



HAL
open science

Discrepancy and sparsity

Mario Grobler, Yiting Jiang, Patrice Ossona de Mendez, Sebastian Siebertz,
Alexandre Vigny

► **To cite this version:**

Mario Grobler, Yiting Jiang, Patrice Ossona de Mendez, Sebastian Siebertz, Alexandre Vigny. Discrepancy and sparsity. *Journal of Combinatorial Theory, Series B*, 2024, 169, pp.96-133. <10.1016/j.jctb.2024.06.001>. <hal-04800291>

HAL Id: hal-04800291

<https://hal.science/hal-04800291v1>

Submitted on 24 Nov 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



HAL Authorization

Discrepancy and Sparsity^{*}

Mario Grobler

University of Bremen, Bremen, Germany

Yiting Jiang

*Université de Paris, CNRS, IRIF, F-75006, Paris, France
and Department of Mathematics, Zhejiang Normal University, China*

Patrice Ossona de Mendez

*Centre d'Analyse et de Mathématiques Sociales (CNRS, UMR 8557), Paris, France
and Computer Science Institute of Charles University, Praha, Czech Republic*

Sebastian Siebertz

University of Bremen, Bremen, Germany

Alexandre Vigny

University of Bremen, Bremen, Germany

Abstract

We study the connections between the notions of combinatorial discrepancy and graph degeneracy. In particular, we prove that the maximum discrepancy over all subgraphs H of a graph G of the neighborhood set system of H is sandwiched between $\Omega(\log \deg(G))$ and $\mathcal{O}(\deg(G))$, where $\deg(G)$ denotes the degeneracy of G . We extend this result to inequalities relating weak coloring numbers and discrepancy of graph powers and deduce a new characterization of bounded expansion classes.

Then, we switch to a model theoretical point of view, introduce pointer structures, and study their relations to graph classes with bounded expansion. We deduce that a monotone class of graphs has bounded expansion if and only if all the set systems definable in this class have bounded hereditary discrepancy.

Using known bounds on the VC-density of set systems definable in nowhere dense classes we also give a characterization of nowhere dense classes in terms of discrepancy.

As consequences of our results, we obtain a corollary on the discrepancy of neighborhood set systems of edge colored graphs, a polynomial-time algorithm to compute ε -approximations of size $\mathcal{O}(1/\varepsilon)$ for set systems definable in bounded expansion classes, an application to clique coloring, and even the non-existence of a quantifier elimination scheme for nowhere dense classes.

Keywords: graph sparsity, first-order logic, discrepancy, degeneracy, weak coloring numbers, definable set systems, class with bounded expansion, nowhere dense class, pointer structures, quantifier elimination, VC-density, epsilon-approximation, clique coloring

1. Introduction and some motivating examples

Discrepancy theory emerged from the study of the irregularities of statistical distributions and number sequences. It developed and became a central tool in computational geometry. Two decades ago, Matoušek [27] initiated the study of *combinatorial discrepancy*, which became a significant subject in its own right. The combinatorial discrepancy measures the inevitable irregularities of set systems and the inherent difficulty to approximate them. (We refer to Section 2 for the formal definitions of the concepts considered in this paper.)

Discrepancy theory offers powerful tools and techniques with many applications in computational geometry, probabilistic algorithms, derandomization, communication complexity, searching, machine learning, pseudorandomness, optimization, computer graphics, and more. Central notions in this theory are also the well known notions of VC-dimension, ε -nets and ε -approximations, the latter corresponding to the expected properties of a pseudorandom set. We refer the reader to the textbooks [9, 12, 28] for a presentation of the discrepancy method and some of its applications.

A structural theory of classes of sparse graphs emerged recently, which is based on the study of densities of shallow minors, generalized coloring numbers, and constrained orientations [35]. In this setting, two central notions are those of *classes with bounded expansion*, which generalize classes excluding a topological minor, and *nowhere dense classes*, which generalize classes locally excluding a topological minor. These classes have strong algorithmic and structural properties. In particular, in a nowhere dense class it can be checked in almost linear time whether a first-order formula is satisfied in a given graph from the class [19]. This last example is only one among others that witness a strong connection between sparsity theory and first-order logic. We refer the interested reader to [35] for a comprehensive study of these classes.

The aim of this paper is to establish a bridge between discrepancy theory and sparsity theory through a study of the combinatorial discrepancy of set systems defined from sparse classes of graphs by means of first-order formulas and further to give new characterizations of degeneracy, bounded expansion, and nowhere denseness in terms of discrepancy of definable set systems. Before proceeding, we take time for a few motivating examples.

Problem 1. Assume that a graph G has the property that for every red/blue coloring of the edges of G there exists a partition (A, B) of the vertex set of G such that the number of red (resp. blue) neighbors in A and B of any vertex differ by at most 1. Does G contain a vertex with small degree?

It follows from our results that such a graph G is 1303-degenerate (see Section 8.1).

*This paper is part of a project that has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement No 810115 – DYNASNET), from the German Research Foundation (DFG) with grant agreement No 444419611, and from the french ANR project HOSIGRA (ANR-17-CE40-0022).



Email addresses: grobler@uni-bremen.de (Mario Grobler), yjiang@irif.fr (Yiting Jiang), pom@ehess.fr (Patrice Ossona de Mendez), siebertz@uni-bremen.de (Sebastian Siebertz), vigny@uni-bremen.de (Alexandre Vigny)

Problem 2. Given a planar graph G , find a small subset F of edges such that, for every pair u, v of distinct vertices of G , the probability that an edge of G belongs to a uv -path of length at most 100 differs from the probability that an edge in F belongs to a uv -path of length at most 100 by at most ε .

We prove that a set F of edges of size $\mathcal{O}(1/\varepsilon)$ with the prescribed properties can be constructed deterministically in polynomial time (see Section 8.2).

Problem 3. Does there exist a constant c , such that the vertices of every map graph¹ G can be colored red or blue, in such a way that the difference between the number of red and blue vertices in every maximal clique of G is at most c ?

Although there are quite a few reasons to believe that such a constant would not exist (it is not even possible in general to color the vertices of a perfect graph red and blue in such a way that no maximal clique is monochromatic [7]) we prove that such a constant c exists for map graphs (see Section 8.3).

Last, we also consider the following (seemingly completely unrelated) problem from sparse finite model theory. It is known that every class of finite graphs with bounded expansion has a quantifier elimination scheme involving unary relations and functions [15]. As it is known that the fixed-parameter tractability of first-order model-checking extends from bounded expansion classes to the more general nowhere dense classes [18, 19], it is natural to ask whether the quantifier elimination scheme also extends. It has been conjectured that this is not the case, but no proof of this fact was known.

Problem 4. Give an example of a nowhere dense class \mathcal{C} of graphs such that there exists no expansion σ of the signature of graphs by unary relation and function symbols with the property that every first-order formula is equivalent on a σ -expansion \mathcal{C}^+ of \mathcal{C} to a quantifier-free first-order σ -formula.

We prove that the class \mathcal{C} of 1-subdivisions of bipartite graphs whose girth exceeds the maximum degree has the above property. Precisely, there is no expansion σ of the signature of graphs by unary relation and function symbols in which the formula $\varphi(x, y)$ expressing that x and y are at distance 2 in the graph is equivalent on a σ -expansion of \mathcal{C} to a quantifier-free first-order σ -formula (see Section 8.4).

All these results are obtained as special cases of general theorems on combinatorial discrepancy of set systems definable in sparse graph classes, which we prove in this paper.

¹A *map graph* is the vertex-face incidence graph of a planar map.

This paper is organized as follows: In Section 2 we recall some needed concepts and notations and present the results obtained in this paper. In Section 3 we relate the notions of degeneracy and discrepancy. This connection is extended in Section 4 to a relation linking generalized coloring numbers and discrepancy. From Section 5 onward, we take a (finite) model theoretic point of view, putting the results obtained in Section 3 in a wider perspective. In Section 6, we translate the quantifier elimination scheme introduced in [15] for classes with bounded expansion into a model theoretic language, which allows us to extend the result obtained in Section 4 to the general setting of set systems definable in graphs of a bounded expansion class. This section is admittedly technical and is basically a formal translation of known results. Thus, the reader unfamiliar with model theory may directly move to Theorems 6.1 and 6.2. We discuss the case of nowhere dense classes in Section 7 and applications in Section 8.

2. Preliminaries and statement of the results

2.1. Combinatorial discrepancy

Let (U, \mathcal{S}) be a *set system*, where \mathcal{S} is a collection of subsets of the *ground set* U . When the ground set is clear from the context, we refer to the set system as \mathcal{S} . The *discrepancy* of a mapping $\chi : U \rightarrow \{-1, 1\}$ on a set $S \in \mathcal{S}$ is $\text{disc}_\chi(S) = \left| \sum_{v \in S} \chi(v) \right|$; the *discrepancy* of χ on \mathcal{S} is the maximum of $\text{disc}_\chi(S)$ over all $S \in \mathcal{S}$, that is, $\text{disc}_\chi(\mathcal{S}) = \max_{S \in \mathcal{S}} \text{disc}_\chi(S)$. The (*combinatorial*) *discrepancy* of \mathcal{S} is the minimum discrepancy of a mapping $\chi : U \rightarrow \{-1, 1\}$ on \mathcal{S} , that is,

$$\text{disc}(\mathcal{S}) = \min_{\chi: U \rightarrow \{-1, 1\}} \max_{S \in \mathcal{S}} \left| \sum_{v \in S} \chi(v) \right|.$$

Thus, the combinatorial discrepancy of a set system measures how balanced a 2-coloring of this system can be.

It is usual to consider bounds for the discrepancy of a set system (U, \mathcal{S}) in terms of $n = \left| \bigcup \mathcal{S} \right|$ (and sometimes in terms of $m = |\mathcal{S}|$). For instance, by a celebrated result of Spencer [41], the discrepancy of a set system is in $\mathcal{O}(\sqrt{n} \log(m/n))$, and in the case where $m = n$ we have $\text{disc}(\mathcal{S}) \leq 6\sqrt{n}$. This latter bound is tight up to the constant. Another important result is the Theorem of Beck and Fiala [4].

Theorem 2.1 (Beck-Fiala Theorem). The discrepancy of a set system with *degree* at most t (that is, each element lies in at most t sets) is less than $2t$.

This theorem was subsequently improved by Bednarchak and Helm [5], who gave an upper bound of $2d - 3$ for $d \geq 3$. Then Bukh [6] proved that the upper bound can be decreased to $2d - \log^* d$ for sufficiently large d .

Many of the techniques to derive upper bounds for the discrepancy are non-constructive and it is difficult to efficiently find optimal discrepancy colorings. Given a set system on n elements and $m \in \mathcal{O}(n)$ sets, it is NP-hard to distinguish whether the system has discrepancy 0 or $\Omega(\sqrt{n})$ [8]. In particular, under the assumption $\text{P} \neq \text{NP}$, one cannot compute a function χ whose discrepancy on the set system is within factor $o(\sqrt{n})$ of the discrepancy of the system. However, in polynomial time one can compute a coloring χ with discrepancy $\mathcal{O}(\sqrt{n})$ [24]. Also, the proof of the Beck-Fiala Theorem is constructive, and gives a polynomial time deterministic algorithm to compute a coloring χ with discrepancy smaller than twice the degree of the set system.

A standard example of set systems with high discrepancy is given by the following.

Example 1 (Sylvester's example). Sylvester inductively constructed Hadamard matrices H_p of order 2^p for every non-negative integer p as follows: $H_0 = (1)$, $H_1 = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, and for $p \geq 1$ $H_{p+1} = \begin{pmatrix} H_p & H_p \\ H_p & -H_p \end{pmatrix} = H_1 \otimes H_p$, where \otimes denotes the Kronecker product of matrices. Let

$$\mathcal{S}_p = \{ \{j : (H_p)_{i,j} = 1\} : 1 \leq i \leq 2^p \}.$$

Then

$$\text{disc}(\mathcal{S}_p) = \Omega(\sqrt{n}),$$

where $n = |\bigcup \mathcal{S}_p|$.

Unfortunately, the discrepancy is known to be a *fragile* notion, as witnessed by the following standard example (see e.g. [3]): let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a set system with ground set U and let $\mathcal{S}' = \{S'_1, \dots, S'_m\}$ be a copy of \mathcal{S} with ground set U' disjoint from U . Then the set system $\mathcal{S}'' = \{S_1 \cup S'_1, \dots, S_m \cup S'_m\}$ has always discrepancy zero, independent of the discrepancy of \mathcal{S} . A more robust notion is the *hereditary discrepancy* of a set system (U, \mathcal{S}) , defined as $\text{herdisc}(\mathcal{S}) = \max_{U' \subseteq U} \text{disc}(\mathcal{S}|_{U'})$, where $\mathcal{S}|_{U'}$ denotes the set system $\{S \cap U' : S \in \mathcal{S}\}$. Moreover, bounding the hereditary discrepancy allows to bound the sizes of ε -nets and ε -approximations (what is not the case for discrepancy). For this reason, we focus on hereditary discrepancy in this paper. Note that the bound in the Beck-Fiala Theorem applies to the hereditary discrepancy as well, as considering the trace over a subset of the universe does not increase the degree of a set system.

2.2. Sparse graph classes.

We consider finite, simple and undirected graphs. For a graph G we write $V(G)$ for its vertex set and $E(G)$ for its edge set. We denote by $\delta(G)$ the *minimum degree* of G , by $\bar{d}(G)$ the *average degree* of G , and by $\omega(G)$ the *clique number* of G .

We denote by $N_G(v)$ the *neighborhood* of v in G , that is, the set of all the vertices adjacent to v in G . Of prime interest is the *neighborhood set system* $\mathcal{S}^E(G)$ of a graph G , which is defined as $\mathcal{S}^E(G) = \{N_G(v) : v \in V(G)\}$.

We write $H \subseteq G$ (resp. $H \subseteq_i G$) if H is a subgraph (resp. an induced subgraph) of G . A graph G is *d-degenerate* if every non-empty induced subgraph of G has minimum degree at most d . The minimum integer d such that a graph G is d -degenerate is the *degeneracy* $\text{deg}(G)$ of G . Hence, $\text{deg}(G) = \max_{H \subseteq_i G} \delta(H)$.

A class \mathcal{C} of graphs is called *monotone* if it is closed under taking subgraphs and *hereditary* if it is closed under taking induced subgraphs. A class \mathcal{C} is *degenerate* if there is an integer d such that all the graphs in \mathcal{C} are d -degenerate.

The *r-subdivision* of a graph G is the graph $G^{(r)}$ obtained by subdividing every edge of G exactly r times. A $\leq r$ -*subdivision* of G is a graph obtained by subdividing each edge of G at most r times. A graph H is a *topological minor* of a graph G at depth r if a $\leq 2r$ -subdivision of H is a subgraph of G . We denote by $G \tilde{\nabla} r$ the set of all the topological minors of G at depth r , and define $\tilde{\nabla}_r(G) = \max\{|E(H)|/|V(H)| : H \in G \tilde{\nabla} r\}$. The two key notions in the theory of sparsity [35] are the notions of *bounded expansion* and *nowhere denseness*, defined as follows.

Definition 2.1. A class \mathcal{C} of graphs has *bounded expansion* if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ with

$$\forall G \in \mathcal{C} \quad \forall H \in G \widetilde{\nabla} r \quad \bar{d}(H) \leq f(r). \quad (1)$$

Example 2. All proper minor-closed classes of graphs and all classes of graphs with bounded maximum degree have bounded expansion.

Definition 2.2. A class \mathcal{C} of graphs is *nowhere dense* if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ with

$$\forall G \in \mathcal{C} \quad \forall H \in G \widetilde{\nabla} r \quad \omega(H) \leq f(r). \quad (2)$$

Note that every class with bounded expansion is nowhere dense but the converse does not hold.

Example 3. The class of all graphs G whose girth exceeds the maximum degree is nowhere dense but does not have bounded expansion.

Bounded expansion and nowhere dense classes enjoy numerous characterizations and applications (see [35]). Among others, let us mention the characterizations based on p -centered colorings [33] and on (generalized) weak coloring numbers [43].

The *depth* of a rooted forest is the maximum number of vertices in a path linking a root to a leaf. The *treedepth* $\text{td}(G)$ of a graph G is the minimum height of a rooted forest Y with vertex set $V(Y) = V(G)$ such that adjacent vertices in G belong to a path of Y linking a root to a leaf. For a positive integer p , a *p -treedepth coloring* of a graph G is a vertex coloring such that every $k \leq p$ color classes induce a subgraph with treedepth at most k . Closely related is the notion of a *p -centered coloring*, which is a coloring of the vertices such that every connected subgraph with at most p colors on the vertices contains some uniquely colored vertex. It is easily checked that every $(p + 1)$ -centered coloring is a p -treedepth coloring (see [32, 35]).

Theorem 2.2 (Nešetřil, Ossona de Mendez [33]). A class \mathcal{C} of graphs has bounded expansion if and only if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that every graph $G \in \mathcal{C}$ admits, for each positive integer p , a p -centered coloring with at most $f(p)$ colors.

We recall the definition of the generalized coloring numbers, which have been introduced by Kierstead and Yang [23] as a generalization of the so-called coloring number. Let G be a graph and let L be a linear ordering of $V(G)$. We say that a vertex u is *weakly d -reachable* from a vertex v if there exists in G a path P of length at most d (possibly 0) linking u and v such that u is the minimum vertex of P with respect to L , and we denote by $\text{WReach}_d[G, L, v]$ the set of all vertices weakly d -reachable from v . The *weak coloring number* $\text{wcol}_d(G)$ is defined as the minimum over all possible linear orderings L of $\max_{v \in V(G)} |\text{WReach}_d[G, L, v]|$.

Theorem 2.3 (Zhu, [43]). A class \mathcal{C} of graphs has bounded expansion if and only if there exists a function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that for every graph $G \in \mathcal{C}$ and every positive integer d we have $\text{wcol}_d(G) \leq f(d)$.

Note that nowhere dense classes can also be characterized in terms of bounds on the weak coloring numbers.

Structural and algorithmic properties of classes with bounded expansion and nowhere dense classes have strong links with first-order logic. In particular, the fixed-parameter linear time first-order model-checking algorithm for bounded expansion classes [15] is based on a quantifier elimination scheme, which will be central to the study conducted in Section 6. This also justifies to extend the study of the discrepancy of neighborhood set systems presented in Sections 3 and 4 to first-order definable set systems.

2.3. Model theory and definable set systems.

In order to establish a bridge between an approach of discrepancy based on structural graph theory and one based on (finite) model theory, we consider (from Section 5 onward) set systems defined from graphs by means of first-order formulas. As a motivating example among others, note that first-order logic is the foundation of query languages over relational databases (see e.g. [25]). In this original setting, queries are formulas and query answers form definable set systems.

For simplicity, we focus on (vertex colored) graphs, and on structures with only unary predicates and unary functions (which we call *pointer structures*). However, all definitions extend to general structures in a straightforward way. We use standard notation from graph theory [11] and finite model theory [25], and we refer to the referenced textbooks for all undefined notation.

Recall that a *signature* σ is a set of relation and function symbols with an attached arity. A σ -*structure* \mathbf{M} is defined by its *domain* M (which is a set), and an interpretation of the symbols in σ as actual relations and functions on M . The *Gaifman graph* of a structure \mathbf{M} is the graph with vertex set M , where two (distinct) vertices $u, v \in M$ are adjacent if they belong together to a tuple of a relation (k -ary functions being interpreted as $(k + 1)$ -ary relations).

The *quantifier rank* of a first-order formula φ is the maximum number of nested quantifiers in φ . For example, the quantifier rank of the formula $\exists x ((\exists y (P(x, y) \vee (\forall z R(x, yz)))) \wedge (\forall t Z(x, t)))$ is three. We say that two σ -structures \mathbf{M} and \mathbf{M}' are *q -equivalent* if they satisfy the same sentences with quantifier rank at most q .

Let G be a graph with adjacency relation $E(x, y)$. A set $S \subseteq V(G)^d$ is *definable* in G with *parameters* if there exists a partitioned formula $\varphi(\bar{x}; \bar{y})$ (with $\bar{x} = (x_1, \dots, x_d)$ and $\bar{y} = (y_1, \dots, y_\ell)$) and an ℓ -tuple $\bar{b} = (b_1, \dots, b_\ell)$ of vertices of G such that a d -tuple $\bar{v} = (v_1, \dots, v_d)$ belongs to S if and only if $G \models \varphi(\bar{v}, \bar{b})$. For $\bar{b} \in V(G)^\ell$, let $\varphi(G, \bar{b}) = \{\bar{v} \in V(G)^d : G \models \varphi(\bar{v}, \bar{b})\}$ be the subset of d -tuples of vertices of G defined by φ for the parameter \bar{b} . The *set system defined by $\varphi(\bar{x}; \bar{y})$ on G* is the set system $(V(G)^{|\bar{x}|}, \mathcal{S}^\varphi(G))$, where $\mathcal{S}^\varphi(G) = \{\varphi(G, \bar{b}) : \bar{b} \in V(G)^{|\bar{y}|}\}$. In particular, the neighborhood set system $\mathcal{S}^E(G)$ of a graph G is defined by the partitioned formula $E(x; y)$.

Example 4 (Sylvester's graphs). Let S_p be the bipartite graph with 2^{p+1} vertices, whose adjacency matrix is obtained from the Hadamard matrix H_p (described in Example 1) by replacing each entry -1 by 0 . Then we have

$$\text{disc}(\mathcal{S}^E(S_p)) = \Omega\left(\left|\bigcup \mathcal{S}^E(S_p)\right|^{1/2}\right),$$

since $\mathcal{S}^E(S_p)$ is the set system obtained from the set system \mathcal{S}_p introduced in Example 1 by duplicating every set (as H_p is symmetric).

Note that we allow formulas without parameter variables. In such a case, the set system defined by a formula $\rho(\bar{x})$ on G is the singleton $\{\rho(G)\}$, where $\rho(G) = \{\bar{v} \in V^{|\bar{x}|} : G \models \rho(\bar{v})\}$. More generally, for a finite set Φ of formulas $\varphi_i(\bar{x}, \bar{y})$ (with same $|\bar{x}|$ but possibly different $|\bar{y}|$), the set system defined by Φ on G is the set system $(V(G)^{|\bar{x}|}, \mathcal{S}^\Phi(G))$, where $\mathcal{S}^\Phi(G) = \bigcup_i \mathcal{S}^{\varphi_i}(G)$.

2.4. VC-dimension and shatter functions.

The Vapnik–Chervonenkis dimension (VC-dimension) is a classical measure for the complexity of set systems [42]. We say that a subset X of the ground set U of a set system (U, \mathcal{S}) is *shattered* by \mathcal{S} if $\mathcal{S}|_X$ contains all the subsets of X . The *VC-dimension* of \mathcal{S} is the maximum size of a subset shattered by \mathcal{S} . Classes \mathcal{C} of graphs with the property that for all first-order formulas φ , there exists a constant d such that the system $\mathcal{S}^\varphi(G)$ has VC-dimension bounded by d for all $G \in \mathcal{C}$ play a key role in model theory: they are called *dependent classes* [39]. Classes whose monotone closure is dependent are exactly nowhere dense classes [1].

A more precise description of the complexity of a set system \mathcal{S} is given by its (*primal*) *shatter function* $\pi_{\mathcal{S}}$, which is defined as follows: for an integer m , we define $\pi_{\mathcal{S}}(m)$ to be the maximum over all subsets $X \subseteq U$ of size m of the number of distinct sets in the collection $\mathcal{S}|_X$. By the famous Sauer-Shelah Lemma [37, 38, 42] we have $\pi_{\mathcal{S}}(k) \in \mathcal{O}(m^d)$ if the VC-dimension of \mathcal{S} is at most d .

A strong connection between these notions and combinatorial discrepancy is given by the next fundamental result, which is known to be optimal [2].

Theorem 2.4 (Matoušek [26] and Matoušek, Welzl, and Wernisch [29]). Let \mathcal{S} be a set system, let n be the size of the ground set of \mathcal{S} , and let d, C be constants, such that $\pi_{\mathcal{S}}(m) \leq Cm^d$ for all $m \leq n$. Then the discrepancy of \mathcal{S} is bounded by

$$\text{disc}(\mathcal{S}) = \mathcal{O}(n^{1/2-1/2d}), \text{ if } d > 1, \text{ and } \text{disc}(\mathcal{S}) = \mathcal{O}(\log^{5/2} n), \text{ if } d = 1.$$

Each set system \mathcal{S} with ground set U can be represented by its *incidence graph* $I_{\mathcal{S}}$ with parts U and \mathcal{S} , where $u \in U$ is adjacent to $S \in \mathcal{S}$ if $u \in S$. The *dual* of a set system \mathcal{S} with ground set U is the set system \mathcal{S}^* , whose incidence graph is obtained from the incidence graph of \mathcal{S} by exchanging the parts. (Note that this corresponds to the notion of hypergraph duality.)

The *dual shatter function* $\pi_{\mathcal{S}}^*(m)$ of a set system \mathcal{S} is defined as the primal shatter function of the dual set system \mathcal{S}^* . The discrepancy of a set system can also be bounded in terms of the dual shatter function (where the bound is also tight [26]). As noticed by Matoušek [26], the next theorem follows from a slight modification of the bound obtained by Matoušek, Welzl, and Wernisch [29] for the discrepancy of set systems defined by half-spaces, by using (in the abstract setting) the packing lemma due to Haussler [20], instead of the elementary lemma concerning the volume of an r -ball in an arrangement of hyperplanes due to Chazelle and Welzl.

Theorem 2.5. Let \mathcal{S} be a set system, let n be the size of the ground set of \mathcal{S} , and let d, C be constants, such that $\pi_{\mathcal{S}}^*(m) \leq Cm^d$ for all $m \leq n$. Then the discrepancy of \mathcal{S} is bounded by

$$\text{disc}(\mathcal{S}) = \mathcal{O}(n^{1/2-1/2d} \sqrt{\log n}).$$

Bounded expansion and nowhere dense classes can be characterized by shatter functions:

Theorem 2.6 (Pilipczuk, Siebertz, Toruńczyk [36]). Let \mathcal{C} be a monotone class of graphs. Then

- the class \mathcal{C} has bounded expansion if and only if for every formula $\varphi(\bar{x}; \bar{y})$ there exists a constant C with $\pi_{\mathcal{S}^\varphi(G)}(m) \leq Cm^{|\bar{y}|}$ and $|\bigcup \mathcal{S}^\varphi(G)|_A \leq C|A|^{|\bar{y}|}$ for every $G \in \mathcal{C}$ and every $A \subseteq V(G)$;
- the class \mathcal{C} is nowhere dense if and only if for every formula $\varphi(\bar{x}; \bar{y})$ and for every $\varepsilon > 0$ there exists a constant C with $\pi_{\mathcal{S}^\varphi(G)}(m) \leq Cm^{|\bar{y}|+\varepsilon}$ and $|\bigcup \mathcal{S}^\varphi(G)|_A \leq C|A|^{|\bar{y}|+\varepsilon}$ for every $G \in \mathcal{C}$ and every $A \subseteq V(G)$.

From this result, one can derive a bound on the hereditary discrepancy of sets systems definable in nowhere dense and bounded expansion classes. In the case of a nowhere dense class, the derived bound is established in Lemma 7.1; in the bounded expansion case, our result will establish a constant upper bound, clearly improving the bound derived from Theorems 2.4 and 2.6.

2.5. Our results.

In Section 3, we prove that the notions of discrepancy and degeneracy are deeply linked. Precisely, we prove that the maximum discrepancy over all subgraphs H of a graph G of the neighborhood set system of H is sandwiched between $\Omega(\log \deg(G))$ and $\mathcal{O}(\deg(G))$ (see Theorem 3.1). In Section 4, we extend this result to inequalities relating weak coloring numbers and discrepancy of graph powers. Precisely, the maximum hereditary discrepancy over all subgraphs H of the power G^d of a graph G of the neighborhood set system of H is sandwiched between $\Omega(\log \text{wcol}_{\lceil d/2 \rceil}(G))$ and $\mathcal{O}(\text{wcol}_d(G)^2)$ (Theorem 4.1), and we deduce a first characterization of bounded expansion classes in terms of discrepancy (Corollary 4.1).

In order to extend these results further, we switch to a model theoretic point of view. In Section 5, we introduce pointer structures, which are structures with only unary relations and unary functions, and prove that set systems definable by a quantifier-free formula in such structures have bounded hereditary discrepancy (Theorem 5.1). As a corollary, we deduce that every set system (induced by a finite ground set) definable in an infinite set has bounded discrepancy (Corollary 5.1) and that the neighborhood set systems of the graphs in a set-defined class have bounded hereditary discrepancy (Theorem 5.2). Then, we introduce the first-order theories $\text{Th}_{\sigma, \rho}^{\text{TF}}$ for pointer structures, mimicking the characteristic properties of bounded expansion classes and prove that they have quantifier elimination (Theorem 6.1). From this, we deduce a characterization of bounded expansion classes as the monotone classes whose definable set systems have bounded hereditary discrepancy (Theorem 6.2).

Then, using bounds on the VC-density of set systems definable in nowhere dense classes proved in [36], we give a characterization of nowhere dense classes in terms of discrepancy (Theorem 7.1). We believe that our upper bounds on discrepancy can be improved in this case, and we propose a conjecture for the optimal bound (Conjecture 5).

In Section 8, we provide some corollaries on edge colored graphs (Corollary 8.1), ε -approximations (Corollary 8.2), clique coloring (Corollary 8.3), and quantifier elimination schemes (Corollary 8.4), which allows us to solve the motivating problems presented in the introduction.

Finally, in Section 9, we discuss some possible extensions of this work.

3. Discrepancy and Degeneracy

In this section, we relate the degeneracy of a graph G to the discrepancy of the neighborhood set system $\mathcal{S}^E(G)$ of G . Our main result is the following theorem, which directly follows from Lemmas 3.1 and 3.2 proved below.

Theorem 3.1. For every graph G we have

$$\frac{\log_2(\pi \deg(G))}{4} - 2 \leq \max_{H \subseteq G} \text{disc}(\mathcal{S}^E(H)) < 3 \deg(G). \quad (3)$$

About the upper bound. The bipartite graph G with a part of size n and a part of size $\binom{n}{d}$ corresponding to all possible neighborhoods of size d in the part of size n is such that $\max_{H \subseteq G} \text{disc}(\mathcal{S}^E(H)) \geq d = \deg(G)$. Hence, the upper bound of Theorem 3.1 is tight up to a constant factor.

About the lower bound. As Spencer's bound $\text{disc}(\mathcal{S}) \leq 6\sqrt{n}$ for every set system \mathcal{S} on a ground set of size n and $\mathcal{O}(n)$ many sets is known to be tight up to a constant factor (see Example 4), we have $\max_{H \subseteq K_n} \text{disc}(\mathcal{S}^E(H)) \in \Omega(\deg(K_n)^{1/2})$. It is possible that the lower bound given in Theorem 3.1 might be improved to $\Omega(\deg(G)^c)$ for some positive constant c .

About the maximum over subgraphs. In the statement, we consider $\max_{H \subseteq G} \text{disc}(\mathcal{S}^E(H))$, which is a monotone form of the discrepancy. Instead of $\text{disc}(\mathcal{S}^E(H))$ we could have considered $\text{herdisc}(\mathcal{S}^E(H))$. However, this would have led to the same result, as for every graph G we have

$$\max_{H \subseteq G} \text{herdisc}(\mathcal{S}^E(H)) = \max_{H \subseteq G} \text{disc}(\mathcal{S}^E(H)).$$

Indeed, as $\text{herdisc}(\mathcal{S}^E(H)) \geq \text{disc}(\mathcal{S}^E(H))$ we get that $\max_{H \subseteq G} \text{herdisc}(\mathcal{S}^E(H))$ is at least $\max_{H \subseteq G} \text{disc}(\mathcal{S}^E(H))$. For the reverse inequality, it will be sufficient to prove $\text{herdisc}(\mathcal{S}^E(G)) \leq \max_{H \subseteq G} \text{disc}(\mathcal{S}^E(H))$. Let $V' \subseteq V(G)$ be such that $\text{herdisc}(\mathcal{S}^E(G)) = \text{disc}(\mathcal{S}^E(G)|_{V'})$, let $W \subseteq V(G)$ be the set of all neighbors in G of some vertex in V' , and let H be the subgraph of G with vertex set $V' \cup W$, where we keep all the edges of G with some endpoint in V' . Then $\mathcal{S}^E(G)|_{V'} \subseteq \mathcal{S}^E(H)$, thus $\text{herdisc}(\mathcal{S}^E(G)) \leq \text{disc}(\mathcal{S}^E(H))$.

We now prove that the discrepancy of the neighborhood set system is linearly bounded by the degeneracy.

Lemma 3.1. Let G be a graph. Then $\text{herdisc}(\mathcal{S}^E(G)) < 3 \deg(G)$.

Proof. Let $d = \deg(G)$ and let \vec{G} be an orientation of G with maximum out-degree d . Let $N^-(v)$ and $N^+(v)$ denote, respectively, the in-neighborhood and the out-neighborhood of a vertex v . Consider the set system $\mathcal{S}_1 = \{N^-(v) : v \in V(G)\}$. Every vertex v in $V(G)$ belongs to at most d sets in \mathcal{S}_1 , for if $v \in N^-(u)$, then $u \in N^+(v)$. According to the Beck-Fiala Theorem, for every subset X of vertices there exists a function $\chi : X \rightarrow \{-1, 1\}$ such that for every vertex v we have $|\sum_{u \in N^-(v) \cap X} \chi(u)| < 2d$. As $|N^+(v) \cap X| \leq d$ we have $|\sum_{u \in N_G(v) \cap X} \chi(u)| < 3d$. Hence we have $\text{disc}(\mathcal{S}^E(G)|_X) < 3 \deg(G)$. \square

Remark 3.1. Using Bukh's improvement [6], if $\deg(G)$ is sufficiently large, then the bound can be decreased to $3 \deg(G) - \log^* \deg(G)$.

Lemma 3.2. Let G be a graph and let $c = \max_{H \subseteq G} \text{disc}(\mathcal{S}^E(H))$. Then G is $\frac{16^{c+2}}{\pi}$ -degenerate.

Proof. In the following we consider G as a vertex-labeled graph and all subgraphs as vertex-labeled graphs (that is, we do not identify isomorphic subgraphs). First assume that G is bipartite. Let A and B be the two parts of the bipartition with $|A| \geq |B|$. For every subgraph $H \subseteq G$ denote by \mathcal{S}_H the set system $\{N_H(v) : v \in A \cap V(H)\}$. Note that $\mathcal{S}_H \subseteq \mathcal{S}^E(H)$ and $\bigcup \mathcal{S}_H \subseteq B$. Hence $\text{disc}(\mathcal{S}_H) \leq c$ with witness coloring $\gamma_H : B \rightarrow \{-1, 1\}$.

Let $d = \text{deg}(G)$, $n = |B|$, and $m = |E(G)|$. The graph G has 2^m spanning subgraphs and 2^n ways to color B (with colors in $\{-1, 1\}$). Hence, there exists a coloring γ such that the set \mathcal{F}_γ of all spanning subgraphs H of G with $\text{disc}_\gamma(\mathcal{S}_H) \leq c$ has size at least $2^m/2^n$.

For $v \in A$ and $\alpha \in \{-1, 1\}$, define $N_G^\alpha(v) = N_G(v) \cap \gamma^{-1}(\alpha)$. Then for every spanning subgraph H of G the inequality $\text{disc}_\gamma(\mathcal{S}_H) \leq c$ rewrites as

$$\forall v \in A \quad \left| |N_H^{-1}(v)| - |N_H^1(v)| \right| \leq c.$$

As G is bipartite, these conditions on the neighborhoods of the vertices in A are independent and the number of graphs H in \mathcal{F}_γ is the product over all vertices $v \in A$ of the number of pairs $(X_v, Y_v) \subseteq N_G^{-1}(v) \times N_G^1(v)$ with $\left| |X_v| - |Y_v| \right| \leq c$.

Let $d^{-1}(v) = |N_G^{-1}(v)|$ and $d^1(v) = |N_G^1(v)|$. By considering supersets of $N_G^{-1}(v)$ and $N_G^1(v)$ obtained by adding c dummy elements, one easily checks that the number of pairs $(X_v, Y_v) \subseteq N_G^{-1}(v) \times N_G^1(v)$ with $\left| |X_v| - |Y_v| \right| \leq c$ is bounded by $\sum_{k=0}^{m(v)} \binom{d^{-1}(v)+c}{k} \binom{d^1(v)+c}{k}$, where $m(v) = \min(d^{-1}(v), d^1(v)) + c$. Thus,

$$|\mathcal{F}_\gamma| \leq \prod_{v \in A} \sum_{k=0}^{m(v)} \binom{d^{-1}(v)+c}{k} \binom{d^1(v)+c}{k}.$$

▷ *Claim 1.* Let $a \leq b$ be positive integers and let $s = \lceil (a+b)/2 \rceil$. Then

$$\sum_{k=0}^a \binom{a}{k} \binom{b}{k} \leq \binom{2s}{s}.$$

Proof of the claim. By replacing b by $b+1$ if necessary, we can assume that $a+b$ is even. Assume first that $a < b$, hence, $a+1 \leq b-1$. Then for every $0 \leq k \leq a$ we have $\frac{a+1}{a+1-k} \cdot \frac{b-k}{b} \geq 1$. Thus,

$$\binom{a+1}{k} \binom{b-1}{k} = \frac{a+1}{a+1-k} \binom{a}{k} \frac{b-k}{b} \binom{b}{k} \geq \binom{a}{k} \binom{b}{k}.$$

It follows that we can reduce to the case $a = b$. Then $\sum_{k=0}^a \binom{a}{k} \binom{b}{k} = \sum_{k=0}^s \binom{s}{k}^2 = \binom{2s}{s}$, which is the special case $m = n = r = s$ of the Chu-Vandermonde identity $\sum_{k=0}^r \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$. ◁

Now we use the following upper bound.

▷ *Claim 2* ([10, Lemma 17.5.1]). For $0 < k < n$ we have

$$\binom{n}{k} \leq \sqrt{\frac{n}{\pi k(n-k)}} 2^{nH(k/n)},$$

where $H(p) = -p \log_2(p) - (1-p) \log_2(1-p)$, which is the standard entropy function. ◁

As $H(1/2) = 1$ we have $\binom{2s}{s} \leq \frac{4^s}{\sqrt{\pi s/2}}$. Let $h(v) = \lceil \frac{d(v)}{2} + c \rceil \geq \delta(G)/2$. We have

$$\begin{aligned} |\mathcal{F}_\gamma| &\leq \prod_{v \in A} \sum_{k=0}^{m(v)} \binom{d^{-1}(v) + c}{k} \binom{d^1(v) + c}{k} \\ &\leq \prod_{v \in A} \binom{2h(v)}{h(v)} \leq \prod_{v \in A} \frac{2^{2h(v)}}{\sqrt{\pi h(v)/2}} \\ &\leq \prod_{v \in A} \frac{2^{d(v)+2c+1}}{\sqrt{\pi \delta(G)/4}} \leq \prod_{v \in A} \left(\frac{4^{c+1}}{\sqrt{\pi \delta(G)}} 2^{d(v)} \right) \\ &\leq \left(\frac{4^{c+1}}{\sqrt{\pi \delta(G)}} \right)^{|A|} 2^m. \end{aligned}$$

As $|\mathcal{F}_\gamma| \geq \left(\frac{1}{2}\right)^n 2^m$ we get $\frac{4^{c+1}}{\sqrt{\pi \delta(G)}} \geq 2^{-\frac{n}{|A|}}$. Hence, as $n = |B| \leq |A|$, we get $\delta(G) \leq \frac{64}{\pi} \cdot 16^c$.

As this holds for every induced subgraph, we get $\deg(G) \leq \frac{64}{\pi} \cdot 16^c$.

Now, if G is not bipartite, we get that the degeneracy is at most $\frac{16^{c+2}}{\pi}$, as G includes a bipartite subgraph whose degeneracy is at least one fourth of the degeneracy of G (first construct a bipartite subgraph G' with at least half of the edges of G , then find a subgraph whose minimum degree is at least the edge density of G' , see e.g. [35]). ◻

4. Discrepancy and Generalized Coloring Numbers

In this section, we extend the result of Section 3 by considering the (generalized) weak coloring numbers. Our main result is the following.

Theorem 4.1. Let G be a graph and let d be a positive integer. Then

$$\begin{aligned} \frac{\log_2(\text{wcol}_{\lceil d/2 \rceil}(G))}{6(d+1)} - \frac{\log_2(d+1)}{3} - \frac{3}{2} &\leq \max_{d' \leq d} \max_{H \subseteq G} \text{herdisc}(\mathcal{S}^E(H^{d'})) \\ &< (2d \text{wcol}_{d-1}(G) + 1) \text{wcol}_d(G). \end{aligned}$$

Note that, as $\text{wcol}_1(G) = \deg(G) + 1$, Theorem 4.1 yields (for $d = 1$)

$$\frac{\log_2(\deg(G) + 1)}{12} - \frac{11}{6} \leq \max_{H \subseteq G} \text{herdisc}(\mathcal{S}^E(H)) < 3 \deg(G) + 3,$$

to be compared with Theorem 3.1.

As a corollary of Theorem 4.1 we obtain the following characterization of classes with bounded expansion.

Corollary 4.1. Let \mathcal{C} be a monotone class of graphs. Then \mathcal{C} has bounded expansion if and only if the hereditary discrepancy of $\mathcal{S}^E(G^k)$ is bounded on \mathcal{C} for each positive integer k .

The proof of Theorem 4.1 will follow from Lemmas 4.1 and 4.2, which we will prove next.

Lemma 4.1. Let G be a graph and d a positive integer. Then

$$\text{herdisc}(\mathcal{S}^E(G^d)) < (2d \text{wcol}_{d-1}(G) + 1) \text{wcol}_d(G).$$

Proof. Let L be a linear ordering such that $\max_{v \in V(G)} |\text{WReach}_d[G, L, v]| = \text{wcol}_d(G)$. For brevity, in the following we write $\text{WReach}_i[v]$ for $\text{WReach}_i[G, L, v]$. For a vertex $v \in V(G)$ we define $W_0(v) = \{v\}$ and, for $1 \leq i \leq d$, $W_i(v) = \text{WReach}_i[v] \setminus \text{WReach}_{i-1}[v]$. Hence $(W_0(v), \dots, W_d(v))$ is a partition of $\text{WReach}_d[v]$. In particular, $\sum_{i=0}^d |W_i(v)| = |\text{WReach}_d[v]| \leq \text{wcol}_d(G)$. We further define, for $0 \leq i \leq d$, $\text{WReach}_i^*[v] = \{u : v \in \text{WReach}_i[u]\}$.

For every vertex v , we denote by $B_d[v]$ the set of all vertices at distance at most d from v . Note that $B_d[v] = N_{G^d}(v) \cup \{v\}$. For each pair (u, v) of vertices at distance at most d (including pairs (u, u)) we fix a shortest path $P_{u,v}$ linking u and v , and we denote by $m_{u,v}$ the minimum vertex of $P_{u,v}$ with respect to L . Note that if $\text{dist}(m_{u,v}, v) = i$, then $m_{u,v} \in W_i(v) \cap \text{WReach}_{d-i}[u]$, as otherwise there would exist a shorter path linking u and v . If $W_i(v) \cap \text{WReach}_{d-i}[u]$ is not empty, then $u \in B_d[v]$. Thus,

$$B_d(v) = \bigcup_{i=0}^d \bigcup_{z \in W_i(v)} \text{WReach}_{d-i}^*[z].$$

Let $\mathcal{S}' = \{\text{WReach}_i^*[z] : z \in V(G), 1 \leq i \leq d\}$. If $u \in \text{WReach}_i^*[z]$, then $z \in \text{WReach}_i[u]$, thus every vertex belongs to at most $d \cdot \text{wcol}_d(G)$ sets $\text{WReach}_i^*[z]$. According to the Beck-Fiala Theorem, for every subset X of vertices there exists a function $\chi : X \rightarrow \{-1, 1\}$ such that $|\text{disc}_\chi(\text{WReach}_i^*[z] \cap X)| \leq 2d \cdot \text{wcol}_d(G) - 1$ for all $z \in V(G)$ and $1 \leq i \leq d$.

As $\bigcup_{z \in W_d(v)} \text{WReach}_0^*[z] = W_d(v) \subseteq \text{WReach}_d[v]$, for every vertex $v \in V(G)$, we have

$$\begin{aligned} \text{disc}_\chi(N_{G^d}(v)) &\leq \left(\sum_{i=0}^{d-1} \sum_{z \in W_i(v)} \text{disc}_\chi(\text{WReach}_{d-i}^*[z] \cap X) \right) + \text{disc}_\chi((\text{WReach}_d[v] \setminus \{v\}) \cap X) \\ &\leq \text{wcol}_{d-1}(G)(2d \text{wcol}_d(G) - 1) + (\text{wcol}_d(G) - 1) \\ &< (2d \text{wcol}_{d-1}(G) + 1) \text{wcol}_d(G). \end{aligned} \quad \square$$

Lemma 4.2. Let G be a graph and let d be a positive integer. Then there exists a subgraph H of G and an integer $d' < 2d$ with

$$\text{herdisc}(\mathcal{S}^E(H^{d'})) \geq \frac{1}{12d} \log_2(\text{wcol}_d(G)) - \frac{1}{3} \log_2 d - \frac{11}{6}.$$

Proof. We make use of the following inequalities relating the weak coloring numbers and shallow topological minor average degrees to d -admissibility² [40, Theorem 4.1.3] and [17].

$$\begin{aligned} \text{wcol}_d(G) &\leq \text{adm}_d(G)^d \\ \text{adm}_d(G) &\leq 6d \lceil \widetilde{\text{V}}_{d-1}(G) \rceil^3. \end{aligned}$$

Let $\alpha = \left(\frac{1}{6d}\right)^{1/3} \text{wcol}_d(G)^{1/3d}$. Let M be a shallow topological minor of G at depth $(d-1)$ witnessing $\widetilde{\text{V}}_{d-1}(G) \geq \alpha$. Each edge of M appears in G as a path of length $i+1$, for some $0 \leq i \leq 2(d-1)$. Thus, for some $1 \leq d' < 2d$, the graph G contains the exact $(d'-1)$ -subdivision H_0 of a graph M_0 with $|E(M_0)|/|V(M_0)| \geq \alpha/2d$. It follows that $\deg(M_0) \geq \alpha/d$. According to Theorem 3.1, M_0 has a subgraph M with $\text{disc}(\mathcal{S}^E(M)) \geq \frac{1}{4} \log_2(\pi\alpha/d) - 2$. The $(d'-1)$ -subdivision H of M is a subgraph of H_0 , thus a subgraph of G , and $\text{disc}(\mathcal{S}^E(M)) \leq \text{herdisc}(\mathcal{S}^E(H^{d'}))$. Hence, the result follows, as

$$\begin{aligned} \frac{1}{4} \log_2(\pi\alpha/d) - 2 &= \frac{1}{4} \left(\log_2(\pi/d) - \frac{1}{3} \log_2(6d) + \frac{1}{3d} \log_2(\text{wcol}_d(G)) \right) - 2 \\ &= \frac{1}{12d} \log_2(\text{wcol}_d(G)) - \frac{1}{3} \log_2 d + \left(\frac{\log_2(\pi)}{4} - \frac{\log_2 6}{12} - 2 \right) \\ &\geq \frac{1}{12d} \log_2(\text{wcol}_d(G)) - \frac{1}{3} \log_2 d - \frac{11}{6}. \end{aligned}$$

□

Proof of Theorem 4.1. If $H \subseteq G$ and $d' \leq d$, then $\text{wcol}_{d'}(H) \leq \text{wcol}_d(G)$. Thus, we deduce the upper bound of Theorem 4.1 from Lemma 4.1.

According to Lemma 4.2 applied to G and $\lceil d/2 \rceil$ (instead of d), there exists a subgraph H of G and an integer $d' \leq d$ with

$$\begin{aligned} \text{herdisc}(\mathcal{S}^E(H^{d'})) &\geq \frac{1}{12\lceil d/2 \rceil} \log_2(\text{wcol}_{\lceil d/2 \rceil}(G)) - \frac{1}{3} \log_2 \lceil d/2 \rceil - \frac{11}{6} \\ &\geq \frac{1}{6(d+1)} \log_2(\text{wcol}_{\lceil d/2 \rceil}(G)) - \frac{1}{3} \log_2(d+1) - \frac{3}{2}. \end{aligned}$$

□

5. A model theoretic approach to degeneracy: pointer structures

A *pointer structure* is a unary structure, that is, a structure with a (possibly infinite) signature σ that consists of unary relation and unary function symbols. It will be convenient to assume that the function symbols in σ are the elements of a monoid \mathcal{F}_σ . We fix the signature σ and we will consider only σ -structures satisfying the theory T_σ that consists of the sentences $\forall x f(g(x)) = h(x)$, where f and g range over \mathcal{F}_σ and $h = f \circ g$. Note that this is not really a restriction, but rather a way to simplify the notations.

²We refer the interested reader to [17] for a definition of d -admissibility. Note that the definition of $\text{adm}_d(G)$ and $\text{wcol}_d(G)$ used in our paper and the two quoted papers differs by 1 from the original definition of [14].

Observation 5.1. For every d -degenerate graph G there is a pointer structure \mathbf{M} with d unary functions such that $\mathcal{S}^E(G) = \mathcal{S}^\eta(\mathbf{M})$, where $\eta(x, y)$ is the quantifier-free formula

$$\eta(x, y) := \neg(x = y) \wedge \bigvee_{i=1}^d ((f_i(x) = y) \vee (f_i(y) = x)).$$

Proof. If G is d -degenerate, then G has an orientation \vec{G} with maximum out-degree d . Consider such an orientation \vec{G} of G and an arbitrary labeling of the arcs of \vec{G} by integers between 1 and d such that the arcs with the same origin have pairwise different labels. We define \mathbf{M} as the pointer structure with domain $V(G)$ such that for every vertex v and every integer $i \in [d]$ we have $f_i(v) = w$ if (v, w) is an arc of \vec{G} with label i , and $f_i(v) = v$ if no arc with origin v has label i . That $G \models E(u, v)$ is equivalent to $\mathbf{M} \models \eta(u, v)$ is straightforward. \square

For a pointer structure \mathbf{M} (with signature σ) and a subset A of the domain of \mathbf{M} , we define the pointer structure $\mathbf{M}[A]$ *weakly induced* by A on \mathbf{M} as the pointer structure with domain A (and signature σ), such that for every relation symbol $P \in \sigma$, every function symbol $f \in \sigma$, and all distinct u and v in A , we have

$$\begin{aligned} \mathbf{M}[A] \models P(u) &\iff \mathbf{M} \models P(u) \\ \mathbf{M}[A] \models f(u) = v &\iff \mathbf{M} \models f(u) = v \end{aligned}$$

(It follows that if $u \in A$ and the image by f of u in \mathbf{M} is not in A , then u is a fixed point of f in $\mathbf{M}[A]$.) Note that the Gaifman graph of $\mathbf{M}[A]$ is the subgraph of the Gaifman graph of \mathbf{M} induced by A .

5.1. Hereditary discrepancy of QF-definable sets systems

The goal of this section is to prove the following theorem.

Theorem 5.1. For every finite set $\Phi = \{\varphi_1(\bar{x}; \bar{y}), \dots, \varphi_p(\bar{x}; \bar{y})\}$ of quantifier free partitioned σ -formulas (with same $|\bar{x}|$ but possibly different $|\bar{y}|$) there exists a constant C_Φ such that for every σ -structure \mathbf{M} we have

$$\text{herdisc}(\mathcal{S}^\Phi(\mathbf{M})) \leq C_\Phi.$$

The proof goes by the following key steps. One of the building blocks is the Beck-Fiala Theorem. However, we cannot directly apply this result as it requires set systems to have bounded degree. We generalize the theorem to set systems that are Boolean combinations of set systems with bounded degree (Lemmas 5.1 and 5.2). Finally, we prove that we can decompose set systems defined by quantifier free formulas into Boolean combinations of set systems that have bounded degree (Lemma 5.3), which concludes the proof of Theorem 5.1.

The two following lemmas extend the Beck-Fiala Theorem to handle set systems built from sets in a set system of bounded degree.

Lemma 5.1. Let (U, \mathcal{S}_0) be a set system of degree at most t . Then the set system (U, \mathcal{S}_1) , where \mathcal{S}_1 is the collection of all intersections of sets in \mathcal{S}_0 (with the convention that U is the empty intersection) has degree at most 2^t .

Proof. As each $v \in U$ belongs to at most t sets S_1, \dots, S_t in \mathcal{S}_0 , it belongs to at most 2^t sets in \mathcal{S}_1 , namely the sets of the form $\bigcap_{i \in I} S_i$, for $I \subseteq [t]$. Thus the set system (U, \mathcal{S}_1) has degree at most 2^t . \square

Lemma 5.2. Let (U, \mathcal{S}_1) and (U, \mathcal{S}) be two set systems over the same ground set U such that \mathcal{S}_1 is closed under taking intersections (including U , the empty intersection) and such that every set in \mathcal{S} is a Boolean combination of at most k sets from \mathcal{S}_1 . Then $\text{herdisc}(\mathcal{S}) \leq 4^k \text{herdisc}(\mathcal{S}_1)$.

Proof. Let $S \in \mathcal{S}$. By assumption, S is a Boolean combination of at most k sets S_1, \dots, S_k in \mathcal{S}_1 . This Boolean combination can be written as a disjoint union of at most 2^k terms of the form $T_I = \bigcap_{i \in I} S_i \cap \bigcap_{j \in [k] \setminus I} (U \setminus S_j)$, where $I \subseteq [k]$. Let $\mathbb{1}_Y : U \rightarrow \{0, 1\}$ denote the indicator function of a subset Y of U . For a subset J of $[k]$, let $S_J = \bigcap_{j \in J} S_j$. Note that $S_J \in \mathcal{S}_1$. Then we have

$$\mathbb{1}_{T_I} = \mathbb{1}_{S_I} \cdot \prod_{j \in [k] \setminus I} (\mathbb{1}_U - \mathbb{1}_{S_j}) = \sum_{J \subseteq [k] \setminus I} (-1)^{|J|} \mathbb{1}_{S_{I \cup J}}.$$

To see this, consider some $u \in U$. Assume first that u does not appear in any of the sets S_j for $j \in [k] \setminus I$. Then the only term in the sum with $\mathbb{1}_{S_{I \cup J}}(u) \neq 0$ is that with $J = \emptyset$, so we obtain exactly $\mathbb{1}_{S_I}$. Otherwise, the set $K = \{j \in [k] \setminus I : u \in S_j\}$ is non-empty and $u \in S_J$ if and only if $J \subseteq K$. Thus we count u exactly $\sum_{J \subseteq K} (-1)^{|J|} = \sum_{i=0}^{|K|} \binom{|K|}{i} (-1)^i = (1-1)^{|K|} = 0$ times by the binomial theorem.

Hence, for any $X \subseteq U$ we have $|\chi(T_I \cap X)| \leq \sum_{J \subseteq [k] \setminus I} |\chi(S_{I \cup J} \cap X)| \leq 2^k \cdot \text{herdisc}(\mathcal{S}_1)$. As $\chi(S \cap X)$ is the sum of at most 2^k terms of the form $\chi(T_I \cap X)$ we deduce $\text{herdisc}(\mathcal{S}) \leq 4^k \text{herdisc}(\mathcal{S}_1)$. \square

We now turn to Lemma 5.3, which is the main ingredient from the logical point of view.

Lemma 5.3. Let $\varphi(\bar{x}; \bar{y})$ be a quantifier-free σ -formula. There exists a finite set

$$\Psi = \{\rho_i(\bar{x}) : 1 \leq i \leq k\} \cup \{\psi_j(\bar{x}; \bar{z}) : 1 \leq j \leq \ell\}$$

of quantifier-free σ -formulas (with same $|\bar{x}|$, but possibly different $|\bar{z}|$ for the formulas ψ_j) where each $\psi_j(\bar{x}, \bar{z})$ has the form

$$\psi_j(\bar{x}, \bar{z}) := \bigwedge_{r=1}^{|\bar{z}|} (f_{\alpha_j, r}(x_{i_j, r}) = z_r)$$

and there is an integer p such that for every σ -structure \mathbf{M} , every set in $\mathcal{S}^\varphi(\mathbf{M})$ is a Boolean combination of at most p sets in $\mathcal{S}^\Psi(\mathbf{M})$, and $\mathcal{S}^\Psi(\mathbf{M})$ has degree at most $|\Psi| = k + \ell$.

Note that we do not translate φ into an equivalent formula. An example for the construction can be found below the proof and it may be helpful to consider it in parallel while reading the proof.

Proof. The formula $\varphi(\bar{x}; \bar{y})$ is equivalent (modulo T_σ) to a Boolean combination of formulas of the form

$$\zeta(\bar{x}; \bar{y}) := \rho(\bar{x}) \wedge \eta(\bar{y}) \wedge \bigwedge_{(f, g, i, j) \in \mathcal{F}} f(x_i) = g(y_j),$$

where \mathcal{F} is a finite subset of $\mathcal{F}_\sigma \times \mathcal{F}_\sigma \times [|\bar{x}|] \times [|\bar{y}|]$. This is because we may assume that φ uses no composition of functions, as \mathcal{F}_σ is a monoid, and because every atomic formula can

either use a unary relation symbol (and then is in ρ or η), or express equality of two terms. If both terms are of the form $f(x_i)$ (resp. $g(y_j)$), then this atomic formula is in ρ (resp. in η), otherwise it must be of the form $f(x_i) = g(y_j)$. Moreover, we can assume that each pair (g, j) belongs to at most one tuple $(f, g, i, j) \in \mathcal{F}$ as $(f(x_i) = g(y_j)) \wedge (h(x_{i'}) = g(y_j))$ is equivalent to $(f(x_i) = h(x_{i'})) \wedge (f(x_i) = g(y_j))$. Then for every structure \mathbf{M} and every tuple \bar{b} we have that $\varphi(\mathbf{M}, \bar{b})$ is a Boolean combination of sets of the form $\zeta(\mathbf{M}, \bar{b})$.

Now for every $\zeta(\bar{x}; \bar{y}) = \rho(\bar{x}) \wedge \eta(\bar{y}) \wedge \bigwedge_{(f,g,i,j) \in \mathcal{F}} f(x_i) = g(y_j)$ of the Boolean combination, we add ρ and ψ to Ψ , where we define \bar{z} and $\psi(\bar{x}; \bar{z})$ from $\bigwedge_{(f,g,i,j) \in \mathcal{F}} f(x_i) = g(y_j)$ by replacing each appearing term $g(y_j)$ by a new variable $z_{g,j}$. Note that \bar{z} may be longer than \bar{y} , however its length is at most the number of conjuncts in the big conjunction in ζ . Then ψ has the required form. Moreover, for every tuple \bar{b} , we can define an assignment \bar{c} of \bar{z} (let $c_{g,j} := g(b_j)$) such that

$$\zeta(\mathbf{M}; \bar{b}) = \begin{cases} \rho(\mathbf{M}) \cap \psi(\mathbf{M}; \bar{c}) & \text{if } \mathbf{M} \models \eta(\bar{b}) \\ \emptyset & \text{otherwise.} \end{cases}$$

Finally, observe that for every tuple \bar{a} , there do not exist two distinct tuples \bar{c}, \bar{c}' with $\bar{a} \in \psi(\mathbf{M}, \bar{c}) \cap \psi(\mathbf{M}, \bar{c}')$. If $\bar{a} \in \psi(\mathbf{M}, \bar{c}) \cap \psi(\mathbf{M}, \bar{c}')$, then for all g, j of the conjunction, $c_{g,j} = f(a_i) = c_{g,j}'$, hence $\bar{c} = \bar{c}'$. Hence the degree of each \mathcal{S}^ψ is 1. Obviously, the degree of each \mathcal{S}^ρ is also 1 (as this set system contains only one set). As we have k formulas ρ and ℓ formulas ψ , we have that the degree of $\mathcal{S}^\Psi(\mathbf{M})$ is at most $k + \ell$. \square

Example 5. Let $\varphi(x_1, x_2, x_3; y_1, y_2) = R(x_3) \wedge B(y_1) \wedge (h(x_1) = f(y_2)) \wedge (h(x_2) = g(y_2))$. We replace $f(y_2)$ by z_1 , and $g(y_2)$ by z_2 , and obtain $\rho(\bar{x}) = R(x_3)$ and $\psi(\bar{x}; \bar{z}) = (h(x_1) = z_1) \wedge (h(x_2) = z_2)$.

Let \mathbf{M} be a σ -structure, and let b_1, b_2 be two elements of the universe of \mathbf{M} , from which we derive $c_1 = f(b_2)$ and $c_2 = g(b_2)$. Note that for any tuple \bar{a} , if $\mathbf{M} \models \varphi(\bar{a}; \bar{b})$, then $\mathbf{M} \models \rho(\bar{a}) \wedge \psi(\bar{a}; \bar{c})$. Note that the inverse is not true, for example if $B(b_1)$ is not satisfied. Nonetheless, for any fixed b_1, b_2 , the set $\varphi(\mathbf{M}, \bar{b})$ is either empty or equal to $\rho(\mathbf{M}) \cap \psi(\mathbf{M}, \bar{c})$.

For the degree bound, take two tuples \bar{c} and \bar{c}' such that $\psi(\mathbf{M}, \bar{c}) \cap \psi(\mathbf{M}, \bar{c}') \neq \emptyset$. Take a tuple \bar{a} in the intersection. We have that $h(a_1) = c_1 = c_1'$ and $h(a_2) = c_2 = c_2'$, hence $\bar{c} = \bar{c}'$. So all sets in $\psi(\mathbf{M}, \bar{z})$ are pairwise disjoint.

Example 6. Let $\varphi(x; y) = A(x) \wedge (B(y) \wedge (f(x) = y) \vee \neg B(y) \wedge (f(x) = f(y)))$. Then we can choose $\Psi = \{\rho(x), \psi(x, z)\}$, where $\rho(x) = A(x)$ and $\psi(x, z) = (f(x) = z)$, as for every b we have

$$\varphi(G, b) = \begin{cases} \rho(G) \cap \psi(G, b) & \text{if } b \in B(G) \\ \rho(G) \cap \psi(G, f(b)) & \text{otherwise.} \end{cases}$$

We finally deduce a uniform bound for set systems defined on σ -structures by quantifier-free formulas, the main result of this section.

Proof of Theorem 5.1. According to Lemma 5.3 (applied to each formula in Φ) there exists a finite set Ψ of quantifier free partitioned formulas and integers k, t such that for every σ -structure \mathbf{M} the set system $\mathcal{S}^\Psi(\mathbf{M})$ has degree at most t , and every set in \mathcal{S}^Φ is a Boolean combination of at most k sets in \mathcal{S}^Ψ . Let \mathbf{M} be any σ -structure and let \mathcal{S}_0 be the collection of all intersections of sets in $\mathcal{S}^\Psi(\mathbf{M})$. According to Lemma 5.1, the set system \mathcal{S}_0 has degree at most 2^t . Thus, according to Theorem 2.1 we have $\text{herdisc}(\mathcal{S}_0) < 2^{t+1}$. Then it follows from Lemma 5.2 that $\text{herdisc}(\mathcal{S}^\Phi) < 4^k \text{herdisc}(\mathcal{S}_0) < 2^{2k+t+1}$, so we can choose $C_\Phi = 2^{2k+t+1}$. \square

Note that this theorem is tight in the sense that for every signature σ with a relation or a function of arity at least two there exists a quantifier-free formula with unbounded discrepancy on the class of all σ -structures.

5.2. Pointer structures and set-defined graphs

Pointer structures are intrinsically related to set-defined classes. Recall that the *age* of an infinite graph is the class of all its finite induced subgraphs. A class \mathcal{C} (of finite graphs) is *set-defined* if it is included in the age of an (infinite) graph definable in \mathbb{N} (seen as a model of an infinite set in the empty language) [22]. In other words, \mathcal{C} is set defined if there exists a partitioned formula $\varphi(\bar{x}; \bar{y})$ with $|\bar{x}| = |\bar{y}| = d$ such that every graph in \mathcal{C} is an induced subgraph of the graph with vertex set \mathbb{N}^d in which \bar{u} and \bar{v} are adjacent if and only if $\mathbb{N} \models \varphi(\bar{u}, \bar{v})$. Set-defined classes have bounded VC-dimension (they are actually both semi-algebraic and stable). Set-defined classes of graphs generalize classes of degenerate graphs. Indeed, it is proved in [22] that a class is degenerate if and only if it is *biclique-free* (i.e. excludes some biclique as a subgraph) and set-defined.

Lemma 5.4. If a set system is definable by a quantifier free formula in a pointer structure, then it is included in the neighborhood set system of a set-defined graph. Conversely, every neighborhood set system of a set-defined graph is included in a set system definable by a quantifier free formula in a pointer structure.

Proof. Let $q(\bar{x}; \bar{y})$ be a quantifier-free formula in the language of pointer structures. We can assume that the formula q contains no composition of functions (as we assumed that the functions in the signature form a monoid). Let f_1, \dots, f_a (resp. P_1, \dots, P_b) be the functions (resp. the predicates) appearing in q . Let \mathbf{M} be a pointer structure with universe M . Without loss of generality we assume $M = \{2, \dots, |M| + 2\}$. We will define \mathbf{M} on the subset $X = \{0, 1\} \cup M$ of \mathbb{N} , where we will define a bipartite graph such that the neighborhoods of the elements of one part correspond to the sets defined by q on \mathbf{M} .

We associate to each element u of M the tuples

$$\begin{aligned}\Lambda_0(u) &= (0, u, f_1(u), \dots, f_a(u), 0, p_1(u), \dots, p_b(u)) \text{ and} \\ \Lambda_1(u) &= (1, u, f_1(u), \dots, f_a(u), 0, p_1(u), \dots, p_b(u)),\end{aligned}$$

where $p_i(u) = 1$ if $\mathbf{M} \models P_i(u)$, and $p_i(u) = 0$ otherwise. The tuples $\Lambda_0(u)$ will belong to one part of the bipartite graph and the tuples $\Lambda_1(u)$ will belong to the other part. We use the first coordinate to distinguish the parts.

Then it is easily checked that the formula $q(\bar{x}; \bar{y})$ is equivalent to a quantifier free formula $Q(\bar{s}, \bar{t})$, where \bar{s} consists of all the tuples $\Lambda_0(x_i)$ and \bar{t} consists of all the tuples $\Lambda_1(y_j)$. For example, to check the atomic formula $f_i(x_1) = g_j(y_2)$ we need to check that the $(i + 2)$ nd coordinate of s_1 is equal to the $(j + 2)$ nd coordinate of t_2 (the $i + 2$ nd coordinate of s_1 , which is the tuple $\Lambda_0(x_1)$, contains the value $f_i(x_1)$ and analogously, the $j + 2$ nd coordinate of t_2 , which is the tuple $\Lambda_1(y_2)$, contains the value $f_j(y_2)$). Similarly, $\mathbf{M} \models P_i(x_1)$ if and only if the coordinate $a + 3 + i$ of s_1 contains 1. We use the 0 in coordinate $a + 3$ to test this, by checking if the coordinate $a + 3$ of s_1 is equal to the coordinate $a + 3 + i$ of s_1 .

The set system defined by q on \mathbf{M} is the neighborhood set system of an induced subgraph of the bipartite graph defined on X (as a subset of \mathbb{N}) by \bar{s} adjacent to \bar{t} if $X \models Q(\bar{s}; \bar{t})$.

Conversely, assume that $\varphi(\bar{x}; \bar{y})$ is a first-order formula (in the empty language). We may assume $|\bar{x}| = |\bar{y}| = d > 1$, as we can repeat a coordinate and pad to get $|\bar{x}| = |\bar{y}| > 1$ if needed. We may also assume that φ is quantifier free (as the theory of infinite sets has quantifier elimination, see [21]). Let G be a finite induced subgraph of the graph defined by φ on \mathbb{N} . Let A be the set of vertices of G , which is a finite subset of \mathbb{N}^d . Let \mathbf{M} be the pointer structure with domain $A \cup X$, where X is the set of all elements of \mathbb{N} appearing in a tuple of A , and where $f_i : A \cup X \rightarrow X$ is defined by: $f_i(\bar{v}) = v_i$ if $\bar{v} \in A$ and $f_i(u) = u$ if $u \in X$. Note that an element of $A \cup X$ is a fixed-point of f_i if and only if it belongs to X . It is easily checked that $\mathcal{S}^E(G)$ is included in $\mathcal{S}^q(\mathbf{M})_{|A}$ for some quantifier free formula $q(x, y)$. \square

Corollary 5.1. Let (U, \mathcal{S}) be a set system definable in an infinite set X . Then there exists a constant C such that for every finite subset A of U we have $\text{disc}(\mathcal{S}_{|A}) \leq C$.

Theorem 5.2. For every set-defined class \mathcal{C} there exists a constant C with

$$\text{herdisc}(\mathcal{S}^E(G)) \leq C,$$

for every $G \in \mathcal{C}$.

Proof. The statement is a direct consequence of Lemma 5.4 and Corollary 5.1. \square

We now use Example 4 to give a lower bound on the discrepancy of neighborhood set systems of set-definable graphs in terms of the length of the parameter tuples used in the formula defining the set system.

Example 7. For every non-negative integer d there exists a set system (U, \mathcal{S}) definable in an infinite set X by a formula $\varphi(\bar{x}; \bar{y})$ with $|\bar{x}| = |\bar{y}| = d + 1$, such that $\text{disc}(\mathcal{S}_{|A}) = \Omega(\sqrt{d})$ for some subset A of size 2^{d+1} .

Proof. For a non-negative integer p , let S_p be Sylvester's graph, introduced in Example 4. The above statement will follow from the lower bound on $\text{disc}(\mathcal{S}^E(S_p))$ given there and the property that S_p is an induced subgraph of the graph defined on \mathbb{N} by some formula $\varphi_p(\bar{x}, \bar{y})$, where $|\bar{x}| = |\bar{y}| = p + 1$, which we prove now.

We define inductively the parts U_p and V_p of S_p , as well as the formula φ_p . We let $U_0 = \{1\}$, $V_0 = \{2\}$, and $\varphi_0(x, y) = \neg(x = y)$. Assume S_k is the subgraph of the graph defined by φ_k induced by U_k and V_k . We define

$$\begin{aligned} \varphi_{k+1}(x_1, \dots, x_{k+1}, y_1, \dots, y_{k+1}) &= \varphi_k(x_1, \dots, x_k, y_1, \dots, y_k) \leftrightarrow ((x_{k+1} = x_1) \vee (y_{k+1} = x_1)), \\ U_{k+1} &= \{(x_1, \dots, x_{k+1}) \in \{1, 2\}^{k+1} : x_1 = 1\}, \\ \text{and } V_{k+1} &= \{(y_1, \dots, y_{k+1}) \in \{1, 2\}^{k+1} : y_1 = 2\}. \end{aligned}$$

\square

6. A model theoretic approach to classes with bounded expansion

In this section we translate the quantifier elimination scheme introduced in [15] for classes with bounded expansion into a model theoretic language, which allows us to extend the result obtained in Section 4 to the general setting of set systems definable in graphs of a bounded expansion class. Important model theoretic properties of classes of graphs with bounded expansion (like the existence of a quantifier elimination scheme) are based on the existence of so-called “transitive fraternal augmentations” with bounded in-degree. Here, we work with transitive fraternal augmentations with *bounded out-degree* instead, as arcs (u, v) will represent functions and bounded out-degree corresponds to a bounded number of functions in a structure. This can be formalized as follows (see [35, Section 7.4] and also [33]): A class \mathcal{C} of finite graphs has bounded expansion if and only if there exists a function $F : \mathbb{N} \rightarrow \mathbb{N}$ such that for every $G \in \mathcal{C}$ there exists a directed supergraph \vec{G}^+ and a *fraternity function* $r : E(\vec{G}^+) \rightarrow \mathbb{N}$ satisfying:

- (P1) **founding:** an arc $e = (u, v)$ of \vec{G}^+ is such that $r(e) = 1$ if and only if it links two vertices that are adjacent in G ;
- (P2) **fraternity:** for every two arcs $e_1 = (u, v)$ and $e_2 = (u, w)$ of \vec{G}^+ there is an arc $e_3 = (v, w)$ or (w, v) in \vec{G}^+ with $r(e_3) \leq r(e_1) + r(e_2)$;
- (P3) **boundedness:** every vertex of \vec{G}^+ has at most $F(i)$ outgoing arcs e with $r(e) \leq i$.

Note that the fraternal augmentation algorithm presented in [35] (see also [34]) can be iteratively used to compute a supergraph \vec{G}^+ and a fraternity function r of a finite graph G in time $O(n^4)$ satisfying the properties 1 and 2. Moreover, for every class \mathcal{C} of finite graphs with bounded expansion there exists a function $F_{\mathcal{C}}$ such that for graphs in \mathcal{C} the property 3 will be satisfied by the computed \vec{G}^+ and r (with $F = F_{\mathcal{C}}$). Note that we do not use the finiteness of G in the definition of fraternity functions and we will use all defined notation also for infinite graphs.

We now introduce theories $\text{Th}_{\sigma, \rho}^{\text{TF}}$, which are intrinsically related to the notion of bounded expansion. A *ranked signature* is a pair (σ, ρ) , where σ is a signature and $\rho : \sigma \rightarrow \mathbb{N}$ is a (*signature*) *ranking*. A *ranked pointer structure* is a pointer structure with a ranked signature. For a positive integer i , we denote by σ_i the subset of symbols in σ with rank at most i , and we define the *rank- i shadow* of a σ -structure \mathbf{M} as the σ_i -reduct of \mathbf{M} , that is the σ_i -structure obtained from \mathbf{M} by “forgetting” the functions and relations whose symbols are not in σ_i . Note that the structures \mathbf{M} considered in this section may be infinite.

We consider signatures

$$\begin{aligned} \zeta &= \mathcal{F}_{\sigma} \cup \{M_{\alpha} : \alpha \in \mathbb{N}\} \\ \sigma &= \zeta \cup \{P_{\varphi} : \varphi \in \text{FO}_1(\zeta)\}, \end{aligned}$$

where \mathcal{F}_{σ} is a monoid of function symbols, the symbols M_{α} and P_{φ} are unary predicates, and $\text{FO}_1(\zeta)$ is the class of all (equivalence classes of) first-order formulas on ζ -structures with a single free variable³. Moreover, we require that the rank function ρ on σ is such that every σ_i is finite. Note that this implies that σ is countable.

³By “equivalence classes” we mean that we keep only one representative of each equivalence class for the relation “is logically equivalent to”.

We define the theory $\text{Th}_{\sigma,\rho}^{\text{TF}}$ of fraternal σ -structures by the following axioms, which mirror the properties of fraternity functions.

(A1) **fraternity:** For all functions $f, g \in \mathcal{F}_\sigma$, and for every x there exists a function h (possibly depending on x) with $h(f(x)) = g(x)$ or $h(g(x)) = f(x)$ and $\rho(h) \leq \rho(f) + \rho(g)$.

As each σ_i is finite, this axiom can be expressed by the sentences $\theta_{i,j}^{\text{frat}}$ (for $i, j \in \mathbb{N}$), where

$$\theta_{i,j}^{\text{frat}} := \bigwedge_{f \in \sigma_i} \bigwedge_{g \in \sigma_j} \left(\forall x \bigvee_{h \in \sigma_{i+j}} (h(x) = f(g(x)) \vee (h(x) = g(f(x)))) \right).$$

(A2) **transitive coloring:** The unary predicates M_α with $\rho(M_\alpha) = i$ define a partition of the domain, and for every sequence of $k \leq i$ functions f_1, \dots, f_k where $\rho(f_j) \leq i$ for all $j \leq k$ we have $M_\alpha(x) \wedge (f_1 \circ \dots \circ f_k(x) \neq x) \rightarrow \neg M_\alpha(f_1 \circ \dots \circ f_k(x))$.

(A3) **typing:** For every formula $\varphi \in \text{FO}_1(\zeta)$ the predicate P_φ satisfies $P_\varphi(x) \leftrightarrow \varphi(x)$.

Recall that a p -centered coloring of a graph G is a coloring of the vertices such that every connected subgraph with at most p colors on the vertices contains some vertex with a unique color.

Lemma 6.1. Let \mathbf{M} be a fraternal σ -structure and let p, q be positive integers. Then the unary predicates M_α with $\rho(M_\alpha) = q \cdot (2^{p-1} + 2)$ define a $(p+1)$ -centered coloring of the Gaifman graph of the rank- q shadow of \mathbf{M} .

Proof. Denote by \vec{G} the (possibly infinite) directed graph whose vertex set is the domain of \mathbf{M} , in which there is an arc from a vertex u to a (distinct) vertex v if there exists $f \in \mathcal{F}_\sigma$ with $f(u) = v$ and $\rho(f) \leq q$, and let G be its underlying undirected graph (which is the Gaifman graph of the rank- q shadow of \mathbf{M}). Note that for every pair of distinct vertices u and v , there is at most one arc from u to v and at most one arc from v to u since we consider simple graphs only. Let \vec{G}^+ be the directed graph obtained in a similar way but without the constraint $\rho(f) \leq q$. For each arc $e = (u, v)$ of \vec{G}^+ define $r(e) = \lceil \rho(f)/q \rceil$, where $f \in \mathcal{F}_\sigma$ is such that $f(u) = v$ and $\rho(f)$ is minimal for this property. This implies that the axiom (P1) is satisfied by \vec{G}^+ and r . Then, for every two arcs $e_1 = (u, v)$ and $e_2 = (u, w)$ of \vec{G}^+ , we deduce from axiom (A1) that at least one of (v, w) and (w, v) is an arc of \vec{G}^+ with r -value at most $r(e_1) + r(e_2)$. It follows that the axiom (P2) is satisfied by \vec{G}^+ and r . The axiom (P3) is also satisfied by \vec{G}^+ and r , with $F(i) = |\{f \in \mathcal{F}_\sigma : \rho(f) \leq i\}|$. Then, it follows from [35, Lemma 7.8] that every vertex coloring c with $c(u) \neq c(v)$ whenever there is a directed path of length at most p using only arcs e with $r(e) \leq 2^{p-1} + 2$ linking distinct vertices u and v is a $(p+1)$ -centered coloring of G . Note that the proof of [35, Lemma 7.8] does not involve any argument based on the finiteness of the considered graphs, and thus remains valid in an infinite graph setting. The considered arcs correspond to functions with rank ρ at most $q \cdot (2^{p-1} + 2)$ and we conclude that the predicates M_α with $\rho(M_\alpha) \leq q \cdot (2^{p-1} + 2)$ define such a coloring, as desired. \square

The connection between ranked pointer structures models of $\text{Th}_{\sigma,\rho}^{\text{TF}}$ and classes with bounded expansion follows.

Lemma 6.2. A class \mathcal{C} (of finite graphs) has bounded expansion if and only if there exists a ranked signature (σ, ρ) such that for every graph $G \in \mathcal{C}$ there exists a model \mathbf{M} of $\text{Th}_{\sigma,\rho}^{\text{TF}}$ with the property that G is the Gaifman graph of the rank-1 shadow of \mathbf{M} .

Proof. Assume \mathcal{C} has bounded expansion, $G \in \mathcal{C}$ and \vec{G}^+ is a directed supergraph of G with properties (P1) to (P3). At each vertex u we number arbitrarily the outgoing arcs $e = (u, v)$ with $r(e) = i$ by distinct values $\lambda(e) \in [F(i)]$. We define the mappings $\zeta_{i,j} : V(G) \rightarrow V(G)$ by $\zeta_{i,j}(u) = v$ if $e = (u, v) \in E(\vec{G}^+)$, $r(e) = i$, and $\lambda(e) = j$; $\zeta_{i,j}(u) = u$, otherwise. As the maximum out-degree of arcs with r -value i is $F(i)$, there is a vertex coloring γ_i with at most $2iF(i)^i + 1$ colors such that no directed path of length at most i with arcs e with $r(e) \leq i$ links two vertices with the same γ_i -value. To see this, observe that the i th power of a directed graph with out-degree F has out-degree at most iF^i , such that the underlying undirected graph is $2iF^i$ degenerate and the greedy coloring uses at most $2iF^i + 1$ colors. Let $\sigma_0 = \{f_{i,j} : i \in \mathbb{N}, j \in [F(i)]\} \cup \{M_{i,j} : i \in \mathbb{N}, j \in [(2F(i)^i + 1)]\}$ and $\sigma = \sigma_0 \cup \{M_\psi : \psi \in \text{FO}_1[\sigma_0]\}$. We define the rank ρ on σ by $\rho(f_{i,j}) = \rho(M_{i,j}) = i$ and $\rho(M_\psi) = 1$. Let \mathbf{M} be the σ -structure with domain $V(G)$ where we interpret $f_{i,j}(x)$ by $\zeta_{i,j}(x)$, $M_{i,j}(x)$ by $(\gamma_i(x) = j)$, and $M_\psi(x)$ by $\psi(x)$. By construction, \mathbf{M} is a model of $\text{Th}_{\sigma,\rho}^{\text{TF}}$, and G is the Gaifman graph of the rank-1 shadow of \mathbf{M} .

The converse follows from Lemma 6.1 (for $q = 1$) and the property that classes with bounded p -centered colorings have bounded expansion. \square

Lemma 6.3. Let p be a positive integer, let σ be the signature of a pointer structure, and let $\psi(\bar{x})$ be a formula of the form

$$\psi(\bar{x}) := \exists z_1 \dots \exists z_p \psi'(\bar{x}, \bar{z}),$$

where ψ' is a conjunction of atomic formulas of the form $z_i = f(z_j)$ or $z_i = x_j$ and a Boolean combination of formulas of the form $z_i = z_j$ or $\alpha(z_i)$.

Then there exists a formula $\zeta(\bar{x})$, which is a Boolean combination of quantifier-free formulas $q_i(\bar{x})$ and single variable formulas $\eta_i(x_j)$, such that ψ and ζ are equivalent on σ -structures whose Gaifman graphs have tree-depth at most p .

Proof. By removing all the symbols that are not used in ψ , we can restrict to the case where the signature is finite. It follows from [35, Section 6.8] that for every two integers r and p there is an integer $C(r, p)$ such that every colored graph⁴ with treedepth at most p is r -equivalent to one of its induced subgraphs of order at most $C(r, p)$. The lemma thus follows from the finite case, for which we refer the reader (for instance) to [15]. \square

We continue to prove that $\text{Th}_{\sigma,\rho}^{\text{TF}}$ has quantifier elimination. Following the standard quantifier elimination proof scheme (see e.g. [21, Lemma 2.3.1]), we can reduce to the case of eliminating a single existential quantifier.

Lemma 6.4. Let $\varphi(\bar{x}) := \exists y \varphi'(\bar{x}, y)$ be a formula, where φ' is a quantifier-free σ -formula. Then $\varphi(\bar{x})$ is equivalent to a quantifier-free formula $\psi(\bar{x})$ on all models of $\text{Th}_{\sigma,\rho}^{\text{TF}}$.

Proof. Let $\sigma_1 \subseteq \sigma$ be the (finite) set of symbols used in φ . We consider an arbitrary (possibly infinite) model \mathbf{M} of $\text{Th}_{\sigma,\rho}^{\text{TF}}$ and construct a formula ψ such that $\mathbf{M} \models \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$. Our construction of ψ will be independent of the choice of the model \mathbf{M} . Let p be the minimum integer such that φ' is equivalent to a formula of the form $\exists z_1 \dots \exists z_p \psi'(\bar{x}, y, \bar{z})$, where ψ' is a conjunction of atomic formulas of the form $z_i = f(z_j)$ or $z_i = x_j$ or $z_i = y$ and a Boolean combination of formulas of the form $z_i = z_j$ or $\alpha(z_i)$.

⁴By a colored graph, we mean a graph whose vertices and edges are colored using a finite set of colors.

Let \mathbf{M}_1 be the σ_1 -reduct of \mathbf{M} and let $q = \max\{\rho(f) : f \in \sigma_1\}$, and let $t = q(2^{p-1} + 2)$. According to Lemma 6.1, the predicates M_α with $\rho(\alpha) = t$ define a partition of the domain of \mathbf{M}_1 , such that any p classes weakly induce a pointer structure whose Gaifman graph has tree-depth at most p . Let $T = \{\alpha : \rho(M_\alpha) = t\}$. We rewrite the formula $\exists y (\exists z_1 \dots \exists z_p \psi'(\bar{x}, y, \bar{z}))$ as

$$\bigvee_{t_1 \in T} \dots \bigvee_{t_p \in T} \exists y \exists z_1 \dots \exists z_p \left(M_{t_0}(y) \wedge \left(\bigwedge_{i=1}^p M_{t_i}(z_i) \right) \wedge \psi'(\bar{x}, y, \bar{z}) \right).$$

According to Lemma 6.3 there exist formulas $\zeta_{t_1, \dots, t_p}(\bar{x})$, which are Boolean combinations of quantifier-free formulas and single variable formulas, such that the above formula is equivalent to

$$\bigvee_{t_1 \in T} \dots \bigvee_{t_p \in T} \zeta_{t_1, \dots, t_p}(\bar{x}).$$

By replacing in each ζ_{t_1, \dots, t_p} each single free variable formula $\eta(x)$ by the corresponding predicate P_η , we get a quantifier-free formula $\hat{\zeta}_{t_1, \dots, t_p}(\bar{x})$ that is equivalent to $\zeta_{t_1, \dots, t_p}(\bar{x})$ on every model of $\text{Th}_{\sigma, \rho}^{\text{TF}}$. Thus, we have $\text{Th}_{\sigma, \rho}^{\text{TF}} \models \varphi(\bar{x}) \leftrightarrow \psi(\bar{x})$, where $\psi(\bar{x})$ is the quantifier-free formula $\bigvee_{t_1 \in T} \dots \bigvee_{t_p \in T} \hat{\zeta}_{t_1, \dots, t_p}(\bar{x})$. \square

The main result of this section now follows by an easy induction from Lemma 6.4.

Theorem 6.1. For every ranked signature (σ, ρ) and every formula $\varphi(\bar{x})$ there exists a quantifier free formula $q(\bar{x})$ such that φ and q are equivalent on every model of $\text{Th}_{\sigma, \rho}^{\text{TF}}$.

In other words, the theory $\text{Th}_{\sigma, \rho}^{\text{TF}}$ has quantifier elimination.

We now can derive a model theoretical characterization of bounded expansion classes.

Theorem 6.2. Let \mathcal{C} be a monotone class of (finite) graphs. Then the following are equivalent:

- (i) the class \mathcal{C} has bounded expansion;
- (ii) the hereditary discrepancy of every set system $\mathcal{S}^\varphi(G)$ definable on a monadic expansion of \mathcal{C} is bounded;
- (iii) for each positive integer k , the hereditary discrepancy of $\mathcal{S}^E(G^k)$ for $G \in \mathcal{C}$ is bounded.

Proof. (i) \Rightarrow (ii) as a consequence of Lemma 6.2 and Corollary 5.1, (ii) \Rightarrow (iii) as $\mathcal{S}^E(G^k)$ is definable on \mathcal{C} , and (iii) \Rightarrow (i) as a consequence of Corollary 4.1. \square

7. Discrepancy in nowhere dense classes

Let $\varphi(\bar{x}, \bar{y})$ be a first-order formula and let \mathcal{C} be a nowhere dense class. We now give two bounds on the discrepancy of $\mathcal{S}^\varphi(G)$ (for $G \in \mathcal{C}$), in terms of the lengths of the tuples \bar{x} and \bar{y} , respectively.

Lemma 7.1. Let \mathcal{C} be a nowhere dense class and let $\varphi(\bar{x}, \bar{y})$ be a first-order formula. Then for every $\varepsilon > 0$ there exists a constant C with

$$\text{disc}(\mathcal{S}^\varphi(G)|_A) \leq C n^{\frac{1}{2} - \frac{1}{2|\bar{y}|} + \varepsilon}$$

for every $G \in \mathcal{C}$ and $A \subseteq V(G)^{|\bar{x}|}$, where $n = \left| \bigcup \mathcal{S}^\varphi(G)|_A \right|$.

Proof. Let $m \in \mathbb{N}$ and let M be a subset of $V(G)^{|\bar{x}|}$ with $|M| = m$. Then $M \subseteq U^{|\bar{x}|}$, where $U \subseteq V(G)$ is the set of all the elements appearing in some tuple in M . Obviously $|U| \leq |\bar{x}| \cdot |M|$. According to Theorem 2.6, for every $\varepsilon > 0$ there exists a constant c such that for every $G \in \mathcal{C}$ and every $A \subseteq V(G)$ we have $\left| \mathcal{S}^\varphi(G)|_{A^{|\bar{x}|}} \right| \leq c |A|^{|\bar{y}| + \varepsilon}$. Thus, we have $\left| \mathcal{S}^\varphi(G)|_A \right| \leq c (m |\bar{x}|)^{|\bar{y}| + \varepsilon}$. It follows that the primal shatter function $\pi(m)$ of $\mathcal{S}^\varphi(G)$ (and, more generally, of $\mathcal{S}^\varphi(G)|_A$ for $A \subseteq V(G)^{|\bar{x}|}$) is bounded by $c' m^{|\bar{y}| + \varepsilon}$, where $c' = c |\bar{x}|^{|\bar{y}|}$, and the result follows from Theorem 2.4. \square

Lemma 7.2. Let \mathcal{C} be a nowhere dense class and let $\varphi(\bar{x}, \bar{y})$ be a first-order formula. Then for every $\varepsilon > 0$ there exists a constant C with

$$\text{disc}(\mathcal{S}^\varphi(G)|_A) \leq C n^{\frac{1}{2} - \frac{1}{2|\bar{x}|} + \varepsilon}$$

for every $G \in \mathcal{C}$ and $A \subseteq V(G)^{|\bar{x}|}$, where $n = \left| \bigcup \mathcal{S}^\varphi(G)|_A \right|$.

Proof. Let $\varphi^*(\bar{y}; \bar{x}) = \varphi(\bar{x}; \bar{y})$. As in the proof of Lemma 7.1, the shatter function of $\mathcal{S}^{\varphi^*}(G)$ is bounded by $c m^{|\bar{x}| + \varepsilon}$ for some constant c . As $\mathcal{S}^{\varphi^*}(G)$ is the dual of the set system of $\mathcal{S}^\varphi(G)$, the dual shatter function $\pi^*(m)$ of $\mathcal{S}^\varphi(G)$ is bounded by $c m^{|\bar{x}| + \varepsilon}$ as well, and the result follows from Theorem 2.5. \square

Theorem 7.1. For a monotone class of graphs \mathcal{C} the following are equivalent:

- (1) \mathcal{C} is nowhere dense;
- (2) for every monadic lift \mathcal{C}^+ of \mathcal{C} , for every formula $\varphi(\bar{x}; \bar{y})$ in the language of \mathcal{C}^+ , for every $\alpha > \frac{1}{2} - \frac{1}{2 \min(|\bar{x}|, |\bar{y}|)}$, and for $G \in \mathcal{C}$, we have

$$\text{herdisc}(\mathcal{S}^\varphi(G^+)) \in \mathcal{O}\left(\left| \bigcup \mathcal{S}^\varphi(G^+) \right|^\alpha\right);$$

- (3) for every positive integer k , every $\varepsilon > 0$, and for $G \in \mathcal{C}$, we have

$$\text{disc}(\mathcal{S}^E(G^k)) \in \mathcal{O}(|G|^\varepsilon);$$

- (4) there exists a positive integer r such that for every partitioned formula $\varphi(\bar{x}; \bar{y})$ with $|\bar{x}| = |\bar{y}| = r$ and for $G \in \mathcal{C}$ we have $\text{disc}(\mathcal{S}^\varphi(G)) \in o\left(\left| \bigcup \mathcal{S}^\varphi(G) \right|^{1/2}\right)$.

Proof. The proof follows from the following implications:

(1) \Rightarrow (2) This immediately follows from Lemmas 7.1 and 7.2.

(2) \Rightarrow (3) This immediately follows from the special case of the set system defined by a formula $\varphi(x, y)$ expressing that the distance between x and y is at most k .

(3) \Rightarrow (1) (by contradiction): Assume the class \mathcal{C} is not nowhere dense. As \mathcal{C} is monotone there exists a positive integer k such that the class \mathcal{C} contains the $(k-1)$ -subdivision of all graphs. In particular, \mathcal{C} contains the $(k-1)$ -subdivision of all the graphs S_p (defined in Example 4). Let G_p be the $(k-1)$ -subdivision of S_p and let $A_p \subseteq V(G_p)$ be the set of the principal vertices of G_p (i.e. the vertices of S_p). Then $\text{disc}(\mathcal{S}^E(G^k)) = \Omega(|A_p|^{1/2}) = \Omega(|G|^{1/4})$. Hence $\neg(1) \Rightarrow \neg(3)$.

(2) \Rightarrow (4) This is trivial as we can always choose $\alpha < 1/2$ in (2).

(4) \Rightarrow (1) (by contradiction): Assume the class \mathcal{C} is not nowhere dense. As above, \mathcal{C} contains the $(k-1)$ -subdivision of all the graphs S_p (defined in Example 4), for some positive integer k .

For $p > 1$, consider the graph G_p obtained from S_p by adding a new vertex s adjacent to all the vertices in one of the parts of S_p , and a vertex r adjacent only to s . Let $\mu_A(x)$ be the formula expressing that x has degree greater than one and is adjacent to a vertex adjacent to a vertex with degree one, and let $\mu_B(x)$ be the formula expressing that no neighbor of x is adjacent to a vertex with degree one.

For $d \in \mathbb{N}$ we define the formula $\zeta_d(\bar{x}; \bar{y})$ with $|\bar{x}| = |\bar{y}| = d$ inductively, as follows (where $\bar{x}' = (x_1, \dots, x_{d-1})$ and $\bar{y}' = (y_1, \dots, y_{d-1})$): $\zeta_1(x, y) := \mu_A(x) \wedge \mu_B(y) \wedge E(x, y)$ and, for $d > 1$ we define $\zeta_d(\bar{x}, \bar{y}) := \left(\zeta_{d-1}(\bar{x}', \bar{y}') \leftrightarrow E(x_d, y_d) \right) \wedge \bigwedge_{i=1}^d (\mu_A(x_i) \wedge \mu_B(y_i))$.

It is easily checked that for $p > 1$ the graph defined on $V(G_p)^d$ where \bar{a} is adjacent to \bar{b} if $G_p \models \zeta_d(\bar{a}, \bar{b})$ is isomorphic to the union of G_{dp} and isolated vertices. In particular, we have $\text{disc}(\mathcal{S}^{\zeta_d}(G_p)) \in \Omega(\sqrt{|V(G_p)|^d})$.

For $p > 2$, let H_p be the $(k-1)$ -subdivision of G_p , and let $\hat{\zeta}_d(\bar{x}; \bar{y})$ be the formula obtained from ζ_d by replacing $E(x, y)$ by a formula expressing that $\text{dist}(x, y) = d$, and by conditioning all the vertices to have degree at least 3. Then $\mathcal{S}^{\hat{\zeta}_d}(H_p) = \mathcal{S}^{\zeta_d}(G_p)$. Thus, $\text{disc}(\mathcal{S}^{\hat{\zeta}_d}(H_p)) \in \Omega\left(\left|\bigcup \mathcal{S}^{\varphi}(H_p)\right|^{1/2}\right)$. As \mathcal{C} contains all the graphs H_p , we deduce $\neg(1) \Rightarrow \neg(4)$. \square

Conjecture 5. A monotone class \mathcal{C} is nowhere dense if and only if every for every partitioned formula $\varphi(\bar{x}; \bar{y})$, for every $\varepsilon > 0$, and for $G \in \mathcal{C}$ we have $\text{herdisc}(\mathcal{S}^{\varphi}(G)) \in \mathcal{O}\left(\left|\bigcup \mathcal{S}^{\varphi}(G)\right|^{\varepsilon}\right)$.

8. Applications

We now consider the problems introduced in Section 1 in the light of the results obtained in the previous sections.

8.1. Simultaneous neighborhood discrepancy for red and blue edges

Let G be a graph. To a 2-coloring $\gamma : E(G) \rightarrow \{1, 2\}$ we associate the set system $\mathcal{S}^\gamma(G)$ whose members are the 1-neighborhoods and the 2-neighborhoods of the vertices in G , where by i -neighborhood of a vertex v (with $i \in \{1, 2\}$) we mean the set of all the vertices u adjacent to v by an edge with γ -color i . For every subgraph $H \subseteq G$ we can consider the coloring γ with $\gamma(e) = 1$ if $e \in E(H)$. Then $\mathcal{S}^E(H) \subseteq \mathcal{S}^\gamma(G)$. It follows that

$$\max_{H \subseteq G} \text{disc}(\mathcal{S}^E(H)) \leq \max_{\gamma} \text{disc}(\mathcal{S}^\gamma(G)).$$

On the other hand, let G^γ be the bipartite graph with parts V and $W = V \times \{1, 2\}$, where u is adjacent to (v, i) if $uv \in E(G)$ and uv is colored i . It is easily checked that $\text{deg}(G^\gamma) \leq \text{deg}(G)$. Moreover, it is easily checked that $\mathcal{S}^\gamma(G)$ is included in $\mathcal{S}^E(G^\gamma)$. Thus, any upper bound on $\text{disc}(\mathcal{S}^E(G^\gamma))$ based on its degeneracy provides an upper bound on $\text{disc}(\mathcal{S}^\gamma(G))$ in terms of $\text{deg}(G)$.

From this and Theorem 3.1, we deduce the following (partial) answer to Problem 1.

Corollary 8.1. For every graph G we have

$$\frac{\log_2(\pi \text{deg}(G))}{4} - 2 \leq \max_{\gamma} \text{disc}(\mathcal{S}^\gamma(G)) < 3 \text{deg}(G), \quad (4)$$

where the maximum runs over 2-colorings $\gamma : E(G) \rightarrow \{1, 2\}$ and $\mathcal{S}^\gamma(G)$ is the set system, whose members are the 1-neighborhoods and the 2-neighborhoods of the vertices in G .

Problem 1. Assume that a graph G has the property that for every red/blue coloring of the edges of G there exists a partition (A, B) of the vertex set of G such that the number of red (resp. blue) neighbors in A and B of any vertex differ by at most 1. Does G contain a vertex with small degree?

According to Corollary 8.1, we deduce from $\max_{\gamma} \text{disc}(\mathcal{S}^\gamma(G)) \leq 1$ that the degeneracy of G is at most $2^{12}/\pi < 1304$.

8.2. Epsilon-nets and epsilon-approximations.

An important application of the notion of discrepancy is that of an ε -net. A subset N of the ground set U of a set system (U, \mathcal{S}) is an ε -net for \mathcal{S} if N intersects all the sets $S \in \mathcal{S}$ that have at least $\varepsilon |U|$ elements. A related concept is that of an ε -approximation. A subset $A \subseteq U$ is an ε -approximation for \mathcal{S} if $\left| \frac{|A \cap S|}{|A|} - \frac{|S|}{|U|} \right| \leq \varepsilon$ for every $S \in \mathcal{S}$. The notions of ε -nets and ε -approximations play a key role in the approximation of general sets from the systems by smaller subsets. Upper bounds for the sizes of ε -nets and ε -approximations of a set system can be derived from bounds on its hereditary discrepancy. Matoušek, Welzl, and Wernisch proved [29, Lemma 2.2] that if (U, \mathcal{S}) is a set system with $U \in \mathcal{S}$ and f is a function such that $\text{disc}(\mathcal{S}|_X) \leq f(|X|)$ for all $X \subseteq U$, then for every integer $t \geq 0$ there exists an ε -approximation A for \mathcal{S} with $|A| = \lceil \frac{|U|}{2^t} \rceil$ and $\varepsilon \leq \frac{2}{|U|} \cdot \sum_{i=0}^{t-1} 2^i f(\lceil \frac{|U|}{2^i} \rceil) < 2 \text{herdisc}(\mathcal{S}) \frac{2^t}{|U|}$. In particular, every set system (U, \mathcal{S}) has an ε -approximation of size at most $\lceil \frac{2 \text{herdisc}(\mathcal{S})}{\varepsilon} \rceil$. Furthermore, this construction is efficient and an ε -approximation with the stated size can be computed in polynomial time. Thus, as definable set systems can be constructed in polynomial time in bounded expansion classes (by using the model checking algorithm of [15]), we get the following.

Corollary 8.2. For every class \mathcal{C} with bounded expansion, there exists an integer c and an algorithm that computes, for a partitioned formula $\varphi(\bar{x}; \bar{y})$, a graph $G \in \mathcal{C}$, and a real number and $\varepsilon > 0$, an ε -approximation of $\mathcal{S}^\varphi(G)$ in time $\mathcal{O}(|V(G)|^c)$, where the constant hidden in the \mathcal{O} -notation depends on \mathcal{C} , φ , and ε .

As the class of planar graphs has bounded expansion, we deduce the following answer to Problem 2.

Problem 2. Given a planar graph G , find a small subset F of edges such that, for every pair u, v of distinct vertices of G , the probability that an edge of G belongs to a uv -path of length at most 100 differs from the probability that an edge in F belongs to a uv -path of length at most 100 by at most ε .

Consider the formula $\varphi(\bar{x}; \bar{y})$ with $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$, which asserts that x_1 and x_2 are adjacent, and that there exists a path of length at most 100 linking y_1 and y_2 including the edge x_1x_2 . For any fixed bounded expansion class \mathcal{C} , like the class of planar graphs, there exists a constant c such that for every $G \in \mathcal{C}$ we have $\text{herdisc}(\mathcal{S}^\varphi(G)) \leq c$. It follows that the set system $(E(G), \mathcal{S}^\varphi(G))$ has an ε -approximation with size $\lceil \frac{2c}{\varepsilon} \rceil$, which can be computed in polynomial time.

8.3. Clique coloring

A *clique coloring* of a graph G is a coloring of the vertices of G such that no maximal clique is monochromatic [13]. If a hereditary class of graphs has a bounded clique coloring number, it is obviously (exponentially) χ -bounded. However, while it was conjectured [13] that every perfect graph is 3-colorable, Charbit et al. [7] showed they do not have a bounded clique coloring number.

As a variant of clique coloring, one can ask whether a graph admits a vertex coloring by k colors in such a way that no maximal clique K contains more than $\lceil (1 - \varepsilon) |K| \rceil$ vertices⁵ of same color (for some fixed integer k and positive real ε). For instance, an easy induction shows that cographs do not have such a coloring in general, while classes with bounded shrubdepth (and, more generally, classes with structurally bounded expansion) do.

In this context, discrepancy colorings appear as a very strong property, by requiring that the vertices can be 2-colored in such a way that in every maximal clique the difference between the numbers of vertices of each color is bounded by some constant c . Even classes with bounded shrubdepth do not have this property. However, we shall prove the following.

Corollary 8.3. For every class \mathcal{C} with bounded expansion and every integer k there is a constant c such that for every graph $G \in \mathcal{C}$, the set system formed by all maximal cliques of G^k has hereditary discrepancy at most c .

Proof. According to [30], there exists an integer p (depending on \mathcal{C} and k) such that for every $G \in \mathcal{C}$ and every maximal clique K of G there exist $v_1, \dots, v_p \in K$ with $K = \bigcap_{i=1}^p N_{G^k}[v_i]$. On the other hand, given p vertices u_1, \dots, u_p it is easily checked that the property that $\bigcap_{i=1}^p N_{G^k}[u_i]$ induces a maximal clique of G^k can be checked by a first-order sentence $\kappa(x_1, \dots, x_p)$. Hence the set system of all maximal cliques of G^k (plus the empty set) is definable in G as $\mathcal{S}^\varphi(G)$, where $\varphi(\bar{x}; \bar{y}) := (\bar{x} = \bar{y}) \wedge \kappa(\bar{y})$. \square

⁵Note that the rounding allows to introduce a threshold, meaning that small cliques (of size at most $\lceil 1/\varepsilon \rceil$) are allowed to be monochromatic.

Problem 3. Does there exist a constant c , such that the vertices of every map graph G can be colored red or blue, in such a way that the difference between the number of red and blue vertices in every maximal clique of G is at most c ?

Let G be a map graph, that is, the vertex-face incidence graph of a planar map. By definition, G is an induced subgraph of the square H^2 of a planar graph H . According to Corollary 8.3, the hereditary discrepancy of the set system of the maximal cliques of H^2 is bounded by some constant c . As every maximal clique of G is the trace on $V(G)$ of some maximal clique of H^2 we deduce that the set system of all maximal cliques of G has bounded discrepancy.

8.4. Quantifier elimination

Our results on hereditary discrepancy unexpectedly allow us to answer Problem 4.

Problem 4. Give an example of a nowhere dense class \mathcal{C} of graphs such that there exists no expansion σ of the signature of graphs by unary relation and function symbols with the property that every first-order formula is equivalent on a σ -expansion \mathcal{C}^+ of \mathcal{C} to a quantifier-free first-order σ -formula.

Corollary 8.4. Let \mathcal{C} be the class of the 1-subdivisions of all bipartite graphs, whose girth is greater than their maximum degree. The class \mathcal{C} is nowhere dense and such that there does not exist an expansion σ of the signature of graphs by unary relations and functions, an expansion F of graphs into σ -structures and a quantifier-free formula $q(x, y)$ in the language of σ -structures with

$$\forall G \in \mathcal{C} \forall a, b \in V(G) \quad (G \models \varphi(a, b)) \iff (F(G) \models q(a, b)), \quad (5)$$

where $\varphi(a, b)$ expresses that a and b are at distance 2 in G .

Proof. Let \mathcal{C}_0 be the class of all bipartite graphs whose girth is greater than the maximum degree. It is well known that this class is monotone, nowhere dense, but has unbounded degeneracy. Let \mathcal{C} be the class of the 1-subdivisions of the graphs in \mathcal{C}_0 . This class is obviously also nowhere dense.

Let σ_0 be the signature consisting of two unary functions f_1 and f_2 . According to Observation 5.1, we can associate to each $G \in \mathcal{C}$ (which is the 1-subdivision of a bipartite graph H with parts V_1 and V_2) the σ_0 -structure \mathbf{M} defined as follows: if v is a principal vertex (i.e. a vertex of H) then $f_1(v) = f_2(v) = v$, while if v is the subdivision vertex of the edge $\{u_1, u_2\} \in V_1 \times V_2$ then $f_1(v) = u_1$ and $f_2(v) = u_2$. Let \mathcal{D} be the class of all the σ_0 -structures \mathbf{M} obtained from the graphs $G \in \mathcal{C}$.

Assume towards a contradiction that there is an expansion σ of the signature of graphs by unary relations and functions, an expansion F of graphs into σ -structures and a quantifier-free formula $q(x, y)$ in the language of σ -structures such that equation (5) holds. Then this defines an expansion σ' of σ_0 , an expansion F' of σ_0 structures into σ' -structures, and a quantifier free formula q' , such that

$$\forall \mathbf{M} \in \mathcal{D} \forall a, b \in M \quad (\mathbf{M} \models \varphi'(a, b)) \iff (F'(\mathbf{M}) \models q'(a, b)),$$

where $\varphi'(u, v)$ expresses that u and v are at distance 2 in the Gaifman graph of \mathbf{M} .

According to Theorem 5.1 there is a constant C such that $\text{disc}(\mathcal{S}^{q'}(F'(\mathbf{M}))) \leq C$ for all $\mathbf{M} \in \mathcal{D}$. However, it is easily checked that if \mathbf{M} is the σ_0 -structure corresponding to the 1-subdivision G of a graph H we have $\mathcal{S}^E(H) \subseteq \mathcal{S}^q(G) = \mathcal{S}^{q'}(\mathbf{M})$. According to Theorem 6.2 the set systems $\mathcal{S}^E(H)$ with $H \in \mathcal{C}_0$ have unbounded discrepancy, what contradicts $\text{disc}(\mathcal{S}^{q'}(\mathbf{M})) = \text{disc}(\mathcal{S}^{q'}(F'(\mathbf{M}))) \leq C$. \square

9. Concluding Remarks

As mentioned in Section 5.2, the set systems definable by a quantifier-free formula in pointer structures are basically the same as the neighborhood set systems of set-defined graphs.

A k -copy of a graph G is the graph obtained from k copies of G by adding an edge between any two clone vertices. A *simple (first-order) interpretation* \mathbf{I} is a pair of formulas $(\eta(x, y), \nu(x))$; it maps a (possibly colored) graph G into the graph $\mathbf{I}(G)$ with vertex set $\nu(G) = \{v \in G : G \models \nu(v)\}$ and edge set $\eta(G) \cap \nu(G)^2 = \{\{u, v\} \in \nu(G)^2 : G \models \eta(u, v) \vee \eta(v, u)\}$ (where the formulas η and ν can use the colors). A *(first-order) transduction* \mathbf{T} is a pair (k, \mathbf{I}) , where k is an integer and \mathbf{I} is a simple interpretation; it maps a graph G to the class of all graphs that can be obtained from G by computing a k -copy, then coloring this k -copy in an arbitrary way, then applying the transduction \mathbf{I} . We refer the interested reader to [16, 31] for a discussion on first-order transductions.

For a monotone class \mathcal{C} , it follows from [22] and Theorem 6.2 that the four following properties are equivalent:

1. \mathcal{C} has bounded expansion;
2. every transduction of \mathcal{C} is set-defined;
3. every transduction of \mathcal{C} is linearly χ -bounded;
4. every set system definable in \mathcal{C} has bounded hereditary discrepancy.

The equivalence between the first three items is known to hold for biclique-free classes of graphs. We conjecture that this extends to the equivalence with the last item.

Conjecture 6. A biclique-free class of graphs has bounded expansion if and only if every set system definable in \mathcal{C} has bounded hereditary discrepancy.

An intriguing related problem is the following, for which a positive answer would imply conjecture 6.

Problem 7. Is it true that for every non-degenerate biclique-free class \mathcal{C} of graphs there exists a transduction \mathbf{T} such that $\mathbf{T}(\mathcal{C})$ includes a monotone non-degenerate class of graphs?

References

- [1] Hans Adler and Isolde Adler. Interpreting nowhere dense graph classes as a classical notion of model theory. *European Journal of Combinatorics*, 36:322–330, 2014.
- [2] Noga Alon, Lajos Rónyai, and Tibor Szabó. Norm-graphs: variations and applications. *Journal of Combinatorial Theory, Series B*, 76(2):280–290, 1999.
- [3] Nikhil Bansal. Algorithmic aspects of combinatorial discrepancy. In *A Panorama of Discrepancy Theory*, pages 425–457. Springer, 2014.
- [4] József Beck and Tibor Fiala. “Integer-making” theorems. *Discrete Applied Mathematics*, 3(1):1–8, 1981.
- [5] Debe Bednarchak and Martin Helm. A note on the Beck–Fiala theorem. *Combinatorica*, 17(1):147–149, 1997.
- [6] Boris Bukh. An improvement of the Beck–Fiala theorem. *Combinatorics, Probability and Computing*, 25(3):380–398, 2016.
- [7] Pierre Charbit, Irena Penev, Stéphan Thomassé, and Nicolas Trotignon. Perfect graphs of arbitrarily large clique-chromatic number. *Journal of Combinatorial Theory, Series B*, 116:456–464, 2016.
- [8] Moses Charikar, Alantha Newman, and Aleksandar Nikolov. Tight hardness results for minimizing discrepancy. In *Proceedings of the twenty-second annual ACM-SIAM symposium on Discrete Algorithms*, pages 1607–1614. SIAM, 2011.
- [9] Bernard Chazelle. *The discrepancy method: randomness and complexity*. Cambridge University Press, 2001.
- [10] T.M. Cover and J.A. Thomas. *Elements of Information Theory*. Wiley, 2nd edition, 2006.
- [11] Reinhard Diestel. *Graph Theory, 4th Edition*, volume 173 of *Graduate texts in mathematics*. Springer, 2012.
- [12] Michael Drmota and Robert F Tichy. *Sequences, discrepancies and applications*. Springer, 2006.
- [13] Dwight Duffus, Bill Sands, Norbert Sauer, and Robert E Woodrow. Two-colouring all two-element maximal antichains. *Journal of Combinatorial Theory, Series A*, 57(1):109–116, 1991.
- [14] Z. Dvořák. Constant-factor approximation of domination number in sparse graphs. *European J. Combin.*, 34(5):833–840, 2013.
- [15] Zdeněk Dvořák, Dan Král, and Robin Thomas. Testing first-order properties for subclasses of sparse graphs. *Journal of the ACM*, 60:5 Article 36, 2013.
- [16] J. Gajarský, S. Kreutzer, J. Nešetřil, P. Ossona de Mendez, M. Pilipczuk, S. Siebertz, and S. Toruńczyk. First-order interpretations of bounded expansion classes. *ACM Transactions on Computational Logic*, 21(4):Article 29, 2020.
- [17] Martin Grohe, Stephan Kreutzer, Roman Rabinovich, Sebastian Siebertz, and Konstantinos Stavropoulos. Coloring and covering nowhere dense graphs. *SIAM Journal on Discrete Mathematics*, 32(4):2467–2481, 2018.
- [18] Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. In *Proceedings of the 46th Annual ACM Symposium on Theory of Computing, STOC ’14*, pages 89–98, New York, NY, USA, 2014. ACM.
- [19] Martin Grohe, Stephan Kreutzer, and Sebastian Siebertz. Deciding first-order properties of nowhere dense graphs. *J. ACM*, 64(3):17:1–17:32, 2017.
- [20] David Haussler. Sphere packings for subsets of the Boolean n -cube with bounded Vapnik–Chervonenkis dimension. *Journal of Combinatorial Theory, Series A*, 69(2):217–232, 1995.
- [21] Wilfrid Hodges et al. *A shorter model theory*. Cambridge university press, 1997.
- [22] Yiting Jiang, Jaroslav Nešetřil, Patrice Ossona de Mendez, and Sebastian Siebertz. Regular partitions of gentle graphs. *Acta Mathematica Hungarica*, 161(2):719–755, 2020. Special issue dedicated to Endre Szemerédi’s 80th birthday.
- [23] Hal A. Kierstead and Daqing Yang. Orderings on graphs and game coloring number. *Order*, 20:255–264, 2003.
- [24] Avi Levy, Harishchandra Ramadas, and Thomas Rothvoss. Deterministic discrepancy minimization via the multiplicative weight update method. In *International Conference on Integer Programming and Combinatorial Optimization*, pages 380–391. Springer, 2017.
- [25] Leonid Libkin. *Elements of finite model theory*. Springer Science & Business Media, 2013.
- [26] Jiří Matoušek. Tight upper bounds for the discrepancy of half-spaces. *Discrete & Computational Geometry*, 13(3-4):593–601, 1995.
- [27] Jiří Matoušek. Combinatorial discrepancy. In *Geometric Discrepancy*, pages 101–135. Springer, 1999.
- [28] Jiří Matoušek. *Geometric discrepancy: An illustrated guide*, volume 18. Springer Science & Business Media, 2009.
- [29] Jiří Matoušek, Emo Welzl, and Lorenz Wernisch. Discrepancy and ϵ -approximations for bounded VC-dimension. *Combinatorica*, 13(4):455–466, 1993.
- [30] J. Nešetřil, P. Ossona de Mendez, M. Pilipczuk, and X. Zhu. Clustering powers of sparse graphs. *Electronic Journal of Combinatorics*, 27(4):P4.17, 2020.
- [31] J. Nešetřil, P. Ossona de Mendez, and S. Siebertz. Structural properties of the first-order transduction quasiorder. arXiv:2010.02607 [math.CO], 2021. accepted at CSL 2022.
- [32] Jaroslav Nešetřil and Patrice Ossona de Mendez. Tree depth, subgraph coloring and homomorphism bounds. *European Journal of Combinatorics*, 27(6):1022–1041, 2006.

- [33] Jaroslav Nešetřil and Patrice Ossona de Mendez. Grad and classes with bounded expansion I. Decompositions. *European Journal of Combinatorics*, 29(3):760–776, 2008.
- [34] Jaroslav Nešetřil and Patrice Ossona de Mendez. Grad and classes with bounded expansion II. Algorithmic aspects. *European Journal of Combinatorics*, 29(3):777–791, 2008.
- [35] Jaroslav Nešetřil and Patrice Ossona de Mendez. *Sparsity (Graphs, Structures, and Algorithms)*, volume 28 of *Algorithms and Combinatorics*. Springer, 2012. 465 pages.
- [36] Michał Pilipczuk, Sebastian Siebertz, and Szymon Toruńczyk. On the number of types in sparse graphs. In *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 799–808. ACM, 2018.
- [37] Norbert Sauer. On the density of families of sets. *Journal of Combinatorial Theory, Series A*, 13(1):145–147, 1972.
- [38] Saharon Shelah. A combinatorial problem; stability and order for models and theories in infinitary languages. *Pacific Journal of Mathematics*, 41(1):247–261, 1972.
- [39] Saharon Shelah. *Classification theory: and the number of non-isomorphic models*. Elsevier, 1990.
- [40] S. Siebertz. *Nowhere Dense Classes of Graphs: Characterisations and Algorithmic Meta-Theorems*, volume 5. Universitätsverlag der TU Berlin, 2016.
- [41] Joel Spencer. Six standard deviations suffice. *Transactions of the American mathematical society*, 289(2):679–706, 1985.
- [42] Vladimir Naumovitch Vapnik and Alexei Iakovlevitch Červonenkis. On the uniform convergence of relative sequences of events to their probabilities. *Theory Probab. Appl.*, 16:264–280, 1971.
- [43] Xuding Zhu. Colouring graphs with bounded generalized colouring number. *Discrete Math.*, 309(18):5562–5568, 2009.