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► **To cite this version:**

Yves Andre. On the direct summand conjecture and beyond. ICBS Beijing 2024, Jul 2024, Beijing, China. [⟨hal-04799967⟩](#)

HAL Id: hal-04799967

<https://hal.science/hal-04799967v1>

Submitted on 23 Nov 2024

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On the direct summand conjecture and beyond

Yves André

Abstract

Up to a finite covering, a sequence of nested subvarieties of an affine algebraic variety just looks like a flag of vector spaces (Noether’s normalization); understanding this “up to” is a primary motivation for a fine study of finite coverings.

The direct summand conjecture is such a general statement concerning finite coverings of regular schemes.

We review this and other related statements, which were recently settled in full generality using methods coming from p -adic Hodge theory (perfectoids), and emphasize their geometric significance.

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1 Introduction

1.1 Flatness and purity

In the context of base change and descent in algebraic geometry, it is traditional to focus on the notion of faithful flatness; however, the key concept turns out to be “purity”, as was already known in the 70’s and then forgotten.

A ring map $R \xrightarrow{\alpha} S$, is said to be *flat*, *resp. faithfully flat*, *resp. pure*, if the base change functor

$$\alpha^* = - \otimes_R S : Mod_R \rightarrow Mod_S$$

is *exact*, *faithful exact*, *faithful* (so that *faithfully flat* = *flat* + *pure*).

Among the three properties, the third one has always been the poor relative. Here are some known criteria:

- α is pure $\Leftrightarrow \alpha$ is universally injective (i.e. remains injective after any base-change).
- If α splits in Mod_R , it is pure. Conversely, if α is pure, it splits in Mod_R if either α is finitely presented in Mod_R or R is a complete Noetherian local ring (Lazard, Fedder).
- If there exists an S -module M which is faithfully flat over R , then α is pure (Bourbaki).

1.2 Descent and purity

Let us turn to “reverse base change”, that is, descent. A descent datum on an S -module N is an isomorphism

$$N \otimes_R S \xrightarrow{\phi} S \otimes_R N$$

satisfying the usual cocycle relation.

Descent data form a category $DD(\alpha)$ (objects: (N, ϕ)), and the base change functor α^* factors as:

$$Mod_R \xrightarrow{C_\alpha} DD(\alpha) \xrightarrow{\text{forget } \phi} Mod_S.$$

Grothendieck’s descent theorem asserts that α *faithfully flat* $\Rightarrow C_\alpha$ *is an equivalence*.

It was discovered by J.-P. Olivier that flatness is actually irrelevant:

$$\alpha \text{ is pure} \Leftrightarrow C_\alpha \text{ is an equivalence} \Leftrightarrow C_\alpha \text{ is fully faithful.}$$

This remarkable result, published only as an announcement in a 1970 CRAS note [21], was long overlooked. A reconstruction of the proof is proposed in [3].

2 Geometric interpretation of purity

2.1 Grothendieck topologies

There is no need to comment about the geometric meaning of flatness in Algebraic Geometry¹, but one may wonder about the geometric meaning of purity. It turns out that the answer is best expressed in terms of Grothendieck topologies.

We recall that on a category \mathcal{C} with initial object and fibered products (such as the category Aff_k of affine schemes over a commutative ring k),

- a Grothendieck pretopology: class of “covering families” $(Y_i \rightarrow X)$ in \mathcal{C} , containing all isomorphisms, stable by base change and by composition.
- a Grothendieck topology is a Grothendieck pretopology such that any family refined by a covering family is a covering family.

¹but see the last remark below...

2.2 Examples: canonical, fpqc, descent topologies

Examples: • the *fpqc pretopology* on Aff_k : covering maps $Y \rightarrow X$ are faithfully flat maps.

• the *fpqc topology* on Aff_k : $Y \rightarrow X$ is a covering map if and only if there exists $Z \rightarrow Y$ such that the composition $Z \rightarrow X$ is faithfully flat.

• the *effective descent topology* on Aff_k : covering families: those for which the corresponding functors C_α 's are equivalences.

• the *canonical topology* on \mathcal{C} is the finest Grothendieck topology such that all representable presheaves are sheaves (*i.e.* for all covering families $(Y_i \rightarrow X)$, $F(X) \rightarrow \prod F(Y_i) \rightrightarrows \prod F(Y_i \times_X Y_j)$ is exact).

In the case of Aff_k , Olivier proved (*cf.* [3] for details):

Proposition 2.1 *i)* $\text{Spec } S \rightarrow \text{Spec } R$ is a canonical covering if and only if $R \rightarrow S$ is pure.

ii) On Aff_k , the canonical topology coincide with the effective descent topology.

On the other hand, M. Raynaud and L. Gruson proved that the fpqc topology is strictly coarser than the canonical topology [22].

3 Finite coverings

3.1 A general question

Let us come back to *finite coverings* (= finite surjective morphisms), and focus on the following

General question: *what is the behaviour of finite coverings with respect to the canonical or the fpqc topology?*

Let us examine the question through examples, varying the assumptions and the requirements.

3.2 Non canonical finite coverings

• Examples of *non-canonical finite coverings* $Y \rightarrow X$ (with X affine Noetherian)?

- The normalization of any non-normal excellent scheme is an example (e.g. two crossing lines: $k[x, y]/(xy) \rightarrow k[x] \times k[y]$ is not pure, since its reduction mod $(x - y)$ sends $x \neq 0$ to 0).

- Example with X normal: $Y = \mathbb{A}_k^4 \rightarrow X = \mathbb{A}_k^4/(\mathbb{Z}/4)$, where $\text{char } k = 2$ and $\mathbb{Z}/4$ acts cyclically on the variables.

The point is that here X is not Cohen-Macaulay (M.J. Bertin [5]), whereas a finite canonical covering $\text{Spec } S \rightarrow \text{Spec } R$ descends the Cohen-Macaulay property.

- Going further, is there any non-canonical finite covering with X regular? The answer is no:

Theorem 3.1 *Any finite covering of a regular scheme is canonical (hence an effective descent morphism).*

Due to the above interpretation of purity, this is geometric translation of the direct summand conjecture (M. Hochster '69):

any finite extension S of a regular ring R splits in Mod_R ,

(proved in [12] in the presence of a base field - by commutative algebra techniques - passing to the inseparable closure and using Frobenius in characteristic p ; [1] in the remaining mixed characteristic case - by techniques coming from p -adic Hodge theory: perfectoids.)

3.3 Canonical finite coverings, non fpqc

• Example of a canonical finite covering $Y \rightarrow X$ which is not a covering for the fpqc topology: The quadric cone.

Let k be a field of characteristic $\neq 2$, $S := k[x, y]$ with $\mathbb{Z}/2$ -action $(x, y) \mapsto (-x, -y)$. Then $S = k[x, y]^+ \oplus k[x, y]^-$, thus $R := k[x, y]^+ = k[x^2, xy, y^2] \rightarrow S$ is pure.

Proposition 3.2 [3] *$\text{Spec } S \rightarrow \text{Spec } R$ is not a covering for the fpqc topology. What is more, no S -module is faithfully flat over R .*

(This is a counterexample to the converse of Bourbaki's purity criterion.)

- Going further, is there any example with X regular? The answer is no:

Theorem 3.3 *Any finite covering Y of a regular scheme X is a covering for the fpqc topology: namely, there exists a faithfully flat composed map $Z \rightarrow Y \rightarrow X$.*

(proved by Hochster, Huneke [14] in the presence of a base field; [1] in general, using perfectoid methods).

Remarks 3.4 1) $Y \rightarrow X$ is a covering for the fpqc pretopology if and only if Y is Cohen-Macaulay.

2) Theorem 3.3 is much stronger than Theorem 3.1: e.g. unlike Theorem 3.1, it is quite non-trivial in characteristic 0 (reduction to characteristic p and ultraproducts). characteristic 3) For a local domain of characteristic p or mixed characteristic $(0, p)$, there is a canonical choice for $T = \mathcal{O}(Z)$, namely

$$T = \widehat{R}^+,$$

the p -adic completion of the absolute integral closure R^+ of R (Hochster-Huneke-Liubeznik [14][16] in the presence of a base field. Bhatt [7] in mixed characteristic, using prismatic methods).

4) In contrast to Theorem 3.3, some finite coverings of $X = \mathbb{A}_k^3$ are not coverings for the fppf topology: in general, the flat covering space Z is not of finite type, not even noetherian (see also [6]).

Theorem 3.3 is a geometric translation of the existence of Cohen-Macaulay algebras (reduce to S complete local, finite extension of regular R ; then an S -algebra T is Cohen-Macaulay if every secant sequence in S becomes regular in T ; which is equivalent to: T is faithfully flat over R).

3.4 Motivations and applications

Motivation and applications come from

- the “homological conjectures” in Commutative Algebra, *cf.* [13],
- the study of singularities and birational geometry in mixed characteristic (Ma, Schwede, et al) [20].

In characteristic 0, for a resolution of singularities $\pi : \tilde{Y} \rightarrow Y = \text{Spec } S$, Grauert-Riemenschneider + local duality implies $\mathbb{H}_y^i(R\pi_*\mathcal{O}_{\tilde{Y}}) = 0$ for $i < \dim Y$; *i.e.* $R\pi_*\mathcal{O}_{\tilde{Y}}$ behaves like a Cohen-Macaulay S -algebra “in the derived category”.

Ma and Schwede’s idea, in mixed characteristic, is to replace $R\pi_*\mathcal{O}_{\tilde{Y}}$ by the Cohen-Macaulay $\mathcal{O}(Y)$ -algebra $\mathcal{O}(Z)$.

4 Splinters

Definition 4.1 *A noetherian affine scheme $X = \text{Spec } R$ is a splinter if any finite covering of X is canonical. Equivalently: if any finite extension $R \rightarrow S$ splits in Mod_R (whence the name).*

Here are some of their properties.

- Any splinter is normal.
- Theorem 3.1 says that any *regular* X is a splinter.

Are there any other? The answer is yes:

- In characteristic 0, any normal X is a splinter (by the “tr/deg” trick)
- the quadric cone² $\text{Spec } k[x^2, xy, y^2]$ in characteristic $\neq 2$ is a splinter.

The main supply of splinters comes from:

- X is a splinter if there exists a canonical covering Y which is regular .

A weakly functorial version of Theorem 2 [2] then implies that X is Cohen-Macaulay. In particular, *if S is a regular ring and R is a direct summand of S (as an R -module), then R is Cohen-Macaulay* (in fact $X = \text{Spec } R$ has only rational singularities, *cf.* details in [3]).

- *Splinters in positive or mixed characteristic are always Cohen-Macaulay (a consequence of the Hochster-Huneke-Bhatt theorem about \widehat{R}^+).*

Remarks 4.2 1) *The proofs of the last two properties in mixed characteristic use advanced tools from perfectoid and prismatic theory.*

- 2) *What if we replace the canonical topology by the fpqc topology?*

²and more generally any so-called F -regular X in characteristic p

It seems likely that only regular schemes are “fpqc analogs of splinters”.

3) In EGA, one finds many properties of schemes and morphisms which descend along faithfully flat morphisms (i.e. with respect to the fpqc pretopology). Which properties descend with respect to the fpqc topology (not only with respect to the fpqc pretopology)? Here are two examples (cf. [3]):

- In characteristic p , X is regular if and only if the Frobenius map $F_X : X \rightarrow X$ is faithfully flat (Kunz [17]), i.e. is a covering for the fpqc pretopology. Enhancement: X is regular if and only if F_X is covering for the fpqc topology - in the context of the previous remark, this implies that fpqc analogs of splinters X are regular if F_X is finite.

- Regularity descends with respect to the fpqc topology (Ma) - in the context of the previous remark, this implies that the “main supply of splinters” cannot provide non-regular fpqc analogs of splinters.

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