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# Representation of a Noisy Transmission Line

Martin Bucher and Daniel Molnar  
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## Abstract

We analyse a lossy transmission line and the Johnson-Nyquist noise generated therein. A representation as a noisy two-port with a voltage and a current noise sources on one end of a noiseless two-port is given. An expression for the noise properties is given for an arbitrary temperature profile along the transmission line. Compatibility of the general expression found here with expectations for special cases calculable using thermodynamics is demonstrated.

## Index Terms

noise, Gaussian noise, colored noise, circuit noise, linear circuits, transmission line noise, cable noise, transmission line measurements

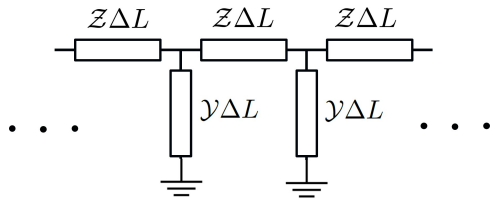


Fig. 1. A cable of uniform along its length is represented as an alternating sequence of series and shunt elements chained together.

## I. MOTIVATION

Here we investigate how to model and to predict the noise generated in a lossy transmission line at high accuracy and in the case where the temperature is not uniform along the cable length. For many applications the level of detail developed in this paper is not needed. Our motivation is to better model noise in radio astronomy experiments that aim to measure the so-called 21 cm global signal. (See for example [1] and references therein for a discussion.) These experiments seek to measure predicted deviations of order a few tens of mK from the perfect 2.75 K CMB (Cosmic Microwave Background), which must be separated from synchrotron emission from our own Galaxy (separable on account of its smooth frequency dependence) whose Rayleigh-Jeans brightness temperature at the frequencies of interest is of order  $10^3$ – $10^4$  K around the Galactic plane. In contrast to measurements of the CMB anisotropies, where the data consist of measurements of small temperature differences between different directions in the sky, global 21 cm experiments have more the character of an absolute measurement of electrical noise. A detection of a 21cm absorption trough centered around 78 MHz deeper than the theoretically expected has been claimed [2], but it is presently unclear whether this feature is a genuine sky signal

or an instrumental artefact. Other experiments are underway to verify this claim and detect the elusive 21 cm global signal. (See for example [3] and references therein.) The case will remain open until a convincing argument can be put forward that the instrumental setup has been correctly modelled at the requisite accuracy.

Since the amplifier chain cannot be calibrated absolutely, thermal (Johnson-Nyquist) noise sources (i.e., resistors at known temperatures) and calibrated noise sources (e.g., standard noise diodes) having various source impedances are used to calibrate the amplifier gain [4]–[7] and measure the four real noise parameters of the LNA [8]. In order to characterize these quantities at the required accuracy, the contribution of the Johnson-Nyquist noise generated within the connecting transmission lines (i.e., cables) due to resistive losses needs to be calculated.

In this paper, we consider only Johnson-Nyquist noise, and linearity is always assumed. We note that if the conductors of the transmission line also carry a DC bias or power supply current, additional noise may be generated due to resistance fluctuations, and this excess noise adds in quadrature. Such excess noise whose power is proportional to the square of the DC current typically has a red spectrum, with  $P \sim \nu^{-\alpha}$  where  $\alpha \approx 1$ . The pure power law here is only an approximation: unlike for the case of Johnson-Nyquist noise, resistance fluctuations do not follow a precise universal functional form. Nothing universal may be said about such excess noise, and we are not considering this type of noise in the current work. We also assume the absence of long-range spatial correlations in the Johnson-Nyquist noise.

## II. GENERAL REPRESENTATION OF A CABLE

A cable may be modelled as an infinite sequence of infinitesimal series and shunt elements chained together, as illustrated in Fig. 1. These may most conveniently be represented in the transfer matrix (or *ABCD*-matrix) representation (see e.g., [9]). From the representation of a cable element of infinitesimal length, the transfer matrix for a cable of finite length  $L$  can be derived using the usual rules of matrix

exponentiation. We write

$$\begin{aligned}\mathcal{T}_{series}(\Delta L) &= \begin{pmatrix} 1 & \mathcal{Z}(\Delta L) \\ 0 & 1 \end{pmatrix} + O(\Delta L^2) \\ &= \begin{pmatrix} 1 & +j\bar{\mathcal{X}}(\Delta L) \\ 0 & 1 \end{pmatrix} + O(\Delta L^2), \\ \mathcal{T}_{shunt}(\Delta L) &= \begin{pmatrix} 1 & 0 \\ \mathcal{Y}(\Delta L) & 1 \end{pmatrix} + O(\Delta L^2) \\ &= \begin{pmatrix} 1 & 0 \\ +j\bar{\mathcal{B}}(\Delta L) & 1 \end{pmatrix} + O(\Delta L^2)\end{aligned}\quad (1)$$

where  $\mathcal{Z}$  and  $\mathcal{Y}$  are impedance and admittance per unit length, respectively. Here we define complex versions of the reactance and susceptance per unit length as  $\bar{\mathcal{X}} = \mathcal{X} - j\mathcal{R}$  and  $\bar{\mathcal{B}} = \mathcal{B} - j\mathcal{G}$  in order to facilitate comparison with lossless case. We work in a narrow frequency band so the frequency dependence is implicit. Such a representation is always possible when the cable respects reciprocity, in which case a 2-port always has a  $\pi$  or  $T$  representation. Homogeneity along the length of the cable is also assumed, as is linearity. For a finite length  $L$  of cable, we obtain the transfer matrix

$$\begin{aligned}\mathcal{T}(L) &= \exp \left[ \begin{pmatrix} 0 & j\bar{\mathcal{X}} \\ j\bar{\mathcal{B}} & 0 \end{pmatrix} L \right] \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\bar{\mathcal{B}}/\bar{\mathcal{X}}} \end{pmatrix} \exp \left[ +j\sqrt{\bar{\mathcal{B}}\bar{\mathcal{X}}}L\sigma_1 \right] \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\bar{\mathcal{X}}/\bar{\mathcal{B}}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\bar{\mathcal{B}}/\bar{\mathcal{X}}} \end{pmatrix} \left[ \cos[(k-j\gamma)L]\sigma_0 + j\sin[(k-j\gamma)L]\sigma_1 \right] \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\bar{\mathcal{X}}/\bar{\mathcal{B}}} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\bar{\mathcal{B}}/\bar{\mathcal{X}}} \end{pmatrix} \left[ \frac{e^{+jkL+\gamma L} + e^{-jkL-\gamma L}}{2}\sigma_0 + \frac{e^{+jkL+\gamma L} - e^{-jkL-\gamma L}}{2}\sigma_1 \right] \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{\bar{\mathcal{X}}/\bar{\mathcal{B}}} \end{pmatrix}\end{aligned}\quad (2)$$

where  $\sqrt{\bar{\mathcal{B}}\bar{\mathcal{X}}} = \bar{k} = k - j\gamma$ . We may rewrite

$$\mathcal{T} = \begin{pmatrix} \cos(\bar{k}L) & +jZ_c \sin(\bar{k}L) \\ +jZ_c^{-1} \sin(\bar{k}L) & \cos(\bar{k}L) \end{pmatrix}. \quad (3)$$

In terms of the elements of the transfer matrix, the impedance matrix  $\mathbf{Z}$  is given by

$$\mathbf{Z} = \begin{pmatrix} \frac{A}{C} & \frac{AD-BC}{C} \\ \frac{1}{C} & \frac{D}{C} \end{pmatrix}, \quad (4)$$

so that for the above

$$\mathbf{Z} = Z_c \begin{pmatrix} -j \cot(\bar{k}L) & -j \csc(\bar{k}L) \\ -j \csc(\bar{k}L) & -j \cot(\bar{k}L) \end{pmatrix}. \quad (5)$$

For an infinitely long cable, as  $L \rightarrow \infty$ ,  $\cot(\bar{k}L) \rightarrow +j$  and  $\csc(\bar{k}L) \rightarrow +j$  because  $\text{Im}(\bar{k}) < 0$ . Consequently, in this limit

$$\mathbf{Z} \rightarrow \begin{pmatrix} Z_c & 0 \\ 0 & Z_c \end{pmatrix}. \quad (6)$$

Note that  $Z_c$  in the general case has a non-vanishing imaginary part, so that no reflectionless matching with a lossless transmission line is possible.

For a finite section of cable it is useful to express the cable behavior in terms of an  $S$ -matrix with respect to some real reference characteristic impedance  $\bar{Z}_c$  (e.g.,  $\bar{Z}_c = 50 \Omega$ ), so that

$$\mathbf{S} = \frac{\mathbf{Z} - \bar{Z}_c \mathbf{I}}{\mathbf{Z} + \bar{Z}_c \mathbf{I}}. \quad (7)$$

Here  $Z_c = \sqrt{\mathcal{Z}/\mathcal{Y}} = \sqrt{\bar{\mathcal{X}}/\bar{\mathcal{B}}}$  acts as a characteristic impedance of the lossy transmission line, which in the lossy case is not real. We see that the cable is characterized by four real parameters. Any symmetric reciprocal cable is completely characterized by the above parameterization.

Unlike in the time domain Telegrapher's equation where the series and shunt resistances are independent of frequency (as are the series inductance and shunt capacitance), here no assumption is made that the propagation can be adequately modelled by a PDE in  $z$  and  $t$ . In reality, as a consequence of the skin effect, the effective series resistance increases with frequency, and the attenuation due to dielectric loss (represented as shunt resistance) increases even more rapidly with frequency. Here no particular dependence on frequency is assumed for the effective series impedance and shunt admittance.

### III. THE NOISY CABLE

The description above is so far noiseless. We have not yet associated Johnson-Nyquist noise with the resistive elements. Assuming a physical temperature  $T$ , we replace each series resistor in the infinitesimal equivalent circuit with a resistor and a random voltage source in series, and similarly we replace each shunt resistor with the same shunt resistor and a random current source in parallel (Thévenin and Norton equivalent).

This is not a convenient representation, as in general we would wish to view the cable as a noisy two-port, moving all the distributed noise sources to the output end of the cable. How to move noise sources between ports for the general case of an  $N$ -port is described in detail in Ref. [11] This can be done simply using the form of the transfer matrix found above, so that

$$\begin{aligned}\begin{pmatrix} v_{out}(L) \\ i_{out}(L) \end{pmatrix} &= \int_0^L dL' \\ &\times \begin{pmatrix} T_{vv}(L'_L) & T_{vi}(L'_L) \\ T_{iv}(L'_L) & T_{ii}(L'_L) \end{pmatrix} \begin{pmatrix} \frac{\delta v(L')}{dL'} \\ \frac{\delta i(L')}{dL'} \end{pmatrix}. \end{aligned}\quad (8)$$

Here  $\delta v(L')$  and  $\delta i(L')$  are the series voltage and shunt current sources associated with the interval  $[L', L' + dL']$ . So far these are infinitesimal complex numbers. We have not so far made

any assumptions regarding how these are determined (e.g., from Johnson-Nyquist noise). For the correlation functions, we obtain

$$\begin{pmatrix} \langle v_{out}(L)v_{out}(L)^* \rangle & \langle v_{out}(L)i_{out}(L)^* \rangle \\ \langle i_{out}(L)v_{out}(L)^* \rangle & \langle i_{out}(L)i_{out}(L)^* \rangle \end{pmatrix} = \int_0^L dL' \int_0^L dL'' \mathbf{T}(L; L') \begin{pmatrix} \langle \delta v(L')\delta v(L'')^* \rangle & \langle \delta v(L')\delta i(L'')^* \rangle \\ \langle \delta i(L')\delta v(L'')^* \rangle & \langle \delta i(L')\delta i(L'')^* \rangle \end{pmatrix} \mathbf{T}^\dagger(L; L'') \quad (9)$$

In the case of spatially white noise (which here follows from the hypothesis of translation invariance), this simplifies to

$$\begin{pmatrix} \langle v_{out}(L)v_{out}(L)^* \rangle & \langle v_{out}(L)i_{out}(L)^* \rangle \\ \langle i_{out}(L)v_{out}(L)^* \rangle & \langle i_{out}(L)i_{out}(L)^* \rangle \end{pmatrix} = \int_0^L dL' \mathbf{T}(L; L') \begin{pmatrix} W_{vv}(L') & W_{vi}(L') \\ W_{iv}(L') & W_{ii}(L') \end{pmatrix} \mathbf{T}^\dagger(L; L') \quad (10)$$

where

$$\langle \delta a(L') \delta b(L'')^* \rangle = \delta(L' - L'') W_{ab}(L'). \quad (11)$$

Assuming Johnson noise in the resistive elements of the infinitesimal equivalent circuit for bandwidth  $B$  and in the

absence of correlation in the noise of the shunt and series resistances, we obtain

$$\begin{aligned} W_{vv} &= 4\mathcal{R}k_B T(L)B, \\ W_{ii} &= 4\mathcal{G}k_B T(L)B, \\ W_{vi} &= W_{iv} = 0. \end{aligned} \quad (12)$$

The cross term vanishes by reflection symmetry about the cable cross section. Here we have written  $T(L)$  in the place of  $T$  in order to admit the possibility of a varying temperature along the cable.

$$\begin{aligned} \begin{pmatrix} \langle v_{out}(L)v_{out}(L)^* \rangle & \langle v_{out}(L)i_{out}(L)^* \rangle \\ \langle i_{out}(L)v_{out}(L)^* \rangle & \langle i_{out}(L)i_{out}(L)^* \rangle \end{pmatrix} &= 4k_B B \int_0^L dx T(x) \\ &\times \begin{pmatrix} \cos[\bar{k}x] & jZ_c \sin[\bar{k}x] \\ j \sin[\bar{k}x]/Z_c & \cos[\bar{k}x] \end{pmatrix} \begin{pmatrix} \mathcal{R} & 0 \\ 0 & \mathcal{G} \end{pmatrix} \begin{pmatrix} \cos[\bar{k}^*x] & -j \sin[\bar{k}^*x]/Z_c^* \\ -jZ_c^* \sin[\bar{k}^*x] & \cos[\bar{k}^*x] \end{pmatrix} \\ &= 4k_B B \int_0^L dx T(x) \left[ \mathcal{R} \begin{pmatrix} \cos[\bar{k}x] \cos[\bar{k}^*x] & -j \frac{\cos[\bar{k}x] \sin[\bar{k}^*x]}{Z_c^*} \\ +j \frac{\sin[\bar{k}x] \cos[\bar{k}^*x]}{Z_c} & \frac{\sin[\bar{k}x] \sin[\bar{k}^*x]}{Z_c Z_c^*} \end{pmatrix} \right. \\ &\quad \left. + \mathcal{G} \begin{pmatrix} Z_c Z_c^* \sin[\bar{k}x] \sin[\bar{k}^*x] & +jZ_c \sin[\bar{k}x] \cos[\bar{k}^*x] \\ -jZ_c^* \cos[\bar{k}x] \sin[\bar{k}^*x] & + \cos[\bar{k}x] \cos[\bar{k}^*x] \end{pmatrix} \right] \end{aligned} \quad (13)$$

The above general formula can be evaluated analytically for a uniform physical temperature along the length of cable as well as for some simple  $T(L)$  profiles. Otherwise numerical integration is required. In the case of uniform physical temperature along the cable's length, i.e.,  $T(L) = T_0$ , eqn.(13) can be evaluated analytically to give

$$\begin{aligned} \begin{pmatrix} \langle v_{out}(L)v_{out}(L)^* \rangle & \langle v_{out}(L)i_{out}(L)^* \rangle \\ \langle i_{out}(L)v_{out}(L)^* \rangle & \langle i_{out}(L)i_{out}(L)^* \rangle \end{pmatrix} &= 4\mathcal{R}k_B T_0 B \begin{pmatrix} \left[ \frac{\sinh(2\gamma L)}{4\gamma} + \frac{\sin(2kL)}{4k} \right] & \frac{1}{Z_c^*} \left[ + \frac{\sinh^2(\gamma L)}{2\gamma} - \frac{j \sin^2(kL)}{2k} \right] \\ \frac{1}{Z_c} \left[ + \frac{\sinh^2(\gamma L)}{2\gamma} + \frac{j \sin^2(kL)}{2k} \right] & \frac{1}{Z_c Z_c^*} \left[ \frac{\sinh(2\gamma L)}{4\gamma} - \frac{\sin(2kL)}{4k} \right] \end{pmatrix} \\ &+ 4\mathcal{G}k_B T_0 B \begin{pmatrix} Z_c Z_c^* \left[ \frac{\sinh(2\gamma L)}{4\gamma} - \frac{\sin(2kL)}{4k} \right] & Z_c \left[ + \frac{\sinh^2(\gamma L)}{2\gamma} + \frac{j \sin^2(kL)}{2k} \right] \\ Z_c^* \left[ + \frac{\sinh^2(\gamma L)}{2\gamma} - \frac{j \sin^2(kL)}{2k} \right] & \left[ \frac{\sinh(2\gamma L)}{4\gamma} + \frac{\sin(2kL)}{4k} \right] \end{pmatrix}. \end{aligned} \quad (14)$$

The correlation matrix in eqn. (14) may be transformed to the travelling wave representation [12], [13] with respect to a reference characteristic impedance  $Z_{ref}$  where  $v_{L,R} = (v \mp Z_{ref}i)/2\sqrt{\text{Re}(Z_{ref})}$ . Setting  $Z_{ref} = Z_c$ , we obtain

$$\begin{pmatrix} \langle v_L(L)v_L(L)^* \rangle & \langle v_L(L)v_R(L)^* \rangle \\ \langle v_R(L)v_L(L)^* \rangle & \langle v_R(L)v_R(L)^* \rangle \end{pmatrix} = \frac{4\mathcal{R}k_B T_0 B}{4\text{Re}(Z_{ref})} \begin{pmatrix} \frac{1 - \exp(-2\gamma L)}{2\gamma} & \frac{-j(\exp(-2jkL) - 1)}{2k} \\ +j(\exp(+2jkL) - 1) & \frac{\exp(+2\gamma L) - 1}{2\gamma} \end{pmatrix} \\ + \frac{4\mathcal{G}k_B T_0 B |Z_c|^2}{4\text{Re}(Z_{ref})} \begin{pmatrix} \frac{1 - \exp(-2\gamma L)}{2\gamma} & \frac{-j(\exp(-2jkL) - 1)}{2k} \\ +j(\exp(+2jkL) - 1) & \frac{\exp(+2\gamma L) - 1}{2\gamma} \end{pmatrix}. \quad (15)$$

We observe that as the cable becomes long, the amplitude of the travelling noise wave propagating into the cable diverges while the noise wave emanating from the cable approaches a constant amplitude. This entering divergent wave is needed to produce the wave incident on the other end of the cable. The divergence is exactly such that this wave has an amplitude approaching a constant by the time it reaches the other end at  $x = L$  when  $L$  is large.

#### IV. COMPATIBILITY WITH THERMODYNAMIC EXPECTATIONS

In the above discussion, a section of lossy cable at temperature  $T$  was represented as a noisy 2-port whose noise properties were inferred by replacing each resistor in the infinitesimal equivalent circuit with the resistor-source combination suggested by the Johnson-Nyquist formula [14]–[16]. Here we use as a starting point the S-matrix representation for a finite cable section and derive the constraints imposed by considerations of thermodynamic equilibrium. For the sequence of ports (one-port)<sub>1</sub>-(two-port)-(one-port)<sub>2</sub> connected together and all at the same temperature, the noise sources must have values such that there is no net power transfer across the two junction. We verify that the cable model with noise sources as computed in the previous section satisfies these conditions.

We first review and generalize the thermodynamic argument. In the classic derivation of Johnson-Nyquist noise, a resistor of resistance  $R$ , which may be understood as a 1-port, is connected to a transmission line with  $Z_c = R$ . The TEM modes of the transmission line, which may be decomposed into left-moving and right-moving modes, are assumed to be in a thermal state at a temperature  $T$  equal to the temperature of the resistor. As indicated in Fig. 2, the left-moving waves in this setup are completely absorbed by the resistor, which by design has been perfectly matched. In order to obtain zero net power transfer, the right-moving noise source must on the average replenish the left-moving power flux at each frequency; consequently  $\langle |v_L|^2 \rangle = R k_B T (\Delta B)$ . This argument generalizes to the case where  $0 < |\Gamma| \leq 1$ , in which case we obtain

$$\langle |v_L|^2 \rangle = (1 - |\Gamma|^2) R k_B T (\Delta B). \quad (16)$$

We now generalize to a thermal 2-port, as illustrated in Fig. 3. Let us for the moment turn off the outgoing travelling

wave noise sources of the two-port model  $v_{1L}$  and  $v_{2R}$  and calculate the power deficit of the return power flux assuming that the external noise sources  $v_{1R}$  and  $v_{2L}$  are thermal sources. We obtain the following fractional power deficits

$$\frac{P_{1,return}}{P_{1,incident}} = 1 - |S_{11}|^2 - |S_{21}|^2, \\ \frac{P_{2,return}}{P_{2,incident}} = 1 - |S_{22}|^2 - |S_{12}|^2. \quad (17)$$

Consequently,

$$\langle |v_{1L}|^2 \rangle = (1 - |S_{11}|^2 - |S_{21}|^2) R k_B T (\Delta B), \\ \langle |v_{2R}|^2 \rangle = (1 - |S_{22}|^2 - |S_{12}|^2) R k_B T (\Delta B). \quad (18)$$

The same argument generalizes to describe the noise of a thermal  $N$ -port.

It may be verified that the expressions in eqns. (14) and (15) satisfy the thermodynamic requirements derived in this section when all components are at the same temperature.

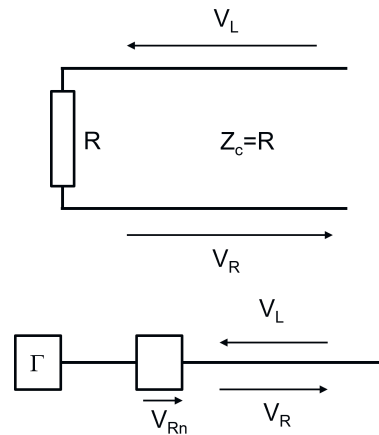


Fig. 2. **1-Port Thermal Noise.** Panel (a) illustrates the classic Johnson-Nyquist noise argument where the left-moving waves in a thermal state of the transmission line are completely absorbed by the resistor, which in turn through its internal thermal fluctuations generates a right-moving wave such that the left-moving and right-moving power fluxes cancel. In panel (b) this argument is generalized to an arbitrary 1-port with a not necessarily vanishing reflection coefficient  $\Gamma$  and an outgoing travelling wave noise source.

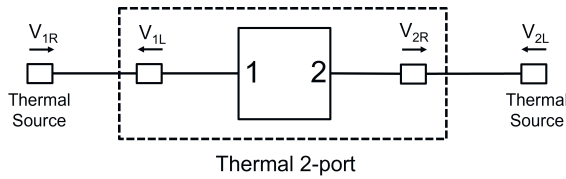


Fig. 3. **2-Port Thermal Noise.** The outgoing travelling wave sources in the 2-port generate the missing power absorbed within the thermal 2-port.

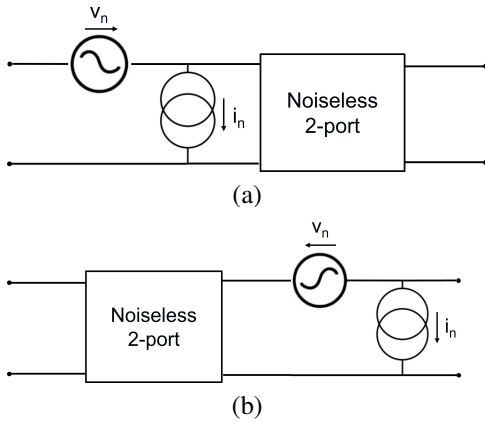


Fig. 4. Even though noise is generated everywhere along the cable, one can represent the noise generated within the cable by an equivalent circuit consisting of a noiseless 2-port with current and voltage sources localized at either end of the cable, as illustrated in (a) and (b). A noiseless cable is represented by a 2-port here.

## V. DISCUSSION

The matrix

$$\begin{pmatrix} 1 & \delta\mathcal{Z} \\ \delta\mathcal{G} & 1 \end{pmatrix} \quad (19)$$

expresses the most general possible infinitesimal transfer matrix consistent with symmetry when the ends are exchanged ( $1 \leftrightarrow 2$ ) and reciprocity, which is equivalent to requiring that the matrix have unit determinant. A linear perturbation to the diagonal would violate reciprocity or reflection symmetry. We see that a four-parameter model follows from these very general considerations. This modelling is without approximation.

In principle measurement of the reflection coefficient for two different finite lengths of cable with the same boundary condition, for example the pair  $\Gamma(L_1)^{(open)}$  and  $\Gamma(L_2)^{(open)}$ , or the pair  $\Gamma(L_1)^{(shorted)}$  and  $\Gamma(L_2)^{(shorted)}$ , suffices to determine  $\mathcal{Z}$  and  $\mathcal{Y}$  of the cable, as there are four real parameters to be determined. Alternatively, one could measure the 2-port S parameters of a single length of cable. The S-parameters can be converted to T or Z matrices by applying the standard conversion formulas. These four real parameters completely determine the noise properties assuming that the temperature profile along the cable is known.

To the extent that the loss tangent depends weakly on frequency, the dielectric loss (in dB per unit length) increases linearly with frequency (as opposed to linearly with the square root of the frequency, as is the case for the skin depth). Thus at higher frequencies dielectric loss dominates. Teflon (R) [or PTFE (polytetrafluoroethylene)] has a low loss tangent

compared to other less expensive dielectrics suitable at lower frequencies when dielectric loss is subdominant.

We have presented a general model of a lossy cable and the thermal noise generated within in which the cable is represented by a lossless 2-port with a voltage and current noise sources on one end. We showed that in the case where the cable is at a uniform temperature the resulting noise properties are consistent with thermodynamic expectations. However, the formula derived also extends to general case where the temperature profile along the cable is not uniform.

Whereas many models of lossy cables are parameterized by three real parameters, which at a given fixed frequency could for example be taken to be a phase velocity, real characteristic impedance, and attenuation coefficient, our model includes four real parameters because the characteristic impedance in the lossy case is in general complex (with a non-vanishing imaginary part). These four parameters can be readily be measured, for example using a VNA, and these parameters together with the temperature profile completely determine the noise properties of a section of cable provided that linearity holds and there are no DC bias currents.

Supplementary material with some examples implementing eqn. (13) as well as a code may be found at: [https://github.com/DannyMolnar/NoisyCable\\_paper](https://github.com/DannyMolnar/NoisyCable_paper)

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