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LARGE DEVIATIONS OF THE EMPIRICAL MEASURES OF A STRONG-FELLER MARKOV PROCESS INSIDE A SUBSET AND QUASI-ERGODIC DISTRIBUTION

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ABSTRACT. In this work, we establish, for a strong Feller process, the large deviation principle for the occupation measure conditioned not to exit a given subregion. The rate function vanishes only at a unique measure, which is the so-called quasi-ergodic distribution of the process in this subregion. In addition, we show that the rate function is the Dirichlet form in the particular case when the process is reversible. We apply our results to several stochastic processes such as the solutions of elliptic stochastic differential equations driven by a rotationally invariant α -stable process, the kinetic Langevin process, and the overdamped Langevin process driven by a Brownian motion.

Dedicated to Patrick Cattiaux

AMS 2020 Subject classifications.

Key words and Phrases. Large deviation principle, quasi-stationary distribution, quasi-ergodic distribution, killed Markov process, stable and kinetic processes.

1. INTRODUCTION

1.1. Setting and purpose of this work. Let $(X_t)_{t \geq 0}$ be a *càdlàg* Markov process valued in a Polish space \mathcal{S} , defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in E})$, where \mathbb{P}_x means that $\mathbb{P}_x(X_0 = x) = 1$ for each $x \in \mathcal{S}$. Given a nonempty open subset \mathcal{D} of \mathcal{S} , consider the first exiting time of the process from \mathcal{D} :

$$\sigma_{\mathcal{D}} := \inf\{t \geq 0, X_t \in \mathcal{D}^c\}. \quad (1.1)$$

A natural question in population processes [20, 52, 12] and in metastability in molecular dynamics [49, 28, 29, 47], is to investigate the long time behavior of the law of the process $(X_t)_{t \geq 0}$ conditioned to stay inside \mathcal{D} , i.e. to study the quantity

$$\lim_{t \rightarrow +\infty} \mathbb{P}_{\nu}[X_t \in \cdot \mid t < \sigma_{\mathcal{D}}],$$

where ν is a given initial distribution on \mathcal{D} and $\mathbb{P}_{\nu}(\cdot) = \int_{\mathcal{D}} \mathbb{P}_x(\cdot) \nu(dx)$ (under \mathbb{P}_{ν} , the distribution of X_0 is ν). Intuitively the limit distribution $\mu_{\mathcal{D}}$ should satisfy

$$\mu_{\mathcal{D}}(\cdot) = \mathbb{P}_{\mu_{\mathcal{D}}}[X_t \in \cdot \mid t < \sigma_{\mathcal{D}}], \quad \forall t > 0.$$

This is exactly the definition of the *quasi-stationary distribution* (q.s.d. in short) of the process in \mathcal{D} , see e.g. [20]. Considering the killed semigroup

$$P_t^{\mathcal{D}}(x, A) := \mathbb{P}_x[X_t \in A, t < \sigma_{\mathcal{D}}], \quad \forall A \in \mathcal{B}(\mathcal{S}), \quad (1.2)$$

where $\mathcal{B}(\mathcal{S})$ is the Borel σ -field of \mathcal{S} , then $\mu_{\mathcal{D}}$ is a q.s.d. if and only if

$$\mu_{\mathcal{D}} P_t^{\mathcal{D}} = \lambda(t) \mu_{\mathcal{D}}, \quad \lambda(t) = \mathbb{P}_{\mu_{\mathcal{D}}}(t < \sigma_{\mathcal{D}}),$$

i.e. $\mu_{\mathcal{D}}$ is a positive left eigen-measure of the killed Dirichlet semigroup $P_t^{\mathcal{D}}$. In our previous work [35], we gave a quite general framework for the existence, uniqueness of $\mu_{\mathcal{D}}$, as well as for the exponential convergence of $\mathbb{P}_{\nu}[X_t \in \cdot | t < \sigma_{\mathcal{D}}]$ to $\mu_{\mathcal{D}}$ as $t \rightarrow +\infty$. From a statistical point of view, it is also very natural to consider the limit behavior of the conditional distribution $\mathbb{P}_{\nu}[L_t \in \cdot | t < \sigma_{\mathcal{D}}]$ of the empirical distributions (or occupation measures)

$$L_t = \frac{1}{t} \int_0^t \delta_{X_s} dx \quad (1.3)$$

as $t \rightarrow +\infty$, where δ_x is the Dirac measure at the point x . Quite curiously, the empirical distribution L_t , knowing that $\{t < \sigma_{\mathcal{D}}\}$, will not converge to the q.s.d. $\mu_{\mathcal{D}}$ (unlike in the case where $\mathcal{D} = \mathcal{S}$ if $(X_t, t \geq 0)$ is ergodic), but to the so-called *quasi-ergodic distribution* (q.e.d. in short) $\pi_{\mathcal{D}}$ which is defined by

$$\pi_{\mathcal{D}} = \varphi \mu_{\mathcal{D}}$$

where φ is the right positive eigenfunction satisfying $P_t^{\mathcal{D}}\varphi = \lambda(t)\varphi$ and $\mu_{\mathcal{D}}(\varphi) = \int_{\mathcal{S}} \varphi d\mu_{\mathcal{D}} = 1$.

The purpose of this work is to establish the large deviation principle (L.D.P. in short) of $\mathbb{P}_{\nu}[L_t \in \cdot | t < \sigma_{\mathcal{D}}]$ with some rate function $I_{\mathcal{D}}$ which vanishes only at a unique measure which is the q.e.d. $\pi_{\mathcal{D}}$. This gives quite precise information about how it approaches to the q.e.d. $\pi_{\mathcal{D}}$ (for instance exponentially fast in probability, see Corollary 1).

We should emphasize that the true history is much more delicate than what said roughly above. Indeed $(\lambda(t), \mu_{\mathcal{D}}, \varphi)$ is in general not unique: even for the one-dimensional Ornstein-Uhlenbeck process with $\mathcal{S} = \mathbb{R}$, $\mathcal{D} = (0, +\infty)$, the uniqueness of the q.s.d. fails, see [50]. Our work consists to find a rich family of initial distribution ν (covering at least all Dirac measures $\delta_x, x \in \mathcal{D}$) so that the intuitive picture above holds. As the rate function $I_{\mathcal{D}}$ of the L.D.P. of L_t is usually interpreted as some (minus)-entropy functional, the q.e.d. $\pi_{\mathcal{D}}$ where $I_{\mathcal{D}}$ vanishes can be understood as the quasi-equilibrium of maximal entropy.

1.2. Organization of this work. This work is organized as follows. In Section 2.1, we introduce the conditions we will impose on the process $(X_t, t \geq 0)$ which are mainly those of [35] adapted to the killed Feynman-Kac semigroups we consider. We then give in Section 2.2 the main result of this work which is Theorem 2 about the large deviations of $\mathbb{P}_{\nu}[L_t \in \cdot | t < \sigma_{\mathcal{D}}]$ on $\mathcal{P}(\mathcal{D})$ equipped with the τ -topology (stronger than the usual weak convergence topology), see also Theorem 1 about the spectral gap of the killed Feynman-Kac semigroup. We also provide the identification of the rate function in the reversible case, see Corollary 2. Section 3 is devoted to the proof of Theorem 1, and we prove Theorem 2 in Section 4. In Section 5 we prove the identification of the rate function in the reversible case and finally provides examples in Section 6.

2. MAIN RESULTS

2.1. Framework: notations and assumptions. Let $(X_t, t \geq 0)$ be a time homogeneous Markov process valued in a metric complete separable (say Polish) space \mathcal{S} , with càdlàg paths and satisfying the strong Markov property, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, (\mathbb{P}_x)_{x \in \mathcal{S}})$ where $\mathbb{P}_x[X_0 = x] = 1$ for all $x \in \mathcal{S}$ (and where the filtration satisfies the usual condition). Let $\mathcal{B}(\mathcal{S})$ be the Borel σ -algebra of \mathcal{S} , $b\mathcal{B}(\mathcal{S})$

the Banach space of all bounded and Borel measurable functions $f : \mathcal{S} \rightarrow \mathbb{R}$ equipped with the sup-norm

$$\|f\| = \sup_{x \in \mathcal{S}} |f(x)|.$$

We also denote by $\mathcal{C}_b(\mathcal{S})$ the space of bounded continuous real-valued functions over \mathcal{S} . Given an initial distribution ν on \mathcal{S} , we write $\mathbb{P}_\nu(\cdot) = \int_{\mathcal{S}} \mathbb{P}_x(\cdot) \nu(dx)$. For $\mathcal{A} \subset \mathcal{S}$, we denote by $\mathbf{1}_{\mathcal{A}}$ the indicator function of \mathcal{A} . The transition probability semigroup of $(X_t, t \geq 0)$ is denoted by $(P_t, t \geq 0)$. We say that P_t is strong Feller if $P_t f$ is continuous on \mathcal{S} for any $f \in b\mathcal{B}(\mathcal{S})$. We denote by $\mathcal{P}(\mathcal{S})$ the space of probability measures on \mathcal{S} . For any measure ν , transition kernel $P(x, dy)$, and function f on $(\mathcal{S}, \mathcal{B}(\mathcal{S}))$, we write

$$\nu(f) = \langle \nu, f \rangle = \int_{\mathcal{S}} f d\nu, \quad Pf(x) = \int_{\mathcal{S}} f(y) P(x, dy), \quad \text{and } (\nu P)(f) = \nu(Pf).$$

We recall that the space $\mathbb{D}([0, T], \mathcal{S})$ of \mathcal{S} -valued càdlàg paths defined on $[0, T]$, equipped with the Skorokhod topology, is a Polish space, see e.g. [4].

For a continuous time Markov process, often what is given is its generator \mathfrak{L} , not its transition semigroup $(P_t, t \geq 0)$, which is unknown in general. We say that a continuous function f belongs to the extended domain $\mathbb{D}_e(\mathfrak{L})$ of \mathfrak{L} , if there is some measurable function g on \mathcal{S} such that $\int_0^t |g|(X_s) ds < +\infty$, \mathbb{P}_x -a.e. for all $x \in \mathcal{S}$ and

$$M_t(f) := f(X_t) - f(X_0) - \int_0^t g(X_s) ds \tag{2.1}$$

is a \mathbb{P}_x -local martingale for all $x \in \mathcal{S}$. Such a function g , denoted by $\mathfrak{L}f$, is not unique in general. But it is unique up to the equivalence of *quasi-everywhere* (q.e.) that we recall: two functions g_1, g_2 are said to be equal q.e., if $g_1 = g_2$ almost everywhere in the (resolvent) measure $R_1(x, \cdot) = \int_0^{+\infty} e^{-t} P_t(x, \cdot) dt$ for every $x \in \mathcal{S}$.

We will work in the following framework, which is slightly different from our previous work [35]. More precisely, we consider the following assumptions on the non-killed process:

- (C1) For each $t > 0$, P_t is strong Feller.
- (C2) For every $T > 0$, $x \mapsto \mathbb{P}_x(X_{[0, T]} \in \cdot)$ is continuous from \mathcal{S} to the space $\mathcal{P}(\mathbb{D}([0, T], \mathcal{S}))$ equipped with the weak convergence topology.
- (C3) There exist a continuous function $\mathbf{W} : \mathcal{S} \rightarrow [1, +\infty[$, belonging to the extended domain $\mathbb{D}_e(\mathfrak{L})$, two sequences of positive constants (r_n) and (b_n) where $r_n \rightarrow +\infty$, and an increasing sequence of compact subsets (K_n) of \mathcal{S} and some constant $p > 1$, such that

$$-\mathfrak{L}\mathbf{W}^p(x) \geq r_n \mathbf{W}^p(x) - b_n \mathbf{1}_{K_n}(x), \quad q.e.$$

Let us now introduce the setting for the killed process. Let \mathcal{D} be an non empty and open subset of \mathcal{S} and $\sigma_{\mathcal{D}}$ be the first exiting time of \mathcal{D} defined in (1.1). The transition semigroup of the killed process $(X_t, 0 \leq t < \sigma_{\mathcal{D}})$ is given by (1.2), or equivalently for $t \geq 0$ and $x \in \mathcal{D}$,

$$P_t^{\mathcal{D}} f(x) = \mathbb{E}_x[\mathbf{1}_{t < \sigma_{\mathcal{D}}} f(X_t)], \tag{2.2}$$

for $f \in b\mathcal{B}(\mathcal{D})$. This semigroup is often called the Dirichlet semigroup.

We now turn to the conditions on the Dirichlet semigroup $(P_t^{\mathcal{D}}, t \geq 0)$.

- (C4) For a measure-separable¹ class \mathcal{C} of bounded and continuous functions on \mathcal{D} , $P_t^\mathcal{D} f$ is continuous on \mathcal{D} for any $f \in \mathcal{C}$.
- (C5) There exists $t_0 > 0$ such that for all $t \geq t_0$, for all $x \in \mathcal{D}$ and nonempty open subsets O of \mathcal{D} , $P_t^\mathcal{D}(x, O) > 0$. Moreover, there exists $x_0 \in \mathcal{D}$ such that $\mathbb{P}_{x_0}[\sigma_\mathcal{D} < +\infty] > 0$.

Notice that (C1), (C3) are slightly stronger than [35, (C1), (C3)].

Running Example. A prototypical example of reversible dynamics satisfying (C1) \rightarrow (C5) is the solution to the so-called overdamped Langevin process

$$dy_t = \mathbf{c}(y_t)dt + dB_t, \quad (2.3)$$

where $(B_t, t \geq 0)$ is a standard Brownian motion over \mathbb{R}^d . It can indeed be checked that that when $\mathbf{c} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ is locally Lipschitz such that

$$\lim_{|x| \rightarrow +\infty} \mathbf{c}(x) \cdot \frac{x}{|x|} = -\infty, \quad (2.4)$$

the conditions (C1) \rightarrow (C5) are satisfied for the strong solution $(y_t, t \geq 0)$ of (2.3) on any subdomain \mathcal{D} (i.e. non empty, open and connected) of \mathbb{R}^d with the Lyapunov function

$$\mathbf{W}(x) = e^{a|x|(1-\chi(x))},$$

where $a > 0$ and $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ equals 1 in a neighborhood of 0 in \mathbb{R}^d . Such a claim can be proved using e.g. the techniques developed in [34]. We refer to Section 6 for more complicated examples arising from statistical physics of processes satisfying (C1) \rightarrow (C5).

2.2. Main results. In this section, we state the main result of this work, which is Theorem 2 below. Before, we need a result on the spectral gap of the killed (outside \mathcal{D}) Feynman-Kac semigroup of $(X_t, t \geq 0)$, which has its own interest. This is the purpose of Theorem 1 stated in Section 2.2.1. In a nutshell, we need this control on Feynman-Kac semigroup as our approach for large deviations is based on Gärtner-Ellis theorem. It thus relies on a control of a log-Laplace transform which can be recasted in a Feynman-Kac framework.

2.2.1. Spectral gap of Feynman-Kac semigroup on weighted function spaces. In this section we study the existence of the spectral gap of the killed Feynman-Kac semigroup of $(X_t, t \geq 0)$ on a weighted space of measures. In order to introduce our main object of interest, we need to introduce the potential V for which we will suppose throughout the paper without further mention

(H_V) $V \in b\mathcal{B}(\mathcal{D})$, i.e. V is a bounded and measurable function.

The killed (outside \mathcal{D}) Feynman-Kac semigroup is given by

$$P_t^{\mathcal{D}, V} f(x) = \mathbb{E}_x \left[f(X_t) e^{\int_0^t V(X_s) ds} \mathbf{1}_{t < \sigma_\mathcal{D}} \right], f \in b\mathcal{B}(\mathcal{D}), t \geq 0, x \in S. \quad (2.5)$$

Note that the generator of this killed semigroup is (formally) the Schrödinger operator $\mathfrak{L}_{\mathcal{D}, V} = \mathfrak{L}_\mathcal{D} + V$ where $\mathfrak{L}_\mathcal{D}$ is the generator of $(P_t^\mathcal{D}, t \geq 0)$. Note also that $P_t^\mathcal{D} = P_t^{\mathcal{D}, 0}$. We refer to the classical textbook [24] for the theory of Feynman-Kac semigroups (see also [54, 26]).

¹Here measure-separability means: if $\mu(f) = \nu(f)$ for all $f \in \mathcal{C}$, the two positive measures μ, ν on \mathcal{D} are the same.

Under the condition **(C3)**, we will consider $P_t^{\mathcal{D},V}$ as bounded operators on the weighted Banach space $b_{\mathbf{W}}\mathcal{B}(\mathcal{D})$, where $b_{\mathbf{W}}\mathcal{B}(\mathcal{D})$ is defined by:

$$b_{\mathbf{W}}\mathcal{B}(\mathcal{D}) := \left\{ f : \mathcal{D} \rightarrow \mathbb{R} \text{ measurable s.t. } \|f\|_{\mathbf{W}} := \sup_{x \in \mathcal{D}} \frac{|f(x)|}{\mathbf{W}(x)} < +\infty \right\}. \quad (2.6)$$

Indeed, by [35, Proposition 5.1] and **(C3)**, we have using also the fact that $\mathbf{W} \geq 1$:

$$\begin{aligned} \mathfrak{L}\mathbf{W} &= \mathfrak{L}(\mathbf{W}^p)^{1/p} \leq \frac{1}{p}(\mathbf{W}^p)^{\frac{1}{p}-1} \mathfrak{L}\mathbf{W}^p \leq \frac{1}{p} \mathbf{W}^{1-p} (-r_n \mathbf{W}^p - b_n \mathbf{1}_{K_n}) \\ &\leq -r_n^* \mathbf{W} + b_n^* \mathbf{1}_{K_n} \end{aligned}$$

where $r_n^* = r_n/p$ and $b_n^* = b_n/p$. Consequently, $e^{-b_n^* t} \mathbf{W}(X_t)$ is a supermartingale. Hence, one has that:

$$P_t^{\mathcal{D}} \mathbf{W} \leq e^{b_n^* t} \mathbf{W},$$

and then that:

$$P_t^{\mathcal{D},V} \mathbf{W}(x) \leq e^{\|V\|t} P_t^{\mathcal{D}} \mathbf{W}(x) \leq e^{(b_n^* + \|V\|)t} \mathbf{W}(x).$$

In conclusion, we have that

$$\|P_t^{\mathcal{D},V}\|_{\mathbf{W}} := \sup\{\|P_t^{\mathcal{D},V} f\|_{\mathbf{W}}, \|f\|_{\mathbf{W}} \leq 1\} \leq e^{(b_n^* + \|V\|)t}.$$

Large deviations of $\mathbb{P}_\nu[L_t \in \cdot | t < \sigma_{\mathcal{D}}]$ are closely related to the spectral properties of the killed Feynman-Kac semigroup $(P_t^{\mathcal{D},V}, t \geq 0)$, see indeed [27] and [59]. For this reason, we will need following result about the spectral gap of the Feynman-Kac semigroup $P_t^{\mathcal{D},V}$ on $b_{\mathbf{W}}\mathcal{B}(\mathcal{D})$, which generalizes [35, Theorem 2.2] from the case $V \equiv 0$ to general real-valued bounded V .

Theorem 1. *Assume **(C1)** \rightarrow **(C5)** and **(H_V)**. For any given bounded potential $V \in b\mathcal{B}(\mathcal{D})$, consider the log spectral radius of $P_1^{\mathcal{D},V}$ on $b_{\mathbf{W}}\mathcal{B}(\mathcal{D})$:*

$$\Lambda_{\mathcal{D}}(V) = \lim_{t \rightarrow +\infty} \frac{1}{t} \log \|P_t^{\mathcal{D},V}\|_{\mathbf{W}}. \quad (2.7)$$

Then:

1. For any $t > 0$, there is only one probability measure $\mu_{\mathcal{D},V}$ such that $\mu P_t^{\mathcal{D},V} = c(t) \mu_{\mathcal{D},V}$ for some constant $c(t)$ and $\mu_{\mathcal{D},V}(\mathbf{W}) < +\infty$. Moreover, $\forall t > 0$, $c(t) = e^{\Lambda_{\mathcal{D}}(V)t}$, $\Lambda_{\mathcal{D}}(V) < \sup_{\mathcal{D}} V$, and $\mu_{\mathcal{D},V}$ is independent of $t > 0$ and charges all non-empty open subsets O of \mathcal{D} .
2. There is a unique continuous function $\varphi_{\mathcal{D},V}$ bounded by $c\mathbf{W}$ (for some constant $c > 0$) such that $\mu_{\mathcal{D},V}(\varphi_{\mathcal{D},V}) = 1$ and

$$P_t^{\mathcal{D},V} \varphi_{\mathcal{D},V} = e^{\Lambda_{\mathcal{D}}(V)t} \varphi_{\mathcal{D},V} \text{ on } \mathcal{D}, \forall t \geq 0. \quad (2.8)$$

Moreover, $\varphi_{\mathcal{D},V} > 0$ everywhere on \mathcal{D} .

3. There exist $\delta > 0$ and $C \geq 1$ such that for all $f \in b_{\mathbf{W}}\mathcal{B}(D)$ and $t > 0$:

$$|e^{-\Lambda_{\mathcal{D}}(V)t} P_t^{\mathcal{D},V} f - \mu_{\mathcal{D},V}(f) \cdot \varphi_{\mathcal{D},V}| \leq C e^{-\delta t} \|f\|_{\mathbf{W}} \cdot \mathbf{W} \text{ on } \mathcal{D}. \quad (2.9)$$

Note also that Item **3** in Theorem 1 implies that the Feynman-Kac operator $P_t^{\mathcal{D},V}$ ($t > 0$) on $b_{\mathbf{W}}\mathcal{B}(D)$ has a spectral gap near its spectral radius $e^{\Lambda_{\mathcal{D}}(V)t}$ and its spectral projection is the mapping $f \mapsto \varphi_{\mathcal{D},V} \mu_{\mathcal{D},V}(f)$, which is one-dimensional. Such a result is of independent interest. Note that $\mu_{\mathcal{D},0}$ is the (unique) q.s.d. of the Markov process $(X_t, t \geq 0)$ in

$$\mathcal{P}_{\mathbf{W}}(\mathcal{D}) := \{\nu \in \mathcal{P}(\mathcal{D}), \nu(\mathbf{W}) < +\infty\}.$$

Remark. The notion of q.s.d. can also be extended to killed renormalized Feynman–Kac semigroup, see e.g. [33, Definition 1]. Item **1** in Theorem 1 implies that $\mu_{\mathcal{D},V}$ is the unique q.s.d. of the killed (in \mathcal{D}) renormalized Feynman–Kac semigroup associated with (2.5) in $\mathcal{P}_{\mathbf{W}}(\mathcal{D})$.

We will prove Theorem 1 in Section 3 using [35, Theorem 3.5 and Theorem 4.1]. To prove the spectral gap of $P_t^{\mathcal{D},V}$ ($t > 0$) on $b_{\mathbf{W}}\mathcal{B}(D)$ we will use the non-compact parameter β_w which was introduced in [61].

2.2.2. Related literature on long time behavior of Feynman-Kac semigroups. Non-killed Feynman-Kac semigroups have been widely studied in the literature and we refer for instance to [23, 7, 43, 18, 57, 39, 37, 16, 17, 2] for the study of these semigroups in L^p spaces, see also [63, 58, 19, 5]. We also mention [32] for a very recent investigation of the long time behavior of non-killed Feynman-Kac semigroups and its numerical approximations, see also [21]. Impossible not to refer to [8, 9] for pioneering works in the case $V = 0$ using ultracontractivity, which is linked in the reversible case to an adapted version of assumption **(C3)**. We also refer to [13, 14, 25] and references therein for general conditions for ergodicity of non conservative Markov semigroups, see also the classical textbook [53]. We finally refer to the recent work [33] where we study the basic properties and the long time behavior of killed Feynman-Kac semigroups of several models, arising from statistical physics, with very general singular Schrödinger potentials. As already mentioned above, the main goal of this work is to derive a L.D.P. for $\mathbb{P}_\nu[L_t \in \cdot | t < \sigma_{\mathcal{D}}]$, this is the purpose of the next section.

2.2.3. Large deviations. The space $\mathcal{P}(\mathcal{S})$ of probability measures on \mathcal{S} equipped with the weak convergence topology is a Polish space, whose Borel σ -field is denoted by $\mathcal{B}(\mathcal{P}(\mathcal{S}))$. We say that a subset B of $\mathcal{P}(\mathcal{S})$ is measurable if $B \in \mathcal{B}(\mathcal{P}(\mathcal{S}))$. The weak convergence topology is written

$$\beta_n \xrightarrow{w} \beta.$$

Notice that for any bounded measurable $V : \mathcal{S} \rightarrow \mathbb{R}$, the functional $\beta \in \mathcal{P}(\mathcal{S}) \rightarrow \beta(V)$ is $\mathcal{B}(\mathcal{P}(\mathcal{S}))$ -measurable, by the regularity of probability measures on the Polish space \mathcal{S} . The empirical distribution L_t given by (1.3) is a random variable valued in $\mathcal{P}(\mathcal{S})$. A probability measure $\beta \in \mathcal{P}(\mathcal{D})$ is identified with the probability measure on \mathcal{S} with $\beta(\mathcal{D}^c) = 0$ (i.e. $\mathcal{P}(\mathcal{D}) \equiv \{\beta \in \mathcal{P}(\mathcal{S}); \beta(\mathcal{D}^c) = 0\} = \{\beta \in \mathcal{P}(\mathcal{S}); \beta(\mathcal{D}) = 1\}$), and a function on \mathcal{D} is identified with the function $\mathbf{1}_{\mathcal{D}}f$ on \mathcal{S} . We consider also on $\mathcal{P}(\mathcal{S})$ the τ -topology $\sigma(\mathcal{P}(\mathcal{S}), b\mathcal{B}(\mathcal{S}))$, i.e. the weakest topology such that $\beta \mapsto \beta(f)$ is continuous for each $f \in b\mathcal{B}(\mathcal{S})$, which is stronger than the weak convergence topology. The τ -convergence is written

$$\beta_n \xrightarrow{\tau} \beta.$$

The main result of this work is the following.

Theorem 2. *Assume **(C1)** \rightarrow **(C5)**, and let $\nu \in \mathcal{P}_{\mathbf{W}}(\mathcal{D})$. Then:*

- A.** *Conditioned to be inside the set \mathcal{D} , $\mathbb{P}_\nu[L_T \in \cdot | T < \sigma_{\mathcal{D}}]$, as T goes to infinity, satisfies the L.D.P. on $\mathcal{P}(\mathcal{D})$ w.r.t. the τ -convergence topology, with speed T and with the rate function*

$$I_{\mathcal{D}}(\beta) = \sup_{V \in b\mathcal{B}(\mathcal{D})} \{\beta(V) - (\Lambda_{\mathcal{D}}(V) - \Lambda_{\mathcal{D}}(0))\}, \quad \beta \in \mathcal{P}(\mathcal{D}). \quad (2.10)$$

More precisely:

- a. The rate function $I_{\mathcal{D}}$ is good or inf-compact, i.e. the level set $\{I_{\mathcal{D}} \leq L\}$ of $I_{\mathcal{D}}$ is compact in $(\mathcal{P}(\mathcal{D}), \tau)$ for any constant $L \in \mathbb{R}^+$;
- b. For any open measurable subset \mathbf{G} of $(\mathcal{P}(\mathcal{S}), \tau)$,

$$\liminf_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}_{\nu} [L_T \in \mathbf{G} | T < \sigma_{\mathcal{D}}] \geq - \inf_{\beta \in \mathbf{G}} I_{\mathcal{D}}(\beta); \quad (2.11)$$

- c. For any closed measurable subset \mathbf{F} of $(\mathcal{P}(\mathcal{D}), \overset{\tau}{\rightarrow})$,

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \mathbb{P}_{\nu} [L_T \in \mathbf{F} | T < \sigma_{\mathcal{D}}] \leq - \inf_{\beta \in \mathbf{F}} I_{\mathcal{D}}(\beta); \quad (2.12)$$

B. Furthermore

$$I_{\mathcal{D}}(\beta) = 0 \iff \beta = \varphi_{\mathcal{D}} \mu_{\mathcal{D}} \quad (2.13)$$

where $\mu_{\mathcal{D}} = \mu_{\mathcal{D},0}$, $\varphi_{\mathcal{D}} = \varphi_{\mathcal{D},0}$ is the right positive eigenfunction of the killed Dirichlet semigroup $P_t^{\mathcal{D}}$ so that $\mu_{\mathcal{D}}(\varphi_{\mathcal{D}}) = 1$. We call $\pi_{\mathcal{D}} := \varphi_{\mathcal{D}} \mu_{\mathcal{D}}$ the q.e.d. of the process in \mathcal{D} .

Recall that in the second item in Theorem 2, $\mu_{\mathcal{D}}$ is the q.s.d. of the Markov process in \mathcal{D} . We give in Section 6 several examples of processes arising from statistical physics satisfying **(C1)** \rightarrow **(C5)**. We then have the following corollary of Theorem 2.

Corollary 1. Assume **(C1)** \rightarrow **(C5)**, and let $\nu \in \mathcal{P}_{\mathbf{W}}(\mathcal{D})$. For any measurable τ -neighborhood \mathcal{N} of the q.e.d. $\pi_{\mathcal{D}} := \varphi_{\mathcal{D}} \mu_{\mathcal{D}}$, there are constants $C, \delta > 0$ such that

$$\mathbb{P}_{\nu} [L_T \notin \mathcal{N} | T < \sigma_{\mathcal{D}}] \leq C e^{-\delta T}, \quad \forall T > 0. \quad (2.14)$$

In other words, it is a conditional law of large numbers: conditioned not to leave \mathcal{D} before time T (i.e. knowing the event $\{T < \sigma_{\mathcal{D}}\}$), the empirical distribution L_T converges to $\pi_{\mathcal{D}}$ (in the τ -topology) exponentially fast in probability as $T \rightarrow +\infty$.

2.2.4. *Donsker-Varadhan entropy functional.* We conjecture that the rate function $I_{\mathcal{D}}(\beta)$ in the L.D.P. above should be

$$I_{\mathcal{D}}(\beta) = J(\beta) - \inf_{\beta \in \mathcal{P}(\mathcal{S}): \beta(\mathcal{D})=1} J(\beta) \quad (2.15)$$

for all $\beta \in \mathcal{P}(\mathcal{D})$, where $J(\beta)$ is the Donsker-Varadhan entropy functional given by

$$J(\beta) = \inf_{\mathbb{Q}} H_{\mathcal{F}_{[0,1]}}(\mathbb{Q} | \mathbb{P}_{\beta}), \quad \forall \beta \in \mathcal{P}(\mathcal{S}). \quad (2.16)$$

Here the infimum is taken over all probability distributions \mathbb{Q} on the path space $\mathbb{D}(\mathbb{R}_+, \mathcal{S})$ so that $X_t(\omega) = \omega(t)$, $\omega = (t \mapsto \omega(t)) \in \mathbb{D}(\mathbb{R}_+, \mathcal{S})$ is a stationary stochastic processes with the marginal distribution $\mathbb{Q}_0(\cdot) = \mathbb{Q}(X_0 \in \cdot) = \beta$ (given); and $\mathcal{F}_{[0,1]} = \sigma(\omega(t); t \in [0, 1])$; and for the sub σ -field \mathcal{G} , $\mathbb{P}|_{\mathcal{G}}$ is the restriction on \mathcal{G} of probability measure \mathbb{P} , and

$$H_{\mathcal{G}}(\mathbb{Q} | \mathbb{P}) := \begin{cases} \int \log \frac{d\mathbb{Q}|_{\mathcal{G}}}{d\mathbb{P}|_{\mathcal{G}}} d\mathbb{Q}, & \text{if } \mathbb{Q}|_{\mathcal{G}} \ll \mathbb{P}|_{\mathcal{G}} \\ +\infty, & \text{otherwise} \end{cases}$$

is the usual relative entropy of $\mathbb{Q}|_{\mathcal{G}}$ w.r.t. $\mathbb{P}|_{\mathcal{G}}$. In other words $H_{\mathcal{F}_{[0,1]}}(\mathbb{Q} | \mathbb{P}_{\beta})$ is the relative entropy *per unit time* of the stationary process law \mathbb{Q} w.r.t. our Markov process law \mathbb{P}_{β} with the same initial distribution β , and it is the rate function for the level-3 (or path level) large deviations of the Markov process. See [27] or [60] for other variational expressions of $J(\beta)$.

In the general setting of Theorem 2, we can only prove one half of (2.15): $I_{\mathcal{D}}(\beta) \geq J(\beta) + \Lambda_{\mathcal{D}}(0)$, see (4.6). However we can prove it in the reversible case, see indeed the next section.

2.2.5. *The reversible case.* In this section we assume in addition that:

(C6) The semigroup $(P_t, t \geq 0)$ is symmetric on $L^2(\mathcal{S}, \pi)$ where π is the unique invariant probability measure of our Markov process $(X_t, t \geq 0)$, namely for all $t \geq 0$:

$$\langle P_t f, g \rangle_{\pi} = \langle f, P_t g \rangle_{\pi} := \int_{\mathcal{S}} f P_t g d\pi, \quad \forall f, g \in L^2(\mathcal{S}, \pi),$$

i.e. $(X_t, t \geq 0)$ is reversible under \mathbb{P}_{π} .

Corollary 2. *Assume (C1) to (C6). Assume also that $(P_t, t \geq 0)$ is topologically irreducible² on \mathcal{S} . Then, the rate function $I_{\mathcal{D}}$ given by (2.10) has the following expression: for any $\beta \in \mathcal{P}(\mathcal{D})$,*

$$I_{\mathcal{D}}(\beta) + \lambda_{\mathcal{D}} = \begin{cases} \mathcal{E}(\sqrt{h}, \sqrt{h}), & \text{if } \beta = h\pi \ll \pi \\ +\infty, & \text{otherwise,} \end{cases} \quad (2.17)$$

where

$$\mathcal{E}(f, f) = \begin{cases} \langle \sqrt{-\mathfrak{L}}f, \sqrt{-\mathfrak{L}}f \rangle_{\pi}, & \text{if } f \text{ belongs to } \mathbb{D}_{L^2(\mathcal{S}, \pi)}(\sqrt{-\mathfrak{L}}), \\ +\infty, & \text{otherwise,} \end{cases}$$

is the Dirichlet form (where $\mathbb{D}_{L^2(\mathcal{S}, \pi)}(\sqrt{-\mathfrak{L}})$ is the domain of $\sqrt{-\mathfrak{L}}$ in $L^2(\mathcal{S}, \pi)$) and

$$\lambda_{\mathcal{D}} := -\Lambda_{\mathcal{D}}(0) = \inf \left\{ \mathcal{E}(f, f); \int_{\mathcal{S}} f^2 d\pi = 1, f \mathbf{1}_{\mathcal{D}^c} = 0, \pi\text{-a.e.} \right\}$$

is the Dirichlet eigenvalue.

Note indeed the right hand side of (2.17) is the closed expression of the Donsker-Varadhan entropy $J(\beta)$ in the reversible case, and $\lambda_{\mathcal{D}} = \inf_{\{\beta \in \mathcal{P}(\mathcal{S}), \beta(\mathcal{D})=1\}} J(\beta)$.

The large deviation principle for killed Feynman-Kac semigroups of symmetric Markov processes is an open topic³. It has also been considered recently in [42], with additive functional taking into account the jumps of the process, see also [6, 64, 38, 41, 15] and references therein. We also mention [10] for Feynman-Kac representation formulas and L.D.P. of solutions to deterministic models of phenotypic adaptation in the small mutations and large time regime.

Running example. Assumptions **(C1)** to **(C6)** are satisfied for the process solution to (2.3) when (2.4) holds and $\mathbf{c} = -\nabla U$, for some $U : \mathbb{R}^d \rightarrow \mathbb{R}_+$. In this case, the invariant probability measure π is the so-called Gibbs measure $e^{-2U(x)} dx / Z$, where $Z = \int_{\mathbb{R}^d} e^{-2U(y)} dy$.

²I.e. for some $t_0 > 0$ and all $t \geq t_0$, $P_t(x, O) > 0$ for all $x \in \mathcal{S}$ and all non-empty subsets O of \mathcal{S} .

³It could definitely have attracted Patrick Cattiaux, but he was distracted too often by part of the authors to have time to do so.

3. PROOF OF THEOREM 1

Proof of Theorem 1. We mention that the proof Theorem 1 is slightly different from the one we performed in [33] as here V is not assumed to be continuous. Assume **(C1)**→**(C5)** and **(H_V)**. The proof of Theorem 1 is divided into several steps.

Step 1. In this step we prove that the essential spectral radius of $P_t^{\mathcal{D},V}$ on $b\mathbf{W}\mathcal{B}(\mathcal{D})$ is 0. To this end, we use, as in [35, Theorem 3.5 and Theorem 4.1], the non-compact parameters β_w and β_τ (introduced in [61]) of a positive bounded kernel $Q(x, dy)$ over \mathcal{S}

$$\beta_w(Q) := \inf_{K \subset \subset \mathcal{S}} \sup_{x \in \mathcal{S}} Q(x, K^c) \text{ and } \beta_\tau(Q) := \sup \limsup_{(A_n) \rightarrow \emptyset} \sup_{x \in \mathcal{S}} Q(x, A_n) \quad (3.1)$$

where $K \subset \subset \mathcal{S}$ means that K is a compact subset of \mathcal{S} and where the supremum above is taken over all sequences $(A_n) \subset \mathcal{B}(\mathcal{S})$ decreasing to \emptyset .

For any $t > 0$ fixed, set

$$Q_0(x, dy) := \frac{\mathbf{W}(y)}{\mathbf{W}(x)} P_t^{\mathcal{D}}(x, dy) \text{ and } Q(x, dy) = \frac{\mathbf{W}(y)}{\mathbf{W}(x)} P_t^{\mathcal{D},V}(x, dy).$$

By [35, proof of Theorem 3.5],

$$\beta_\tau(\mathbf{1}_K Q_0) = 0, \quad \forall K \subset \subset \mathcal{S}.$$

Since $Q \leq e^{t \sup_x |V|(x)} Q_0$, we have also

$$\beta_\tau(\mathbf{1}_K Q) = 0, \quad \forall K \subset \subset \mathcal{S}.$$

Hence by [61, Theorem 3.5], the essential spectral radius $r_{\text{ess}}(Q|_{b\mathcal{B}})$ of Q on $b\mathcal{B}$ (see [61] for definition), is given by

$$r_{\text{ess}}(Q|_{b\mathcal{B}}) = \inf_{n \geq 1} [\beta_w(Q^n)]^{1/n}. \quad (3.2)$$

By [35, Theorem 3.5], $\beta_w(Q_0) = 0$ and therefore

$$\beta_w(Q) \leq e^{t \sup_x V(x)} \beta_w(Q_0) = 0.$$

That implies, by (3.2),

$$r_{\text{ess}}(P_t^{\mathcal{D},V}|_{b\mathbf{W}\mathcal{B}}) = r_{\text{ess}}(Q|_{b\mathcal{B}}) \leq \beta_w(Q) = 0,$$

which is the desired result.

Step 2. In this step we prove that $P_t^{\mathcal{D},V}$ is strongly Feller. To this end, we show that for any $f \in b\mathcal{B}(\mathcal{D})$, $P_t^{\mathcal{D},V} f$ is continuous on \mathcal{D} . For any $\varepsilon \in (0, t)$ and for any $f \in b\mathcal{B}(\mathcal{D})$, consider for $x \in \mathcal{D}$,

$$Q_\varepsilon f(x) := \mathbb{E}_x \left[\mathbf{1}_{t < \sigma_{\mathcal{D}}} f(X_t) e^{\int_\varepsilon^t V(X_s) ds} \right] = P_\varepsilon^{\mathcal{D}}(P_{t-\varepsilon}^{\mathcal{D},V} f)(x).$$

By [35, Lemma 5.2], $P_\varepsilon^{\mathcal{D}}$ is strong Feller, and thus, the function $Q_\varepsilon f = P_\varepsilon^{\mathcal{D}}(P_{t-\varepsilon}^{\mathcal{D},V} f)$ is continuous. Since on the other hand, it holds:

$$\begin{aligned} & \sup_{x \in \mathcal{D}} |P_t^{\mathcal{D},V} f(x) - Q_\varepsilon f(x)| \\ &= \sup_{x \in \mathcal{D}} \left| \mathbb{E}_x \left[\mathbf{1}_{t < \sigma_{\mathcal{D}}} f(X_t) e^{\int_\varepsilon^t V(X_s) ds} (e^{\int_0^\varepsilon V(X_s) ds} - 1) \right] \right| \\ &\leq (e^{\varepsilon \|V\|} - 1) \|f\| e^{t \|V\|} \end{aligned}$$

which goes to zero as $\varepsilon \rightarrow 0^+$. Hence, we conclude that the function $P_t^{\mathcal{D},V} f$ is continuous on \mathcal{D} .

Step 3. We now conclude the proof of Theorem 1. By the generalized Perron-Frobenius type theorem [35, Theorem 4.1], there is a unique couple (μ, φ) where $\mu \in \mathcal{P}_{\mathbf{W}}(\mathcal{D})$ is a probability measure on \mathcal{D} , charging all non-empty open subsets of \mathcal{D} , and $\varphi \in b_{\mathbf{W}}\mathcal{B}(\mathcal{D})$ is a continuous and everywhere positive function on \mathcal{D} with $\mu(\varphi) = 1$ such that

$$\mu P_1^{\mathcal{D},V} = e^\Lambda \mu, \quad P_1^{\mathcal{D},V} \varphi = e^\Lambda \varphi,$$

where Λ is the spectral radius of $P_1^{\mathcal{D},V}$ and

$$\|e^{-\Lambda n} P_n^{\mathcal{D},V} f - \varphi \mu(f)\|_{\mathbf{W}} \leq C e^{-\delta n} \|f\|_{\mathbf{W}}, \quad \forall f \in b_{\mathbf{W}}\mathcal{B}(\mathcal{D}), \quad n \in \mathbb{N}. \quad (3.3)$$

From this exponential convergence and using the semigroup property, it is quite easy to extend it to whole semigroup $(P_t^{\mathcal{D},V}, t \geq 0)$ (for the details, see [35, proof of Theorem 5.3]), to finally deduce that all the assertions of Theorem 1 hold (with $\Lambda = \Lambda_{\mathcal{D}}(V)$) except the inequality $\Lambda_{\mathcal{D}}(V) < \sup_{\mathcal{D}} V$ which remains to be proved.

Note that $\Lambda_{\mathcal{D}}(V) \leq \sup_{\mathcal{D}} V$ follows from the equality

$$\mathbb{E}_{\mu} \left[e^{\int_0^t V(X_s) dx} \mathbf{1}_{t < \sigma_{\mathcal{D}}} \right] = \mu P_t^{\mathcal{D},V}(\mathbf{1}) = e^{\Lambda_{\mathcal{D}}(V)t} \mu(\mathbf{1}), \quad \forall t \geq 0.$$

In addition, the fact that $\Lambda_{\mathcal{D}}(V) < \sup_{\mathcal{D}} V$ follows from the last condition in **(C5)**. Indeed, if in contrary $\Lambda_{\mathcal{D}}(V) = \sup_{\mathcal{D}} V$, then for all $t \geq 0$,

$$\mathbb{E}_{\mu} [e^{\int_0^t V(X_s) dx} \mathbf{1}_{t < \sigma_{\mathcal{D}}}] = e^{t \sup_{\mathcal{D}} V}.$$

Since μ charges all non empty subset of \mathcal{D} and because the function $x \mapsto e^{\Lambda_{\mathcal{D}}(V)t} \mathbf{1} - P_t^{\mathcal{D},V}(\mathbf{1})$ is non negative and continuous, we get that for all $x \in \mathcal{D}$ and $t \geq 0$,

$$e^{t \sup_{\mathcal{D}} V} = \mathbb{E}_x [e^{\int_0^t V(X_s) dx} \mathbf{1}_{t < \sigma_{\mathcal{D}}}] \leq e^{t \sup_{\mathcal{D}} V} \mathbb{P}_x[t < \sigma_{\mathcal{D}}],$$

so that $\mathbb{P}_x[t < \sigma_{\mathcal{D}}] = 1$, i.e. $\mathbb{P}_x[\sigma_{\mathcal{D}} = +\infty] = 1$, a contradiction with the last condition in **(C5)**. Hence, it holds $\Lambda_{\mathcal{D}}(V) < \sup_{\mathcal{D}} V$. \square

4. PROOF OF THEOREM 2

4.1. A generalized Gärtner-Ellis theorem. Our main tool in the proof of Theorem 2 is the following well known generalized Gärtner-Ellis theorem ([27, 59]) that we recall.

Theorem 3. *Let $(\mathbb{P}_T)_{T>0}$ be a sequence of probability distribution on $\mathcal{P}(\mathcal{S})$, such that:*

(1) *For any potential $V \in b\mathcal{B}(\mathcal{D})$,*

$$\mathbf{\Lambda}(V) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log \int_{\mathcal{P}(\mathcal{D})} e^{T\beta(V)} \mathbb{P}_T(d\beta). \quad (4.1)$$

(2) *The mapping $V \mapsto \mathbf{\Lambda}(V)$ is Gateaux differentiable on $b\mathcal{B}(\mathcal{D})$, i.e. the mapping $t \mapsto \mathbf{\Lambda}(V + tV_1)$ is differentiable at $t = 0$ for any potential $V, V_1 \in b\mathcal{B}(\mathcal{D})$.*

(3) *\mathbb{P}_T is exponentially $*$ -tight, namely, $\forall L > 0, \exists \mathbf{K}$ compact in $(\mathcal{P}(\mathcal{D}), \tau)$ such that for any measurable neighborhood \mathcal{N} of \mathbf{K} ,*

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P}_T[L_T \notin \mathcal{N}] \leq -L.$$

Then, \mathbb{P}_T satisfies the L.D.P. on $(\mathcal{P}(\mathcal{D}), \tau)$ with speed T and the rate function

$$I(\beta) = \sup \{ \beta(V) - \mathbf{\Lambda}(V), \quad V \in \mathcal{C}_b(\mathcal{D}) \}, \quad \beta \in \mathcal{P}(\mathcal{D}),$$

which is the Legendre transform $\mathbf{\Lambda}^$ of the Cramer functional $\mathbf{\Lambda}$.*

Remark 1. In the theorem above, without the exponential $*$ -tightness condition (3), the local L.D.P. below holds: for any $\beta \in \mathcal{P}(\mathcal{D})$,

(1) for any measurable neighborhood \mathcal{N} of β (in $\mathcal{P}(\mathcal{D})$),

$$\liminf_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P}_T(\mathcal{N}) \geq -I(\beta); \quad (4.2)$$

(2) for any $a < I(\beta)$, there is a measurable neighborhood \mathcal{N} of β ,

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P}_T(\mathcal{N}) \leq -a. \quad (4.3)$$

The exponential $*$ -tightness is well-adapted to the non-metrisable topology (such as the τ -topology), and it is equivalent to the usual exponential tightness for large deviations of a sequence of probability distributions on Polish spaces (see [59]).

Proof of Theorem 2. Assume **(C1)** \rightarrow **(C5)**. The proof is divided into several steps and consists in using Theorem 3 above.

Step 1. In this step we prove the equality (4.4). Let

$$\mathbb{P}_T(\cdot) = \mathbb{P}_\nu[L_T \in \cdot | T < \sigma_{\mathcal{D}}].$$

Let $\Lambda_{\mathcal{D}}(V)$ be the minus Dirichlet eigenvalue given by (2.7), $\mu_{\mathcal{D},V}$ the left eigen-probability distribution, $\varphi_{\mathcal{D},V}$ the right eigenfunction, of $P_t^{\mathcal{D},V}$ over $b_{\mathbf{W}}\mathcal{B}(\mathcal{D})$ which are given in Theorem 1. We have by Theorem 1,

$$\begin{aligned} \int_{\mathcal{P}(\mathcal{D})} e^{T\beta(V)} \mathbb{P}_T(d\beta) &= \frac{\mathbb{E}_\nu[e^{\int_0^T V(X_t) dt} \mathbf{1}_{T < \sigma_{\mathcal{D}}}] }{\mathbb{P}_\nu[T < \sigma_{\mathcal{D}}]} \\ &= e^{(\Lambda_{\mathcal{D}}(V) - \Lambda_{\mathcal{D}}(0))T} \frac{e^{-\Lambda_{\mathcal{D}}(V)T} \nu(P_T^{\mathcal{D},V} \mathbf{1})}{\nu(e^{-\Lambda_{\mathcal{D}}(0)T} P_T^{\mathcal{D}} \mathbf{1})} \\ &= e^{(\Lambda_{\mathcal{D}}(V) - \Lambda_{\mathcal{D}}(0))T} \frac{\nu(\varphi_{\mathcal{D},V}) \mu_{\mathcal{D},V}(\mathbf{1}) + O(e^{-\delta T})}{\nu(\varphi_{\mathcal{D},0}) \mu_{\mathcal{D}}(\mathbf{1}) + O(e^{-\delta T})} \end{aligned}$$

Thus, one has that $\Lambda(V)$ is given by:

$$\Lambda(V) = \lim_{T \rightarrow +\infty} \frac{1}{T} \log \int_{\mathcal{P}(\mathcal{D})} e^{T\beta(V)} \mathbb{P}_T(d\beta) = \Lambda_{\mathcal{D}}(V) - \Lambda_{\mathcal{D}}(0). \quad (4.4)$$

Step 2 (Gateaux-differentiability). By Theorem 1, we recall that the operator $P_1^{\mathcal{D},V}$ has a spectral gap near its spectral radius $e^{\Lambda(V)}$ on $b_{\mathbf{W}}\mathcal{B}(\mathcal{D})$ and its eigen-projection is one-dimensional. Given two potentials $V, V_1 \in b\mathcal{B}(\mathcal{D})$, since for any $f \in b_{\mathbf{W}}\mathcal{B}(\mathcal{D})$, the mapping

$$\lambda \in \mathbb{C} \mapsto P_1^{\mathcal{D},V+\lambda V_1} f$$

valued in the Banach space $b_{\mathbf{W}}\mathcal{B}(\mathcal{D})$, is analytic on a neighborhood of 0 in \mathbb{C} , we deduce by the perturbation theory of operators [40, Chapter 7, Theorems 1.7 and 1.8]), that the mapping

$$\lambda \mapsto e^{\Lambda_{\mathcal{D}}(V+\lambda V_1)} \text{ (the spectral radius of } P_1^{\mathcal{D},V+\lambda V_1} \text{)}$$

is also analytic on a neighborhood of 0 in \mathbb{C} . By (4.4), this shows item (2) in Theorem 3, namely that $V \in b\mathcal{B}(\mathcal{D}) \mapsto \Lambda(V)$ is Gateaux differentiable.

Let us recall that those first two steps yield the local L.D.P. for $\mathbb{P}_\nu[L_T \in \cdot | T < \sigma_{\mathcal{D}}]$ with the rate function $I_{\mathcal{D}}$, see indeed Remark 1.

Step 3 (exponential *tightness). We now prove item (3) in Theorem 3. Let us first recall that it follows from [60], that under the assumptions **(C1)**, **(C2)**, **(C3)**, $\mathbb{P}_\nu[L_T \in \cdot]$ satisfies, on $(\mathcal{P}(\mathcal{S}), \tau)$, the good upper bound of large deviations with Donsker-Varadhan's rate function $J(\beta)$ given by (2.16). Thus for any measurable τ -closed subset \mathbf{F} of $\mathcal{P}(\mathcal{D})$,

$$\mathbf{F} = \mathbf{F} \cap \{\beta \in \mathcal{P}(\mathcal{S}); \beta(\mathcal{D}) = 1\} \text{ is closed in } (\mathcal{P}(\mathcal{S}), \tau),$$

and

$$\{L_T \in \mathbf{F}, T < \sigma_{\mathcal{D}}\} \subset \{L_T \in \mathbf{F} \cap \{\beta \in \mathcal{P}(\mathcal{S}); \beta(\mathcal{D}) = 1\}\},$$

and thus we get the following upperbound:

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P}_\nu[L_T \in \mathbf{F}, T < \sigma_{\mathcal{D}}] \leq - \inf_{\beta \in \mathbf{F}} J(\beta).$$

Since by Theorem 1 (with $V = 0$), we have

$$\lim_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P}_\nu[T < \sigma_{\mathcal{D}}] = \Lambda_{\mathcal{D}}(0),$$

we thus finally deduce that

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P}_\nu[L_T \in \mathbf{F} | T < \sigma_{\mathcal{D}}] \leq - \inf_{\beta \in \mathbf{F}} J(\beta) - \Lambda_{\mathcal{D}}(0). \quad (4.5)$$

This upper bound, together with the local L.D.P. remarked above, yields the following inequality:

$$I_{\mathcal{D}}(\beta) \geq J(\beta) + \Lambda_{\mathcal{D}}(0) = J(\beta) - \lambda_{\mathcal{D}}, \quad \forall \beta \in \mathcal{P}(\mathcal{D}). \quad (4.6)$$

Note that this is half of the equality (2.15).

Now for any $L > 0$, consider the set \mathbf{K} defined by

$$\mathbf{K} := \{\beta \in \mathcal{P}(\mathcal{S}); \beta(\mathcal{D}) = 1, J(\beta) \leq L - \Lambda_{\mathcal{D}}(0)\} \subset \mathcal{P}(\mathcal{D}).$$

It is a compact subset of $(\mathcal{P}(\mathcal{D}), \tau)$. We then obtain using the upper bound of large deviations above that for any measurable τ -open neighborhood \mathcal{N} of \mathbf{K} ,

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{P}_\nu[L_T \notin \mathcal{N} | T < \sigma_{\mathcal{D}}] \leq - \inf_{\beta \notin \mathcal{N}} J(\beta) - \Lambda_{\mathcal{D}}(0) \leq -L.$$

This is the desired exponential *-tightness.

Step 4. To finish the proof, it remains to show (2.13). Notice that $I(\beta) = 0$ if and only if $\beta \in \partial\Lambda_{\mathcal{D}}(0)$ (the sub-differential of $\Lambda_{\mathcal{D}}$ at $V = 0$). Since $\Lambda_{\mathcal{D}}$ is Gateaux-differentiable, the level set $\{I = 0\}$ is a singleton. Thus it remains to prove that $\pi_{\mathcal{D}} = \varphi_{\mathcal{D}}\mu_{\mathcal{D}} \in \partial\Lambda_{\mathcal{D}}(0)$ or equivalently that

$$\Lambda_{\mathcal{D}}(V) \geq \pi_{\mathcal{D}}(V), \quad \forall V \in b\mathcal{B}(\mathcal{D}). \quad (4.7)$$

To this end, we have at first by Jensen's inequality:

$$\begin{aligned} \log \mathbb{E}_\nu \left[e^{\int_0^T V(X_t) dt} | T < \sigma_{\mathcal{D}} \right] &\geq \mathbb{E}_\nu \left[\int_0^T V(X_t) dt | T < \sigma_{\mathcal{D}} \right] \\ &= \frac{\mathbb{E}_\nu \left[\int_0^T \mathbf{1}_{t < \sigma_{\mathcal{D}}} V(X_t) \mathbf{1}_{T < \sigma_{\mathcal{D}}} dt \right]}{\mathbb{P}_\nu[T < \sigma_{\mathcal{D}}]} \\ &= \frac{\int_{\mathcal{D}} \int_0^T P_t^{\mathcal{D}} V P_{T-t}^{\mathcal{D}} \mathbf{1} dt d\nu}{\int_{\mathcal{D}} P_T^{\mathcal{D}} \mathbf{1} d\nu}. \end{aligned}$$

In addition, by Theorem 1, we have

$$e^{-\Lambda_{\mathcal{D}}(V)T} \int_{\mathcal{D}} P_T^{\mathcal{D}} \mathbf{1} d\nu = \nu(\varphi_{\mathcal{D}}) \mu_{\mathcal{D}}(\mathbf{1}) + O(e^{-\delta T}) \int_{\mathcal{D}} \mathbf{W} d\nu = \nu(\varphi_{\mathcal{D}}) + O(e^{-\delta T}),$$

and (recall that the potential V is bounded)

$$\begin{aligned} & e^{-\Lambda_{\mathcal{D}}(V)T} \int_{\mathcal{D}} \int_0^T P_t^{\mathcal{D}} (V P_{T-t}^{\mathcal{D}} \mathbf{1}) dt d\nu \\ &= \int_0^T \int_{\mathcal{D}} e^{-\Lambda_{\mathcal{D}}(V)t} P_t^{\mathcal{D}} (V \varphi_{\mathcal{D}} + O(e^{-\delta(T-t)}) |V| \cdot \mathbf{W}) d\nu dt \\ &= \int_0^T \int_{\mathcal{D}} [\varphi_{\mathcal{D}} \mu_{\mathcal{D}}(V \varphi_{\mathcal{D}}) + O(e^{-\delta T}) \mathbf{W}] d\nu dt \\ &= T \pi_{\mathcal{D}}(V) \cdot \nu(\varphi_{\mathcal{D}}) + O(1). \end{aligned}$$

Thus we obtain

$$\begin{aligned} \Lambda_{\mathcal{D}}(V) &\geq \limsup_{T \rightarrow +\infty} \frac{1}{T} \log \mathbb{E}_{\nu} \left[e^{\int_0^T V(X_t) dt} \mid T < \sigma_{\mathcal{D}} \right] \\ &\geq \lim_{T \rightarrow +\infty} \frac{1}{T} \frac{e^{-\Lambda_{\mathcal{D}}(V)T} \int_{\mathcal{D}} \int_0^T P_t^{\mathcal{D}} V P_{T-t}^{\mathcal{D}} \mathbf{1} dt d\nu}{e^{-\Lambda_{\mathcal{D}}(V)T} \int_{\mathcal{D}} P_T^{\mathcal{D}} \mathbf{1} d\nu} \\ &= \lim_{T \rightarrow +\infty} \frac{1}{T} \frac{T \pi_{\mathcal{D}}(V) \cdot \nu(\varphi_{\mathcal{D}}) + O(1)}{\nu(\varphi_{\mathcal{D}}) + O(e^{-\delta T})} \\ &= \pi_{\mathcal{D}}(V). \end{aligned}$$

This is the desired result (4.7). The proof of Theorem 2 is thus complete. \square

Remark 2. A quite natural approach for the large deviations of $\mathbb{P}_{\nu}[L_T \in \cdot \mid T < \sigma_{\mathcal{D}}]$ is to use Varadhan-Ellis principle by approximation (this was suggested by J.D. Deuschel). Let us consider for $n \geq 1$, the potential functions V_n defined by:

$$V_n(x) = -n \mathbf{1}_{\mathcal{D}^c}(x).$$

The killed semigroup $P_t^{\mathcal{D}} f(x)$ can be approximated by the (non-killed) Feynman-Kac semigroup

$$P_t^{V_n} f(x) = \mathbb{E}_x \left[\exp \left(\int_0^t V_n(X_s) ds \right) f(X_t) \right].$$

Under **(C1)** to **(C3)** together with the topological irreducibility of the process on the whole space \mathcal{S} , for each $n \geq 1$ fixed, as V_n is bounded, by the L.D.P. in [60] and Varadhan-Ellis principle, the family

$$Q_n(L_T \in \cdot) := \frac{\mathbb{E}_{\nu} \left[\mathbf{1}_{[L_T \in \cdot]} \exp \left(\int_0^t V_n(X_s) ds \right) \right]}{\mathbb{E}_{\nu} \left[\exp \left(\int_0^t V_n(X_s) ds \right) \right]}$$

satisfies the L.D.P. on $(\mathcal{P}(\mathcal{S}), \tau)$ with the rate function

$$I_n(\beta) = J(\beta) - \Lambda(V_n), \quad \forall \beta \in \mathcal{P}(\mathcal{S}).$$

On the one hand, if we now let $n \rightarrow +\infty$, $Q_n(L_T \in \cdot)$ converges to the target distribution: $\mathbb{P}_{\nu}[L_T \in \cdot \mid T < \sigma_{\mathcal{D}}]$ and this is satisfying. However, on the other hand, it remains now to exchange the limit order. For the upper bound of L.D.P., it is enough to prove that $\Lambda(V_n) \rightarrow \Lambda(0)$ (which is already quite difficult). But the main difficulty with this

approach follows from the lower bound of L.D.P.. Without further assumptions on the subset \mathscr{D} , such as the connectedness in the case when the paths of $(X_t, t \geq 0)$ are a.s. continuous, it is easy to see that the L.D.P. in Theorem 2 with a convex rate function fails.

5. PROOF OF COROLLARY 2

Proof of Corollary 2. Assume **(C1)** \rightarrow **(C6)**. The proof of Corollary 2 is divided into several steps.

Step 1 (Preparation). In the framework of **(C1)** \rightarrow **(C3)**, under the extra condition that P_t is topologically irreducible on \mathscr{S} , our Markov process $(X_t, t \geq 0)$ admits a unique invariant probability measure π , which charges all non-empty subsets of \mathscr{S} , and $P_t(x, dy) = p_t(x, y)\pi(dy)$ is absolutely continuous w.r.t. π for all $t \geq 2t_0$, see indeed [60]. Thus, we have

$$J(\beta) < +\infty \implies \beta \ll \pi, \quad (5.1)$$

a fact which was noted in [60]. Let us now consider the quadratic form \mathcal{E} defined by

$$\mathcal{E}(f, f) = \langle \sqrt{-\mathfrak{L}}f, \sqrt{-\mathfrak{L}}f \rangle_{L^2(\mathscr{S}, \pi)},$$

with domain $\mathbb{D}(\mathcal{E}) = \mathbb{D}_{L^2(\mathscr{S}, \pi)}(\sqrt{-\mathfrak{L}})$. The quadratic form \mathcal{E} is the so-called Dirichlet form of the (non-killed) reversible Markov semigroup $(P_t, t \geq 0)$ over \mathscr{S} . It is well known ([27]) that for $\beta = h\pi \in \mathcal{P}(\mathscr{S})$,

$$J(\beta) = \begin{cases} \mathcal{E}(\sqrt{h}, \sqrt{h}), & \text{if } \sqrt{h} \in \mathbb{D}(\mathcal{E}); \\ +\infty, & \text{otherwise.} \end{cases} \quad (5.2)$$

Step 2 (Rayleigh's principle). Given a potential function $V \in b\mathcal{B}(\mathscr{D})$, the semigroup $P_t^{\mathscr{D}, V}$ is symmetric on $L^2(\mathscr{D}, \mathbf{1}_{\mathscr{D}}\pi)$. Its log spectral radius on $L^2(\mathscr{D}, \mathbf{1}_{\mathscr{D}}\pi)$, defined by

$$\Lambda_{\mathscr{D}, 2}(V) := \lim_{T \rightarrow +\infty} \frac{1}{T} \log \|P_t^{\mathscr{D}, V}\|_{L^2(\mathscr{D}, \mathbf{1}_{\mathscr{D}}\pi)},$$

is always not greater than its log spectral radius on in the Banach space $b_{\mathbf{W}}\mathcal{B}(\mathscr{D})$, i.e. than $\Lambda_{\mathscr{D}}(V)$ (by the spectral decomposition). On the other hand, take an initial distribution $\nu = h\pi$ so that $h\mathbf{1}_{\mathscr{D}^c} = 0$, π -a.e., $h \in L^2(\mathscr{S}, \pi)$ and $\nu(\mathbf{W}) < +\infty$. Then, we have by Theorem 1:

$$\Lambda_{\mathscr{D}, 2}(V) \geq \lim_{T \rightarrow +\infty} \frac{1}{T} \log \nu(P_t^{\mathscr{D}, V} \mathbf{1}) = \Lambda_{\mathscr{D}}(V).$$

Therefore, it holds for all $V \in b\mathcal{B}(\mathscr{D})$:

$$\Lambda_{\mathscr{D}}(V) = \Lambda_{\mathscr{D}, 2}(V). \quad (5.3)$$

The quadratic Dirichlet form associated with $P_t^{\mathscr{D}, V}$ on $L^2(\mathscr{D}, \mathbf{1}_{\mathscr{D}}\pi)$ is defined by (see indeed [51]), for all $f \in \mathbb{D}(\mathcal{E}_{\mathscr{D}, V}) = \{g \in \mathbb{D}(\mathcal{E}); \mathbf{1}_{\mathscr{D}^c}g = 0, \pi\text{-a.e.}\}$,

$$\mathcal{E}_{\mathscr{D}, V}(f, f) = \mathcal{E}(f, f) - \int_{\mathscr{D}} V f^2 d\pi.$$

By Rayleigh's principle, one has:

$$\begin{aligned} & \Lambda_{\mathcal{D},2}(V) \\ &= \sup\{-\mathcal{E}_{\mathcal{D},V}(f, f); f \in \mathbb{D}(\mathcal{E}_{\mathcal{D},V}), \pi(f^2) = 1\} \\ &= \sup\left\{\int_{\mathcal{S}} V f^2 d\pi - \mathcal{E}(f, f); f \in \mathbb{D}(\mathcal{E}), \pi(f^2) = 1 \text{ and } f \geq 0, \mathbf{1}_{\mathcal{D}^c} f = 0, \pi\text{-a.e.}\right\} \end{aligned}$$

where we have used the fact that $\mathcal{E}_{\mathcal{D},V}(|f|, |f|) \leq \mathcal{E}_{\mathcal{D},V}(f, f)$. Thus by (5.3) and (5.1), we deduce that

$$\Lambda_{\mathcal{D}}(V) = \Lambda_{\mathcal{D},2}(V) = \sup\left\{\int_{\mathcal{S}} V d\beta - J(\beta); \beta \in \mathcal{P}(\mathcal{D})\right\}.$$

As $\Lambda(V) = \Lambda_{\mathcal{D}}(V) - \Lambda_{\mathcal{D}}(0)$ (see (4.4)) and since by definition $\Lambda_{\mathcal{D}}(0) = -\lambda_{\mathcal{D}}$, we get by Legendre-Fenchel's theorem,

$$I_{\mathcal{D}}(\beta) = \Lambda^*(\beta) = (\Lambda_{\mathcal{D}})^*(\beta) + \Lambda_{\mathcal{D}}(0) = J(\beta) - \lambda_{\mathcal{D}}, \quad \forall \beta \in \mathcal{P}(\mathcal{D}).$$

This is the desired result. The proof of Corollary 2 is thus complete. \square

6. EXAMPLES

In this section we give some examples, arising from statistical physics, of processes satisfying **(C1)** \rightarrow **(C5)**.

6.1. Kinetic Langevin process driven by a Brownian motion. Let $U : \mathbb{R}^d \rightarrow [1, +\infty]$ be measurable function and consider the phase space

$$\mathcal{S} = \{U < +\infty\} \times \mathbb{R}^d.$$

Let us consider the so-called kinetic Langevin process $(X_t = (x_t, v_t), t \geq 0)$ which is the solution to the stochastic differential equation in \mathcal{S} :

$$dx_t = v_t dt, \quad dv_t = -\nabla U(x_t) dt - \gamma v_t dt + dB_t, \quad (6.1)$$

where $\gamma > 0$ is the friction parameter and $(B_t, t \geq 0)$ is d -dimensional standard Brownian motion. The validity of the conditions **(C1)** \rightarrow **(C5)** have been shown in:

- a. In [35] when the potential U is only \mathcal{C}^1 over \mathbb{R}^d , namely when ∇U is continuous over \mathbb{R}^d (in this case $\mathcal{S} = \mathbb{R}^d \times \mathbb{R}^d$), and when $\mathcal{D} = \mathcal{O} \times \mathbb{R}^d$ where \mathcal{O} is a \mathcal{C}^2 subdomain of \mathbb{R}^d .
- b. In [36] when the potential U models the singular interactions between the particles and when $\mathcal{D} = \mathcal{O} \times \mathbb{R}^d$ where \mathcal{O} is a subdomain of $\{U < +\infty\}$ with \mathcal{C}^2 boundary inside $\{U < +\infty\}$. More precisely, in this case, $d = kN$ (with $k, N \geq 1$), N is the number of \mathbb{R}^k -particles, and U has the form

$$U(x^1, \dots, x^N) = \sum_{i=1}^N V_c(x^i) + \sum_{1 \leq i < j \leq N} V_I(x^i - x^j), \quad x_i \in \mathbb{R}^k,$$

where $V_I : \mathbb{R}^k \rightarrow \mathbb{R} \cup \{+\infty\}$ is e.g. the Coulomb potential, the Riesz potential, or the Lennard-Jones potential.

More recently, the authors proved in [34] that in both cases **a** and **b** above, the conditions **(C4)** and **(C5)** actually hold with any such subdomain \mathcal{D} of the phase space \mathcal{S} (i.e. without any assumption on the regularity of \mathcal{O}). We refer to [48, 3, 11] for related results

6.2. SDE driven by a rotationally invariant stable processes. In this section, we prove Theorem 4 below for the process solution to the elliptic stochastic differential equation (6.4) driven by a rotationally invariant stable processes. This theorem aims at showing that such processes satisfy under mild assumptions the conditions **(C1)** \rightarrow **(C5)**.

6.2.1. Definition of the process and assumptions. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space (where the filtration satisfies the usual condition). Let us consider a (Lévy) rotationally invariant α -stable process $(L_t^\alpha, t \geq 0)$ on \mathbb{R}^d ($0 < \alpha < 2, d \geq 1$), see e.g. [1, Example 3.3.8 and Section 4.3.4]. We denote by F_α its Lévy measure and we recall that it is pure jump process where:

$$F_\alpha(dz) = \frac{C_\alpha}{|z|^{d+\alpha}} dz, \quad C_\alpha > 0. \quad (6.2)$$

Recall also that for all $t \geq 0$, L_t^α admits moments of order $q \in [0, \alpha)$. Let $\beta > 1$ and $U : \mathbb{R}^d \rightarrow [1, +\infty)$ be a \mathcal{C}^2 function such that for some $R_* > 1$ and $c_* > 0$,

$$\nabla U(x) \cdot x \geq c_* |x|^{2\beta} \text{ for all } |x| > R_*. \quad (6.3)$$

Remark that in the case $U(x) = |x|^k$ (for large $|x|$), condition (6.3) is verified if $k > 2$ with $\beta = k/2$. Let $(X_t(x), t \geq 0)$ be the solution (see Corollary (3)) of the Lévy driven elliptic stochastic differential equation

$$dX_t = -\nabla U(X_t) dt + dL_t^\alpha, \quad X_0 = x \in \mathbb{R}^d. \quad (6.4)$$

For a non empty subset \mathcal{D} of \mathbb{R}^d , we recall that $\sigma_{\mathcal{D}} = \inf\{t \geq 0, X_t \notin \mathcal{D}\}$. In the rest of this section, we check that the process $(X_t, t \geq 0)$ satisfies **(C1)** \rightarrow **(C5)**. In what follows, $B(x, r)$ is the open ball of \mathbb{R}^d centered at x of radius $r > 0$. Let us mention that one can easily adapt our analysis to non gradient vector field in (6.4).

6.2.2. On Assumptions (C1) and (C3). The infinitesimal generator of (6.4) is given by, for $\psi \in \mathcal{C}_c^2(\mathbb{R}^d)$ (see e.g. [1, Section 6.7]),

$$\mathcal{L}^X \psi(x) = -\nabla U(x) \cdot \nabla \psi(x) + \int_{\mathbb{R}^d} [\psi(x+z) - \psi(x) - \nabla \psi(x) \cdot z \mathbf{1}_{\{|z| \leq 1\}}] F_\alpha(dz).$$

In the following $\theta > 0$ is small enough such that

$$2\beta\theta < \alpha \wedge 1. \quad (6.5)$$

Consider a smooth function $\mathbf{V} : \mathbb{R}^d \rightarrow [1, +\infty)$ such that for $|x| > 1$, $\mathbf{V}(x) = 2 + |x|^{\beta\theta}$. Then, for $p > 1$, define the function \mathbf{W} by

$$\mathbf{W} = \mathbf{V}^{1/p}. \quad (6.6)$$

Proposition 1. *Assume (6.3). Then, for any $p > 1$, (C3) is satisfied with \mathbf{W} defined by (6.6).*

Proof. Recall that $\beta > 1$. Let $x \in \mathbb{R}^d$. If $|x| > \max(R_*, 1)$, we have $\mathbf{V}(x) = 2 + |x|^{\beta\theta}$ and in this case one has $\nabla \mathbf{V}(x) = \beta\theta x |x|^{\beta\theta-2}$. Therefore, using (6.3), one has $-\nabla U(x) \cdot \nabla \mathbf{V}(x) \leq -c_* \beta\theta |x|^{\beta\theta+2\beta-2}$ and thus, since $2\beta > 2$, one has when $|x| \rightarrow +\infty$, $-\nabla U(x) \cdot \nabla \mathbf{V}(x) / \mathbf{V}(x) \rightarrow -\infty$. Let us now consider for $x \in \mathbb{R}^d$,

$$\mathcal{A}_1(x) := \int_{|z| \leq 1} [\mathbf{V}(x+z) - \mathbf{V}(x) - \nabla \mathbf{V}(x) \cdot z] F_\alpha(dz).$$

Since Hess \mathbf{V} is bounded over \mathbb{R}^d , it holds for all $x, z \in \mathbb{R}^d$: $|\mathbf{V}(x+z) - \mathbf{V}(x) - \nabla \mathbf{V}(x) \cdot z| \leq C|z|^2$, for some $C > 0$. Hence, one has for all $x \in \mathbb{R}^d$, $|\mathcal{A}_1(x)| \leq C \int_{|z| \leq 1} |z|^2 F_\alpha(dz)$. Thus, $\mathcal{A}_1(x)$ is well defined and $\mathcal{A}_1(x)/\mathbf{V}(x) \rightarrow 0$ as $|x| \rightarrow +\infty$. Let us now consider the quantity

$$\begin{aligned} \mathcal{A}_2(x) &:= \int_{|z| > 1} [\mathbf{V}(x+z) - \mathbf{V}(x)] F_\alpha(dz) \\ &= \int_{|z| > 1} [\mathbf{V}(x+z) - \mathbf{V}(x)] \mathbf{1}_{|x+z| \geq 1 \text{ and } |x| \geq 1} F_\alpha(dz) \\ &\quad + \int_{|z| > 1} [\mathbf{V}(x+z) - \mathbf{V}(x)] \mathbf{1}_{|x+z| < 1 \text{ or } |x| < 1} F_\alpha(dz) =: \mathcal{A}_2^a(x) + \mathcal{A}_2^b(x). \end{aligned}$$

Let $|z| \geq 1$ and $x \in \mathbb{R}^d$. Assume that $x \in \mathbb{R}^d$ is such that $|x+z| \geq 1$ and $|x| \geq 1$ so that $\mathbf{V}(x+z) = 2 + |x+z|^{\beta\theta}$ and $\mathbf{V}(x) = 2 + |x|^{\beta\theta}$. Since $\beta\theta < 1$, using the inequality $(a+b)^{\beta\theta} \leq a^{\beta\theta} + b^{\beta\theta}$ for $a, b \geq 0$, we have thanks to (6.2) and (6.5) that in this case: $\mathcal{A}_2^a(x) \leq \int_{|z| > 1} [(|x| + |z|)^{\beta\theta} - |x|^{\beta\theta}] F_\alpha(dz) \leq \int_{|z| > 1} |z|^{\beta\theta} F_\alpha(dz) < +\infty$. Assume now that $x \in \mathbb{R}^d$ is such that $|x+z| < 1$ so that $\mathbf{V}(x+z) \leq c_2 := \sup_{|y| \leq 1} \mathbf{V}$. We then have since $\mathbf{V} \geq 0$:

$$\mathcal{A}_2^b(x) \leq \int_{|z| > 1} [c_2 - \mathbf{V}(x)] F_\alpha(dz) \leq c_2 \int_{|z| > 1} F_\alpha(dz) < +\infty,$$

Assume now that $x \in \mathbb{R}^d$ is such that $|x| < 1$. Since for some $L > 0$, $\mathbf{V}(y) \leq L(2 + |y|^{\beta\theta})$, we then have using again that $\mathbf{V} \geq 0$:

$$\mathcal{A}_2^b(x) \leq L \int_{|z| > 1} [2 + (|x| + |z|)^{\beta\theta}] F_\alpha(dz) \leq 2L \int_{|z| > 1} F_\alpha(dz) + L \int_{|z| > 1} (1 + |z|^{\beta\theta}) F_\alpha(dz).$$

Consequently $\mathcal{A}_2^b(x) < +\infty$. In all cases, we have that $\mathcal{A}_2(x)/\mathbf{V}(x) \rightarrow 0$ as $|x| \rightarrow +\infty$ (note also that $|\mathcal{A}_2| \leq C(\mathbf{V} + 1)$ for some $C > 0$). In conclusion, we have proved that for some $m > 0$, $|\mathcal{L}^X \mathbf{V}(x)| \leq m(|x|^{\beta\theta + 2\beta - 2} + 1)$ and $\mathcal{L}^X \mathbf{V}(x)/\mathbf{V}(x) \rightarrow -\infty$ as $|x| \rightarrow +\infty$. As $\mathbf{V} \in \mathbb{D}_e(\mathfrak{L}^X)$, where $\mathbb{D}_e(\mathfrak{L}^X)$ is the extended domain of the process (6.4), and $\mathfrak{L}^X \mathbf{V} = \mathcal{L}^X \mathbf{V}$ (see indeed the end of the proof of Corollary 3 below), we have thus proved that **(C3)** holds. \square

Corollary 3. *Assume (6.3). Then, for all $x \in \mathbb{R}^d$, there exists a unique pathwise solution $(X_t(x), t \geq 0)$ to (6.4) which moreover defines a strong Markov process. Moreover, a.s. $(X_t(x), t \geq 0) \in \mathbb{D}(\mathbb{R}_+, \mathbb{R}^d)$.*

Proof. The proof leading Equation (6.7) below is rather standard. We do it for sake of completeness. The computations carried out in the proof of Proposition 1 shows that for any $t \geq 0$, $\int_0^t |\mathcal{L}^X \mathbf{V}(x_s)| ds$ is a.s. finite for any càdlàg process $(x_s, s \geq 0)$ (as such a process is a.s. bounded over $[0, t]$). Note also that since the number of jumps is at most countable, it holds a.s. $\int_0^t \mathcal{L}^X \mathbf{V}(x_{s-}) ds = \int_0^t \mathcal{L}^X \mathbf{V}(x_s) ds$. Let $c_1 > 0$ such that $\mathcal{L}^X \mathbf{V} \leq c_1 \mathbf{V}$ over \mathbb{R}^d . Set for $R \geq 0$, $\sigma_R := \inf\{t \geq 0, \mathbf{V}(X_t) \geq R\}$. Note that $\mathcal{V}_R := \inf\{t \geq 0, X_t \notin \mathcal{V}_R\}$ where $\mathcal{V}_R = \{x \in \mathbb{R}^d, \mathbf{V}(x) < R\}$ is an open bounded (say by $c_R > 0$) subset of \mathbb{R}^d . Consider the unique strong solution of (6.4) up to time σ_R and set $\sigma_\infty := \lim_{R \rightarrow +\infty} \sigma_R = \sup_{R > 0} \sigma_R$ which is its lifetime. Let us prove that $\mathbb{P}_x[\sigma_\infty = +\infty] = 1$ for all $x \in \mathbb{R}^d$. For $x \in \mathcal{V}_R$, using Itô formula [1, Theorem 4.4.7] on

the interval $[0, t \wedge \sigma_R(x)]$, one has:

$$\begin{aligned} M_{t \wedge \sigma_R}^{\mathbf{V}}(x) &:= \mathbf{V}(X_{t \wedge \sigma_R}(x)) - \mathbf{V}(x) - \int_0^{t \wedge \sigma_R} \mathcal{L}^X \mathbf{V}(X_{s^-}(x)) ds \\ &= \int_0^{t \wedge \sigma_R} \int_{\mathbb{R}^d} [\mathbf{V}(X_{s^-}(x) + z) - \mathbf{V}(X_{s^-}(x))] \tilde{N}_\alpha(ds, dz) \\ &= \int_0^t \int_{\mathbb{R}^d} \mathbf{1}_{s < \sigma_R} [\mathbf{V}(X_{s^-}(x) + z) - \mathbf{V}(X_{s^-}(x))] \tilde{N}_\alpha(ds, dz), \end{aligned}$$

where \tilde{N}_α is the compensated Poisson random measure of $(L_t^\alpha, t \geq 0)$. Since $\mathbf{V}(y) \leq L(2 + |y|^{\beta\theta})$ and $2\beta\theta < 1$ (see (6.5)), we have for all $Y \in \mathbb{R}^d$,

$$\begin{aligned} &\int_{|z|>1} \mathbf{1}_{\mathbf{V}(Y) \leq R} |\mathbf{V}(Y + z) - \mathbf{V}(Y)|^2 F_\alpha(dz) \\ &\leq 2 \int_{|z|>1} \mathbf{1}_{\mathbf{V}(Y) \leq R} \mathbf{V}^2(Y + z) F_\alpha(dz) + 2R^2 \int_{|z|>1} F_\alpha(dz) \\ &\leq 2L^2 \int_{|z|>1} [4 + 2c_R^{2\beta\theta} + 2|z|^{2\beta\theta}] F_\alpha(dz) + 2R^2 \int_{|z|>1} F_\alpha(dz). \end{aligned}$$

Moreover, since $|\nabla \mathbf{V}|$ is bounded over \mathbb{R}^d , there exists $C > 0$ such that for all $Y \in \mathbb{R}^d$,

$$\int_{|z| \leq 1} \mathbf{1}_{\mathbf{V}(Y) \leq R} |\mathbf{V}(Y + z) - \mathbf{V}(Y)|^2 F_\alpha(dz) \leq C \int_{|z| \leq 1} |z|^2 F_\alpha(dz) < +\infty.$$

Hence, because $\mathbf{1}_{s < \sigma_R} \leq \mathbf{1}_{\mathbf{V}(X_{s^-}) \leq R}$, one deduces that $\mathbb{E}_x[\int_0^t \int_{\mathbb{R}^d} \mathbf{1}_{s < \sigma_R} |\mathbf{V}(X_{s^-} + z) - \mathbf{V}(X_{s^-})|^2 F_\alpha(dz) ds] < +\infty$. Consequently, for all $R > 0$, the process $(M_{t \wedge \sigma_R}^{\mathbf{V}}(x), t \geq 0)$ is a martingale. Thus,

$$\begin{aligned} \mathbb{E}_x[\mathbf{V}(X_{t \wedge \sigma_R})] &\leq \mathbf{V}(x) + \mathbb{E}_x \left[\int_0^{t \wedge \sigma_R} \mathcal{L}^X \mathbf{V}(X_{s^-}) ds \right] \\ &\leq \mathbf{V}(x) + c_1 \mathbb{E}_x \left[\int_0^{t \wedge \sigma_R} \mathbf{V}(X_{s^-}) ds \right] \\ &= \mathbf{V}(x) + c_1 \mathbb{E}_x \left[\int_0^{t \wedge \sigma_R} \mathbf{V}(X_s) ds \right] \leq \mathbf{V}(x) + c_1 \mathbb{E}_x \left[\int_0^t \mathbf{V}(X_{s \wedge \sigma_R}) ds \right]. \end{aligned}$$

By Grönwall's inequality [31, Theorem 5.1 in Appendixes], we have $\mathbb{E}_x[\mathbf{1}_{\sigma_R \leq t} \mathbf{V}(X_{t \wedge \sigma_R})] \leq \mathbb{E}_x[\mathbf{V}(X_{t \wedge \sigma_R})] \leq \mathbf{V}(x) e^{c_1 t}$. Since $\mathbf{V}(X_{\sigma_R}) \geq R$, it then holds

$$\mathbb{P}_x[\sigma_R \leq t] \leq \frac{e^{c_1 t}}{R} \mathbf{V}(x), \quad \forall t \geq 0. \quad (6.7)$$

This proves that $\sigma_\infty(x)$ is a.s. infinite. Note also that we have proved that $\mathbf{V} \in \mathbb{D}_e(\mathfrak{L}^X)$ and hence $\mathbf{W} \in \mathbb{D}_e(\mathfrak{L}^X)$ (see [35, Proposition 5.1]). The strong Markov property follows from the (pathwise) uniqueness by standard considerations. \square

We also mention [66, Theorem 3.1] for existence and uniqueness of solutions to α -stable driven stochastic differential equations in a similar setting where the analysis relies there on the fact that a rotationally invariant α -stable process is (in law) a subordinated Brownian motion. We also refer to [44] where the exponential ergodicity of a Markov process defined as the solution to a SDE with jump noise is investigated.

In the following, we denote by $(P_t, t \geq 0)$ the semigroup of the process $(X_t, t \geq 0)$, where we recall that $P_t f(x) = \mathbb{E}_x[f(X_t)]$, for all $f \in b\mathcal{B}(\mathbb{R}^d)$.

Proposition 2. *For all $t > 0$, P_t is strong Feller.*

Proof. Let $R > 0$ and let $U_R : \mathbb{R}^d \rightarrow \mathbb{R}$ be a C^2 function such that $U_R = U$ on the closure of \mathcal{V}_R and such that all its derivatives of order less than 2 are bounded. Let $(X_s^R, s \geq 0)$ be the solution of $dX_t^R = -\nabla U_R(X_t^R)dt + dL_t^\alpha$. For all $R > 0$, by [65, Theorem 1.1] and standard approximation theorem [22, Lemma 7.1.5], $x \in \mathbb{R}^d \mapsto \mathbb{E}_x[f(X_t^R)]$ is continuous for all $t > 0$ and $f \in b\mathcal{B}(\mathbb{R}^d)$. Let $x_n \rightarrow x \in \mathbb{R}^d$. Consider a compact set K and $R_0 > 0$ such that $x_n, x \in K \subset \mathcal{V}_{R_0}$ for all $n \geq 1$. Since $(X_s, s \geq 0)$ and $(X_s^R, s \geq 0)$ coincides before their first exit time from \mathcal{V}_R , we have for all $f \in b\mathcal{B}(\mathbb{R}^d)$, $t \geq 0$, $R \geq R_0$, and $n \geq 1$,

$$\begin{aligned} & |\mathbb{E}_x[f(X_t)] - \mathbb{E}_{x_n}[f(X_t)]| \\ & \leq |\mathbb{E}[f(X_t^R(x)) - f(X_t^R(x_n))] \mathbf{1}_{t < \sigma_R(x) \wedge \sigma_R(x_n)}]| + 2\|f\| \mathbb{P}[\sigma_R(x) \wedge \sigma_R(x_n) \leq t] \\ & \leq |\mathbb{E}[f(X_t^R(x)) - f(X_t^R(x_n))]| + 4\|f\| \mathbb{P}[\sigma_R(x) \wedge \sigma_R(x_n) \leq t]. \end{aligned}$$

Note that $\sup_{n \geq 1} \mathbb{P}[\sigma_R(x) \wedge \sigma_R(x_n) \leq t] \leq \mathbb{P}[\sigma_R(x) \leq t] + \sup_{n \geq 1} \mathbb{P}[\sigma_R(x_n) \leq t] \leq 2e^{c_1 t} \sup_K \mathbf{V}/R \rightarrow 0$ as $R \rightarrow +\infty$ by (6.7). Hence $|\mathbb{E}_x[f(X_t)] - \mathbb{E}_{x_n}[f(X_t)]| \rightarrow 0$ as $n \rightarrow +\infty$, the desired result. \square

Let us mention that the strong Feller property of solutions to Lévy driven stochastic differential equation has been extensively investigated in the literature, see e.g. [55, 56, 46, 30, 67, 62] and references therein.

6.2.3. On Assumption (C2).

Proposition 3. *The process $(X_t, t \geq 0)$ solution to (6.4) satisfies (C2).*

Proof. Let $T > 0$ be fixed. Recall that for $r > 0$, $\mathcal{V}_R = \{x \in \mathbb{R}^d, \mathbf{V}(x) < R\}$ is an open bounded subset of \mathbb{R}^d and $\sigma_R = \inf\{t \geq 0, X_t \notin \mathcal{V}_R\}$. Let $x_n \rightarrow x \in \mathbb{R}^d$. One can assume that for some $R_0 > 0$, $x_n, x \in \mathcal{V}_R$ for all $R \geq R_0$ and $n \geq 0$. In the following we assume that $R \geq R_0$. By Gronwall Lemma [31, Theorem 5.1 in Appendixes] we have for all $R \geq 0$, when $T < \sigma_R(x) \wedge \sigma_R(x_n)$, $\sup_{s \in [0, T]} |X_t(x) - X_t(x_n)| \leq |x - x_n| e^{b_R T}$, where $b_R := \sup_{y \in \mathcal{H}_R} |\text{Hess } U(y)|$. Thus, one has for all $\epsilon > 0$ and all $R > 0$ fixed, as $n \rightarrow \infty$,

$$\mathbf{P}_n(R) = \mathbb{P}\left[\sup_{s \in [0, T]} |X_t(x) - X_t(x_n)| \geq \epsilon, T < \sigma_R(x) \wedge \sigma_R(x_n)\right] \rightarrow 0.$$

Consequently, it holds for any $\epsilon > 0$:

$$\begin{aligned} \mathbb{P}\left[\sup_{s \in [0, T]} |X_t(x) - X_t(x_n)| \geq \epsilon\right] & \leq \mathbf{P}_n(R) + \mathbb{P}[T \geq \sigma_R(x) \wedge \sigma_R(x_n)] \\ & \leq \mathbf{P}_n(R) + \mathbb{P}[T \geq \sigma_R(x)] + \mathbb{P}[T \geq \sigma_R(x_n)] \\ & \leq \mathbf{P}_n(R) + 2R_0 \frac{e^{c_1 T}}{R}, \end{aligned}$$

where we have used (6.7) to get the last inequality and the fact that $\mathbf{V}(x_n) < R_0$ and $\mathbf{V}(x) < R_0$. Let us now consider $\delta > 0$. Pick $R_\delta \geq R_0$ such that $2R_0 e^{c_1 T}/R_\delta \leq \delta/2$. For this fix $R_\delta > 0$, $\mathbf{P}_n(R_\delta) \rightarrow 0$ as $n \rightarrow +\infty$, and thus, there exists $N_\delta \geq 1$ such that for all $n \geq N_\delta$, $\mathbf{P}_n(R_\delta) \leq \delta/2$. Therefore, one has that $\mathbb{P}[\sup_{s \in [0, T]} |X_t(x) - X_t(y)| \geq \epsilon] \leq \delta$ for all $n \geq N_\delta$. We have thus proved that $X_{[0, T]}(x_n) \rightarrow X_{[0, T]}(x)$ in probability as $n \rightarrow +\infty$ for the supremum norm over $[0, T]$. Thus $X_{[0, T]}(x_n) \rightarrow X_{[0, T]}(x)$ in probability as $n \rightarrow +\infty$ also for the distance of $\mathbb{D}([0, T], \mathbb{R}^d)$, see e.g. [4, Section 12]. Therefore,

$\mathbb{P}_{x_n}[X_{[0,T]} \in \cdot]$ converges weakly to $\mathbb{P}_x[X_{[0,T]} \in \cdot]$ in $\mathcal{P}(\mathbb{D}([0, T], \mathbb{R}^d))$, which is precisely **(C2)**. The proof is thus complete. \square

6.2.4. *On Assumption (C5)*. Let \mathcal{D} be an open subset of \mathbb{R}^d . Recall that $(P_t^\mathcal{D}, t \geq 0)$ denotes the semigroup of the killed process $(X_t, t \geq 0)$, i.e. $P_t^\mathcal{D}f(x) = \mathbb{E}_x[f(X_t)\mathbf{1}_{t < \sigma_\mathcal{D}}]$, for all $f \in b\mathcal{B}(\mathcal{D})$ (see (2.2)). We have the following result.

Proposition 4. *Let \mathcal{D} be a subdomain of \mathbb{R}^d . Let $T > 0$ and $x, z \in \mathcal{D}$. Then, for all $\epsilon > 0$ and $T > 0$,*

$$P_T^\mathcal{D}(x, B(z, \epsilon)) = \mathbb{P}_x[|X_T - z| < \epsilon, T < \tau_\mathcal{D}] > 0. \quad (6.8)$$

In addition, if $\mathbb{R}^d \setminus \overline{\mathcal{D}}$ is nonempty, then for all $y \in \mathcal{D}$, $\mathbb{P}_y[\sigma_\mathcal{D} < +\infty] > 0$. Thus, Assumption (C5) is satisfied.

Proof. Fix $T > 0$ and $\epsilon > 0$. Let \mathcal{O} be a bounded subdomain of \mathcal{D} with closure included in \mathcal{D} and such that $x, z \in \mathcal{O}$. The proof is divided into three steps.

Step A. Preliminary analysis. Let $\mathbf{c} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ be a globally Lipschitz vector field such that $\mathbf{c} = -\nabla U$ on the closure of \mathcal{O} . Let $(X_s^*, s \geq 0)$ be the solution of $dX_t^* = \mathbf{c}(X_t^*)dt + dL_t^\alpha$. Since $(X_s, s \geq 0)$ and $(X_s^*, s \geq 0)$ coincides before their first exit time from \mathcal{O} , Equation (6.8) holds if

$$\mathbb{P}_x[\{|X_T^* - z| < \epsilon\} \cap \{\overline{\text{Ran } X_{[0,T]}^*} \subset \mathcal{O}\}] > 0. \quad (6.9)$$

Let us prove (6.9). In the rest of the proof, we adopt the notation of [45, Section 2.2]. Note here that $r \equiv 0$, $b = \mathbf{c}$, $\sigma \equiv 1$, and $c(x, u) \equiv u$. In view of [45, Theorem 2.1] and (6.9), the goal is to construct $\phi \in \mathbf{S}_{0,T,x}^{\text{const}}$ such that

$$\overline{\text{Ran } \phi} \subset \mathcal{O}, \phi_0 = x, \text{ and } |\phi_T - z| < \epsilon/2. \quad (6.10)$$

If such a curve ϕ exists, by [45, Theorem 2.1], it holds for all $\epsilon_0 > 0$,

$$\mathbb{P}_x[d_T(X^*, \phi) < \epsilon_0] > 0, \quad (6.11)$$

where d_T is the Skorokhod metric of $\mathbb{D}([0, T], \mathbb{R}^d)$, see [4, Section 12]. It is not difficult to construct $\phi \in \mathbf{S}_{0,T,x}^{\text{const}}$ satisfying (6.10) using the simple procedure described in [45, Equation (7)].

Step B. Construction of the curve ϕ . We will construct ϕ with $f_t \equiv 0$. First note that for any $r' > 0$ and $a \neq b \in \mathbb{R}^d$, $J(a, B(b, r')) = F(B(b - a, r')) \in (0, +\infty)$ if $r' < |b - a|$. Therefore $a \neq b \Rightarrow b \in \text{supp}(J(a, \cdot))$ (i.e. the jump from a to b is admissible). Define

$$\tilde{\mathbf{b}}(y) = \mathbf{c}(y) - \mathbf{a}, \text{ where } \mathbf{a} = - \int_{|w| \leq 1} w_L F_\alpha(dw),$$

and w_L is the orthogonal projection of z on the vector space $L = \{\ell \in \mathbf{R}^d, \int_{|w| \leq 1} |w \cdot \ell| F_\alpha(dw) < +\infty\}$.

Assume $x \neq z$ (the case $x = z$ is treated similarly). Consider two disjoint open balls $B(x, \epsilon')$ and $B(z, \epsilon')$ whose closures are included in \mathcal{O} for any $\epsilon' \in (0, \epsilon'_0)$ for some $\epsilon'_0 \in (0, \epsilon/4)$. Fix such a $\epsilon' > 0$.

Initialization. Let ϕ be the solution of $\dot{\phi}_s = \tilde{\mathbf{b}}(\phi_s)$ with $\phi_0 = x$. Choose $t_1 \in (0, T)$ such that $\phi_t \in \overline{B(x, \epsilon'/2)}$ for all $t \in [0, t_1)$ and $\phi_{t_1^-} \in \overline{B(x, \epsilon'/2)}$. Then, pick $x_1 \in B(z, \epsilon'/2)$ and set $\phi_{t_1} = x_1 \neq x$ (this jump is admissible according to the previous discussion).

Second step. We let ϕ_t evolve again according to the flow $\dot{u}_s = \tilde{\mathbf{b}}(u_s)$ on $[t_1, t_2)$ with initial condition x_1 . If $\tilde{\mathbf{b}}(x_1) = 0$, we stop the procedure because $\phi_t = x_1$ for all $t \geq t_1$. Otherwise, there exists $t_2 > t_1$ such that $\phi_t \in \overline{B}(z, \epsilon'/2)$ for all $t \in [t_1, t_2)$ and $x_1 \neq \phi_{t_2^-} \in \overline{B}(z, \epsilon'/2)$. If one can choose $t_2 > T$, we stop the construction of ϕ . Otherwise, we come back to x_1 setting

$$\phi_{t_2} := x_1$$

and we then consider this point as the initial value of the Cauchy problem $\dot{u}_s = \tilde{\mathbf{b}}(u_s)$ on the time interval $[t_2, t_2 + t_2 - t_1)$.

Iteration. Then, one repeats this procedure a finite number of times to construct ϕ over $[0, t_1) \cup [t_1, t_2) \cup [t_2, 2t_2 - t_1) \cup [2t_2 - t_1, 3t_2 - 2t_1) \cup \dots [nt_2 - (n-1)t_1, (n+1)t_2 - nt_1)$. By choice of $\epsilon' > 0$, the resulting ϕ has the desired properties, i.e. $\phi \in \mathbf{S}_{0,T,x}^{\text{const}}$ satisfies (6.10).

Step C. End of the proof of (6.9). Assume that $\epsilon_0 > 0$ is small enough (say $\epsilon_0 \in (0, \epsilon_\phi)$, $\epsilon_\phi \in (0, \epsilon)$) such that $d_T(f, \phi) < \epsilon_0/2$ implies that $\overline{\text{Ran}} f \subset \mathcal{O}$ (note that in particular $|f_T - z| \leq |f_T - \phi_T| + |\phi_T - z| \leq d_T(f, \phi) + \epsilon/2 < \epsilon$). Then, using (6.11) with such a small $\epsilon_0 > 0$ yields $\mathbb{P}_x[\{|X_T^* - z| < \epsilon\} \cap \{\overline{\text{Ran}} X_{[0,T]}^* \subset \mathcal{O}\}] > 0$, which is exactly (6.9). Therefore, (6.8) is satisfied. The second statement in Proposition 4 is then easy to obtain with the same analysis. This concludes the proof of Proposition 4. \square

6.2.5. On Assumption (C4).

Lemma 1. *For all compact subset K of \mathbb{R}^d and $\delta > 0$, $\lim_{t \rightarrow 0^+} \sup_{x \in K} \mathbb{P}_x[\sigma_{B(x,\delta)} \leq t] = 0$.*

One way to prove Lemma 1 is to study the trajectories of the process (6.4), as this done in the proof of [34, Lemma 2.4]. We give here another proof based on the Itô formula which is inspired from the computations leading to (6.7).

Proof. Let K be a compact subset of \mathbb{R}^d . Let $\Psi : \mathbb{R}^d \rightarrow [0, 1]$ be a smooth function such that $\Psi = 0$ on $B(0, \rho/2)$ and $\Psi = 1$ on $B^c(0, \rho)$. Note that Ψ , $\nabla \Psi$ and $\text{Hess} \Psi$ are uniformly bounded over \mathbb{R}^d . Set $\Psi_x(z) = \Psi(z-x)$. Let K_δ be the (closed) δ -neighborhood of K . In addition, for any $x, z \in \mathbb{R}^d$,

$$\begin{aligned} |\mathcal{L}^X \Psi_x(z)| &\leq |\nabla U(z) \cdot \nabla_z \Psi_x(z)| + 2 \int_{|y|>1} F_\alpha(dy) + \sup_{w \in \mathbb{R}^d} |\text{Hess}_w \Psi_x(w)| \int_{|y|\leq 1} |y|^2 F_\alpha(dy) \\ &\leq |\nabla U(z) \cdot \nabla_z \Psi_x(z)| + C_0, \end{aligned}$$

for some $C_0 > 0$ independent of $x, z \in \mathbb{R}^d$. Since for $x \in K$, $\nabla_z \Psi_x(z) = 0$ for all $z \notin K_\delta$, we deduce that $\sup_{x \in K, z \in K_\delta} |\mathcal{L}^X \Psi_x(z)| < +\infty$. On the other hand, by Itô formula, the process $(M_t^{\Psi_x}(x), t \geq 0)$ is a martingale, where $M_t^{\Psi_x}(x) := \Psi_x(X_t(x)) - \Psi_x(x) - \int_0^t \mathcal{L}^X \Psi_x(X_{s^-}(x)) ds$. In particular $\Psi_x \in \mathbb{D}_e(\mathfrak{L}^X)$ and $\mathfrak{L}^X \Psi_x = \mathcal{L}^X \Psi_x$. Thus, since in addition $\Psi_x(x) = 0$, we have using the optional stopping theorem,

$$\begin{aligned} \mathbb{E}_x[\Psi_x(X_{t \wedge \sigma_{B(x,\delta)}})] &\leq \mathbb{E}_x \left[\int_0^{t \wedge \sigma_{B(x,\delta)}} \mathcal{L}^X \Psi_x(X_{s^-}(x)) ds \right] \\ &\leq t \sup_{x \in K, z \in K_\delta} |\mathcal{L}^X \Psi_x(z)|. \end{aligned}$$

Notice that we have used above that when $x \in K$ and $s < \sigma_{B(x,\delta)}$, $X_{s^-}(x) \in K_\delta$ and thus for all $x \in K$ and $s < \sigma_{B(x,\delta)}$, $|\mathcal{L}^X \Psi_x(X_{s^-}(x))| \leq \sup_{x \in K, z \in K_\delta} |\mathcal{L}^X \Psi_x(z)|$. Note that

$|X_{\sigma_{B(x,\delta)}}(x) - x| \geq \delta$. Hence, $\Psi_x(X_{\sigma_{B(x,\delta)}}(x)) = 1$ and

$$\mathbb{P}_x[\sigma_{B(x,\delta)} \leq t] = \mathbb{E}_x[\mathbf{1}_{\sigma_{B(x,\delta)} \leq t} \Psi_x(X_{\sigma_{B(x,\delta)}})] \leq t \sup_{x \in K, z \in K_\delta} |\mathcal{L}^X \Psi_x(z)|.$$

This ends the proof of the lemma. \square

Using **(C1)**, Lemma 1, and the same arguments as those used to prove [19, Theorem 2.2], we deduce the following result.

Corollary 4. *Let \mathcal{D} be any non empty open subset of \mathbb{R}^d and $t > 0$. Then, for any $t > 0$, $P_t^\mathcal{D}$ is strongly Feller. In particular **(C4)** holds.*

In conclusion, we have thus proved that, when $\beta > 1$ and (6.3) holds, the process $(X_t, t \geq 0)$ solution to (6.4) satisfies **(C1)** \rightarrow **(C5)** with the Lyapunov function defined in (6.6). Hence, we have the following result.

Theorem 4. *Let $\beta > 1$ and assume (6.3). Let \mathcal{D} be any subdomain of \mathbb{R}^d such that $\mathbb{R}^d \setminus \overline{\mathcal{D}}$ is nonempty. Then, the empirical distribution of the process solution to (6.4) (see Corollary 3) satisfies all the assertions of Theorem 1, Theorem 2 and Corollary 1 with the Lyapunov function defined in (6.6).*

Note. When \mathcal{D} is bounded, one can modify U outside $\overline{\mathcal{D}}$ so that it satisfies (6.3) and then, as \mathbf{W} is bounded over $\overline{\mathcal{D}}$ (see (6.6)), all the assertions of Theorem 1, Theorem 2 and Corollary 1 hold on the whole space $b\mathcal{B}(\mathcal{D})$ and for all $\nu \in \mathcal{P}(\mathcal{D})$.

Note. Notice that the fact that \mathcal{D} is connected is not necessary to get the result of Proposition 4 (this is the main difference with solutions to SDE driven by a Brownian motion), and thus Theorem 4 holds when e.g. \mathcal{D} is a finite union of disjoint subdomains of \mathbb{R}^d such that $\mathbb{R}^d \setminus \overline{\mathcal{D}}$ is nonempty.

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