



HAL
open science

Note on the coincidence of two henselisations

María Emilia Alonso García, Henri Lombardi, Stefan Neuwirth

► **To cite this version:**

María Emilia Alonso García, Henri Lombardi, Stefan Neuwirth. Note on the coincidence of two henselisations. Journal of Algebra and Its Applications, 2024, <10.1142/S021949882650194X>. <hal-04803641>

HAL Id: hal-04803641

<https://hal.science/hal-04803641v1>

Submitted on 26 Nov 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY 4.0 - Attribution - International License

Note on the coincidence of two henselisations

María Emilia Alonso García[†], Henri Lombardi[‡], and Stefan Neuwirth[‡]

[†]Departamento de Álgebra, Geometría y Topología, Facultad de Ciencias Matemáticas,
Universidad Complutense Madrid, Spain, mariemi@mat.ucm.es.

[‡]Université de Franche-Comté, CNRS, UMR 6623, LmB, 25000 Besançon, France,
henri.lombardi@univ-fcomte.fr, stefan.neuwirth@univ-fcomte.fr.

Abstract

We compare two henselisations of a residually discrete valuation domain. Our constructive proof that a certain natural morphism is an isomorphism is also a proof in classical mathematics. Although this isomorphism is implicitly accepted as obvious in the literature, it seems that no proof was previously available.

This note is written in Bishop’s style of constructive mathematics (Bishop 1967, Bishop and Bridges 1985, Bridges and Richman 1987, Mines, Richman, and Ruitenburg 1988, Lombardi and Quitté 2015, Yengui 2015).

Terminology, notation.

This paragraph clarifies the usual constructive terminology for a number of classical notions. From a constructive point of view, the development of good definitions¹ is a decisive part of the work which consists in extracting the constructive content of a concrete result obtained by means of a classical proof.

A ring is *normal* when every principal ideal is integrally closed. In classical mathematics, this constructive definition is shown to be equivalent to the more usual definition according to which the ring becomes an integrally closed domain by localisation at any prime ideal.

A ring is *connected* if every idempotent is equal to 0 or 1.

Let \mathbf{K} be a discrete field. A \mathbf{K} -vector space E is *finite* over \mathbf{K} (respectively *strictly finite* over \mathbf{K}) if it is a \mathbf{K} -vector space of finite type (respectively if we know a finite basis of E over \mathbf{K}). More generally, if \mathbf{A} is a ring, an \mathbf{A} -module E is *finite* over \mathbf{A}

¹A “good definition” must have a clear constructive meaning, apply to cases usually dealt with in current literature, and be equivalent in classical mathematics to the most common definition. Naturally, this is not always a simple matter. Several “good” constructively inequivalent definitions can apply to a single classical notion and it is sometimes useful to consider them.

if it is an \mathbf{A} -module of finite type, and *strictly* finite over \mathbf{A} if it is an \mathbf{A} -projective module of finite type.

A \mathbf{K} -algebra \mathbf{L} is *étale* if it is strictly finite and if the trace form is nondegenerate; equivalently, it is strictly finite and generated by separable elements over \mathbf{K} . More generally, if \mathbf{A} is a ring, an \mathbf{A} -algebra \mathbf{B} is *strictly étale* if it is strictly finite and if the trace form is nondegenerate. See for example [Lombardi and Quitté \(2015, Definitions VI-1.1 and VI-5.1, Theorems VI-1.4, VI-1.7, VI-5.4, and VI-5.5\)](#).

The *Jacobson radical* of a ring \mathbf{A} , denoted by $\text{Rad}(\mathbf{A})$, is the ideal formed by the $x \in \mathbf{A}$ such that $1 + x\mathbf{A} \subseteq \mathbf{A}^\times$ (the multiplicative group of units).

A *pp-ring* (or *quasi-integral ring*) is a ring in which the annihilator of any element is generated by an idempotent. A quasi-integral ring is normal if, and only if, it is integrally closed in its total ring of fractions. An integral domain is none other than a connected quasi-integral ring.

Let \mathbf{A} be a *local ring*, i.e., for any $x \in \mathbf{A}$, x or $1 - x$ is a unit. Its Jacobson radical $\text{Rad}(\mathbf{A})$, also denoted by \mathfrak{m} or $\mathfrak{m}_{\mathbf{A}}$, is equal to the ideal formed by the nonunits. It is called a *Heyting field* if $\text{Rad}(\mathbf{A}) = \{0\}$. The residual field of \mathbf{A} is the quotient ring $\mathbf{A}/\text{Rad}(\mathbf{A})$; \mathbf{A} is *residually discrete* when the residual field is a discrete field; this is equivalent to having a test for “ $x \in \mathbf{A}^\times$ or $x \in \text{Rad}(\mathbf{A})$?” in \mathbf{A} .

A morphism in the category of local rings is a ring morphism $\varphi: \mathbf{A} \rightarrow \mathbf{C}$ which reflects units, i.e. such that $\varphi(x) \in \mathbf{C}^\times$ implies $x \in \mathbf{A}^\times$. Such a morphism is sometimes called *local*, to distinguish it from mere ring morphisms. In the category of residually discrete local rings, the morphism $\varphi: \mathbf{A} \rightarrow \mathbf{C}$ is said to define \mathbf{C} as an *\mathbf{A} -local-algebra*.²

A *Hensel code* in a residually discrete local ring $(\mathbf{A}, \mathfrak{m})$ is a pair $(f, a) \in \mathbf{A}[X] \times \mathbf{A}$ with f monic, $f(a) \in \mathfrak{m}$ and $f'(a) \in \mathbf{A}^\times$.

For a residually discrete local ring \mathbf{A} and a Hensel code (f, a) , a *Hensel zero of code* (f, a) in an \mathbf{A} -local-algebra $\varphi: \mathbf{A} \rightarrow \mathbf{C}$ is an element $\alpha \in \mathbf{C}$ which verifies $f(\alpha) = 0$ and $\alpha - \varphi(a) \in \text{Rad}(\mathbf{C})$ (we then have $f'(\alpha) \in \mathbf{C}^\times$). This zero is necessarily unique. A *special polynomial* is a polynomial $g(X) = X^n - X^{n-1} + a_{n-2}X^{n-2} + \cdots + a_0$ with the a_k 's in \mathfrak{m} ; in this case, $(g, 1)$ is a Hensel code whose Hensel zero is *the special zero* of g . A polynomial $f \in \mathbf{A}[X]$ such that $(f, 0)$ is a Hensel code is called a *Hensel polynomial*; in this case a Hensel zero of code $(f, 0)$ is simply called a Hensel zero of f in \mathbf{C} .

A residually discrete local ring \mathbf{A} is said to be *henselian* if every Hensel polynomial has a Hensel zero in \mathbf{A} .

A *discrete valued field* (\mathbf{K}, \mathbf{V}) is a discrete field \mathbf{K} with a subring \mathbf{V} which verifies: $xy = 1 \in \mathbf{K} \implies x \in \mathbf{V}$ or $y \in \mathbf{V}$. In this case \mathbf{V} is a normal local ring. We also

²“ \mathbf{A} -local-algebra” must be read as a single word: it is an abbreviation in which the word “local” refers to both the object \mathbf{C} and the morphism φ .

require that \mathbf{V} be residually discrete. This is equivalent to requiring that we have a test for “ $\exists x \in \mathbf{V}$ such that $a = xb$?” , where $a, b \in \mathbf{K}$.

Let \mathbf{A} be an arbitrary commutative ring and $f \in \mathbf{A}[X]$ a monic polynomial; we shall use the notation $\boxed{\mathbf{A}_{[f]} = \mathbf{A}[X]/\langle f \rangle}$ and $\boxed{\mathbf{A}_{\{f\}} = \mathbf{A}_{[f]}[1/f']}$.

Coquand and Lombardi (2016, Theorem 6.3) state the following helpful non-trivial result.

Theorem 6.3. *If \mathbf{A} is a normal ring and $f \in \mathbf{A}[X]$ a monic polynomial, then $\mathbf{A}_{\{f\}}$ is also a normal ring.*

Presentation of the problem

We answer a natural question that arises from the articles Kuhlmann and Lombardi 2000 and Alonso García, Lombardi, and Perdry 2008.

The henselisation of a residually discrete local ring $(\mathbf{A}, \mathfrak{m})$, denoted by $(\mathbf{A}^h, \mathfrak{m}^h)$, is constructed in Alonso García, Lombardi, and Perdry 2008. It solves the universal problem for the full subcategory of henselian residually discrete local rings in the category of residually discrete local rings. In other words the construction is functorial and the functor “henselisation” is the left adjoint functor to the inclusion functor.

The henselisation of a discrete valued field (\mathbf{K}, \mathbf{V}) , denoted by $(\mathbf{K}^H, \mathbf{V}^H)$, is constructed in Kuhlmann and Lombardi 2000. It solves the universal problem for the full subcategory of henselian discrete valued fields in the category of discrete valued fields.

Since \mathbf{V} is a residually discrete local ring, it is natural to compare the two henselisations \mathbf{V}^h and \mathbf{V}^H .

This problem does not seem to be addressed in the classical literature, perhaps because the answer seems obvious. But it has not been obvious to us, hence this note. It also answers the problem from the point of view of classical mathematics, since our constructive definitions are equivalent to the usual definitions in classical mathematics.

Given the universal property that characterises $(\mathbf{V}^h, \mathfrak{m}^h)$, and since $(\mathbf{V}^H, \mathfrak{m}^H)$ is a residually discrete henselian local ring, we have a unique local morphism of \mathbf{V} -algebras, viz. $\varphi: \mathbf{V}^h \rightarrow \mathbf{V}^H$. Our result is the following.

Theorem 1. *Let (\mathbf{K}, \mathbf{V}) be a discrete valued field with henselisation $(\mathbf{K}^H, \mathbf{V}^H)$. Let $(\mathbf{V}^h, \mathfrak{m}^h)$ be the henselisation of the residually discrete local ring $(\mathbf{V}, \mathfrak{m})$. The unique local morphism of \mathbf{V} -algebras $\varphi: \mathbf{V}^h \rightarrow \mathbf{V}^H$ is an isomorphism.*

The construction of the henselisation of a residually discrete local ring

An elementary step in the construction of the henselisation \mathbf{A}^h of a residually discrete local ring $(\mathbf{A}, \mathfrak{m})$ is as follows. Let f be a Hensel polynomial. Consider $\mathbf{A}[x] = \mathbf{A}_{[f]}$, $S = 1 + \mathfrak{m} + \langle x \rangle$, and $\mathbf{A}_f := S^{-1}\mathbf{A}[x]$. The ring \mathbf{A}_f is a residually discrete local ring with $\text{Rad}(\mathbf{A}_f) = \mathfrak{m}\mathbf{A}_f$ and $\frac{x}{1}$ is a Hensel zero of f in \mathbf{A}_f . The natural morphism $\mathbf{A} \rightarrow \mathbf{A}_f$ is local, faithfully flat, and we identify \mathbf{A} with a subring of \mathbf{A}_f . This morphism solves the universal problem for the Hensel code $(f, 0)$ in the category of \mathbf{A} -algebras which are residually discrete local rings.

We obtain the henselisation \mathbf{A}^h as a filtered colimit of extensions of type $\mathbf{A} \rightarrow \mathbf{A}_f$. We have a filtered colimit because of the universal property satisfied by the morphism $\mathbf{A} \rightarrow \mathbf{A}_f$. We thus obtain \mathbf{A}^h which is a residually discrete local ring of radical $\text{Rad}(\mathbf{A}^h) = \mathfrak{m}\mathbf{A}^h$, and the morphism $\mathbf{A} \rightarrow \mathbf{A}^h$ yields the henselian residually discrete local ring “generated by \mathbf{A} ” in the sense of the left adjoint functor to the inclusion functor in the category of residually discrete local rings.

Special polynomials

The following lemma, directly inspired by [Alonso García, Lombardi, and Perdry \(2008, Lemma 5.3 and Proposition 5.4\)](#), tells us that in the construction of the henselisation, we can restrict ourselves to the case where we add a special zero of a special polynomial.

Lemma 2. *Let $(\mathbf{A}, \mathfrak{m})$ be a residually discrete local ring and $f(X) = a_0 + a_1X + \dots + a_nX^n \in \mathbf{A}[X]$ with $a_0 \in \mathfrak{m}$ and $a_1 \in \mathbf{A}^\times$. We define the special polynomial*

$$\begin{aligned} g(X) &= X^n - X^{n-1} + a_0(a_1^{-2}a_2X^{n-2} - a_0a_1^{-3}a_3X^{n-3} + \dots + (-1)^na_0^{n-2}a_1^{-n}a_n) \\ &= X^n - X^{n-1} + a_0\ell(X) \quad \text{with } \ell(X) \in \mathbf{A}[X] \end{aligned}$$

and $g_1(X) = g(X + 1)$: g_1 is a Hensel polynomial with Hensel zero $\mu \in \mathfrak{m}\mathbf{A}_{g_1}$ and the special zero of g is $\delta = 1 + \mu$. The polynomial f has a zero γ in $a_0\mathbf{A}_{g_1} \subseteq \mathfrak{m}\mathbf{A}_{g_1} \subseteq \mathfrak{m}\mathbf{A}^h$, with $f'(\gamma)$ a unit in \mathbf{A}_{g_1} , and γ is the unique zero of f in $\mathfrak{m}\mathbf{A}_{g_1}$. If f is monic, f is a Hensel polynomial, $\mathbf{A}_f = \mathbf{A}_{g_1}$, and γ is the Hensel zero of f .

Proof. The following equality occurs in $\mathbf{A}[X, 1/X]$:

$$a_0g(X) = X^n f\left(\frac{-a_0a_1^{-1}}{X}\right). \quad (*)$$

Note that $\delta \in \mathbf{A}_{g_1}^\times$. Let $\gamma = -a_0(a_1\delta)^{-1} \in \mathfrak{m}\mathbf{A}_{g_1}$. Applying $(*)$ we get $-a_0g(\delta) = \delta^n f(\gamma)$, so $f(\gamma) = 0$. Moreover $f'(\gamma) \in \mathbf{A}_{g_1}^\times$ because $f'(0) \in \mathbf{A}^\times$.

Uniqueness is straightforward without the need to assume that f be monic. \square

Equality to 0 in the henselisation of a discrete valued field

Consider now the discrete valued field (\mathbf{K}, \mathbf{V}) and let $\mathbf{A} = \mathbf{V}$. Consider the natural morphism $\theta: \mathbf{V}_{g_1} \rightarrow \mathbf{V}^{\text{H}}$. Suppose $(\mathbf{K}^{\text{ac}}, \mathbf{V}^{\text{ac}})$ is an extension of (\mathbf{K}, \mathbf{V}) to an algebraic closure.³ The advantage of considering a special polynomial is that all its zeros in \mathbf{K}^{ac} , with the exception of the special zero, are in $\text{Rad}(\mathbf{V}^{\text{ac}}) = \mathfrak{m}\mathbf{V}^{\text{ac}}$.⁴

Consider $\alpha = p(\mu) \in \mathbf{V}_{g_1}$ for $p \in \mathbf{V}[X]$ and its image $\alpha' = p(\mu')$, where μ' is the Hensel zero of g_1 in \mathbf{V}^{H} .⁵ Kuhlmann and Lombardi (2000, beginning of the proof of proposition 2.3) explain how to test if $\alpha' = 0$. We calculate the characteristic polynomial $q(T) \in \mathbf{V}[T]$ of $p(x)$ in the \mathbf{K} -algebra $\mathbf{K}[x] = \mathbf{K}_{[g_1]}$. The zeros of q in an algebraic closure of \mathbf{K} are the $\alpha'_i = p(\mu'_i)$ where the μ'_i are the zeros of g_1 (let $\mu'_1 = \mu'$ and $\alpha'_1 = \alpha'$). If $q(0) \neq 0$, $\alpha' \neq 0$ because $q(\alpha') = 0$.⁶ If $q(0) = 0$ we still don't know whether $\alpha' = 0$. We then calculate the characteristic polynomial $r(T) \in \mathbf{V}[T]$ of $xp(x)$ in the \mathbf{K} -algebra $\mathbf{K}[x]$. The μ'_i for $i \geq 2$ are units, their valuation is zero. The zeros of r in \mathbf{K}^{ac} are the $\mu'_i \alpha'_i$. The valuation of the zeros of r is therefore the same as that of the zeros of q , except in general the valuation of $\alpha' = \alpha'_1$ which increases by $v(\mu')$ when $v(\alpha')$ is finite, i.e. when $\alpha' \neq 0$. On the other hand $\infty + v(\mu') = \infty$ and we therefore obtain that $\alpha' = 0$ if, and only if, the valuation of the zeros has not changed. The valuation of the zeros of a polynomial is given by Newton's polygon method. We therefore have an explicit test for $\alpha' = 0$.

The Newton polygon of a polynomial $q \in \mathbf{V}[X]$ with respect to v also provides information about its zeros in extensions. An “isolated slope” of this polygon corresponds to a *v-isolated* zero in \mathbf{K}^{H} ; Kuhlmann and Lombardi (2000, Proposition 2.2) give a precise description of this zero which ends as follows. In summary, a zero α corresponding to an isolated slope of a Newton polygon can always be made explicit either as an element of \mathbf{K} , or in the form $(a\delta + b)/(c\delta + d)$, where δ is the special zero of a special polynomial $g(X)$, $a, b, c, d \in V$, $(c\delta + d) \neq 0$ and $(ad - bc) \neq 0$.

Remark 3. In the following we use the fact that, since \mathbf{V} is integrally closed, \mathbf{V}_{g_1} has no zerodivisor (Coquand and Lombardi 2016, Theorem 6.3).⁷ In the end, we deduce that \mathbf{V}_{g_1} not only has no zerodivisor but is in fact integral. In the appendix, we give two independent constructive proofs of the fact that \mathbf{V}_{g_1} is integral and

³A constructive use of the purely ideal structure $(\mathbf{K}^{\text{ac}}, \mathbf{V}^{\text{ac}})$ is obtained by considering the dynamic algebraic structure defined by adding the positive diagram of $(\mathbf{A}, \mathfrak{m})$ to the dynamic theory of algebraically closed discrete valued fields. This application of the dynamic method is explained in more detail in Kuhlmann and Lombardi 2000.

⁴This last equality follows from Kuhlmann and Lombardi 2000.

⁵This is not restrictive, as we want to test only equality to 0.

⁶Then $\alpha \neq 0$ because $\alpha = 0$ implies $\alpha' = 0$. Note that here, at this stage of the proof, $\alpha \neq 0$ simply means $\neg(\alpha = 0)$, as we do not yet know whether we have a zero test in \mathbf{V}^{H} .

⁷By Theorem 6.3, the ring $\mathbf{V}_{\{f\}}$ is normal. \mathbf{V}_{g_1} is a localisation of $\mathbf{V}_{\{f\}}$, so that it is also normal, and as it is local it has no zerodivisor.

integrally closed. ■

We shall also need the following observation.

Lemma 4. *If $\eta \in \mathbf{K}[\delta]$, where δ is a special zero of a special polynomial, there exists an exponent m such that $\delta^m \eta$ is a v -isolated zero of a polynomial $Q \in \mathbf{V}[X]$.*

Proof. Result given in [Kuhlmann and Lombardi 2000](#), proof of proposition 2.3. □

Proof of Theorem 1

This paragraph proves the desired isomorphism. We keep the above notation.

Injectivity

For injectivity, it is sufficient to prove it for an elementary step in the construction of \mathbf{V}^h described in [Alonso García, Lombardi, and Perdry 2008](#).

In the present case with $\mathbf{A} = \mathbf{V}$ we obtain in particular a unique \mathbf{V} -local morphism $\theta: \mathbf{V}_{g_1} \rightarrow \mathbf{V}^H$. It sends the Hensel zero μ of g_1 in \mathbf{V}_{g_1} to the Hensel zero μ' of g_1 in \mathbf{V}^H and $\alpha = p(\mu)$ to $\alpha' = p(\mu')$.

We have to show that θ is injective, i.e. to show that $\alpha' = 0 \implies \alpha = 0$.

If $q(0) \neq 0$ then $\alpha' \neq 0$ therefore $\neg(\alpha = 0)$. Otherwise let's write $q(T) = T^m q_1(T)$ with $m > 0$ and $q_1(0) \neq 0$. If $\alpha' = 0$, then $q_1(\alpha') \neq 0$. This implies $q_1(\alpha) \neq 0$, and as $q(\alpha) = \alpha^m q_1(\alpha) = 0$, we get $\alpha = 0$. Now we know that the natural morphism $\theta: \mathbf{V}_{g_1} \rightarrow \mathbf{V}^H$ is injective and that \mathbf{V}_{g_1} itself is integral.

Surjectivity

In the following we can therefore identify any ring \mathbf{V}_f with its image in \mathbf{V}^h , and \mathbf{V}^h with its image in \mathbf{V}^H . To ensure that φ is surjective, it suffices to show that for any Hensel zero $\mu \in \mathbf{K}^H$ of a polynomial $g_1 \in \mathbf{V}[X]$, any element η of

$$\mathbf{W} = \{ \eta \in \mathbf{K}[\mu] \mid v(\eta) \geq 0 \} \subseteq \mathbf{V}^H$$

is in the image of \mathbf{V}^h .

Note that this implies that \mathbf{K}^H is the field of fractions of \mathbf{V}^h . However, this does not tell us that $\mathbf{V}^H \subseteq \mathbf{V}^h$. To obtain this inclusion we must use Lemma 4. It shows that $\delta^m \eta$ is in the image of a certain $\mathbf{V}_u \subseteq \mathbf{V}^h$. As $1/\delta$ is also in \mathbf{V}^h , this shows that $\eta \in \mathbf{V}^h$. This completes the proof of the surjectivity of φ .

We have completed the proof of the fact that the canonical morphism is an isomorphism. We now turn to the appendices.

Appendix 1

Here is another constructive proof, different from the one given in the above proof, for the fact that \mathbf{V}_f is an integrally closed ring.

Alonso García, Lombardi, and Neuwirth (2021) consider the situation of a Hensel polynomial f for a residually discrete local ring $(\mathbf{A}, \mathfrak{m}_{\mathbf{A}})$ when \mathbf{A} is integral. The assumption is that $(\mathbf{A}, \mathfrak{m}_{\mathbf{A}})$ is an integral local ring dominated by a valuation ring \mathbf{V} of the field of fractions \mathbf{K} of \mathbf{A} (so $\mathfrak{m}_{\mathbf{A}} = \mathbf{A} \cap \mathfrak{m}_{\mathbf{V}}$). We consider a step of the construction of each of the henselisations \mathbf{A}^h and \mathbf{V}^H . We thus have a residually discrete local ring $\mathbf{A}_f \subseteq \mathbf{A}^h$ obtained by adding the Hensel zero β of f and a valuation ring \mathbf{W} extension of \mathbf{V} obtained by adding the Hensel zero β' of f .

Precisely $\mathbf{W} = \{ \zeta \in \mathbf{K}[\beta'] \mid v(\zeta) \geq 0 \}$.⁸ Consider the morphism $\theta_f: \mathbf{A}_f \rightarrow \mathbf{W}$ which extends the inclusion morphism $\mathbf{A} \rightarrow \mathbf{V}$. The paper shows that $\text{Ker}(\theta_f)$ is a *minimal detachable prime ideal* of \mathbf{A}_f . We apply this result to the present situation where $\mathbf{A} = \mathbf{V}$ is self-dominated: the kernel $\text{Ker}(\theta_f)$ of the natural morphism $\theta_f: \mathbf{V}_f \rightarrow \mathbf{W}$ is a minimal detachable prime ideal.

\mathbf{V}_f is a residually discrete local ring, localisation of $\mathbf{V}_{\{f\}}$, which is normal by Theorem 6.3 because \mathbf{V} is a valuation domain. So \mathbf{V}_f is a local normal ring and therefore has no zerodivisor: this gives the result used in the previous proof. In other words the ideal $\{0\}$ of \mathbf{V}_f is a prime ideal (in the sense that the quotient ring has no zerodivisors). We conclude by saying that since $\text{Ker}(\theta_f)$ is a minimal detachable prime ideal and $\{0\}$ is a prime ideal, $\{0\}$ is detachable and \mathbf{V}_f is integral.

Appendix 2

We now give a third, more direct, proof for the fact that the ring $\mathbf{V}_f \subseteq \mathbf{V}^h$ is integrally closed. The calculations underlying the three proofs are undoubtedly more similar than is apparent at first glance.

In fact Lemma 11 below and the analogous Theorem 6.3 in Coquand and Lombardi 2016 are both based on the formula of Tate⁹ in Lemma 5.

Lemma 5 (Tate). *Let \mathbf{A} be a commutative ring and $f \in \mathbf{A}[X]$ a monic polynomial of degree d . Let us write*

$$f(X) = f(Y) + (X - Y)g(X, Y) \text{ with } g(X, Y) = \sum_{j=0}^{d-1} g_j(Y)X^j$$

⁸ $\mathbf{K}[\beta']$ is constructed as a discrete valued field according to a procedure specific to discrete valued fields. It is a finite separable extension of \mathbf{K} , but its dimension as a \mathbf{K} -vector space is not generally known. A similar phenomenon occurs when we construct the real closure of a discrete ordered field.

⁹A classical result is that if a ring \mathbf{A} is a normal domain (i.e. integrally closed), the same applies to $\mathbf{A}[X]$. A difficulty from an algorithmic point of view is that the usual proof of the previous result no longer works when we assume only that \mathbf{A} has no zerodivisor and not that it is integral. This was the primary motivation for the article Coquand and Lombardi 2016.

and let us denote by $\text{tr}: \mathbf{A}_{[f]} \rightarrow \mathbf{A}$ the trace form of the extension $\mathbf{A}_{[f]}$. Then we have, for all $b \in \mathbf{A}_{[f]} = \mathbf{A}[x]$,

$$f'(x)b = \sum_{j=0}^{d-1} \text{tr}(g_j(x)b) x^j. \quad (1)$$

Note that $b, f'(x), g_j(x) \in \mathbf{A}_{[f]}$ and $\text{tr}(g_j(x)b) \in \mathbf{A}$.

Proof. See [Raynaud \(1970, Chapter VII, corollary page 74\)](#) or [Appendix 3](#). \square

[Remark 6](#) prepares the corollary which follows.

Remark 6. Let $h \in \mathbf{A}[X]$ be a monic polynomial of degree d and $\Delta = uh + vh' = \pm \text{Res}(h, h')$ be the discriminant of h . Then h' is a unit in $\mathbf{A}_{[h]}$ if, and only if, Δ is a unit in \mathbf{A} ([Lombardi and Quitté 2015, Basic elimination lemma III-7.5](#)). Let us suppose that \mathbf{A} is an integral domain with field of fractions \mathbf{K} and that h is separable over \mathbf{K} . We clearly have $\mathbf{A}_{[h]} \subseteq \mathbf{K}_{[h]}$ (respectively free over \mathbf{A} and \mathbf{K} with basis $1, x, \dots, x^{d-1}$). Since h', v , and Δ are units in $\mathbf{K}_{[h]}$, they are regular in $\mathbf{A}_{[h]}$. We have $\mathbf{K}_{[h]} = \mathbf{K}_{\{h\}} = \mathbf{K}_{[h]}[\frac{1}{\Delta}]$. Furthermore we have $\mathbf{A}_{\{h\}}[\frac{1}{v}] = \mathbf{A}[\frac{1}{\Delta}][h] \subseteq \mathbf{K}_{[h]}$. Thus the natural morphism $\mathbf{A}_{[h]} \rightarrow \mathbf{K}_{[h]}$ factorises into

$$\mathbf{A}_{[h]} \rightarrow \mathbf{A}_{\{h\}} \rightarrow \mathbf{A}_{\{h\}}[\frac{1}{v}] = \mathbf{A}[\frac{1}{\Delta}][h] \rightarrow \mathbf{K}_{[h]} = \mathbf{K}_{\{h\}}, \quad (2)$$

where the morphisms are canonical injections which are localisation morphisms obtained by inverting regular monoids (successively $h'^{\mathbb{N}}$, $v^{\mathbb{N}}$, and the nonzero elements of \mathbf{A}). \blacksquare

Corollary 7. *Let \mathbf{A} be integrally closed and \mathbf{K} its field of fractions. Let h be a separable monic polynomial of $\mathbf{A}[X]$. Then $\mathbf{A}_{\{h\}}$ is normal quasi-integral, with $\mathbf{K}_{[h]} = \mathbf{K}_{\{h\}}$ as total ring of fractions.*

Proof. If $d = \deg h$, $\mathbf{A}_{[h]}$ is free of dimension d over \mathbf{A} and $\mathbf{K}_{[h]} = \mathbf{K} \otimes_{\mathbf{A}} \mathbf{A}_{[h]}$ is an étale \mathbf{K} -algebra of dimension d . We therefore have $\mathbf{A}_{[h]} \subseteq \mathbf{K}_{[h]}$; then $\mathbf{A}_{\{h\}} = \mathbf{A}_{[h]}[1/h'] \subseteq \mathbf{K}_{[h]} = \mathbf{K}_{\{h\}}$. To show that $\mathbf{A}_{\{h\}}$ is integrally closed, we start by showing that any element $b \in \mathbf{K}_{[h]}$ integral over $\mathbf{A}_{[h]}$ is in $\mathbf{A}_{\{h\}}$. Let $b \in \mathbf{K}_{[h]}$ be integral over $\mathbf{A}_{[h]}$. As $\mathbf{A}_{[h]}$ is integral over \mathbf{A} , b is integral over \mathbf{A} . We have $h'b \in \mathbf{A}_{[h]}$ according to [Lemma 5](#) (Equation (1) with h in place of f and \mathbf{K} in place of \mathbf{A}): since b is integral over \mathbf{A} and $h_j(x) \in \mathbf{A}$ for each j , $\text{tr}(h_j(x)b)$ is integral over \mathbf{A} , therefore in \mathbf{A} . Consequently $h'b \in \mathbf{A}_{[h]}$ and $b \in \mathbf{A}_{\{h\}}$.

In particular, since any idempotent of $\mathbf{K}_{[h]}$ is integral over any subring, it belongs to $\mathbf{A}_{\{h\}}$. Let c be an arbitrary element of $\mathbf{A}_{\{h\}} \subseteq \mathbf{K}_{[h]}$. If $e^2 = e$ and $\langle e \rangle = \text{Ann}(c)$ in $\mathbf{K}_{[h]}$, a fortiori $e^2 = e$ and $\langle e \rangle = \text{Ann}(c)$ in $\mathbf{A}_{\{h\}}$. So $\mathbf{A}_{\{h\}}$ is quasi-integral and its total ring of fractions is $\mathbf{K}_{[h]}$. Thus $\mathbf{A}_{\{h\}}$ is quasi-integral and integrally closed in its total ring of fractions, so it is normal. \square

Lemma 8. *Let $f \in \mathbf{A}[X]$ be a monic polynomial. The pairs of complementary idempotents of $\mathbf{A}_{[f]}$ are in bijective correspondence with the factorisations $f = gh$ with g, h monic and $\langle g, h \rangle = \langle 1 \rangle$. For such a factorisation we obtain a natural isomorphism $\mathbf{A}_{[f]} \simeq \mathbf{A}_{[g]} \times \mathbf{A}_{[h]}$.*

Proof. If $gu + hv = 1$ in $\mathbf{A}_{[f]} = \mathbf{A}[X]$, the factorisation corresponds to the idempotents $e = gu$ and $1 - e = hv$. If \mathbf{A} is a discrete field \mathbf{K} , an idempotent $e(x)$ corresponds to the element $g(x)$, where $g(X) = \text{pgcd}(f(X), e(X))$ in $\mathbf{K}[X]$. [Alonso García, Lombardi, and Perdry \(2008\)](#) give a proof in the case of a ring. We will use this lemma with a discrete field only. \square

Lemma 9. *Let \mathbf{K} be a discrete field, $f \in \mathbf{K}[X]$ a monic polynomial, and $b \in \mathbf{K}_{[f]}$. We have a factorisation $f = gh$ as in Lemma 8 with b a unit in $\mathbf{K}_{[h]}$ and nilpotent in $\mathbf{K}_{[g]}$.*

Proof. The algebra $\mathbf{B} := \mathbf{K}_{[f]}$ is strictly finite zerodimensional. In a reduced zerodimensional ring, for any element b there is an idempotent e such that $\langle b \rangle = \langle e \rangle$, b is a unit in $\mathbf{B}[1/e] \simeq \mathbf{B}/\langle 1 - e \rangle$ and nilpotent in $\mathbf{B}/\langle e \rangle$. Apply Lemma 8. \square

Lemma 10. *Let \mathbf{K} be a discrete field and $f \in \mathbf{K}[X]$ be a monic polynomial. Then $\mathbf{K}_{\{f\}} \simeq \mathbf{K}_{\{h\}} = \mathbf{K}_{[h]}$ for a monic separable polynomial h which divides f . In particular the \mathbf{K} -algebra $\mathbf{K}_{\{f\}}$ is étale.*

Proof. Apply Lemma 9 with $b = f'(x)$. We have $f = gh$ in $\mathbf{K}[X]$, $f' = gh' + hg'$, $\mathbf{K}_{[f]} \simeq \mathbf{K}_{[g]} \times \mathbf{K}_{[h]}$. The \mathbf{K} -algebra $\mathbf{K}_{\{f\}}$ is isomorphic to $\mathbf{K}_{[h]}$ because f' is a unit in $\mathbf{K}_{[h]}$ and nilpotent in $\mathbf{K}_{[g]}$. In the \mathbf{K} -algebra $\mathbf{K}_{[h]}$, g and $f' = h'g$ are units, so h' is also a unit and h is separable over \mathbf{K} . Thus $\mathbf{K}_{\{h\}} = \mathbf{K}_{[h]}$ is an étale \mathbf{K} -algebra. \square

Note that $\mathbf{K}_{\{f\}}$ is zero when f divides a power of f' in $\mathbf{K}[X]$.

Lemma 11. *Let \mathbf{A} be a normal domain and $f \in \mathbf{A}[X]$ a monic polynomial. Then $\mathbf{A}_{\{f\}}$ is a normal quasi-integral ring.*

Proof. Let $\mathbf{K} = \text{Frac } \mathbf{A}$. We apply Lemma 9 with $b = f'$ and Lemma 10, we write $f = gh$ with monic polynomials g and h in $\mathbf{K}[X]$. We have $g, h \in \mathbf{A}[X]$ because \mathbf{A} is integrally closed. Moreover g monic divides a power of f' in $\mathbf{K}[X]$ so in $\mathbf{A}[X]$ (the division takes place entirely in $\mathbf{A}[X]$). So g is a unit in $\mathbf{A}_{\{f\}}$, which implies $h = 0$ in $\mathbf{A}_{\{f\}}$ because $gh = f = 0$. We therefore obtain

$$\mathbf{A}_{\{f\}} = \mathbf{A}_{[gh]}[\frac{1}{f'g}] \simeq \mathbf{A}_{[gh]}[\frac{1}{g}][\frac{1}{h'}] \simeq \mathbf{A}_{[h]}[\frac{1}{g}][\frac{1}{h'}] = \mathbf{A}_{\{h\}}[1/g].$$

Relative to the natural inclusion $\mathbf{A}_{[h]} \rightarrow \mathbf{K}_{[h]}$, the elements h', f' , and g , units in $\mathbf{K}_{[h]}$, are regular in $\mathbf{A}_{[h]}$. We therefore obtain the canonical injective morphisms of localisation at regular monoids shown below:

$$\mathbf{A}_{[h]} \rightarrow \mathbf{A}_{\{h\}} \rightarrow \mathbf{A}_{\{f\}} \rightarrow \mathbf{K}_{[h]} \simeq \mathbf{K}_{\{f\}}.$$

We conclude that $\mathbf{A}_{\{h\}}$ is normal quasi-integral, and therefore also its localisation $\mathbf{A}_{\{f\}} \simeq \mathbf{A}_{\{h\}}[1/g]$. \square

Lemma 12. *Let \mathbf{A} be an integrally closed residually discrete local ring and $f \in \mathbf{A}[X]$ a monic polynomial. Then \mathbf{A}_f is an integrally closed residually discrete local ring.*

Proof. We know that \mathbf{A}_f is a residually discrete local ring. As a localisation of $\mathbf{A}_{\{f\}}$, it is a normal quasi-integral ring. And every local ring is connected. \square

Appendix 3

In this appendix, we recall a proof of Tate's lemma given in [Coquand and Lombardi 2016](#) which is simpler than the one of [Raynaud \(1970, Chapter VII, §1\)](#).

Proof of Lemma 5. We have $\mathbf{A}_{[f]} = \mathbf{A}[x] = \mathbf{A}[X]/\langle f \rangle$. Let $\mathbf{B} = \mathbf{A}[x_1, \dots, x_d]$ be the universal decomposition algebra of \mathbf{A} for f , where $f(X) = \prod_{i=1}^d (X - x_i)$ ([Lombardi and Quitté 2015, Definition III-4.1](#)). Each $\mathbf{A}[x_i] \subseteq \mathbf{B}$ is isomorphic to $\mathbf{A}[x]$ and we can take $x = x_1$. We write $b = h(x) \in \mathbf{A}_{[f]}$. For all $\ell(x) \in \mathbf{A}_{[f]}$ we have $\text{tr}(\ell(x)) = \sum_{i=1}^d \ell(x_i) \in \mathbf{A}$.¹⁰ We have $g(Y, Y) = f'(Y)$ and $g(x_1, x_i) = g(x_i, x_1) = 0$ if $i \neq 1$: in fact $g(X, x_1) = \prod_{i=2}^d (X - x_i)$ because $(X - x_1)g(X, x_1) = f(X) - f(x_1) = f(X) - \prod_{i=1}^d (X - x_i)$. So

$$f'(x_1)b = g(x_1, x_1)h(x_1) = \sum_{j=0}^{d-1} g_j(x_1)h(x_1)x_1^j$$

and for $i \neq 1$

$$0 = g(x_1, x_i)h(x_i) = \sum_{j=0}^{d-1} g_j(x_i)h(x_i)x_1^j.$$

Summation gives the Tate formula announced. \square

References

María Emilia Alonso García, Henri Lombardi, and Hervé Perdry. Elementary constructive theory of Henselian local rings. *Math. Log. Q.*, 54:253–271, 2008. doi:[10.1002/malq.200710057](#). 3, 4, 6, 9

María Emilia Alonso García, Henri Lombardi, and Stefan Neuwirth. On a theorem by de Felipe and Teissier about the comparison of two henselisations in the non-Noetherian case. *J. Algebra*, 570:587–594, 2021. doi:[10.1016/j.jalgebra.2020.11.020](#). 7

¹⁰See for example [Lombardi and Quitté \(2015, Lemma III-5.12\)](#).

- Errett Bishop. *Foundations of constructive analysis*. McGraw-Hill, New York, 1967. [1](#)
- Errett Bishop and Douglas Bridges. *Constructive analysis*. Grundlehren der mathematischen Wissenschaften, 279. Springer, Berlin, 1985. doi:[10.1007/978-3-642-61667-9](#). [1](#)
- Douglas Bridges and Fred Richman. *Varieties of constructive mathematics*. London Mathematical Society Lecture Note Series, 97. Cambridge University Press, Cambridge, 1987. doi:[10.1017/CBO9780511565663](#). [1](#)
- Thierry Coquand and Henri Lombardi. Some remarks about normal rings. In Dieter Probst and Peter Schuster, editors, *Concepts of proof in mathematics, philosophy, and computer science*, Ontos Mathematical Logic, 6, pages 141–149. De Gruyter, Berlin, 2016. doi:[10.1515/9781501502620-008](#). [3](#), [5](#), [7](#), [10](#)
- Franz-Viktor Kuhlmann and Henri Lombardi. Construction du hensélisé d'un corps valué. *J. Algebra*, 228:624–632, 2000. doi:[10.1006/jabr.2000.8289](#). [3](#), [5](#), [6](#)
- Henri Lombardi and Claude Quitté. *Commutative algebra: constructive methods. Finite projective modules*. Algebra and applications, 20. Springer, Dordrecht, 2015. URL <https://arxiv.org/abs/1605.04832>. Translated from the French (Calvage & Mounet, Paris, 2011, revised and extended by the authors) by Tania K. Roblot. [1](#), [2](#), [8](#), [10](#)
- Ray Mines, Fred Richman, and Wim Ruitenburg. *A course in constructive algebra*. Universitext. Springer, New York, 1988. doi:[10.1007/978-1-4419-8640-5](#). [1](#)
- Michel Raynaud. *Anneaux locaux henséliens*. Lecture Notes in Mathematics, 169. Springer, Berlin, 1970. doi:[10.1007/BFb0069571](#). [8](#), [10](#)
- Ihsen Yengui. *Constructive commutative algebra: projective modules over polynomial rings and dynamical Gröbner bases*. Lecture Notes in Mathematics, 2138. Springer, Cham, 2015. doi:[10.1007/978-3-319-19494-3](#). [1](#)