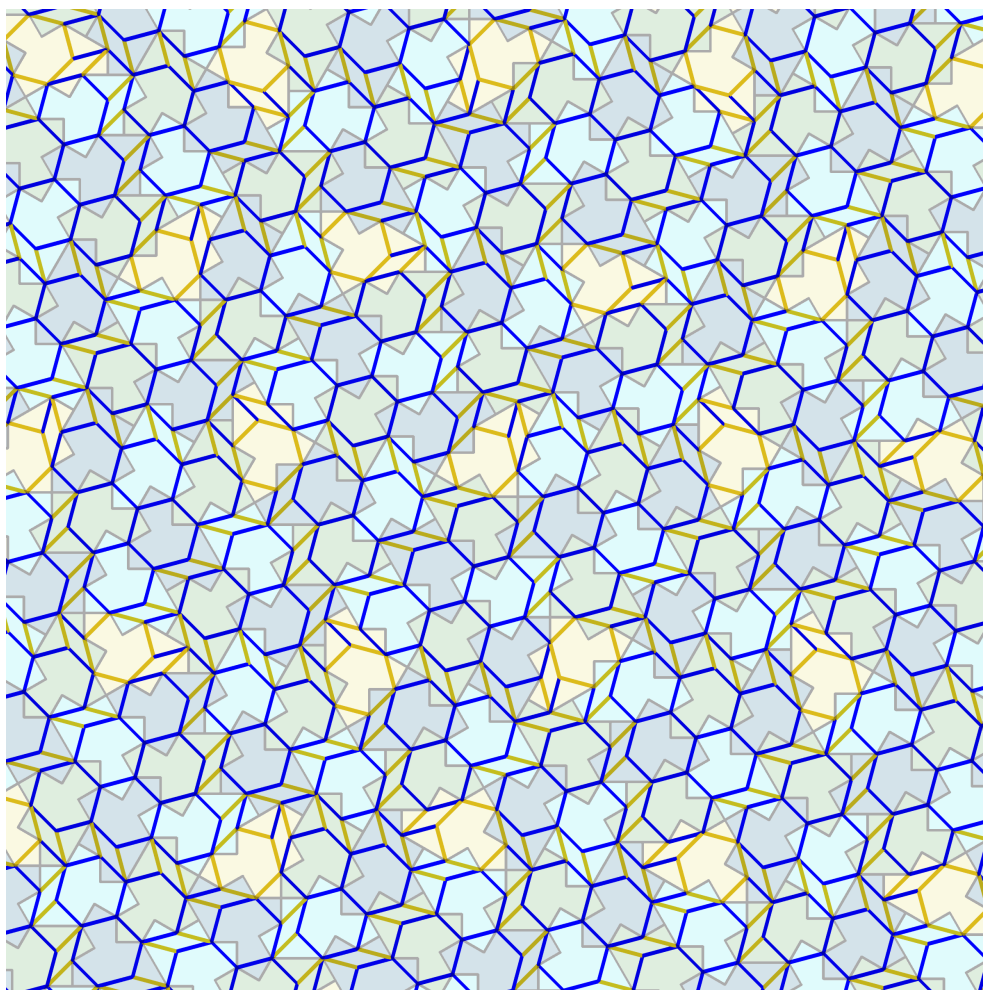


Figure 8

Let us show the decorations only on a different and bigger patch:

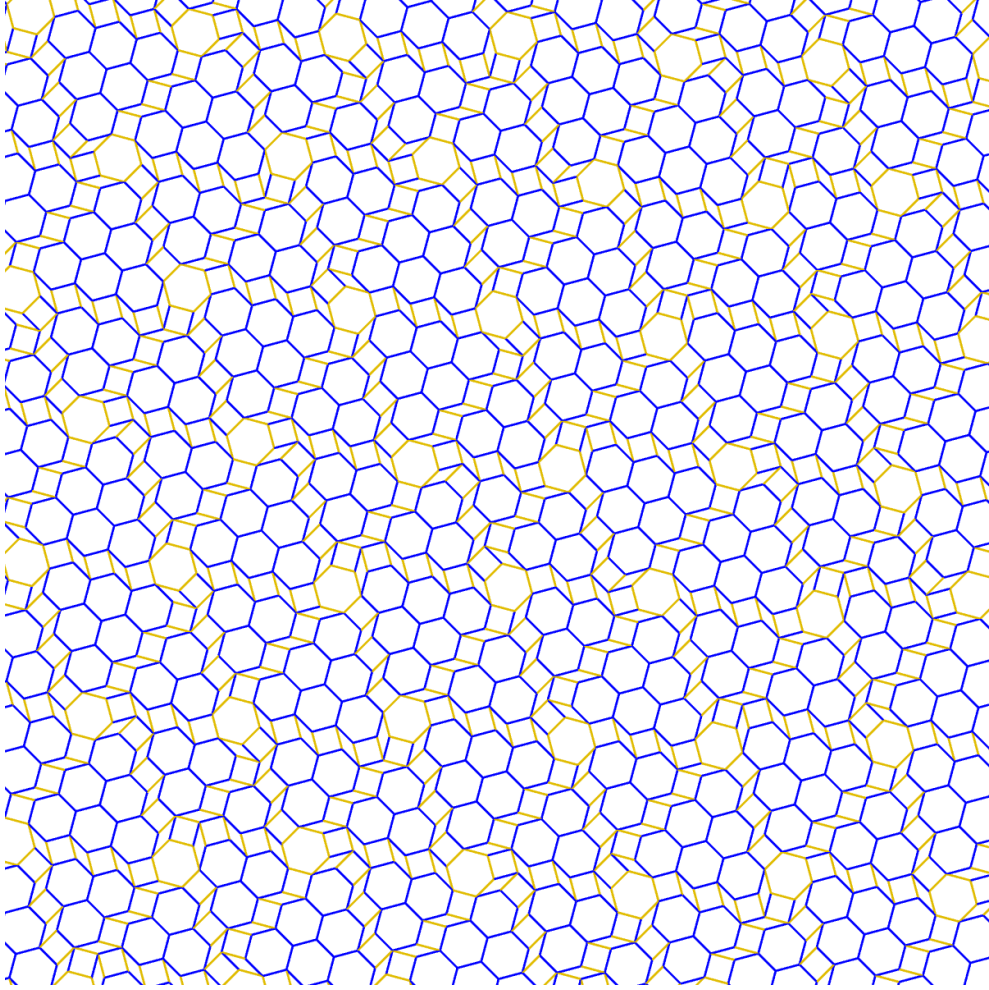


Figure 9

We observe that the blue graph seems to have only bounded connected components made of hexagons that are isolated or arranged in triangles, with or without antennae.

Definition. We will call *cc* the connected components of the blue graph.

Each *cc* seems to be in contact with three yellow hexagons and each yellow hexagon seems to be in contact with 6 blue components, one at each of its corners. It is as if the blue *cc*'s correspond to the triangles in a regular triangular tessellation of the plane which would have been deformed (such a correspondence is qualified as *combinatorial*), and the yellow hexagons to the vertices of such a tessellation.

The blue and yellow segments all have the same length. If we multiply the

length of the yellow segments by a common factor to make them bigger we get:

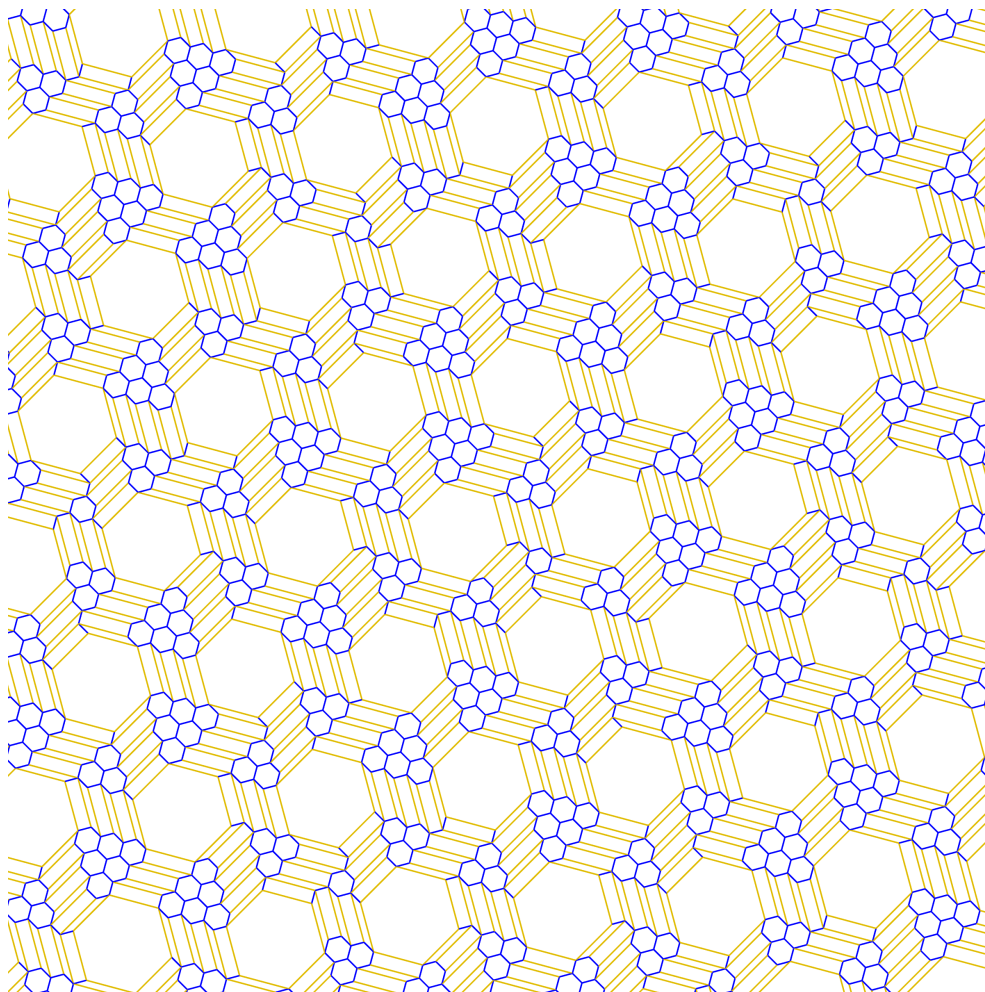


Figure 10

In a way dual to the previous, the yellow hexagons would correspond to a regular hexagonal tessellation of the plane and the blue cc's to its vertices.

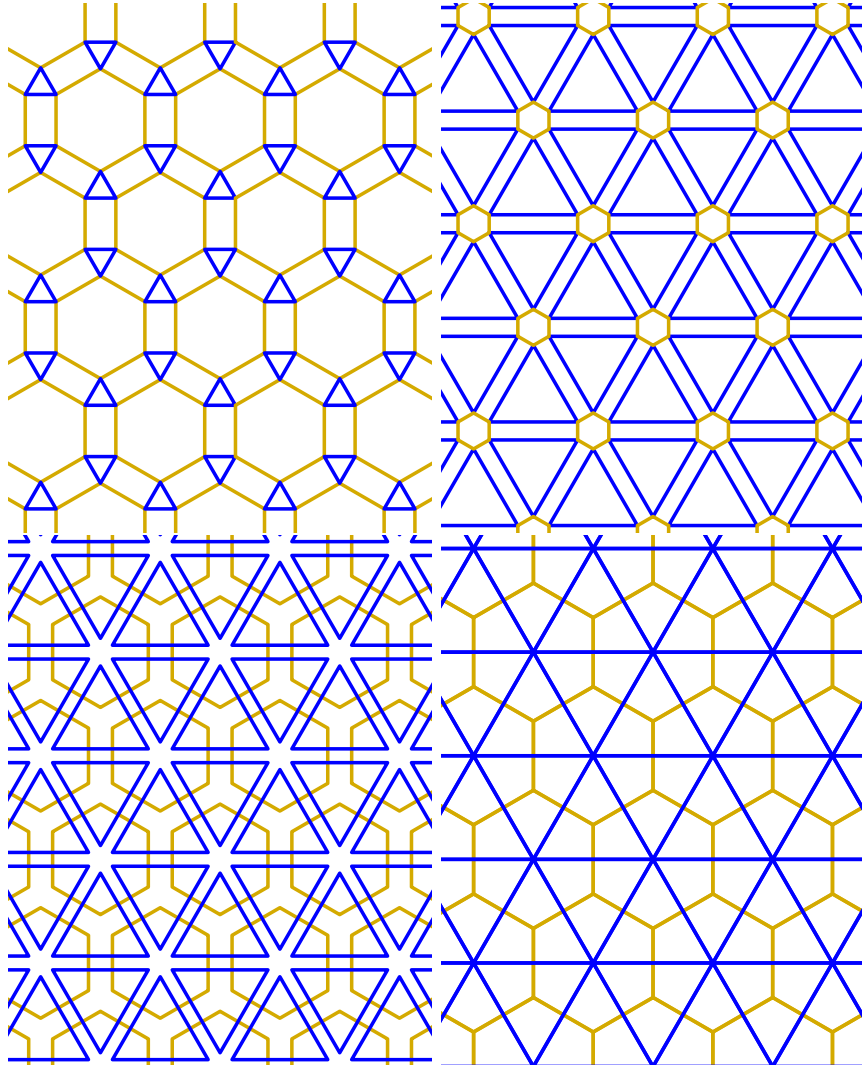


Figure 11 – Four ways to represent duality between blue triangles and yellow hexes.

In this article we will call **honeycomb** the regular hexagonal tessellation of the whole plane.

When squeezing the blue segments the yellow hexes seem to compact into a honeycomb. If instead we squeeze the yellow segments, the blue hexes seem to

compact into a honeycomb too.

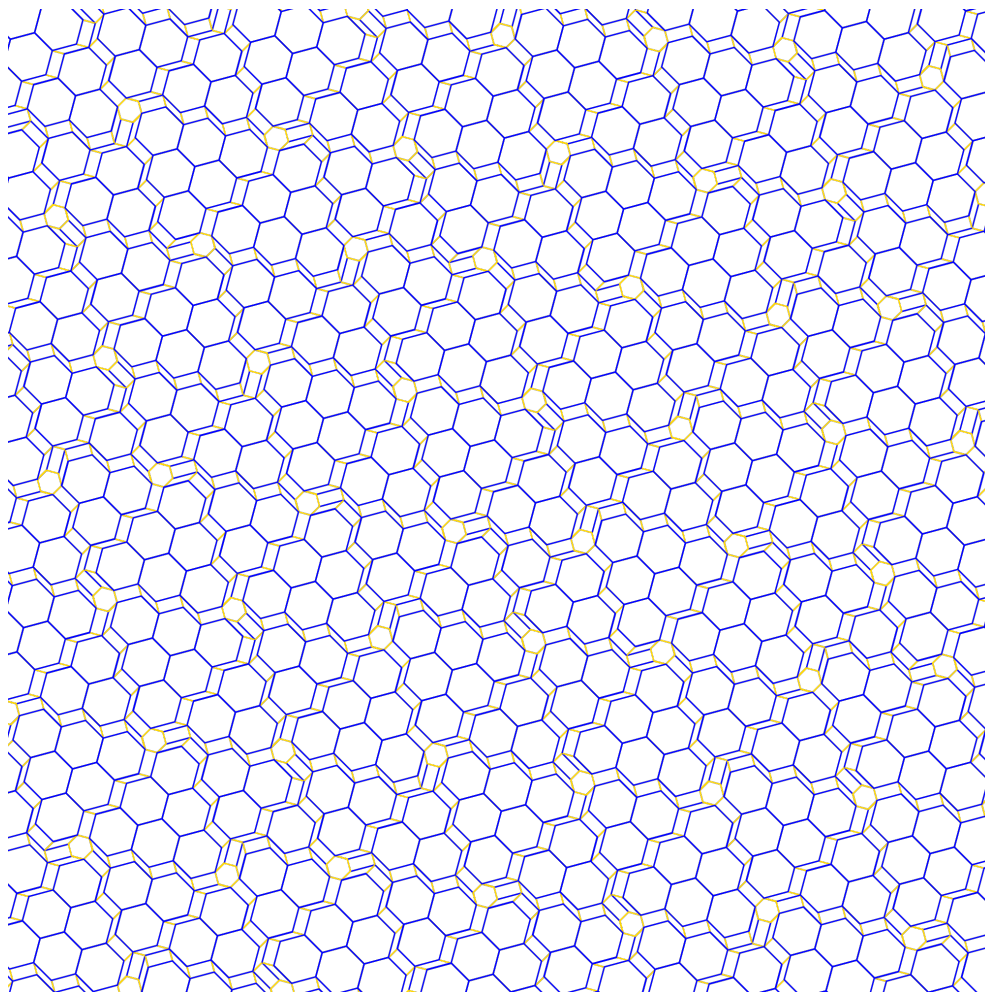


Figure 12

Let us call **cluster** the maximal groups of blue hexes that touch each other. Clusters are subsets of cc's, possibly strict subsets. We observe that each cluster comes into sets of $T_1 = 1$, $T_2 = 3$ or $T_3 = 6$ hexes arranged in a triangle. We will denote them T1, T2, T3 where T stands for triangle. Each cc seems to contain exactly one cluster.

These observations will be proved in Proposition 7. If they are true, clusters and cc's are in one-to-one correspondence, and also in one-to-one correspondence with the vertices (corners) of a yellow honeycomb. Let us call *HY* the latter tessellation. Each T2 or T3 cluster come in two possible orientations, and in fact this seems to follow the orientation of the triangle it corresponds to in the dual tessellation to *HY*.

This hints at the fact that the T1 clusters also have a natural orientation, which we will justify too.

Colouring the regular blue hex tiling w.r.t such triangle sets gives pictures like:

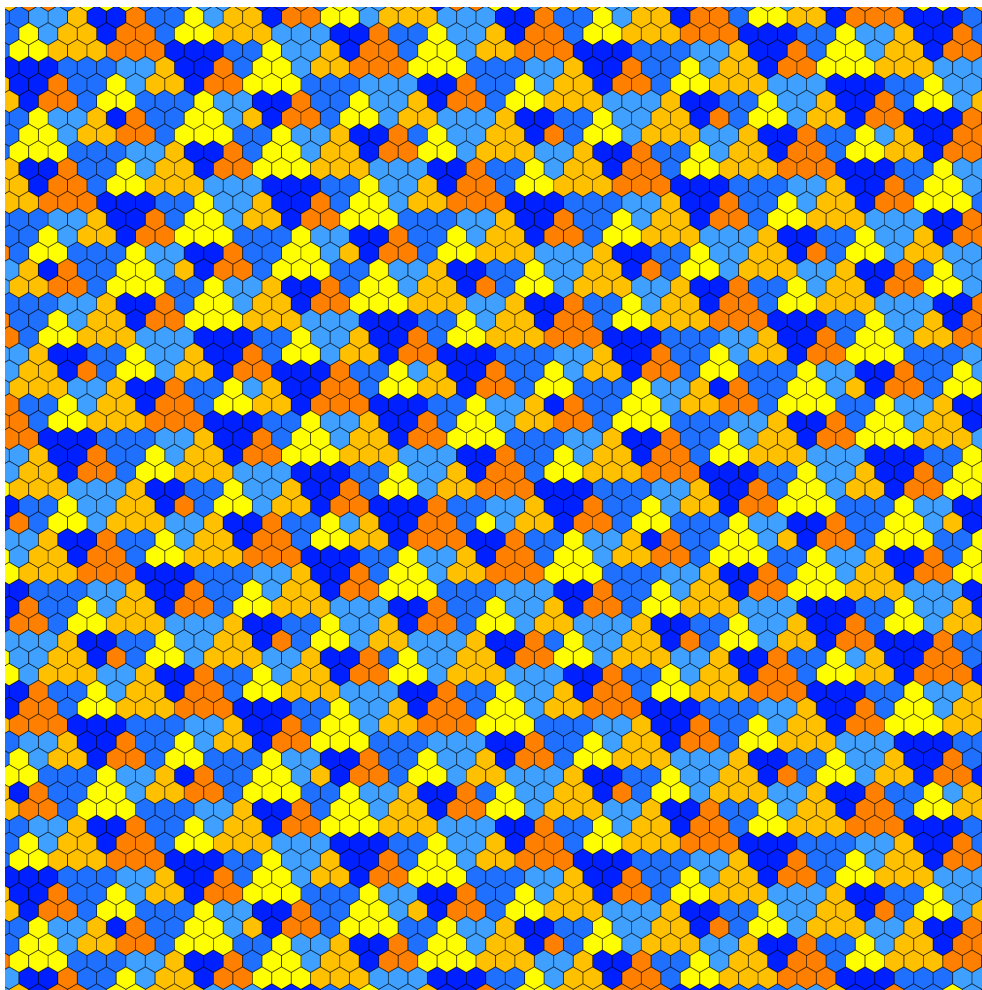


Figure 13

In this article, we investigate the self-similar structure of this partition.

The key word self-similar is not that much used in the field as *hierarchical structure*. Many proofs of existence of whole plane tilings, for the Spectre and Hat monotiles [Smi+23a; Smi+23b] and before them of many other non-periodic tilesets, including Penrose tiles, are done via substitution systems, where a finite arrangement of shapes is used and replaced by an new arrangement of the same shapes, and the procedure repeated to cover large pieces of the plane. The proof of aperiodicity is usually done by reversing the procedure and proving that any whole plane tiling can be de-substituted. But this is also the case of the periodic tiling by the square for instance. A key point in aperiodicity is *uniqueness* of the way the de-substitution can be done, and tilings with this property are called *uniquely hierarchical*. In general, unique hierarchy automatically provides a substitution system.

For the Spectre and the Hat, substitution systems and/or unique hierarchy has been proved in several ways, both by the original authors and by others. See

for instance the works of Shigeki Akiyama, Yoshiaki Araki, Erhard Künzel and James Smith, some of which are available as [AA23; Smi24]. Unpublished work of Künzel include a substitution system in terms of tree structure. Unpublished work of Pieter Mostert uses edge substitution.

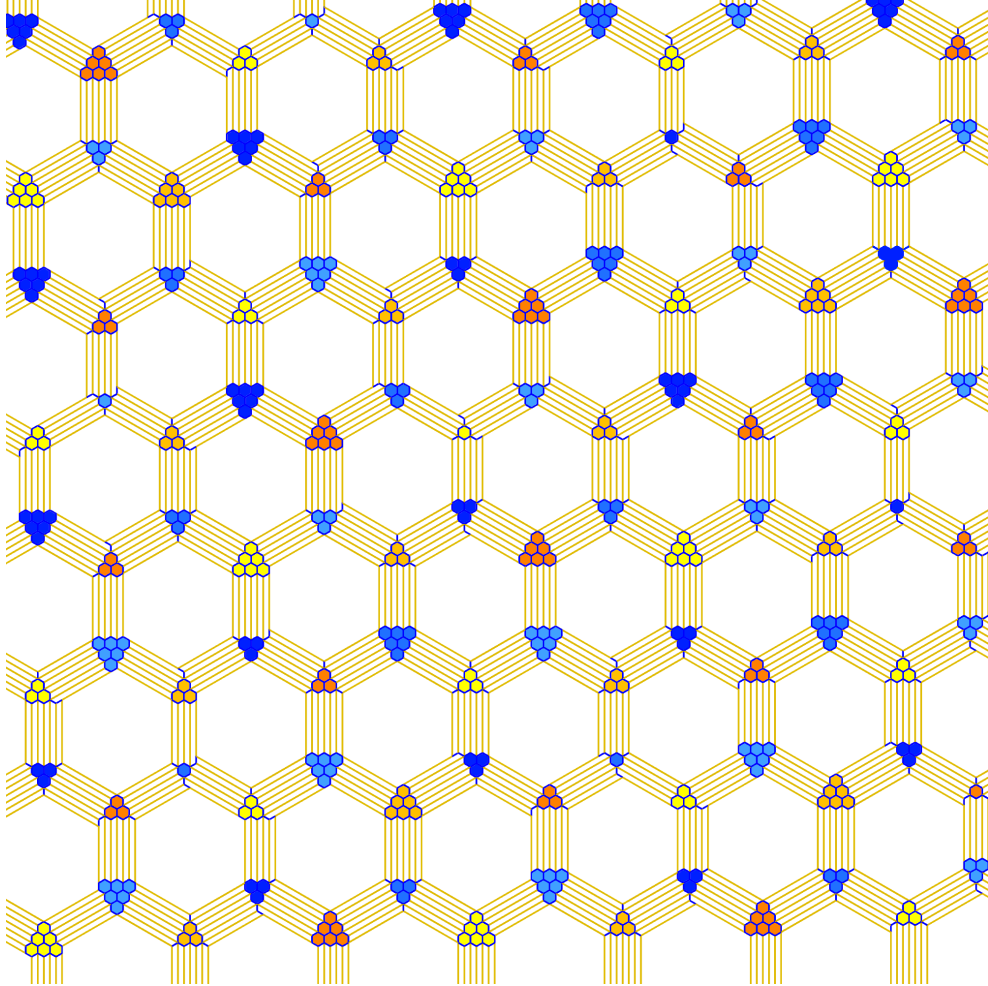


Figure 14 – This picture uses the colour scheme of Figure 13 but burst as in Figure 12, and here the blue lines have been turned in one direction and the yellow lines in the other direction, for better alignment. Incidentally it has some aesthetic value.

In [Smi+23a], the authors partitioned the tiles in any whole plane tiling by the Spectre into patches of 8 or 9 tiles, called H8 and H9, each containing exactly one odd tile. They form up to deformation a honeycomb, arranged exactly like HY . The blue Tn partition and the H8/H9 partition are sort of dual to each other. We investigate this duality further in Section 2.14.3

When passing from a tiling with the Spectre to the yellow/blue graph tracing hexes, squares and rhombs, we a priori lost information. The orientation of the Spectre associated to a hex in a Tn or to a yellow hex is only retained up to a

rotation of a sixth of a turn: we only know if it is even or odd (i.e. its class modulo 2 in $\mathbb{Z}/12\mathbb{Z}$). But maybe the arrangement of the nearby hexes forces its value? We will see that this is the case, provided we use the yellow hexes and retain how they are attached to nearby clusters/cc's.

Then we compacted the blue hexes into a partition of the honeycomb in sets of type T1, T2, T3. Doing so we removed the yellow hexes from the picture and a priori lost two more pieces of information: which blue cc's have antennae and how the yellow hexes are attached to the blue cc's. During the compaction, the 6 corners of a yellow hex are merged into a single vertex of the blue honeycomb. What we can do to keep a piece of information is to locate where these vertices are: call them **dots**. It turns out that their position is enough to recover the two pieces of information above: antennae and yellow hex position in the yellow/blue decorated tiling graph.

But what if we do not put the dots?

Question. Can the tiling be deduced from the partition only?

We will see in Section 4 that the answer is: not always. The process cannot be purely local. And we may end up with situations where there are more than one solution.

But first we can observe (and we will prove) that there is another constraint on the orientation of the tiles. Recall that the Tn clusters seem to be in correspondence with the triangles of a the tessellation of the plane by regular triangles, dual to HY (see Figures 10 and 14 again). These triangles come in two orientations and the T2 and T3 seem to mimic it. Let us call **point up** and **point down** these two orientations. For a fixed orientation of the triangle, it seems that an even tile has its hex in a Tn cluster, and the orientation of the tile seems to be constrained to only three of the 6 possible orientations, differing by multiples of a third of a turn, if the corresponding triangle points up; or to the three other even orientations, if the triangle points down. By the way, this gives a natural interpretation of pointing up or down for the T1 clusters too.

Here of an extract is the condensed hex decomposition of a whole plane Spectre tiling, with supplementary information: clusters, dots making the condensed yellow hexes, and a line in the blue hexes indicating the orientation of the underlying spectre as on the figure below:

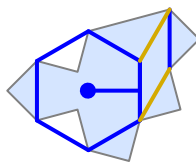


Figure 15

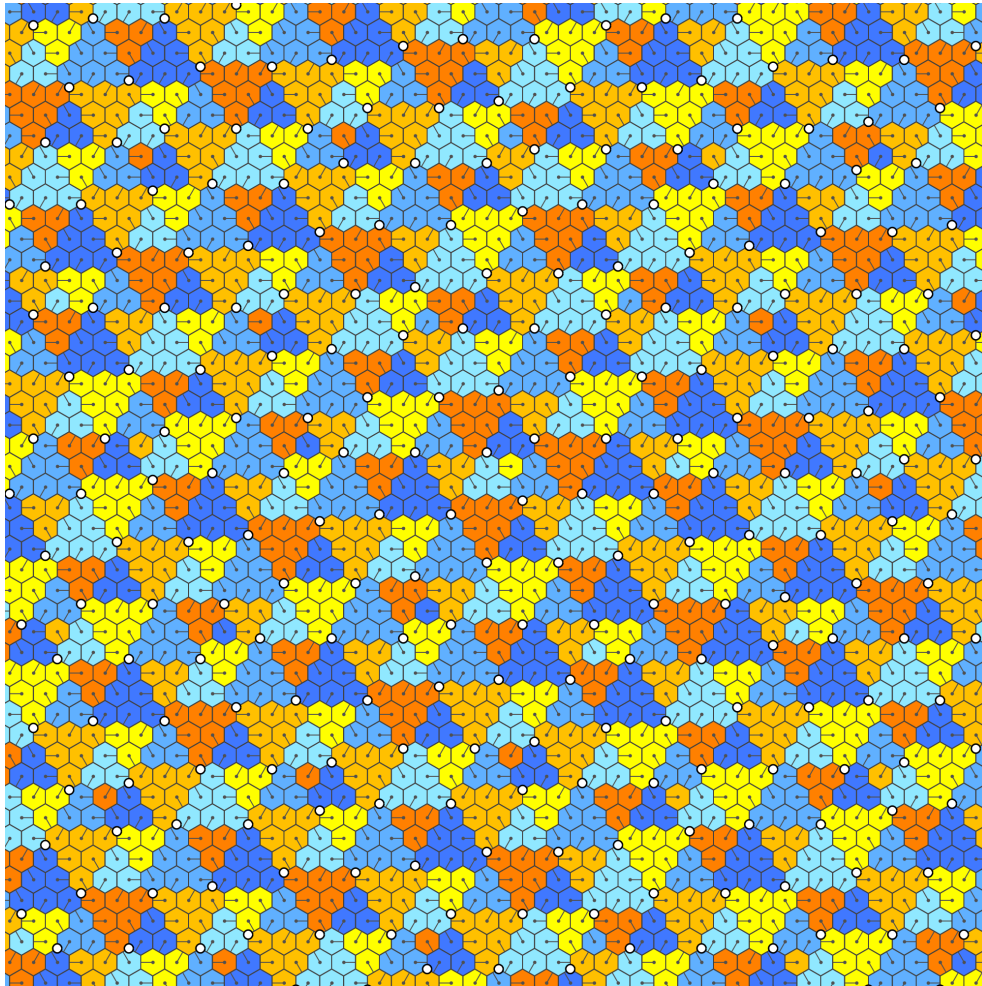


Figure 16

The authors of [Smi+23a] proved that tilings with the Spectre are (uniquely) hierarchical. More precisely the H8 and H9 patches are arranged in a way so that they can be themselves grouped in bigger patches that follow the same kind of arrangement, so can be grouped themselves further, infinitely many times for a whole plane tiling.⁴ The even Spectre tiles themselves can be added at the bottom of this hierarchy, but the odd tiles have to be disregarded.⁵ However, as centres of the H8 and H9 patches, they correspond in some way to the first level of patches.

It is thus to be expected that the dots in Figure 16 actually come into their own triangular clusters of 1, 3 or 6 dots. And we can observe that if we colour them according a bipartite colouring of the vertices of the blue hex graph, then they seem to be coming in such clusters, which moreover have a nice geometrical property: *they trace the vertices of regular triangles*. In the picture below, we removed some information for more visibility.

⁴There is a subtlety here: the H9 patches have to be marked as distinct according to the way they touch their neighbouring patches, yielding 8 different kinds of H9, see Section 4 of [Smi+23a], especially Figures 4.1 and 4.2.

⁵The authors of [Smi+23a] decided to merge them with a specific adjacent even tile yielding a piece they called the *mystic* and that plays the role of an H8 patch while the other even tiles play the role of an H9.

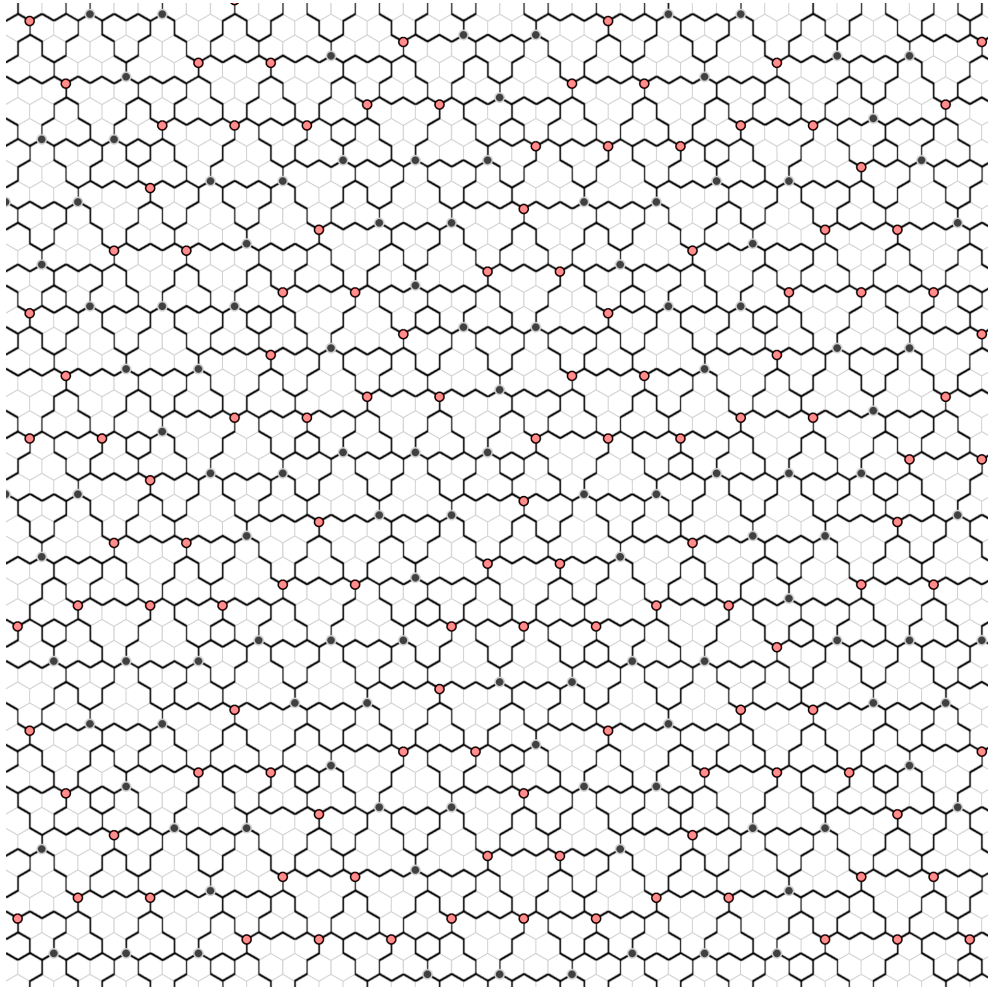


Figure 17 – We invite the reader to spot the regular triangles traced by dots of a given colour. A solution is given in the next figure.

The triangular clusters in which the yellow hexagons group seem to be *combinatorially* (as opposed to *geometrically*) dual to a, say green, honeycomb, i.e. they can be associated to the vertices of such a graph: see Figure 18 (more on this in Section 2.10.3). If we choose the green honeycomb HG to be in the same scale and orientation as the blue one HB , there seems to be a formula relating, for a given yellow hex h :

- the vertex b in HB of the dot associated to h
- the yellow hex centre y of h in HY
- the vertex g in HG associated to the yellow cluster h is in.

If we use complex numbers $a, b, c \in \mathbb{C}$ to locate these points, the formula should look like

$$b = 3.y + j.g + \text{cst}$$

where $j \in \mathbb{C}$ is the principal 3rd root of unity, cst is some constant depending on the position of the origin in HB , HY and HG , and provided we chose the orientation of HB , HY and HG appropriately. We prove this in Proposition 56

In particular the green coordinate could be deduced as a simple linear combination of the blue and yellow coordinates.

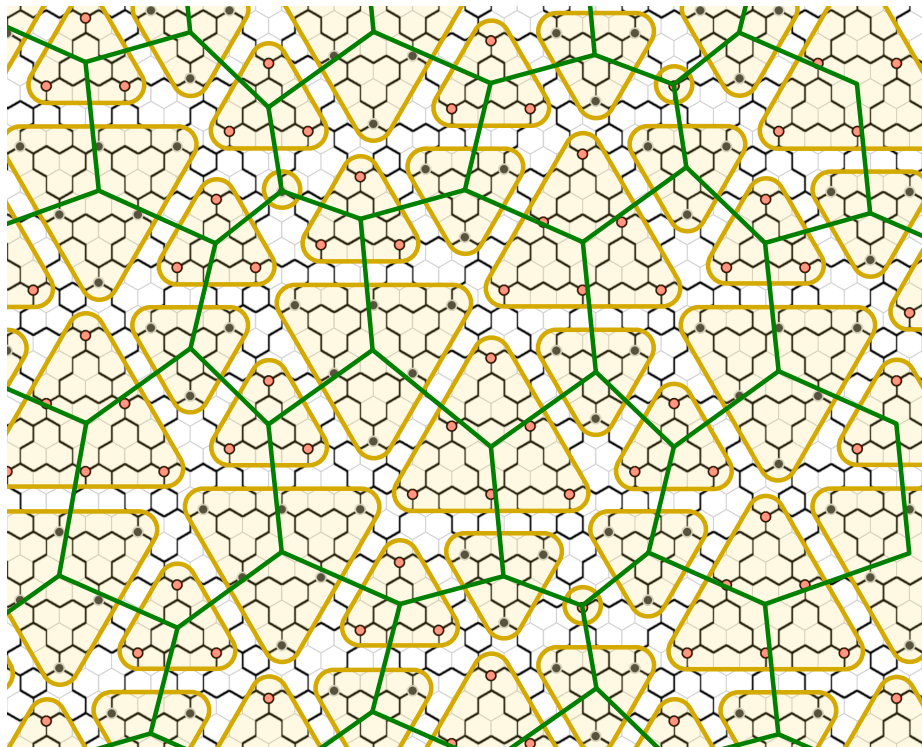


Figure 18

1.1 Acknowledgements

I would like to thank, besides the authors of [Smi+23a; Smi+23b], David Smith, Joseph Samuel Myers, Craig S. Kaplan, and Chaim Goodman-Strauss, discoverers of this wonderful tile, the following people with whom I had many discussions: Nan Ma who boosted my interest in the subject by explaining us how to lift the tiling to 4D space (not used in the present article), and with which I started a more in-depth study of the tiling: see [CNM23]. Pieter Mostert, who revealed to me the importance of the hexagons and some insight about the cut-and-project methods (not used in the present article). Several persons in a newsgroup discussion forum, including Yoshiaki Araki, George Baloglou, Maurizio Paolini, Joshua Socolar and Dale Walton.

The free program `Inkscape` was extremely useful in producing the many pictures of the present document. The figures make extensive use of colour and I apologize to the colour-blind people, for they will have to use specific computer programs of physical filters to distinguish some of them.

This research was possible thanks to the great freedom left by the CNRS to its researchers, and the facilities of the Toulouse Mathematics Institute at Université de Toulouse.

For the Spectre and the Hat, existence and aperiodicity of tilings has been proved in several ways, both by the authors of [Smi+23b; Smi+23a] and by oth-

ers. The original articles used computer assisted searches in some parts. See for instance the works of Shigeki Akiyama, Yoshiaki Araki, Erhard Künzel and James Smith, some of which are available in [AA23; Smi24], for examples of non computer-assisted proofs of existence for Hat and Spectre, and of aperiodicity for Hat tilings. We believe that having a substitution system, which proves existence of whole plane tilings with a monotile, is a strong step towards the proof of aperiodicity of all tilings with this monotile. Yet, to our knowledge, there was no announced non computer assisted proof of aperiodicity of Spectre tilings before the present article.

2 Analysis

From now on all angles are expressed in terms of fractions of a full turn. For instance $1/12$ means $1/12$ -th of a turn, i.e. $360^\circ/12 = 30^\circ$.

We call **honeycomb** a tessellation of the whole plane by regular hexagons.

The word **orientation** will often be used to designate the bearing of an object, i.e. the amount it has been turned by, with respect to a reference position. This is not to be confused with other classical uses of this term in mathematics: for instance given a segment, we can give it a preferred direction, i.e. decide which of its vertices is first and which is last. Here we call this a **directed** segment instead of *oriented* segment. With this terminology, a segment's orientation is a number modulo $1/2$ (of a turn) and an oriented segment's orientation is a number modulo 1.

2.1 The Spectre and the D2 and D3 tiles

People observed that the Spectre shape splits into two shapes, both mirror symmetric and rotationally symmetric of order respectively 3 and 2. They are sometimes called D3 and D2 by reference to their symmetry group and we will use this denomination here.

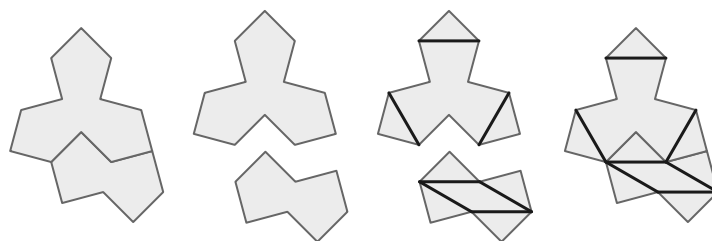


Figure 19 – Splitting of the Spectre into pieces D3 (top) and D2 (bottom). Splitting of the Spectre decoration.

Any tiling by the Spectre (with or without reflections) gives a tiling by D3 and D2, as on Figure 20. But the converse is false: notwithstanding the fact that D3 or D2 alone can tile the plane, there are also tilings mixing the two but for which there are on average more tiles of one type than the other, as on Figure 21, whereas to come from a Spectre tiling, the D3 and D2 must come in pairs (attached in a specific way).

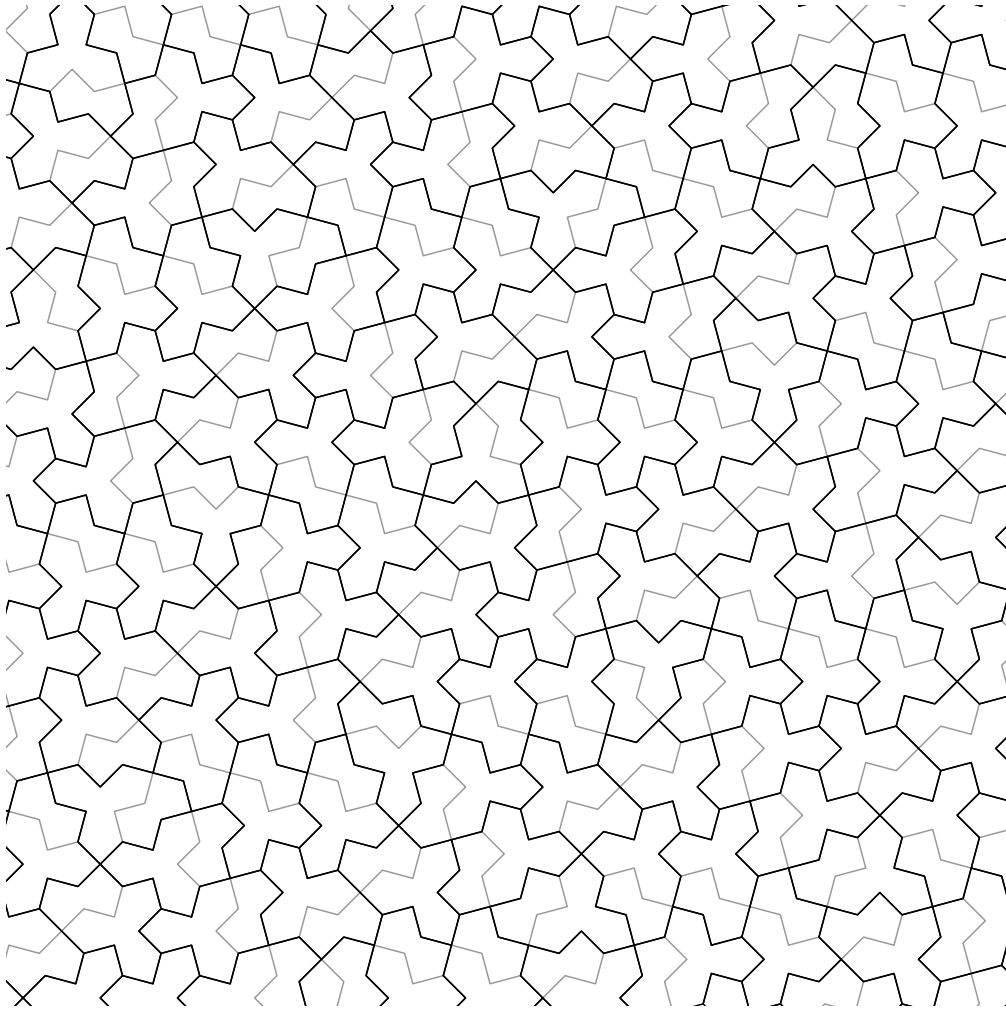


Figure 20 – D3 and D2 tiling coming from a whole plane Spectre tiling.

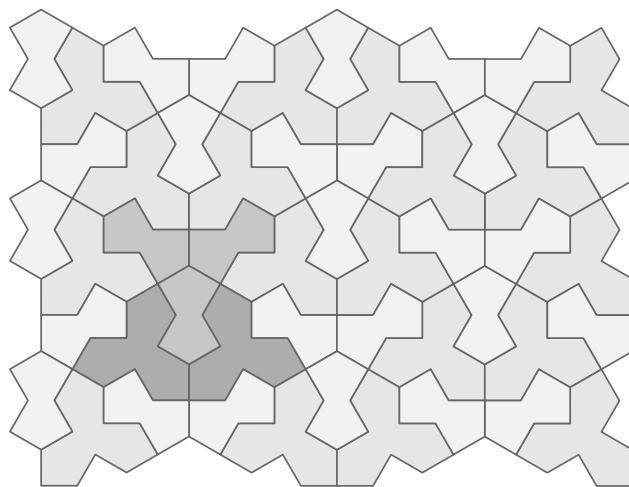


Figure 21 – A periodic tiling with D3 and D2, for which there are on a fundamental domain (indicated in darker shades) 3 tiles of shape D2 for 2 tiles of shape D3.

Shape D3 can be constructed from a regular hexagon by placing along each edge the long side of an isosceles right triangle (angles $1/8, 1/8, 1/4$). They are alternatively placed inside and outside and we call them in/out dents. The same holds for D2, starting from a rhomb (with small angle $1/12$) but its 4 dents are placed outside.

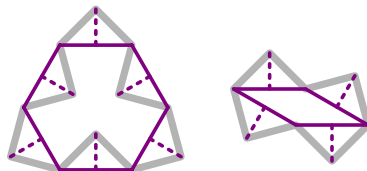


Figure 22

The internal angles of the shapes D3 and D2 are all equal to $1/4, 3/4$ and $1/3$ for the first and $1/4, 1/3$ and $2/3$ for the second. In any D3 and D2 tiling, the vertices must match with vertices, edges with edges. Moreover a vertex with angle $k/4$ can only match with a $k'/4$ and a $k/3$ with a $k'/3$.

In particular the inward dents of a D3 can only receive an outward dent of a D3 or D2, as illustrated below:

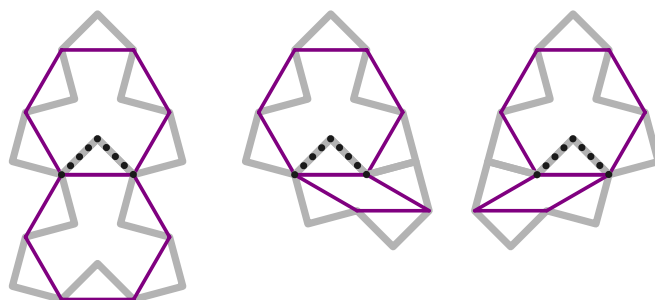


Figure 23 – The three possible ways an inward dent of a D3 can be completed.

The rhomb from which D2 is constructed is also its decoration. The decoration of D3 traces only three sides of the hexagon from which it is constructed. However by the above, in any whole plane tiling by D3 and D2, the decoration of the adjacent tiles filling the three inward dents must complete the hexagon (see Figures 19 and 23).

Notice that on both shapes D3 and D2, the vertices whose angles are multiple of $1/4$ alternate with those whose angles are multiple of $1/3$. Note also that the decoration vertices are exactly those vertices of the two shapes whose angle is a multiple of $1/3$.

Proposition 1. *Consider any whole plane tiling by the shapes D3 and D2. Their decorations trace a graph cutting the plane into regular hexagons associated to the D3 pieces, rhombs associated to the D2 pieces, and squares, filled with 4 dents.*

Proof. Any cut part P which is not the hex of a D3 or the rhomb of a D2 is covered by outward dents, which are isosceles right triangles. The long side of these triangles are part of the graph, the other two are not. Such piece P can only be a square covered by 4 dents. \square

We call this graph the **decoration graph** and the corresponding tiling into hexes, rhombs and squares the **decoration tiling**. Since all polygons of this graph have an even number of sides, it follows that it carries a *bipartite colouring*, unique up to colour renaming. We call **improved graph** the decoration graph together with a bipartite colouring.

Proposition 2. *In a whole plane tiling by D3 and D2, the colour of a vertex v of the improved graph is determined by the class modulo 2 of k , where $k/12$ designates the orientation⁶ of any segment ending on v on the boundary of any tile D3 or D2 having v as vertex.*

Proof. Indeed the edges of the decoration graph correspond to right angles on the boundary of the shapes D3 and D2. Jumping from one vertex to the other one along such an edge hence changes the orientation index k by adding/subtracting 3. \square

Like for the Spectre, in any whole plane tiling with the shapes D3 and D2 the pieces can only come in orientations differing by a multiple of $1/12$. The decorations of D3 and D2 are composed of segments of the same length, that come in orientations differing by a multiple of $1/3$ in D3 and of $1/12$ in D2, and when we turn and assemble them, they all differ by multiples of $1/12$. In particular they come into two classes (or one if only D3 is used) modulo $1/6$, which we can colour blue and yellow.

The pieces of type D3 and D2 come respectively in 4 and 6 orientations, and the colouring of their marking only depends on this orientation, as on the figure below:

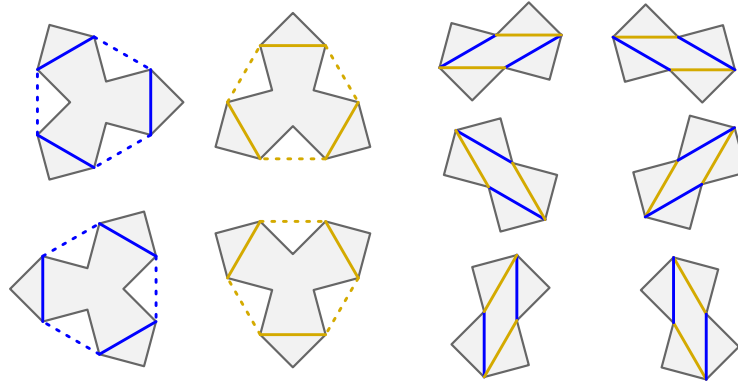


Figure 24

By Proposition 2, the bipartite colouring of the vertices of the improved graph also only depends on the orientation of the pieces, see the figure below. Dents are on the right of the blue segments when followed from their black dot to their white dot, and on the left of the yellow segments followed the same way: from black to white dot.

⁶Relative to any fixed reference direction.

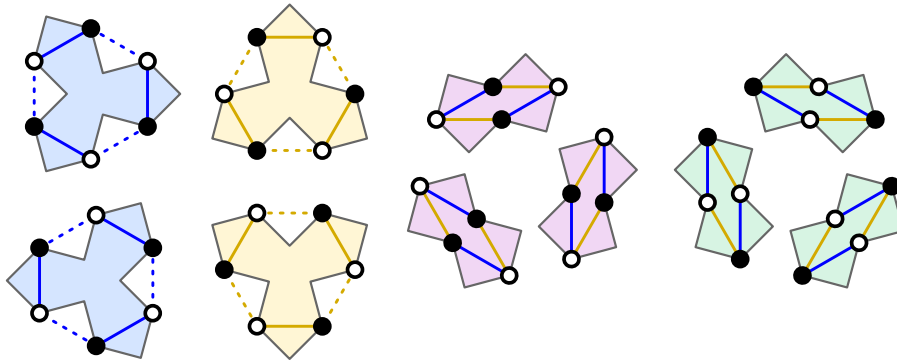


Figure 25

The improved decoration for the D3 and D2 tiling of Figure 21 looks as follows:

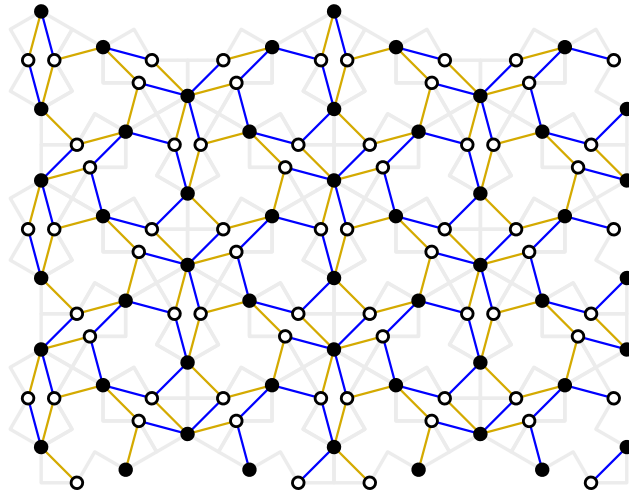


Figure 26 – Improved decoration graph for a whole plane tiling by D3 and D2 not coming from a Spectre tiling. The orientation is turned by $1/24$ compared to Figure 25.

We conclude this section by a summary of simple properties of the decoration tiling into hexes, rhombs and squares. These tiles can only touch at vertices or along a common edge. Call two tiles **adjacent** if they share an edge.

Proposition 3. *For the decoration of a D3 and D2 tiling: no two rhombs can be adjacent; no two squares can be adjacent.*

Proof. This is an immediate consequence of the orientation of dents. \square

Proposition 4. *Two blue hexes in contact must touch along an edge.*

Proof. Indeed if they were to touch only at a vertex, since blue hexes are parallel, this means that there would remain two angles of $1/6$, which can each only be filled by two acute angles of rhombs. But rhombs cannot share an edge (Proposition 3). \square

Two adjacent hexes not only have parallel edges but their vertices colours are identical by translation. This reflects that two D3 with one filling an inward dent of the other have the same orientation.

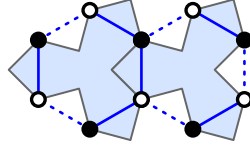


Figure 27

2.2 Spectre tilings

2.2.1 Clusters

Consider now a whole plane tiling by the Spectre, and the associated improved decoration, as on Figure 28.

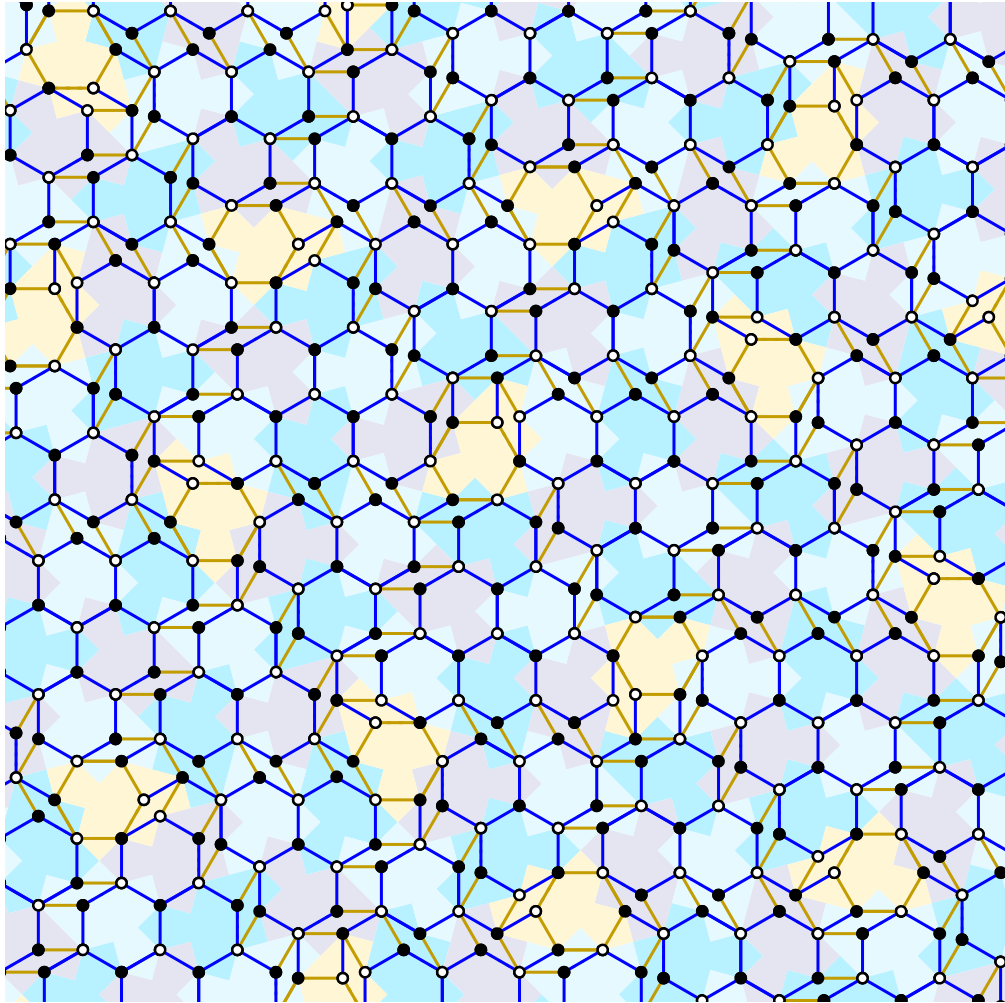


Figure 28

Note that for the corresponding D3, D2 tiling, there must be a pairing between adjacent D3 and D2 tiles, i.e. a pairing of the hexagons and the adjacent rhombs. Because the Spectre cannot be reflected, pink D2 (third column) in Figure 25 can only be associated to blue D3 and the green D2 (fourth column) to yellow D3 so we may as well colour them as follows:

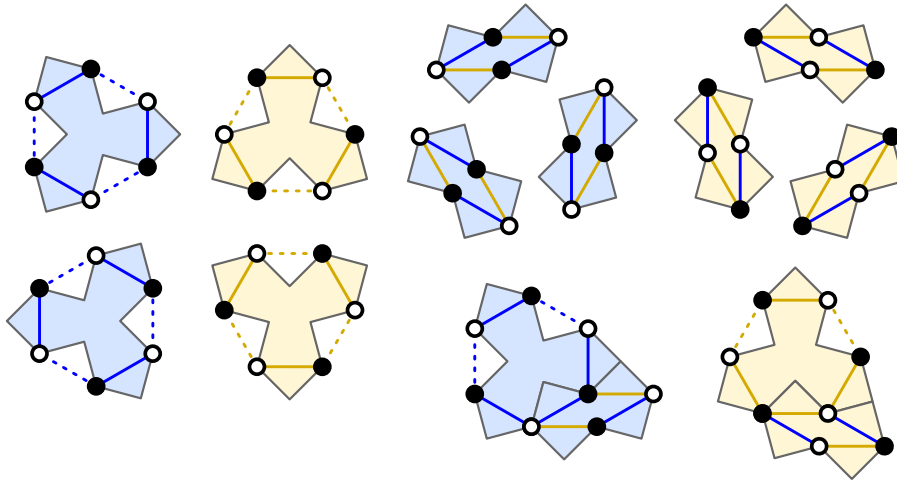


Figure 29 – Orientations match those of Figure 28

We now study how blue hexes can cluster. Recall that by Proposition 4, two blue hexes in contact must touch along an edge.

Definition 5. We call clusters equivalence classes of blue hexes under adjacency.

Let us start with a simple completion rule:

Lemma 6. *For any edge shared between two blue hexes, the black dot of this edge is surrounded by 3 blue hexes. In other words: if we have*

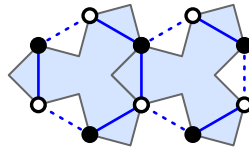


Figure 30

then we have the darker piece below:

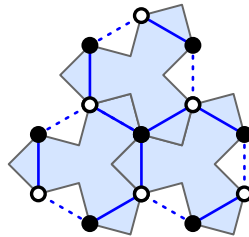


Figure 31

Proof. Indeed in the figure below, the pink inward dent can only be filled in by a blue D3 or a yellow D2, and only in one way in each case.

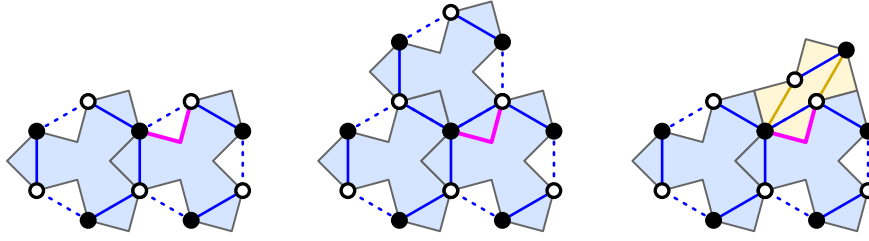


Figure 32

But in the second case, the companion yellow D3 of the yellow D2 has no room to be placed (this is easier to see on the decoration graph: where the yellow rhomb is in contact with the two hexes, there only remains room for squares, so we cannot fit its companion yellow hex). \square

Proposition 7. *A blue hex cluster is necessarily a T1, T2 or T3.*

Proof. If a cluster is not a T1 then there are at least two adjacent blue hexes. By Lemma 6, the cluster is necessarily triangular (or is infinite containing arbitrarily large triangular arrangements). This triangle has a side made of at most 3 hexes, otherwise there is at least a hex inside that is completely surrounded by hexes, preventing it to have a paired rhomb.

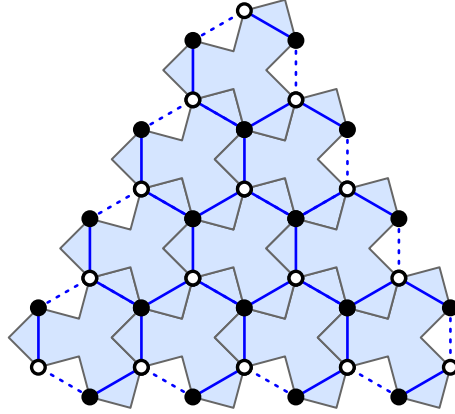


Figure 33 – The central D3 cannot be paired with a D2.

\square

Here are the three possible clusters, up to rotation:

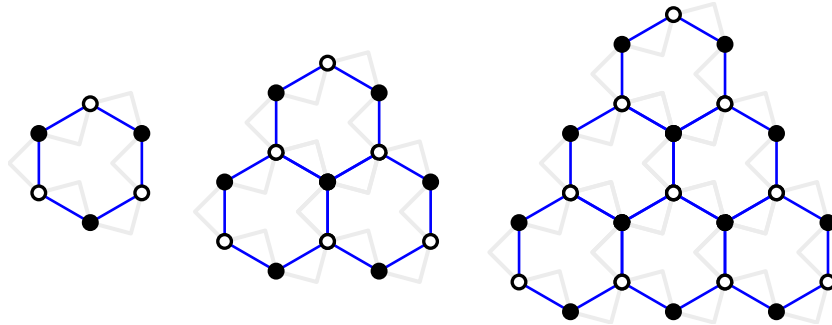


Figure 34 – Clusters T1, T2 and T3, up to rotation

Each D3 corresponding to these hexes has a paired D2, and this pair completely surrounds a black dot.

Blue hexes with bipartite colouring can come in two orientations differing by $1/3$, that correspond to the two possible orientations of the underlying blue D3. This means that the clusters T1, T2 and T3 also come in two possible orientations. For T2 and T3, this orientation can be deduced from the orientation of the hex arrangement, but for T1 this is invisible if we only look at the blue hexagon. By abuse of language we say that the clusters in Figure 34 point up, though they also point in two other directions. And we will say that the other orientation points down.

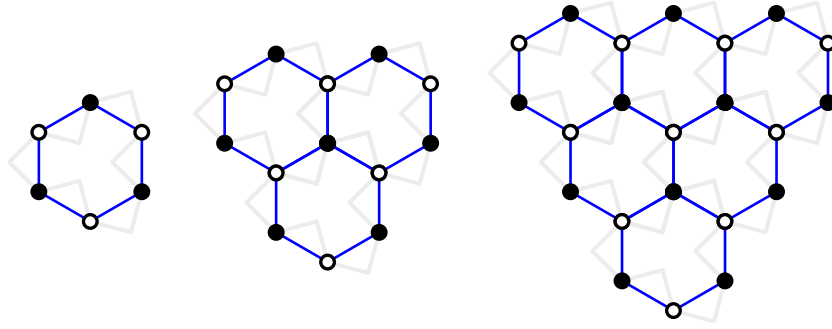


Figure 35 – Down pointing clusters.

2.3 From cluster to cc

2.3.1 Visual aids for Spectre tilings

In this section we indicate boundary shapes that can be visually identified and used to exclude situation, or force the presence of a given tile in a given orientation.

The reader should bear in mind the following: the Spectre can be seen as a 14-gon with equal length edges with vertices of two types: 7 with angles multiples of $1/3$ and 7 with angles multiple of $1/4$ (one of them flat). Then in a whole plane tiling by the Spectre, for two tiles in contact, each vertex of one tile touches a vertex of the other one, and their type is the same.

Proposition 8. *The following polygonal line cannot be included, even after rotation, in the union of boundaries of tiles in a whole plane tiling by the Spectre, where the red lines have the same length as the Spectre edge (seen as a 14-gon).*

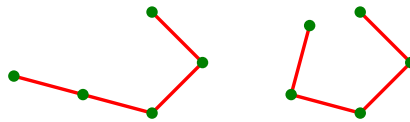


Figure 36

Proof. A direct inspection shows that the Spectre cannot fit the $1/3$ angle at the central vertex. (On the left, only the reflected Spectre fits.) \square

Proposition 9. *In each of the figures below, if the polygonal line on the left of the arrow is included in the union of boundaries of tiles in a whole plane tiling by the Spectre, then a tile must be placed as the red tile on the right of the arrow.*

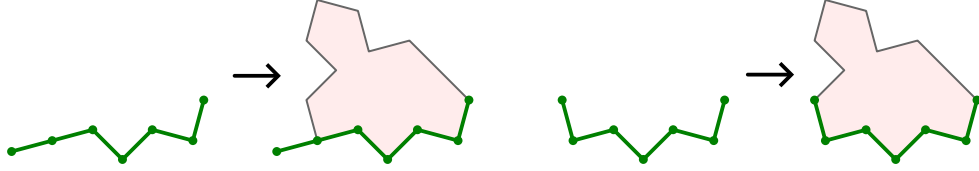


Figure 37

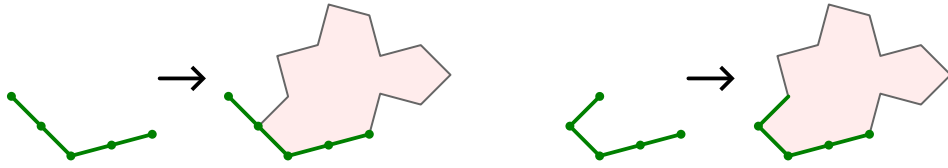


Figure 38

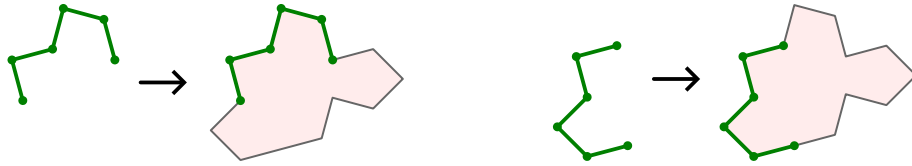


Figure 39

Proof. In the first two situations, every other way to fill the dent at the central vertex gives rise to an outline forbidden by Proposition 8 (the red polygonal lines in the figure below).

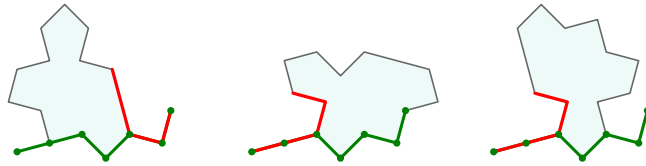


Figure 40

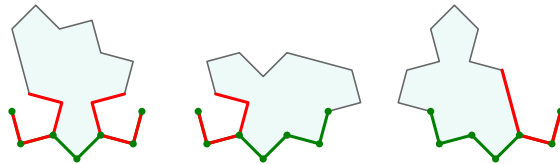


Figure 41

In the next two situations, there is only one way to fit a tile at the $1/3$ angle of the central vertex.

In the last two, the only other way to fill the circled vertex is as follows:

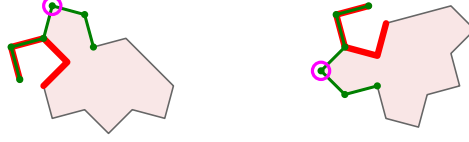


Figure 42

□

2.3.2 Properties of odd tiles

Proposition 10. *No whole plane tiling by the Spectre can have only even or only odd tiles.*

Proof. By rotation, is enough to prove the claim for even tiles. By contradiction, assume this is possible and consider the decoration graph and also the induced D3 and D2 tiling. In this case we have that the yellow edges belong only to D2 pieces. The dent on these yellow edges can only belong to squares. So each D2 is part of an infinitely long chain as on the left column of Figure 43.

The squares cannot be filled by pieces of type D2 because the dent must be supported by a blue segment and the piece would have to be oriented as on the second column and hence cannot fit.

So they are filled by pieces of type D3 as on the third column. But then we get clusters of blue hexes that are too big and contradict Proposition 7.

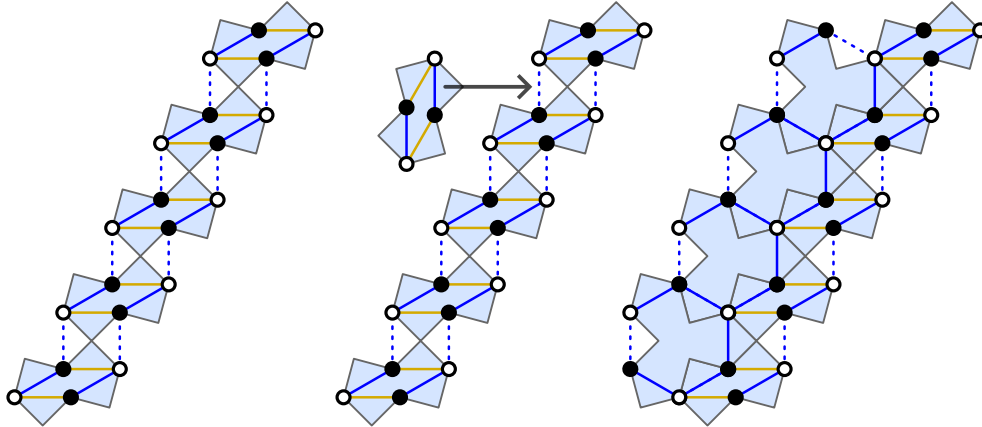


Figure 43 – A chain of 5 pieces D2 placed as on the left column (blue D2 with touching dents supported by yellow segments) cannot appear in a whole plane tiling of the plane by Spectres.

□

The central argument in Figure 43 will be generalized in Lemma 35.

We reprove (and improve) the forced environment of odd tiles explained in Section 4 of [Smi+23a]. The proof of the following key proposition goes through the examination of many cases. It is not obvious how to optimize that and I did

not manage to take a significant advantage of the tile decorations for that. It would be nice to have a more conceptual proof of Proposition 11.

Remark. By seeing a draft of the present notes, the author of [Smi24] informed us that for Hats tilings, the latter article includes a conceptual proof of the existence of a parity class which cannot form clusters with more than one tile, from which a neighbouring environment is somewhat easy to deduce. It is quite possible that one could adapt these arguments for Spectre tilings.

Proposition 11. *Consider in a whole plane tiling by the Spectre, two tiles of different parity and in contact along at least one edge. Then they are, up to a rotation, part of the following arrangement, with one of them being the red tile. The non-red tiles all have a parity opposite to the red one.*

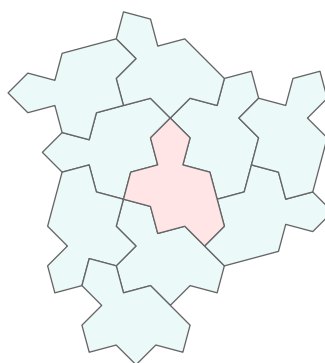


Figure 44

Since the proof is a bit long, we moved its content in Section 3.1. It makes an extensive use of the visual aids of Section 2.3.1.

This allows us to reprove one of a key observations of [Smi+23a] (for this we could have stopped at Figure 192 in the previous proof).

Corollary 12. *In every whole plane tiling by the Spectre, there is a parity class for which the tiles are isolated.*

Proof. We proceed as in [Smi+23a], Section 4, page 13. Every two tiles in the tiling can be connected by a chain of adjacent tiles. Two adjacent tiles of different parity must be in the configuration of Proposition 11 up to rotation, and we call (in this proof) *iso* the one of the two tiles corresponding to the red central tile of the configuration. Only one of the two tiles is *iso* since in the configuration, all cyan tiles adjacent to the red one are also adjacent to another cyan tile, and all cyan tiles of the configuration have the same parity, opposite to the red. Now consider any chain of adjacent tiles. In it, the *iso* tiles are isolated: the next and previous tiles cannot be *iso*, and they have a different parity from the *iso*. Moreover for two consecutive tiles in the chain of different parity, one must be *iso*. By contraposition, two consecutive non-*iso* tiles must have the same parity. It follows that, along the chain, *iso* tiles must all have the same parity. \square

We can assume that is this parity class is the *odd* one and this is what we do on the rest of this article.

2.3.3 Properties of cc

Consider the blue/yellow decoration graph generated by a tiling with the Spectre. By convention we use blue tiles, with a blue hexagon, for even tiles and yellow tiles, with a yellow hexagon, for odd tiles, where we assume as we just explained, that the odd tiles are the isolated ones.

Definition 13. We call cc the connected components of the blue graph in the decoration graph (i.e. without the yellow segments).

There is the following observation:

Proposition 14. *The only vertices such that all edges sharing it have the same colour, are vertices at the meeting point of three blue hexes.*

Proof. All vertices in an even tile touch at least one blue edge, and by Proposition 11 all odd tiles are isolated. So no vertex can touch only yellow edges.

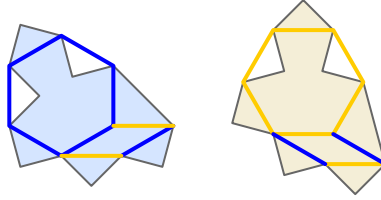


Figure 45

All vertices in an odd tile touch at least one yellow edge, so the only way to get a purely blue vertex is that each tile it touches is a blue tile, touched at one of the blue vertices below, which all belong to a hex.

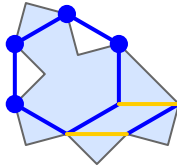


Figure 46

□

Proposition 15. *The following configuration cannot appear, even rotated, in a whole plane tiling by the Spectre.*

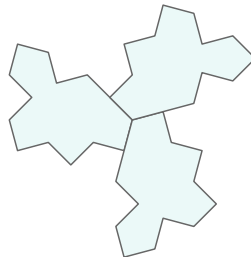


Figure 47

Proof. It forces the following sequence by repeated application of Proposition 9 which ends up in a configuration with non-extendable parts (or we can invoke Proposition 11 at the second frame).

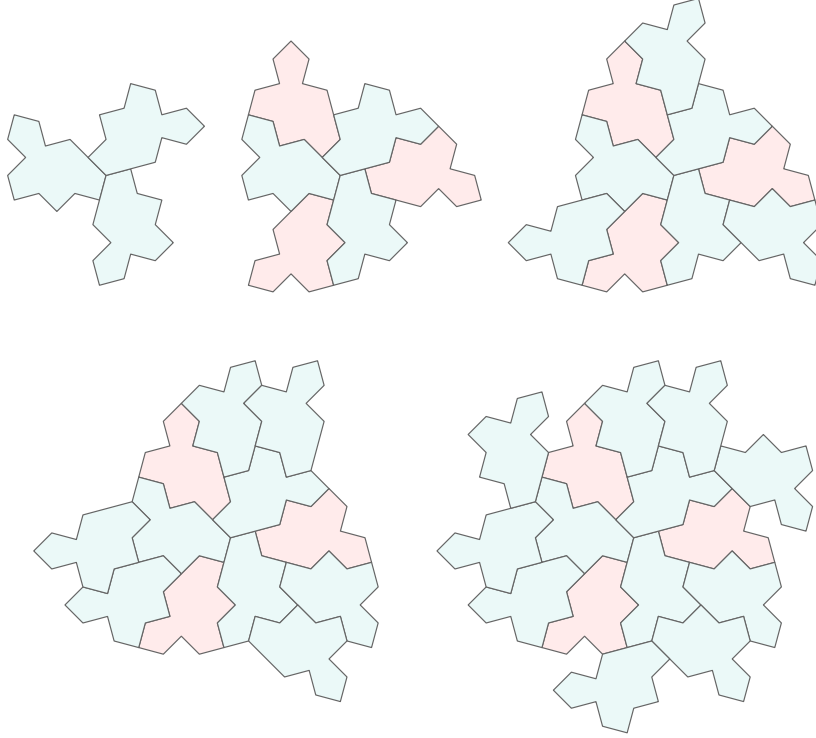


Figure 48

□

Proposition 16. *Each cc contains at least one blue hex.*

Proof. A blue segment not bounding a hex bounds a square. Disappointingly, we have to proceed by inspection of many cases.

First, the blue segment of an even tile that is outlined in red below:

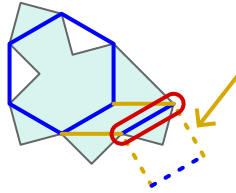


Figure 49

The yellow segment indicated by the arrow will be in a tile, which must be one of the following cases.

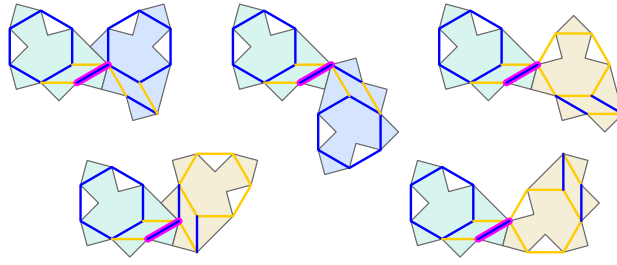


Figure 50

However, not all cases can be realized: two cases have red outlines which fit no tile, and another one forces two odd tiles to be adjacent.

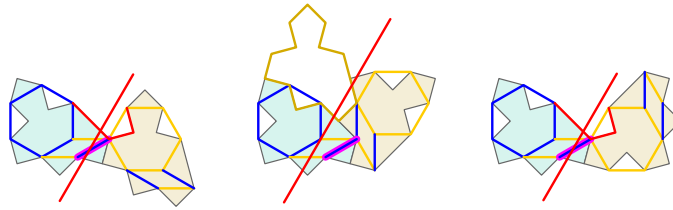


Figure 51

Alternatively we can invoke Proposition 11 to rule out these same four cases.

In the two remaining situations, the leftmost one has a hex touching the initial blue segment, and in the the second one we look at what can fit in the upper right free space :

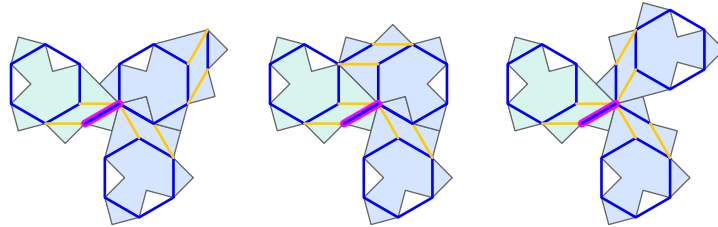


Figure 52

The first two have a hex, and the last one cannot appear by Proposition 15.

We now study the case of the blue segments of an odd tile. By Proposition 11 a yellow tile has a fixed immediate environment, which includes the following two blue tiles:

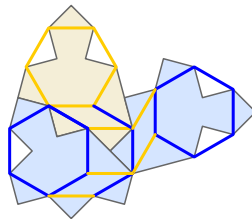


Figure 53

The left blue segment of the odd tile is on a hex so we only need to inspect the right one. The free corner in the common segment of the blue tiles can only complete as follows:

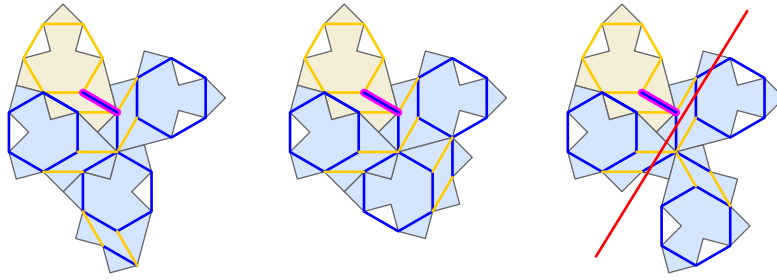


Figure 54

The right one has already been excluded above and the other two have a hex. \square

Proposition 17. *In the D3 and D2 tiling induced by a whole plane tiling by the Spectre, the following arrangement or its rotations cannot appear:*

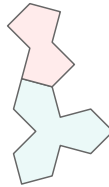


Figure 55

Proof. The D2 and D3 must pair so as to form spectres. The red D2 can only be paired with the red D3 as in the central frame in the figure below. The green inward dent can only be filled by the hatched red D2. But there is no room left for the associated D3 of the latter.

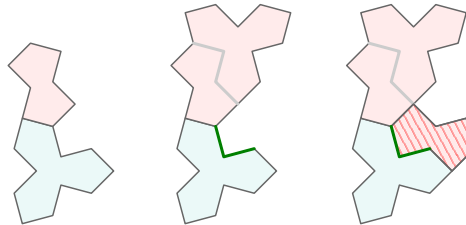


Figure 56

\square

The following consequence extends Lemma 6.

Corollary 18. *If a blue segment protrudes from the black dot of a blue hex, then this segment belongs to two adjacent blue hexes.*

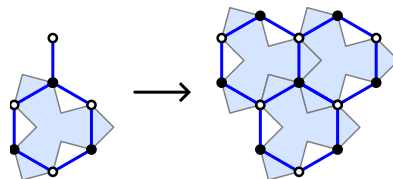


Figure 57

Proof. Only a blue hex or a yellow D2 can be placed to contain this blue segment with the colours of its extremities. The latter has been excluded by Proposition 17, so it is a blue hex. Then we get the two possibilities on the right of the following figure:

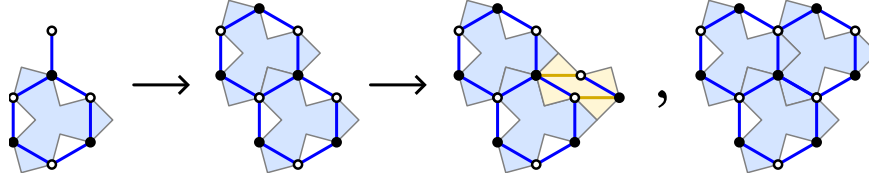


Figure 58

The one with a yellow D2 does not leave room for its associated yellow D3, so we are in the last configuration. \square

Below we show the only two ways a specific arrangement of D3 and D2 can be filled at a specific dent (in green). In one of the two ways, the blue segment with a violet highlight belongs to a blue hex.

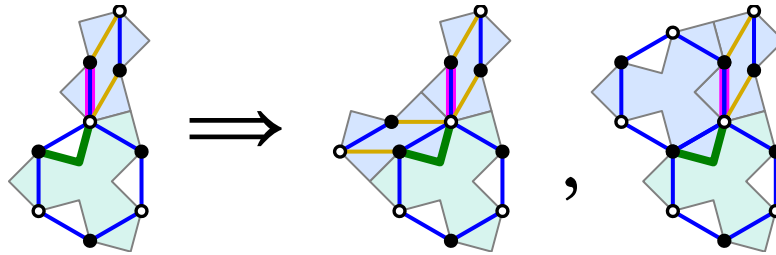


Figure 59

We now inspect all the possible ways a blue segment can protrude from a blue hex without being itself in a hex. By Corollary 18, it has to stem from a white dot of the hex. We get the following possibilities, up to rotations, for the cyan tile (not all may appear in an infinite tiling):

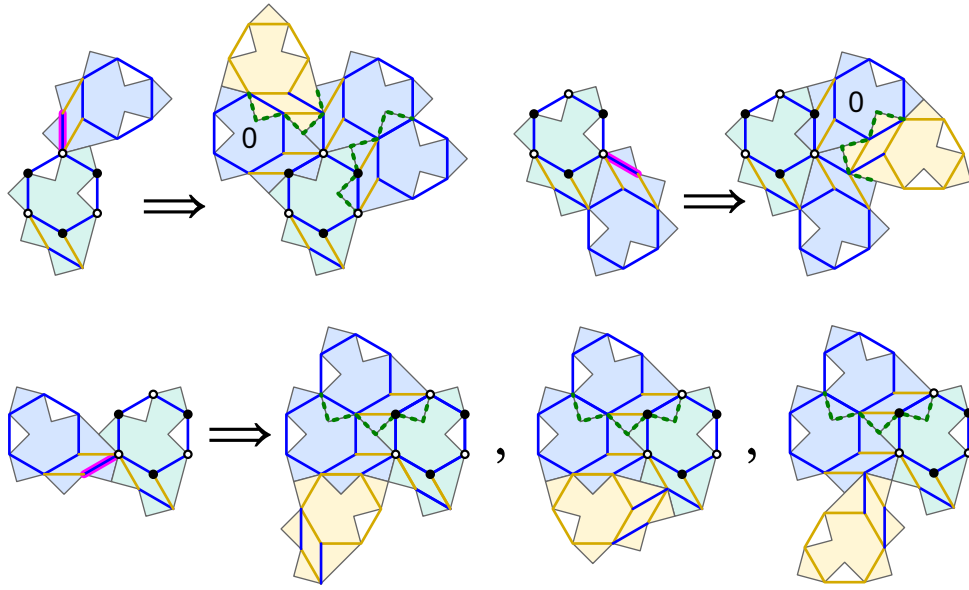


Figure 60 – A list of possible antennae. Not all may actually be allowed in a whole plane tiling by the Spectre.

The tiles marked 0 are deduced from the paragraph above Figure 59 and the fact that the highlighted blue segment shall not be on the boundary of a blue hex.

Recall that we defined *clusters* as maximal sets of adjacent blue hexes and proved that in whole plane tilings they can only come in three possible triangular configurations, that we called T1, T2 and T3. We also noted that each T_n can come in two orientations, related to the orientation of the underlying piece D3.

Definition 19. For a T_n , let us define *tips* as below:

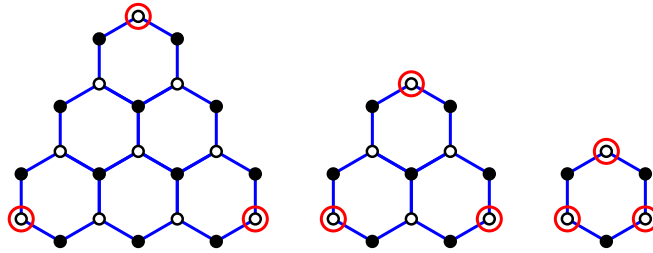


Figure 61 – Position of the tips of a T_n pointing up

In the picture above, the T_n point up and the red circles indicate what we call the tips. They are in particular white dots of the white/black dot colouring of vertices introduced earlier.

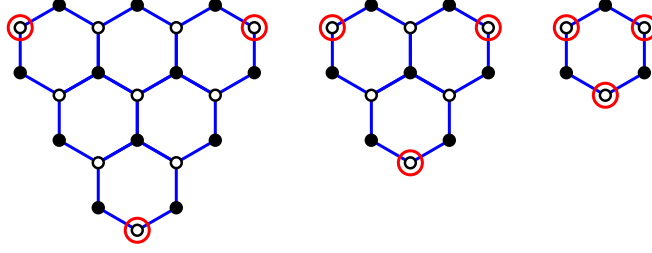


Figure 62 – Position of the tips of a T_n pointing down

From the analysis above it follows that:

Proposition 20. *A cc contains at most one hex cluster and a possible antenna at each of its three tips.*

Proof. Indeed we saw that a cc contains a cluster, that a cluster is a T_n for $n = 1, 2$ or 3 , and Corollary 18 implies that no tip can grow on the sides of a T_n . Figure 60 shows than an antenna touches one blue hex and no more. \square

We already mentioned that blue hexes with the bipartite vertex colouring come in two possible orientations. Actually we can extend this to all blue segments, in a compatible way, using the improved decoration, the one with black/white dots at the vertices of the blue/yellow decorated graph. Blue segments come in three orientations, six if we direct them. Let us direct them from the black dot to the white. The six orientations differ by a rotation by multiples of $1/6$. Let us group them into their two classes modulo rotation by multiples of $1/3$.

Proposition 21. *In the improved decorated graph of a tiling by D_3 and D_2 , two blue edges sharing a vertex and directed as above belong to the same orientation class. Two directed blue edges connected by a yellow edge belong to opposite classes.*

Proof. The first claim follows from the fact that the colouring is bipartite and that the undirected blue edges have only 3 orientations. The second claim follows from the fact that the colouring is bipartite and that in the decoration graph we have, besides the hexagons, only squares and rhombs, where the parallel sides have the same colour, so opposite directions w.r.t. the vertex colouring. \square

2.3.4 Interface between cc

In the sequel, unless explicitly stated, all statements concern whole plane tilings by the Spectre.

Recall that the notion of cc was defined in Definition 13 as the connected components of the blue graph in the decoration graph (i.e. without the yellow segments). Consider a cc. By Proposition 21, all its directed segments belong to the same class modulo rotation by a multiple of $1/3$. We can thus associate a class to the whole cc. Consider now two cc's. We say that they are adjacent if there is a yellow segment linking them. From the same proposition:

Corollary 22. *Two adjacent cc's have opposite classes. The corresponding T_n thus have opposite classes.*

We now proceed with a more precise description of the environment of a cc.

Proposition 23. *Each cc touches three yellow hexagons at three contact vertices related as follows to the three tips of the T_n the cc contains. For each tip the contact vertex is either:*

- *the other end of the antenna if there is one attached to the tip,*
- *the tip itself,*
- *the next vertex on the T_n boundary followed in the counter-clockwise direction.*

Between two such contact vertices, there are 3 interfaces to 3 adjacent cc's. Each interface consists in squares and rhombs in alternation, with parallel yellow edges. They end with two yellow hexagons.

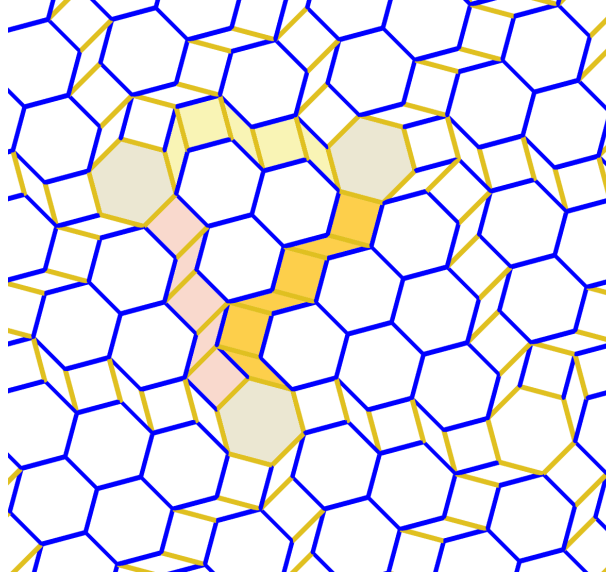


Figure 63 – Example of a cc and its three interfaces to adjacent cc's.

Proof. Possible antennae ends were enumerated in Figure 60. In particular there is always a yellow hex at the end.

Let us study further the environment of a T_n tip. In the next figure we show all the possible top tip neighbourhoods. It is a subset of all the possible neighbourhoods, where we exclude the presence of another hex touching the tip.

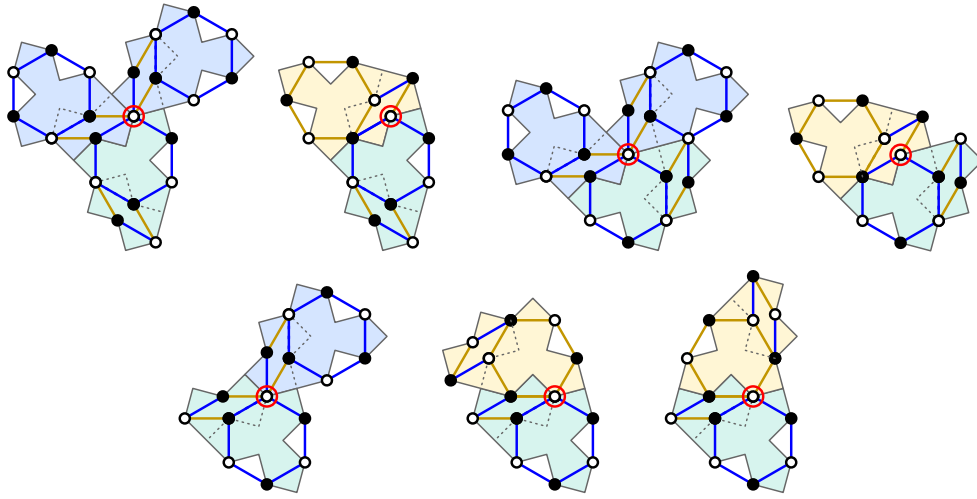


Figure 64 – A list of possible top tip neighbourhoods. The vertex circled in red represent the top tip of an up pointing T_n . Not all those configurations can actually appear in a whole plane tiling: for instance the second one is forbidden by Proposition 11

We see that those which do not have an antenna touch a yellow hex either at the tip or at the next vertex following the blue hex boundary in the counter clockwise direction.

In the figure below, we sum up the possible ways a top tip of an up pointing cluster is connected to a yellow hex, as a consequence of Figures 60 and 64:

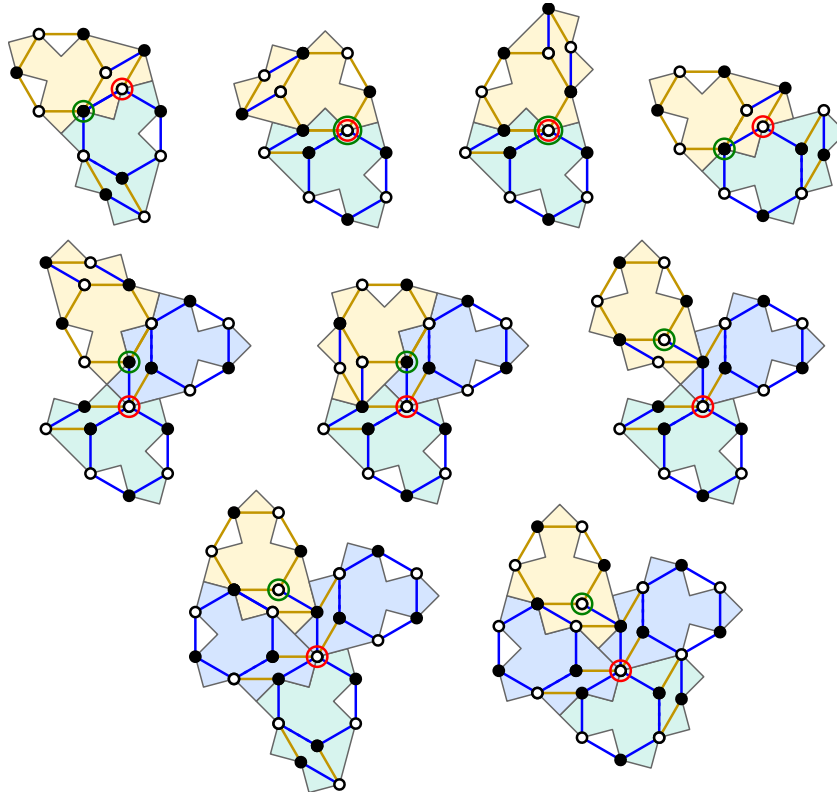


Figure 65 – Top tip circled in red, contact point in green.

Consider one of the three yellow hexes in contact with the cc. The two sides of this yellow hex that are adjacent to the contact point can each be seen as the starting element of a finite sequence of parallel yellow segments attached to the boundary of the cc and ending on another yellow hex in contact.

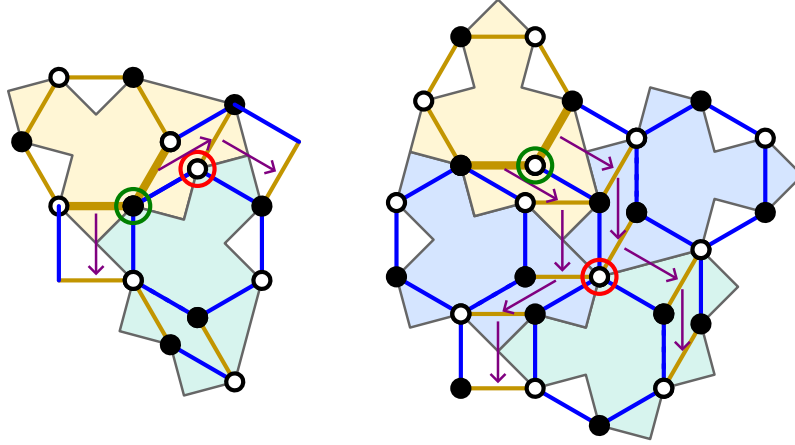


Figure 66 – How we start following the boundary on two examples

This claim follows from induction by following the cc boundary: on Figure 65, inspection of the cases show that the first blue segment in this boundary together with the yellow segment are part of the boundary of a quadrilateral which is either a square or a rhomb. The opposite sides of this quadrilateral are parallel segments of the same colours. Opposite to the yellow segment, there can be a square or a rhomb or a hex. No two squares or two rhombs can follow each other (Proposition 3). \square

2.4 A list of cc

We will enumerate all possible types of cc and their interfaces (this information includes the contact points with the yellow hexes). We will also see that a T3 never has antennae while a T2 has between 1, 2 or 3 of them, and a T1 necessarily has 3.

In the sequel we use the following **marking** in the hexagons to indicate which pieces of type D3 and D2 pair to form Spectres:

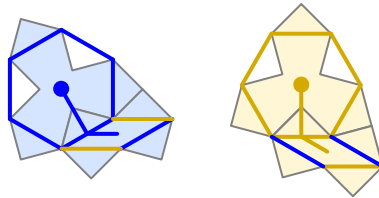


Figure 67 – Marking of the hexagon of a spectre. The marking is a dot at the centre of the hexagon plus a broken line that links the centre of the hexagon to the midpoint of the side of the hexagon in common with the rhomb of the associated D2, then to the centre of the rhomb.



Figure 68 – Left: We removed the Spectre, and hence its composing D3 and D2, from the previous picture, but maintained the decoration (hex+rhomb) and the marking. Right: we only kept the hexagon and the marking, which now has a small segment sticking out.

We find interesting to look at the interpretation of Figure 44 in terms of marked hexagons.

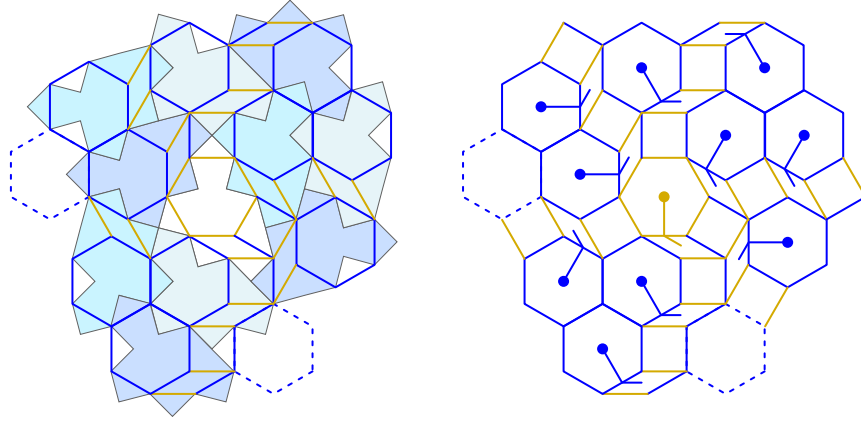


Figure 69 – Environment of an odd tile, from the point of view of the decoration graph. The two dashed hexes are respectively deduced from Lemma 6 and the description of antennae in Proposition 16 and fig. 60 (or more simply a direct inspection of what can fit there).

The next six statements, from Proposition 24 to Proposition 29, have their proofs moved to Section 3.2.

Proposition 24. *Every cc of a T1 is necessarily as on the picture below:⁷*

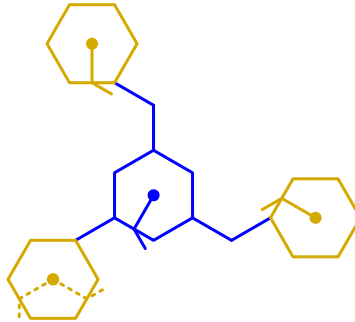


Figure 70

⁷Actually we will see in Corollary 48 that the lower left arrow can only have one orientation.

Proposition 25. *A T2 with markings as below cannot occur.*

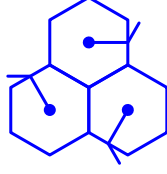


Figure 71

Proposition 26. *A T2 with markings as below left has a cc as below right:*

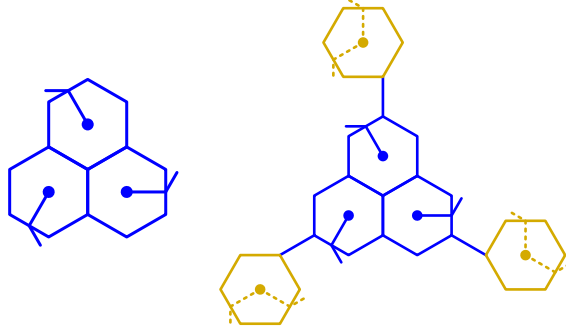


Figure 72

Proposition 27. *The cc of a T2 can only be as in the previous proposition or as follows:*

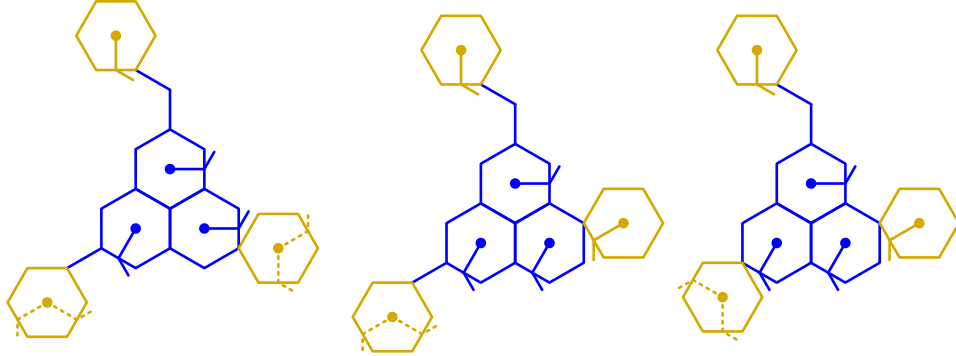


Figure 73

Lemma 28. *A T3 is environed as follows, in terms of the D3 and D2 induced tiling (equivalently, in terms of the yellow/blue decoration graph):*

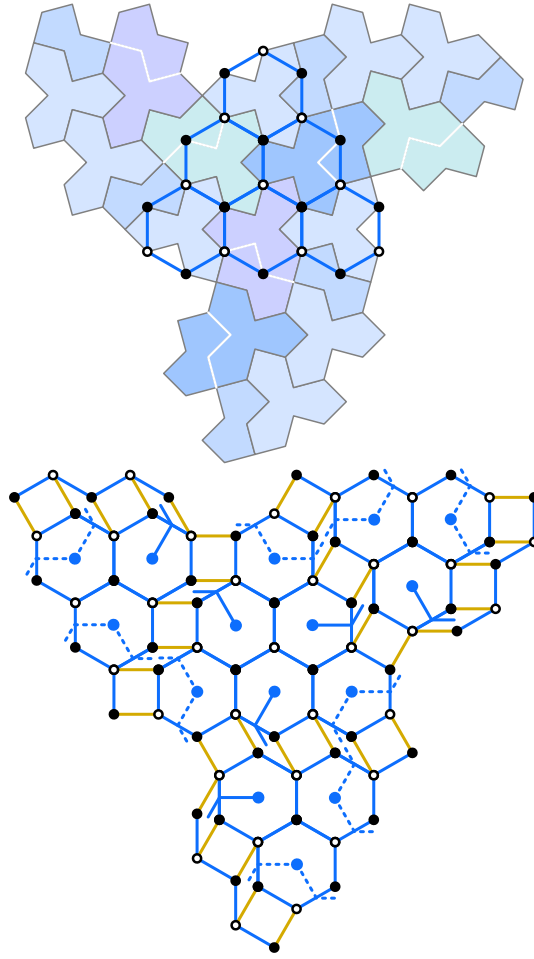


Figure 74 – Two equivalent depictions of the environment of a T3. Some of the D2 and D3 have a determined pairing into a Spectre (such pairs are filled with the same colour and their common boundary is in white). For others (in two pale blue shades), we do not know in advance. All the D3 and D2 and Spectre are “blue” in our previous denomination.

Proposition 29. *A T3 has no antenna: it is its own cc. It can only be as follows on its lower left corner, and similarly for the other two by rotation.*

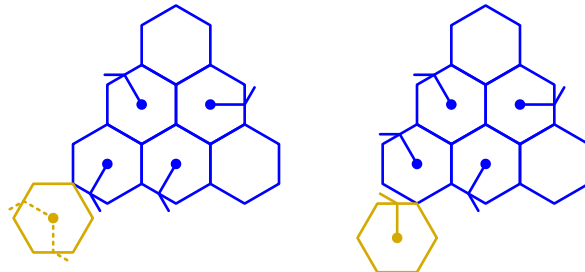


Figure 75

As already remarked, in each case $n = 1, 2$ or 3 , the yellow hex associated to a tip of a T_n cluster is either at the end of an antenna if there is one, or at the

tip, or at a vertex on the boundary of the cluster, situated one blue segment away from the tip, counter-clockwise.

From this, we get the following possibilities for the cc's and their *interfaces* as defined in Proposition 23. Not all those possibilities may appear in an actual tiling of the whole plane by the Spectre. This is one of the main figures of the article and we will refer to it often.

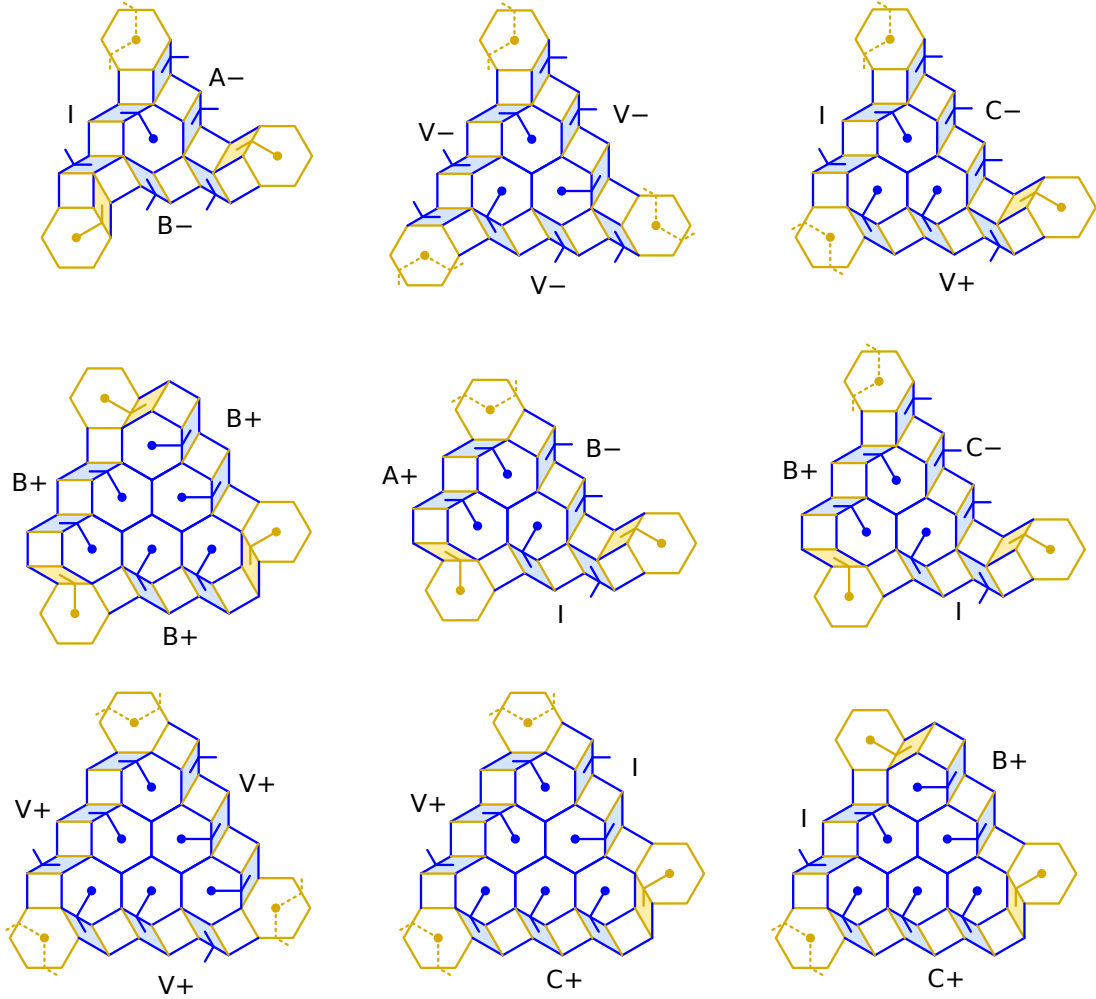


Figure 76 – In this list, we marked interfaces between adjacent cc's with a letter corresponding to their shape. The plus or minus is there to indicate that a + can only fit a – and vice-versa. The 'I' has no sign because it fits with itself. We also indicated which rhombs are blue and which are yellow. The blue rhombs that match external blue hexes have a small marking. Note that the the yellow hexes with a solid yellow marking are those linked to a yellow rhomb of one of the interfaces. The other ones have a dotted marking indicating the two indents of the underlying D3 that are still free for a matching yellow rhomb.

Two neighbouring cc's in the decoration graph of a whole plane Spectre tiling must have a common interface, in the sense that two blue hex and a chain of rhombs and squares linking them must be the same. In other words, the interfaces in the pieces of Figure 76 must superimpose.

Let us sum up what we obtained:

Theorem 30. *Consider the hex, squares and rhombs tiling associated via the decoration graph to a whole plane tiling by the Spectre. It decomposes as a union of pieces as in Figure 76, rotated by multiples of $1/6$, superimposing at their interfaces and yellow hexes.*

Conversely:

Theorem 31. *Any arrangement of the pieces above such that*

1. *the pieces cover the plane,*
2. *the blue hexagons of distinct pieces are disjoint,*
3. *the interfaces overlap in a matching way (disregarding the yellow hex markings): A+ with A−, B+ with B−, C+ with C−, V+ with V− and I with I, (then the yellow hexagons match)*
4. *each yellow hex has been given a marking by a solid (not dotted) yellow marking by at least one piece; no yellow hex has been given two different solid yellow marking,*

gives rise to a tiling by Spectres.

Proof. Indeed the blue hex/rhomb association always match when the letters and sign follow the rules: below we show the interfaces facing each other. Bear in mind that the number of hexes behind a given interface could depend on the piece: for instance B+ can bound cc's with a T3 and cc's with a T2, B− can bound cc's with a T1 and cc's with a T2, etc.

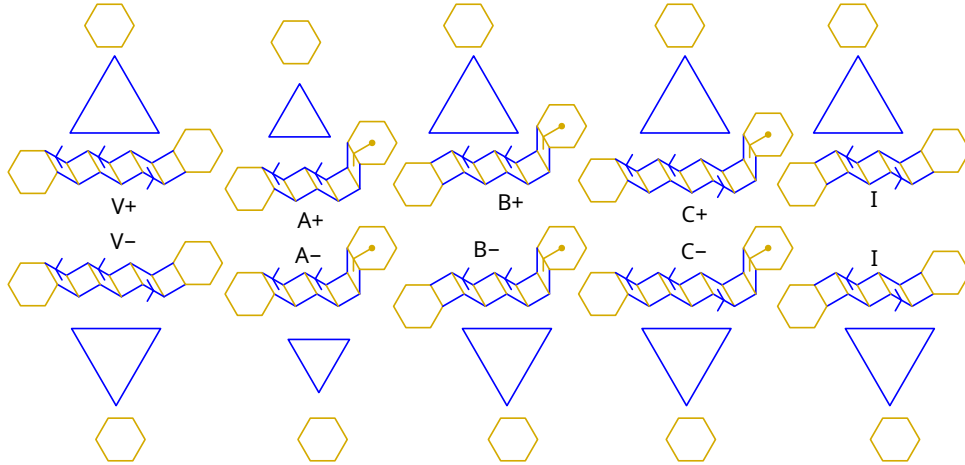


Figure 77 – List of interfaces and their correspondence

Let us be more explicit. In the figure above, the triangles represent cc's. Consider an interface of a cc and its rhombs, with markings pointing towards the cc (towards the triangle in the sketches), or away. We know that in the cc, there is a hex matching with each rhomb whose marking points towards the triangle. The rhombs whose marking points away will point towards the adjacent triangle, for which we thus know that there is a matching cc.

Now replace all hexes by D3's, whose orientations are given by the marking (it points to an indent of the D3), and replace all rhombs by D2's. They will be disjoint and if the conditions of the theorem are satisfied, we pair all D3 and D2 and get disjoint Spectres.

There remains to check that the Spectres cover the whole plane. In our situation the whole plane is tiled by hexes, rhombs and squares, so there remains to check that the squares are covered. All squares belong to an interface and inspection of all the pieces in Figure 76 (take the hexagon orientation into account, knowing that the marking points to an indent of the D3) show that at least three of their four sides are covered by a dent: the only side which may fail to be is the outer part of the interface (alternatively, the way the pieces were made already ensures that). But since that part is the inner part of the same interface for the adjacent cc, it is covered too. \square

Rule 4 is important: in the figure below the central yellow hex has no matching yellow rhomb. In other examples, there could be more than one yellow rhomb for only one hex.

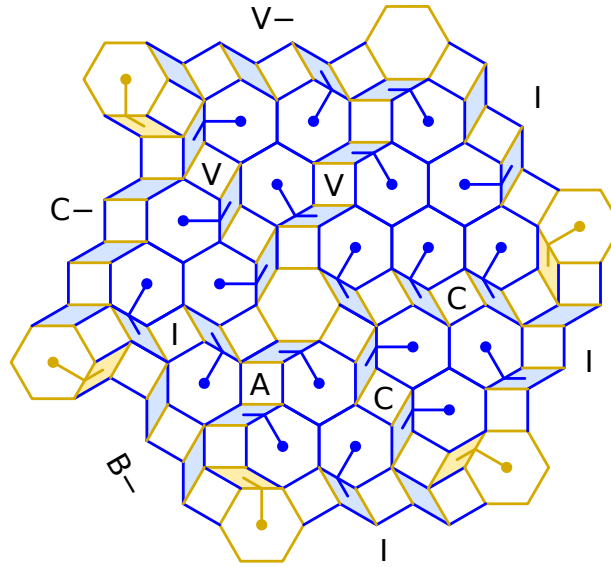


Figure 78

2.5 Contracted graph and partition of the honeycomb

2.5.1 For a general D3 and D2 tiling

This section is not strictly needed for the analysis done here and can be safely skipped.

Given a D2/D3 tiling and its yellow/blue decoration graph, we may contract every yellow hex to a point, and every square and rhomb to a blue segment, by

collapsing each parallel yellow lines as below:

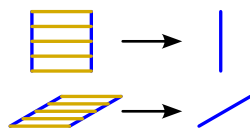


Figure 79

All the collapsing classes on an example:

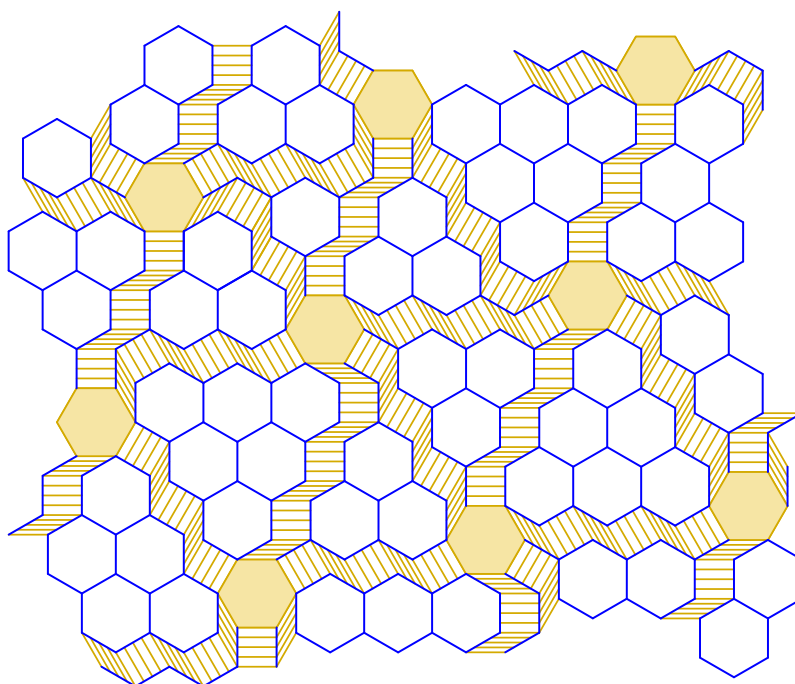


Figure 80

Collapsing only the following parallel strand:

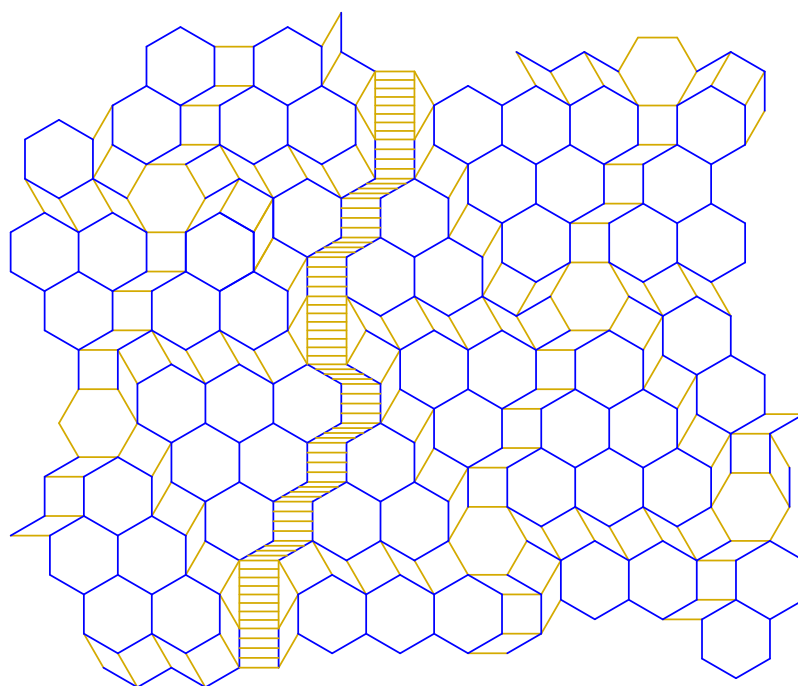


Figure 81

gives:

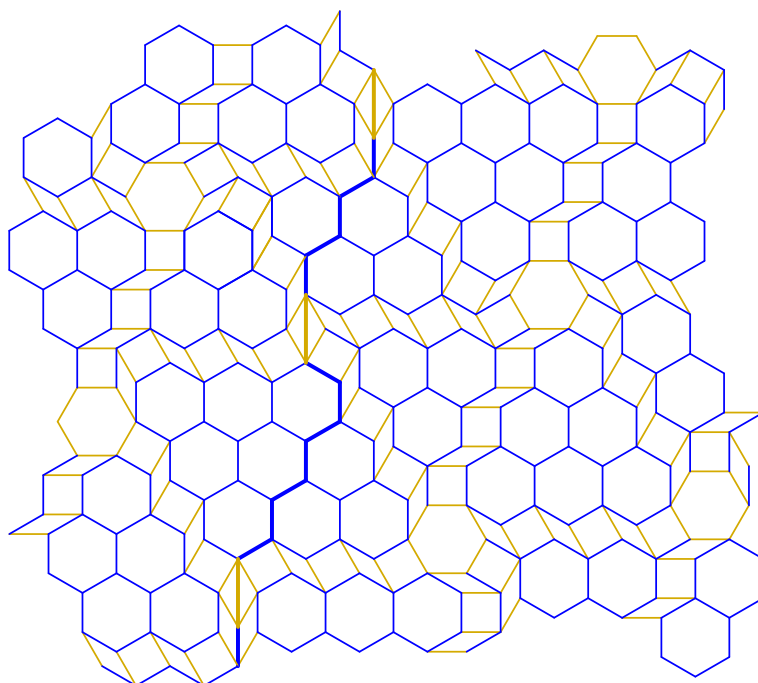


Figure 82

The contracted graph is a subset of the honeycomb. It may or may not be a strict subset. For instance, if one starts from a tiling made solely of D2 tiles...

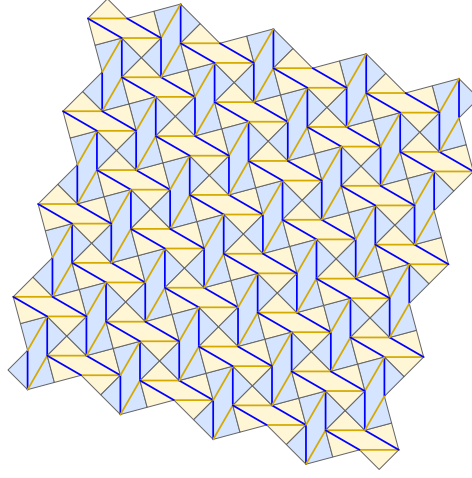


Figure 83

... then after collapsing all the yellow segments, there remains only one broken blue line.

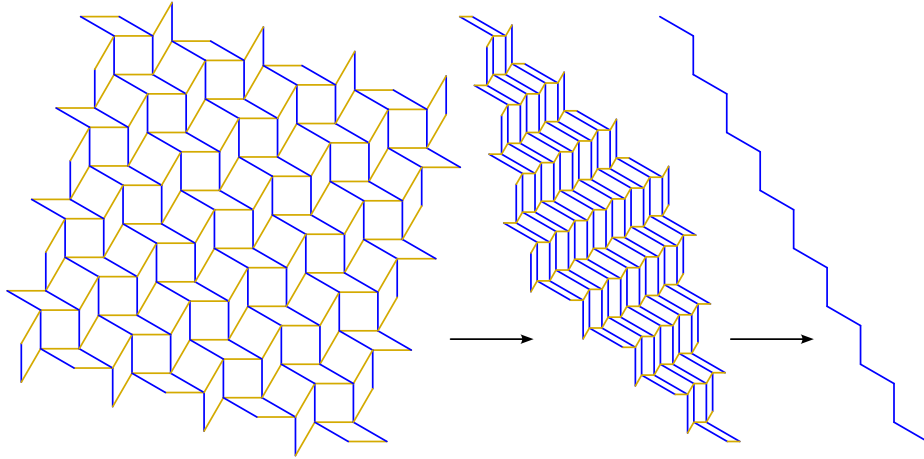


Figure 84

On the decoration graph a whole plane tiling by D2/D3's, each fracture extends both ways to infinity. Because we can proceed fracture by fracture, the quotient actually naturally maps to the plane so that every blue segment and every blue hex of the non-quotiented graph is injected in the collapsed graph by a translation.

Remark. A word of caution: the white/black vertex colouring of the improved decoration graph does not pass to the quotient. However the quotient being a subset of a hexagonal lattice, whose vertices has its own bipartite colouring.

The quotient map can be seen as a map from the plane to the plane, sending every blue segment or hex to a blue segment or a hexagon of some hexagonal tiling of the plane (honeycomb).

Definition 32. This projection is called the **yellow contraction**.

Proposition 33. *The yellow contraction maps different blue hexagons of the decoration graph to different hexagons of the honeycomb.*

Proof. We saw that the yellow contraction can be obtained by collapsing fractures one after the other. In such collapses, the blue hexagons (their interior) remain disjoint. \square

As we have seen, surjectivity depends on the D3/D2 tiling under study. We will see that it is surjective if it comes from a Spectre tiling.

Similarly we define the **blue contraction**, mapping yellow segments and hexes to (a subset of) a yellow hexagonal tiling of the plane. Every cc is contracted to a point by the blue contraction, hence the cc is a connected component of the preimage of a vertex by the blue contraction. Actually, the cc must be the whole preimage of the vertex: this can be proved by uncollapsing the fractures one by one: preimages remain connected throughout the process, and actually *simply connected*. (This is also a consequence of Moore's theorem relating plane quotients and end of isotopies.)

2.5.2 For Spectre tilings

In this section we assume that the D3 and D2 tiling comes from a Spectre tiling. Consider the decoration graph and the tessellation into hexes, squares and rhomb it induces. Consider as above the yellow contraction: it induces a mapping of the plane hosting the decoration graph to the plane hosting a pure regular hexagonal tessellation (honeycomb).

Proposition 34. *The yellow contraction induces a bijective correspondence between the blue hexes of the decoration graph and the hexagons of the honeycomb.*

Proof. Injectivity has already been seen in Proposition 33. For surjectivity it is enough to prove that for any hexagon H that an image of a hex, any adjacent hexagon H' is also an image of a hex. Let B be a blue hex whose image is H . The edge of H along which H' is adjacent, corresponds to an edge e of B . If there is a hex adjacent to B along e then we are done. Otherwise, from e starts a chain of rhombs and squares, whose blue edges are parallel to e . Such a chain cannot be infinitely long by the lemma below, so must end on a hexagon. The image of this hexagon will then be H' . \square

Above, we used the following result:

Lemma 35. *Consider a chain of adjacent squares and rhomb with all blue edges parallel or all yellow edges parallel. The number of elements of this chain is bounded for Spectre tilings.*

Proof. We will give a proof that is independent of Corollary 12 and Proposition 11, so we can assume all parallel edges are yellow.

Because dents protrude from rhombs and must come inside squares, squares and rhombs must alternate along the chain. We will call **consecutive** rhombs of the chain that touch via a single vertex, i.e. those that are adjacent to the same square of the chain. Consider the colour of the rhombs, i.e. the colour, yellow or blue, of the associated D2 according to Figure 29.

Because we are coming from a Spectre tiling, yellow rhombs must have one dent in a yellow hex, so if there is one, it terminates the chain. We saw in the proof of Proposition 10 that the chain cannot contain 5 consecutive blue rhombs (Figure 43). It follows that the chain is limited to 6 rhombs and 5 squares.⁸ \square

A consequence of Proposition 34 is that the partition of the blue hexes into clusters induces a partition of the honeycomb into triangular groups of 1, 3 and 6 hexagons. Figure 13 showed an example of contracted graph, where we used colouring to keep track of the Tn . Figure 17 showed an example where we used thick lines instead, and added the information of the position of the contracted yellow hexes using dots (in this picture the dots have been coloured using the bipartite colouring of the vertices of the honeycomb).

We claim that this partition plus the dots are enough information to recover the initial Spectre tiling. We can also determine which partitions together with dots come from a Spectre tiling.

First, on the figure below, on the left we present the cc's together with the three yellow hex they touch, contracted to a dot. (The fourth will never appear in a situation coming for a whole plane Spectre tiling but this does not matter at this point.) We still consider that the boundary of the cc is the union of broken lines between dots, that we still call an interface.

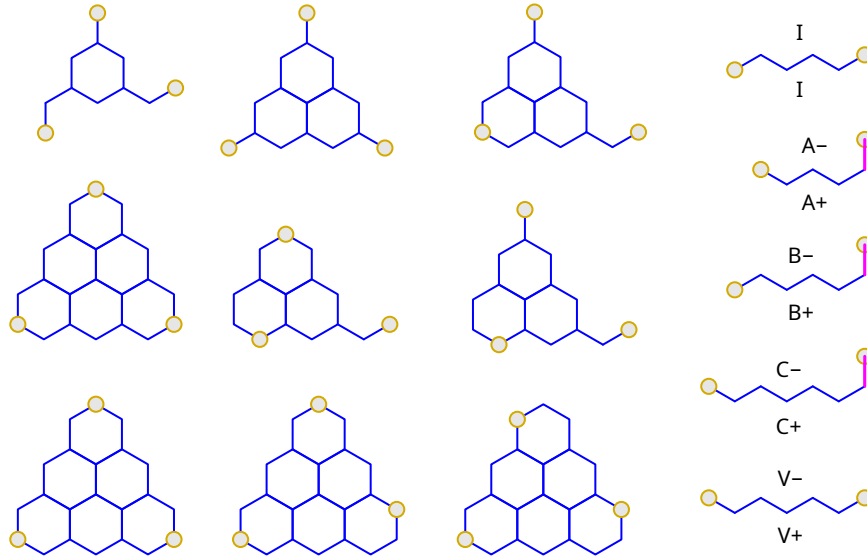


Figure 85

On the right of the figure above, we stress the following important fact:

Lemma 36. *the nature of an interface only depends on its shape as a broken line.*

It follows from pure observation of Figures 76 and 77. Note that on three of the possible interface pairs, we highlighted a particular segment in pink: the ones of type A, B and C. It follows directly from Theorems 30 and 31 that:

⁸Actually they are shorter: we saw in Figure 76 that for one colour of the parallel direction there is at most 4 rhombs and 3 squares, for the other, it follows from the same figure and Figure 69 that there is at most 2 rhombs and 1 square.

Proposition 37. *The data of a partition P of the honeycomb \mathcal{H} and a collection D of dots on some of the vertices of \mathcal{H} , come from a Spectre tiling if and only if:*

1. *It is an assemblage of pieces of Figure 85, where two pieces in contact either only touch on one dot of each piece, or they are in contact along a whole interface between two dots.*
2. *Any dot in D has one and only one of the three segments of \mathcal{H} ending on it which is a pink segment of an interface pair as on Figure 85.*

Moreover, since in Figure 76, the interface completely determine the rhomb/hex pairing for the blue hexes, and the yellow rhomb/hex pairing is determined by the unique A, B or C interface going to it:

Proposition 38. *Given data as in the previous proposition, the Spectre tiling is completely determined.*

We end this section by a property that will be used to prove a relation later between various coordinates (Proposition 56). We make use of complex numbers. It can be safely skipped since it is not used for most of the statements. The proof is immediate.

Lemma 39. *Let $\rho = e^{2\pi i/6} \in \mathbb{C}$ denote the principal primitive 6th root of unity. Assume that the centre of the hexagons of the blue honeycomb forms the lattice $\Lambda = \mathbb{Z} + \rho\mathbb{Z}$, and hence the sides of the hexagons are of the form $\rho^k c$ with⁹ $c = 1/(1 + \rho)$. Then the vectors on the right of Figure 85 are respectively*

- *for I-type, to $(2 + 3\rho) \times c$;*
- *for A-type, to $(1 + 3\rho) \times c$;*
- *for B-type, to $(1 + 4\rho) \times c$;*
- *for C-type, to $(2 + 4\rho) \times c$;*
- *for V-type, to $(3 + 3\rho) \times c$.*

For each pair of dots on any valid hex partition with dots, the vector between two dots on an interface is the product by a power of ρ and of the vector given above for the interface type I, A, ..., V.

For type A, we used that $\rho^2 = \rho - 1$ to transform the expression $2 + 2\rho + \rho^2$ into $1 + 3\rho$. We did similarly for types B and C.

2.6 Equivalent tilesets

By Theorems 30 and 31, tiling the whole plane with the Spectre is combinatorially equivalent to tiling the whole plane with the following marked regular triangles following the rules:

1. the interfaces match (disregarding the yellow hex arrows),
2. each vertex is in contact with one and only one yellow segment, where

⁹Alternatively we can take $c = 1/(\rho + \rho^2) = i\sqrt{3}$.

the two yellow segments superimposed in $A + /A-$, $B + /B-$ and $C + /C-$ interfaces are counted as one.

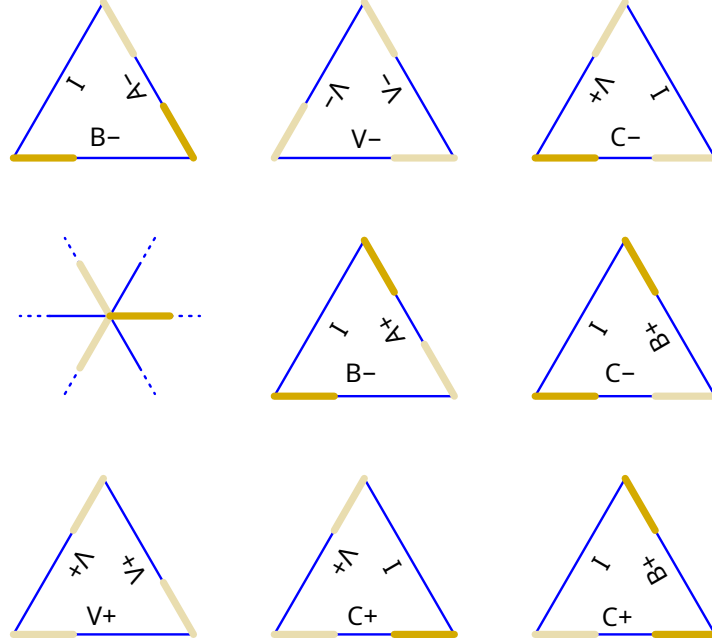


Figure 86 – Some triangles have been rotated for faster visual lookup of markings.

Alternatively we can use the following tiling, which will enforce the yellow edge rule.

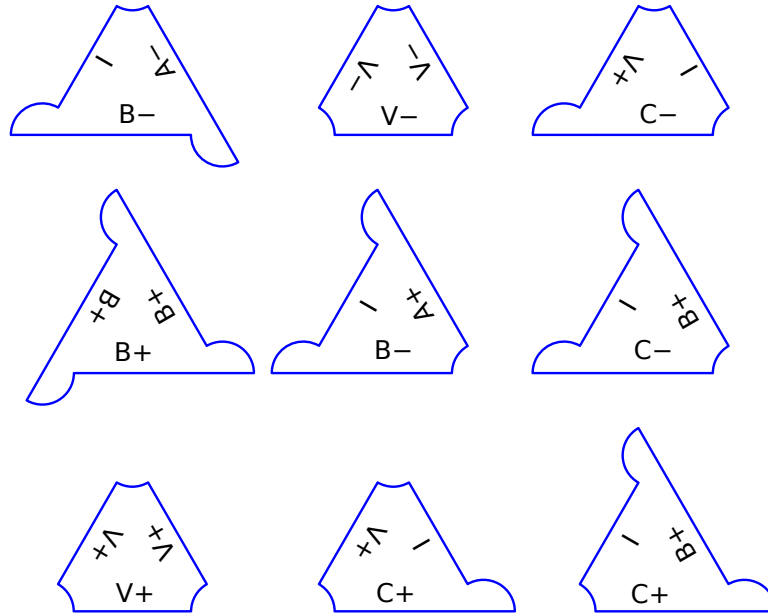


Figure 87 – A correct tiling with these tiles have to tile the plane in the classical sense and respect the markings.

or:

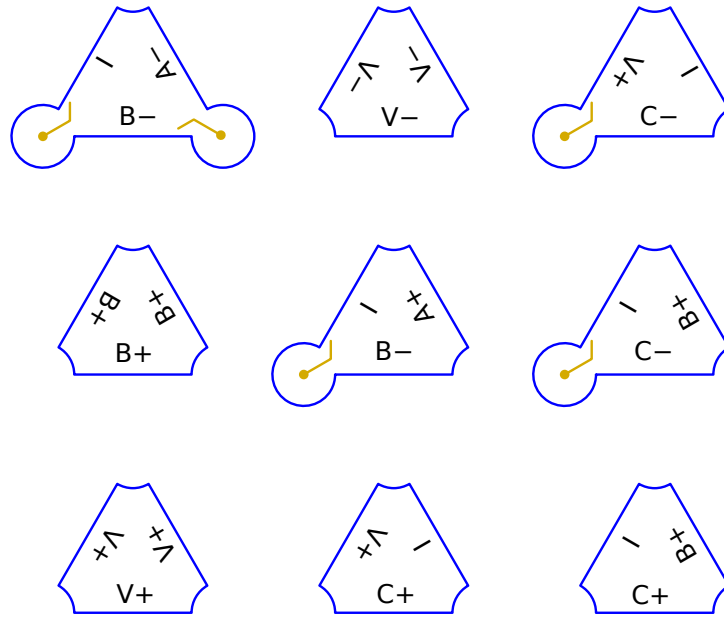


Figure 88

or:

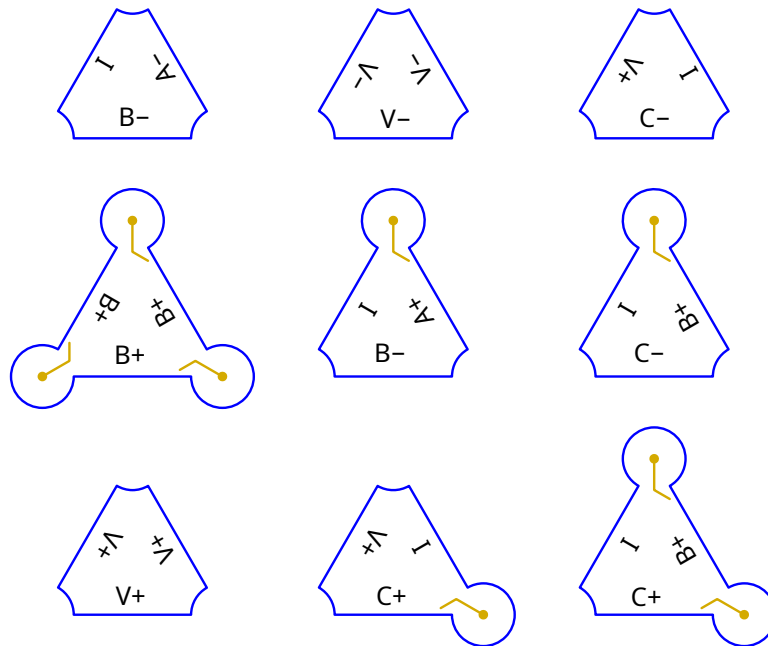


Figure 89

Tiling with all those tilesets are combinatorially equivalent. Note that we can add some supplementary marking as below to ease laying them out:

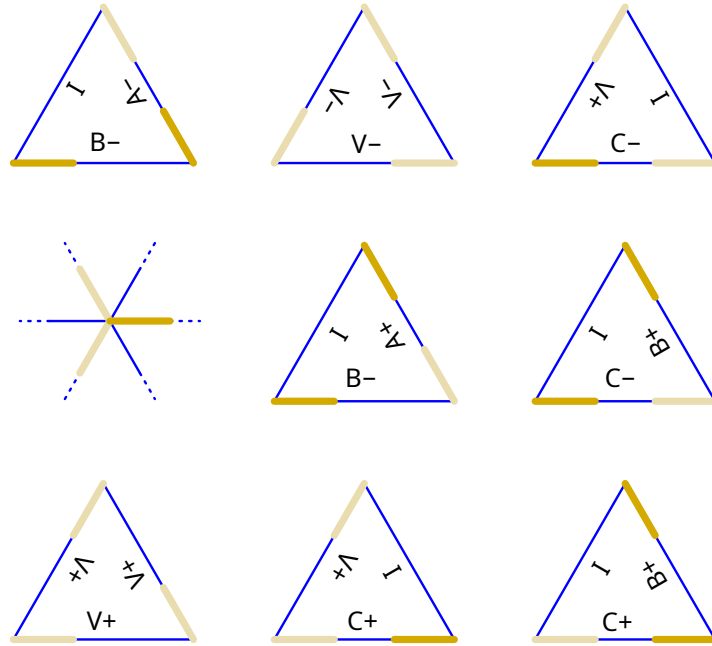


Figure 90 – Tilings with the pieces of Figure 86 automatically respect the beige markings, and at each vertex, stem *alternatively* unmarked and marked half-segments. Recall that to correspond to a Spectre tiling, we must also follow the rule that every vertex should have exactly one yellow marking, as shown on the centre left.

It can be directly checked that the interface labels must respect the new introduced colour. It can also be deduced using Theorem 31, and the fact that it corresponds to the orientation of the yellow hexes.

With these easier depictions to deal with, let us first prove:

Proposition 40. *The triangle marked B+, B+, B+ cannot appear in a whole plane tiling.*

Proof. Let us use the tileset of Figure 86 On the bottom line can only go two types of marked triangle but the one on the left creates a vertex with two yellow lines.

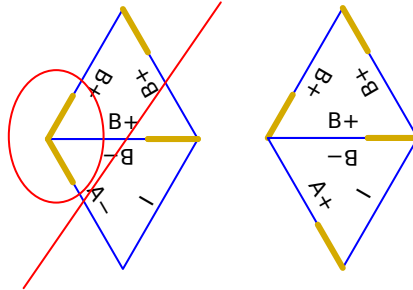


Figure 91

By symmetry this forces the 3 adjacent tiles to the initial one. Since there is only one A– we have:



The diagram shows a portion of a hexagonal lattice. Blue lines represent standard bonds, while yellow lines highlight specific bonds. These highlighted bonds are labeled with signs and letters: $A+$, $A-$, $B+$, $B-$, $C+$, and $C-$. A red circle highlights a central vertex where three highlighted bonds meet, labeled $C-$ and $+A$.

☐

2.7 Triangle vertex neighbourhood

Among the several proposed equivalent tilesets presented before, we choose to use the following pieces:

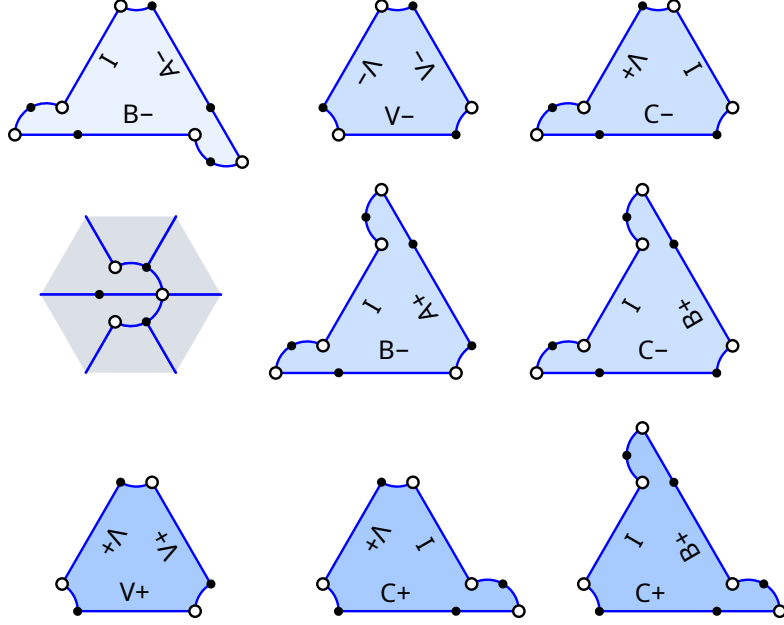


Figure 94 – The lightness of the blue shade has been chosen to reflect the size of the T_n of the corresponding cc: light blue: T1, medium blue: T2, dark blue: T3. The white and black dots must match with dots of the same colour and represent the orientation of the yellow hex (they correspond to the white and black dots on the improved decoration of the yellow hexes, and the white dots also correspond to the beige and yellow markings of Figure 90). Each triangle “vertex” must be as on the centre left. We recall that the white dots are a visual aid and are not necessary: a whole plane tiling with the pieces above, disregarding the white dots, automatically has to match them.

These triangles will arrange as a regular triangular lattice with circular sectors added/removed. The vertices of the underlying triangular lattice are not indicated in the figure above but are “implied”. We need a name to distinguish them from the actual vertices of the pieces in the figure.

Definition 41. We call **triangle vertices** the vertices of the triangles that the pieces are built from (before addition/removal of the circular sectors).

The complete list of compatible arrangements around a triangle vertex is not very long to figure out by hand (not all may appear in a complete plane tiling):

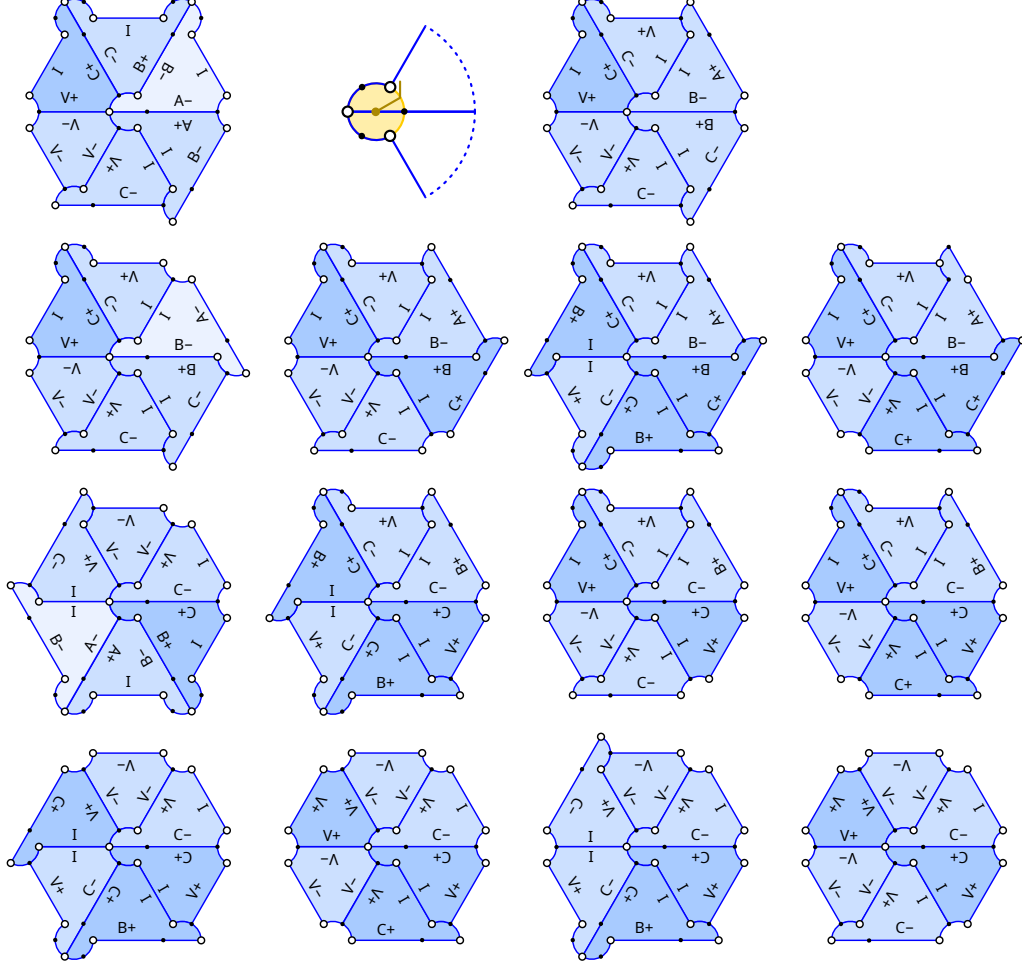


Figure 95 – In this figure all the central yellow hexes have the same orientation of the associated marking (telling where is the paired yellow rhomb), depicted on top.

2.8 Yellow clusters T'_n

2.8.1 Definitions

To prove that tilings with these triangles are uniquely hierarchical, we will identify the next generation of triangles as spotted in the introduction. We will call them T'_n . They are supposed to be like the triangular clusters T_n but for the yellow hexes. Observation guides us into considering candidates following the rules:

1. A V-type interface should link two yellow hexes in the same cluster.
2. An I-type interface should link two adjacent cluster.

So we start enforcing this as terminology:

Definition 42. In a whole plane tiling by the pieces of Figure 94, we declare as **V-connected** any two triangle vertices (yellow hex) that are linked by a V-type segment. Components for this connection relation will be called **yellow clusters**.

Recall that the vertices correspond to yellow hexes, which is why we chose that name for this higher level cluster notion. We will prove in the present section that yellow clusters come in triangular sets $T'n$ for $n = 1, 2$ and 3 .

Definition 43. We declare two components **adjacent** if they are linked by an I-type triangle edge.

For the moment this definition is a bit artificial. A priori a cluster could be adjacent to itself. We will see that this cannot be the case. Keep also in mind that two hex clusters with two hexes that are in contact but linked by another type of triangle edge may or may not be adjacent according to the definition above.

2.8.2 Shape of yellow clusters

Note that there is only one triangle having a $V-$ side, and it is the one of type $V-, V-, V-$. So any two V-connected vertices are on the edge of some $V-, V-, V-$ triangle.

Note also that any tile with a $V+$ side has a $V-, V-, V-$ tile attached to this side. In particular the tile of type $V+, V+, V+$ has a $V-, V-, V-$ tile on each side:

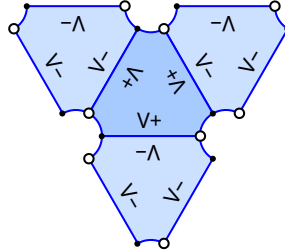


Figure 96

Proposition 44. *Two triangles of type $V-, V-, V-$ that share a triangle vertex are necessarily part of a configuration as above. Such a configuration shares no triangle vertex with any other type V interface. The triangle vertices of the configuration thus form a $T'3$.*

Proof. Because the white and black dots must alternate around a vertex, two $V-, V-, V-$ triangles sharing a triangle vertex must be oriented similarly, thus must be adjacent to a common triangle. The latter must have two $V+$ interfaces, so can only be of type $V+, V+, V+$. The third edge of the latter must match a $V-$ and there is only the piece $V-, V-, V-$ that has a $V-$ interface.

If there would be another V-type edge connecting to one of the triangle vertices of the arrangement, it would imply by the same argument the presence of a $V+, V+, V+$ piece in contact with the arrangement as on the figure below left. Then there would be a $V-, V-, V-$ piece as on below right. But the triangle vertex hole marked in red cannot be filled.

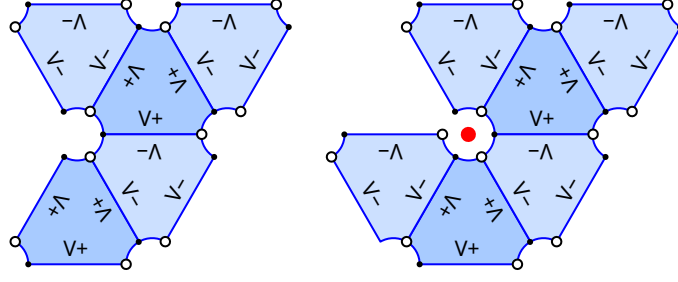


Figure 97

□

Proposition 45. *If a triangle of type $V-$, $V-$, $V-$ does not share a triangle vertex with another tile of the same type, then it shares no triangle vertex with any other type V edge. Its triangle vertices thus form a $T'2$.*

Proof. If such an edge existed, it would be a $V+/V-$ pair, so another triangle with a $V-$ would be there. Since the only triangle with a $V-$ is the $V-, V-, V-$ one, this contradicts the assumption. □

It follows that yellow clusters can only be triangular of type $T'n$ for $n = 1, 2$ and 3 , as it was the case for Tn blue hex clusters.

Like for Tn , the $T'n$ come in two orientations, which we also call **point up** and **point down**. Like for Tn , the orientation is linked to the orientation modulo $1/3$ of the marking of the disk at the centre of its hexagons (i.e. *in fine* of the yellow hex of the decorated graph that this disk represents): this comes from the dot marking on Figure 94 for the $(V+, V+, V+)$ and the $(V-, V-, V-)$ pieces. (Alternatively we can argue that on Figure 76, the V interfaces all have ending yellow hexes in the same class modulo $1/3$ of the orientation, and this is logically equivalent to the fact that V interfaces have an even number of edges on the cc, so link vertices of the same colour in a bipartite colouring of the blue honeycomb.) This allows to extend the notion of orientation to $T'1$ clusters too.

The I -type interfaces change the orientation class modulo $1/3$: to prove it one can adapt any of the three arguments given for V above. It follows in particular that: a cluster is not adjacent to itself.

We will characterize better the adjacency of yellow clusters and prove it mimics a regular triangle tiling, like the blue cluster did.

2.9 First properties of the triangular tiling

Below we change the information on the tiles and we reorder for even faster visual lookup:

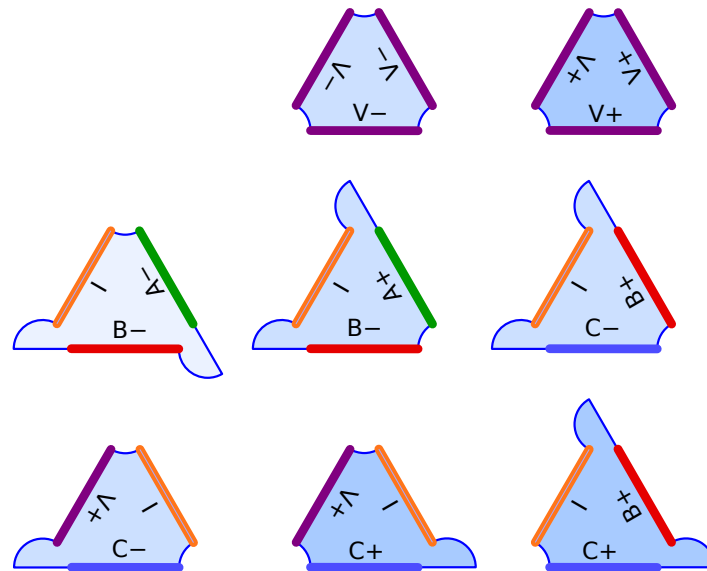


Figure 98

In this case the local vertex arrangements look like this:

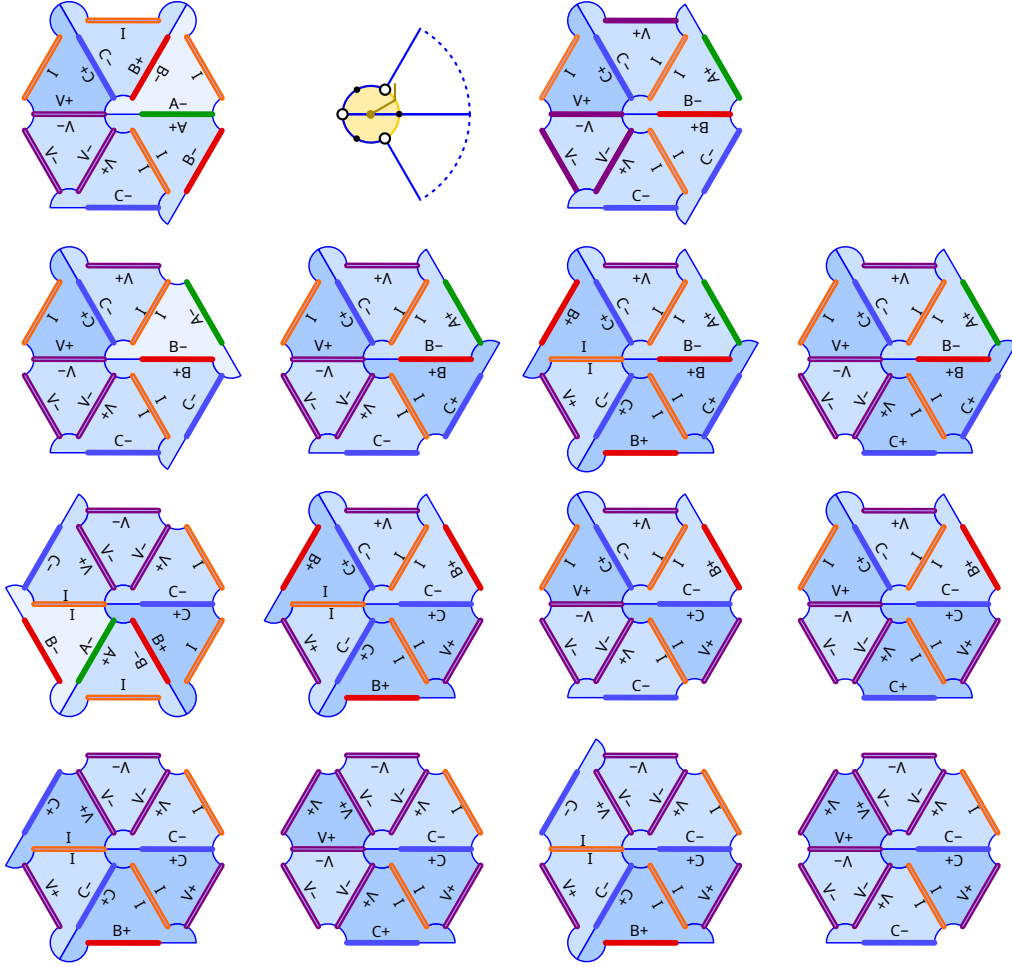


Figure 99

Remark 46. In the sequel we are going to make a lot of deductions from those triangle tilings. Several of these may be analogue to those made earlier with Spectre tiles up to Section 2.4 (some the proofs are in Section 3) and there may be a way to unify them by starting at a lower level, see Section 2.14.1.

Proposition 47. *The only possible arrangement around the unique T1 triangle,*

i.e. the one marked $(B-, A-, I)$, is as follows:

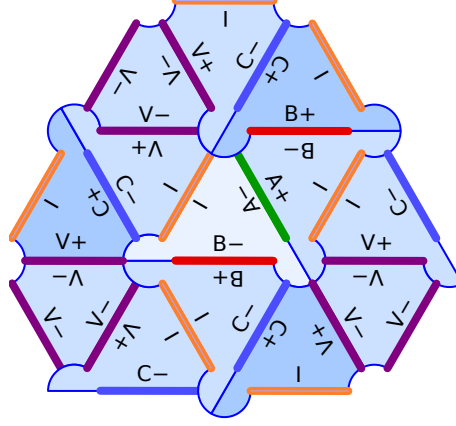


Figure 100

Proof. This follows from Figure 99. Alternatively we can proceed as follows.

Starting from the $A-$ and going around the neighbouring tiles in clockwise order, we see that for the first 5 neighbours, there is each time only one tile that can fit the previously placed tiles: for the first one this is because there is only one triangle with an $A+$. For the next, because there is only one tile with an I interface that has both ends empty of half disks (any half disk would overlap with the already present ones). Etc.

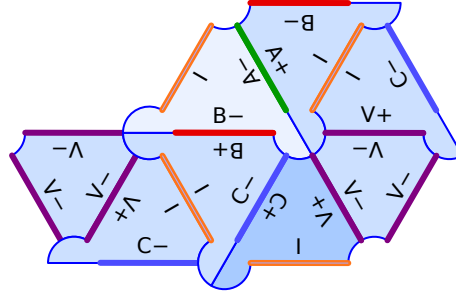


Figure 101

The next tile should have a $V+$ interface with a free end on its right (left and right ends of an interface are determined by rotating the triangle so that the interface is at its bottom). There are only two possibilities. One of them is $V+$, $V+$, $V+$, but it implies the presence of a $V-$, $V-$, $V-$ piece on its right, creating a contact between an $V-$ and the I of the initial triangle. So we get the other one, $V+$, $C+$, I . The next two tiles, still in the clockwise order, are uniquely determined by the interfaces too.

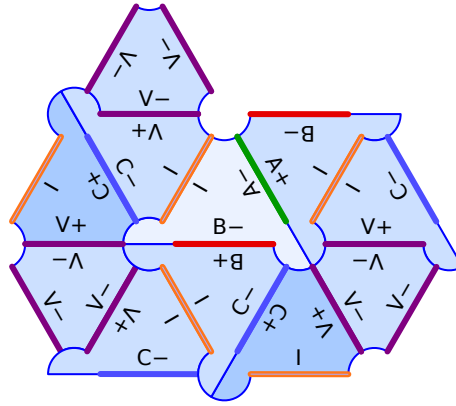


Figure 102

Then on the free $B-$, we need a tile with a half-circle on the right of the corresponding $B+$, for otherwise the hole would stay empty. There is only one such triangle: $B+$, I , $C+$, and then the last one is determined. \square

Corollary 48. *When the first configuration of Figure 76 appears in a whole plane tiling, the orientations of the markings of all yellow hexagons are determined:*

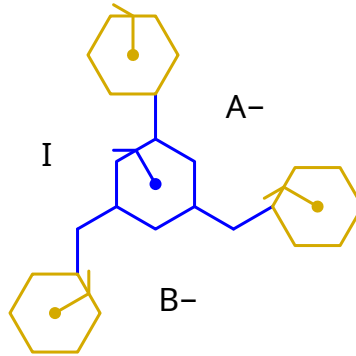


Figure 103

Looking only at the type of interfaces, we realize that collapsing the V type edges in the previous diagram leads to a hexagon of dots related by I -type edges:

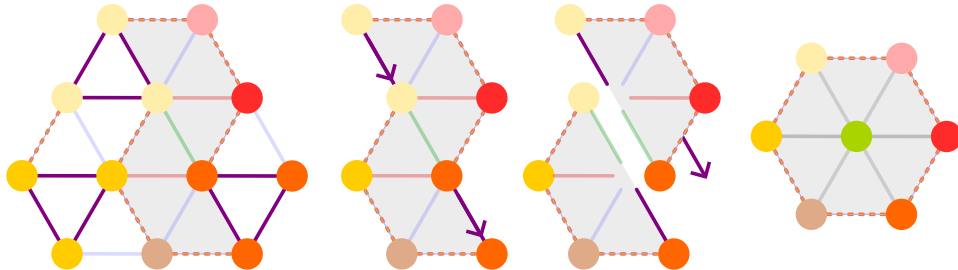


Figure 104 – Dotted lines are I -type interfaces, violet ones are V -type.

These will play the next generation of hexagons after yellow and blue. Among the 6 greyed out tiles, the 4 ones having a B interface (red segment) form a parallelogram with 6 triangle vertices, one for each corner of the big hexagon.

Lemma 49. *The following configuration does not appear:*

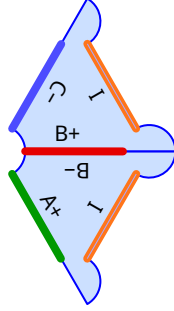


Figure 105

Proof. It appears only once as an internal segment of the list of Figure 99 (the second configuration), upside-down. As a consequence, its left vertex (right vertex in Figure 99) circular hole cannot be filled for the configuration would appear elsewhere in the list. This can also be directly checked by trying to fill this vertex neighbourhood. \square

This excludes the second configuration in Figure 99. Also, given that on Figure 98, the third and eighth tile cannot match their B interface because of two half circles would overlap, the previous lemma forces the pairing between the B-type interfaces: on Figure 98, tile 3 and tile 5 on one hand, tiles 4 and 8 on the other hand.

Corollary 50. *Any A or B interface extends into Figure 100.*

2.10 Packed tilesets

2.10.1 Packs

Thanks to Corollary 50, we can pack tiles into a smaller tileset and use a simpler notation as follows:

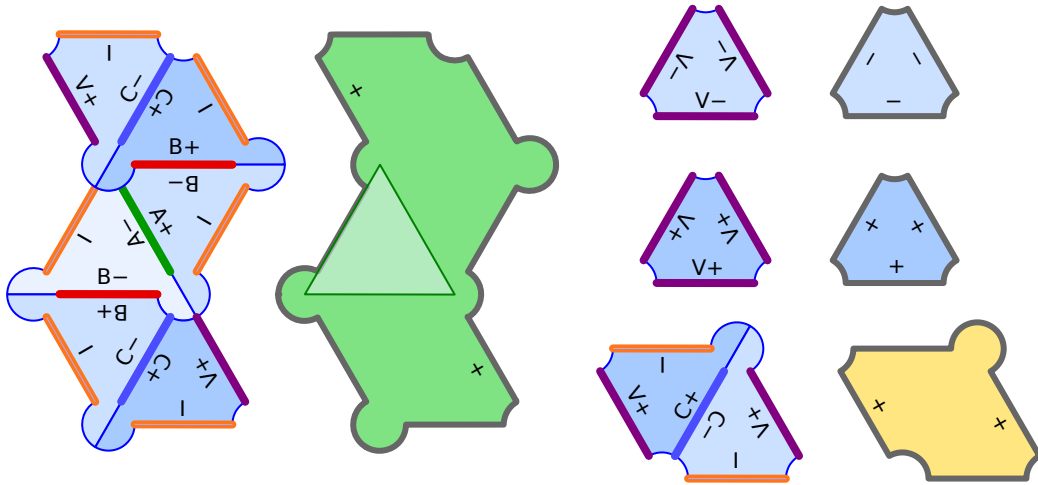


Figure 106

Theorem 51. *Whole plane tilings with those pieces, respecting that $+$ and $-$ must go together, are equivalent to whole plane tilings with any of the triangle tilesets of Figures 87, 94 and 98, so to whole plane Spectre tilings. This corresponds to cutting the plane along the V (violet) and I (orange) type edges.*

Proof. First note that no two distinct pack can intersect since their boundary is only made of V and I while internally they only have A, B or C edges.

Let us prove that the triangle tiling is covered by packs. We have already seen that triangles with a B-type edge must be part of a green pack. Triangles that have no B-type edge and which are not of type $(V-, V-, V-)$ or $(V+, V+, V+)$, are of two types: $(I, V+, C+)$ and $(I, V+, C-)$. Consider a triangle t of of these two types and the triangle that is in contact with it through the C-type edge: either this triangle is the other type $(I, V+, C+)$ or $(I, V+, C-)$, or it has a B-type interface. In the first case t is in a yellow pack, in the second case in a green pack.

The converse statement is immediate. \square

We can equivalently use the following set with only 3 pieces and no markings:

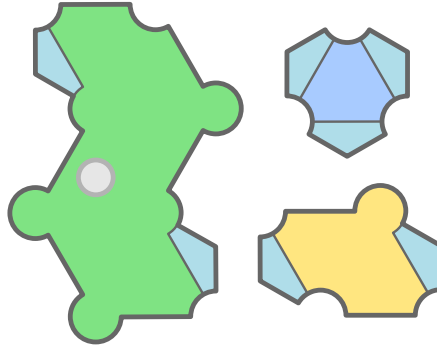


Figure 107

Proposition 47 implies that the big piece is always environed as follows (the environment is actually determined further; see for instance Lemma 67):

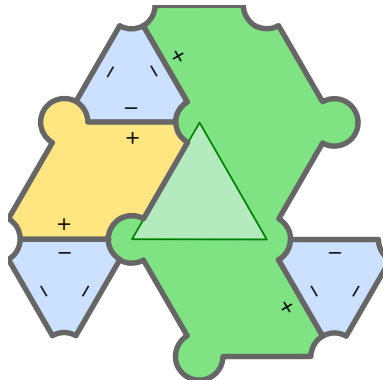


Figure 108

At some point we will use the following decorations on the packed tileset.

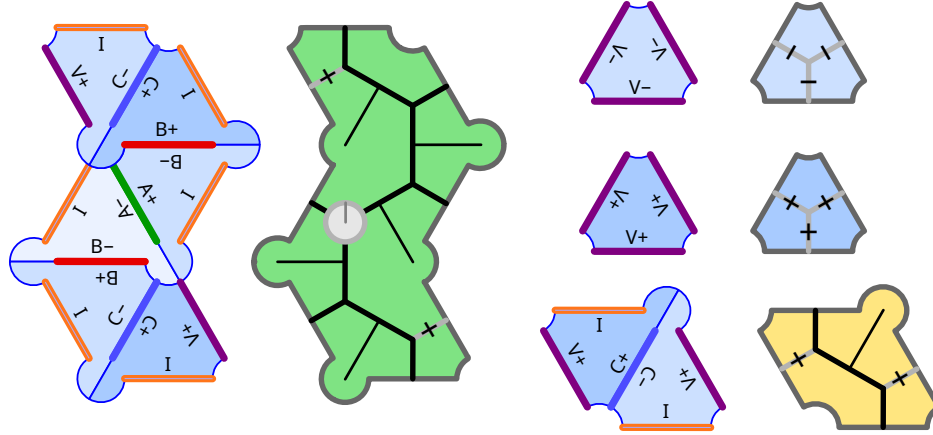


Figure 109

The inner thick lines, in black and in gray, once the packed tiles are assembled in some way, trace hexagons that represent the yellow hexagons represented by the triangle vertices of the triangle pieces composing the packed tiles, **rotated by $1/12$ in the clockwise direction**. The gray thick lines stem from, and are orthogonal to, the centre of the the edges of type V of the underlying triangles. The black thick lines will trace the outline of what will be yellow hexagon clusters: we expect to see triangular clusters of type 1, 2 and 3.

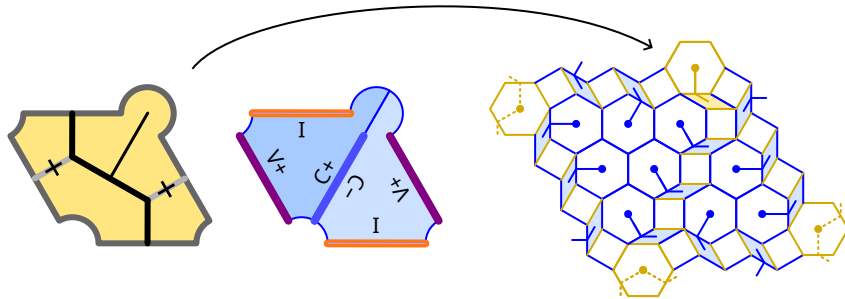


Figure 110 – Example of what a yellow pack represents in terms of hexes of the original decoration graph and Spectre tiling. The arrow shows the yellow hex corresponding to the circle. Note the rotation of $1/12$ between the marking of that circle and that of the yellow hex.

The gray dot is in the centre of the triangle of type (A–, I, B–), the light blue one in Figures 94 and 98, which is the only one which represents a cc with a T1 blue hex cluster (recall that the triangle pieces represent the blue cc's of the yellow/blue decoration graph, which all have a T_n blue hex cluster).

The thin black lines is then the yellow hex marking, rotated by the same amount of $1/12$ clockwise. It is also the extension of the seam between two half circles in the triangular tileset (Figure 94). The thin gray line on the gray dot represents the orientation of the blue hexagon marking of the corresponding blue hex T1 cluster, also rotated by $1/12$ in the clockwise direction.

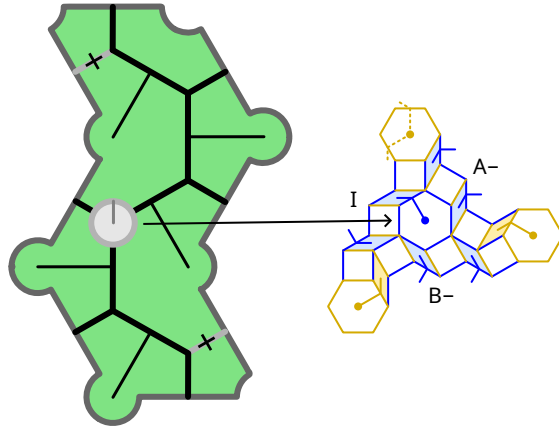


Figure 111 – The gray dot represents a blue hex. Note the $1/12$ rotation between the markings.

We will sometimes omit the thin black and gray lines on the decorated packed tiles of Figure 109.

2.10.2 Adjacency and higher level honeycomb

Let us introduce a third honeycomb into the game, which we call the **green honeycomb**. Since we oriented the blue and the yellow honeycombs the same way, it seems appropriate to orient the green one the same way, which is what we do in the sequel.

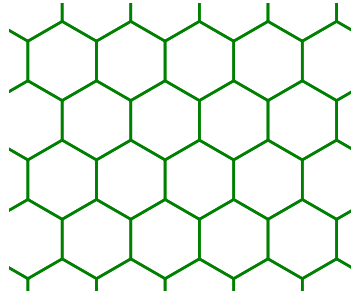


Figure 112 – Green honeycomb

Recall that we called *adjacent* two yellow clusters having two yellow hexagons connected by an I-type interfaces in the underlying blue triangle tileset. This interface is supported by a blue triangle edge, and we look at the vector of this edge, directed by a choice of first and second yellow cluster. Given a whole plane tiling by the blue triangle tileset, we will use these vectors to build a correspondence from each clusters to a *vertex* of this green honeycomb. We sum this up in the following proposition:

Proposition 52. *There is a bijection from the set of yellow clusters to the set of vertices of the green honeycomb such that adjacent vertices correspond to adjacent yellow clusters. Moreover: the yellow cluster points up or down exactly like the triangle of the dual tessellation (to the green honeycomb) that contains the vertex; for two adjacent yellow clusters, the vector between the corresponding vertices of*

the honeycomb has a direction that is rotated by $1/12$ in the anticlockwise direction compared to the direction of the vector supporting the I-type interface.

The rest of this section is a proof of the proposition. Its mechanism reveals interesting insights and more properties of the above correspondence.

Recall that the yellow honeycomb is *visible* as the thick black and gray lines on the decorated packed triangles, and as such is a tessellation of the whole plane dual to the tessellation by the regular triangles supporting the triangle pieces (in the way of the fourth image of Figure 11). In particular the interfaces link yellow hexagon centres to adjacent yellow hexagon centres.

First note that an I-type interface relates two yellow hexes whose orientations are necessarily in different classes modulo $1/3$ of a turn.

Recall that all yellow clusters have been proved to be triangular with 1, 3 or 6 hexagons. We define yellow cluster tips exactly as we defined blue cluster tips in Definition 19. We define **sides** of a cluster as the part of their boundary between two tips. Since tips are yellow hexagon vertices, they are at triangle centres.

Lemma 53. *Each green pack intersect exactly 6 different yellow clusters. Cluster tips are all in green packs. Green packs contain exactly one tip of each of the six yellow clusters it intersects.*

Proof. The first claim comes from yellow clusters being triangular and examination of the thick line in the green pack (see the figure below).

Note that the topmost tip of an up pointing yellow triangular cluster is its only boundary vertex such that the two boundary edges going to it are directed as on the figure below.



Figure 113

An analogue statement holds for the three tips of up or down pointing triangular clusters.

Now examination of the clusters intersected by the different pack pieces show that their direction up or down is determined in advance (if one rotates them by an odd multiple of $1/6$ then one must permute u and d). This follows for instance from the dot markings of Figure 94 but in the case of the green and the yellow pack, this can also be deduced from the orientation class of the clusters with two hexes crossed by the pack, then by the rule that I interfaces shall invert. Then using the rule above, we can identify the tips.

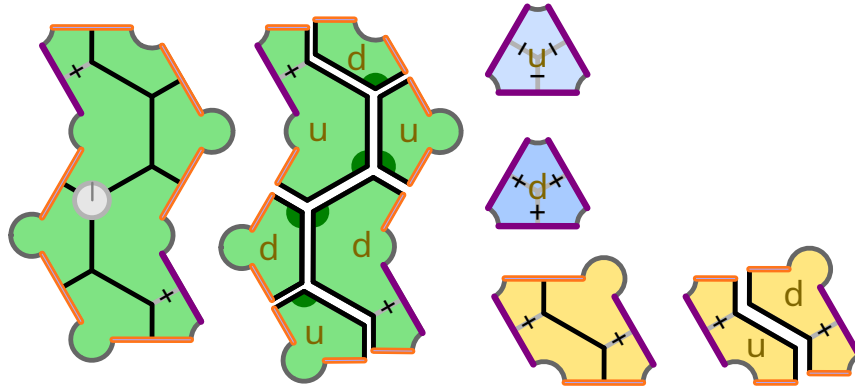


Figure 114 – u: up, d: down, dark-green: yellow cluster tip

□

Corollary 54. *Each yellow cluster meets exactly three green packs. There is at least one I-type interface through each side of a cluster.*

Proof. The first claim is an easy logical consequence of the previous proposition and the fact clusters have three tips. The second claim above follows from examination of the green pack again. □

But there is more to that: examine the picture below.

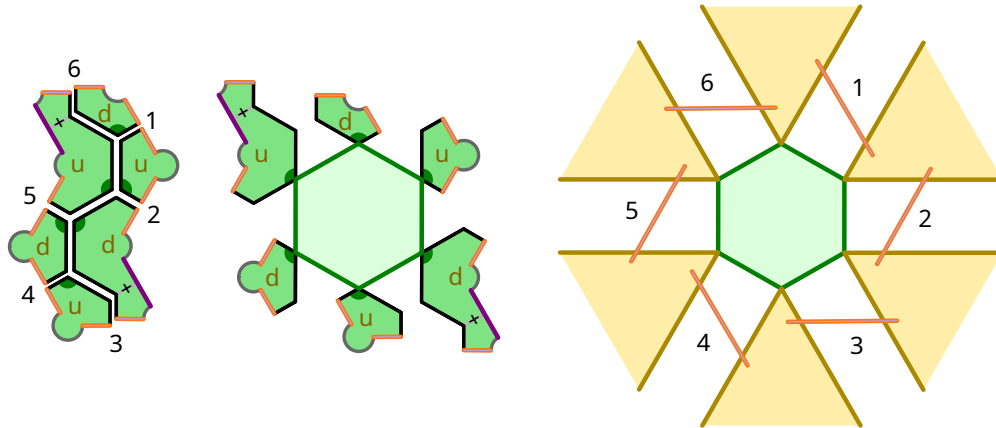


Figure 115

We can associate a green hexagon to the green pack, so that the yellow triangular clusters are related to the hexagons exactly as in the dual regular hexagonal and triangular tessellations are, as on the figure below.

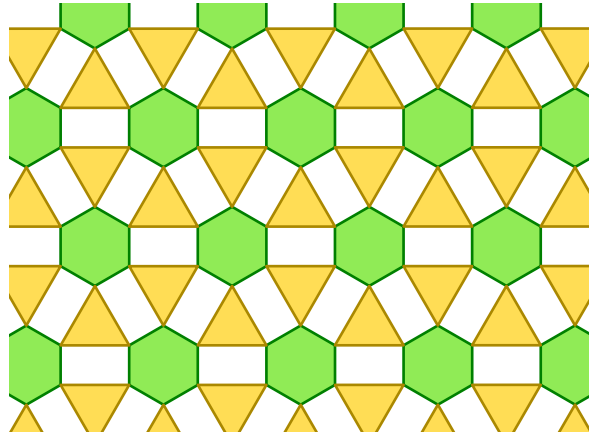


Figure 116

The claim below may now seem obvious, yet we feel it requires an argument.

Proposition 55. *The whole set of yellow clusters and green packs are combinatorially arranged, using Figure 115, exactly as on Figure 116*

Proof. There is no ambiguity on the orientation of the hexagons and of the triangles and identification of their respective tips.

Walking around between the hexes and triangles in Figure 116 corresponds without ambiguity to a walk in the green packs and yellow clusters, and vice-versa.

One main thing is to check that going around a white rectangle in Figure 116 corresponds to coming back to the same pack or clusters. But the intersection of the boundaries of two adjacent yellow packs is a zigzag broken line and must enter green packs near each end. The claim follows.

The other main ingredient is simple connectivity of the plane. Because of this, any walk starting from a green hex (or yellow triangle) and coming back to itself can combinatorially decomposed into elementary loops around hexes, triangles and rectangles. Similarly any walk starting from a pack (or yellow cluster) and coming back to itself can be combinatorially decomposed into elementary loops. Since these loops do not break the correspondence, we get that it is a bijection.

Another approach is to build a piecewise linear map from the plane with the pack tiling to the plane with the hexagons, triangles and rectangles (or the other direction). Such a map will be a covering and since the plane is simply connected, a homeomorphism. \square

By collapsing the yellow triangle in Figure 116 we get the green honeycomb. By the proposition above, this gives a bijective mapping associating a vertex of the green honeycomb to a yellow cluster, such that adjacency in the honeycomb graph corresponds to adjacency as defined before, i.e. via the presence of an I-type edge of the blue triangle pieces connecting yellow clusters. Note that for each such adjacency, the I edge direction must be rotated by $1/12$ in the counterclockwise direction to give the direction of the corresponding hexagon edge in the green cluster.

2.10.3 Coordinate relation between three levels of honeycombs

We prove here a relation that was guessed in the introduction. It will not be needed for most statements in this article.

We use the same notations as in Lemma 39: $\rho = e^{2\pi i/6} \in \mathbb{C}$, we assume that the centre of the hexagons of the blue honeycomb forms the lattice $\Lambda = \mathbb{Z} + \rho\mathbb{Z}$, and hence the sides of the hexagons are of the form $\rho^k c$ with $c = 1/(1 + \rho)$. Below we use a yellow and a green honeycomb, and we also assume that their hexagon centres are the points of the lattice Λ . The vertices of the honeycombs are a subset of $c \times \Lambda$.

We denote HB , HY and HG the blue, yellow and green honeycombs.

Proposition 56. *Consider the data of a partition of HB , with dots, associated to a whole plane tiling by the Spectre. To each dot we associated a hexagon in HY . We call $y \in \Lambda$ the coordinate of its centre and $b \in c \times \Lambda$ the coordinate of the dot in HB . The yellow hexagon belongs itself to a yellow cluster $T'n$ which we associated to a vertex of HG . Denote $g \in c \times \Lambda$ its coordinate. We choose any dot as a reference and let b_0 , y_0 and g_0 denote the associated values. Then we have the following relation for any other dot:*

$$b - b_0 = 3 \times (y - y_0) + \rho^2 \times (g - g_0).$$

(Note that $\rho^2 = j = e^{2\pi i/3}$ is the principal primitive third root of unity.)

Proof. The caption of the figure below recalls the scale and orientation convention, which is the same for each of the three honeycombs.

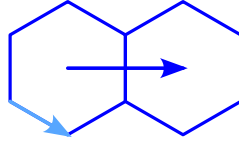


Figure 117 – Dark blue vector has affix 1 and light blue has affix $c = 1/(1 + \rho)$.

On the figures below the light green vector has affix: 3 in HB , 1 in HY , 0 in HG ; the dark green vector has affix: $\rho \times (3 + 2\rho) \times c$ in HB , ρ in HY , ρc in HG . For the blue coordinates, this was already computed in Lemma 39, realizing that the light green is associated to a V-type interface, and the dark green one to type I.

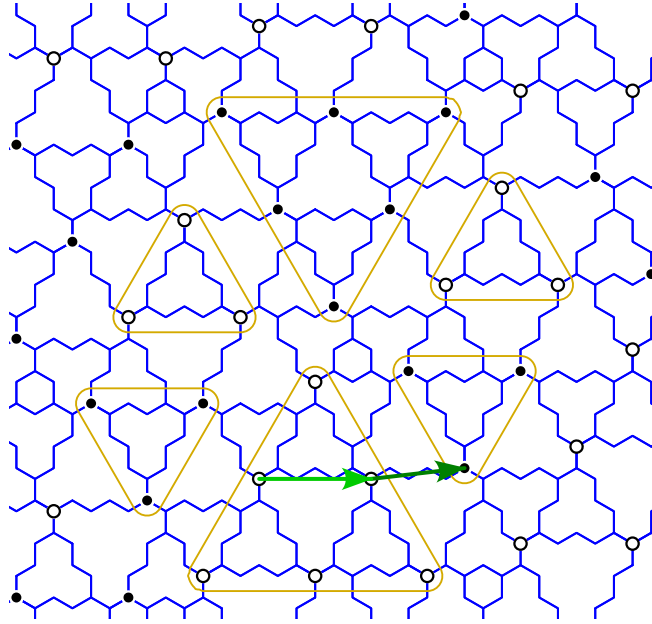


Figure 118

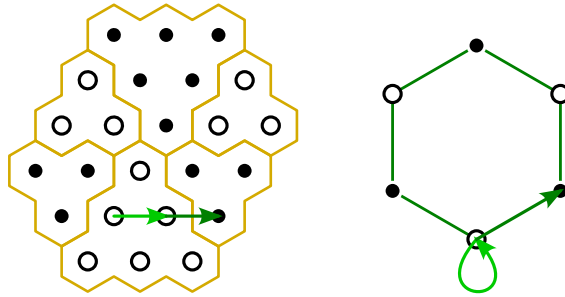


Figure 119 – Note that yellow shows reflected possible arrangements compared to possible arrangements of blue or green honeycombs

More generally, going from one yellow hex to an adjacent yellow hex in the same cluster moves the centre by some ρ^k , $k \in \mathbb{Z}$ (which we may take between 0 and 5 included but this does not matter here), while the blue coordinate will move by $\rho^k \times 3$ and the green one will not move. By definition, two adjacent yellow clusters have a pair of adjacent yellow hexes such that the corresponding dots in the blue honeycomb are extremities of an I-type interface. If going from the first to the second is through vector ρ^k in the yellow honeycomb, then it is by vector $\rho^k(2 + 3\rho)c$ in the blue and vector $\rho^{k+1}c$ in the green.

In both cases, the quantity $b - 3y - \rho^2g$ is invariant. Since any two yellow hexes can be related by a finite sequence of moves as above, we are done. \square

Note the angle $1/12$ rotation between the I-type link in the yellow honeycomb and the green one, which was already noticed in the previous section. Be also aware that not all two adjacent yellow hexes of adjacent yellow clusters are related by an I-type move, yet at least one pair in each adjacent clusters are, as the figure below shows on an example.

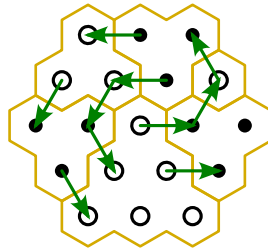


Figure 120 – All the I-type links for some situation (actually occurring in whole plane tilings).

2.11 Environment of the yellow clusters

Recall that we proved in Section 2.8 that every yellow cluster is a $T'n$, $n = 1, 2$ or 3 . We now start the moderately tedious task of classifying all the possible environments of the $T'n$ (yellow hex clusters).

2.11.1 Some visual rules

We sum up here some configurations that are either impossible or have some implied continuations.

Lemma 57. *Consider a connected partial tiling with the packs. Consider a triangle vertex for which among its 6 possible neighbouring triangles, only two are not covered, and they are contiguous. If the free interfaces have no + or – sign, then the non-covered part can only be filled by a green piece, with the central dot being one of the four dots indicated in the figure below.*

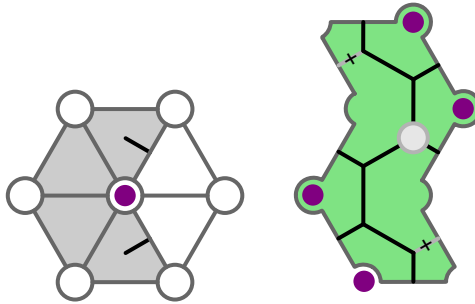


Figure 121 – The thick black lines indicate that the free interface has no + nor – marking. The white circles, including the centre one, may or may not be already filled. (A posteriori the rightmost one cannot.)

This is immediate by examination of all possibilities, and does not take too much time since we have only 4 possible packs pieces.

Lemma 58. *The following configuration can only be filled by a yellow piece:*

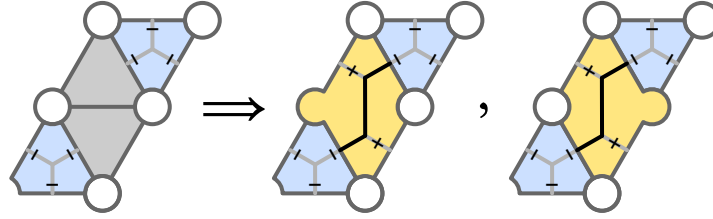


Figure 122

Proof. Again it is a simple examination. Trying to fit a $+, +, +$ piece on some gray triangle implies the other gray triangle is a $-, -, -$ but then some interface is in contact with another $-, -, -$, contradicting the rule that no two $-$ interfaces can fit. If instead one tries to fit the $+$ of a green piece on one of the initial $-, -, -$ piece, the other $-, -, -$ piece touches it on a side without a sign, which is also forbidden. \square

For convenience, we reproduce here Figure 108 and the conclusion of the discussion around this figure:

Lemma 59. *Any green pack is environed as follows in any pack tiling covering a sufficiently big neighbourhood of it (independent of the tiling):*

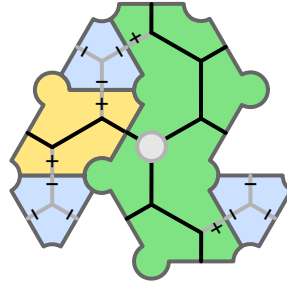


Figure 123

The claim about the neighbourhood independence follows from the fact that in the proof of Proposition 47, which the lemma is deduced from, we did not need to look at the tiling beyond some radius around the central triangle.

2.11.2 $T'1$

We already noted that a $T'1$ is the centre of either the 5th or 8th configuration of Figure 99.

Proposition 60. *The fifth configuration of Figure 99 cannot occur in a whole plane tiling.*

Proof. See Section 3.3. \square

So a $T'1$ is always the centre of the 8th configuration of Figure 99, which we recall below left. Below right we indicate the corresponding green packed tiles configuration it induces.

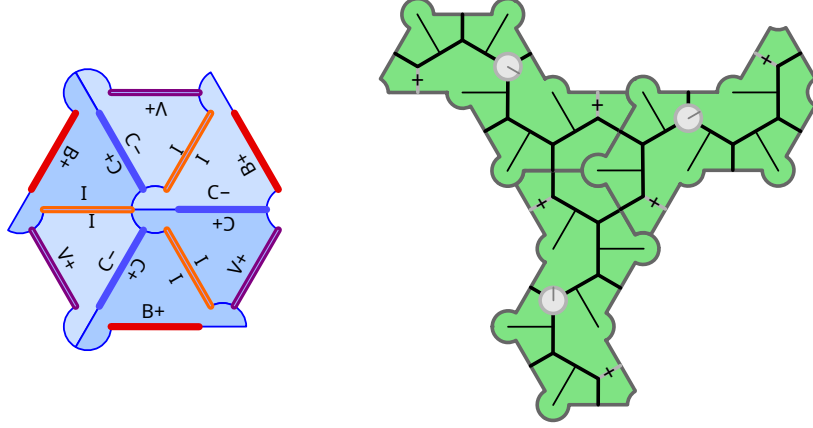


Figure 124 – The $T'1$ is the yellow hex represented by the circular arc at the centre.)

This immediately resonates with the right part of Figure 103, which we recall here, rotated then mirror-reflectied on the right. What resonates with a yellow hex in the figure below is a gray dot in the figure above.

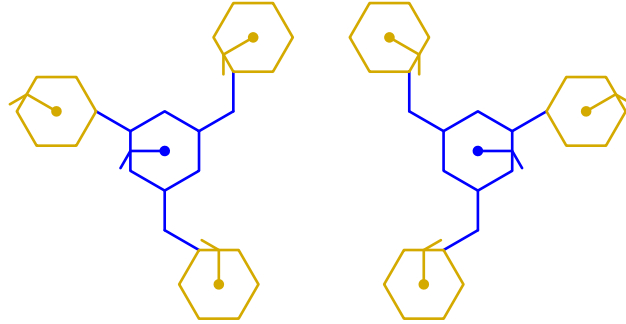


Figure 125 – Left: a specific cc type; right: its reflection

2.11.3 $T'2$

Let now us study the environment of a $T'2$. By the analysis of Section 2.8 it is associated to a $-,-,-$ triangular piece of the packed tiset of Figure 109. Recall that by Proposition 44 the $+,+,+$ triangle cannot touch it for a $T'2$.

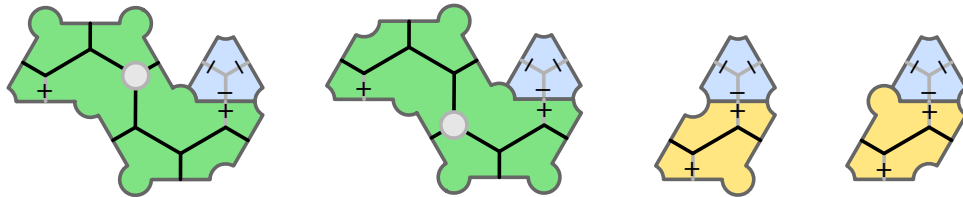


Figure 126 – What can be in contact with one side of a $T'2$

The two diagonal sides of the yellow pack cannot be in contact with another yellow pack, for that would force the presence of a $(-, -, -)$ piece sharing a triangle vertex with the original one, and we would get a $T'3$:

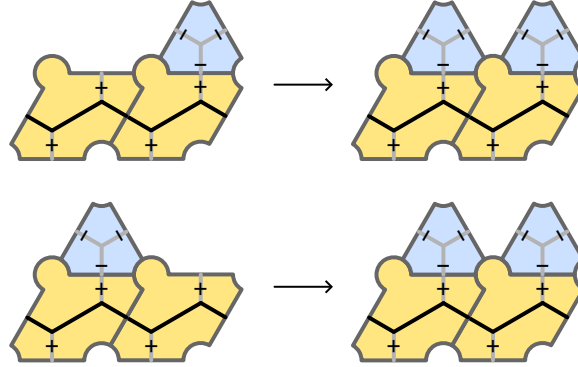


Figure 127

Similarly, the situation below-left is impossible:

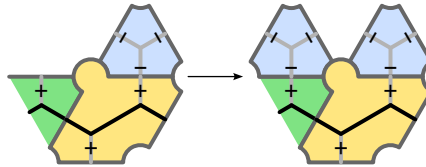


Figure 128

The following is impossible too, because the red dot cannot be filled:

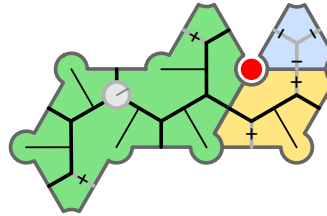


Figure 129

Focusing on the lower left corner, we necessarily are in one of the 6 following cases, where the reader will note that the lower left tip of the $T'2$ is occupied by a green triangle, and also that the green piece appears in the 6 possible orientations.

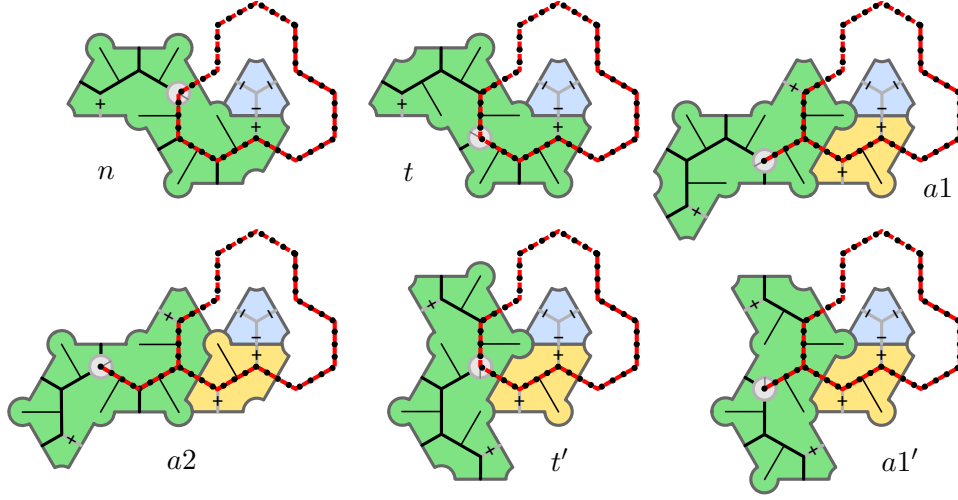


Figure 130 – The 6 possibilities have been given labels.

The immediate environment of the central blue triangle is a copy of three of the situations of Figure 130, rotated by 0, $1/3$ and $2/3$.

We now invite the reader to focus on the circles of position highlighted below:

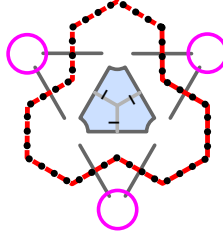


Figure 131

The configurations in the central column of Figure 130 occupy two of these sites, so no two central column configurations can occur, and in fact no two of the following can occur in clockwise order: (n or t or t') then (t or t' or $a1$ or $a1'$).

In the next statement we exclude a few other possibilities. Nearly all concern the presence of a two segments long antenna $a2$ and something else.

Lemma 61. *Consider two tips following each other in the clockwise order around a T'2 cluster of a whole plane tiling by the packs. Then their respective types cannot be in the following list (each item is to be read in the clockwise order): (n, n), ($a2, a2$), ($a2, n$), ($a1, a2$), ($a1', a2$).*

Proof. See Section 3.3. □

All this restricts the possibilities to exactly the same as in Propositions 26 and 27, mirrored, which we recall below. Indeed, using the nomenclature above, we get that no two n nor two $a2$ can be present by the first two case of Figure 131. Every other excluded case concern an $a2$ (length two antenna) and tell us respectively that, in clockwise order, it cannot be followed by an n , nor preceded by $a1$ or $a1'$. Recall also that one cannot have in clockwise order (n or t or t') then (t or t' or $a1$ or $a1'$). So n is followed by $a2$ and t or t' by n or $a2$. Hence, with no n

nor a_2 , there can only be a_1 or a_1' . We thus only get up to circular permutation, in clockwise order: with an n : $(n, a_2, t \text{ or } t' \text{ or } a_1 \text{ or } a_1')$ i.e. the last two cases of Figure 132; with no n but an a_2 : $(t \text{ or } t', a_2, a_1 \text{ or } a_1')$ i.e. the first case; with no n nor a_2 : $(a_1 \text{ or } a_1', a_1 \text{ or } a_1', a_1 \text{ or } a_1')$, i.e. the second case. In each case the hex orientation (visible as thin black or gray lines on Figure 130) match those of Figure 132.

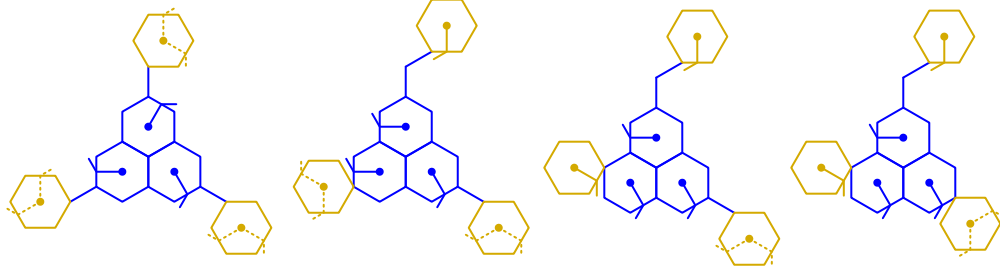


Figure 132 – Mirrored versions of the figures of Propositions 26 and 27

2.11.4 $T'3$

Lemma 62. *The environment of a $T'3$ is as on the figure below:*

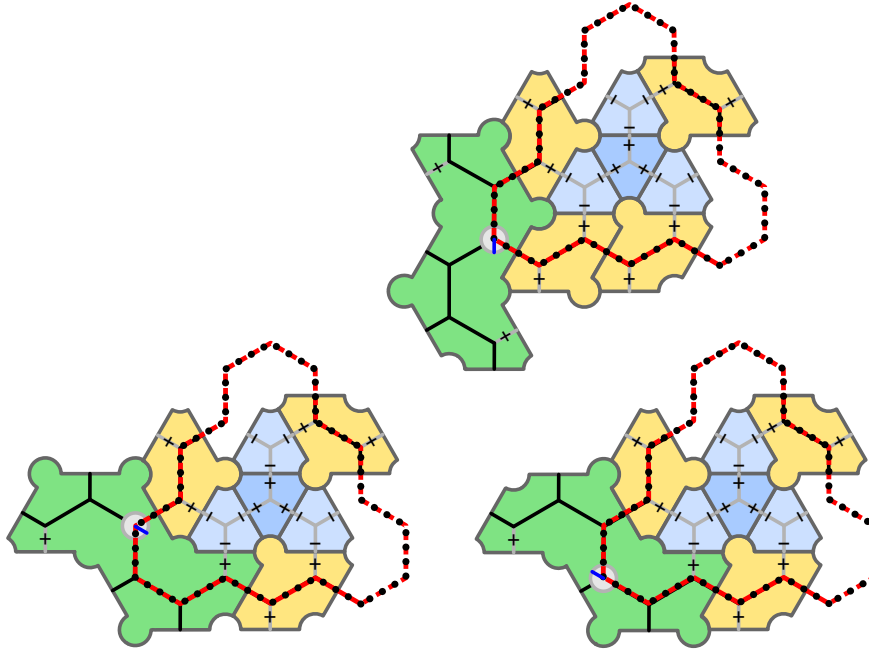


Figure 133

Proof. See Section 3.3. □

All this restricts the possibilities to exactly the same as in Proposition 29,

mirrored. For convenience we reproduce this figure here:

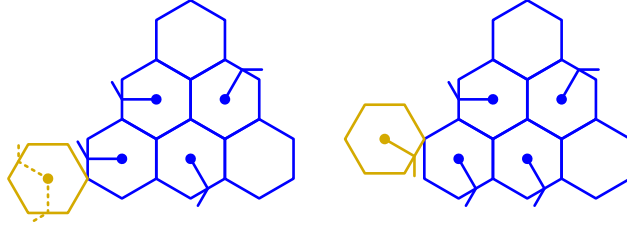


Figure 134 – Mirrored and rotated copies of the two situations of Figure 75

2.12 Hierarchical structure

We can thus reprove the results in [Smi+23a]:

Corollary 63. *Whole plane tilings by the Spectre, without reflections, are uniquely hierarchical.*

Proof. A whole plane tiling induces a yellow/blue graph which by Theorem 30 decomposes into pieces enumerated in Figure 76 with superimposing interfaces.

Such a tiling has been seen to be combinatorially equivalent to a tiling with triangular pieces given in Figure 94. Then the triangles have been proven (Theorem 51) to pack into shapes given by Figure 106. These shapes have been given decorations in Figure 109, which trace in the plane the same type of figure as Figure 17.

Let us mark as pink the central thick black line of the green pack as on the figure below:

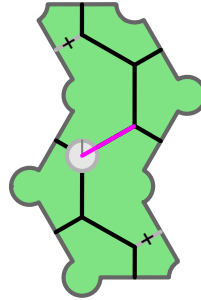


Figure 135

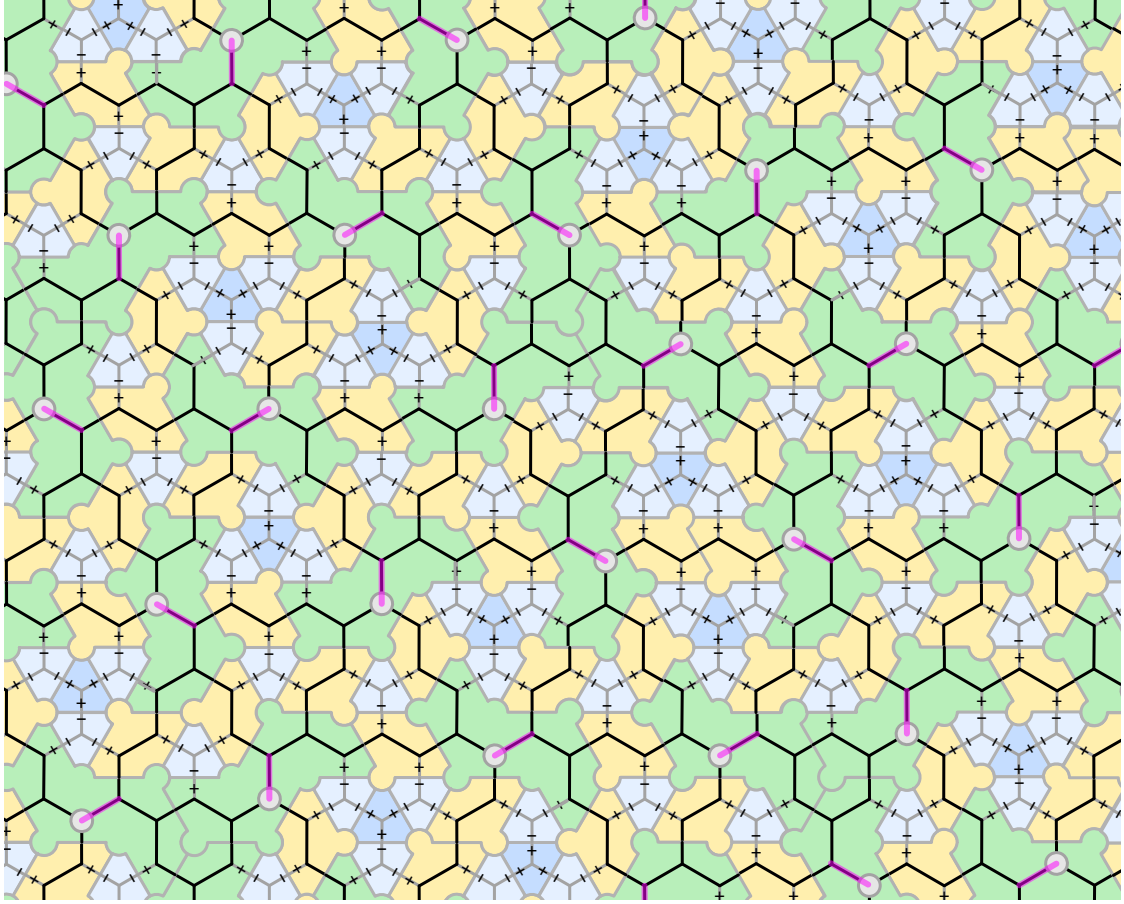


Figure 136 – Example of decorated pack tiling, for some whole plane tiling by the Spectre.

The thick black and gray lines of those decorations cut the plane into big hexagons and the thick black lines group those hexagons into clusters $T'1$, $T'2$, $T'3$, which can be extended within the green packs to reach the big grey dot. Those extended clusters, which we will also call **big cc**, look exactly like the reflection of the blue connected components of the yellow/blue graph that we enumerated in Section 2.4 and called **cc**.

We can then invoke in Section 2.5 the Proposition 37 (the second rule is respected, using the pink segment and the fact that in the list of possibilities made in Sections 2.11.2 to 2.11.4 they always correspond to the pink segment of Figure 85, repeated below), which implies that the extended clusters above correspond combinatorially to some reflected Spectre tiling. Then the whole analysis can be carried out again, which proves the hierarchical nature of the tiling. In every step, the grouping is unique and in the combinatorial equivalence between different kind of tilesets, the correspondence is unique. So the tiling is uniquely hierarchical.

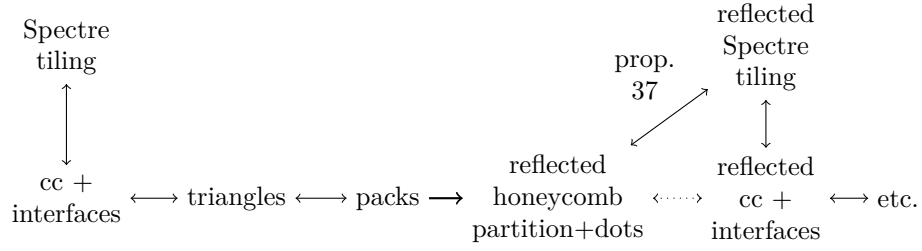


Figure 137

Alternatively, we can bypass the use of Spectres, because the way Proposition 37 is proved is... via the dotted equivalence. \square

Moreover we can invert the thick arrow in Figure 137:

Proposition 64. *Any whole plane or partial honeycomb partition with dots, which is decomposable as an assembly of reflections of the pieces of Figure 138 such that:*

- *pieces are rotated by a multiple of $1/6$ and assembled by superimposing their boundary interfaces from dot to dot (they must have the same shape and dots must map to dots),*
- *hexes do not overlap and each element of the partition is the set of hexes of a piece.*
- *every dot has one and only one of its three incoming segments that is pink in the list on the right of interface pairs of the figure,*

gives rise to a whole plane or partial tiling by the packed pieces of Figure 109.

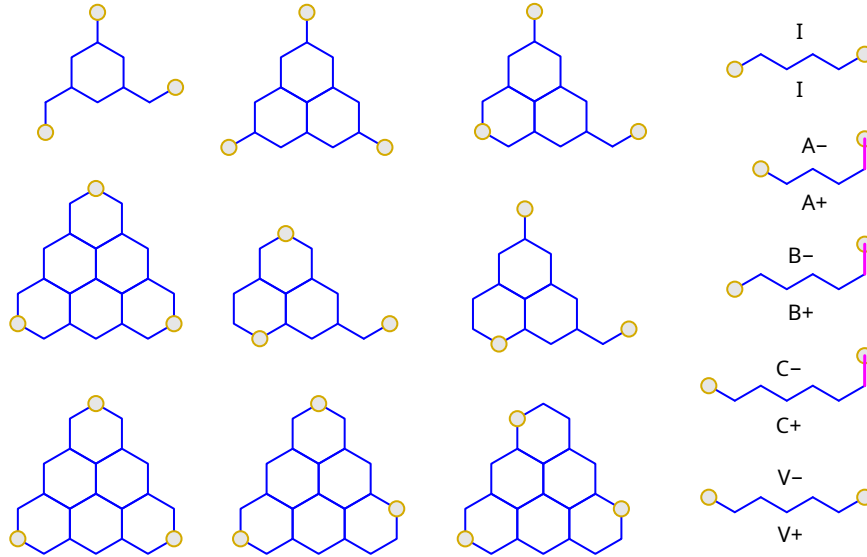


Figure 138 – This is a copy of Figure 85. It shows a list of possible cc shapes (with one actually never appearing) together with dots marking where yellow hexagons touch them, for decorated graphs associated to Spectre tilings. But here we do not assume initially that there is a Spectre tiling; only that we have a

Proof. First we recall the possible tip types for the cc's, according to the position of the associated dot, and the presence or not of an antenna. The tip is associated to the triangle cluster, assumed to point up in the figure below, and we focus here on the lower left tip of the triangle.



Figure 139 – Tip types for a lower left tip. Cluster tip circled in green. Dot position indicated by a gray circle, its orientation by a short blue line.

If we are to cover the dot by a green pack respecting the dot orientation, in can only be the as follows.

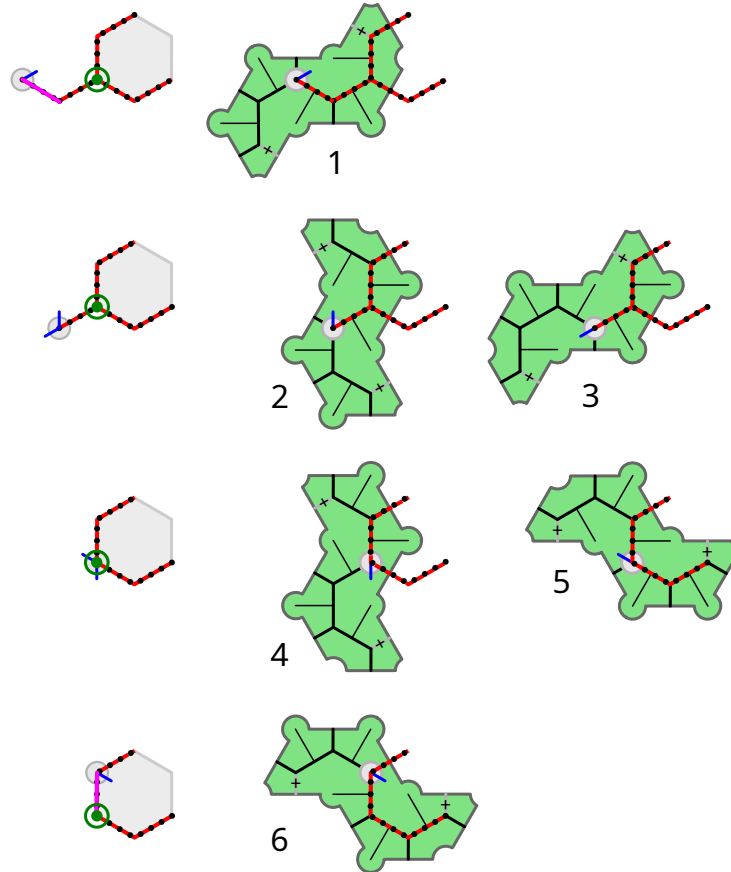


Figure 140

It is remarkable that the dot, hence the pack, comes in the 6 possible orientation once and only once in this figure.

Consider now a dot whose orientation, given by the marking or equivalently the pink edge, we arbitrarily fix as on the figure below with the blue line oriented down. Then there are exactly the 6 possibilities below, where the tip comes in the 6 possible orientation once and only once in this figure.

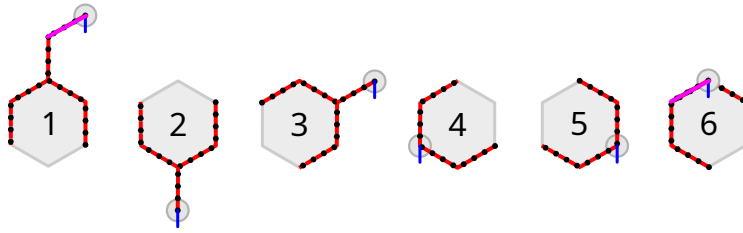


Figure 141

On the figure below we realize that the dot must be environed by a unique assembly. Moreover this assembly uses each possibilities above once.

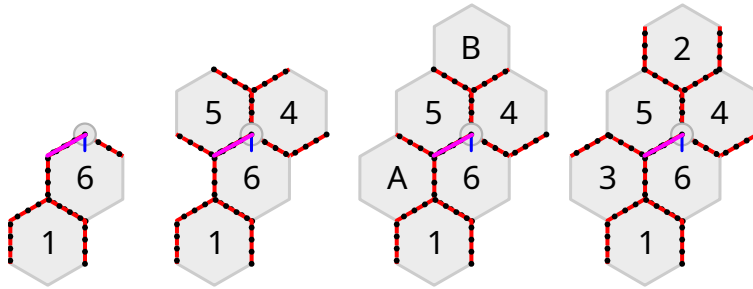


Figure 142 – First, because of the pink segment rule, 1 and 6 must assemble as on the left picture. The other two hexagons touching the dot can only be 4 and 5 as on the second picture. Then the hexagons called A and B can only be 3 and 2.

We now examine the effect of placing the green pack on this configuration:

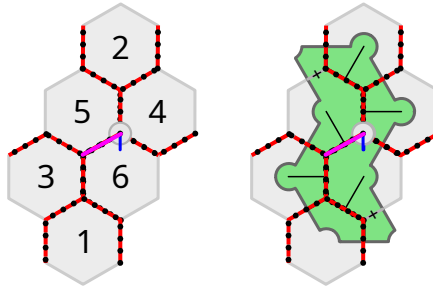


Figure 143

Keeping this in mind we will proceed as follows: first for each dot we put the green pack as above. We claim that two different dots give disjoint green packs. Indeed note first that on Figure 143, the only vertices of the honeycomb that a green pack covers belong to the boundary of the reflected cc piece containing its dot. Then on Figure 140 we realize that:

Fact 1: the vertices of the cluster that are covered by the pack are in 4 cases the tip plus the next vertex along the cluster boundary in the clockwise direction, and in cases 5 and 6 the two preceding ones.

On a T2 and a T3, these vertices are disjoint. On a T1 the three tips have antennae and thus we also have disjointness. We also must check disjointness of the disks at triangle corners, i.e. at hex centres. Actually disks from the green packs are only placed at those hexes that have their cluster tip as a vertex. So

for the three dots of a given reflected cc piece, they are different hexagons, unless we are in the case of a T1. In that case, since the only reflected cc piece in our collection with a T1 is as on the figure below, for which in the two allowed cases¹⁰ there is no overlap:

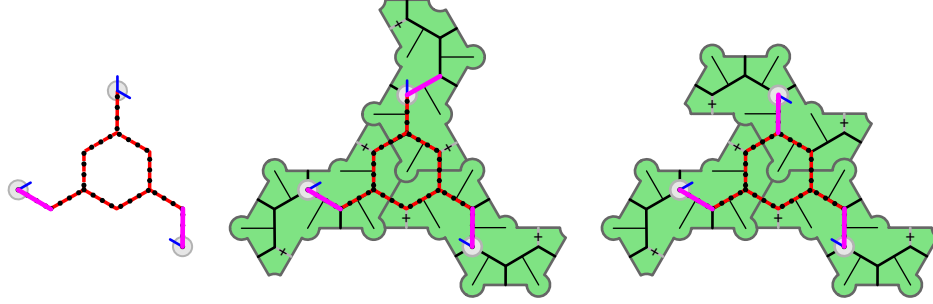


Figure 144

Second, for each T2 we put a $-,-,-$ tile in the centre and for each T3 we put a $+,+,+$ one, surrounded by three $-,-,-$ tiles:

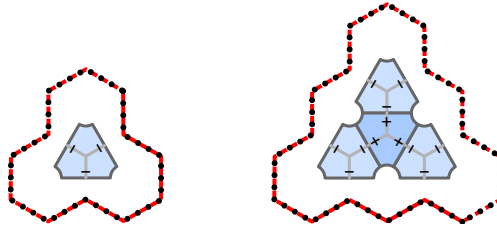


Figure 145

There remains to place yellow tiles. For each T3 clusters we place three ones as follows:

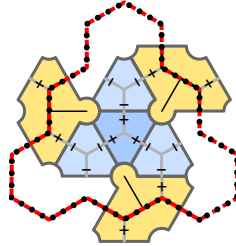


Figure 146

Note that they cannot overlap the vertices covered by the green, by Fact 1 above. Then we place a yellow tile as follows for each type 1 tip of a reflected cc piece with an T2, on the side preceding it, in the clockwise order. We show the three allowed cases in the figure: the yellow piece is placed the same way in each.

¹⁰One of which actually never appears if the situation of the proposition covers the whole plane, but that does not matter here.

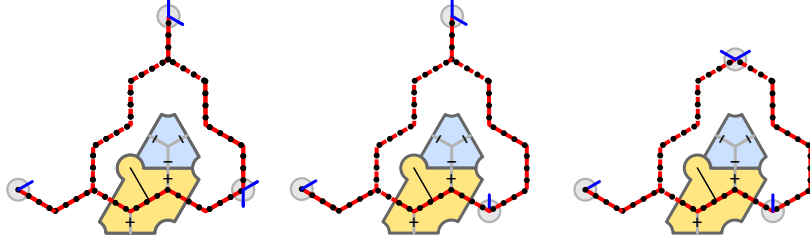


Figure 147

Using Fact 1 again, it covers honeycomb vertices disjoint from the green packs. It is also disjoint from the already placed yellow tiles: it is obvious for the disk and the honeycomb vertices they cover belong to an interface that does not match with those of the reflected cc pieces of T3 type if we take the dot orientation into account: the interface would have to be of type I but the orientation of the two left dots on the figure below are both imposed and do not match.

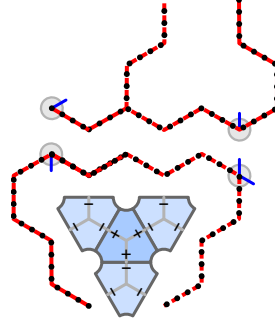


Figure 148 – The interface of a reflected cc piece with a T2 and preceding a dot at the end of a length 2 antenna, cannot match with a T3. The only interface shapes that could match would have the shapes above, but then the left dots orientation are different.

All hex centres are now covered by disks, and only once: the only dots not covered by the green packs are the non-tip hexes of T3 clusters, covered by yellow packs, and those of tips of type 1, covered exactly once in the case of T1 by Figure 144, in the case of T2 covered by the yellow piece of Figure 147, and not appearing for T3. \square

An alternative proof is by looking at one of the substitution systems proposed in Figures 149 and 150 and carefully check that at the interfaces, we get no superimposition and we cover everything.

Remark. Another alternative proof could be designed using another interpretation of packs made in Section 2.14.1: considering the data of the proposition, follow the diagonal arrow of Figure 137 to get a reflected Spectre tiling. Then prove that the associated hex centres draw packs as on Figure 167. We will not go through this approach here.

By Proposition 64, we can traverse Figure 137 and thus the hierarchy in the other direction. In particular we reprove¹¹ the companion to the previous corol-

¹¹The fact that the proposition is enough is classical: it goes by extracting subsequences converging on bounded sets and performing a diagonal argument.

lary, also a results of [Smi+23a]:

Corollary 65. *There are whole plane tilings by the Spectre, without reflections.*

In the next section we make the procedure of Proposition 64 explicit and we propose several equivalent substitution systems.

2.13 Substitution systems

Substitution systems are an explicit way of going backward in the hierarchy, i.e. refining a tiling into a tiling with smaller/more pieces.

2.13.1 From cc to reflected triangle packs

One key step in going backwards is Proposition 64. Either its proof, or a careful inspection of the the list of possible environments determined in Sections 2.11.2 to 2.11.4, give several substitution systems that can work. If we allow overlaps it is easy. We are most interested in substitution systems without overlaps, and it turns out to be possible: for this we attach the green packs only to the T'_n to

type 6 points in the nomenclature of Figure 140:

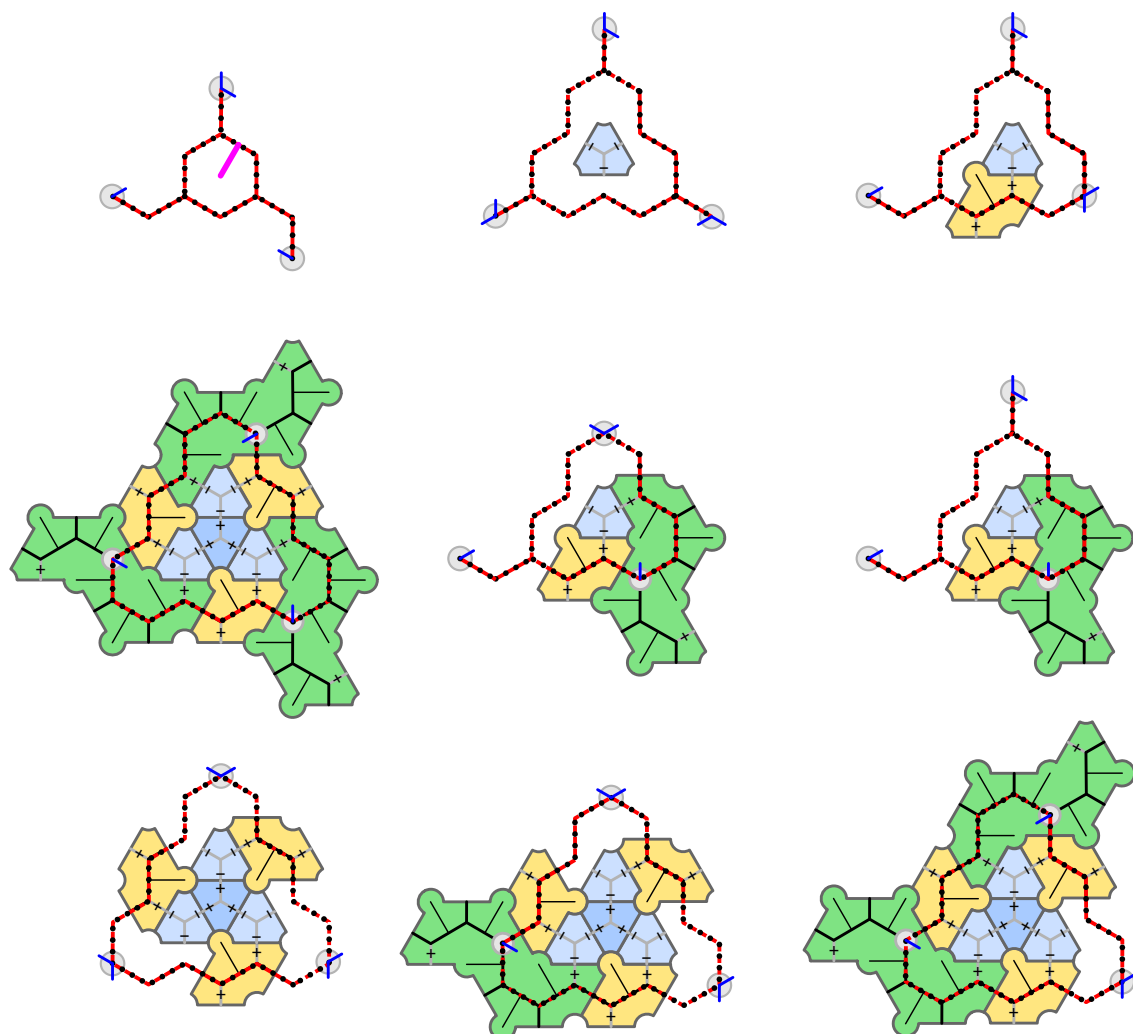


Figure 149

We get the following variant, somewhat simpler, if we attach the green packs

to the length two antenna ends instead:

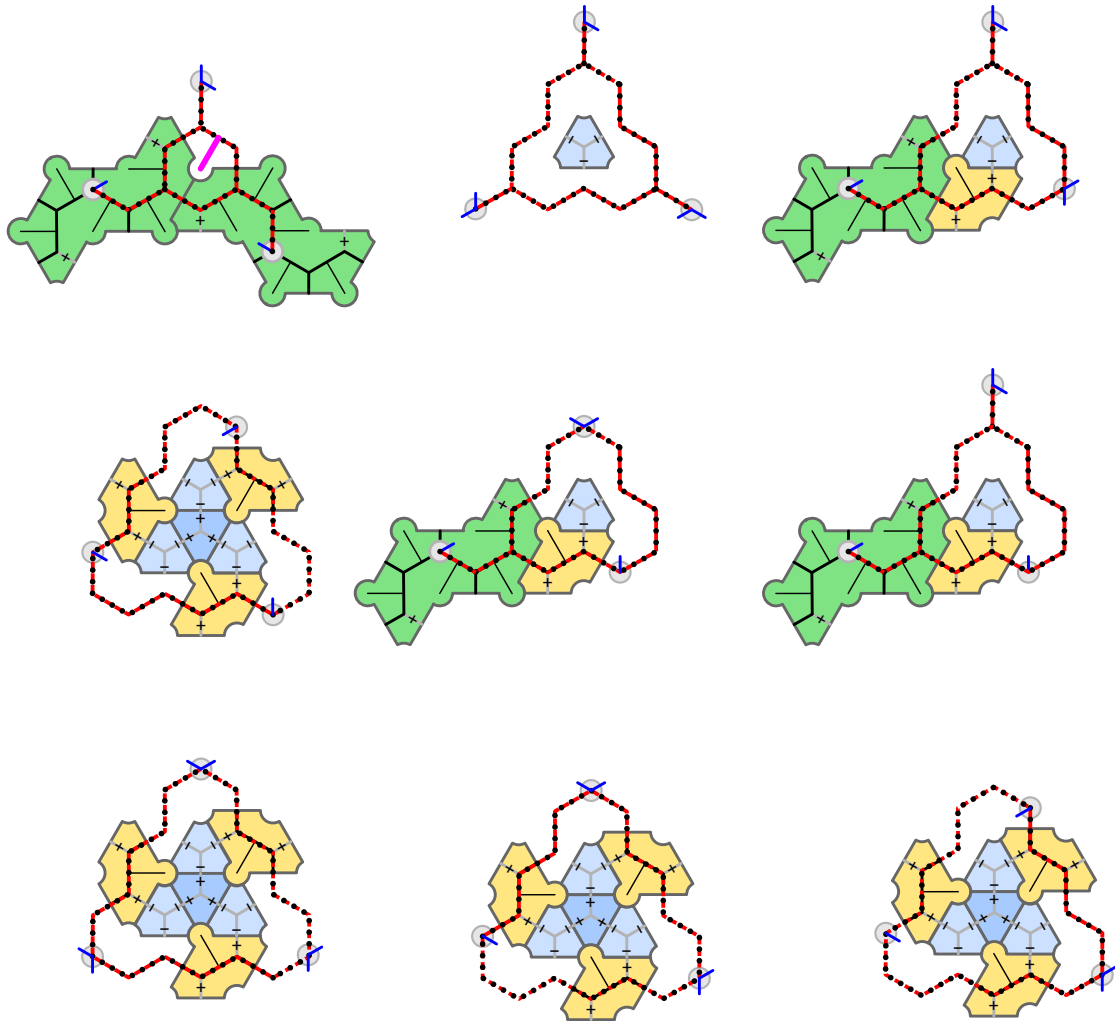


Figure 150

Conversely, the following figure shows the compacted cc's of the triangle pieces of the packs. Initially this is not a substitution, since this is what the packs represent; however, since we are reversing the procedure, this can be thought of,

and used, as a substitution.

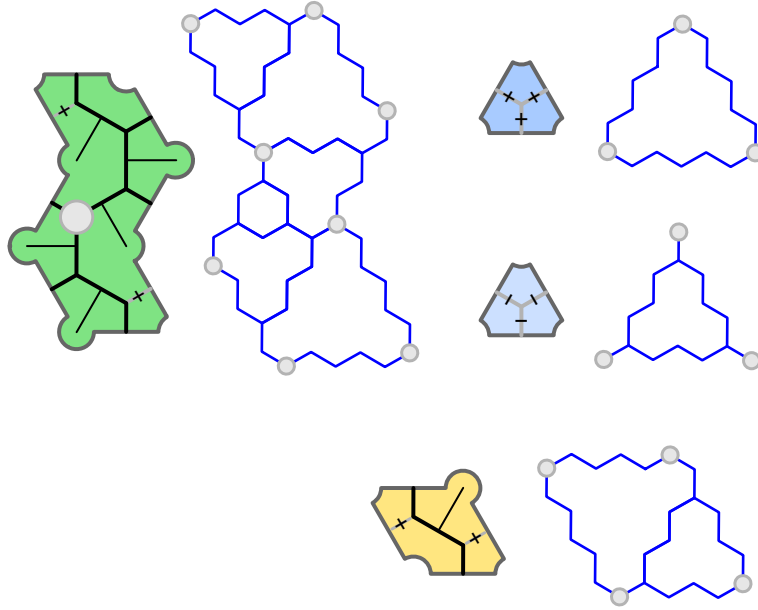


Figure 151

The substitution is then obtained by alternating the two substitutions above (and making reflections).

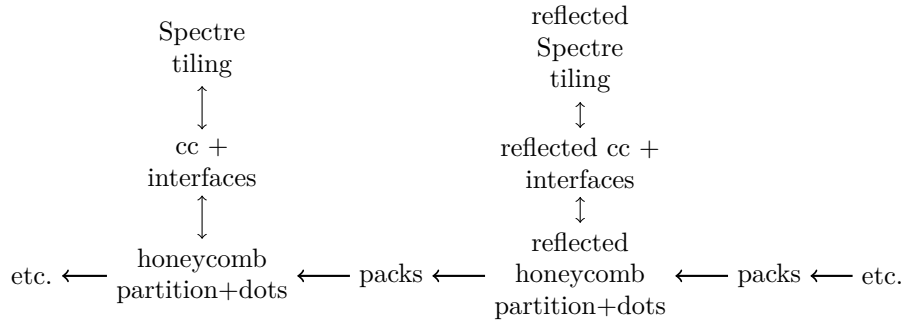


Figure 152 – The two-step substitution procedure

On Figure 151, it is apparent that to the four packed pieces correspond in the honeycomb polygons traced by the marked points: an octagon with a half-turn symmetry, two regular triangles, and a parallelogram. Actually the octagon is a regular hexagon cut in two halves along a diagonal, where the two halves have been translated away from each other and linked. It will remain of this type under successive substitutions. The two equilateral triangles remain equilateral (and identical) and the parallelogram will remain a parallelogram.

To get a Spectre tiling in the end we need to go up in Figure 152. One can start from a compacted cc tessellation of all or part of the plane and then space them out using yellow segments oriented as in the interfaces of Figure 76. Then the rhomb/hex pairing is completely determined, and by replacing in each pair

the hex by an D3 and the rhomb by an D2 on gets a Spectre tiling. An example is illustrated in Figure 110.

This amounts to assembling the following replacement of the cc's:

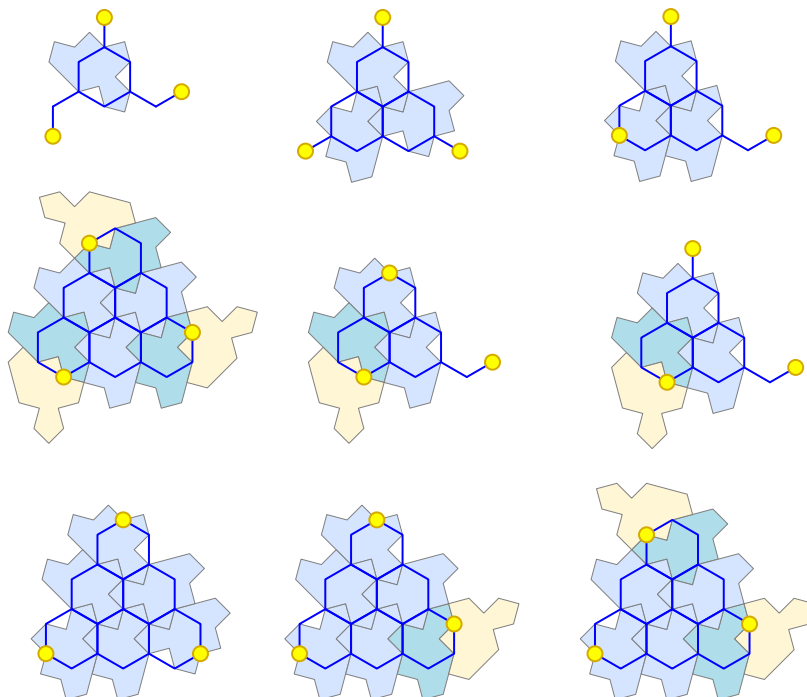


Figure 153

Or equivalently with:

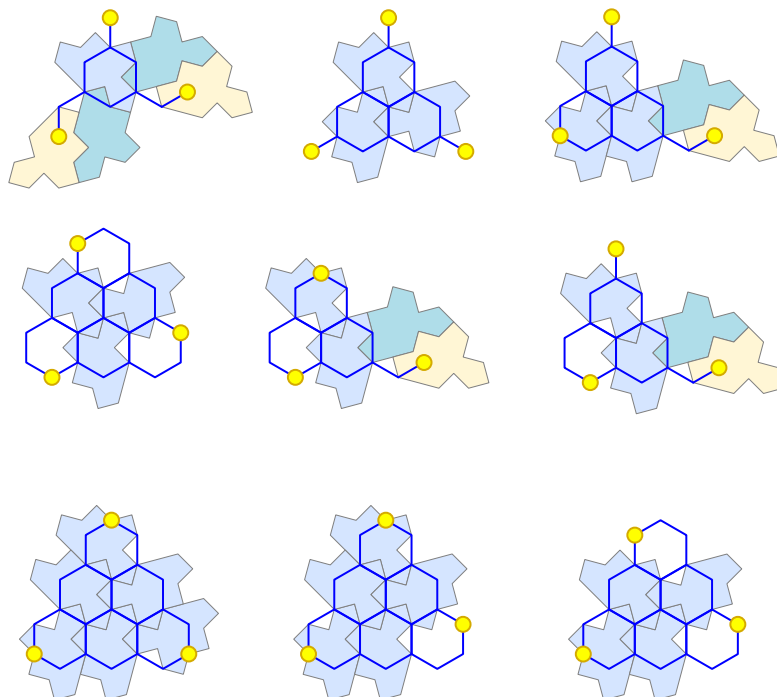


Figure 154

But we can also use another approach: see Section 2.14.

By chaining the two substitutions of Figures 149 and 151 in one order or its opposite, one can express the substitution either in terms of packed triangle pieces only, or in terms of honeycomb partition + dots only.

2.13.2 In terms of triangle packs only

Substitution rule in terms of the 4 packed triangle pieces:

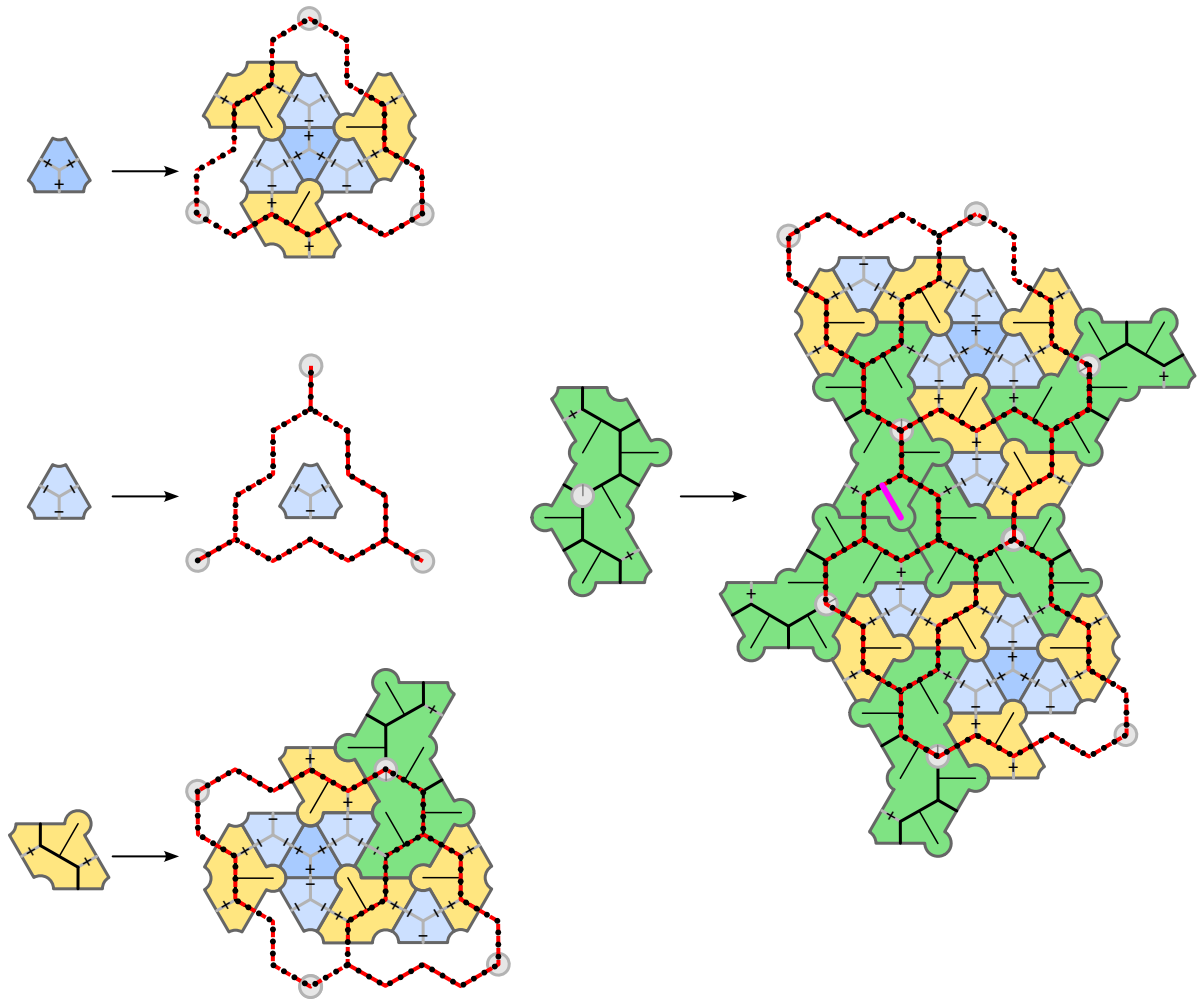


Figure 155 – Keeping track of the dots help to assemble the substituted pieces

We see that we can also use the three pieces of Figure 107.

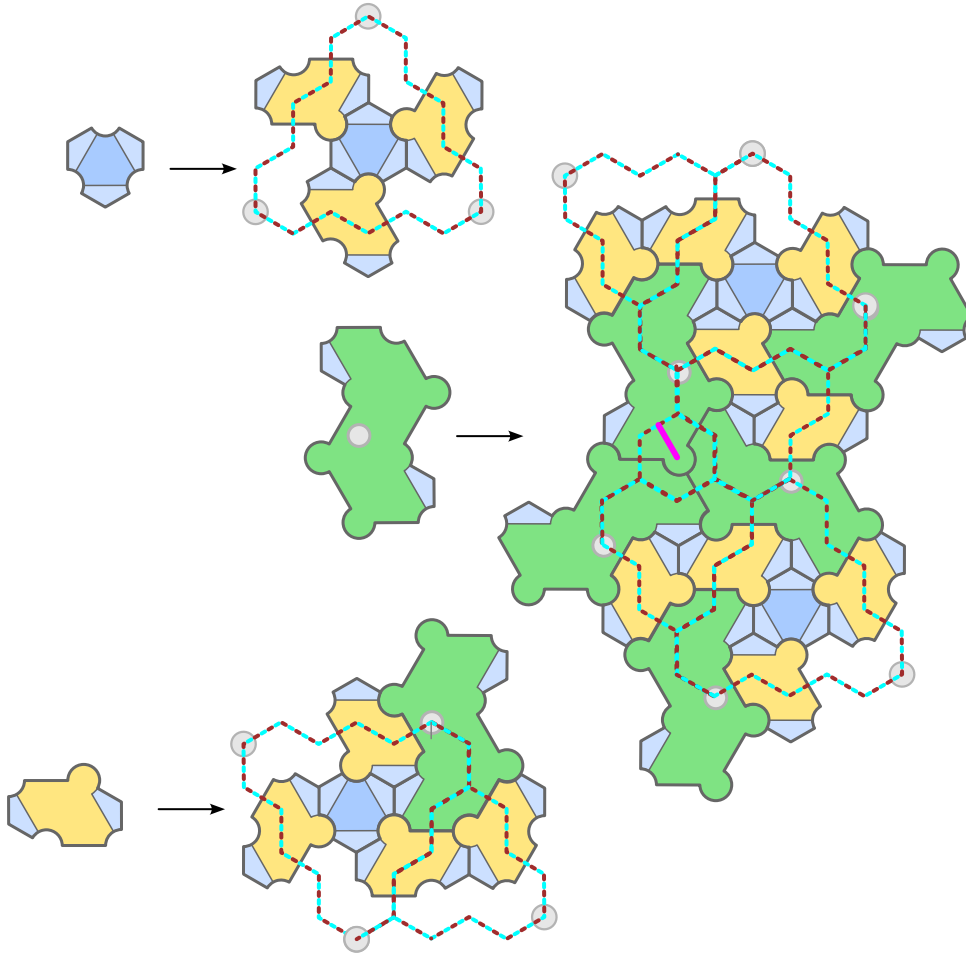


Figure 156

These last two sets are reminiscent of the tile grouping made in the non-chiral tiling of [Smi+23b], the H, T, P, F system of what they called meta-tiles. See more in Section 2.14.

2.13.3 Variants

There are many possible variants.

We can for instance factor each green piece with a yellow one, leading to the

turquoise piece and the substitution below:

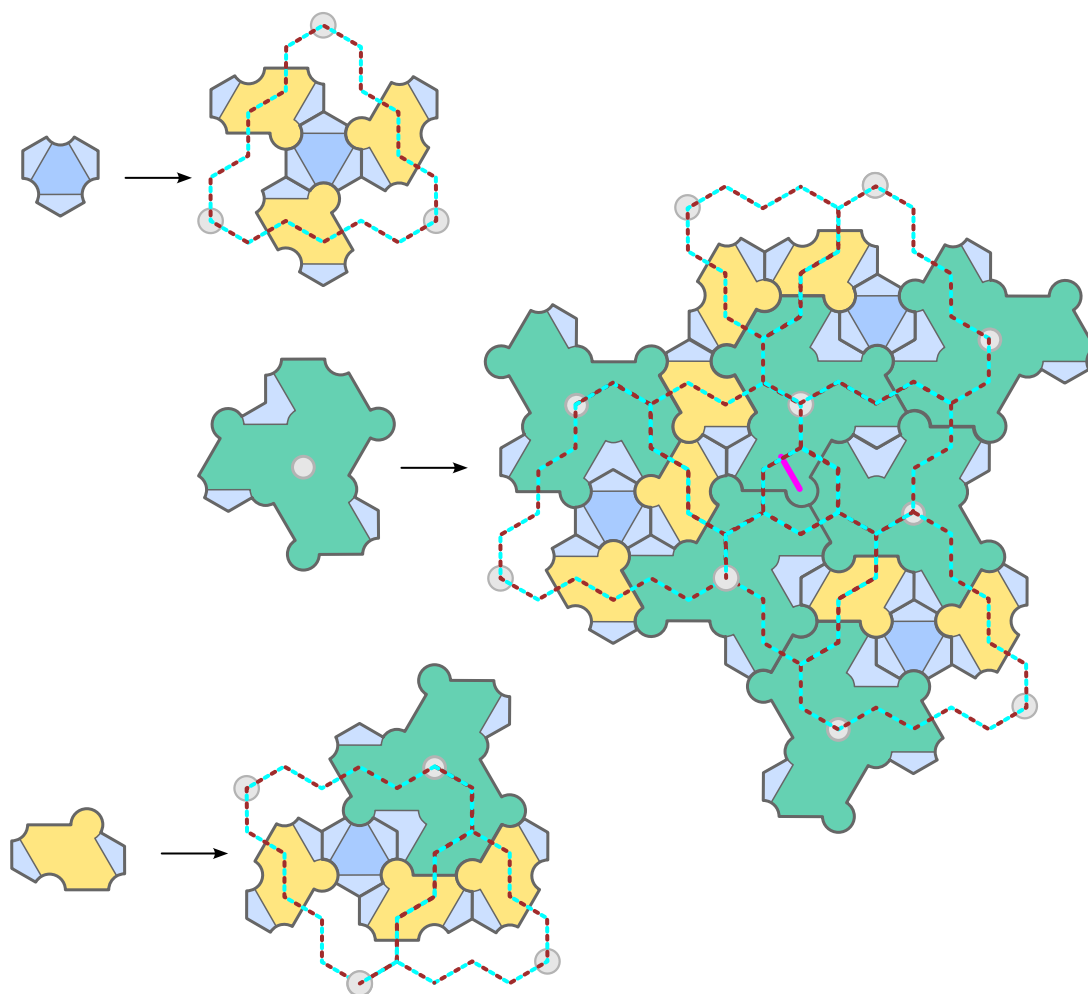


Figure 157

We could instead start from Figure 155 and group a yellow and a light blue piece together.



Figure 158

There still remains 4 tiles in this case:

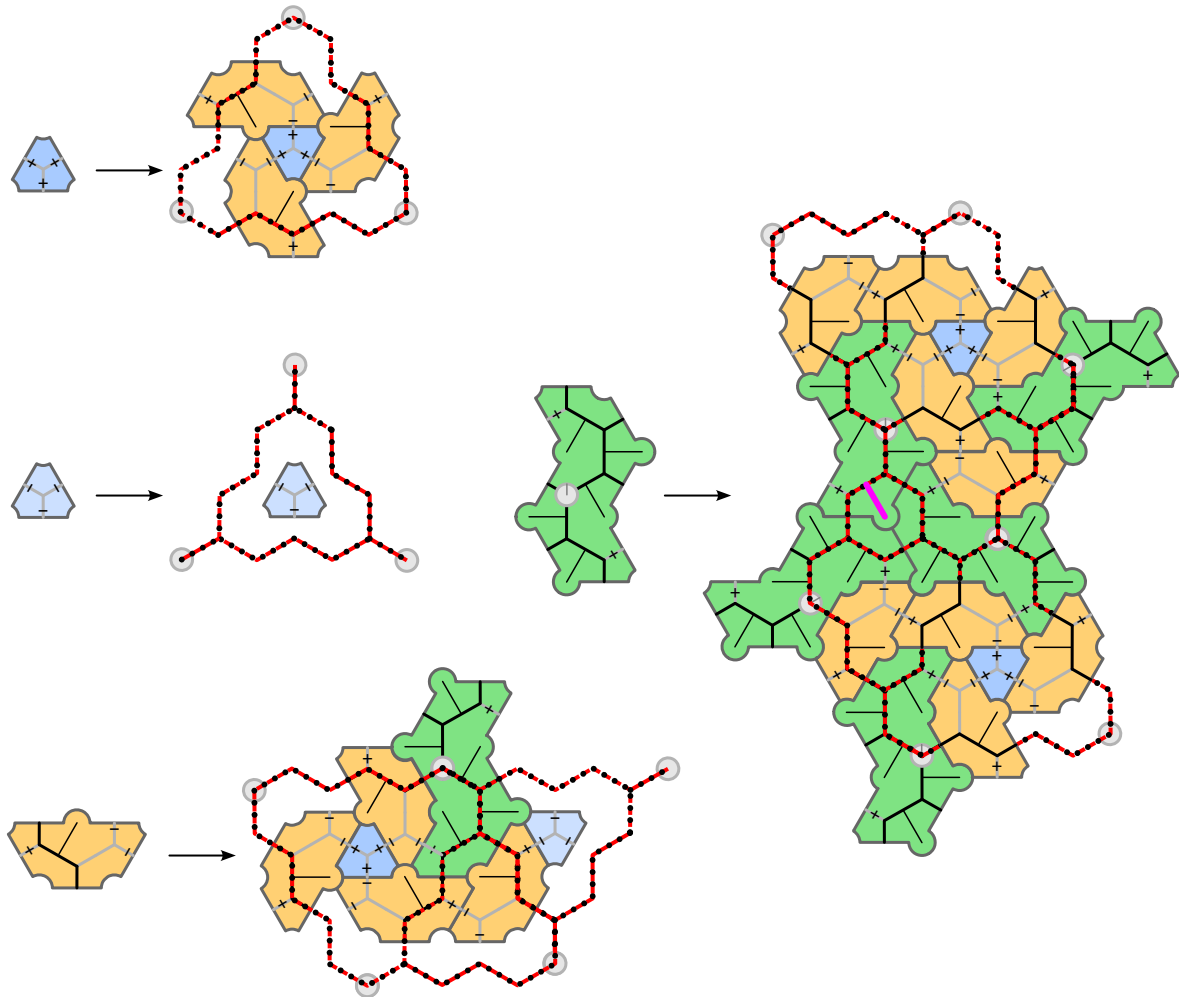


Figure 159

It is also possible to group tiles as follows (2 variants):

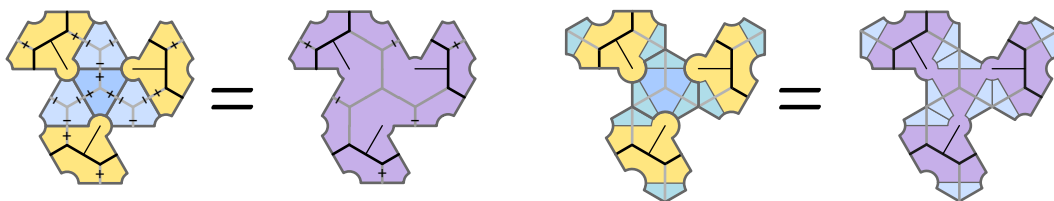


Figure 160

But we will not show the resulting substitutions here.

It is likely that there are a lot of other possible variants.

2.13.4 In terms of honeycomb partition and dots only

Finally, we give the substitution rules in terms of T_n +dots only: from Figure 149

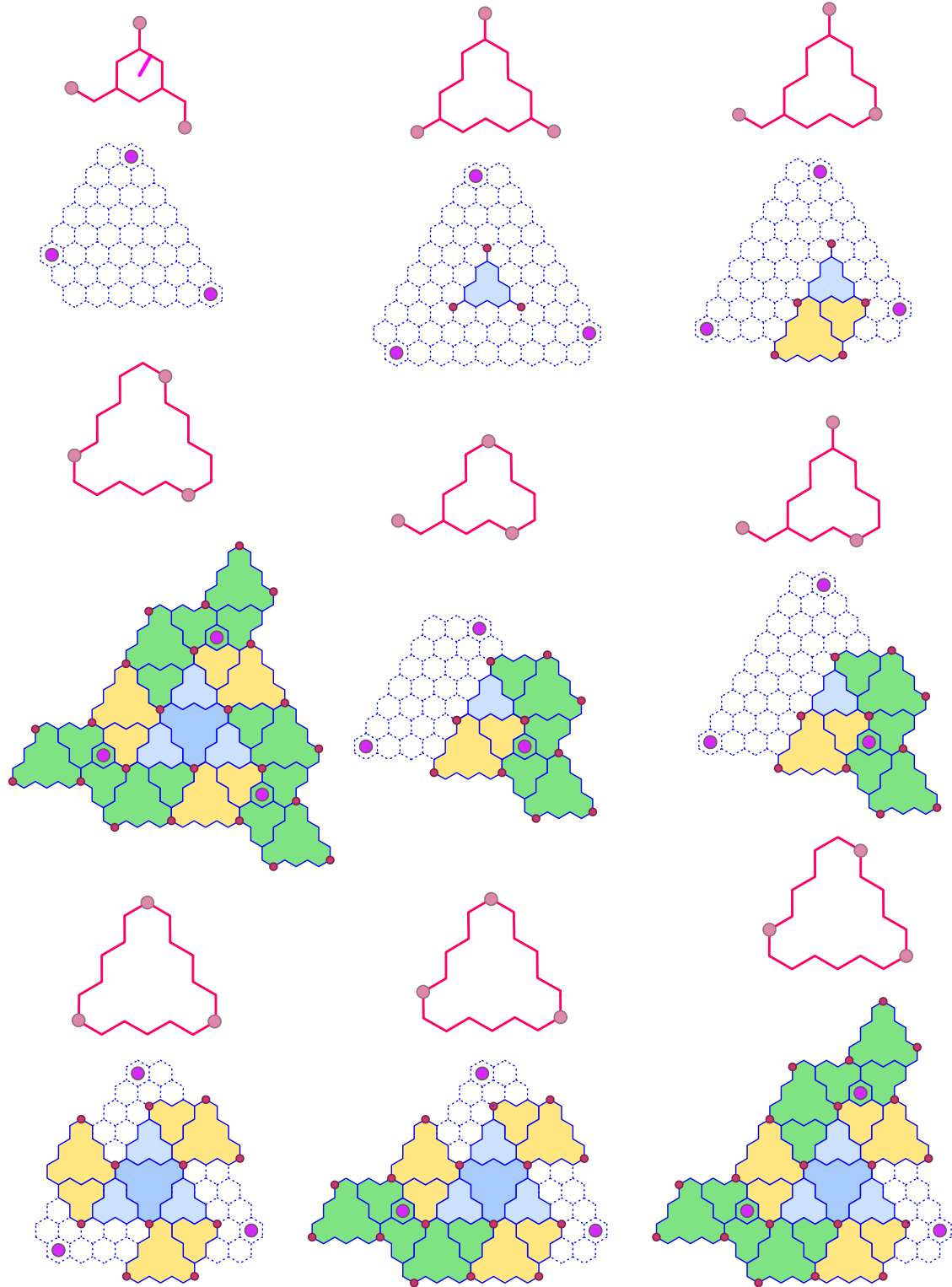


Figure 161 – The 4th configuration never appears in a whole plane tiling.

Or, from the variant Figure 150

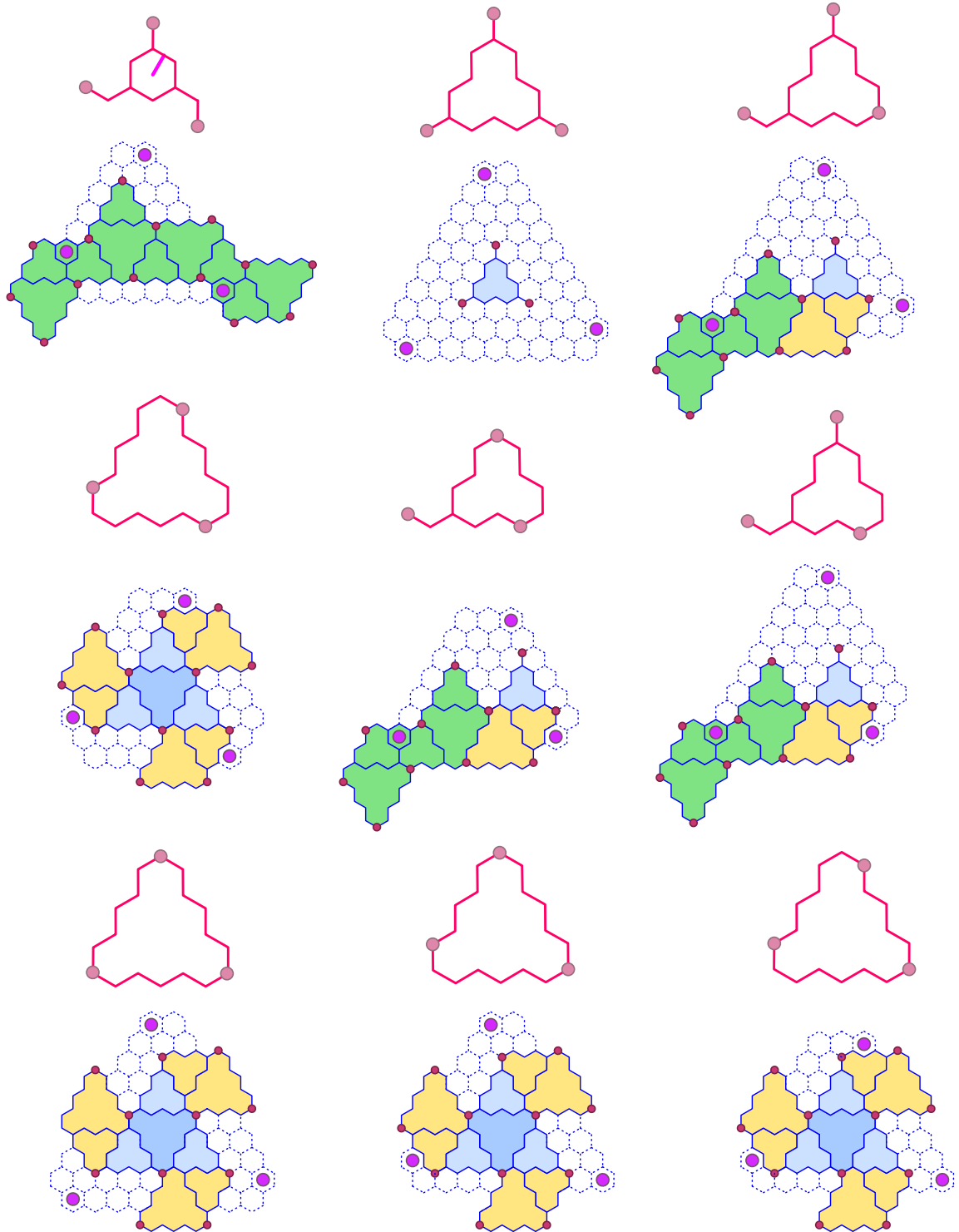


Figure 162 – The 4th configuration never appears in a whole plane tiling.
Consider any of these two substitutions and call it S . Consider a whole plane

tiling. Its cc's (with dots) can be grouped in packs on which decoration form a new cc system, which itself can be grouped in packs, etc. We proved that the initial cc's are necessarily obtained by the substitution S from the bigger ones one level above. On the figure, it can be seen that the 4th kind of cc with dots does not appear after the substitution. This already proves that no whole plane tiling has the 4th kind of cc.

Proposition 66. *In a whole plane tiling by the Spectre, all cc types of Figure 76 other than the 4th appear. All triangle packs of Figure 106 appear in their triangle representation.*

Proof. It makes the argument easier to consider the second substitution system. Numbering the types of cc from 1 to 9, and calling G , Y , B , D the packs (Green with 6 triangles, Yellow with 2, Blue and Dark blue), we get that in the first substitution: from Figure 150 $1 \rightarrow G$, $2 \rightarrow B$, $3 \rightarrow G, Y, B$, $5 \rightarrow Y, B, D$, $6 \rightarrow G, Y, B$, $7 \rightarrow Y, B, D$, $8 \rightarrow Y, B, D$, $9 \rightarrow Y, B, D$, and from Figures 76 and 106, the packs are composed of triangles representing: for G : 1, 3, 5, 6, 8, 9; for Y : 3 and 8; for B : 2; for D : 7. Starting from anywhere but 2 and B , we see that the set of descendants is everything (apart 4) at every generation. Starting from 2 we get only B and from B we get only 2. It is impossible to cover the honeycomb only using pieces of type 2. The claim follows. \square

2.14 Going down one more level

2.14.1 Packs to Spectres

Comparison of Figures 69, 76, 149 and 151 suggests to use the following substitutions from *reflected* packs to Spectre patches.

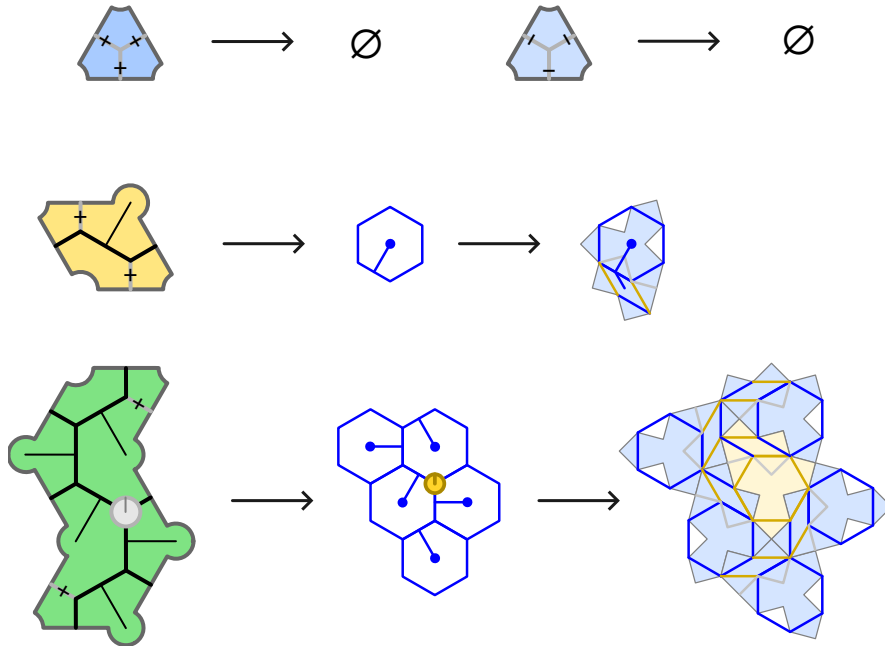


Figure 163 – Note: packs are reflected

to get the following variants of Figures 153 and 154 for the final substitution step after spacing-out cc's:

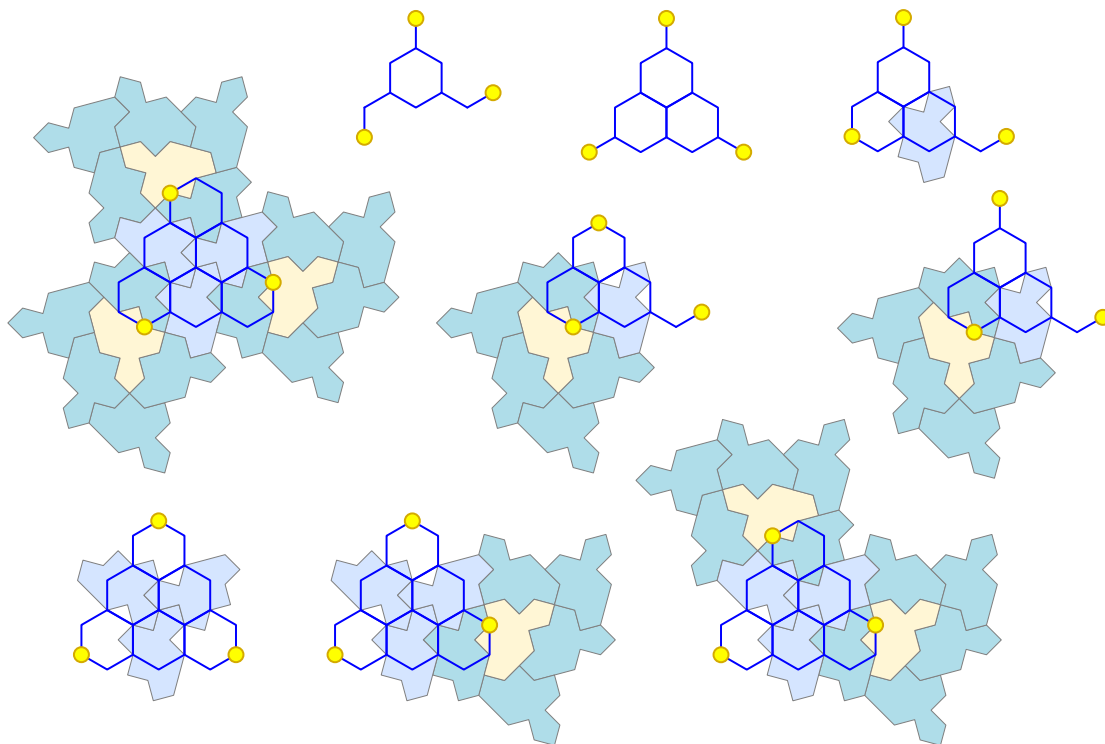


Figure 164

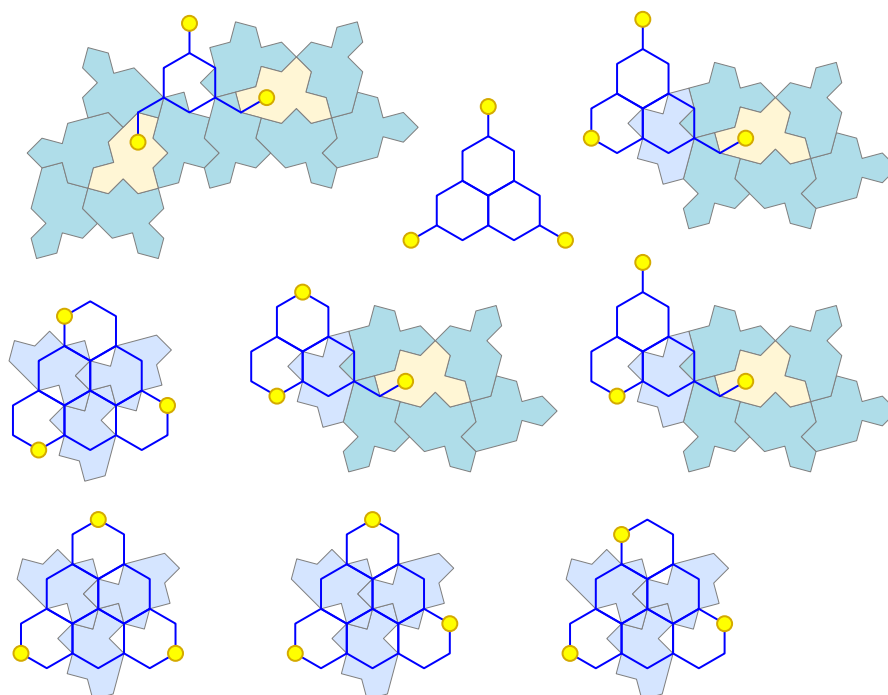


Figure 165

We get the following packs on an example of the Spectre tiling (compare with Figure 136):

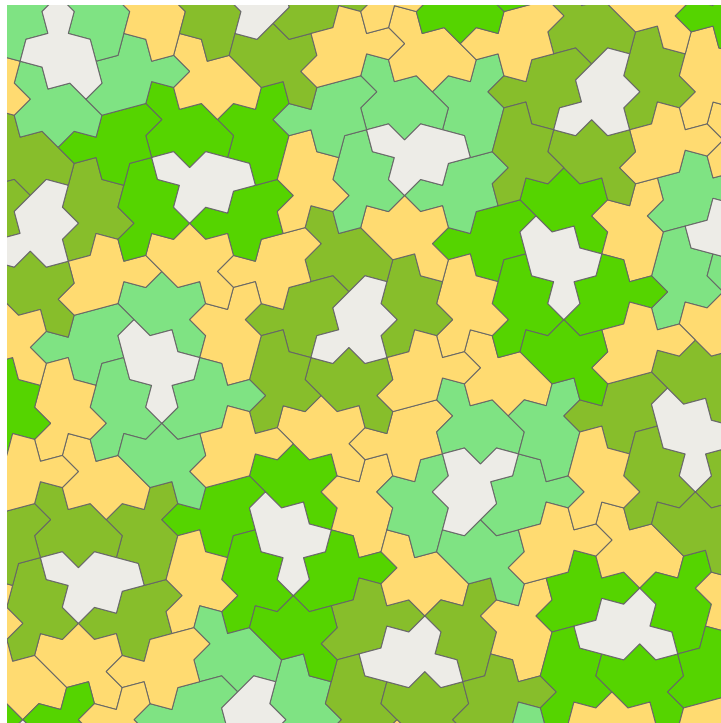


Figure 166 – Here gray tiles are the odd ones and are part of the green packs.

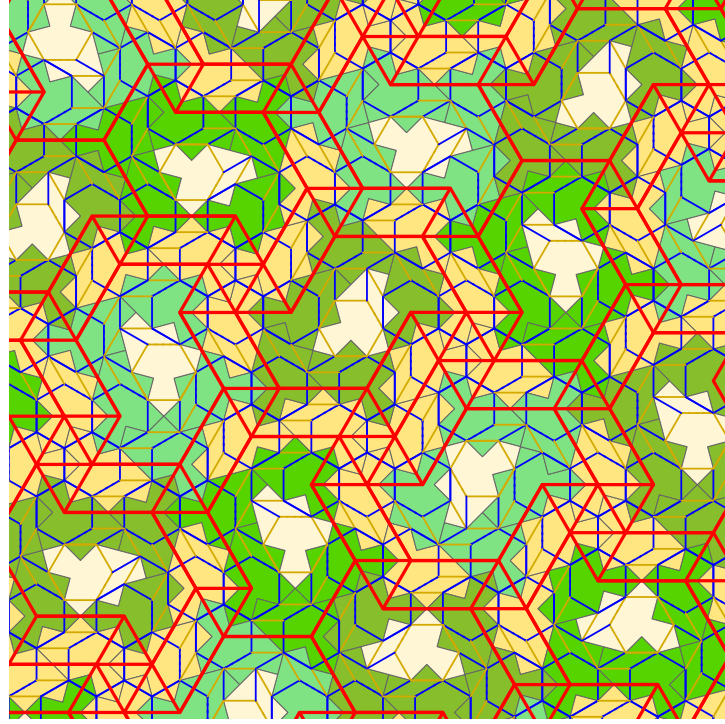


Figure 167 – Adding the yellow/blue decoration graph and tracing out the triangle patches, using blue hex centres as vertices:

We interpreted packs as a Spectre packing instead of packing of triangles representing cc's. To complete this approach, it would be nice to interpret the triangles in terms of meeting points of 3 even Spectres of D3 and an optional odd one. Based on Figure 167, it seems that the following association can be made for the packs (it extends Figure 163; if used as a substitution it gives compatible superimpositions).

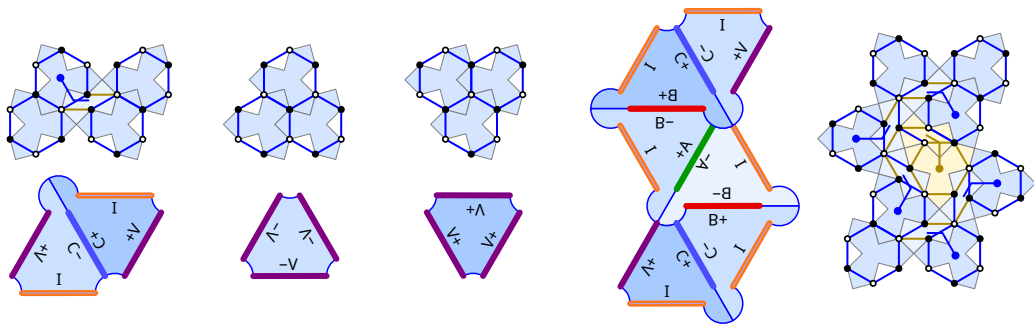


Figure 168

Yielding the following association for the triangles:

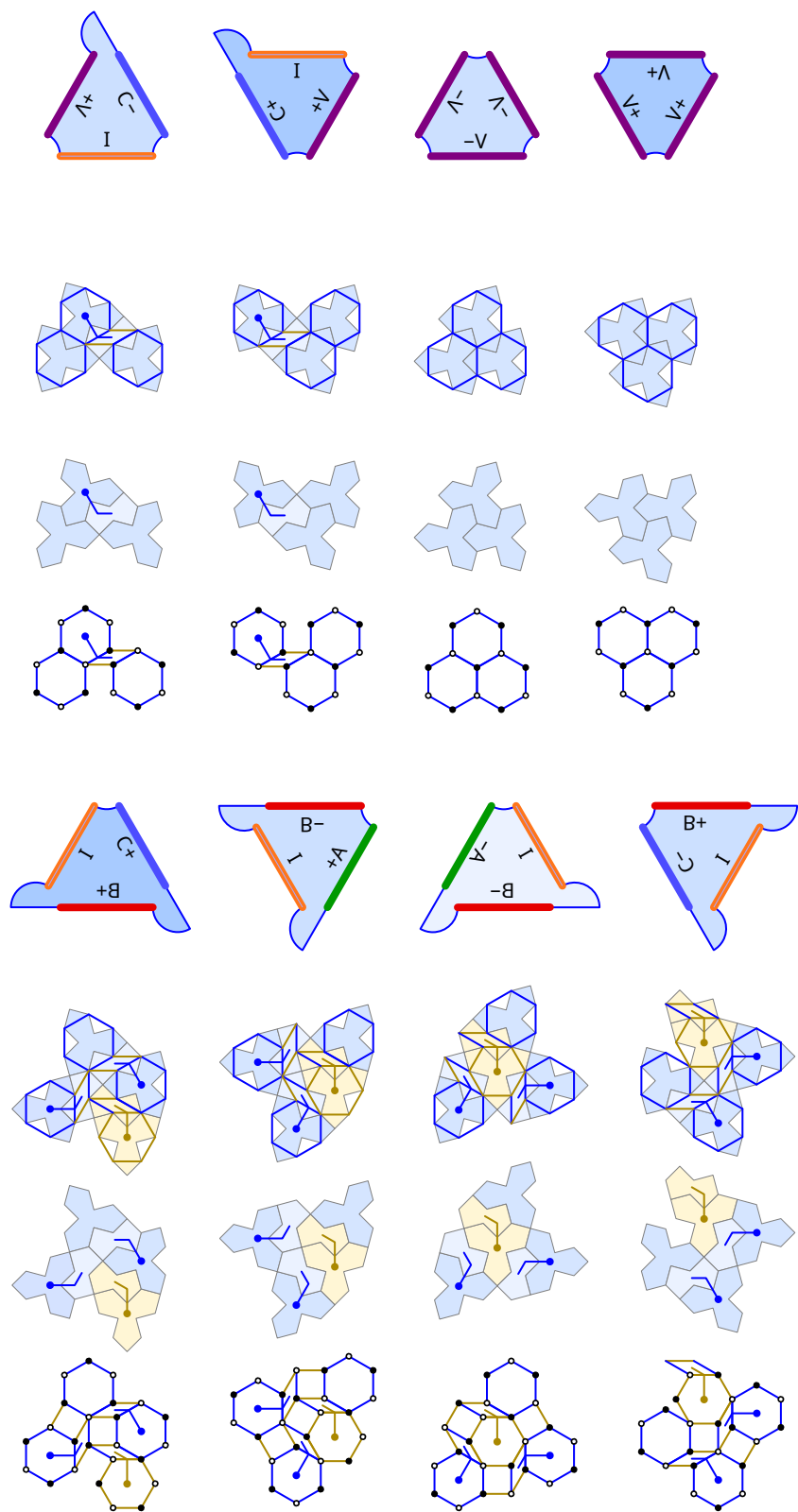


Figure 169

Each triangle is associated to a trio of blue hexes that share a corner under

the yellow contraction (see Section 2.5). We could probably have started from an exhaustive study of that kind of triples instead of the blue cc's, and performed basically the same study as for the triangle tileset of Section 2.6 (Figure 98), allowing for a shorter deduction path to the substitution structure of the Spectre tiling, as there ought to be repetitions between the sections up to Section 2.4 included, and the study of the properties of the triangle tileset.

2.14.2 One more spectre around the odd

As a variant we can use the packs of Figure 157:

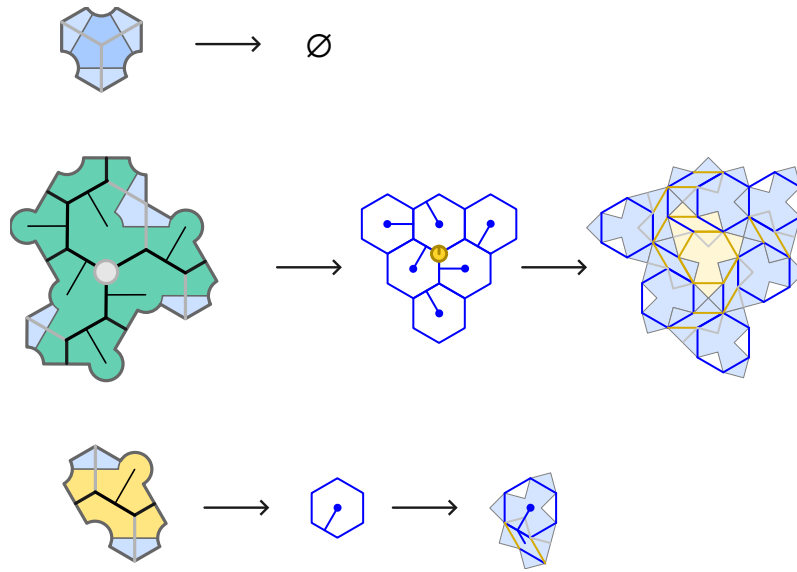


Figure 170 – Packs are reflected

Yielding:

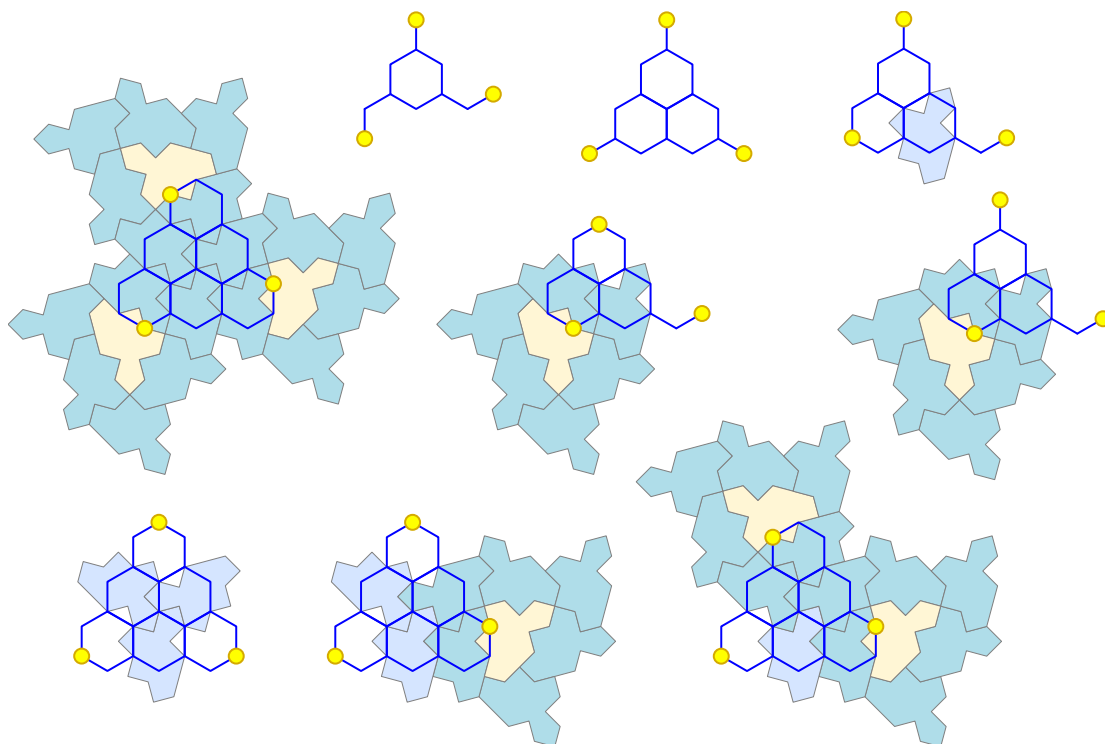


Figure 171

Or equivalently:

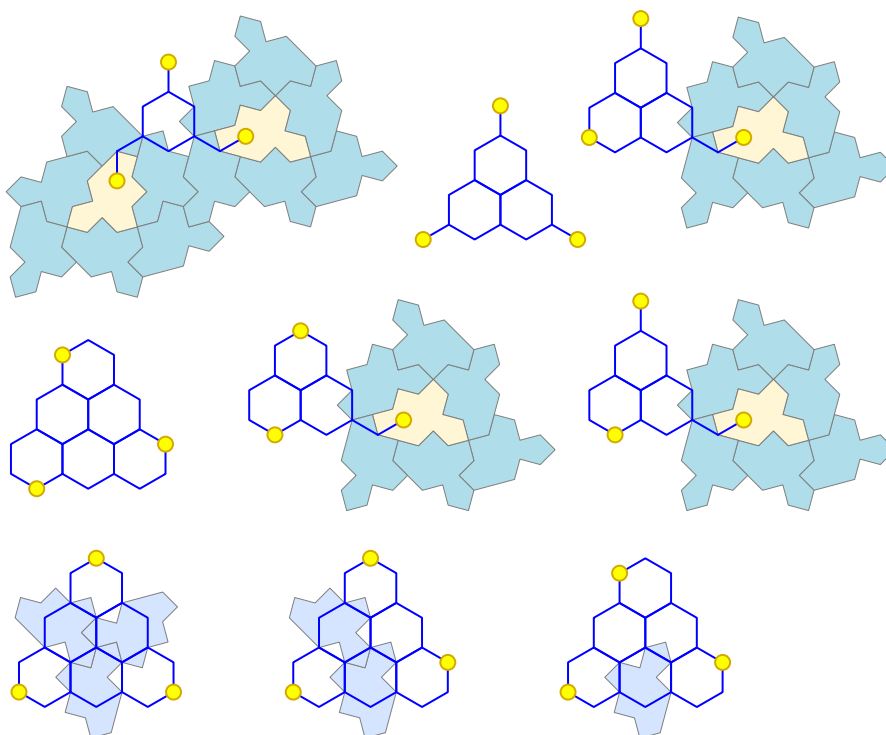


Figure 172

2.14.3 Recognizing the hexagons of [Smi+23a]

We are now very close to the H8/H9 clusters of [Smi+23a]. We reproduce them below for convenience:

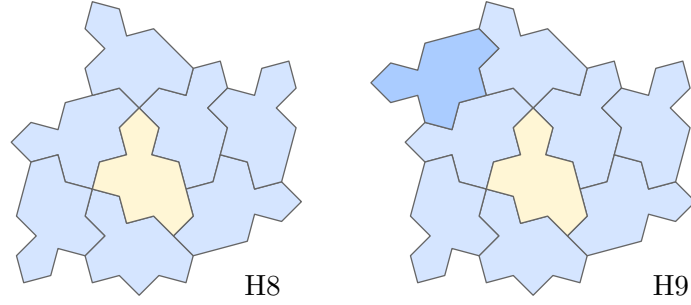


Figure 173 – H9 is H8 with the dark blue tile added

The H9 is one tile less than the environment of Figure 44.

The last version of our clusters above around the yellow Spectre have 7 tiles, including the yellow one. It is however not possible to integrate the one missing to get an H8, in a way that is compatible with our cc shape+dots-based analysis: to do this we need to subdivide shapes of cc into cases, by adding the following information: which orientation (out of two possible) do we have for the yellow hexes at the cc dots for which this orientation is not already known. We have not tried to determine the implication of this modification on the substitution systems proposed earlier.

To conclude Section 2, we propose a way to recognize the nine hex clusters Γ , Δ , Θ , Λ , Ξ , Π , Σ , Φ and Ψ of [Smi+23a] from the triangle assemblies of Figure 99.

First we note that in Figure 99, none of the four cases of the third column can appear in a whole plane tiling. The one on line 1 has been ruled-out by Lemma 49 and the one on line 2 by Proposition 60.

The proof of the following three lemma is to be found in Section 3.4.

For the other two, we start from an improvement on Figure 108:

Lemma 67. *Every green pack is environed as follows:*

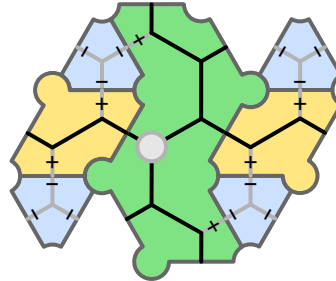


Figure 174

Lemma 68. *The following arrangement cannot occur in a whole plane tiling:*

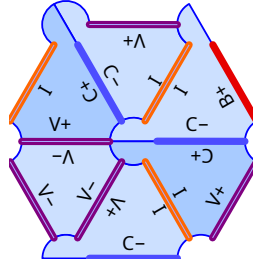


Figure 175

Lemma 69. *The following arrangement cannot occur in a whole plane tiling:*

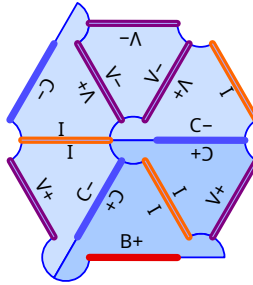


Figure 176

Lemma 70. *All the other vertex environments of Figure 99, i.e. those that are not in the third column, occur in whole plane tilings.*

Proof. Let us number the remaining environments:

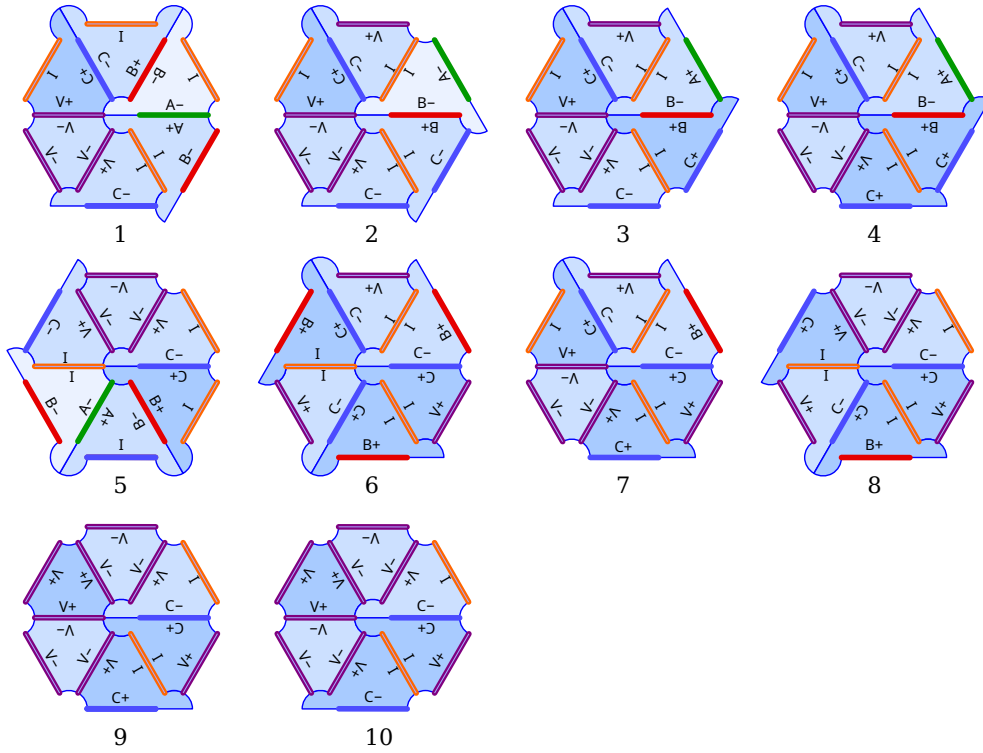


Figure 177

We can interpret them in terms of the packs, which are visually easier to spot:

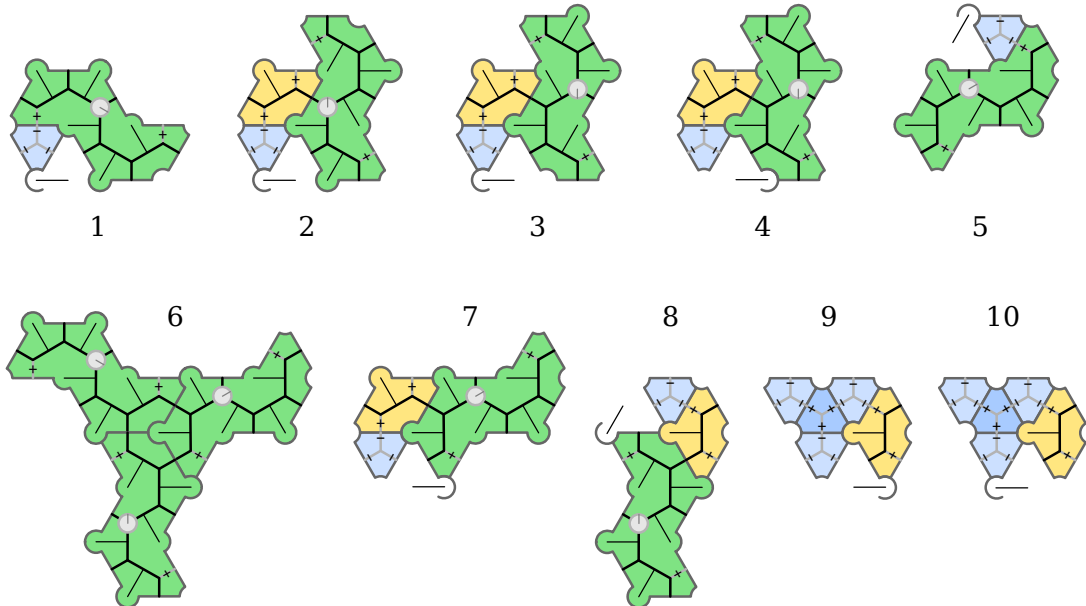


Figure 178

We look at the the second iteration of a yellow pack and find an instance of

each of the situations of the previous figure:

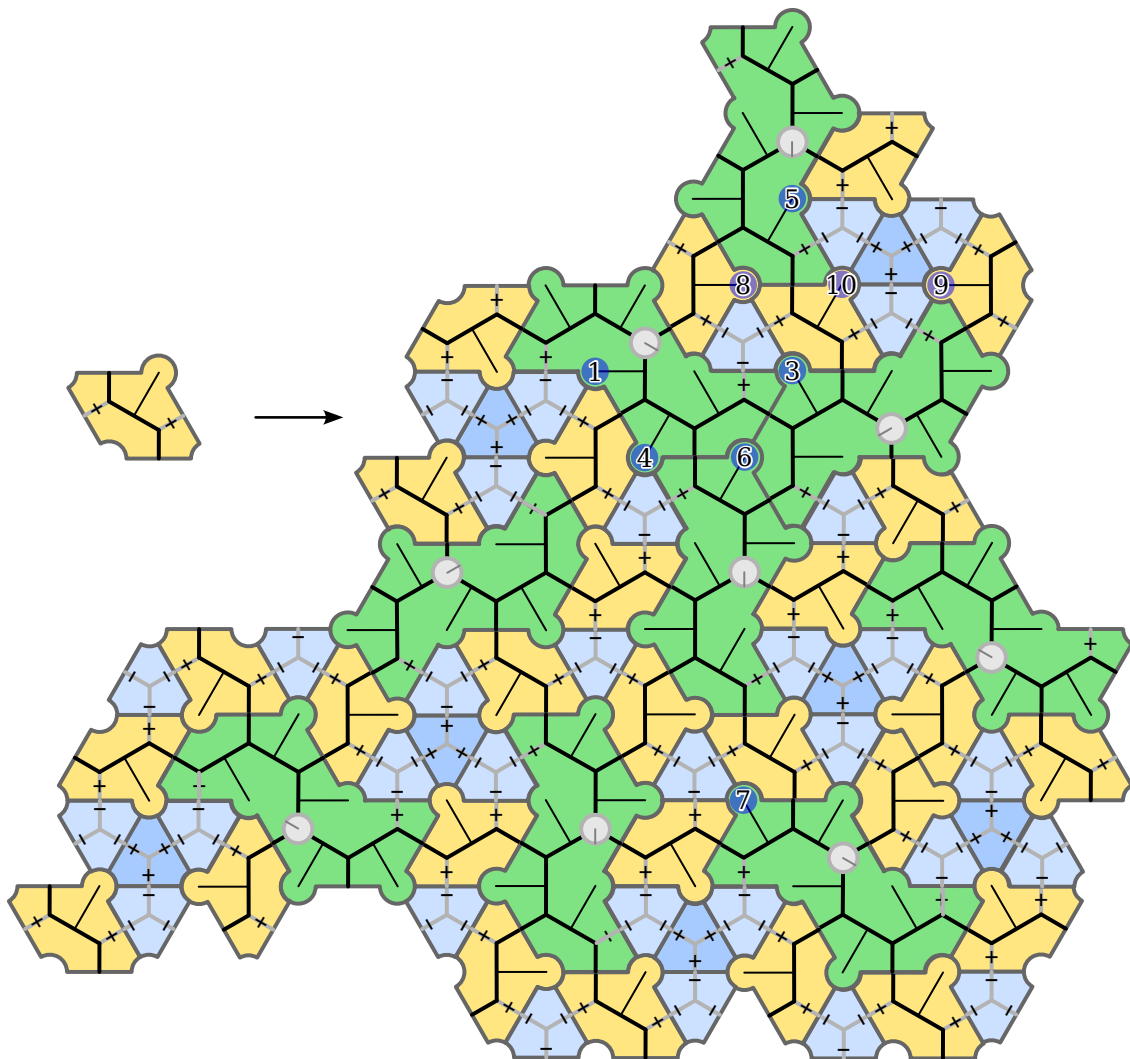


Figure 179

□

Using Figures 69 and 76, we are able to identify the cluster types for the remaining ten arrangements. On the figure we rotated the arrangements by a half-turn, for easier visual identification with Figure 4.1 of [Smi+23a]. Curiously, Cluster Φ appears twice.

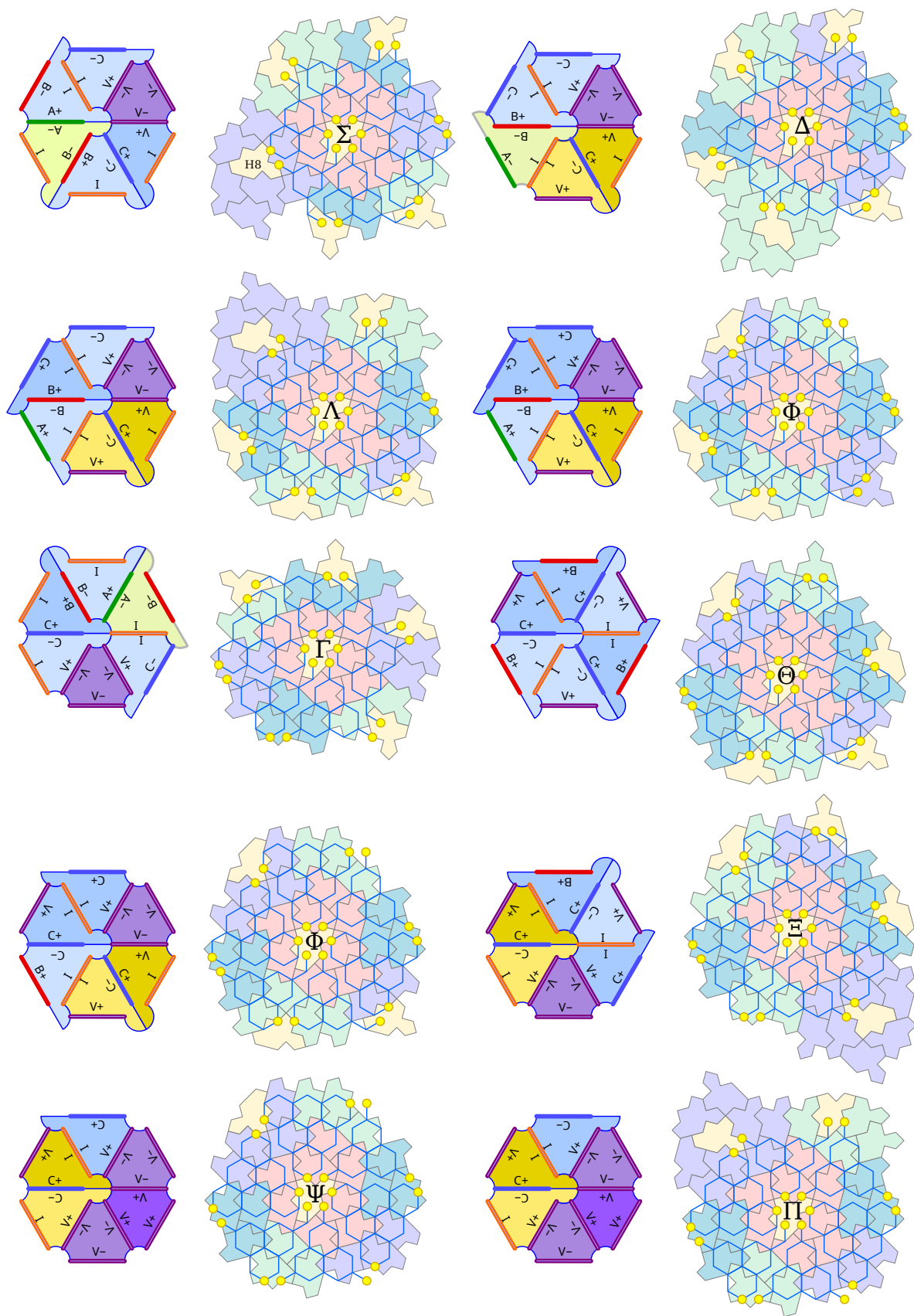


Figure 180

3 Deferred proofs

3.1 Odd tile environment

Here we prove Proposition 11.

Proposition 71. *The following arrangement cannot appear in a whole plane tiling by the Spectre.*

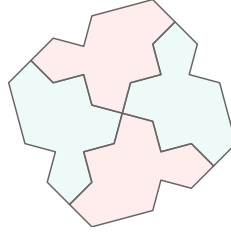


Figure 181

Proof. In the figure below we look at the $1/4$ angle of the inward dent on the upper right. It can be filled in 4 ways. Each leads immediately or by repeated use of Proposition 9 to an unfillable outline as per Proposition 8.

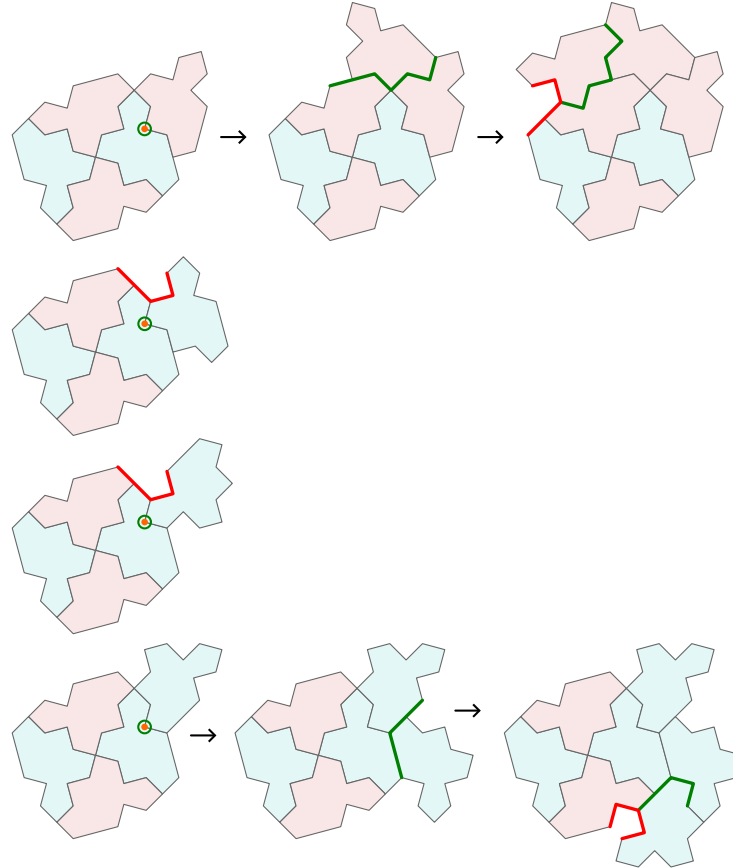


Figure 182

□

Proposition 72. *The following arrangements cannot appear in a whole plane tiling by the Spectre.*

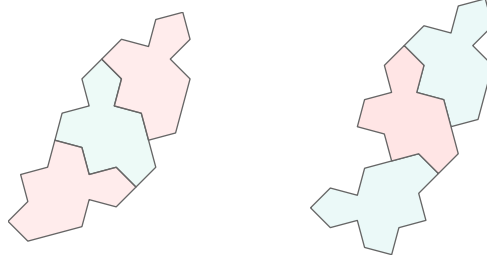


Figure 183

Proof. We first treat the left configuration. In the figure below we look at the $1/4$ angle of some inward dent. It can be filled in 4 ways. Each leads immediately or by use of Proposition 9 to an unfillable outline as per Proposition 8 or the forbidden situation of Proposition 71.

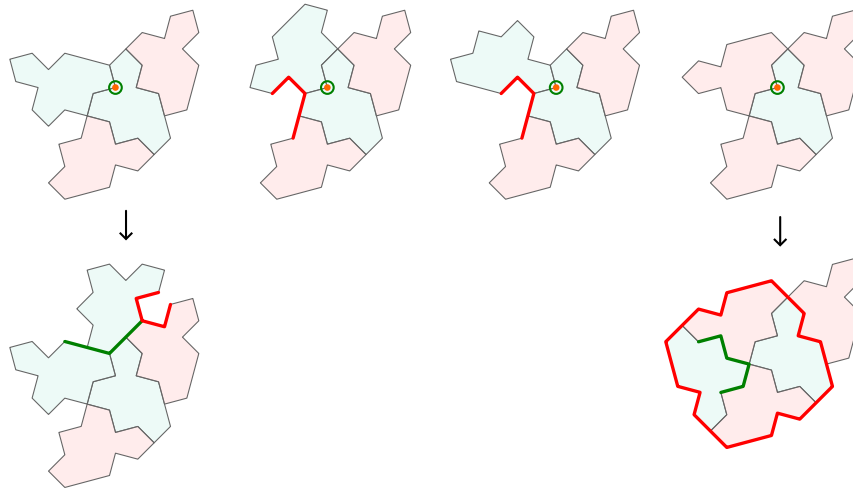


Figure 184

For the right configuration:

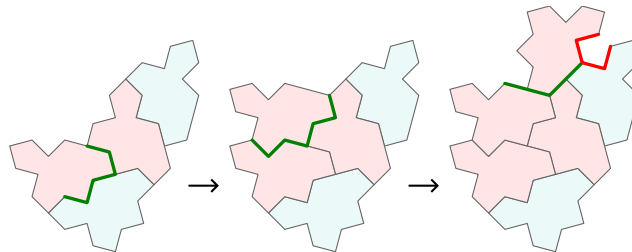


Figure 185

□

Proof of Proposition 11. First there are, up to rotation by a multiple of $1/12$, only 3 ways two tiles of different parity can be in contact along at least one edge:

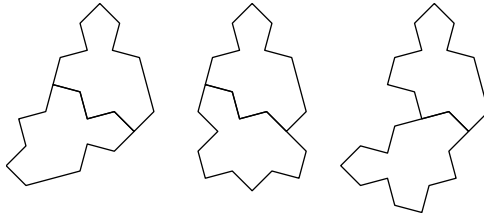


Figure 186

or 6 ways if we only perform turns by multiples of $1/6$:

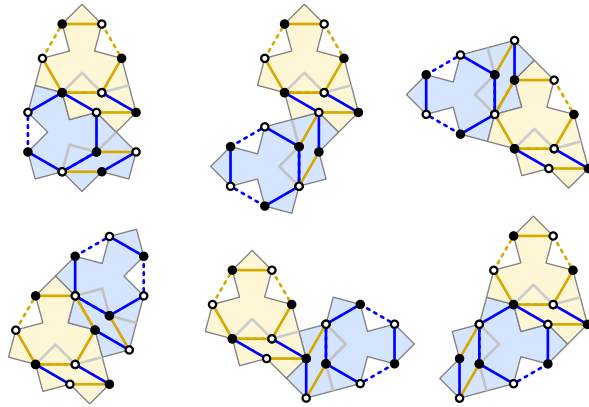


Figure 187

This is proved by fixing the orientation of the yellow tile, then for each of the 6 possible orientations of the blue tile, trying to place it adjacently.

We now prove that for each of these three configurations, one of the two tiles is like the red tile in the figure below, up to a rotation of a multiple of $1/12$. (Note that the other of the two tiles may or may not be one of the blue tiles in the figure below.)

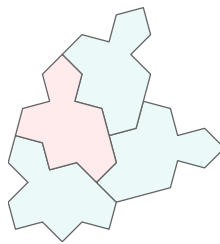


Figure 188

To prove this claim, we first study the central case of Figure 186 (which has the same shape as the *mystic* of [Smi+23a]):

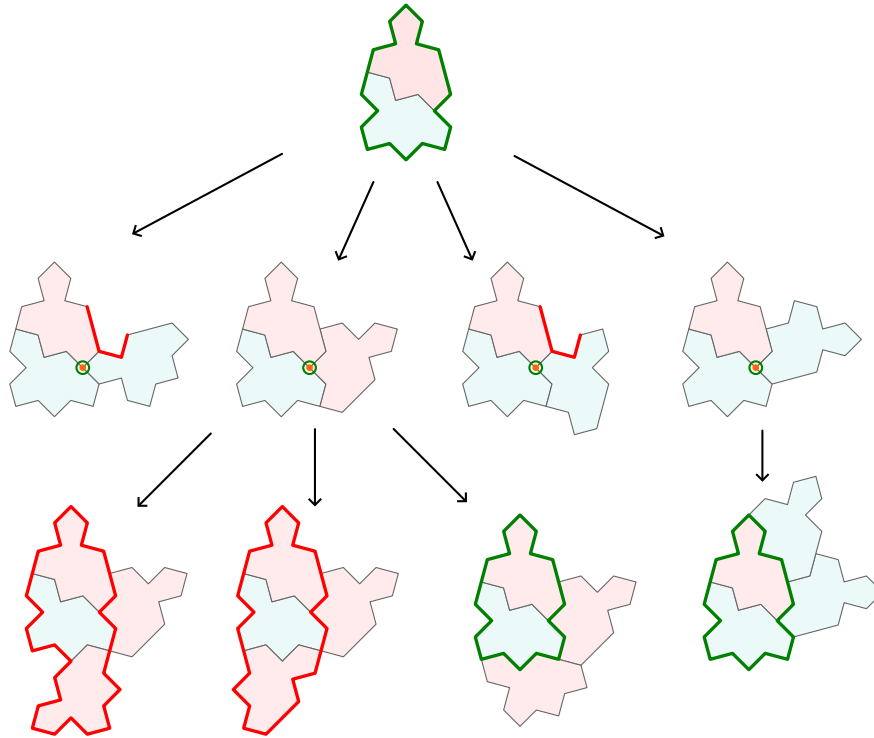


Figure 189

In the figure above, only two cases are possible. The one on the right is exactly as in Figure 188, as is the one left to it after swapping the colours and rotating by $-1/12$.

In the right case of Figure 186, we see below that we must have a mystic shape, and can use the previous figure to get the second deduction in the diagram below:

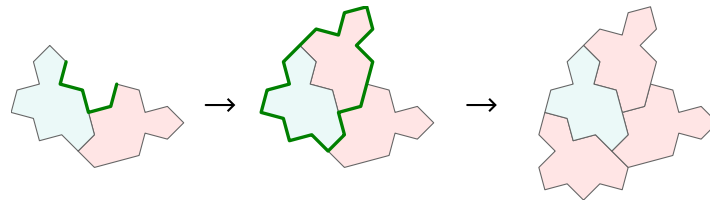


Figure 190

Last, in first case of Figure 186, we get, using Figure 189 when we see a mystic:

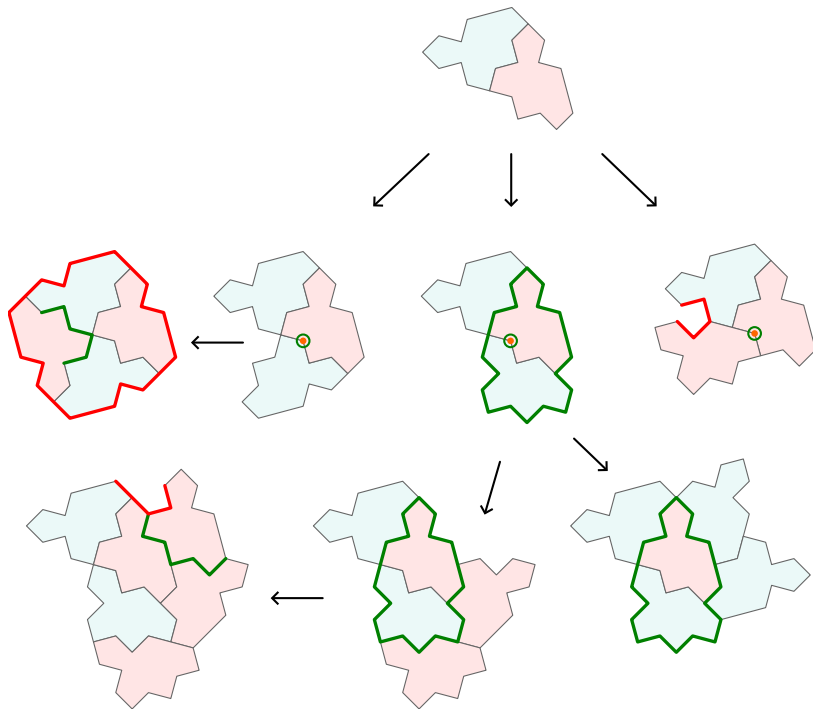


Figure 191

This proves the claim.

From the configuration of Figure 188 we deduce:

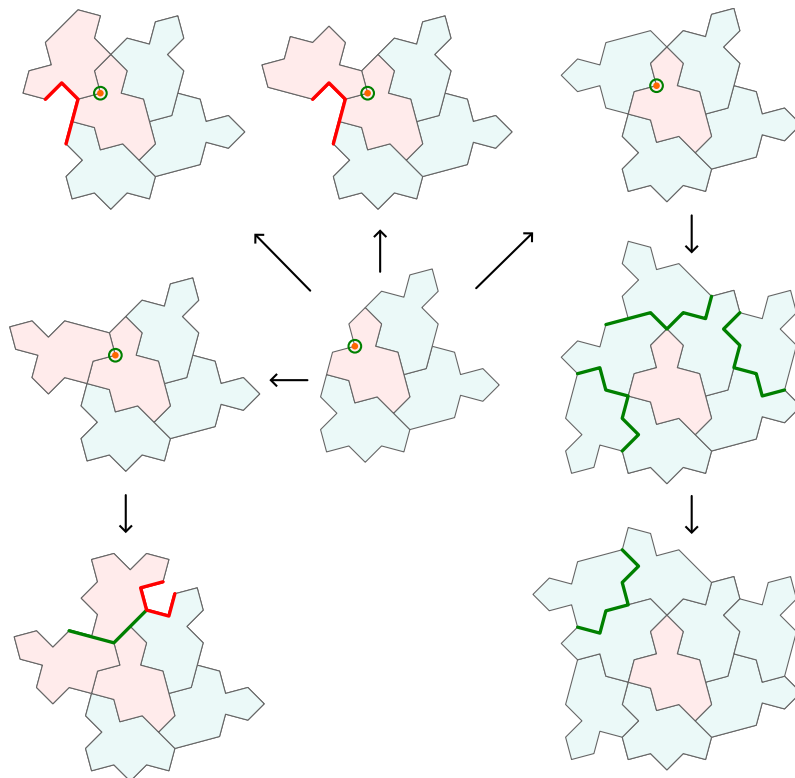


Figure 192

Which is almost the whole wanted picture, except one tile, for which the deduction is a bit long. Note that *this last tile is not needed to deduce that whole plane tilings with the Spectre necessarily have a parity class whose tiles are all isolated*.

The deduction goes as follows: if it were not the case, the only other possibility at the marked indent below would be the tile marked 0.

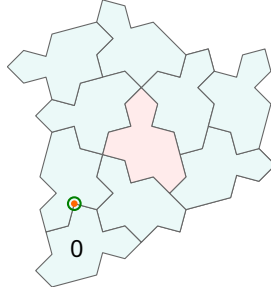


Figure 193

If this were the case, let us look at another indent and at the 4 possible ways to fill it and what this implies. Numbers indicate in which order the tiles can be deduced using Proposition 9.

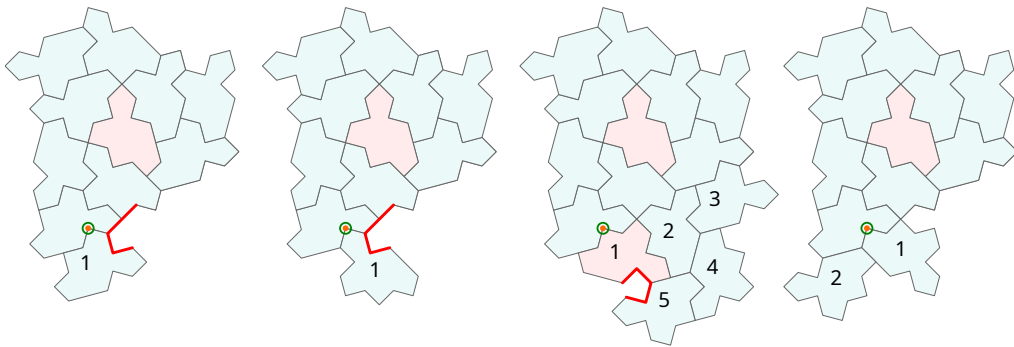


Figure 194

The last case is ruled out as follows:

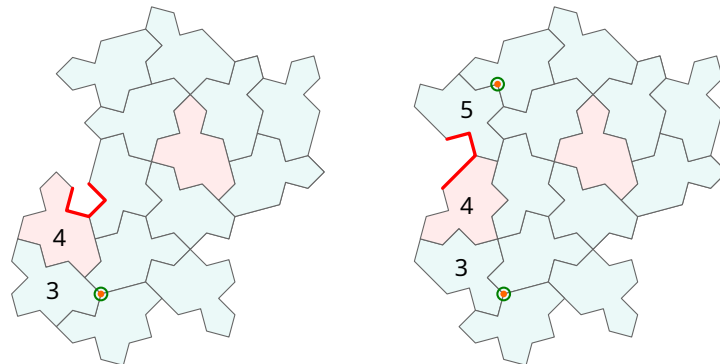


Figure 195

□

3.2 Cc types

The proposition below is Proposition 24.

Proposition. *Every cc of a T1 is necessarily as on the picture below:¹²*

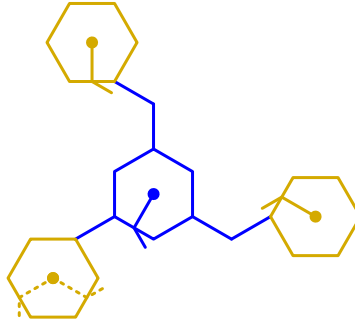


Figure 196

Proof. Let us fix the orientation of the Spectre that the T1 is associated to:

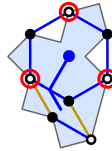


Figure 197

With this orientation, we are in the first two possibilities of Figure 64. Knowing that the second configuration cannot appear, the top tip can only be:

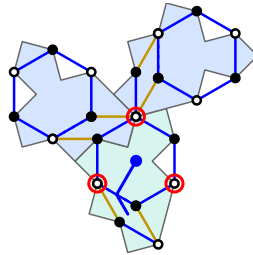


Figure 198

which by repeated use of Proposition 9 can only complete into

¹²Actually we will see in Corollary 48 that the lower left arrow can only have one orientation.

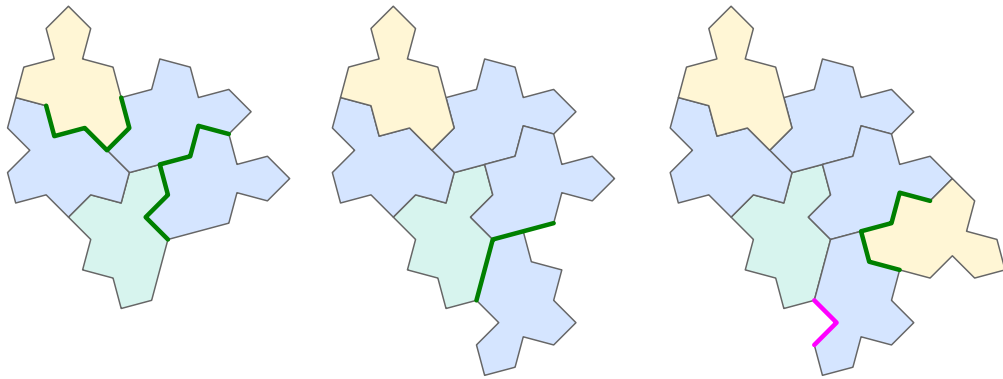


Figure 199

and then looking at what can fit in the pink dent above and completing two more tiles, we get the following two possibilities:

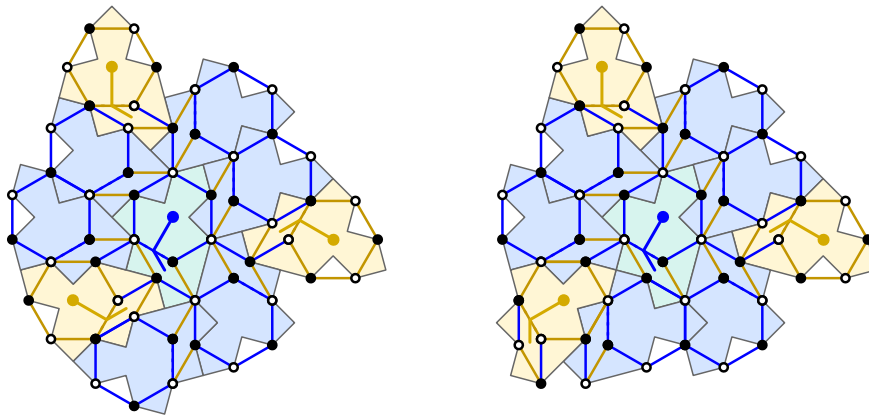


Figure 200

□

The proposition below is Proposition 25.

Proposition. *A T2 with markings as below cannot occur.*

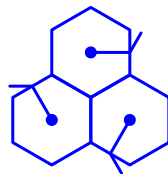


Figure 201

Proof. Initially, there is an order 3 rotational symmetry. The inward dent of the blue segment highlighted in red below can only have two ways to be filled, because no other blue hex can touch it by definition of a T2:

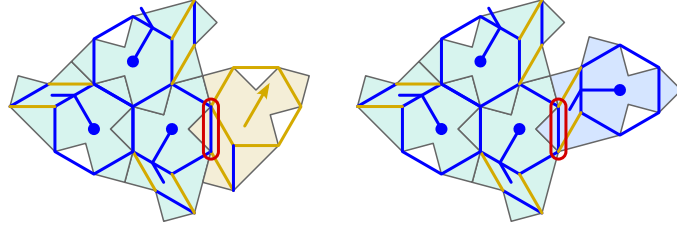


Figure 202

The one of the left contradicts the orientation of the bottom-most green tile in Proposition 11 at the tile outlined in red below left. The one on the right implies the configuration below right, which yields the same contradiction at the tile outlined in red.

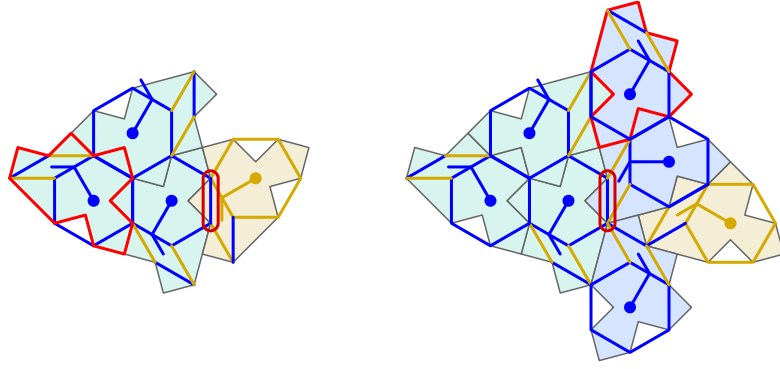


Figure 203

□

The proposition below is Proposition 26.

Proposition. *A T2 with markings as below left has a cc as below right:*

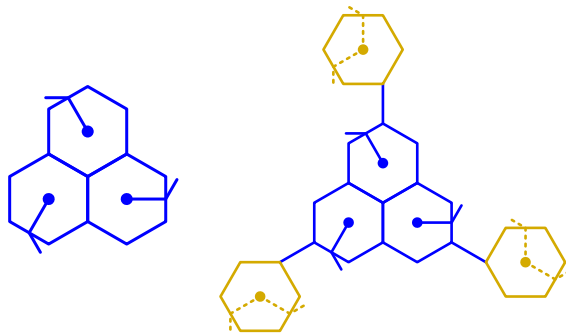


Figure 204

Proof. We have the following series of implications, with the red boundaries being places where no tile can fit.

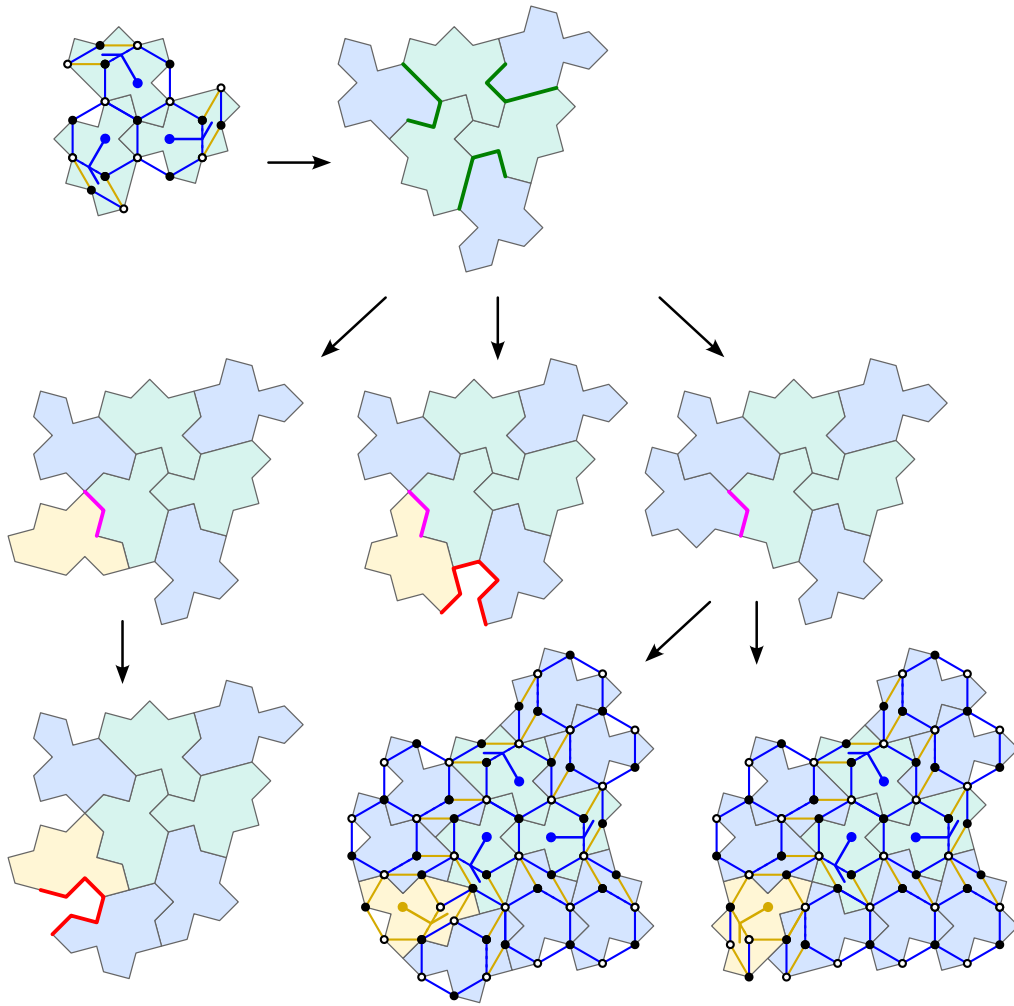


Figure 205

In the end there are two possibilities for the lower left corner of the T2, and the other two corners have the same possibilities by rotation. \square

The proof of Proposition 27 is by a similar kind of analysis and is omitted here.

The lemma below is Lemma 28

Lemma. *A T3 is environed as follows, in terms of the D3 and D2 induced tiling:*

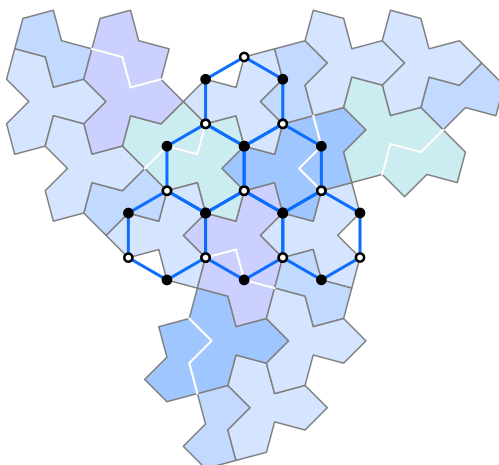


Figure 206

Proof. We claim that the pink inward dents below must be filled as follows:

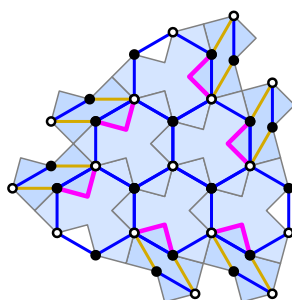


Figure 207

Indeed the dent filling it must be supported by a blue segment. By definition of a T3 we cannot use a blue D3 (it has a blue hex) so we are left with a yellow D2 or a blue D2. But the yellow one does not fit, as the following figure shows:

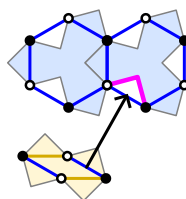


Figure 208

We can immediately see that three D3 have only one adjacent D2 so must be paired with them:

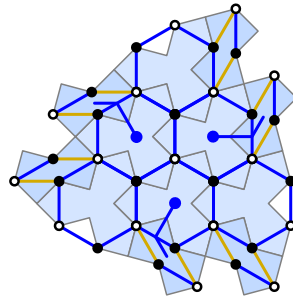


Figure 209

In the following configuration, the pink hollow can only be filled by a D3, as the only other fitting piece is a yellow D2 whose paired D3 cannot fit.

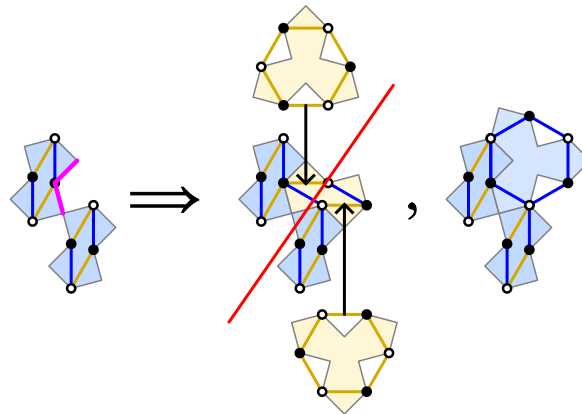


Figure 210

So we get:

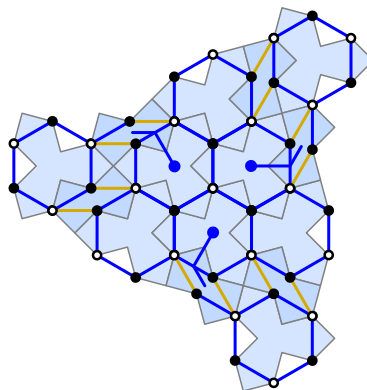


Figure 211

The following configuration on the left leads to the presence of an D2 (in orange) which cannot be paired.

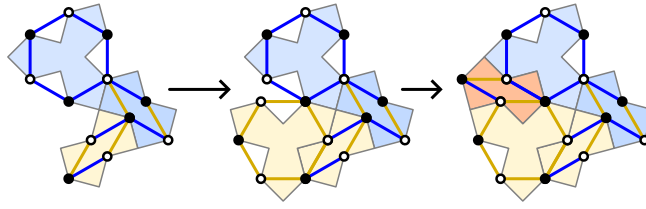


Figure 212

As a consequence we get:

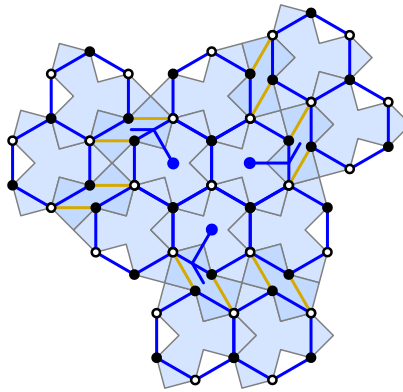


Figure 213

Now by the completion rule of Lemma 6 we get the left frame below, and then the cyan tile can only have their associated D2 placed as below right, and this allows to add one more D2 by the principle of Figure 208

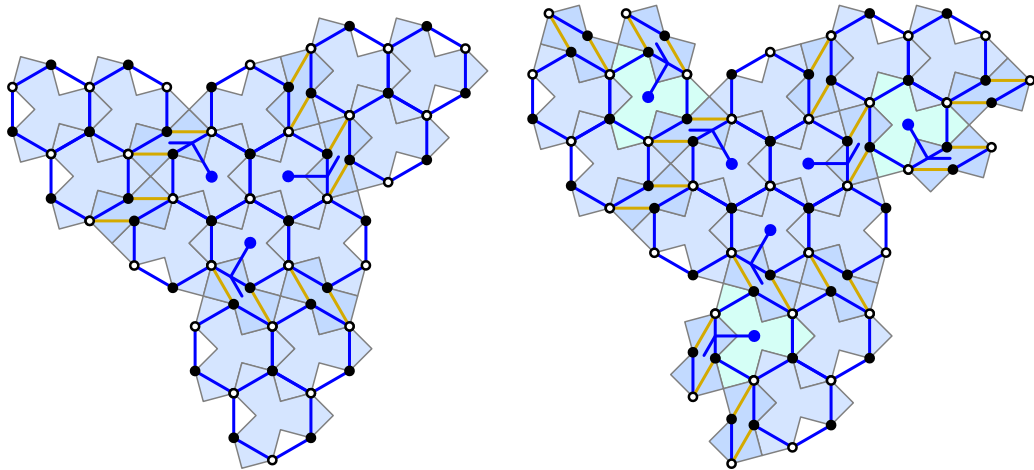


Figure 214

□

The proposition below is Proposition 29

Proposition. *A T3 has no antenna: it is its own cc. It can only be as follows*

on its lower left corner, and similarly for the other two by rotation.

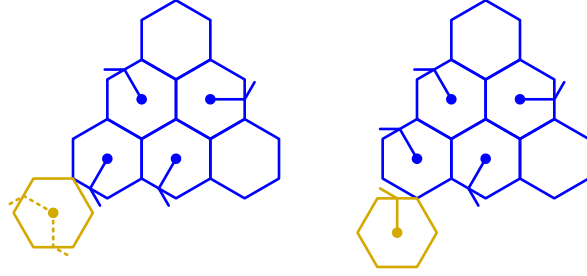


Figure 215

Proof. The Spectre containing the lower left blue hex can be in two positions. For each, let us see what this implies on the environment given by Figure 206.

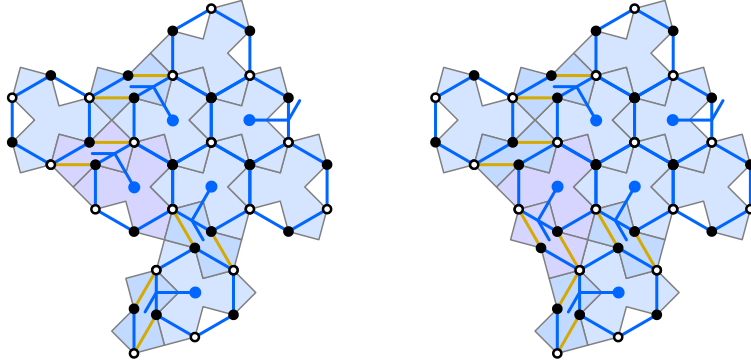


Figure 216

They imply:

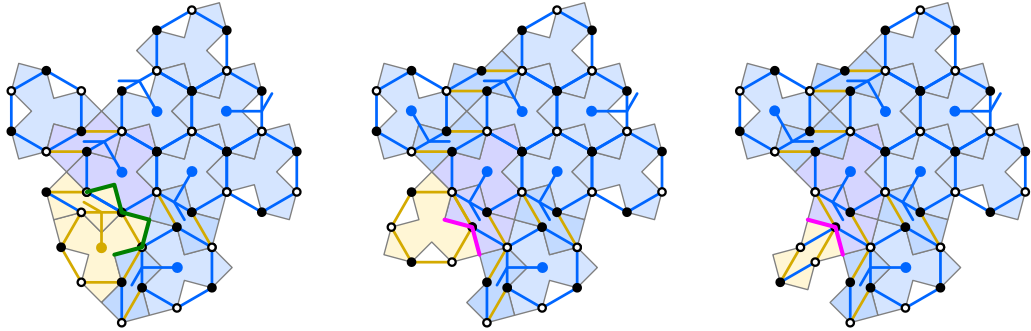


Figure 217

The central case implies the first two below, and the right case implies the right one: first the paired yellow D3 can only fit on one side of the yellow D2, then the the mauve tile is a forced blue tile, but its position with respect to the yellow tile contradicts Proposition 11.

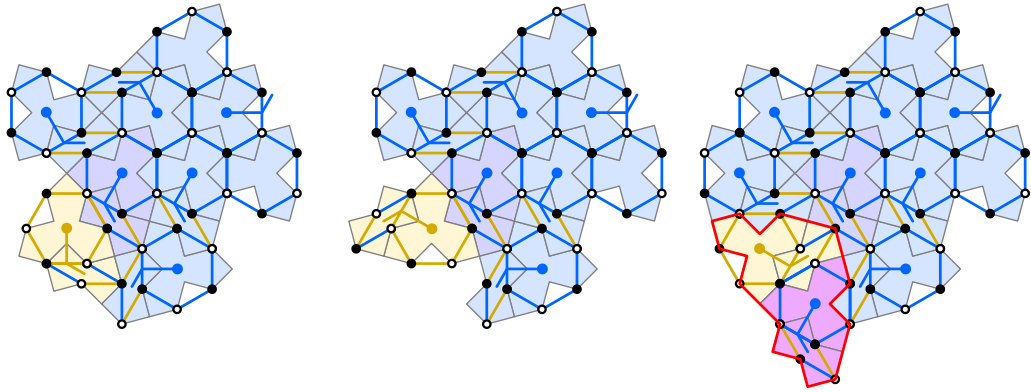


Figure 218

□

3.3 Environment of T'_n clusters

We will make implicit or explicit use of the visual aids of Section 2.11.1, which we encourage the reader to consult again.

The statement below is a copy of Proposition 60.

Proposition. *The fifth configuration of Figure 99 cannot occur in a whole plane tiling.*

Proof. We have the following sequence of deductions

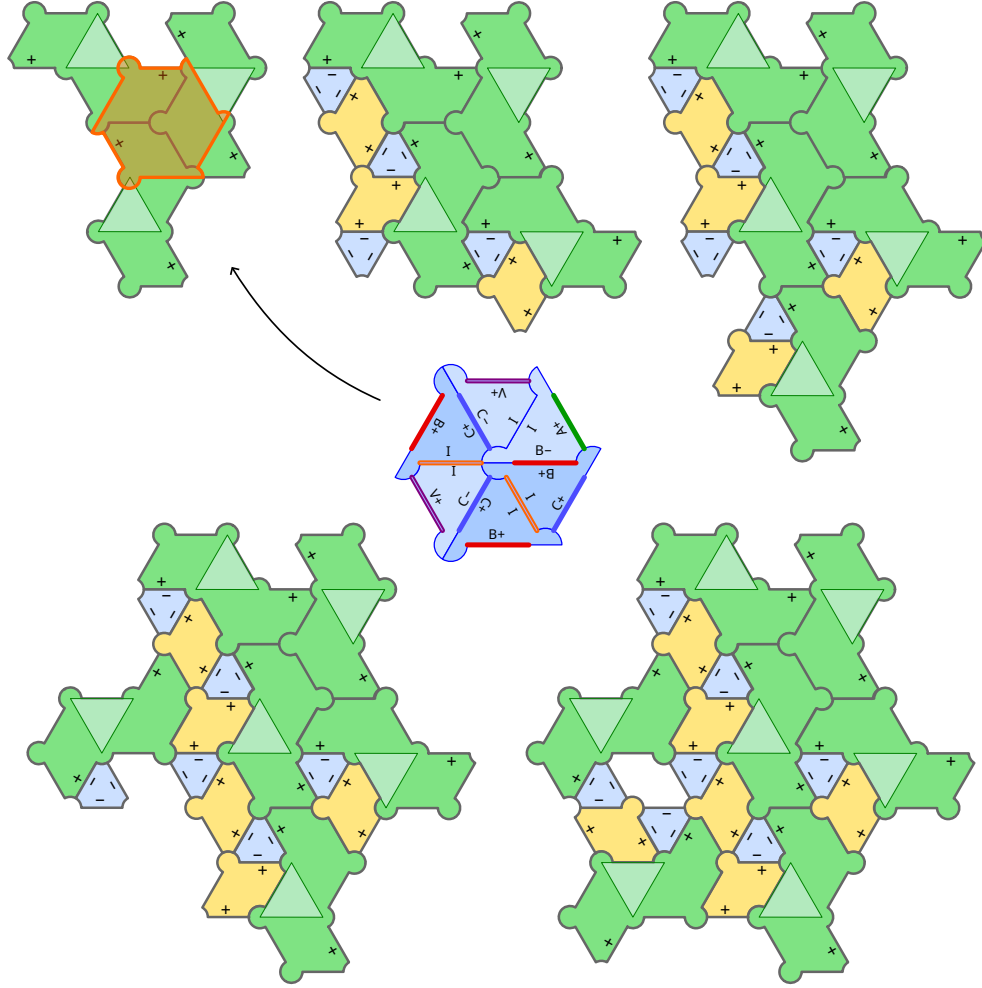


Figure 219

The last one leaves a hole that cannot be filled. □

The following statement is a copy of Lemma 61.

Lemma. *Consider two tips following each other in the clockwise order around a $T'2$ cluster of a whole plane tiling by the packs. Then, using the nomenclature of Figure 130, their respective types cannot be in the following list (each item is to be read in the clockwise order): (n, n) , $(a2', a2')$, $(a2', n)$, $(a1, a2')$, $(a1', a2')$.*

Proof. We have the following deductions for each case, each leading to an impossibility indicated in red:

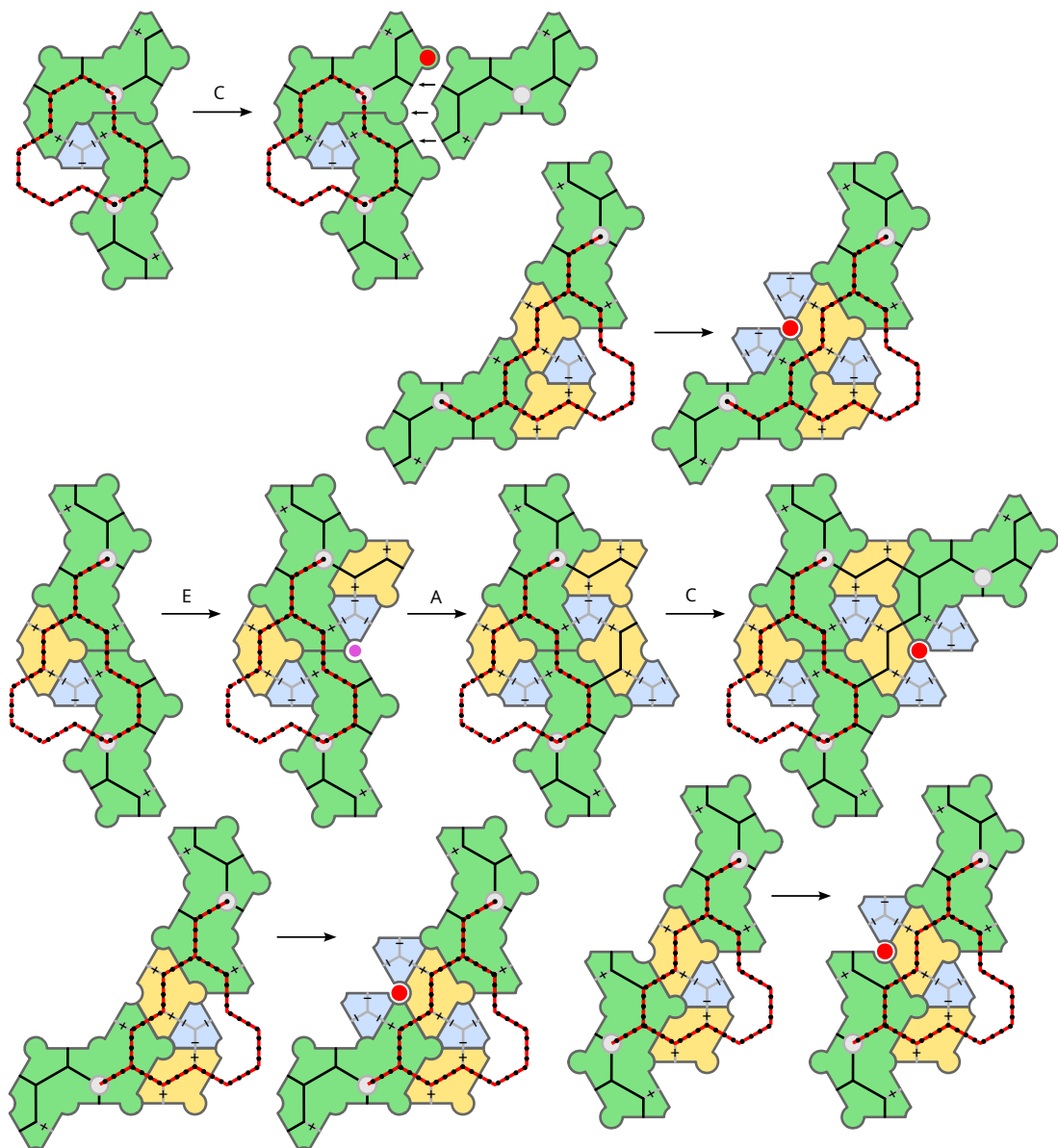


Figure 220 – The implication with an E is deduced from the visual rule of Lemma 59, those with a C from that of Lemma 57. For the A , the mauve dot can only be filled by a yellow piece, since no green piece can fit that would fill it.

□

The statement below is a copy of Lemma 62.

Lemma. *The environment of a $T'3$ is as on the figure below:*

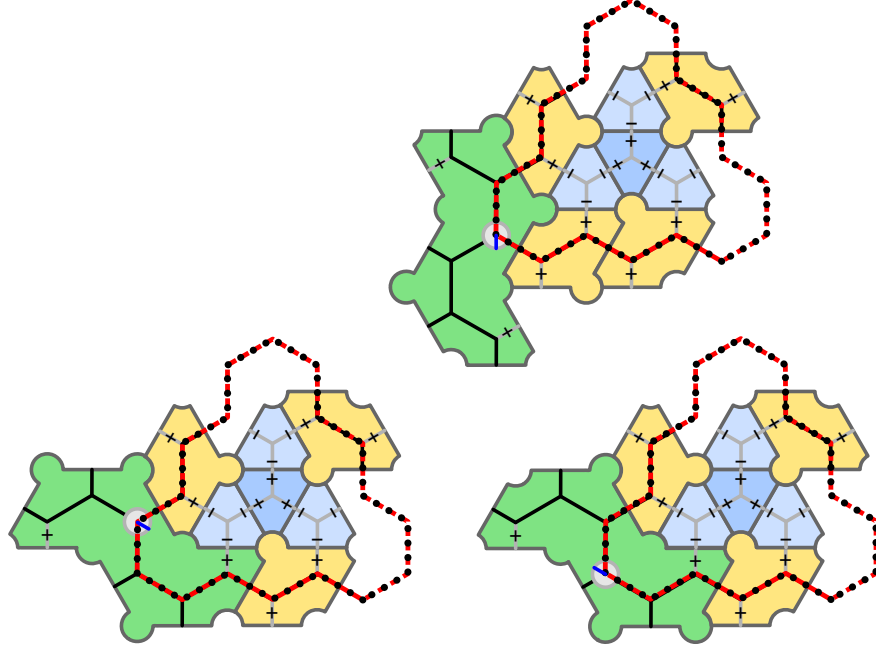


Figure 221

Proof. First, the environment of a $T'3$, described using the triangle tileset and the packed tileset, must be as follows:

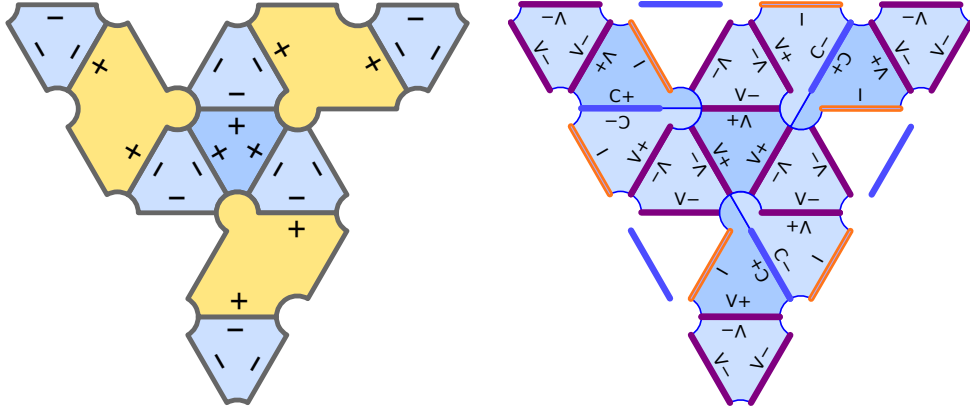


Figure 222

With the packed tileset, it is quite easy to prove: the highlighted segments in the following figure cannot match a green pack, because there is no room (an already placed $-,-,-$ triangle is taking the room). They cannot match a $+,+,+$ triangle because of Proposition 44. So this must be a yellow pack. The other $+$ side of the yellow pack must match the only pack with a $-$, namely the $-,-,-$ triangle.

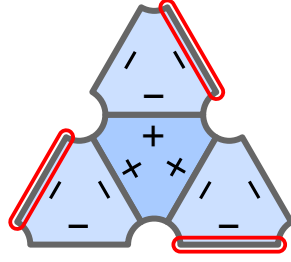


Figure 223 – Let us focus on the highlighted sides.

Similarly as for the T'2, from the T'3 environment, we get the situations depicted below, where red dots also mean an impossibility, and the explanations are below the figure.

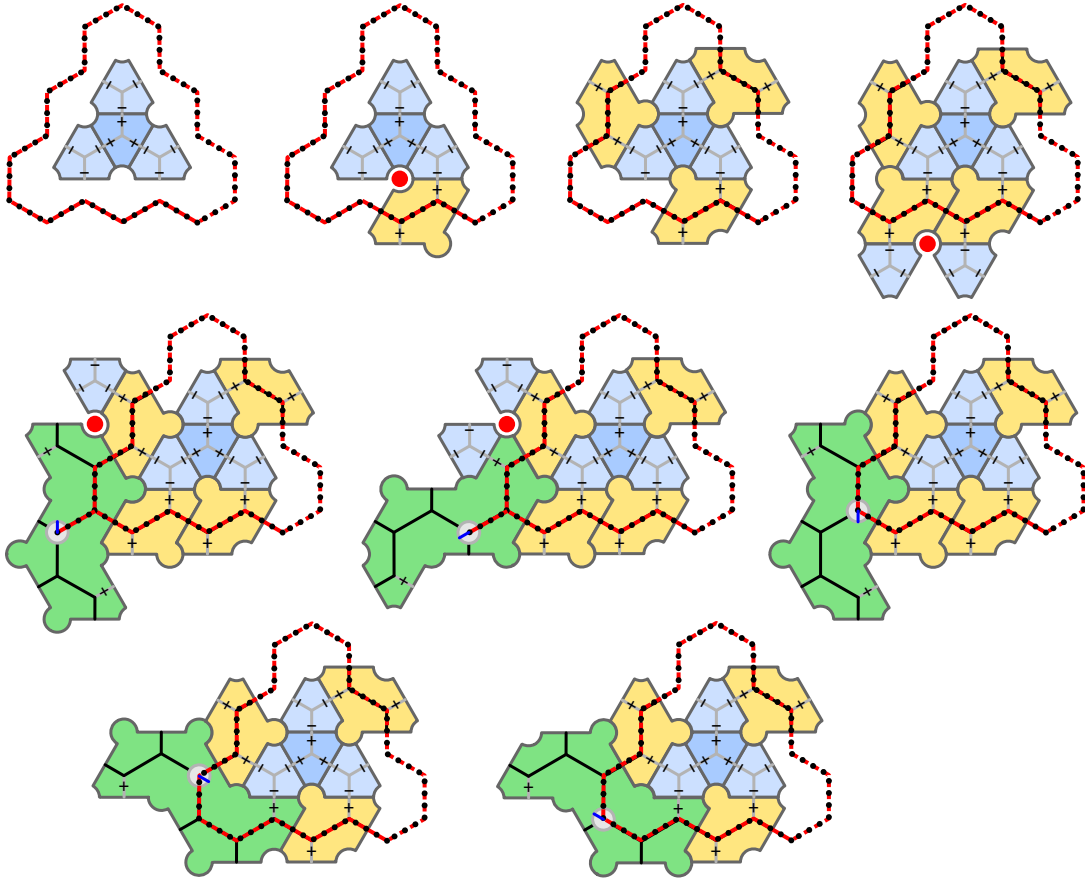


Figure 224

The image is the initial T'3. We saw the right '-' segment of the bottom part has to be in contact with one of the two + of a yellow one. The second image shows that one of these yellow + does not leave a possibility to complete the tiling. So we must be in the situation of the third image. The fourth image and the second row explore (im)possibilities in the case the left '-' segment of the bottom part of the initial configuration is in contact with a yellow +: only one case is not ruled out. The last row explores the case where it is in contact with a green +. There, no case is ruled out. \square

3.4 Recognizing the hexagons of [Smi+23a]

The statement below is a copy of Lemma 67.

Lemma. *Every green pack is environed as follows:*

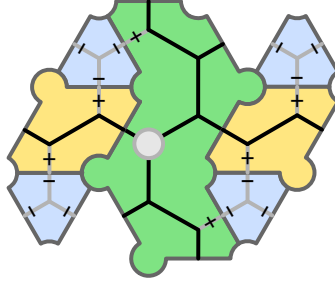


Figure 225

Proof. Otherwise we would have the following set of deductions, starting from Figure 108.

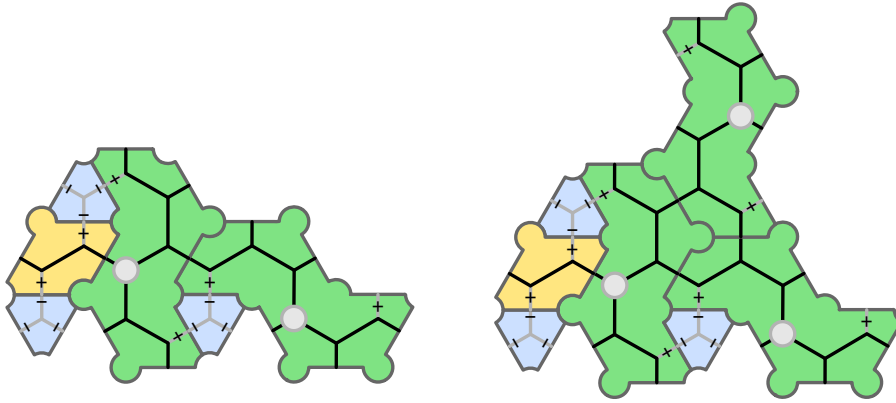


Figure 226

Leading to a configuration of green tiles that was excluded in Figure 219. □

The following statement is a copy of Lemma 68.

Lemma. *The following arrangement cannot occur in a whole plane tiling:*

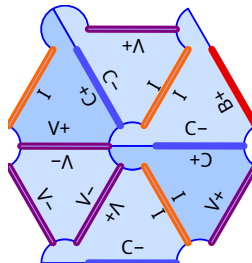


Figure 227

Proof. If the arrangement appears then there must be packs as follows:

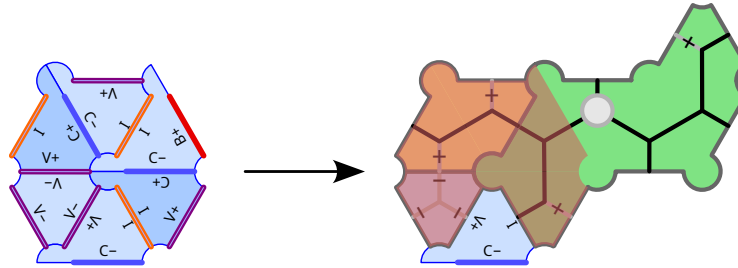


Figure 228

The bottom triangle can only be filled by the tips highlighted in dark green in the figure below (the ones in red have the wrong triangle type).

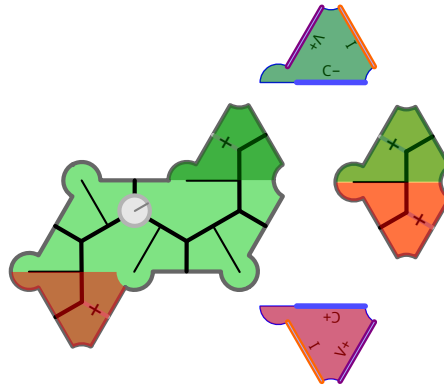


Figure 229

If the bottom is filled by a yellow piece then this leads to the following impossibility:

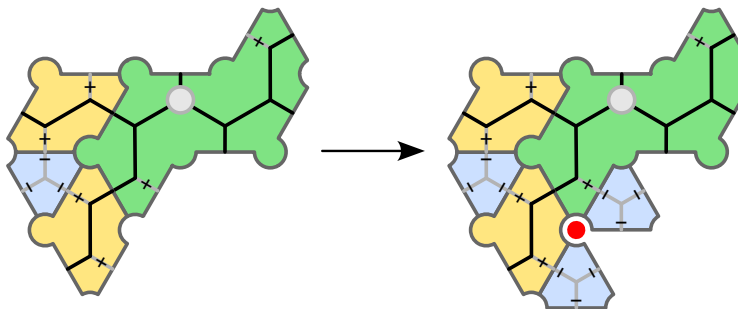


Figure 230

So it is filled by a green pack. We then have the following set of deductions:

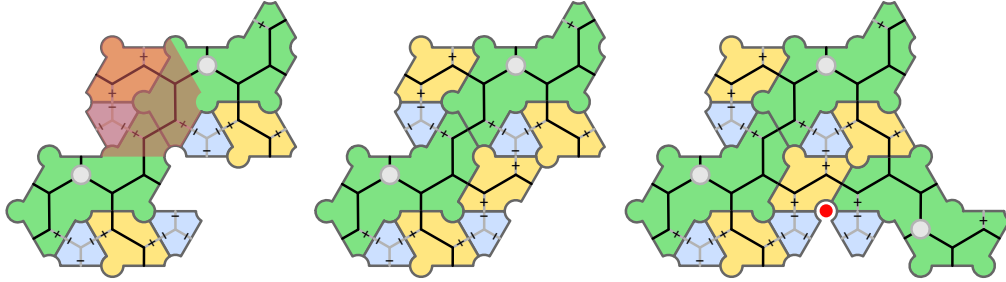


Figure 231 – For the first frame we used Figure 225. For the second and third one the visual rules of Section 2.11.1.

Leading to a non-fillable disc in red. □

The following statement is a copy of Lemma 69.

Lemma. *The following arrangement cannot occur in a whole plane tiling:*

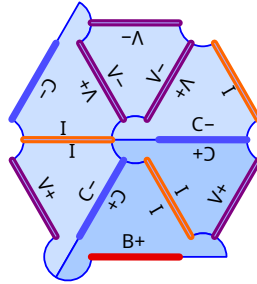


Figure 232

Proof. Otherwise we would have the following set of deductions where we start

by an alternative:

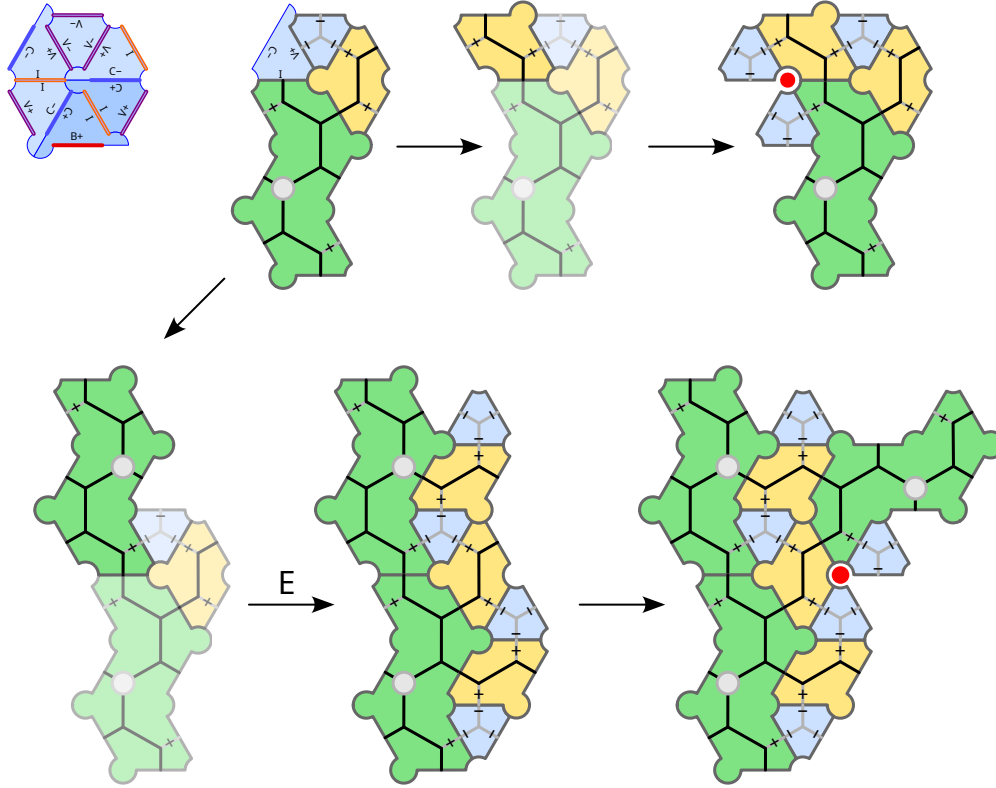


Figure 233 – For the arrow with an E indicates we used Figure 225

Each alternative lead to to a disc indicated in red and that is impossible to fill. \square

4 Double level hierarchy on the honeycomb partition does not require dots

In this section we come back to the partition of the honeycomb into triangular clusters, as on Figure 13. Recall that hexagons in the honeycomb correspond to blue hexes of the decoration graph associated to a Spectre tiling. We explained in Section 2 that if we add the position of the dots corresponding to contracted yellow hexes to the picture of the honeycomb partition, it has a unique hierarchical structure.

Can we do without the dots?

Deducing the position of the dots from the sole partition is not as trivial as one would think. In particular it may require to look at arbitrary large patches.

Let us be more precise. In the partition, the clusters can be thought of as related as are triangles on a regular triangular lattice. There is one dot corresponding to every vertex of this lattice, i.e. to every circular group of 6 adjacent clusters. There are no other dots. We observe only 3 kinds of cluster groups on the honeycomb, up to a rotation of $1/6$:

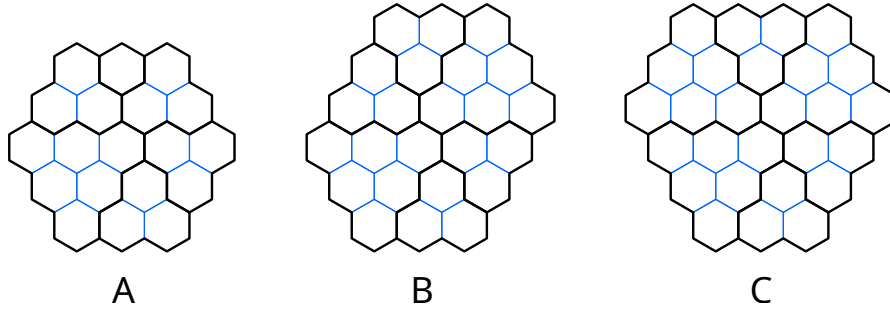


Figure 234

In group *C* the corresponding dot can only be placed on one vertex, but in the other two, they have two possibilities:

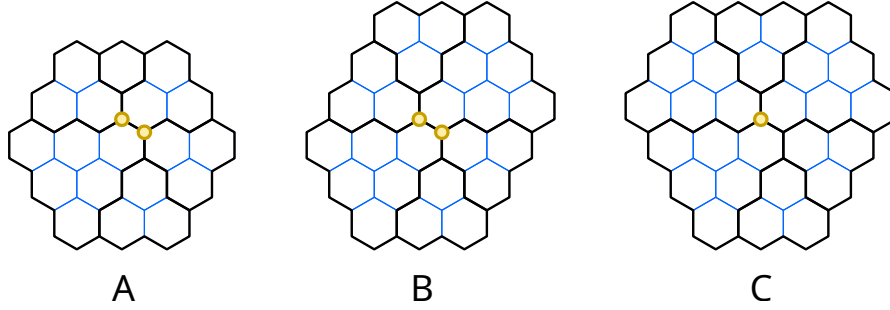


Figure 235

These claims can be deduced from the study of Section 2: for instance from the list of possible vertex environments in Lemma 70 and the cc's that the triangle pieces represent, see Figure 76.

Looking at larger groups of clusters around a group of 6 adds constraints to the position of the dot. Unfortunately, determining the placement of a dot on a group A or B may require to look at arbitrary larger groups depending on the situation: this will follow from the end of this section.

Remarkably, it is possible to describe the *second iterate* of the hierarchy/substitution uniquely in terms of the partition, i.e. without the dots.

To explain this, let us first note that in the partitioned honeycomb with dots, from Figure 100 in Proposition 47, we get that the environment of a T1 cc is as follows, up to a rotation of a multiple of $1/6$:

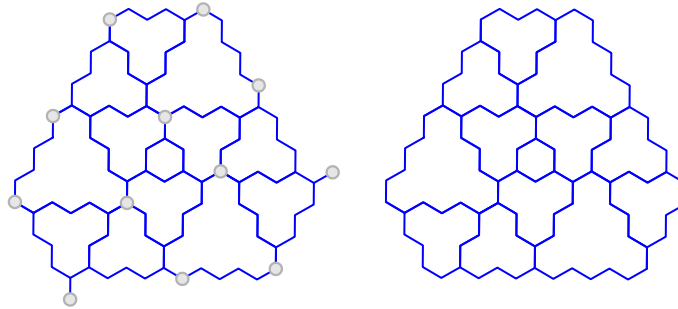


Figure 236 – On the right, we only retained the clusters: the figure then gets an order 3 rotation symmetry

In Figures 161 and 162 the T1 are the hexagons containing the purple dots. Let us place the cluster group of the right figure above for each of the 8 occurring (i.e. disregarding the 4th) situations of these figures. For each purple dot, it is possible to know the orientation of the group (there are only 2 possibilities because of the order 3 symmetry) from the orientation of the dot before the substitution (indicated in Figures 149 and 150): this is because it is either known or has two known possibilities that differ from $1/3$.

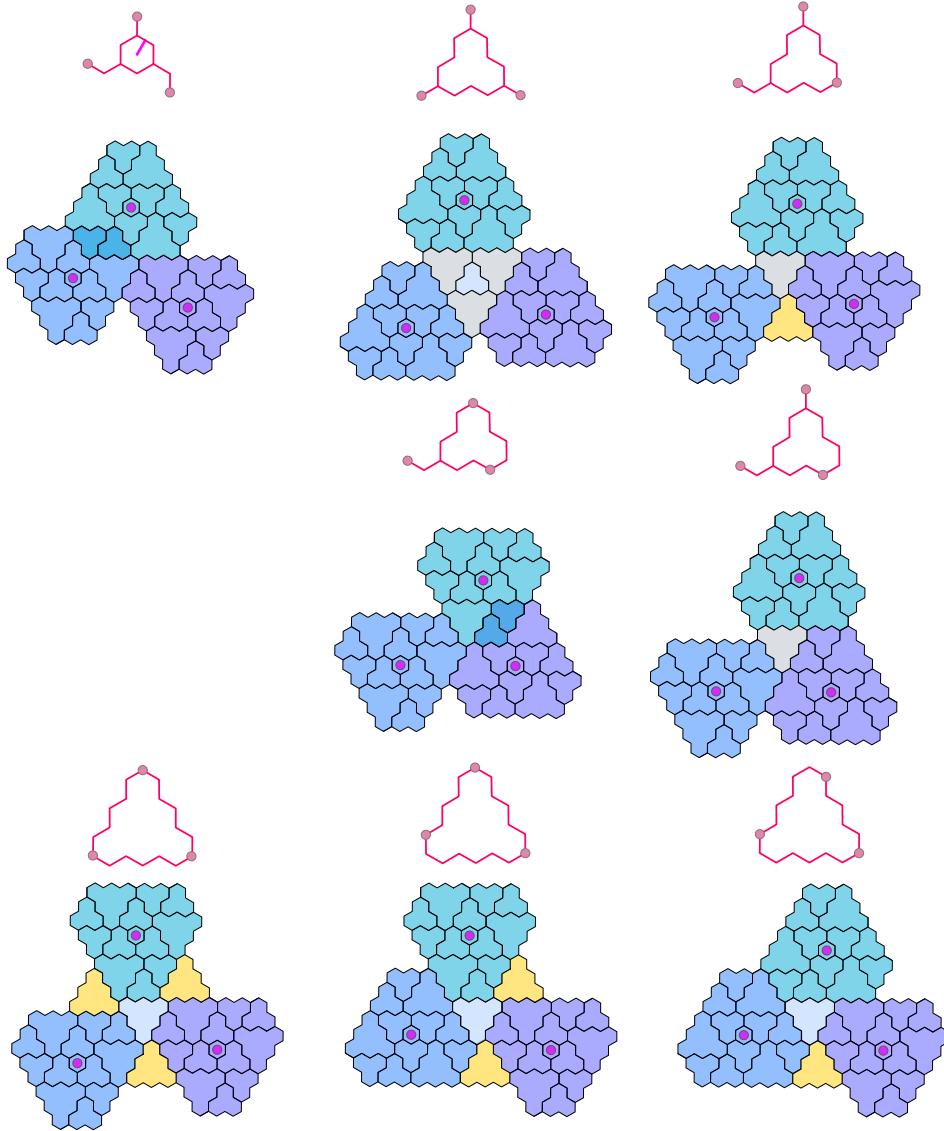


Figure 237 – We placed neighbourhoods at purple dots. Where they overlap, the shade is darker. We added yellow and light blue clusters coming from Figures 161 and 162. Then we placed gray clusters in the centre as the only possible ones filling the gaps and respecting Figure 234.

Now the purple dots correspond to hexes two levels up (call them **green hexes**). The V-type interfaces of the cc's one level up (in red on the last figure) link dots whose corresponding “green” hex two levels up is in the same green clusters, while

I-type interfaces link dots whose corresponding green hexes are in adjacent green clusters: see Definitions 42 and 43 and figs. 104 and 106 and the whole analysis we made in of the hierarchy (but apply all this one level up). Figure 237 implies it is possible to deduce the blue honeycomb partition from the green honeycomb partition two levels above. More precisely the green honeycomb clusters will provide groups of 1, 3 or 6 purple dots in a triangle configuration on the blue honeycomb, with the corresponding blue hexes being separated by a determined vector (associated to V-type interfaces): using complex numbers, if two adjacent hexagons of the blue honeycomb are separated by vector 1, and thus the whole blue honeycomb by vectors in $\Lambda = \mathbb{Z} + \rho\mathbb{Z}$ where $\rho = e^{i2\pi/6}$ is a primitive cube root of 1, then the vector is in $(8 + \rho) \times \Lambda$. The way the triangle points is also known. Adjacent green honeycomb clusters are separated by a vector that is known too, 7 times a power of ρ which is determined by which side of the two green clusters touch. Note also that the partition of the green honeycomb into green clusters follows from the vectors between nearby T1 in the blue honeycomb: the V-type and I-type interfaces are characterized by this vector being respectively in $(8 + \rho) \times \rho^{\mathbb{Z}}$ and $7 \times \rho^{\mathbb{Z}}$. The other interfaces have incompatible vectors : A gives $(5 + \rho^{-1}) \times \rho^{\mathbb{Z}}$, B gives $(6 + 2 \times \rho^{-1}) \times \rho^{\mathbb{Z}}$ and C gives $(8 + \rho^{-1}) \times \rho^{\mathbb{Z}}$.

Finally we explain how to get dot ambiguity from a given partition. Start from the environment of a T1 as on Figure 236. The partition has order 3 rotation symmetry, while the dots cannot. Perform the order two substitution for the partition an arbitrary number of times. Note that a T1 lies at the centre again, and that the obtained arrangement of clusters must keep the order three symmetry, while as we noted, the dots cannot have this symmetry. There will be differences in the dots near the outer boundary of the big arrangement, and knowing one may determine the orientation at the centre. This way we can already make a whole plane tiling for which the blue honeycomb partition has order three symmetry around an T1, whose orientation is thus not determined in advanced. But better: knowing that there are non-symmetric situations with a T1 whose orientation is forced in Figure 237, it is possible to construct from any whole plane tiling, using an even number of substitution, a whole plane partitions orientation that at some T1 is only determined if one looks at a big enough situation.

We end with the following picture, showing occurrences of the shape in Figure 236, and the way the same shape may appear two levels up in some whole plane tiling.

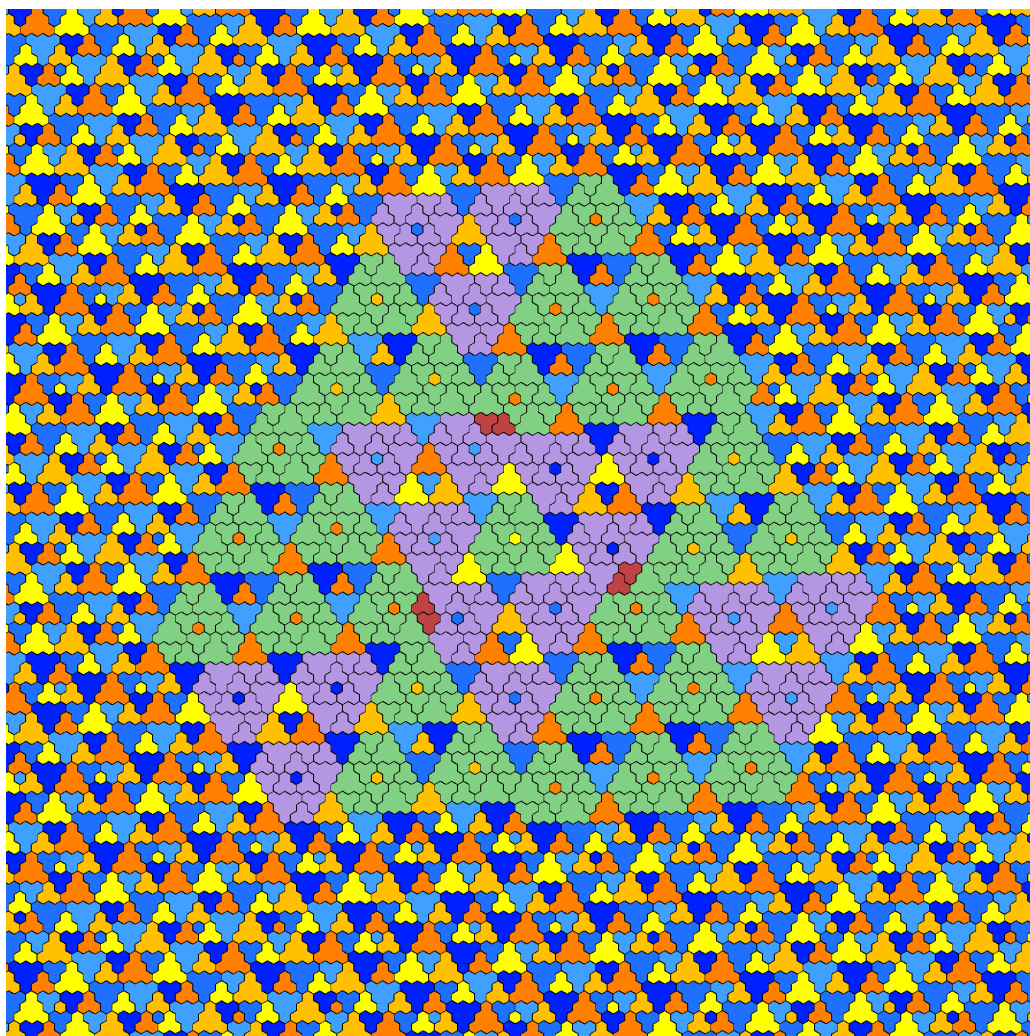


Figure 238

References

- [AA23] Shigeki Akiyama and Yoshiaki Araki. *An alternative proof for an aperiodic monotile*. 2023. arXiv: 2307.12322 [math.MG]. URL: <https://arxiv.org/abs/2307.12322>.
- [CNM23] Arnaud Chéritat, Nan Ma, and Pieter Mostert. *4D lift of the tilings by the Smith et al. aperiodic monotile*. <https://www.math.univ-toulouse.fr/~cheritat/2023-monotile/4D-lift/>. 2023.
- [Smi+23a] David Smith, Joseph Samuel Myers, Craig S. Kaplan, and Chaim Goodman-Strauss. *A chiral aperiodic monotile*. 2023. arXiv: 2305.17743 [math.CO].
- [Smi+23b] David Smith, Joseph Samuel Myers, Craig S. Kaplan, and Chaim Goodman-Strauss. *An aperiodic monotile*. 2023. arXiv: 2303.10798 [math.CO].

- [Smi24] James Smith. *Turtles, Hats and Spectres: Aperiodic structures on a Rhombic tiling*. 2024. arXiv: 2403.01911 [math.MG]. URL: <https://arxiv.org/abs/2403.01911>.