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Fairness and Consensus in an Asynchronous Opinion Model for Social Networks

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Abstract

We introduce a DeGroot-based model for opinion dynamics in social networks. A community of agents is represented as a weighted directed graph whose edges indicate how much agents influence one another. The model is formalized using labeled transition systems, henceforth called *opinion transition systems (OTS)*, whose states represent the agents' opinions and whose actions are the edges of the influence graph. If a transition labeled (i, j) is performed, agent j updates their opinion taking into account the opinion of agent i and the influence i has over j . We study (*convergence to*) *opinion consensus* among the agents of strongly-connected graphs with influence values in the interval $(0, 1)$. We show that consensus cannot be guaranteed under the standard *strong fairness* assumption on transition systems. We derive that consensus is guaranteed under a stronger notion from the literature of concurrent systems; *bounded fairness*. We argue that bounded-fairness is too strong of a notion for consensus as it almost surely rules out random runs and it is not a constructive liveness property. We introduce a weaker fairness notion, called *m-bounded fairness*, and show that it guarantees consensus. The new notion includes almost surely all random runs and it is a constructive liveness property. Finally, we consider OTS with *dynamic influence* and show convergence to consensus holds under *m-bounded fairness* if the influence changes within a fixed interval $[L, U]$ with $0 < L < U < 1$. We illustrate OTS with examples and simulations, offering insights into opinion formation under fairness and dynamic influence.

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Related Version *Technical Report with proofs*: <https://arxiv.org/abs/2312.12251> [7]

Supplementary Material *Software (Source python code for simulations)*: <https://github.com/promueva/Fairness-and-Consensus-in-Opinion-Models>

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1 Introduction

Social networks have a strong impact on *opinion formation*, often resulting in polarization. Broadly, the dynamics of opinion formation in social networks involve users expressing their opinions, being exposed to the opinions of others, and potentially adapting their own views based on these interactions. Modeling these dynamics enables us to glean insights into how opinions form and spread within social networks.



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44 The models of social learning aim to capture opinion dynamics in social networks [36].
 45 The DeGroot model [14] is one of the most prominent formalisms for social learning and
 46 opinion formation dynamics, and it remains a continuous focus of study in social network
 47 theory [21]. A given community is represented as a weighted directed graph, known as the
 48 *influence graph*, whose edges indicate how much individuals (*agents*) influence one another.
 49 Each agent has an opinion represented as a value in $[0, 1]$, indicating the strength of their
 50 agreement with an underlying proposition (e.g., “*AI poses a threat to humanity*”). Agents
 51 repetitively revise their opinions by averaging them with those of their contacts, taking into
 52 account the influence each contact holds. (There is empirical evidence validating the opinion
 53 formation through averaging of the model in controlled sociological experiments, e.g., [10].) A
 54 fundamental theoretical result of the model states that the agents will *converge to consensus*
 55 if the influence graph is *strongly connected* and the agents have non-zero self-influence (*puppet*
 56 *freedom*) [21]. The significance of this result lies in the fact that consensus is a central
 57 problem in social learning. Indeed, the inability to reach consensus is a sign of a polarized
 58 community.

59 Nevertheless, the DeGroot model makes at least two assumptions that could be overly
 60 constraining within social network contexts. Firstly, it assumes that all the agents update
 61 their opinions simultaneously (*full synchrony*), and secondly, it assumes that the influence of
 62 agents remains the same throughout opinion evolution (*static influence*). These assumptions
 63 may hold in some controlled scenarios and render the model tractable but in many real-world
 64 scenarios individuals do not update their opinions simultaneously [29]. Instead, opinion
 65 updating often occurs *asynchronously*, with different agents updating their opinions at
 66 different times. Furthermore, individuals may gain or lose influence through various factors,
 67 such as expressing contrarian or extreme opinions [20].

68 In this paper, we introduce an *asynchronous* DeGroot-based model with *dynamic influence*
 69 to reason about opinion formation, building upon notions from concurrency theory. The
 70 model is presented by means of labeled transition systems, here called *opinion transition*
 71 *systems (OTS)*. The states of an OTS represent the agents’ opinions, and the actions (labels)
 72 are the edges of the influence graph. All actions are *always* enabled. If a transition labeled
 73 with an edge (i, j) is chosen, agent j updates their opinion by averaging it with the opinion
 74 of agent i weighted by the influence that this agent carries over j . A *run* of an OTS is an
 75 infinite sequence of (chosen) transitions.

76 We shall focus on the problem of convergence to opinion consensus in runs of the OTS,
 77 *assuming* strong connectivity of the influence graph and puppet freedom. For consensus to
 78 make sense, all agents should have the chance to update their opinions. Therefore, we need
 79 to make *fairness* assumptions about the runs. In concurrency theory, this means requiring
 80 that some actions be performed sufficiently often.

81 We first show that contrary to the DeGroot model, consensus *cannot* be guaranteed for
 82 runs of OTS even under the standard *strong fairness* assumption (i.e., that each action occurs
 83 infinitely often in the run) [22, 27]. This highlights the impact of asynchronous behavior on
 84 opinion formation.

85 We then consider the well-known notion of *bounded fairness* in the literature on verification
 86 of concurrent systems [16]. This notion requires that every action must be performed not
 87 just eventually but within some bounded period of time. We show that bounded-fairness
 88 guarantees convergence to consensus. This also gives us insight into opinion formation
 89 through averaging, i.e., preventing unbounded delays of actions (opinion updates) is sufficient
 90 for convergence to consensus.

91 Nevertheless, bounded fairness does not have some properties one may wish in a fairness

92 notion. In particular, it is not a *constructive liveness* property in the sense of [34, 33].
93 Roughly speaking, a fairness notion is a constructive liveness property if, while it may require
94 that a particular action is taken sufficiently often, it should not prevent any other action
95 from being taken sufficiently often. Indeed, we will show that preventing unbounded delays
96 implies preventing some actions from occurring sufficiently often.

97 Furthermore, bounded-fairness is not *random inclusive*. A fairness notion is random
98 inclusive if any random run (i.e., a run where each action is chosen independently with
99 non-zero probability) is *almost surely* fair under the notion. We find this property relevant
100 because we wish to apply our results to other asynchronous randomized models whose runs
101 are random and whose opinion dynamics can be captured as an OTS.

102 We therefore introduce a new weaker fairness notion, called *m-bounded fairness*, and show
103 that it guarantees consensus. The new notion is shown to be a constructive liveness property
104 and random inclusive. We also show that consensus is guaranteed under *m-bounded fairness*
105 even if we allow for *dynamic influence* as long as all the changes of influence are within a
106 fixed interval $[L, U]$ with $0 < L < U < 1$.

107 All in all, we believe that asynchronous opinion updates and dynamic influence provide
108 us with a model more faithful to reality than the original DeGroot model. The fairness
109 assumptions and consensus results presented in this paper show that the model is also
110 tractable and that it brings new insights into opinion formation in social networks. To the
111 best of our knowledge, this is the first work using fairness notions from concurrency theory
112 in the context of opinion dynamics in social networks.

113 Furthermore, since *m-bounded fairness* is random inclusive, our result extends with
114 dynamic influence the consensus result in [17] for distributed averaging with randomized
115 gossip algorithms. Distributed averaging is a central problem in other application areas, such
116 as decentralized computation, sensor networks and clock synchronization.

117 **Organization.** The paper is organized as follows: In Section 2, we introduce OTS and
118 the consensus problem. Initially, to isolate the challenges of asynchronous communication in
119 achieving consensus, we assume static influence. In Section 3, we identify counter-examples,
120 graph conditions, and fairness notions for consensus to give some insight into opinion dynamics.
121 In Section 4, we introduce a new notion of fairness and state our first consensus theorem.
122 Finally, in Section 5, we add dynamic influence and give the second consensus theorem.

123 The detailed proofs are included in a related technical report [7]. The Python code
124 used to produce OTS examples and simulations in this paper can be found in the following
125 repository: <https://github.com/promueva/Fairness-and-Consensus-in-Opinion-Models>.

126 **2 The Model**

127 In the standard DeGroot model [14], agents update their opinion *synchronously* in the
128 following sense: at each time unit, all the agents (individuals) update simultaneously their
129 current opinion by listening to the current opinion values of those who influence them. This
130 notion of updating may be unrealistic in some social network scenarios, as individuals may
131 listen to (or read) others' opinions at different points in time.

132 In this section, we introduce an opinion model where individuals update their beliefs
133 asynchronously; one agent at a time updates their opinion by listening to the opinion of one
134 of their influencers.

135 **2.1 Opinion Transition Systems**

136 In social learning models, a *community* is typically represented as a directed weighted
 137 graph with edges between individuals (agents) representing the direction and strength of the
 138 influence that one has over the other. This graph is referred to as the *Influence Graph*.

139 ► **Definition 1** (Influence Graph). *An influence graph is a directed weighted graph $G = (A, E, I)$
 140 with $A = \{1, \dots, \mathbf{n}\}$, $\mathbf{n} > 1$, the vertices, $E \subseteq A^2 - Id_A$ the edges (where Id_A is the identity
 141 relation on A) and $I : E \rightarrow (0, 1]$ the weight function.*

142 The vertices in A represent \mathbf{n} agents of a given community or network. The set of edges
 143 E represents the (direct) influence relation between agents; i.e., $(i, j) \in E$ means that agent i
 144 influences agent j . The value $I(i, j)$, for simplicity written $I_{(i,j)}$ or I_{ij} , denotes the strength
 145 of the influence: a higher value means stronger influence.

146 Similar to the DeGroot-like models in [21], we model the evolution of agents' opinions
 147 about some underlying *statement* or *proposition*, such as, for example, “*human activity has*
 148 *little impact on climate change*” or “*AI poses a threat to humanity*”.

149 The *state of opinion* (or *belief state*) of all the agents is represented as a *vector* in $[0, 1]^{|A|}$.
 150 If \mathbf{B} is a state of opinion, $\mathbf{B}[i]$ denotes the *opinion* (*belief*, or *agreement*) value of agent $i \in A$
 151 regarding the underlying proposition: the higher the value of $\mathbf{B}[i]$, the stronger the agreement
 152 with such a proposition. If $\mathbf{B}[i] = 0$, agent i completely *disagrees* with the underlying
 153 proposition; if $\mathbf{B}[i] = 1$, agent i completely *agrees* with the underlying proposition.

154 The opinion state is updated as follows: Starting from an initial state, at each time unit,
 155 one of the agents, say j , updates their opinion taking into account the influence and the
 156 opinion of one of their contacts, say i . Intuitively, in social network scenarios, this can be
 157 thought of as having an agent j read or listen to the opinion of one of their influencers i and
 158 adjusting their opinion $\mathbf{B}[j]$ accordingly.

159 The above intuition can be realized as a *Labelled Transition System* (LTS) whose set of
 160 states is $S = [0, 1]^{|A|}$ and set of *actions* is E .

161 ► **Definition 2** (OTS). *An Opinion Transition System (OTS) is a tuple $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$
 162 where $G = (A, E, I)$ is an influence graph, $\mathbf{B}_{\text{init}} \in S = [0, 1]^{|A|}$ is the initial opinion state,
 163 and $\rightarrow \subseteq S \times E \times S$ is a (labelled) transition relation defined thus: $(\mathbf{B}, (i, j), \mathbf{B}') \in \rightarrow$, written
 164 $\mathbf{B} \xrightarrow{(i,j)} \mathbf{B}'$, iff for every $k \in A$,*

$$165 \quad \mathbf{B}'[k] = \begin{cases} \mathbf{B}[j] + (\mathbf{B}[i] - \mathbf{B}[j])I_{ij} & \text{if } k = j \\ \mathbf{B}[k] & \text{otherwise} \end{cases} \quad (1)$$

166 If $\mathbf{B} \xrightarrow{e} \mathbf{B}'$ we say that \mathbf{B} evolves into \mathbf{B}' by performing (choosing or executing) the action e .

167 A labeled transition $\mathbf{B} \xrightarrow{(i,j)} \mathbf{B}'$ represents the opinion evolution from \mathbf{B} to \mathbf{B}' when
 168 choosing an action represented by the edge (i, j) . As a result of this action, agent j updates
 169 their opinion as $\mathbf{B}[j] + (\mathbf{B}[i] - \mathbf{B}[j])I_{ij}$, thereby moving closer to the opinion of agent i .
 170 Alternatively, think of agent i as pulling the opinion of agent j towards $\mathbf{B}[i]$. The higher the
 171 influence of i over j , I_{ij} , the closer it gets. Intuitively, if $I_{ij} < 1$, it means that agent j is
 172 *receptive* to agent i but offers certain *resistance* to fully adopting their opinion. If $I_{ij} = 1$,
 173 agent j may be viewed as a *puppet* of i who disregards (or forgets) their own opinion to
 174 adopt that of i .

175 ► **Remark 3.** In Def. 1, we do not allow edges of the form (j, j) . In fact, allowing them
 176 would *not* present us with any additional technical issues, and the results in this paper

would still hold. The reason for this design choice, however, has to do with clarity about the intended intuitive meaning of a transition. Suppose that $\mathbf{B} \xrightarrow{(i,j)} \mathbf{B}'$. Since $\mathbf{B}'[j] = \mathbf{B}[j] + (\mathbf{B}[i] - \mathbf{B}[j])I_{ij} = \mathbf{B}[j](1 - I_{ij}) + \mathbf{B}[i]I_{ij}$, agent j gives a weight of I_{ij} to the opinion of i and of $(1 - I_{ij})$ to *their own opinion*. Therefore, the weight that j gives to their opinion may change depending on the agent i . Thus, allowing also a fixed weight I_{jj} of agent j to their own opinion may seem somewhat confusing to some readers. Furthermore, for any $\mathbf{B} \in S$ we would have $\mathbf{B} \xrightarrow{(j,j)} \mathbf{B}$ regardless of the value I_{jj} thus making the actual value irrelevant. Notice also we do not require the sum of the influences over an agent to be 1.

2.2 Runs and Consensus

We are interested in properties of opinion systems, such as convergence to consensus and fairness, which are inherent properties of infinite runs of these systems.

► **Definition 4** (e-path, runs and words). *An execution path (e-path) of an OTS $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$, where $G = (A, E, I)$, is an infinite sequence $\pi = \mathbf{B}_0 e_0 \mathbf{B}_1 e_1 \dots$ (also written $\mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$) such that $\mathbf{B}_t \xrightarrow{e_t} \mathbf{B}_{t+1}$ for each $t \in \mathbb{N}$. We say that e_t is the action performed at time t and that \mathbf{B}_t is the state of opinion at time t . Furthermore, if $\mathbf{B}_0 = \mathbf{B}_{\text{init}}$ then the e-path π is said to be a run of M .*

An ω -word of M is an infinite sequence of edges (i.e., an element of E^ω). The sequence $w_\pi = e_0 e_1 \dots$ is the ω -word generated by π . Conversely, given an ω -word $w = e'_0 e'_1 \dots$ the (unique) run that corresponds to it is $\pi_w = \mathbf{B}_{\text{init}} \xrightarrow{e'_0} \mathbf{B}_1 \xrightarrow{e'_1} \dots$.

► **Remark 5.** The uniqueness of the run that corresponds to a given ω -word is derived from the fact that an OTS is a deterministic transition system¹. This gives us a one-to-one correspondence between ω -words and runs, which allows us to abstract away from opinion states when they are irrelevant or clear from the context. In fact, throughout the paper, we will use the terms ω -words and runs of an OTS interchangeably when no confusion arises. It is also worth noting that in OTS, any action (edge) can be chosen at any point in an execution path; that is, *all actions are enabled*.

Consensus is a property of central interest in social learning models [21]. Indeed, failure to reach a consensus is often an indicator of polarization in a community.

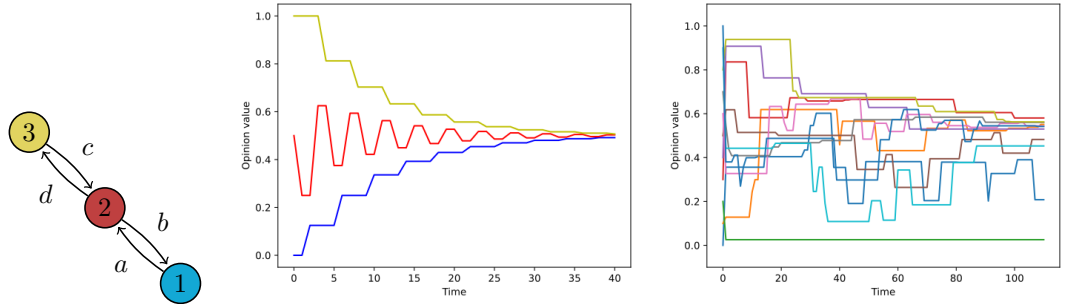
► **Definition 6** (Consensus). *Let $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$ be an OTS with $G = (A, E, I)$ and $\pi = \mathbf{B}_{\text{init}} \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be a run. We say that an agent $i \in A$ converges to an opinion value $v \in [0, 1]$ in π if $\lim_{t \rightarrow \infty} \mathbf{B}_t[i] = v$. The run π converges to consensus if all the agents in A converge to the same opinion value in π .*

Furthermore, \mathbf{B} is said to be a consensual state if it is a constant vector; i.e., if there exists $v \in [0, 1]$ such that for every $i \in A$, $\mathbf{B}[i] = v$.

► **Example 7.** Let $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$ where G is the influence graph in Fig. 1a and $\mathbf{B}_{\text{init}} = (0, 0.5, 1)$. If we perform a on \mathbf{B}_{init} we obtain $\mathbf{B}_{\text{init}} \xrightarrow{a} \mathbf{B}_1 = (0.0, 0.25, 1.0)$.

Consider the word $w = (abcd)^\omega$. Then $\pi_w = \mathbf{B}_{\text{init}} \xrightarrow{a} (0.0, 0.25, 1.0) \xrightarrow{b} (0.125, 0.25, 1.0) \xrightarrow{c} (0.125, 0.625, 1.0) \xrightarrow{d} (0.125, 0.625, 0.8125) \xrightarrow{a} \dots$. Fig. 1b suggests that π_w indeed converges to consensus (to opinion value 0.5). A more complex example of the evolution of opinions from a randomly generated graph with eleven agents is illustrated in Fig. 1c.

¹ While the actions in a run can be seen as being chosen non-deterministically by a scheduler, an OTS is a *deterministic* transition system in the sense that given a state \mathbf{B} and an action e , there exists a unique state \mathbf{B}' such that $\mathbf{B} \xrightarrow{e} \mathbf{B}'$.



(a) OTS with influence graph with agents $A = \{1, 2, 3\}$, edges $E = \{a, b, c, d\}$, influence $I_e = 1/2$ for all $e \in E$ and $\mathbf{B}_{\text{init}} = (0, 0.5, 1)$. (b) Opinion evolution for the run that corresponds to $(abcd)^\omega$ of the OTS in Fig. 1a. Each plot corresponds to the opinion evolution of the agent with the same color. (c) Opinion evolution of a run of an OTS with a $G = (A, E, I)$, $A = \{1, \dots, 11\}$, $I_e = 0.5$ for each $e \in E$, $\mathbf{B}_{\text{init}} = (0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0)$. Each edge of G was generated with prob. 0.3. The edges of the (partial) run were uniformly chosen from E .

■ **Figure 1** Run examples for OTS in Fig. 1a and randomly-generated OTS in Fig. 1c

217 The examples above illustrate runs that may or may not converge to consensus. In the
 218 next section, we identify conditions on the influence and topology of graphs and on the runs
 219 that guarantee this central property of opinion models.

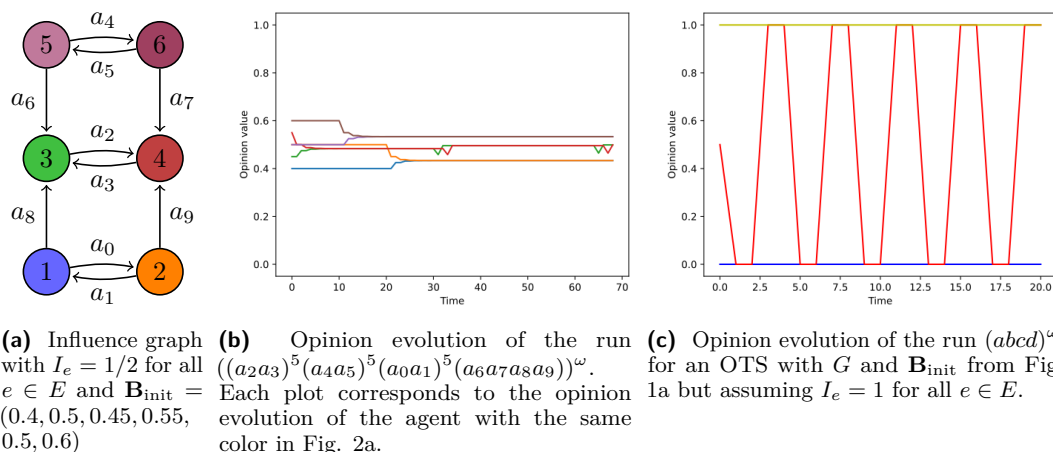
220 3 Strong Connectivity, Puppet-Freedom and Fairness

221 In this section, we discuss graph properties, as well as fairness notions and criteria from the
 222 literature on concurrent systems that give us insight into how agents converge to consensus in
 223 an OTS. For simplicity, we assume an underlying OTS $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$ with an influence
 224 graph $G = (A, E, I)$. We presuppose basic knowledge of graph theory and formal languages.

225 3.1 Strong Connectivity

226 As in the DeGroot model, if there are (groups of) agents in G that do not influence each other
 227 (directly or indirectly) and their initial opinions are different, these groups may converge
 228 to different opinion values. Consider the example in Fig. 2 where the groups of agents
 229 $G_1 = \{1, 2\}$ and $G_2 = \{5, 6\}$ do not have external influence (directly or indirectly), but
 230 influence the group $G_3 = \{3, 4\}$. Each group is strongly connected within; their members
 231 influence each other. The agents in G_1 converge to an opinion, and so do the agents in G_2 ,
 232 but to a different one. Hence, the agents in both groups cannot converge to consensus. The
 233 agents in G_3 do not even converge to an opinion because they are regularly influenced by
 234 the dissenting opinions of G_1 and G_2 .

235 The above can be prevented by requiring *strong connectivity*, i.e., there must be a path
 236 in G from any other to any other. Recall that a *graph path* from i to j of length m in G
 237 is a sequence of edges of E of the form $(i, i_1)(i_1, i_2) \dots (i_{m-1}, j)$, where the agents in the
 238 sequence are distinct. We shall refer to graph paths as *g-paths* to distinguish them from
 239 e-paths in Def. 4. We say that agent i influences agent j if there is a g-path from i to j in G .
 240 The graph G is *strongly connected* iff there is a g-path from any agent to any other in G .
 241 Hence, in strongly-connected graphs, all agents influence one another.



■ **Figure 2** Run examples for OTS in Fig. 1a and Fig. 2a

3.2 Puppet-Freedom

242

243 Nevertheless, too much influence may prevent consensus. If $\mathbf{B} \xrightarrow{(i,j)} \mathbf{B}'$ and $I_{ij} = 1$, agent
 244 j behaves as a *puppet* of i forgetting their own opinion and adopting that of j . Fig. 2c
 245 illustrates this for the strongly-connected graph in Fig. 1a but with $I_{ij} = 1$ for each $(i, j) \in E$:
 246 Agents 1 and 3 use Agent 2 as a puppet, constantly swaying his opinion between 0 and 1.
 247 We therefore say that the influence graph G is *puppet free* if for each $(i, j) \in E$, $I_{ij} < 1$.

3.3 Strong Fairness

248

249 In an OTS, if G is strongly connected but a given edge is never chosen in a run (or not
 250 chosen sufficiently often), it may amount to not having all agents influence each other in
 251 that run, hence preventing consensus. For this reason, we make some fairness assumptions
 252 about the runs.

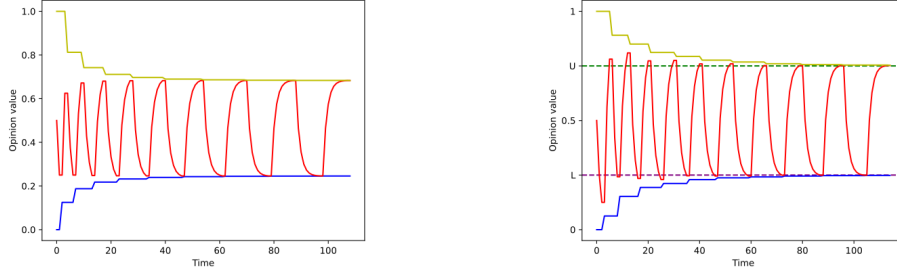
253 In the realm of transition systems, fairness assumptions rule out some runs, typically
 254 those where some actions are not chosen sufficiently often when they are enabled sufficiently
 255 often. There are many notions of fairness (see [5, 19, 25] for surveys), but strong fairness
 256 is perhaps one of the most representative. As noted above, every action $e \in E$ is always
 257 enabled in every run of an OTS. Thus, in our context, strong fairness of a given OTS run
 258 (ω -word) amounts to requiring that every action e occurs infinitely often in the run.

259 ► **Definition 8** (Strong fairness). *Let w be an ω -word of an OTS. We say that w is strongly
 260 fair if every $e \in E$ occurs in every suffix of w .*

261 Notice that the graph from Ex. 7 is strongly connected and puppet free, and the ω -word
 262 $w = (abcd)^\omega$ is indeed strongly fair and converges to consensus. Nevertheless, puppet freedom,
 263 strong fairness, and strong connectivity are not sufficient to guarantee consensus.

264 ► **Proposition 9.** *There exists $(G, \mathbf{B}_{\text{init}}, \rightarrow)$, where G is strongly connected and puppet free,
 265 with a strongly-fair run that does not converge to consensus.*

266 The proof of the existence statement in Prop. 9 is given next.



(a) Opinion evolution of the OTS from Fig. 1a for the ω -word $u = (a^n bc^n d)_{n \in \mathbb{N}^+}$ (b) Opinion evolution of the OTS from Fig. 1a for $U = 0.75$, $L = 0.25$ and the ω -word w from Cons. 10.

■ **Figure 3** Run examples for OTS in Fig. 1a.

267 ► **Construction 10** (Counter-Example to Consensus). Let $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$ be an OTS where
 268 G is the strongly-connected puppet-free influence graph in Fig. 1a and \mathbf{B}_{init} is any state of
 269 opinion such that $\mathbf{B}_{\text{init}}[1] < \mathbf{B}_{\text{init}}[2] < \mathbf{B}_{\text{init}}[3]$. We have $A = \{1, 2, 3\}$ and $E = \{a, b, c, d\}$.
 270 We construct an ω -word w such that π_w does not converge to consensus with the following
 271 infinite iterative process. Let U and L be such that $\mathbf{B}_{\text{init}}[1] < L < \mathbf{B}_{\text{init}}[2] < U < \mathbf{B}_{\text{init}}[3]$.

272 Process: (1) Perform a non-empty sequence of a 's with as many a 's as needed until
 273 the opinion of Agent 2 becomes smaller than L . (2) Perform the action b . (3) Perform a
 274 non-empty sequence of c 's with as many c 's as needed until the opinion of Agent 2 becomes
 275 greater than U . (4) Perform the action d . The result of this iteration is a sequence of the
 276 form a^+bc^+d . Repeat steps 1–4 indefinitely.

277 The above process produces the ω -sequence $w = w_1 \cdot w_2 \cdot \dots$ of the form $(a^+bc^+d)^\omega$, where
 278 each $w_i = a^{n_i}bc^{m_i}d$ is the result of the i -th iteration of the process and $n_i > 0$ and $m_i > 0$
 279 are the number of a 's and c 's in such interaction. (The evolution of the opinion of run π_w ,
 280 with $U = 0.75$, $L = 0.25$ and $\mathbf{B}_{\text{init}} = (0, 0.5, 1)$ is illustrated in Fig. 3b)

281 Since each action $e \in E$ appears infinitely often in w , w is strongly fair. Furthermore,
 282 right after each execution of Step 2, the opinion of Agent 1 gets closer to L , but it is still
 283 smaller than L since the opinion of Agent 2 at that point is smaller than L . For symmetric
 284 reasons, the opinion of Agent 3 gets closer to U , but it is still greater than U . Consequently,
 285 the opinion of Agent 1 is always below L , while the opinion of Agent 3 is always above U
 286 with $L < U$. Therefore, they cannot converge to the same opinion.

287 Another ω -word for the OTS in Fig. 1a exhibiting a behavior similar to w in Cons. 10,
 288 but whose proof of non-convergence to consensus seems more involved, is $u = (a^n bc^n d)_{n \in \mathbb{N}^+} =$
 289 $u_1 \cdot u_2 \cdot \dots$, where each $u_n = a^n bc^n d$. (see Fig. 3a). The delay in both w and u to execute d
 290 after b grows unboundedly due to the growing number of c 's. More precisely, let $\#e(v)$ be
 291 the number of occurrences of $e \in E$ in a finite sequence v .

292 ► **Proposition 11.** Let $w = w_1 \cdot w_2 \cdot \dots$ be the ω -word from Cons. 10 where each w_m has the
 293 form a^+bc^+d . Then for every $m \in \mathbb{N}$, there exists $t \in \mathbb{N}$ such that $\#c(w_{m+t}) > \#c(w_m)$.

294 The above proposition states that the number of consecutive c 's in w grows unboundedly,
 295 and hence so does the delay for executing d right after executing b . To prevent this form of
 296 unbounded delay, we recall in the next section some notions of fairness from the literature
 297 that require, at each position of an ω -word, every action to occur within some bounded
 298 period of time.

3.4 Bounded Fairness

We start by introducing some notation to give a uniform presentation of some notions of fairness from the literature. We assume $|E| > 1$; otherwise, all the fairness notions are trivial.

A word w is a possibly infinite sequence over E . A *subword* of w is either a suffix of w or a prefix of some suffix of w . Let κ be an ordinal from the set $\omega + 1 = \mathbb{N} \cup \{\omega\}$ where ω denotes the first infinite ordinal. A κ -word is a word of length κ . Recall that each ordinal can be represented as the set of all strictly smaller ordinals. We can then view a κ -word $w = (e_i)_{i \in \kappa}$ as a function $w : \kappa \rightarrow E$ such that $w(i) = e_i$ for each $i \in \kappa$. A κ -word w is *complete* if $w(\kappa) = E$ (where $w(\kappa)$ denotes the image of the function w). A κ -window u of w is a *subword* of w of length κ . Thus, if $\kappa = \omega$ then u is a suffix of w , and if $\kappa \in \mathbb{N}$, u can be thought of as a *finite observation* of κ consecutive edges in w . We can now introduce a general notion of fairness parametric in κ .

► **Definition 12** (κ -fairness, bounded-fairness). *Let w be an ω -word over E and $\kappa \in \omega + 1$: w is κ -fair if every κ -window of w is complete. Furthermore, w is bounded fair if it is k -fair for some $k \in \mathbb{N}$.*

Notice that the notion of strong fairness in Def. 8 is obtained by taking $\kappa = \omega$; indeed, w is ω -fair iff every $e \in E$ occurs infinitely often in w . Furthermore, if $\kappa = k$ for some $k \in \mathbb{N}^+$, then we obtain the notion of k -fairness from [16]². Intuitively, if w is k -fair, then at any position of w , every $e \in E$ will occur within a window of length k from that position.

It is not difficult to see that ω -fairness is strictly weaker than bounded-fairness, which in turn is strictly weaker than any k -fairness with $k \in \mathbb{N}$. Let $F(\kappa)$ be the set of all ω -words over E that are κ -fair. We have the following sequence of strict inclusions.

► **Proposition 13.** *For every $k \in \mathbb{N}$, $F(k) \subset F(k + 1) \subset (\bigcup_{\kappa \in \mathbb{N}} F(\kappa)) \subset F(\omega)$.*

► **Example 14.** Let us consider the *fair word* w from Cons. 10, the counter-example to consensus. From Prop. 11, the delay for executing action d immediately after executing action b increases without bound. Thus, for every k , there must be a non-complete k -window u of w such that d does not occur in u . Consequently, w is not bounded fair.

Not only does bounded fairness rule out the counter-example in Cons. 10, but it also guarantees consensus, as shown later, for runs of OTS with strongly-connected, puppet-free influence graphs. Nevertheless, it may be too strong of a requirement for consensus. We, therefore, introduce a weaker notion that satisfies the following criteria and guarantees consensus.

Some Fairness Criteria

Let us briefly discuss some fairness criteria and desirable properties that justify our quest for a weaker notion of fairness that guarantees consensus. An in-depth discussion about criteria for fairness notions, from which we drew some inspiration, can be found in [34, 33, 19, 5].

Machine Closure. Following [1, 26] one of the most important criteria that a notion of fairness must meet is *machine closure* (also called *feasibility* [5]). Fairness properties are properties of infinite runs; hence, a natural requirement is that any finite partial run must have the chance to be extended to a fair run. Thus, we say that a notion of fairness is *machine closed* if every finite word u can be extended to a fair ω -word $u \cdot w$.

² This notion is different from the notion of k -fairness from [9]

340 Clearly, k -fairness with $k \in \mathbb{N}$ is not machine closed; e.g., the word $c^k d$ with $E = \{c, d\}$
 341 cannot be extended to a k -fair ω -word. Nevertheless, bounded fairness is machine closed: Each
 342 k -word u can be extended to a $(k + m)$ -fair word $u \cdot (e_1 \dots e_m)^\omega$ assuming $E = \{e_1, \dots, e_m\}$.

343 **Constructive Liveness.** According to [34], a notion of fairness may require that a
 344 particular action is taken sufficiently often, but it should not prevent any other actions from
 345 being taken sufficiently often. This concept is formalized in [34, 33] in a game-theoretical
 346 scenario, reminiscent of a Banach-Mazur game [28], involving an infinite interaction between
 347 a scheduler and an opponent. The opponent initiates with a word w_0 , then the scheduler
 348 appends a finite word w_1 to w_0 . This pattern continues indefinitely, resulting in an ω -word
 349 $w = w_0 \cdot w_1 \cdot w_2 \dots$. A given fairness notion is said to be a *constructive liveness* property if,
 350 regardless of what the opponent does, the scheduler can guarantee that the resulting ω -word
 351 is fair under the given notion.

352 The notion of *bounded fairness is not a constructive liveness property*. If an ω -word
 353 is bounded fair, it is k -fair for some $k \geq |E| > 1$. Let $c \in E$ and take as the strategy of
 354 the opponent to choose in each of their turns $w_n = c^n$. Since $|E| > 1$, then w_{2k} cannot
 355 be a complete k -window. Therefore, the resulting $w = w_0 \cdot w_1 \cdot w_2 \dots$ is not bounded fair,
 356 regardless of the strategy of the scheduler.

357 It is worth noticing that the above opponent's strategy is reminiscent of our procedure to
 358 construct an ω -sequence in Cons. 10 using the unbounded growth of c 's to prevent consensus.

359 **Random Words.** Consider a word $e_0 e_1 \dots$ where each edge or action $e_n = (i, j)$ is chosen
 360 from E *independently* with probability $p_{(i,j)} > 0$. Let us refer to such kinds of sequences
 361 as *random* words. We then say that a given notion of fairness is *random inclusive* if every
 362 random ω -word is *almost surely* (i.e., with probability one) fair under the given notion.

363 It follows from the Second Borel–Cantelli lemma³ that every random word is *almost*
 364 *surely* strongly fair. Nevertheless, the notion of bounded fairness fails to be random inclusive:
 365 If a word is bounded fair, it is k -fair for some $k \geq |E|$, and thus it needs to have the form
 366 $w_0 \cdot w_1 \dots$ where each w_m is a complete k -window. Since $1 < |E|$, the probability that a
 367 random k window is complete is strictly smaller than 1. Therefore, the probability of a
 368 random word having an infinite number of *consecutive* complete k -windows is 0.

369 Random words are important in simulations of our model (see Fig. 1c). Furthermore,
 370 having a notion of fairness that is random inclusive and guarantees consensus will allow us
 371 to derive and generalize consensus results for randomized opinion models, such as gossip
 372 algorithms [17]. We elaborate on this in the related work. We now introduce our new notion
 373 of fairness.

374 4 A New Notion of Bounded Fairness

375 A natural way to relax bounded fairness to satisfy constructive liveness and random inclusion
 376 is to require that the complete k -windows need only appear infinitely often: i.e., an ω word
 377 w is said to be *weakly bounded* fair if there exists $k \in \mathbb{N}$ such that every suffix of w has a
 378 k -window. Nevertheless, as it will be derived later, weak bounded fairness is not sufficient to
 379 guarantee consensus.

380 It turns out that, to guarantee consensus, it suffices to require that a large enough number

³ The lemma states that if the sum of the probabilities of an infinite sequence of events $E_0 E_1 \dots$ that are independent is infinite, then the probability of infinitely many of those events occurring is 1 [31]. Here, each event E_k expresses that the edge e_k occurs at time k and these events are independent because each edge (i, j) in a random word is chosen independently with probability $p_{(i,j)} > 0$.

381 m of consecutive complete k -windows appear infinitely often. These consecutive windows are
 382 referred to as multi-windows.

383 ► **Definition 15** ((m, κ) multi-window). Let w be an ω -word over E , $m \in \mathbb{N}^+$ and $\kappa \in \omega + 1$.
 384 We say that w has an (m, κ) multi-window if there exists a subword u of w of the form
 385 $u = w_1 \cdot w_2 \cdot \dots \cdot w_m$ where each w_i is a κ -window of w . Furthermore, if each w_n in u is
 386 complete, we say that w has a complete (m, κ) multi-window. If it exists, the word u is called
 387 an (m, κ) multi-window of w .

388 Notice that because of the concatenation of windows in Def. 15, by construction, no
 389 ω -word has a (m, ω) multi-window with $m > 1$: If $\kappa = \omega$ then $m = 1$. In this case, the
 390 multi-window is just a window of infinite length of w , i.e., a suffix of w .

391 ► **Definition 16** ((m, κ) -fairness). Let w be an ω -word over E , $m \in \mathbb{N}^+$ and $\kappa \in \omega + 1$. We
 392 say that w is (m, κ) -fair if every suffix of w has a complete (m, κ) multi-window. We say
 393 that w is m -consecutive bounded fair, or m -bounded fair, if it is (m, k) -fair for some $k \in \mathbb{N}$.

394 Clearly, w is ω -fair iff it is $(1, \omega)$ -fair, and w is weakly bounded fair iff it is 1-bounded ω -
 395 fair. Let $F(m, \kappa)$ and $F(\kappa)$ be the sets of ω -words that are (m, κ) -fair and κ -fair, respectively.
 396 We have the following sequence of strict inclusions (assume $k, m \in \mathbb{N}^+$):

397 ► **Proposition 17.** $F(k) \subset F(m + 1, k) \subset F(m, k) \subset (\bigcup_{\kappa \in \mathbb{N}} F(m, \kappa)) \subset F(1, \omega) = F(\omega)$.

398 **Compliance with Fairness Criteria.** Let us consider the criteria for fairness in the
 399 previous section. The notion of m -bounded fairness is machine closed since bounded fairness
 400 is stronger than m -bounded fairness (Prop. 13 and Prop. 17) and bounded fairness is
 401 machine closed.

402 It is also a constructive liveness property since (m, k) fairness, for $k \geq |E|$, is stronger
 403 than m -bounded fairness (Prop. 17), and it is also a constructive liveness property: A
 404 winning strategy for the scheduler is to choose a complete (m, k) -window at each one of its
 405 turns.

406 Similarly, m -Bounded Fairness is random inclusive since the stronger notion (m, k) -
 407 Fairness is random inclusive for $k \geq |E|$. In a random ω -word $w = w_0 \cdot w_1 \dots$ where each w_n
 408 is a $(m \times k)$ -window, the probability that w_n is a complete (m, k) -multi-window is non-zero
 409 and independent. Thus again, by the Second Borel–Cantelli lemma, almost-surely w has
 410 infinitely many complete (m, k) multi-windows, i.e., it is almost-surely (m, k) -fair.

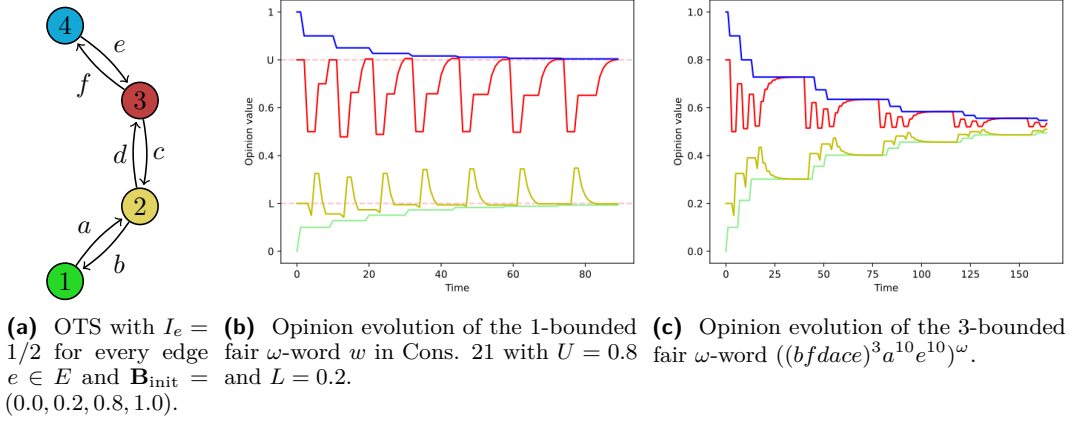
411 4.1 Consensus Theorem

412 We can now state one of our main theorems: m -bounded fairness guarantees consensus in
 413 strongly-connected, puppet-free graphs.

414 ► **Theorem 18** (Consensus under m -bounded fairness). Let $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$ be an OTS
 415 where G is a strongly-connected, puppet-free influence graph. For every run π of M , if w_π is
 416 m -bounded fair and $m \geq |A| - 1$, then π converges to consensus.

417 ► **Remark 19.** A noteworthy corollary of Th. 18 is that, under the same assumptions of
 418 the theorem, if w_π is a bounded fair (a random ω -word), then π converges to consensus (π
 419 almost surely converges to consensus). This follows from the above theorem, Prop. 13, Prop.
 420 17 and the fact that m -bounded fairness is random inclusive.

421 A proof of Th. 18 is given in the technical report [7]. Let us give the main intuitions here.
 422 The proof focuses on the evolution of maximum and minimum opinion values. The sequences



■ **Figure 4** Examples of an m -bounded fair runs. In Fig. 4b and 4c, each plot corresponds to the opinion of the agent with the same color in Fig. 4a.

of maximum and minimum opinion values in a run, $\{\max \mathbf{B}_t\}_{t \in \mathbb{N}}$ and $\{\min \mathbf{B}_t\}_{t \in \mathbb{N}}$, can be shown to be (bounded) monotonically non-increasing and non-decreasing, respectively, so they must converge to some opinion values, say U and L with $L \leq U$.

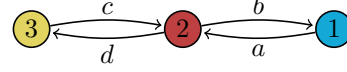
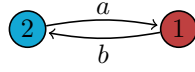
We must then argue that $L = U$ (this implies convergence to consensus of π by the Squeeze Theorem [32]). Since w_π is m -bounded fair with $m \geq |A| - 1$, after performing all the actions of an (m, k) multi-window of w_π , for some $k \geq |E|$, all the agents of A would have influenced each other. In particular, the agents holding the maximum and minimum opinion values, say agents i and j . To see this, notice that since G is strongly connected, there is a path from i to j , $a_1 \dots a_l$ with length $l \leq |A| - 1$. Thus, after performing the first complete k -window of the (m, k) -multi-window, a_1 must be performed, after performing the second complete k -window, a_2 must be performed, and so on. Hence, after performing all the actions of the multi-window, i would have influenced j . It can be shown that their mutual influence causes them to decrease their distance by a positive constant factor (here, the puppet freedom assumption is needed). Since the w_π is m -fair, there are infinitely many (m, k) -windows to be performed, and thus the sequences of maximum and minimum opinion values converge to each other, i.e., $U = L$. ◀

It is worth pointing out that without the condition $m \geq |A| - 1$ in Th. 18, we cannot guarantee consensus. Fig. 4c illustrates an m -bounded fair run, for $m = |A| - 1$, of an OTS with 4 agents that converges to consensus. Nevertheless, the following run construction shows that for $m = |A| - 3$, we can construct an m -bounded fair run that fails to converge to consensus (the run is illustrated in Fig. 4b). It also shows that weak bounded fairness, i.e., 1-bounded fairness, is not sufficient to guarantee convergence to consensus. We do not have a counter-example or a proof for $m = |A| - 2$.

► **Proposition 20.** *There exists $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$, where $G = (A, E, I)$ is a strongly connected, puppet-free graph, with an m -bounded fair ω -word w , $m = |A| - 3$, such that π_w does not converge to consensus.*

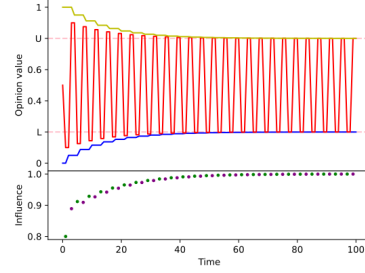
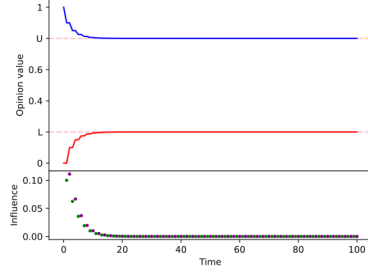
The proof of the above proposition is given in the following construction.

► **Construction 21** (Counter-Example to Consensus for m -bounded fairness with $m \leq |A| - 3$). *Suppose that $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$ where G is the strongly-connected, puppet-free, influence graph in Fig. 4a and \mathbf{B}_{init} is any state of opinion such that $\mathbf{B}_{\text{init}}[1] < \mathbf{B}_{\text{init}}[2] < \mathbf{B}_{\text{init}}[3] < \mathbf{B}_{\text{init}}[4]$.*



(a) $\mathbf{B}_{\text{init}} = (0.0, 1.0)$ and if $\mathbf{B}[1] = \mathbf{B}[2]$ then $I_a^{\mathbf{B}} = I_b^{\mathbf{B}} = 0.5$, otherwise $I_a^{\mathbf{B}} = \left[\frac{U - \mathbf{B}[2]}{2(\mathbf{B}[1] - \mathbf{B}[2])} \right]_0^1$, $I_b^{\mathbf{B}} = \left[\frac{L - \mathbf{B}[1]}{2(\mathbf{B}[2] - \mathbf{B}[1])} \right]_0^1$

(b) $\mathbf{B}_{\text{init}} = (0.0, 0.5, 1.0)$, $I_d^{\mathbf{B}} = I_b^{\mathbf{B}} = 0.5$, if $\mathbf{B}[1] = \mathbf{B}[2]$ then $I_a^{\mathbf{B}} = 0.5$, if $\mathbf{B}[2] = \mathbf{B}[3]$ then $I_c^{\mathbf{B}} = 0.5$, otherwise $I_a^{\mathbf{B}} = \left[\frac{\frac{1}{2}(\mathbf{B}[1] + L) - \mathbf{B}[2]}{\mathbf{B}[1] - \mathbf{B}[2]} \right]_0^1$, $I_c^{\mathbf{B}} = \left[\frac{\frac{1}{2}(\mathbf{B}[3] + U) - \mathbf{B}[2]}{\mathbf{B}[3] - \mathbf{B}[2]} \right]_0^1$



(c) Opinion and influence evolution of the ω -word $(ab)^\omega$. Each plot corresponds to the opinion of the agent with the same color in Fig. 5a. The influences $I_a^{\mathbf{B}}$ and $I_b^{\mathbf{B}}$ are plotted in green and purple.

(d) Opinion and influence evolution of the ω -word $(abcd)^\omega$. Each plot corresponds to the opinion of the agent with the same color in Fig. 5b. The influences $I_a^{\mathbf{B}}$ and $I_c^{\mathbf{B}}$ are plotted in green and purple.

■ **Figure 5** Plots for DOTS in Fig. 5a and Fig. 5b with $U = 0.8$ and $L = 0.2$.⁴

453 We have $A = \{1, 2, 3, 4\}$ and $E = \{a, b, c, d, e, f\}$. We construct an ω -word w such that π_w
 454 does not converge to consensus with the following infinite iterative process. Let U and L be
 455 such that $\mathbf{B}_{\text{init}}[2] \leq L < U \leq \mathbf{B}_{\text{init}}[3]$.

456 Process: (1) Perform the sequence of actions $bfdace$. (2) Perform a sequence of a actions
 457 with as many a 's as needed until the opinion of Agent 2 becomes smaller than L . (3) Perform
 458 a sequence of e 's with as many e 's as needed until the opinion of Agent 3 becomes greater
 459 than U . The result of this iteration is a sequence of the form $bfdace \cdot a^*e^*$. Repeat steps 1-3
 460 indefinitely.

461 The above process produces the ω -sequence $w = v \cdot w_1 \cdot v \cdot w_2 \cdot \dots$ of the form $(bfdace a^*e^*)^\omega$
 462 where $v = bfdace$ and $w_i = a^{m_i}e^{n_i}$ are results of the i -th iteration of the process, and $n_i \geq 0$
 463 and $m_i \geq 0$ are the number of a 's and e 's in each w_i . (The opinion evolution of run π_w ,
 464 with $L = 0.2$, $U = 0.8$ and $\mathbf{B}_{\text{init}} = (0.0, 0.2, 0.8, 1.0)$ is illustrated Fig. 4b)

465 Since the subword v is a complete $(1,6)$ -multi-window and appears infinitely often in w ,
 466 w is m -bounded fair for $m = |A| - 3 = 1$. Furthermore, right after each execution of edge
 467 f in step 1, the opinion of Agent 1 gets closer to L , but it is still smaller than L since the
 468 opinion of Agent 2 at that point is smaller than L . For symmetric reasons, after action b , the
 469 opinion of Agent 4 gets closer to U , but it is still greater than U since the opinion of Agent
 470 3 at that point is greater than U . Consequently, the opinion of Agent 1 is always below L ,
 471 while the opinion of Agent 4 is always above U with $L < U$. Therefore, they cannot converge
 472 to the same opinion.

5 Dynamic Influence

The static weights of the influence graph of an OTS imply that the influence that each individual has on others remains constant throughout opinion evolution. However, in real-life scenarios, the influence of individuals can vary depending on many factors, in particular the state of opinion (or opinion climate). Indeed, individuals may gain or lose influence based on the current opinion trend or for expressing dissenting and extreme opinions, among others.

To account for the above form of dynamic influence, we extend the weight function $I : E \rightarrow (0, 1]$ of the influence graph $G = (A, E, I)$ as a function $I : E \times [0, 1]^{|A|} \rightarrow [0, 1]$ on edges and the state of opinion. The resulting graph is said to have *dynamic influence*.

► **Definition 22** (Dynamic OTS). *A Dynamic OTS (DOTS) is a tuple $(G, \mathbf{B}_{\text{init}}, \rightarrow)$ where $G = (A, E, I)$ has dynamic influence $I : E \times [0, 1]^{|A|} \rightarrow [0, 1]$. We write $I_{ij}^{\mathbf{B}}$ for $I((i, j), \mathbf{B})$. The labeled transition \rightarrow is defined as in Def. 2 but replacing I_{ij} with $I_{ij}^{\mathbf{B}}$ in Eq. 1.*

The notions of runs, words, e-paths, and related notions for DOTS remain the same as those for OTS (Def. 4). Let us consider some examples of dynamic influence.

Confirmation Bias. Under *confirmation bias* [8], an agent j is more influenced by those whose opinion is closer to theirs. The function $I_{ij}^{\mathbf{B}} = 1 - |\mathbf{B}[j] - \mathbf{B}[i]|$ captures a form of confirmation bias; the closer the opinions of i and j , the stronger the influence of i over j .

Bounded Influence. Nevertheless, if we allow dynamic influence that can converge to 0 in a given run $\mathbf{B}_{\text{init}} \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$, i.e., if $\lim_{t \rightarrow \infty} I_{i,j}^{\mathbf{B}^t} = 0$, we may reduce indefinitely influence and end up in a situation similar to non-strong connectivity of the graph, thus preventing consensus as in Section 3.1 (Fig. 2). Analogously, if $\lim_{t \rightarrow \infty} I_{i,j}^{\mathbf{B}^t} = 1$, we may end up in puppet situations preventing consensus like in Section 3.2 (Fig. 2c). Both situations are illustrated in the DOTS in Fig. 5. To prevent them, we bound the dynamic influences.

► **Definition 23** (Bounded Influence). *A DOTS $(G, \mathbf{B}_{\text{init}}, \rightarrow)$ with $G = (A, E, I)$ has bounded influence if there are constants $I_L, I_U \in (0, 1)$ such that for each $\mathbf{B} \in [0, 1]^{|A|}$, $(i, j) \in E$, we have $I_{i,j}^{\mathbf{B}} \in [I_L, I_U]$.*

The previous form of confirmation bias influence $I_{ij}^{\mathbf{B}} = 1 - |\mathbf{B}[j] - \mathbf{B}[i]|$ is not bounded. Nevertheless, the linear transformation $I_L + (I_U - I_L)I_{ij}^{\mathbf{B}}$ can be used to scale any unbounded influence $I_{ij}^{\mathbf{B}}$ into a bounded one in $[I_L, I_U]$ while preserving its shape.

We conclude with our other main theorem, whose proof is given in the technical report [7].

► **Theorem 24** (Consensus with bounded influence). *Let $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$ be a DOTS where G is a strongly-connected, influence graph. Suppose that M has bounded influence. For every run π of M , if w_π is m -bounded fair with $m \geq |A| - 1$, then π converges to consensus.*

The result generalizes Th. 18 to dynamic bounded influence. Therefore, in strongly-connected and dynamic bounded influence graphs, convergence to consensus is guaranteed for all runs that are m -bounded fair, which include each random run almost surely.

6 Conclusions and Related Work

We introduced a DeGroot-based model with asynchronous opinion updates and dynamic influence using labelled transition systems. The model captures opinion dynamics in social

⁴ We use a clamp function for $[0, 1]$ defined as $[r]_0^1 = \min(\max(r, 0), 1)$ for every $r \in \mathbb{R}$.

513 networks more faithfully than the original DeGroot model. The fairness notions studied and
514 the consensus results in this paper show that the model is also tractable and brings new
515 insights into opinion formation in social networks. To our knowledge, this is the first work
516 that uses fairness notions from concurrent systems in the context of DeGroot-based models.

517 There is a great deal of work on DeGroot-based models for social learning (e.g., [4, 13, 12,
518 38, 37, 15, 11]). We discuss work with asynchronous updates and dynamic influence, which
519 is the focus of this paper. The work [15] introduces a version of the DeGroot model in which
520 self-influence changes over time, while the influence on others remains the same. The works
521 [11, 12] explore convergence and stability, respectively, in models where influences change over
522 time. The works mentioned above do not take into account asynchronous communication,
523 whereas this paper demonstrates how asynchronous communication, when combined with
524 dynamic influence, can prevent consensus.

525 Recent works on gossip algorithms [17, 30, 2, 35] study consensus with asynchronous
526 communications for distributed averaging and opinion dynamics. The work in [30] studies
527 *reaching* consensus (in finite time) rather than *converging* to consensus. The works [2, 35]
528 consider undirected cliques rather than directed graphs as influence graphs. The closest
529 work is [17], which states consensus for random runs in directed strongly connected graphs
530 but unlike our case all edges have the same fixed weight $q \in (0, 1)$ (i.e., they assume static
531 influence with the same influence value for all edges). The dynamics of asymmetric gossip
532 updates in [17] can indeed be captured as OTS, and their random runs are almost-surely
533 m -bounded fair. Consequently, our work generalizes the consensus result in [17] by extending
534 it to graphs with (bounded) dynamic influence and whose edges may have different weights.
535 Furthermore, the framework in [17] does not address fairness notions which are the focus
536 and the main novelty of our work.

537 The work [19] discusses probabilistic fairness as a method equally strong as strong fairness
538 to prove liveness properties, where a liveness property is characterized by a set of states such
539 that a run holds this property iff the run reaches a state of this set. However, the property
540 of (*convergence to*) *consensus* (Def. 6) does not correspond to this notion of liveness since
541 it is not about *reaching* a specific set of states but about *converging* to a consensual state.
542 In fact, unless there are puppets or the initial state of a run is already a consensual state,
543 consensus is never reached in finite time in our model.

544 Bounded fair ω -words can be characterized by Prompt Buchi Automata (PBW) [3].
545 Indeed, the set of bounded-fair words of an OTS can be characterized as the language
546 of PBW. Hence, the closure properties of these automata may prove valuable for future
547 developments of our work. It would also be interesting to see in future work whether or not
548 the m -bounded fair words of an OTS can be characterized as the language of a PBW (or of
549 an elegant variant of it).

550 In future work, we plan to study the actual value of consensus in a given system. This may
551 provide information about the most influential agents. We also plan to study how actions
552 can be scheduled (or manipulated), while preserving the fairness assumptions, to converge
553 more quickly or slowly to a consensus, or to a given consensus value. For example, giving
554 priority to edges whose agents have a greater opinion disagreement, while respecting fairness
555 assumptions. We may build on previous work on priorities in concurrent communications [6]
556 for this purpose.

557 Finally, we plan to extend our model with agents that can learn by exchanging beliefs,
558 lies, and information, by building upon our work in concurrent constraint programming (e.g.
559 [24, 23, 18]).

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659 **A** Proof of consensus

660 We prove the consensus of Th. 18 in three steps. (1) We show that the maximum and
661 minimum opinion values converge, respectively, to some values $U, L \in [0, 1]$, $L \leq U$. (2)

662 We establish a conditional proof of $U = L$ by deriving bounds ensuring that the maximum
 663 opinion decreases under the influence of agents with the minimum opinion. The conditions are
 664 strong connectivity, puppet freedom, and a concept we introduce called *recurrent Δ -bound*.
 665 Finally, (3) we show that m -bounded fair runs (on strongly-connected puppet-free graphs)
 666 fulfil these conditions for $m \geq |A| - 1$ and therefore ensure consensus.

667 The complete proofs of the lemmas used next are found in Appendix B. For simplicity,
 668 assume an underlying OTS $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$ with an strongly-connected puppet-free
 669 influence graph $G = (A, E, I)$.

670 **Step 1.** We show that the opinion values in a state is bounded by the extreme opinions
 671 in the previous state.

672 **► Lemma 25** (Opinion evolution is bounded by the extremes). *Let $\mathbf{B} \xrightarrow{e} \mathbf{B}'$ be a transition.*
 673 *Then $\min(\mathbf{B}) \leq \mathbf{B}'[k] \leq \max(\mathbf{B})$ for all $k \in A$.*

674 Notice that monotonicity does not necessarily hold for the opinion of agents (e.g., see Fig.
 675 1c). Nevertheless, it follows from lemma 25 that $\max(\mathbf{B}_t)$ is monotonically non-increasing
 676 and $\min(\mathbf{B}_t)$ is monotonically non-decreasing with respect to t .

677 **► Corollary 26** (Monotonicity of extremes). *Let $\mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be an e-path. $\max(\mathbf{B}_{t+1}) \leq$
 678 $\max(\mathbf{B}_t)$ and $\min(\mathbf{B}_{t+1}) \geq \min(\mathbf{B}_t)$ for all $t \in \mathbb{N}$.*

679 Monotonicity and boundedness of extremes, together with the Monotonic Convergence
 680 Theorem [32], lead us to the existence of limits for opinions of extreme agents.

681 **► Theorem 27** (Limits of extremes). *Let $\mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be an e-path. There exist
 682 $U, L \in [0, 1]$ such that $\lim_{t \rightarrow \infty} \{\max(\mathbf{B}_t)\} = U$ and $\lim_{t \rightarrow \infty} \{\min(\mathbf{B}_t)\} = L$.*

683 Therefore, by the squeeze theorem [32], to prove Th. 18, it suffices to show that
 684 $\{\max(\mathbf{B}_t)\}_{t \geq 0}$ and $\{\min(\mathbf{B}_t)\}_{t \geq 0}$ converge to the same value.

685 **Step 2.** To prove consensus, we now show that $U = L$ for U, L in Th. 27. We say
 686 that an e-path π' is an *e-suffix* of an e-path π if π' is a suffix of π . In what follows we let
 687 $\pi' = \mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be an e-suffix of an strongly-fair run and $w = e_0 e_1 \dots$ be the ω -word
 688 generated by π' .

689 In Lem. 30 we identify a lower bound on how much $\{\max(\mathbf{B}_t)\}_{t \geq 0}$ decreases. The
 690 decrease may occur when an agent of minimum opinion influences all agents.⁵ To characterize
 691 it, we define a function $\Delta_w(i)$ that quantifies how long it takes for an agent to influence
 692 every other. This requires some notation.

693 Recall that a sequence (g -path in this case) $p = a_0 a_1 \dots a_n$ is a *subsequence* of a word
 694 $w = e_0 e_1 \dots$ if there exist indices $i_0 < i_1 < \dots < i_n$ such that $e_{i_0} = a_0, e_{i_1} = a_1, \dots,$
 695 $e_{i_n} = a_n$. We define $\delta_w(p)$ as the length of the smallest prefix w' of w that such that p is a
 696 subsequence of w' .

697 Recall that by definition g -paths are finite (i.e., simple paths in the graph). Let $P_G(i)$ be
 698 the set of all the g -paths starting from i .

699 **► Definition 28** (Δ). *For all $i \in A$, we define $\Delta_w(i) = \max\{\delta_w(p) \mid p \in P_G(i)\}$*

700 Intuitively, $\Delta_w(i)$ is the length of the smallest prefix of w that has all g -paths that start
 701 with agent i as subsequences. Because G is strongly connected, agent i must have influenced
 702 every other agent after the $\Delta_w(i)$ -th action in w .

⁵ A proof of consensus with the maximum opinion influencing all agents can be analogously presented.

703 ▶ **Example 29.** Consider the OTS from Fig. 1a. The path ad is a subsequence of $abcd$.
 704 It holds that $P_G(1) = \{a, ad\}$ and $P_G(3) = \{c, cb\}$. Take the ω -word $w = (abcd)^\omega$. Then,
 705 $\delta_w(a) = 1$, $\delta_w(ad) = 4$, $\delta_w(c) = 3$, and $\delta_w(cb) = 6$; $\Delta_w(1) = 4$ and $\Delta_w(3) = 6$.

706 Now take the ω -word $u = (a^n bc^n d)_{n \in \mathbb{N}^+}$ from the counter-example to consensus in Fig.
 707 3a. Then $\Delta_u(1) = 5$ and $\Delta_u(3) = 7$. But for the suffix $u' = (a^n bc^n d)_{n \geq 10}$ of u , $\Delta_{u'}(1) = 22$
 708 and $\Delta_{u'}(3) = 34$. In this case, $\Delta_{u'}$ increases the later the suffix u' starts in u .

709 The function $\Delta_w(i)$ allows us to express a bound on the maximum opinion in terms of
 710 the opinion of i and the constants $\max(\mathbf{B}_0)$, $|A|$ and the maximum and minimum influences
 711 of the graph. Let $I_{max} = \max_{(i,j) \in E} I_{ij}$ and $I_{min} = \min_{(i,j) \in E} I_{ij}$.

▶ **Lemma 30** (Opinion bound over a network). *Let $\pi' = \mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be an e-suffix of a strongly-fair run and $w = e_0 e_1 \dots$ the ω -word generated by π' . If G is strongly connected, then for all $i \in A$*

$$\max(\mathbf{B}_0) - \max(\mathbf{B}_{\Delta_w(i)}) \geq I_{min}^{|A|} (1 - I_{max})^{\Delta_w(i)} (\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

712 This quantifies a decrement in the opinion of the maximum opinion based on the initial
 713 opinion of an agent $\mathbf{B}_0[i]$. However, this decrease may become unboundedly smaller if $\Delta_w(i)$
 714 grows unboundedly, as in Cons. 10. Therefore, we bound $\Delta_w(i)$ when i is an agent of
 715 minimum opinion. Define $m_{\pi'} \in A$ as the least⁶ agent in A such that $\mathbf{B}_0[m_{\pi'}] = \min(\mathbf{B}_0)$.

716 ▶ **Definition 31** (Δ -bound). *Let π be a strongly fair run and π' an e-suffix of π . We say
 717 $\beta \in \mathbb{N}$ is a Δ -bound of π' if $\Delta_{w_{\pi'}}(m_{\pi'}) \leq \beta$.*

718 *We say $\beta \in \mathbb{N}$ is a recurrent Δ -bound of π if for infinitely many e-suffixes π' of π , β is a
 719 Δ -bound of π' .*

720 Intuitively, a Δ -bound of an e-suffix π' is a length bound for the smallest prefix of $w_{\pi'}$ that
 721 has all g-paths that start with an agent of minimum opinion as subsequences. A recurrent
 722 Δ -bound of a run π is a Δ -bound for infinitely many e-suffixes of π . With this, we apply
 723 Lemma 30 with an agent of minimum opinion to bound the maximum opinion.

724 ▶ **Lemma 32** ($n - \epsilon$ decrement). *Let $\pi = \mathbf{B}_{init} \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be a run with a strongly-
 725 connected G and a recurrent Δ -bound $\beta \in \mathbb{N}$. For all $n \in \mathbb{N}$, there exists an e-suffix
 726 $\pi' = \mathbf{B}_t \xrightarrow{e_t} \mathbf{B}_{t+1} \xrightarrow{e_{t+1}} \dots$ of π such that β is a Δ -bound of π' where*

$$\max(\mathbf{B}_{init}) - \max(\mathbf{B}_{t+\beta}) \geq n * \epsilon$$

727 *with $\epsilon = I_{min}^{|A|} (1 - I_{max})^\beta (U - L)$.*

728 This is enough to prove $U = L$ by contradiction. Suppose $U \neq L$. We have $I_{max} < 1$ by
 729 puppet freedom. Then ϵ is greater than a positive constant. Using Lemma 32, take $n \in \mathbb{N}$
 730 such that $n * \epsilon > \max(\mathbf{B}_{init})$, then $\max(\mathbf{B}_{t+\beta}) < 0$ for some t , a contradiction. Therefore,
 731 $U = L$.

732 If $U = L$, the agents of extreme opinion converge to consensus, and by the squeeze
 733 theorem [32], every agent converges to the same opinion. Thus, any run of an OTS with
 734 strongly-connected puppet-free G and a recurrent Δ -bound converges to consensus.

735 ▶ **Lemma 33** (Consensus with recurrent Δ -bound). *Let $M = (G, \mathbf{B}_{init}, \rightarrow)$ be an OTS where
 736 G is a strongly-connected, puppet-free influence graph. Then for every run π of M , if there
 737 exist $\beta \in \mathbb{N}$ such that β is a recurrent Δ -bound of π , then π converges to consensus.*

⁶ Assume A is ordered under the usual order in the natural numbers.

738 **Step 3.** It remains to show that m -bounded fair runs, with $m \geq |A| - 1$, have a recurrent
739 Δ -bound.

740 ► **Lemma 34** (m -bounded fair runs have a recurrent Δ -bound). *Any bounded m -fair run with*
741 *$m \geq |A| - 1$ has a recurrent Δ -bound $\beta = (|A| - 1) \times k$ for some $k \in \mathbb{N}$.*

742 The intuition is that if π is m -bounded fair, it is (m, k) -fair for some k . This (m, k) -
743 fairness provides a complete (m, k) multi-window for every e-suffix. All g-paths must be a
744 subsequence of any complete (m, k) multi-window with $m \geq |A| - 1$, because a g-path can
745 visit at most $|A| - 1$ agents. This implies $\Delta_{w_{\pi'}, (m_{\pi'})} \leq \beta = (|A| - 1) \times k$ for every e-suffix
746 w' of w that starts with a complete (m, k) multi-window.

747 Therefore, a bounded m -fair run of an OTS with $m \geq |A| - 1$, strong connectivity and
748 puppet freedom also has a recurrent Δ -bound, which implies convergence to consensus.

749 **B Proofs**

750 In this appendix, the reader may find the proofs of the following results:

751 ■ **Proposition 11.**

752 ■ **Proposition 13.**

753 ■ **Proposition 17.**

754 ■ **Theorem 24** (Consensus with bounded influence).

755 ■ **Lemma 25** (Opinion evolution is bounded by the extremes).

756 ■ **Lemma 30** (Opinion bound over a network).

757 ■ **Lemma 32** ($n - \epsilon$ decrement).

758 ■ **Lemma 33** (Consensus with recurrent Δ -bound).

759 ■ **Lemma 34** (m -bounded fair runs have a recurrent Δ -bound).

760 ■ **Proposition 35** (Minimum effort for pulling an agent over a given belief).

761 ■ **Lemma 36** (Opinion bound after one step).

762 ■ **Lemma 37** (Opinion upper bound after n steps).

763 ■ **Lemma 38** (Direct influence bound).

764 ■ **Lemma 39** (Opinion upper bound along a path).

765 ■ **Lemma 41** (epsilon decrement Δ -bound e-suffix).

766 ■ **Lemma 42** (epsilon decrement Δ -bound run).

767 ► **Proposition 11.** *Let $w = w_1 \cdot w_2 \cdot \dots$ be the ω -word from Cons. 10 where each w_m has the*
768 *form a^+bc^+d . Then for every $m \in \mathbb{N}$, there exists $t \in \mathbb{N}$ such that $\#c(w_{m+t}) > \#c(w_m)$.*

769 **Proof.** We will prove this proposition in two steps. First (claim 1), for any $m \in \mathbb{N}$ we
770 will prove, using Prop. 35, that there exists $OE \in \mathbb{N}^+$ such that $\#c(w_m) \leq OE$. Second
771 (claim 2), we will prove, using again Prop. 35 and the fact that **B**[3] converges to U , that
772 there exists $t \in \mathbb{N}^+$ and $UE \in \mathbb{N}^+$ such that $OE < UE$ and $UE \leq \#c(w_{m+t})$. Then, we
773 can conclude that *for each $m \in \mathbb{N}$, there exists $t \in \mathbb{N}^+$ such that $\#c(w_{m+t}) > \#c(w_m)$.*

774 **Claim 1:** Let $L, U \in (0, 1)$ be the two fixed values used for building w , such that
775 $0 < L < \mathbf{B}_0[2] < U < 1$.

776 Let $ua^{\#a(w_m)}b$ be a prefix of w that represents a partial execution of w , that is, $w =$
777 $ua^{\#a(w_m)}bw'$ and $w' = c^{\#c(w_m)}dw_{m+1} \dots$

778 Let **B**[2] and **B**[3] be the opinions of agents 2 and 3 respectively, before starting execution
779 of w' . We know that, by definition, the $\#c(w_m)$ executions of edge c will carry the opinion
780 of agent 2, just over U and below **B**[3]. We don't know the exact value of $\#c(w_m)$, but we

781 can overestimate it easily estimating the minimum effort needed for carrying a opinion of 0
782 over U using the actual opinion of agent 3, $\mathbf{B}[3]$. We call this overestimation OE.

Using proposition 35 we can conclude that

$$\text{OE} = \lceil \log_2 \left(\frac{\mathbf{B}[3] - 0}{\mathbf{B}[3] - U} \right) \rceil = \lceil \log_2 \left(\frac{\mathbf{B}[3]}{\mathbf{B}[3] - U} \right) \rceil$$

783 Because of $\mathbf{B}[2] > 0$ we can conclude that $\text{OE} \geq \#c(w_m)$.

784 **Claim 2:** Now, because of $\lim_{t \rightarrow \infty} \mathbf{B}_t[3] = U$, we can be sure that executing the
785 $w_{m+1}w_{m+2} \dots$ run there will be a moment $m+t$, just after executing the d edge of w_{m+t-1} ,
786 where $\mathbf{B}[3] \leq U + \epsilon$ for any $\epsilon \in (0, \mathbf{B}_{m+t-1}[3] - U)$.

787 That is $w_1w_2 \dots w_mw_{m+1} \dots w_{m+t-1}$ has been executed and $\mathbf{B}[3] \leq U + \epsilon$. Now the
788 edges of $w_{m+t} = a^{p_{m+t}}bc^{q_{m+t}}d$ will be executed. We don't know the exact value of q_{m+t} , but
789 we can underestimate it easily estimating the minimum effort needed for carrying a opinion
790 of L over U at that moment using the actual opinion of agent 3, $\mathbf{B}[3]$. By construction, we
791 know that the partial execution of w until this moment has carried $\mathbf{B}[2]$ under L . We call
792 this underestimation UE.

Using proposition 35 we can conclude that

$$\text{UE} = \lceil \log_2 \left(\frac{\mathbf{B}[3] - L}{\mathbf{B}[3] - U} \right) \rceil = \lceil \log_2 \left(\frac{U + \epsilon - L}{U + \epsilon - U} \right) \rceil = \lceil \log_2 \left(\frac{U + \epsilon - L}{\epsilon} \right) \rceil$$

Because $\mathbf{B}[2] \leq L$ at the beginning of execution of w_{m+t} we can conclude that $q_{m+t} \geq \text{UE}$.
As $q_{m+t} = \#c(w_{m+t})$ then we have

$$\#c(w_{m+t}) \geq \text{UE}$$

793 Then we have $\#c(w_m) \leq \text{OE}$ and $\text{UE} \leq \#c(w_{m+t})$.

Now we choose ϵ such that: $\text{OE} < \text{UE}$, that is, such that

$$\log_2 \left(\frac{\mathbf{B}[3]}{\mathbf{B}[3] - U} \right) < \log_2 \left(\frac{U + \epsilon - L}{\epsilon} \right)$$

794 that is

$$\frac{\mathbf{B}[3]}{\mathbf{B}[3] - U} < \frac{U + \epsilon - L}{\epsilon}$$

795 then

$$\epsilon < \frac{(U - L)(\mathbf{B}[3] - U)}{U}$$

Then, we can conclude that

$$\#c(w_{m+t}) \geq \text{UE} > \text{OE} \geq \#c(w_m)$$

796

797 **► Proposition 13.** For every $k \in \mathbb{N}$, $F(k) \subset F(k+1) \subset (\bigcup_{\kappa \in \mathbb{N}} F(\kappa)) \subset F(\omega)$.

798 **Proof.** We will divide the proof into the following statements to prove:

799 **■** $F(k) \subset F(k+1)$: it will be divided into the following two statements to prove:

- 800 ■ $F(k) \subseteq F(k+1)$: If $w \in F(k)$ then all the k windows of w are complete. Then, every
 801 $e \in E$ will occur within a window of length k from any position of w . Therefore, every
 802 $e \in E$ will also occur within a window of length $k+1$ (or greater) from any position
 803 of w , i.e., all $k+1$ windows of w are complete; thus, $w \in F(k+1)$.
 804 ■ $F(k+1) \not\subseteq F(k)$: Consider a $k+1$ -fair word $w = u \cdot u \cdot \dots$ and the $k+1$ -windows
 805 $u = v \cdot e_i$, such that $v(k) = E'$, where $E' \subset E$ and $E = E' \cup \{e_i\}$, clearly every
 806 $k+1$ -windows in w is complete, but the occurrences of the k -window v in w are not
 807 complete. Therefore, $w \in F(k+1)$ but $w \notin F(k)$.

808 ■ $F(k+1) \subset (\bigcup_{\kappa \in \mathbb{N}} F(\kappa))$: it will be divided into the following two statements to prove:

- 809 ■ $F(k+1) \subseteq (\bigcup_{\kappa \in \mathbb{N}} F(\kappa))$: as $k+1 \in \mathbb{N}$, it is straightforward that $F(k+1) \subseteq (\bigcup_{\kappa \in \mathbb{N}} F(\kappa))$.
 810 ■ $(\bigcup_{\kappa \in \mathbb{N}} F(\kappa)) \not\subseteq F(k+1)$: as $F(k+2) \not\subseteq F(k+1)$ and $F(k+2) \subseteq (\bigcup_{\kappa \in \mathbb{N}} F(\kappa))$,
 811 there is at least a word w such that $w \in F(k+2)$ and hence $w \in \bigcup_{\kappa \in \mathbb{N}} F(\kappa)$ where
 812 $w \notin F(k+1)$.

813 ■ $(\bigcup_{\kappa \in \mathbb{N}} F(\kappa)) \subset F(\omega)$: it will be divided into the following two statements to prove:

- 814 ■ $(\bigcup_{\kappa \in \mathbb{N}} F(\kappa)) \subseteq F(\omega)$: if $w \in (\bigcup_{\kappa \in \mathbb{N}} F(\kappa))$, then $w \in F(k)$ for some $k \in \mathbb{N}$, therefore,
 815 every $e \in E$ will also occur within a window of length k from any position of w , thus,
 816 every $e \in E$ occurs infinitely often in w , hence w is strong fair; $w \in F(\omega)$.
 817 ■ $F(\omega) \not\subseteq (\bigcup_{\kappa \in \mathbb{N}} F(\kappa))$: consider a strong-fair word w , i.e. $w \in F(\omega)$, a set $E = \{e_1, e_2\}$
 818 where $w(i) = e_1$ if i is a power of 2, otherwise, $w(i) = e_2$; as the distance between
 819 consecutive occurrences of e_1 grows unboundedly in w , e_1 will not be within every
 820 window of length k in w for some $k \in \mathbb{N}$. Therefore, $w \notin (\bigcup_{\kappa \in \mathbb{N}} F(\kappa))$.

821

822 ► **Proposition 17.** $F(k) \subset F(m+1, k) \subset F(m, k) \subset (\bigcup_{\kappa \in \mathbb{N}} F(m, \kappa)) \subset F(1, \omega) = F(\omega)$.

823 **Proof.** We will divide it into the following statements to prove:

824 ■ $F(k) \subset F(m+1, k)$: it will be divided into the following two statements to prove:

- 825 ■ $F(k) \subseteq F(m+1, k)$: If $w \in F(k)$ then all the k windows of w are complete. Therefore,
 826 from any position of w , every $e \in E$ will occur within a window of length k ; hence,
 827 from any position of w an infinite number of *consecutive* complete k -windows appear,
 828 thus, every suffix of w has a complete (n, k) multi-window for all $n \in \mathbb{N}^+$, including
 829 when $n = m+1$, then $w \in F(m+1, k)$.
 830 ■ $F(m+1, k) \not\subseteq F(k)$: Consider a word w and $E = \{e_1, e_2\}$ where $w = w_1 \cdot w_2 \cdot w_3 \cdot \dots$
 831 such that every subword w_{2i+1} corresponds to a complete $(m+1, k)$ multi-window and
 832 every subword w_{2i} corresponds to a k window of e_1 , i.e. w can be seen as a sequence
 833 of complete $(m+1, k)$ multi-windows separated between every pair of multi-windows
 834 by a k windows of e_1 . Clearly, $w \in F(m+1, k)$ as every suffix of w has a complete
 835 $(m+1, k)$ multi-window, however, as there are k windows of e_1 in w , it is not true that
 836 from any position of w , every $e \in E$ will occur within a window of length k , therefore
 837 $w \notin F(k)$.

838 ■ $F(m+1, k) \subset F(m, k)$: it will be divided into the following two statements to prove:

- 839 ■ $F(m+1, k) \subseteq F(m, k)$: If $w \in F(m+1, k)$, then from any position of w , $m+1$
840 *consecutive* complete k -windows appear, therefore, clearly from any position of w , m
841 *consecutive* complete k -windows appear, i.e. $w \in F(m, k)$.
- 842 ■ $F(m, k) \not\subseteq F(m+1, k)$: Consider a word w and $E = \{e_1, e_2\}$ where $w = w_1 \cdot w_2 \cdot w_3 \dots$
843 such that every subword w_{2i+1} corresponds to a complete (m, k) multi-window and
844 every subword w_{2i} corresponds to a k window of e_1 , i.e. w can be seen as a sequence
845 of complete (m, k) multi-windows separated between every pair of multi-windows by
846 a k windows of e_1 . Clearly, $w \in F(m, k)$ as every suffix of w has a complete (m, k)
847 multi-window, however, there are no $m+1$ *consecutive* complete k -windows in w ,
848 therefore $w \notin F(m+1, k)$.
- 849 ■ $F(m, k) \subset \bigcup_{\kappa \in \mathbb{N}} F(m, \kappa)$: it will be divided into the following two statements to prove:
- 850 ■ $F(m, k) \subseteq \bigcup_{\kappa \in \mathbb{N}} F(m, \kappa)$: as $k \in \mathbb{N}$, it is straightforward that $F(m, k) \subseteq \bigcup_{\kappa \in \mathbb{N}} F(m, \kappa)$.
851 ■ $(\bigcup_{\kappa \in \mathbb{N}} F(m, \kappa)) \not\subseteq F(m, k)$: Consider a word w and $E = \{e_1, e_2, \dots, e_{k+1}\}$ such that
852 $w = w' \cdot w' \dots$ where $w' = e_1 \cdot e_2 \dots e_{k+1}$. i.e. $w \in F(m, k+1)$. Additionally, as
853 $F(m, k+1) \subseteq \bigcup_{\kappa \in \mathbb{N}} F(m, \kappa)$, $w \in (\bigcup_{\kappa \in \mathbb{N}} F(m, \kappa))$, however, since $|E| = k+1$, it is
854 not possible that there are complete k -windows in w , therefore $w \notin F(m, k)$.
- 855 ■ $(\bigcup_{\kappa \in \mathbb{N}} F(m, \kappa)) \subset F(1, \omega)$: it will be divided into the following two statements to prove:
- 856 ■ $(\bigcup_{\kappa \in \mathbb{N}} F(m, \kappa)) \subseteq F(1, \omega)$: if $w \in (\bigcup_{\kappa \in \mathbb{N}} F(m, \kappa))$, then $w \in F(m, \kappa)$ for some $\kappa \in \mathbb{N}$,
857 therefore, every suffix of w has a complete (m, κ) multi-window. i.e. m *consecutive*
858 complete k -windows appear infinitely often in w . This implies that in every suffix of
859 w , every $e \in E$ occurs. Therefore, every suffix of w can be seen as a complete window
860 of infinite length (ω) , hence $w \in F(1, \omega)$.
- 861 ■ $F(1, \omega) \not\subseteq (\bigcup_{\kappa \in \mathbb{N}} F(m, \kappa))$: consider any $\kappa \in \mathbb{N}$, a word w , a set $E = \{e_1, e_2\}$ where
862 $w(i) = e_1$ if i is a power of 2, otherwise, $w(i) = e_2$; since e_1 and e_2 occur infinitely often
863 in w , every suffix of w can be seen as a complete window of infinite length (ω) , hence
864 $w \in F(1, \omega)$. However, since the distance between consecutive occurrences of e_1 grows
865 unboundedly in w , it is not possible that a complete κ -window occurs infinitely often
866 in w , therefore, there are suffixes of w that don't have a complete (m, κ) multi-window,
867 thus, $w \notin (\bigcup_{\kappa \in \mathbb{N}} F(m, \kappa))$.
- 868 ■ $F(1, \omega) = F(\omega)$: $w \in F(1, \omega)$ corresponds to say that every suffix of w has a complete
869 $(1, \omega)$ multi-window, it is equivalent to say that every $e \in E$ occurs from any position in
870 w , it equates to say that $w \in F(\omega)$.

871

872 ► **Theorem 24** (Consensus with bounded influence). *Let $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$ be a DOTS where*
873 *G is a strongly-connected, influence graph. Suppose that M has bounded influence. For every*
874 *run π of M , if w_π is m -bounded fair with $m \geq |A| - 1$, then π converges to consensus.*

875 **Proof.** We have $I_{ij}^t \in [I_L, I_U]$ by bounded influence. We will reformulate Lemma 36, lemmas
876 37, 38, 39, 30, 41, Lemma 32 and Lemma 33 for consensus with I_L and I_U playing the roles
877 of I_{\min} and I_{\max} .

878 Similarly to Lemma 36, we prove that for any transition $\mathbf{B} \xrightarrow{(i,j)} \mathbf{B}'$ and any $k \in A$, we
879 can show

$$\mathbf{B}'[k] \leq \mathbf{B}[k](1 - I_U) + \max(\mathbf{B})I_U$$

880 Let $\pi' = \mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be an e-suffix of π and $w' = e_0e_1$ the ω -word generated by π' .
 881 Similarly to Lemma 37, we can deduce that for any $i \in A$,

$$\mathbf{B}_n[i] \leq \max(\mathbf{B}_0) - (1 - I_U)^n(\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

Then, similar to Lemma 38, for any $i, j \in A$, if $e_{n+1} = (i, j)$,

$$\mathbf{B}_{n+1}[j] \leq \max(\mathbf{B}_0) - I_L(1 - I_U)^n(\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

882 and this bound can be extended along a path, similar to Lemma 39. Let p be a path in
 883 G that starts with agent i and ends with agent j . Then

$$\mathbf{B}_{\delta_{w'(p)}}[j] \leq \max(\mathbf{B}_0) - I_L^{|p|}(1 - I_U)^{\delta_{w'(p)}}(\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

884 and similar to Lemma 30, we can show that because \mathcal{G} is strongly connected, then for all
 885 $j \in A$ and some $i \in A$

$$886 \quad \max(\mathbf{B}_0) - \mathbf{B}_{\Delta_{w'(i)}}[j] \geq I_L^{|A|}(1 - I_U)^{\Delta_{w'(i)}}(\max(\mathbf{B}_0) - \mathbf{B}_0[i]) \quad (2)$$

887 As proven in Lemma 34, π being m -bounded fair with $m \geq |A| - 1$ implies that there
 888 exists a $\beta \in \mathbb{N}$ such that β is a recurrent Δ -bound of π . Therefore,

889 Similarly to Lemma 41, if the e-suffix $\pi' = \mathbf{B}_t \xrightarrow{e_t} \mathbf{B}_{t+1} \xrightarrow{e_{t+1}} \dots$ of π has β as a bound
 890 Δ , then:

$$\epsilon \leq \max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{t+\beta})$$

891 for $\epsilon = I_L^{|A|}(1 - I_U)^\beta(U - L)$, where $U = \lim_{t \rightarrow \infty} \max(\mathbf{B}_t)$ and $L = \lim_{t \rightarrow \infty} \min(\mathbf{B}_t)$
 892 from Theorem 27.

893 Similarly to Lemma 32, for all $n \in \mathbb{N}$, there exists an e-suffix $\pi' = \mathbf{B}_t \xrightarrow{e_t} \mathbf{B}_{t+1} \xrightarrow{e_{t+1}} \dots$
 894 of π such that β is a Δ -bound of π' where

$$n * \epsilon \leq \max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{t+\beta})$$

895 for $\epsilon = I_L^{|A|}(1 - I_U)^\beta(U - L)$,

896 We are ready to reformulate Lemma 33 for consensus with dynamic influence. It remains
 897 to prove that $U = L$.

898 Suppose, by contradiction, that $U \neq L$. We have $I_U < 1$ by bounded influence. Then
 899 ϵ is greater than or equal to a positive constant. Using Lemma 32, take $n \in \mathbb{N}$ such that
 900 $n * \epsilon > \max(\mathbf{B}_{\text{init}})$, then $\max(\mathbf{B}_{t+\beta}) < 0$ for some t , a contradiction by definition.

Therefore, $U = L$, and by the squeeze theorem [32],

$$\lim_{t \rightarrow \infty} \max(\mathbf{B}_t) = \lim_{t \rightarrow \infty} \min(\mathbf{B}_t) = \lim_{t \rightarrow \infty} \mathbf{B}_t[k] \quad \forall k \in A$$

901 we converge to consensus.

902

903 **► Lemma 25** (Opinion evolution is bounded by the extremes). *Let $\mathbf{B} \xrightarrow{e} \mathbf{B}'$ be a transition.*
 904 *Then $\min(\mathbf{B}) \leq \mathbf{B}'[k] \leq \max(\mathbf{B})$ for all $k \in A$.*

Proof. We want to prove that

$$\mathbf{B}'[k] \leq \max(\mathbf{B})$$

905 Take the transition relation of Definition 2 and let $e = (i, j)$. Either $k \neq j$ or $k = j$.
 906 If $k \neq j$, $\mathbf{B}'[k] = \mathbf{B}[k]$, then $\mathbf{B}'[k] = \mathbf{B}[k] \leq \max(\mathbf{B})$.
 907 If $k = j$,

$$\mathbf{B}'[k] = \mathbf{B}[j] + (\mathbf{B}[i] - \mathbf{B}[j])I_{ij}$$

908 Now note that by definition $\mathbf{B}[i] \leq \max(\mathbf{B})$ and $I_{ij} \leq 1$. Then

$$\mathbf{B}'[k] \leq \mathbf{B}[j] + (\max(\mathbf{B}) - \mathbf{B}[j]) = \max(\mathbf{B})$$

909 as wanted. The proof that $\mathbf{B}'[k] \geq \min(\mathbf{B})$ is analogous.

910

► **Lemma 30** (Opinion bound over a network). *Let $\pi' = \mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be an e -suffix of a strongly-fair run and $w = e_0 e_1 \dots$ the ω -word generated by π' . If G is strongly connected, then for all $i \in A$*

$$\max(\mathbf{B}_0) - \max(\mathbf{B}_{\Delta_w(i)}) \geq I_{\min}^{|A|} (1 - I_{\max})^{\Delta_w(i)} (\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

911 **Proof.** Take any $k \in A$. Because G is strongly connected, there is a path p between i and k .
 912 Apply Lemma 39 to p

$$\mathbf{B}_{\delta_w(p)}[k] \leq \max(\mathbf{B}_0) - I_{\min}^{|p|} (1 - I_{\max})^{\delta_w(p)} (\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

913 Now because this is for all $k \in A$, it follows that

$$\max(\mathbf{B}_{\delta_w(p)}) \leq \max(\mathbf{B}_0) - I_{\min}^{|p|} (1 - I_{\max})^{\delta_w(p)} (\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

914 and by Lemma 25, we know that $\mathbf{B}_{\Delta_w(i)}[j] \leq \max(\mathbf{B}_{\delta_w(p)})$ for all $j \in A$.

$$\mathbf{B}_{\Delta_w(i)}[j] \leq \max(\mathbf{B}_0) - I_{\min}^{|p|} (1 - I_{\max})^{\Delta_w(i)} (\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

915 Recall that by definition 28, $\delta_w(p) \leq \Delta_w(i)$, because p starts with i . Then

$$\max(\mathbf{B}_{\Delta_w(i)}) \leq \max(\mathbf{B}_0) - I_{\min}^{|p|} (1 - I_{\max})^{\Delta_w(i)} (\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

916 Note that $|p| \leq |A|$, therefore

$$\max(\mathbf{B}_{\Delta_w(i)}) \leq \max(\mathbf{B}_0) - I_{\min}^{|A|} (1 - I_{\max})^{\Delta_w(i)} (\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

917 Now by subtracting $\max(\mathbf{B}_0)$ on both sides and multiplying by -1 , we have

$$\max(\mathbf{B}_0) - \max(\mathbf{B}_{\Delta_w(i)}) \geq I_{\min}^{|A|} (1 - I_{\max})^{\Delta_w(i)} (\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

918

► **Lemma 32** ($n - \epsilon$ decrement). *Let $\pi = \mathbf{B}_{\text{init}} \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be a run with a strongly-connected G and a recurrent Δ -bound $\beta \in \mathbb{N}$. For all $n \in \mathbb{N}$, there exists an e -suffix $\pi' = \mathbf{B}_t \xrightarrow{e_t} \mathbf{B}_{t+1} \xrightarrow{e_{t+1}} \dots$ of π such that β is a Δ -bound of π' where*

$$\max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{t+\beta}) \geq n * \epsilon$$

922 with $\epsilon = I_{\min}^{|A|} (1 - I_{\max})^\beta (U - L)$.

923 **Proof.** We proceed by induction on n to prove that for all $n \in \mathbb{N}$, there exists an e-suffix
 924 $\pi' = \mathbf{B}_t \xrightarrow{e_t} \mathbf{B}_{t+1} \xrightarrow{e_{t+1}} \dots$ of π such that β is a Δ -bound of π' where

$$n * \epsilon \leq \max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{t+\beta})$$

925 ■ Base Case ($n = 1$)

926 From Definition 31 and as π has a recurrent Δ -bound β , there exists an e-suffix $\pi' =$
 927 $\mathbf{B}_t \xrightarrow{e_t} \mathbf{B}_{t+1} \xrightarrow{e_{t+1}} \dots$ such that β is a Δ -bound of π' .

As β is a Δ -bound of π' , and by applying Lemma 42 on π and π' , we have the following:

$$\epsilon \leq \max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{t+\beta})$$

928 with $\epsilon = I_{\min}^{|\mathbf{A}|} (1 - I_{\max})^\beta (U - L)$.

929 ■ Inductive Step:

930 As inductive hypothesis, we assume that there is an e-suffix $\pi' = \mathbf{B}_t \xrightarrow{e_t} \mathbf{B}_{t+1} \xrightarrow{e_{t+1}} \dots$
 931 of π such that β is a Δ -bound of π' where

$$n * \epsilon \leq \max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{t+\beta})$$

932 with $\epsilon = I_{\min}^{|\mathbf{A}|} (1 - I_{\max})^\beta (U - L)$.

933 We need to prove that there exists an e-suffix $\pi'' = \mathbf{B}_s \xrightarrow{e_s} \mathbf{B}_{s+1} \xrightarrow{e_{s+1}} \dots$ of π such that
 934 β is a Δ -bound of π'' where

$$(n + 1) * \epsilon \leq \max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{s+\beta})$$

935 with $\epsilon = I_{\min}^{|\mathbf{A}|} (1 - I_{\max})^\beta (U - L)$.

936 To prove it, we consider Definition 31 and the fact that π has a recurrent Δ -bound β .

937 As π has a recurrent Δ -bound β , there are infinitely many e-suffixes of π , such that β is
 938 a Δ -bound of each of them. It implies that there are infinitely many e-suffixes of every
 939 e-suffix of π , such that β is a Δ -bound of each of them.

940 Thus, consider an e-suffix $\pi'' = \mathbf{B}_s \xrightarrow{e_s} \mathbf{B}_{s+1} \xrightarrow{e_{s+1}} \dots$ of π' such that β is a Δ -bound of
 941 π'' and $s \geq t + \beta$.

942 From the inductive hypothesis, we know that:

$$n * \epsilon \leq \max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{t+\beta})$$

943 From Corollary 26 and $s \geq t + \beta$, we know that:

$$\max(\mathbf{B}_{t+\beta}) \geq \max(\mathbf{B}_s)$$

944 Applying Lemma 41 on π'' , we know that:

$$\max(\mathbf{B}_s) \geq \max(\mathbf{B}_{s+\beta}) + \epsilon$$

945 Then $\max(\mathbf{B}_{t+\beta}) \geq \max(\mathbf{B}_{s+\beta}) + \epsilon$ and using the inductive hypothesis, we have:

$$n * \epsilon \leq \max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{t+\beta}) \leq \max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{s+\beta}) - \epsilon$$

946 Therefore:

$$n * \epsilon + \epsilon = (n + 1) * \epsilon \leq \max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{s+\beta})$$

947 As expected.

948

949 ► **Lemma 33** (Consensus with recurrent Δ -bound). *Let $M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$ be an OTS where*
 950 *G is a strongly-connected, puppet-free influence graph. Then for every run π of M , if there*
 951 *exist $\beta \in \mathbb{N}$ such that β is a recurrent Δ -bound of π , then π converges to consensus.*

952 **Proof.** Let $\pi = \mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$. By Theorem 27, there exists $U, L \in [0, 1]$ such that
 953 $U = \lim_{t \rightarrow \infty} \max(\mathbf{B}_t)$ and $L = \lim_{t \rightarrow \infty} \min(\mathbf{B}_t)$.

954 We now prove $U = L$. Let $\beta \in \mathbb{N}$ be a recurrent Δ -bound of π .

955 By Lemma 32, there exists an e-suffix $\pi' = \mathbf{B}_t \xrightarrow{e_t} \mathbf{B}_{t+1} \xrightarrow{e_{t+1}} \dots$ of π such that β is a
 956 Δ -bound of π' where

$$957 \quad n * \epsilon \leq \max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{t+\beta}) \quad (3)$$

958 with $\epsilon \geq I_{\min}^{|\mathcal{A}|} (1 - I_{\max})^\beta (U - L)$ For all $n \in \mathbb{N}$.

959 Now suppose, by contradiction, that $U \neq L$. That, together with $I_{\max} < 1$ from the
 960 puppet freedom property, imply that ϵ is a constant greater than zero. Then $U - L > 0$. Thus,
 961 from (3) there exist some $n \in \mathbb{N}$ such that $n * \epsilon > \max(\mathbf{B}_{\text{init}})$, which implies $\max(\mathbf{B}_{t+\beta}) < 0$,
 962 which a contradiction by definition.

Therefore, $U = L$, and by the squeeze theorem [32],

$$\lim_{t \rightarrow \infty} \max(\mathbf{B}_t) = \lim_{t \rightarrow \infty} \min(\mathbf{B}_t) = \lim_{t \rightarrow \infty} \mathbf{B}_t[k] \quad \forall k \in \mathcal{A}$$

963 which is consensus.

964

965 ► **Lemma 34** (m -bounded fair runs have a recurrent Δ -bound). *Any bounded m -fair run with*
 966 *$m \geq |\mathcal{A}| - 1$ has a recurrent Δ -bound $\beta = (|\mathcal{A}| - 1) \times k$ for some $k \in \mathbb{N}$.*

967 **Proof.** Consider an OTS $(G, \mathbf{B}_{\text{init}}, \rightarrow)$ with a strongly-connected $G = (A, E, I)$. Let π be a
 968 m -bounded fair run with $m \geq |\mathcal{A}| - 1$ and let $w = w_\pi$ be its related ω -word.

969 If w is m -bounded fair, it is (m, k) -fair for some k . Then, every suffix of w has a complete
 970 (m, k) multi-window. Let $w' = w_0 \cdot w_1 \dots w_m \dots$ be one of the infinitely many suffixes that
 971 start when the complete (m, k) multi-window starts. Every w_i is a complete k -window.

972 Let $p = a_1 a_2 \dots a_{|p|-1}$ be any g -path in G .

973 Because w_1 is complete, a_1 must occur in w_1 ; a_2 must occur in w_2 , and so on. $a_{|p|-1}$
 974 must occur in $w_{|p|-1}$. The path p can be length $|\mathcal{A}| - 1$ at most, therefore every edge of p
 975 must occur in order in the complete $(|\mathcal{A}| - 1, k)$ multi-window. This means p is a subsequence
 976 of the multi-window. The size of the multi-window is greater or equal than $(|\mathcal{A}| - 1) * k$,
 977 therefore, $\delta_{w'}(p) \leq (|\mathcal{A}| - 1) * k$ for any p .

978 This includes the g -paths that start with $m_{\pi'}$, then $\Delta_{w'}(m_{\pi'}) \leq (|\mathcal{A}| - 1) * k$.

979 This holds for infinitely many suffixes w' of w , specifically the suffixes that start when
 980 the complete (m, k) multi-window starts.

981

982 ► **Proposition 35** (Minimum effort for pulling an agent over a given opinion). *Suppose that*
 983 *$M = (G, \mathbf{B}_{\text{init}}, \rightarrow)$ where G is the strongly-connected, puppet-free, influence graph in Fig.*
 984 *1a and $\mathbf{B}_{\text{init}} = (0, 0.5, 1)$ as in example 10. Let $i, j \in \mathcal{A}$, $(j, i) \in E$ and $U \in [0, 1]$ such that*
 985 *$\mathbf{B}[i] < U < \mathbf{B}[j]$. Let $t \in \mathbb{N}$, the least number of consecutive executions of edge (j, i) needed*
 986 *for pulling the opinion of agent i over U . Then $t = \lceil \log_2(\frac{\mathbf{B}[j] - \mathbf{B}[i]}{\mathbf{B}[j] - U}) \rceil$.*

Proof. Let $\delta = \mathbf{B}[j] - \mathbf{B}[i]$. The opinion of agent i after t consecutive activations of edge (j, i) is $\mathbf{B}[i] + (\frac{\delta}{2^1} + \frac{\delta}{2^2} + \dots + \frac{\delta}{2^t})$. We need to find t such that this value be over U . That is:

$$\mathbf{B}[i] + (\frac{\delta}{2^1} + \frac{\delta}{2^2} + \dots + \frac{\delta}{2^t}) \geq U$$

987 We know that $\mathbf{B}[i] + (\frac{\delta}{2^1} + \frac{\delta}{2^2} + \dots + \frac{\delta}{2^T}) = \mathbf{B}[i] + \delta(1 - \frac{1}{2^T})$.

Let T be the value where $\mathbf{B}[i] + \delta(1 - \frac{1}{2^T}) = U$, that is, solving T :

$$T = \log_2(\frac{\mathbf{B}[j] - \mathbf{B}[i]}{\mathbf{B}[j] - U})$$

988 Then $t = \lceil T \rceil$ ◀

989 ▶ **Lemma 36** (Opinion bound after one step). *For any transition $\mathbf{B} \rightarrow \mathbf{B}'$, and any $k \in A$*

$$\mathbf{B}'[k] \leq \mathbf{B}[k](1 - I_{max}) + \max(\mathbf{B})I_{max}$$

990 **Proof.** Take the transition relation of Definition 2, where $k \neq j$ implies $\mathbf{B}'[k] = \mathbf{B}[k]$ and
 991 $k = j$ implies $\mathbf{B}'[k] = \mathbf{B}[j] + (\mathbf{B}[i] - \mathbf{B}[j])I_{ij}$. For the first case $\mathbf{B}'[k] = \mathbf{B}[k]$,

992 $\mathbf{B}'[k] \leq \mathbf{B}[k]$

993 $\mathbf{B}'[k] \leq \mathbf{B}[k](1 - I_{max} + I_{max})$

994 $\mathbf{B}'[k] \leq \mathbf{B}[k](1 - I_{max}) + \mathbf{B}[k]I_{max}$

995 $\mathbf{B}'[k] \leq \mathbf{B}[k](1 - I_{max}) + \max(\mathbf{B})I_{max}$ by definition $\mathbf{B}[k] \leq \max(\mathbf{B})$

996

997 For the second case, we have

$$\mathbf{B}'[k] = \mathbf{B}[j] + (\mathbf{B}[i] - \mathbf{B}[j])I_{ij}$$

998 by definition, $\mathbf{B}[i] \leq \max(\mathbf{B})$, therefore

$$\mathbf{B}'[k] \leq \mathbf{B}[j] + (\max(\mathbf{B}) - \mathbf{B}[j])I_{ij}$$

999 In addition, $I_{ij} \leq I_{max}$ by definition; replacing

$$\mathbf{B}'[k] \leq \mathbf{B}[j] + (\max(\mathbf{B}) - \mathbf{B}[j])I_{max}$$

1000 which can be rewritten as

$$\mathbf{B}'[k] \leq \mathbf{B}[k](1 - I_{max}) + \max(\mathbf{B})I_{max}$$

1001 ◀

▶ **Lemma 37** (Opinion upper bound after n steps). *Let $\mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be an e -path. For any $i \in A$*

$$\mathbf{B}_n[i] \leq \max(\mathbf{B}_0) - (1 - I_{max})^n (\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

1002 **Proof.** We use induction for n . For the base case $n = 1$, we claim

$$\mathbf{B}_1[i] \leq \max(\mathbf{B}) - (1 - I_{\max})(\max(\mathbf{B}) - \mathbf{B}[i])$$

1003 which we deduce by applying Lemma 36 to $\mathbf{B} \xrightarrow{e_1} \mathbf{B}_1$:

$$\begin{aligned} 1004 \quad \mathbf{B}_1[i] &\leq \mathbf{B}[i](1 - I_{\max}) + \max(\mathbf{B})(I_{\max}) \\ 1005 &= \mathbf{B}[i] - \mathbf{B}[i](I_{\max}) + \max(\mathbf{B})(I_{\max}) \\ 1006 &= \mathbf{B}[i] + I_{\max}(\max(\mathbf{B}) - \mathbf{B}[i]) \\ 1007 &= \max(\mathbf{B}) - \max(\mathbf{B}) + \mathbf{B}[i] + I_{\max}(\max(\mathbf{B}) - \mathbf{B}[i]) \\ 1008 &= \max(\mathbf{B}) - (\max(\mathbf{B}) - \mathbf{B}[i]) + I_{\max}(\max(\mathbf{B}) - \mathbf{B}[i]) \\ 1009 &= \max(\mathbf{B}) - (1 - I_{\max})(\max(\mathbf{B}) - \mathbf{B}[i]) \end{aligned}$$

1010

1011 as claimed. For the inductive case, we claim

1012

$$\begin{aligned} 1013 \quad \mathbf{B}_n[i] &\leq \max(\mathbf{B}) - (1 - I_{\max})^n(\max(\mathbf{B}) - \mathbf{B}[i]) \implies \\ 1014 \quad \mathbf{B}_{n+1}[i] &\leq \max(\mathbf{B}) - (1 - I_{\max})^{n+1}(\max(\mathbf{B}) - \mathbf{B}[i]) \quad (4) \end{aligned}$$

1015 to prove this, we apply Lemma 36 to $\mathbf{B}_n \xrightarrow{e_{n+1}} \mathbf{B}_{n+1}$:

$$\mathbf{B}_{n+1}[i] \leq \mathbf{B}_n[i](1 - I_{\max}) + \max(\mathbf{B}_n)I_{\max}$$

1016 by Lemma 25, $\max(\mathbf{B}_n) \leq \max(\mathbf{B})$

$$\mathbf{B}_{n+1}[i] \leq \mathbf{B}_n[i](1 - I_{\max}) + \max(\mathbf{B})I_{\max}$$

1017 using the inductive hypothesis, we replace $\mathbf{B}_n[i]$

$$\begin{aligned} 1018 \quad \mathbf{B}_{n+1}[i] &\leq [\max(\mathbf{B}) - (1 - I_{\max})^n(\max(\mathbf{B}) - \mathbf{B}[i])](1 - I_{\max}) + \max(\mathbf{B})I_{\max} \\ 1019 &= \max(\mathbf{B}) - [(1 - I_{\max})^n(\max(\mathbf{B}) - \mathbf{B}[i])](1 - I_{\max}) \\ 1020 &= \max(\mathbf{B}) - (1 - I_{\max})^{n+1}(\max(\mathbf{B}) - \mathbf{B}[i]) \end{aligned}$$

1021

1022 as claimed. Thus, for any $n \in \mathbb{N}$ and $i \in A$,

$$\mathbf{B}_n[i] \leq \max(\mathbf{B}) - (1 - I_{\max})^n(\max(\mathbf{B}) - \mathbf{B}[i])$$

1023 ◀

► **Lemma 38** (Direct influence bound). *Consider the e -path $\mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ with $e_{n+1} = (i, j)$. Then*

$$\mathbf{B}_{n+1}[j] \leq \max(\mathbf{B}) - I_{\min}(1 - I_{\max})^n(\max(\mathbf{B}) - \mathbf{B}[i])$$

1024 **Proof.** For some $\mathbf{B}_n[j]$ and $e_{n+1} = (i, j)$, the transition relation from Definition 2 yields that

$$1025 \quad \mathbf{B}_{n+1}[j] = \mathbf{B}_n[j] + (\mathbf{B}_n[i] - \mathbf{B}_n[j])I_{ij} = \mathbf{B}_n[j](1 - I_{ij}) + \mathbf{B}_n[i]I_{ij} \quad (5)$$

1026 by Lemma 37, we know that

$$\mathbf{B}_n[i] \leq \max(\mathbf{B}) - (1 - I_{\max})^n (\max(\mathbf{B}) - \mathbf{B}[i])$$

1027 replacing $\mathbf{B}_n[i]$ in (5)

$$1028 \quad \mathbf{B}_{n+1}[j] \leq \mathbf{B}_n[j](1 - I_{ij}) + \max(\mathbf{B})I_{ij} - (1 - I_{\max})^n (\max(\mathbf{B}) - \mathbf{B}[i])I_{ij} \quad (6)$$

1029 and to continue, we infer the following

$$\begin{aligned} 1030 \quad & \max(\mathbf{B}) \leq \max(\mathbf{B}) \\ 1031 \quad & \max(\mathbf{B})(1 - I_{ij} + I_{ij}) \leq \max(\mathbf{B}) \\ 1032 \quad & \max(\mathbf{B})(1 - I_{ij}) + \max(\mathbf{B})I_{ij} \leq \max(\mathbf{B}) \\ 1033 \quad & \mathbf{B}_n[j](1 - I_{ij}) + \max(\mathbf{B})I_{ij} \leq \max(\mathbf{B}) \quad \mathbf{B}_n[j] \leq \max(\mathbf{B}) \text{ from Lemma 25} \end{aligned}$$

1034

1035 to replace on (6) which yields

$$\mathbf{B}_{n+1}[j] \leq \max(\mathbf{B}) - I_{ij}(1 - I_{\max})^n (\max(\mathbf{B}) - \mathbf{B}[i])$$

1036 and by definition, $I_{\min} \leq I_{ij}$, therefore

$$\mathbf{B}_{n+1}[j] \leq \max(\mathbf{B}) - I_{\min}(1 - I_{\max})^n (\max(\mathbf{B}) - \mathbf{B}[i])$$

1037 ◀

1038 **► Lemma 39** (Opinion upper bound along a path). *Let $\pi = \mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be an e -path*
 1039 *such that w_π is strongly fair. Let p be a g -path in G from agent i to j . Then*

$$\mathbf{B}_{\delta_w(p)}[j] \leq \max(\mathbf{B}_0) - I_{\min}^{|p|} (1 - I_{\max})^{\delta_w(p)} (\max(\mathbf{B}_0) - \mathbf{B}_0[i])$$

1040 **Proof.** Let $p = p_0 \dots p_k \dots p_{|p|-1}$; $p_k = (i_k, i_{k+1})$. Define

$$\tau(k) = \delta_w(p_0 \dots p_k)$$

1041 Notice that $\tau(k) \in \mathbb{N}$ because $\delta_r(p) \in \mathbb{N}$. $\tau(k)$ is time step when the k -th edge of p is
 1042 activated after $\tau(k - 1)$, i.e. $e_{\tau(k)} = p_k$.

1043 We assert that

$$1044 \quad \mathbf{B}_{\tau(k)}[i_{k+1}] \leq \max(\mathbf{B}_0) - I_{\min}^{k+1} (1 - I_{\max})^{\tau(k)} (\max(\mathbf{B}_0) - \mathbf{B}[i]) \quad (7)$$

1045 proving it by induction over k . For the base case $k = 0$, we claim

$$1046 \quad \mathbf{B}_{\tau(0)}[i_1] \leq \max(\mathbf{B}_0) - I_{\min} (1 - I_{\max})^{\tau(0)} (\max(\mathbf{B}_0) - \mathbf{B}[i]) \quad (8)$$

1047 Consider the e-path $\mathbf{B}_0 \rightarrow \mathbf{B}_1 \rightarrow \mathbf{B}_2 \dots \xrightarrow{p_0} \mathbf{B}_{\tau(0)}$. Applying Lemma 38:

$$1048 \quad \mathbf{B}_{\tau(0)}[i_1] \leq \max(\mathbf{B}_0) - I_{\min}(1 - I_{\max})^{\tau(0)-1}(\max(\mathbf{B}_0) - \mathbf{B}[i]) \quad (9)$$

1049 and we can prove that $(1 - I_{\max})^{\tau(0)-1} \geq (1 - I_{\max})^{\tau(0)}$, therefore we derive

$$1050 \quad \mathbf{B}_{\tau(0)}[i_1] \leq \max(\mathbf{B}_0) - I_{\min}(1 - I_{\max})^{\tau(0)}(\max(\mathbf{B}_0) - \mathbf{B}[i]) \quad (10)$$

1051 as claimed. For the inductive case, we need to prove

$$1052 \quad \mathbf{B}_{\tau(k)}[i_{k+1}] \leq \max(\mathbf{B}_0) - I_{\min}^{k+1}(1 - I_{\max})^{\tau(k)}(\max(\mathbf{B}_0) - \mathbf{B}[i]) \implies$$

$$1054 \quad \mathbf{B}_{\tau(k+1)}[i_{k+2}] \leq \max(\mathbf{B}_0) - I_{\min}^{k+2}(1 - I_{\max})^{\tau(k+1)}(\max(\mathbf{B}_0) - \mathbf{B}[i])$$

1055 Consider the e-path $\mathbf{B}_{\tau(k)} \rightarrow \mathbf{B}_{\tau(k)+1} \rightarrow \mathbf{B}_{\tau(k)+2} \dots \xrightarrow{p_{k+1}} \mathbf{B}_{\tau(k+1)}$ which is an e-suffix of
1056 π . From Lemma 38 we obtain:

$$1057 \quad \mathbf{B}_{\tau(k+1)}[i_{k+2}] \leq \max(\mathbf{B}_0) - I_{\min}(1 - I_{\max})^{\tau(k+1)-\tau(k)-1}(\max(\mathbf{B}_0) - \mathbf{B}_{\tau(k)}[i_{k+1}]) \quad (11)$$

1058 replacing $\mathbf{B}_{\tau(k)}[i_{k+1}]$ given the inductive hypothesis

$$1059 \quad \mathbf{B}_{\tau(k+1)}[i_{k+2}] \leq \max(\mathbf{B}_0) - I_{\min}(1 - I_{\max})^{\tau(k+1)-\tau(k)-1}(\max(\mathbf{B}_0) -$$

$$1060 \quad \left[\max(\mathbf{B}_0) - I_{\min}^{k+1}(1 - I_{\max})^{\tau(k)}(\max(\mathbf{B}_0) - \mathbf{B}[i]) \right])$$

$$1061 \quad = \max(\mathbf{B}_0) - I_{\min}^{k+2}(1 - I_{\max})^{\tau(k+1)-1}(\max(\mathbf{B}_0) - \mathbf{B}[i])$$

1063 and we can prove that $(1 - I_{\max})^{\tau(k+1)-1} \geq (1 - I_{\max})^{\tau(k+1)}$, therefore we derive

$$\mathbf{B}_{\tau(k+1)}[i_{k+2}] \leq \max(\mathbf{B}_0) - I_{\min}^{k+2}(1 - I_{\max})^{\tau(k+1)}(\max(\mathbf{B}_0) - \mathbf{B}[i])$$

1064 as wanted. Thus, (7) is true. Then, for $k = |p| - 1$,

$$1065 \quad \mathbf{B}_{\tau(|p|-1)}[i_{|p|}] \leq \max(\mathbf{B}_0) - I_{\min}^{|p|}(1 - I_{\max})^{\tau(|p|-1)}(\max(\mathbf{B}_0) - \mathbf{B}[i])$$

1066 clearly, $i_{|p|} = j$ and $\tau(|p| - 1) = \delta_w(p)$, therefore

$$1067 \quad \mathbf{B}_{\delta_w(p)}[k] \leq \max(\mathbf{B}_0) - I_{\min}^{|p|}(1 - I_{\max})^{\delta_w(p)}(\max(\mathbf{B}_0) - \mathbf{B}[i])$$

1068 ◀

1069 **► Corollary 40** (L and U are bounds of extremes). *Let $\mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be an e-path*
1070 *and $U, L \in [0, 1]$ such that $\lim_{t \rightarrow \infty} \{\max(\mathbf{B}_t)\} = U$ and $\lim_{t \rightarrow \infty} \{\min(\mathbf{B}_t)\} = L$. Then*
1071 *$\max(\mathbf{B}_t) \geq U$ and $\min(\mathbf{B}_t) \leq L$ for all $t \in \mathbb{N}$.*

1072 ► **Lemma 41** (epsilon decrement Δ -bound e-suffix). *Let $\pi' = \mathbf{B}_0 \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be an e-suffix*
 1073 *of a strongly-fair run, with a Δ -bound $\beta \in \mathbb{N}$, then:*

$$\max(\mathbf{B}_0) - \max(\mathbf{B}_\beta) \geq \epsilon$$

1074 with $\epsilon = I_{\min}^{|A|} (1 - I_{\max})^\beta (U - L)$.

Proof. Considering π' and the agent $m_{\pi'} \in A$, we apply Lemma 30 to bound the maximum opinion of an agent in π' as follows:

$$\max(\mathbf{B}_0) - \max(\mathbf{B}_{\Delta_{w'_\pi}(m_{\pi'})}) \geq I_{\min}^{|A|} (1 - I_{\max})^{\Delta_{w'_\pi}(m_{\pi'})} (\max(\mathbf{B}_0) - \mathbf{B}_0[m_{\pi'}])$$

1075 As $\max(\mathbf{B}_0) \geq U$ and $\mathbf{B}_0[m_{\pi'}] \leq L$ from Corollary 40, then:

$$\max(\mathbf{B}_0) - \max(\mathbf{B}_{\Delta_{w'_\pi}(m_{\pi'})}) \geq I_{\min}^{|A|} (1 - I_{\max})^{\Delta_{w'_\pi}(m_{\pi'})} (U - L)$$

1076 As β is a Δ -bound of π' , $\Delta_{w'_\pi}(m_{\pi'}) \leq \beta$, hence:

$$\max(\mathbf{B}_0) - \max(\mathbf{B}_{\Delta_{w'_\pi}(m_{\pi'})}) \geq I_{\min}^{|A|} (1 - I_{\max})^{\Delta_{w'_\pi}(m_{\pi'})} (U - L) \geq I_{\min}^{|A|} (1 - I_{\max})^\beta (U - L)$$

1077 From Corollary 26 and $\Delta_{w'_\pi}(m_{\pi'}) \leq \beta$, $\max(\mathbf{B}_{\Delta_{w'_\pi}(m_{\pi'})}) \geq \max(\mathbf{B}_\beta)$, consequently:

$$\max(\mathbf{B}_0) - \max(\mathbf{B}_\beta) \geq \max(\mathbf{B}_0) - \max(\mathbf{B}_{\Delta_{w'_\pi}(m_{\pi'})}) \geq \epsilon$$

1078 with $\epsilon = I_{\min}^{|A|} (1 - I_{\max})^\beta (U - L)$, as expected.

1079 ◀

1080 ► **Lemma 42** (epsilon decrement Δ -bound run). *Let $\pi = \mathbf{B}_{\text{init}} \xrightarrow{e_0} \mathbf{B}_1 \xrightarrow{e_1} \dots$ be a run with*
 1081 *strongly-connected G , and an e-suffix $\pi' = \mathbf{B}_t \xrightarrow{e_t} \mathbf{B}_{t+1} \xrightarrow{e_{t+1}} \dots$ of π with a Δ bound $\beta \in \mathbb{N}$,*
 1082 *then:*

$$\max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{t+\beta}) \geq \epsilon$$

1083 with $\epsilon = I_{\min}^{|A|} (1 - I_{\max})^\beta (U - L)$.

Proof. Applying Lemma 41 on π' , we obtain:

$$\max(\mathbf{B}_t) - \max(\mathbf{B}_{t+\beta}) \geq \epsilon$$

1084 with $\epsilon = I_{\min}^{|A|} (1 - I_{\max})^\beta (U - L)$

1085 From Corollary 26, we know $\max(\mathbf{B}_{\text{init}}) \geq \max(\mathbf{B}_t)$, therefore:

$$\max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{t+\beta}) \geq \max(\mathbf{B}_t) - \max(\mathbf{B}_{t+\beta}) \geq I_{\min}^{|A|} (1 - I_{\max})^\beta (U - L)$$

1086 Then:

$$\max(\mathbf{B}_{\text{init}}) - \max(\mathbf{B}_{t+\beta}) \geq \epsilon$$

1087 with $\epsilon = I_{\min}^{|A|} (1 - I_{\max})^\beta (U - L)$, as expected.

1088 ◀