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Intuitionistic modal logics: a minimal setting

Philippe Balbiani^{1*} and Çiğdem Gencer^{1,2†}

¹ Toulouse Institute of Computer Science Research
CNRS-INPT-UT3, Toulouse, France

² Faculty of Arts and Sciences
Aydın University, Istanbul, Turkey

Abstract

We make a clean sweep of the tradition in intuitionistic modal logics by considering a new truth condition of \diamond -formulas saying that in model (W, \leq, R, V) , $\diamond A$ holds at $s \in W$ if there exists $t \in W$ where A holds and such that $s \geq \circ Rt$. While keeping the truth condition of \Box -formulas that is commonly used, we axiomatize validity in the class of all models. The resulting logic is the intuitionistic modal logic that we want to put forward as a candidate for the title of “minimal intuitionistic modal logic”.

1 Syntax and semantics

Let \mathbf{At} be a set of *atoms* (p, q , etc). The set \mathbf{Fo} of all *formulas* (A, B , etc) is defined by $A ::= p | (A \rightarrow A) | \top | \perp | (A \wedge A) | (A \vee A) | \Box A | \diamond A$. For all $A \in \mathbf{Fo}$, $\neg A$ is the abbreviation for $(A \rightarrow \perp)$.

A *Kripke frame* or a *KF* is a structure of the form (W, \leq, R) where W is a nonempty set, \leq is a partial order on W and R is a binary relation on W . Let $\mathcal{C}_{\text{all}}^{\text{kf}}$ be the class of all KFs. A KF (W, \leq, R) is *forward* (respectively: *backward*; *downward*) *confluent* if for all $s, t \in W$, if $s \geq \circ Rt$ then $sR \circ \geq t$ (respectively: for all $s, t \in W$, if $sR \circ \leq t$ then $s \leq \circ Rt$; for all $s, t \in W$, if $s \leq \circ Rt$ then $sR \circ \leq t$). Let $\mathcal{C}_{\text{fc}}^{\text{kf}}$ (respectively: $\mathcal{C}_{\text{bc}}^{\text{kf}}$; $\mathcal{C}_{\text{dc}}^{\text{kf}}$; $\mathcal{C}_{\text{fbc}}^{\text{kf}}$; $\mathcal{C}_{\text{fdc}}^{\text{kf}}$; $\mathcal{C}_{\text{bdc}}^{\text{kf}}$; $\mathcal{C}_{\text{fbdc}}^{\text{kf}}$) be the class of all forward (respectively: backward; downward; forward and backward; forward and downward; backward and downward; forward, backward and downward) confluent KFs. A *valuation on a KF* (W, \leq, R) is a function $V : \mathbf{At} \rightarrow \wp(W)$ associating a \leq -closed subset of W to each atom. Such a function can be extended as a function $V : \mathbf{Fo} \rightarrow \wp(W)$ associating to each $A \in \mathbf{Fo}$ a \leq -closed subset $V(A)$ of W defined as usual when either A is an atom, or the main connective of A is intuitionistic and as follows otherwise: (i) $V(\Box A) = \{s \in W : \text{for all } t \in W, \text{ if } s \leq \circ Rt \text{ then } t \in V(A)\}$; (ii) $V(\diamond A) = \{s \in W : \text{there exists } t \in W \text{ such that } s \geq \circ Rt \text{ and } t \in V(A)\}$. A *relational model* is a couple consisting of a KF and a valuation on that KF. *Truth in a relational model*, *validity in a KF* and *validity on a class of KFs* are defined as usual. For all classes \mathcal{C} of KFs, let $\text{Log}(\mathcal{C})$ be the *logic of* \mathcal{C} .

A *H-modal algebra* or a *HMA* is a structure of the form $(H, \leq_H, \rightarrow_H, \Box_H, \diamond_H)$ where $(H, \leq_H, \rightarrow_H)$ is a Heyting algebra and $\Box_H : H \rightarrow H$ and $\diamond_H : H \rightarrow H$ are operators such that for all $a, b, c \in H$: (i) $\Box_H \top_H = \top_H$; (ii) $\Box_H(a \wedge_H b) = \Box_H a \wedge_H \Box_H b$; (iii) $\diamond_H \perp_H = \perp_H$; (iv) $\diamond_H(a \vee_H b) = \diamond_H a \vee_H \diamond_H b$; (v) if $\diamond_H a \leq_H b \vee_H \Box_H(a \rightarrow_H c)$ then $\diamond_H a \leq_H b \vee_H \diamond_H c$. Let $\mathcal{C}_{\text{all}}^{\text{hma}}$ be the class of all HMAs. A HMA $(H, \leq_H, \rightarrow_H, \Box_H, \diamond_H)$ is *forward* (respectively: *backward*; *downward*) *confluent* if for all $a, b \in H$, $\diamond_H(a \rightarrow_H b) \leq_H (\Box_H a \rightarrow_H \diamond_H b)$ (respectively: $(\diamond_H a \rightarrow_H \Box_H b) \leq_H \Box_H(a \rightarrow_H b)$; $\Box_H(a \vee_H b) \leq_H \diamond_H a \vee_H \Box_H b$). Let $\mathcal{C}_{\text{fc}}^{\text{hma}}$ (respectively: $\mathcal{C}_{\text{bc}}^{\text{hma}}$;

*Email address: philippe.balbani@irit.fr

†Email addresses: cigdem.gencer@irit.fr and cigdemgencer@aydin.edu.tr.

$\mathcal{C}_{\text{dc}}^{\text{hma}}, \mathcal{C}_{\text{fbc}}^{\text{hma}}, \mathcal{C}_{\text{fdc}}^{\text{hma}}, \mathcal{C}_{\text{bdc}}^{\text{hma}}, \mathcal{C}_{\text{fbdc}}^{\text{hma}}$ be the class of all forward (respectively: backward; downward; forward and backward; forward and downward; backward and downward; forward, backward and downward) confluent HMA. A *valuation on a HMA* $(H, \leq_H, \rightarrow_H, \Box_H, \Diamond_H)$ is a function $V : \mathbf{At} \rightarrow H$ associating an element of H to each atom. Such a function can be extended as a function $V : \mathbf{Fo} \rightarrow H$ associating to each $A \in \mathbf{Fo}$ an element $V(A)$ of H defined as usual when either A is an atom, or the main connective of A is intuitionistic and as follows otherwise: (i) $V(\Box A) = \Box_H V(A)$; (ii) $V(\Diamond A) = \Diamond_H V(A)$. An *algebraic model* is a couple consisting of a HMA and a valuation on that HMA. *Truth in an algebraic model, validity in a HMA and validity on a class of HMAs* are defined as usual. For all classes \mathcal{C} of HMAs, let $\text{Log}(\mathcal{C})$ be the *logic of \mathcal{C}* .

2 Axiomatization and completeness

An *intuitionistic modal logic* is a set of formulas closed for uniform substitution, containing the standard axioms of **IPL**, closed with respect to the standard inference rules of **IPL**, containing the axioms $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$, $\Box(p \vee q) \rightarrow ((\Diamond p \rightarrow \Box q) \rightarrow \Box q)$, $\Diamond(p \vee q) \leftrightarrow \Diamond p \vee \Diamond q$ and $\neg \Diamond \perp$ and closed with respect to the inference rules $\frac{p}{\Box p}$, $\frac{p \leftrightarrow q}{\Diamond p \leftrightarrow \Diamond q}$ and $\frac{\Diamond p \rightarrow q \vee \Box(p \rightarrow r)}{\Diamond p \rightarrow q \vee \Diamond r}$. We also consider the axioms (**Af**) $\Diamond(p \rightarrow q) \rightarrow (\Box p \rightarrow \Diamond q)$, (**Ab**) $(\Diamond p \rightarrow \Box q) \rightarrow \Box(p \rightarrow q)$ and (**Ad**) $\Box(p \vee q) \rightarrow \Diamond p \vee \Box q$. Let \mathbf{L}_{\min} be the least intuitionistic modal logic. For all intuitionistic modal logics \mathbf{L} and for all $A \in \mathbf{Fo}$, let $\mathbf{L} \oplus A$ be the least intuitionistic modal logic containing \mathbf{L} and A . Let \mathbf{L}_{fc} (respectively: \mathbf{L}_{bc} ; \mathbf{L}_{dc} ; \mathbf{L}_{fbc} ; \mathbf{L}_{fdc} ; \mathbf{L}_{bdc} ; \mathbf{L}_{fbdc}) be $\mathbf{L}_{\min} \oplus (\mathbf{Af})$ (respectively: $\mathbf{L}_{\min} \oplus (\mathbf{Ab})$; $\mathbf{L}_{\min} \oplus (\mathbf{Ad})$; $\mathbf{L}_{\min} \oplus (\mathbf{Af}) \oplus (\mathbf{Ab})$; $\mathbf{L}_{\min} \oplus (\mathbf{Af}) \oplus (\mathbf{Ad})$; $\mathbf{L}_{\min} \oplus (\mathbf{Ab}) \oplus (\mathbf{Ad})$; $\mathbf{L}_{\min} \oplus (\mathbf{Af}) \oplus (\mathbf{Ab}) \oplus (\mathbf{Ad})$).

Proposition 1. • $\mathbf{L}_{\min} = \text{Log}(\mathcal{C}_{\text{all}}^{\text{kf}}) = \text{Log}(\mathcal{C}_{\text{all}}^{\text{hma}})$;

- $\mathbf{L}_{\text{fc}} = \text{Log}(\mathcal{C}_{\text{fc}}^{\text{kf}}) = \text{Log}(\mathcal{C}_{\text{fc}}^{\text{hma}})$; $\mathbf{L}_{\text{bc}} = \text{Log}(\mathcal{C}_{\text{bc}}^{\text{kf}}) = \text{Log}(\mathcal{C}_{\text{bc}}^{\text{hma}})$; $\mathbf{L}_{\text{dc}} = \text{Log}(\mathcal{C}_{\text{dc}}^{\text{kf}}) = \text{Log}(\mathcal{C}_{\text{dc}}^{\text{hma}})$;
- $\mathbf{L}_{\text{fbc}} = \text{Log}(\mathcal{C}_{\text{fbc}}^{\text{kf}}) = \text{Log}(\mathcal{C}_{\text{fbc}}^{\text{hma}})$; $\mathbf{L}_{\text{fdc}} = \text{Log}(\mathcal{C}_{\text{fdc}}^{\text{kf}}) = \text{Log}(\mathcal{C}_{\text{fdc}}^{\text{hma}})$; $\mathbf{L}_{\text{bdc}} = \text{Log}(\mathcal{C}_{\text{bdc}}^{\text{kf}}) = \text{Log}(\mathcal{C}_{\text{bdc}}^{\text{hma}})$;
- $\mathbf{L}_{\text{fbdc}} = \text{Log}(\mathcal{C}_{\text{fbdc}}^{\text{kf}}) = \text{Log}(\mathcal{C}_{\text{fbdc}}^{\text{hma}})$.

Proposition 2. • **WK** [3] and \mathbf{L}_{\min} are not comparable;

- **WK** [3] is strictly contained in \mathbf{L}_{fc} ;
- \mathbf{L}_{fc} and **FIK** [1] are equal;
- \mathbf{L}_{fbc} and **IK** [2] are equal;
- \mathbf{L}_{fbdc} is strictly contained in **K** — the least normal modal logic.

All in all, \mathbf{L}_{\min} is the intuitionistic modal logic that we want to put forward as a candidate for the title of “minimal intuitionistic modal logic”.

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