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► **To cite this version:**

Philippe Balbiani, Quentin Gougeon, Tinko Tinchev. Characterizing measurable Boolean algebras. Topology, Algebra, and Categories in Logic, Jul 2024, Barcelone, Spain. hal-04800488

**HAL Id: hal-04800488**

**<https://hal.science/hal-04800488v1>**

Submitted on 24 Nov 2024

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# Characterizing measurable Boolean algebras

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In our daily life, we are used to compare things. We sort physical objects (i.e. regions of space) from smaller to bigger, or propositions from less likely to more likely. These relations contribute to our intuitive understanding of reality, and are naturally formalized by a pre-order on a **finite** Boolean algebra  $B$ .<sup>1</sup> This framework is typically *qualitative*, as two elements can only be related in three possible manners: smaller, bigger, or equivalent – without any consideration of degree or magnitude. By contrast, human beings also managed to quantify some of their intuitions through *measurement*, with examples including length, volume, temperature, and probabilities. Quantitative reasoning is a core component of scientific inquiry, and its mathematical foundations have been studied extensively in [KLT71]. Formally, a *measure* on  $B$  is a map  $\mu : B \rightarrow [0, \infty]$  satisfying  $\mu(a \cup b) = \mu(a) + \mu(b)$  whenever  $a \cap b = 0$ . We call  $\mu$  *finite* if in addition we have  $\mu(a) \neq \infty$  for all  $a \in B$ . Obviously, a measure  $\mu$  always induces a relation  $\preceq_\mu$  on  $B$ , defined by  $a \preceq_\mu b \iff \mu(a) \leq \mu(b)$ . The relations of the form  $\preceq_\mu$  will be called *measurable*, and *finitely measurable* in case  $\mu$  is a finite measure. So there is a direct bridge from quantitative to qualitative comparison, but the other way around is more limited, and this raises the question of which conditions on a relation  $\preceq$  are necessary and sufficient for  $\preceq$  to be (finitely) measurable. In the case of finite measures, this problem was solved by Kraft, Pratt and Seidenberg in their 1959 paper [KPS59], and later rewritten by Scott [Sco64] in a clearer manner. We present their result below. Given  $x \in X$  and  $a_1, \dots, a_m \in B$ , we denote by  $\text{count}_x(a_1, \dots, a_m) := \{i \in [1, m] : x \in a_i\}$  the number of  $a_i$ 's that contain  $x$ .

**Theorem 1.** *A relation  $\preceq$  on  $B$  is finitely measurable if and only if the following conditions are satisfied, for all  $m \geq 1$  and for all  $a, b, a_1, \dots, a_m, b_1, \dots, b_m \in B$ :*

- **Positivity:**  $0 \preceq a$ ;
- **Comparability:**  $a \preceq b$  or  $b \preceq a$ ;
- **Cancellation:** if  $\text{count}_x(a_1, \dots, a_m) = \text{count}_x(b_1, \dots, b_m)$  for all  $x \in X$  and  $a_i \preceq b_i$  for all  $i \in [1, m-1]$ , then  $b_m \preceq a_m$ .

However, this result is not fully satisfying for a number of reasons, related to the cancellation conditions. First, they involve the high-level operator  $\text{count}_x(a_1, \dots, a_m)$ , and even though they can be rewritten in a purely Boolean manner [Gär75], they remain quite awkward to read and compute. Second, they come in infinite number, and thus fail to provide a finite axiomatization for various logics of measure, see for instance [Gär75, vdH96, FZ23]. It is surprising, perhaps, that this result has never been improved in sixty years, nor proved to be optimal. In this work, we break this uncomfortable status quo by proposing the following new characterization.

**Theorem 2.** *A relation  $\preceq$  on  $B$  is finitely measurable if and only if the following conditions are satisfied, for all  $a, b, c, d \in B$ :*

- **Comparability:**  $a \preceq b$  or  $b \preceq a$ ;

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\*Speaker.

<sup>1</sup>Here we only consider the case where  $B$  is finite, and thus represent it as a powerset algebra  $(2^X, \cap, \cup, 0, 1, \cdot^c)$ .

- **Linearity:** if  $a \cap c = 0$  and  $a \cup c \preceq b \cup d$  and  $d \preceq c$ , then  $a \preceq b$ .

Let us briefly sketch the proof of Theorem 2. The strategy for the right-to-left implication is to prove that comparability and linearity entail the conditions of Theorem 1. To prove cancellation, assume that  $\text{count}_x(a_1, \dots, a_m) = \text{count}_x(b_1, \dots, b_m)$  for all  $x \in X$ , and  $a_i \preceq b_i$  for all  $i \in [1, m-1]$ . Consider for a moment the case where  $a_1, \dots, a_m, b_1, \dots, b_m$  are all pairwise disjoint. Then, the counting assumption yields  $b_1 \cup \dots \cup b_m \preceq a_1 \cup \dots \cup a_m$ , and by applying linearity  $m-1$  times we arrive at  $b_m \preceq a_m$ . This does not work in the general case, because when  $\text{count}_x(a_1, \dots, a_m) \geq 2$ , the large union  $a_1 \cup \dots \cup a_m$  fails to keep track of the different repetitions of  $x$ . However, we can bypass this issue by ‘duplicating’ the elements of  $X$ . In a critical lemma, we show that we can introduce equivalent copies  $x^1, \dots, x^{2^m}$  of every  $x \in X$ , in a way that preserves positivity, comparability, and a weaker version of linearity. We then tweak the sets  $a_1, \dots, a_m, b_1, \dots, b_m$  by replacing their members with corresponding copies, so that one copy never occurs twice (see the example below).

$$\begin{array}{ll|ll} a_1 = \{x_1, x_3\} & b_1 = \{x_1, x_2\} & a_1^* = \{x_1^1, x_3^1\} & b_1^* = \{x_1^5, x_2^5\} \\ a_2 = \{x_1, x_2\} & b_2 = \{x_3\} & a_2^* = \{x_1^2, x_2^1\} & b_2^* = \{x_3^5\} \\ a_3 = \{x_3\} & b_3 = \{x_1, x_2, x_3\} & a_3^* = \{x_3^2\} & b_3^* = \{x_1^6, x_2^5, x_3^7\} \\ a_4 = \{x_1\} & b_4 = \{x_1\} & a_4^* = \{x_1^3\} & b_4^* = \{x_1^8\} \end{array}$$

It then suffices to apply the previous reasoning to the sets  $a_1^*, \dots, a_m^*, b_1^*, \dots, b_m^*$ .

We also addressed the case of arbitrary measurable relations.

**Theorem 3.** *A relation  $\preceq$  on  $B$  is measurable if and only if the following conditions are satisfied, for all  $a, b, c, d \in B$ :*

- **Comparability:**  $a \preceq b$  or  $b \preceq a$ ;
- **Transitivity:**  $a \preceq b$  and  $b \preceq c$  implies  $a \preceq c$ ;
- **Monotonicity:**  $a \subseteq b$  implies  $a \preceq b$ ;
- **Bounded Linearity:** if  $1 \not\preceq c$  and  $a \cap c = 0$  and  $a \cup c \preceq b \cup d$  and  $d \preceq c$ , then  $a \preceq b$ .

Here the strategy is to apply Theorem 2 to a well-chosen Boolean algebra  $B' \subseteq B$ , and to extend the resulting finite measure to a measure on  $B$  with the desired properties.

Finally, we observe that the conditions of Theorem 2 and Theorem 3 can be checked in time polynomial in the size of  $B$ . In the case of finitely measurable relations, this results in a direct improvement on the polynomial space algorithm of Kraft, Pratt and Seidenberg [KPS59].

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