



**HAL**  
open science

# Asymptotic spreading of KPP reactive fronts in heterogeneous shifting environments II: Flux-limited solutions

Grégoire Nadin, King-Yeung Lam, Xiao Yu

► **To cite this version:**

Grégoire Nadin, King-Yeung Lam, Xiao Yu. Asymptotic spreading of KPP reactive fronts in heterogeneous shifting environments II: Flux-limited solutions. 2024. hal-04799624

**HAL Id: hal-04799624**

**<https://hal.science/hal-04799624v1>**

Preprint submitted on 23 Nov 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Asymptotic spreading of KPP reactive fronts in heterogeneous shifting environments II: Flux-limited solutions

King-Yeung Lam\*, Grégoire Nadin† and Xiao Yu‡

October 21, 2024

## Abstract

We consider the spreading dynamics of the Fisher-KPP equation in a shifting environment, by analyzing the limit of the rate function of the solutions. For environments with a weak monotone condition, it was demonstrated in a previous paper that the rate function converges to the unique Ishii solution of the underlying Hamilton-Jacobi equations. In case the environment does not satisfy the weak monotone condition, we show that the rate function is then characterized by the Hamilton-Jacobi equation with a dynamic junction condition, which depends additionally on the generalized eigenvalue derived from the environmental function. Our results applies to the case when the environment has multiple shifting speeds, and clarify the connection with previous results on nonlocally pulled fronts and forced traveling waves.

## Contents

<b>1</b>	<b>Introduction</b>	<b>2</b>
<b>2</b>	<b>Preliminaries and Statements of Results</b>	<b>4</b>
2.1	Flux-limited solution due to Imbert and Monneau . . . . .	5
2.2	Main results . . . . .	6
2.3	Monotonicity of the flux-limited solutions . . . . .	7
2.4	Generalizations . . . . .	7
2.5	Related optimal control formulations . . . . .	8
2.6	An earlier result: Viscosity solution in the sense of Ishii . . . . .	9
<b>3</b>	<b>Applications and Explicit Formulas</b>	<b>10</b>
<b>4</b>	<b>Preliminary Results</b>	<b>14</b>
4.1	The eigenvalue problem associated with $g(y)$ . . . . .	15
4.2	The continuity of subsolutions . . . . .	16
4.3	Critical slope lemmas . . . . .	17
<b>5</b>	<b>Proof of Main Results</b>	<b>18</b>
5.1	Verification of flux-limited solutions property . . . . .	19
5.2	Equivalence between Ishii solution and FL solution with $A = A_0$ . . . . .	23
5.3	Verification of Kirchhoff junction conditions . . . . .	26
5.4	Proof of main results . . . . .	27

\*Department of Mathematics, The Ohio State University, Columbus, OH 43210, USA (lam.184@osu.edu)

†Institut Denis Poisson, Université d'Orléans, Université de Tours, CNRS, Orléans, France (gregoire.nadin@cnrs.fr)

‡School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China (xy-math19@m.scnu.edu.cn)

<b>A Proof of Proposition 4.2</b>	<b>28</b>
<b>B Proof of Lemmas 4.5 and 4.7</b>	<b>29</b>
<b>C Comparison Principle for problem with Kirchhoff condition</b>	<b>31</b>
<b>D Definition of viscosity solution in sense of Ishii</b>	<b>37</b>

## 1 Introduction

Consider the KPP equation with heterogeneous coefficients

$$\begin{cases} u_t - u_{xx} = u(g(x - c_1 t) - u) & \text{for } (x, t) \in \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}, \end{cases} \quad (1.1)$$

where  $c_1 \in \mathbb{R}$ ,  $g$  is a continuous and positive function on  $\mathbb{R}$  with  $g(\pm\infty) > 0$ , and  $u_0 \in L^\infty(\mathbb{R})$  is nonnegative and compactly supported. This type of equations models the population growth in a shifting habitat. They may arise from the ecological question whether the species can survive in the midst of climate change [48, 7, 41]. In a previous work [40], we considered (1.1) as a special case of a class of integro-differential equations with a distributed time-delay in heterogeneous shifting environments. Under the assumption  $\sup g \leq \max\{g(\pm\infty)\}$ , we utilized the theory of viscosity solutions of Hamilton-Jacobi equations, specifically the uniqueness of viscosity solutions in the sense of Ishii [38], to obtain the complete explicit formulas of rightward spreading speeds for (1.1) in terms of  $c_1$  as well as leftward spreading speeds. However, the assumption  $\sup g \leq \max\{g(\pm\infty)\}$  was crucial in [40], as uniqueness of viscosity solutions in the sense of Ishii can no longer be expected if it was relaxed [3, 28].

To recover uniqueness and further develop the Hamilton-Jacobi approach, we will utilize the notion of flux-limited solution, which was recently introduced by Imbert and Monneau [36, 37] to study Hamilton-Jacobi equations on networks. Throughout our paper, we are mainly working on (1.1) which has one shifting speed  $c_1$ , and we are able to determine the spreading properties of (1.1) for any  $g$  for which  $g(\pm\infty)$  exist. Furthermore, our treatment naturally extends to the case of multiple shifting speeds; see Section 2.4 for the precise statements. Before stating our main results, we provide a brief review on some related works.

The asymptotic speed of spread, or spreading speed in short, is a crucial quantity in spatial ecology that determines the expansion boundary of a population under the joint influence of the diffusion rate and environmental conditions. For simplicity, the diffusion rate has been normalized to 1 in (1.1). In a homogeneous environment, i.e.  $g(\cdot) \equiv g_0$  for a positive constant  $g_0$ , model (1.1) reduces to the classical Fisher-KPP equation. A well known result of Kolmogorov et al. [39] states that there is a number  $c_* = 2\sqrt{g_0} > 0$  such that

$$\lim_{t \rightarrow \infty} \sup_{x \geq ct} u(t, x) = 0 \quad \text{for } c \in (c_*, +\infty), \quad \text{and} \quad \lim_{t \rightarrow \infty} \inf_{0 \leq x \leq ct} u(t, x) > 0 \quad \text{for } c \in (0, c_*).$$

Moreover, the same value also coincides with the minimal wave speed of traveling wave solutions  $U(x - ct)$  of (1.1). This result was later extended to more general nonlinearity and in higher dimensions in [1]. It is also remarkable that in homogeneous environment, the spreading speed can be obtained via local information, where  $c_* = 2\sqrt{g_0}$  is the smallest value of  $c > 0$  such that

$$-c\phi_x + \phi_{xx} + g_0\phi = \Lambda\phi$$

admits a zero eigenvalue.

Since then, spreading speed for various reaction-diffusion models including Fisher-KPP equations are intensively investigated [51, 42, 9, 49, 10, 23, 11]. Among those, an elaborate method

was proposed by Weinberger [51] to establish the existence of spreading speeds for discrete-time order-preserving recursions with a monostable structure and its characterization as the minimal speed of traveling wave solutions. These results were subsequently generalized to monotone semiflows [42, 23]. By combining the Hamilton-Jacobi approach [21] and homogenization ideas [43, 20], Berestycki and Nadin [10, 11] showed the existence of spreading speed for spatially almost periodic, random stationary ergodic, and other general environments, whose speed was characterized as a min-max formula in terms of suitable notions of generalized principal eigenvalues in unbounded domains.

The heterogeneous shifting environment, which is the focus of this article, was introduced by Potapov and Lewis [48] and Berestycki et al. [7] to investigate the impact of shifting climate on the persistence of one or several focal species. As a simple formulation, the temporal-spatial heterogeneity  $x - c_1 t$  was incorporated into various diffusion models including (1.1) for the single species, where  $c_1$  is regarded as the shift velocity of climate. For (1.1), the propagation dynamics have been rigorously explored in [48, 7, 12, 13] for the case of a moving patch of a finite length, and in Li et al. [41] for a retreating semi-infinite patch. The latter problem is a special case of (1.1) in case  $g$  is increasing and  $g(-\infty) < 0 < g(+\infty)$ , where it is proved that the species persists if and only if it can spread faster than the environment with the spreading speed being given by the KPP formula  $c_* = 2\sqrt{g(+\infty)}$ . Hu et al. [35] provided further investigation without assuming  $g(-\infty) < 0$ , and proved that the rightward spreading speed  $c_* = 2\sqrt{g(+\infty)}$  if  $c_1 \leq 2\sqrt{g(+\infty)}$  and  $c_* = 2\sqrt{g(-\infty)}$  if  $c_1 \gg 1$ . A shifting environment can also arise in other ways. Holzer and Scheel [34] considered a partially decoupled reaction-diffusion system of two equations, where a wave solution for the first equation induces a shifting environment for the second one. See also [19, 16, 30, 46] for further results on competition or prey-predator systems. Similar modeling idea was also adapted in Fang et al. [22], where (1.1) was also retrieved from an SIS disease model to study whether pathogen can keep pace with its host. If  $g$  is non-increasing, then (1.1) becomes a special case of the cylinder problem studied by Hamel [32]. Du et al. [18] proposed a free boundary version of (1.1), see also [17]. Yi and Zhao [52, 53] established a general theory on the propagation dynamics without spatial translational invariance. See also [24] for a model with shifting diffusivity. We refer to Wang et al. [50] for a survey on reaction-diffusion models in shifting environments.

Indeed, the shifting habitat does bring about new spreading phenomena in case that the intrinsic growth rate function  $g$  is strictly positive everywhere. When  $0 < \inf g < \sup g \leq \max\{g(\pm\infty)\}$ , the results of [34, 40] clarified that, for a certain range of shift speed  $c_1$ , the initially compactly supported population spreads at a supercritical speed  $c_* > 2\sqrt{g(-\infty)}$  in a phenomenon called non-local pulling [34, 30]. This falls into the biological scenario when the species fails to keep up with the climate shifting, but is still influenced by the presence of a favorable habitat which is at a distance of order  $t$  ahead of the front. When  $\sup g > \max\{g(\pm\infty)\}$ , then Holzer and Scheel [34] proved the existence of forced traveling wave solution, which moves in the same speed as the environment. Subsequently, Berestycki and Fang [8] classified such forced traveling wave solutions and proved global attractivity results.

Our main contribution, in the case of (1.1), where the environment has a single speed  $c_1$ , is to completely determine the existence of rightward spreading speed  $c_*$  and its dependence on the environmental speed  $c_1$ , whenever  $g(\pm\infty)$  exist and  $\inf g > 0$ . Moreover, our framework provides the context in which the spreading results in [34, 40] (where  $c_* < c_1$  with nonlocal pulling) connects with those in [8] (where  $c_* = c_1$ ). Furthermore, our method readily generalizes to the case when the environment has more than one shifting speed (Subsection 2.4).

## Organization of the paper

Our approach is to study the spreading speed via the asymptotic limit of the rate function, following [21] (see also [3, Chapter 29] and [40]). However, the consideration of a shifting habitat leads to a discontinuous Hamiltonian. Also, the rate function has unbounded and discontinuous

initial data since the initial population was compactly supported. In our previous work [40], the solution concept of Ishii was used and the corresponding comparison principle was established. This is discussed in Subsection 2.6. However, the previous results in [40] is not applicable in case  $\sup g > \max\{g(\pm\infty)\}$ , because then the invasion is enhanced by the specific profile of  $g$ , and the solutions in the sense of Ishii are non-unique. To overcome the non-uniqueness of viscosity solution and connect with the results regarding forced waves, we need to incorporate further information of (1.1) in deriving the limiting Hamilton-Jacobi equation. To this end, we recall some results of an eigenvalue problem depending on the coefficient  $g(\cdot)$ . Then in Subsection 2.1, we introduce the concept of a flux-limited solution and prove the comparison principle needed in our context. In Subsection 2.2, we state our main theorems and extensions of our results. We also discuss the viscosity solutions in sense of Ishii and recall some earlier results from [40] in Subsection 2.6.

Section 3 presents the application of Theorem 1, and we place it immediately after stating our main theorem. In this section, we take Theorem 1 for granted and apply it to obtain several explicit formulas for the spreading speed in terms of  $g(+\infty), g(-\infty)$  and  $\Lambda_1$ , where  $\Lambda_1$  is the principal eigenvalue given in (2.5). This provides a general context connecting previous results of [8, 34] concerning forced wave (where  $c_* = c_1$ ) and of [40] concerning nonlocal pulling (where  $c_* < c_1$  but is influenced by the presence of the shifting environment).

In Section 4, we present preliminary results. In particular, we recall the properties of  $\Lambda_1$  in Proposition 4.2 (Subsection 4.1), as well as a few technical results for Hamilton-Jacobi equations.

Section 5 is devoted to the proof of the main results, namely, Proposition 2.3, Corollary 2.4, and Theorem 1. This section only logically depends on Proposition 4.2 (proved in Appendix A), Lemma 4.4 concerning continuity of subsolutions (proved in Subsection 4.2), the critical slope lemmas inspired by Imbert and Monneau [36] (Lemmas 4.5 and 4.7, proved in Appendix B), as well as the comparison principle (proved in Appendix C).

Finally, the appendices presents the proofs of the technical results mentioned above.

## 2 Preliminaries and Statements of Results

The concepts of maximal and minimal spreading speeds are introduced in [33, Definition 1.2] for a single species; see also [27, 46]. In our setting, we define

$$\begin{cases} \bar{c}_* = \inf \{c > 0 \mid \limsup_{t \rightarrow \infty} \sup_{x > ct} u(t, x) = 0\}, \\ \underline{c}_* = \sup \{c > 0 \mid \liminf_{t \rightarrow \infty} \inf_{0 < x < ct} u(t, x) > 0\}, \end{cases} \quad (2.1)$$

where  $\bar{c}_*$  and  $\underline{c}_*$  are the maximal and minimal (rightward) spreading speeds of species  $u$ , respectively. If  $\bar{c}_* = \underline{c}_* > 0$ , we say that the population has the (rightward) spreading speed given by the common value  $c_*$ .

Motivated by the large deviations technique [21, 25], we introduce, for fixed solution  $u$  of (1.1), the scaling  $u^\epsilon(t, x) = u(\frac{t}{\epsilon}, \frac{x}{\epsilon})$  with  $\epsilon > 0$ . The resulting function  $u^\epsilon(t, x)$  satisfies the following equation:

$$\begin{cases} u_t^\epsilon - \epsilon u_{xx}^\epsilon = \epsilon^{-1} u^\epsilon \left( g\left(\frac{x-c_1 t}{\epsilon}\right) - u^\epsilon \right) & \text{for } (t, x) \in (0, \infty) \times \mathbb{R}, \\ u^\epsilon(0, x) = u_0(x/\epsilon) & \text{for } x \in \mathbb{R}. \end{cases} \quad (2.2)$$

Observe that the spreading speed of the population is given by  $c_* > 0$  if and only if

$$\lim_{\epsilon \rightarrow 0} u^\epsilon(t, x) \rightarrow 0 \quad \text{in } C_{loc}(\{x > c_* t\}), \quad \liminf_{\epsilon \rightarrow 0} u^\epsilon(t, x) > 0 \quad \text{in } C_{loc}(\{0 \leq x < c_* t\}). \quad (2.3)$$

To fully characterize these limits, we introduce the following eigenvalue problem:

$$\Phi'' + g(y)\Phi = \Lambda\Phi \quad \text{for } y \in \mathbb{R}. \quad (2.4)$$

In this paper, we define the principal eigenvalue  $\Lambda_1$  of (2.4) as follows:

$$\Lambda_1 := \Lambda_1(g) = \inf \{ \Lambda \in \mathbb{R} : \exists \phi \in C_{loc}^2(\mathbb{R}), \phi > 0, \phi'' + g(y)\phi \leq \Lambda\phi \text{ in } \mathbb{R} \}. \quad (2.5)$$

This and several other notions of principal eigenvalues are analyzed in [14]. We will recall some basic properties of  $\Lambda_1$  and the associated positive eigenfunction in Proposition 4.2.

As we shall see in Section 3, the four quantities  $c_1$ ,  $\Lambda_1$ ,  $g(+\infty)$  and  $g(-\infty)$  completely determine the spreading speed.

## 2.1 Flux-limited solution due to Imbert and Monneau

To determine the exact spreading speed in Theorem 3.2, we will study the rate function  $w^\epsilon(t, x) := -\epsilon \log u^\epsilon(t, x)$ . More precisely, we will show that  $w^\epsilon(t, x) \rightarrow t\hat{\rho}(x/t)$  in  $C_{loc}$ , where the limit  $\hat{\rho}$  is to be interpreted using the notion of *flux-limited solutions* introduced by Imbert and Monneau [36]. This notion is well-adapted to catch the influence of the coefficients along a discontinuity at  $x = c_1$ .

We begin with a few notations regarding the effective Hamiltonian and effective junction condition.

**Definition 2.1.** For each  $A \in \mathbb{R}$ , define the flux-limited junction condition to be

$$F_A(\tilde{p}_+, \tilde{p}_-) = \max\{A, H^-(c_1+, \tilde{p}_+), H^+(c_1-, \tilde{p}_-)\}, \quad (2.6)$$

where  $H^+(c_1\pm, \cdot)$  and  $H^-(c_1\pm, \cdot)$  are

$$H^+(c_1\pm, p) = \begin{cases} -\frac{|c_1|^2}{4} + g(\pm\infty), & \text{for } p \leq c_1/2, \\ -c_1p + p^2 + g(\pm\infty) & \text{for } p > c_1/2, \end{cases} \quad (2.7)$$

$$H^-(c_1\pm, p) = \begin{cases} -c_1p + p^2 + g(\pm\infty) & \text{for } p \leq c_1/2, \\ -\frac{|c_1|^2}{4} + g(\pm\infty) & \text{for } p > c_1/2. \end{cases} \quad (2.8)$$

Note that they are, respectively, the increasing and decreasing parts (in the variable  $p$ ) of

$$H(s, p) = -sp + p^2 + \chi_{\{s > c_1\}}g(+\infty) + \chi_{\{s \leq c_1\}}g(-\infty).$$

The information of the profile of  $g$  can be incorporated into the Hamilton-Jacobi equation by an additional *junction condition* as follows:

$$\begin{cases} \min\{\rho, \rho + H(s, \rho')\} = 0 & \text{for } s > 0, s \neq c_1, \\ \min\{\rho(c_1), \rho(c_1) + F_A(\rho'(c_1+), \rho'(c_1-))\} = 0, \end{cases} \quad (2.9)$$

where  $A$  and  $F_A(\tilde{p}_+, \tilde{p}_-)$  are given in (2.13) and (2.6) respectively. The above equations are to be considered using *piecewise*  $C^1$  test functions whose left and right derivative at  $c_1$  are well defined but maybe unequal:

$$C_{pw}^1 = \{\psi \in C((0, \infty)) : C^1((0, c_1]) \cap C^1([c_1, \infty))\}. \quad (2.10)$$

**Definition 2.2.** Let  $A \in \mathbb{R}$  be given.

- (a) We say that  $\hat{\rho} : (0, \infty) \rightarrow \mathbb{R}$  is a FL-subsolution of (2.9) provided (i)  $\hat{\rho}$  is upper semicontinuous, and (ii) if  $\hat{\rho} - \psi$  (with  $\psi \in C_{pw}^1$ ) attains a local maximum point at some  $s_0 > 0$  such that  $\hat{\rho}(s_0) > 0$ , then

$$\hat{\rho}(s_0) + H(s_0, \psi'(s_0)) \leq 0 \quad \text{in case } s_0 \neq c_1,$$

$$\hat{\rho}(c_1) + F_A(\psi'(c_1+), \psi'(c_1-)) \leq 0 \quad \text{in case } s_0 = c_1.$$

- (b) We say that  $\hat{\rho} : (0, \infty) \rightarrow \mathbb{R}$  is a FL-supersolution of (2.9) provided (i)  $\hat{\rho}$  is lower semicontinuous, (ii)  $\hat{\rho} \geq 0$  for all  $s > 0$ , and (iii) if  $\hat{\rho} - \psi$  (with  $\psi \in C_{pw}^1$ ) attains a local minimum point at some  $s_0 > 0$ , then

$$\begin{aligned} s_0 \neq c_1 &\implies \hat{\rho}(s_0) + H(s_0, \psi'(s_0)) \geq 0; \\ s_0 = c_1 &\implies \hat{\rho}(c_1) + F_A(\psi'(c_1+), \psi'(c_1-)) \geq 0. \end{aligned}$$

- (c) We say that  $\hat{\rho}$  is a FL-solution of (2.9) if it is both FL-subsolution and FL-supersolution of (2.9).

Next, we discuss the uniqueness of FL-solution of (2.9) by first showing the following comparison principle.

**Proposition 2.3.** *Let  $A \in \mathbb{R}$  be given. If  $\underline{\rho}$  and  $\bar{\rho}$  are, respectively, the FL-subsolution and FL-supersolution of (2.9), and such that*

$$\underline{\rho}(0) \leq \bar{\rho}(0) \quad \text{and} \quad \lim_{s \rightarrow +\infty} \frac{\bar{\rho}(s)}{s} = +\infty, \quad (2.11)$$

then  $\underline{\rho}(s) \leq \bar{\rho}(s)$  in  $[0, +\infty)$ .

**Corollary 2.4.** *For each  $A \in \mathbb{R}$ , (2.9) has a unique FL-solution  $\hat{\rho}_A$  which satisfies the following boundary conditions (in a strong sense)*

$$\rho(0) = 0 \quad \text{and} \quad \lim_{s \rightarrow +\infty} \frac{\rho(s)}{s} = +\infty. \quad (2.12)$$

These two results will be proved in Subsection 5.4.

We apply the half-relaxed limit method, due to Barles and Perthame [4, 5], to pass to the (upper and lower) limits of  $w^\epsilon(t, x)$ . Moreover, we can show that  $w^\epsilon(t, x) \rightarrow t\hat{\rho}_A(x/t)$  in  $C_{loc}$ , where the flux limiter  $A$  is identified by

$$A = \Lambda_1 - \frac{|c_1|^2}{4}. \quad (2.13)$$

The spreading speed  $c^*$  will then be fully characterized by  $\hat{\rho}_A$  with the specific flux-limiter  $A$ .

*Remark 2.5.* Note that  $A = \Lambda_1 - \frac{|c_1|^2}{4}$  could be treated as the principal eigenvalue of  $\Psi'' - c_1\Psi' + g(y)\Psi = A\Psi$  in the sense that:

$$A = \inf \left\{ \Lambda \in \mathbb{R} : \exists \phi \in C_{loc}^2(\mathbb{R}), \phi > 0, \phi'' - c_1\phi' + g(y)\phi \leq \Lambda\phi \text{ in } \mathbb{R} \right\}, \quad (2.14)$$

which quantifies the influence of the coefficient  $g(x - c_1t)$  in the moving coordinate  $y = x - c_1t$ .

## 2.2 Main results

We are now in position to state our main result.

**Theorem 1.** *Let  $u$  be a solution of (1.1). Then the following statements hold.*

- (a) *The spreading speed  $c_*$  of  $u$  exists, and is given by*

$$c_* = \hat{s}_A = \sup\{s \in [0, \infty) : \hat{\rho}_A(s) = 0\}, \quad (2.15)$$

where  $\hat{\rho}_A$  is the unique FL-solution of (2.9) with  $A = \Lambda_1 - \frac{c_1^2}{4}$  that also satisfies the boundary conditions (2.12).

- (b) Furthermore, if  $\Lambda_1 = \max\{g(\pm\infty)\}$ , then  $c_* = \hat{s}_{base} = \sup\{s : \hat{\rho}_{base}(s) = 0\}$ , where  $\hat{\rho}_{base}$  is the unique viscosity solution of (2.26)–(2.12) in the sense of Ishii (see Definition D.1).

*Remark 2.6.* In Section 3, we will give explicit formulas of  $c_*$  in terms of  $g(\pm\infty)$ ,  $c_1$  and  $\Lambda_1$ .

*Remark 2.7.* After the research of this work has finished, the preprint of Giletti et al. [29] was brought to our attention. In this work, the authors treated the case when

$$g(t, x) = r_1\chi_{\{x < A(t)\}} + r_2\chi_{\{A(t) \leq x < A(t)+L\}} + r_3\chi_{\{x \geq A(t)+L\}}, \quad (2.16)$$

where  $t \mapsto A(t)$  is either linear or slowly oscillating between two shifting speeds. Interestingly, they obtained the formula of Theorem 3.9 assuming that  $g$  is given by (2.16) with  $A(t) = c_1 t$ .

Furthermore, it was remarked that their construction can be generalized to treat (1.1) provided that  $g(y)$  is constant near  $y = \pm\infty$ . They also conjectured that the last condition may not be necessary. Their proof is based on the direct construction of super/subsolution for the parabolic problem.

Incidentally, our main result can be considered as an affirmative answer of their conjecture, by passing to the limiting Hamilton-Jacobi problem with junction condition. It is worth mentioning that (i) we need only  $A(t) = c_1 t + o(t)$  and (ii) we merely require  $g(\pm\infty)$  exist (but not necessarily constant for  $|x| \gg 1$ ). In particular, the spreading speed can be determined by the value of the eigenvalue  $\Lambda_1$  and the exact shape of  $g$  is not important.

### 2.3 Monotonicity of the flux-limited solutions

Since  $F_A(\tilde{p}_+, \tilde{p}_-)$  is monotone increasing in the variable  $A$ , the effect of the flux limiter  $A$  is as follows.

**Corollary 2.8.** *Let  $A \in \mathbb{R}$  and  $\hat{\rho}_A$  be the unique FL-solution of (2.9)–(2.12).*

- (a) *If  $A \geq A'$ , then  $\hat{\rho}_A(s) \leq \hat{\rho}_{A'}(s)$  for all  $s \geq 0$ . In particular, the free boundary point  $\hat{s}_A$  is monotone increasing with respect to  $A$ , i.e.  $\hat{s}_A \geq \hat{s}_{A'}$ , where*

$$\hat{s}_A := \sup\{s : \hat{\rho}_A(s) = 0\}. \quad (2.17)$$

- (b) *If  $A_0 := \max\{g(\pm\infty)\} - \frac{c_1^2}{4}$  and  $A \leq A_0$ , then  $\hat{\rho}_A$  coincides with the unique viscosity solution of (2.26)–(2.12) in the sense of Ishii. In case  $A > A_0$ , it is a viscosity subsolution (might not be a supersolution) in the sense of Ishii.*

*Proof.* This is a direct consequence of Proposition 2.3, since  $\hat{\rho}_A(0) = \hat{\rho}_{A'}(0) = 0$ ,  $\hat{\rho}_{A'}$  is a FL-supersolution of (2.9) and satisfies (2.12).  $\square$

### 2.4 Generalizations

Our arguments can also be applied in the following setting where  $f = f(t, x, u)$  possesses multiple junction points  $\mathcal{P} = \{c_i\}_{i=1}^n$  for some  $0 < c_1 < c_2 < \dots < c_n$ .

$$\begin{cases} u_t - u_{xx} = f(t, x, u) & \text{for } (x, t) \in \mathbb{R} \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}. \end{cases} \quad (2.18)$$

(H1')  $f(t, x, 0) = 0$  for all  $x \in \mathbb{R}$  and  $t \geq 0$ .

(H2') There exists  $g_i \in L^\infty(\mathbb{R})$  with  $g_i(\pm\infty) > 0$ ,  $i = 1, \dots, n$ , and for some  $\delta_1 > 0$ ,

$$\lim_{t \rightarrow \infty} \text{ess sup}_{|x - c_i t| < \delta_1 t} |f_u(t, x, 0) - g_i(x - c_i t)| = 0.$$



(H3') There exist  $R, \underline{R} : [0, \infty) \rightarrow [0, \infty)$  such that  $\inf \underline{R} > 0$ ,  $R(s) = \underline{R}(s)$  a.e. and

$$R(s) = \limsup_{\substack{\epsilon \rightarrow 0^+ \\ (t,x) \rightarrow (1,s)}} f_u \left( \frac{t}{\epsilon}, \frac{x}{\epsilon}, 0 \right) \quad \text{and} \quad \underline{R}(s) = \liminf_{\substack{\epsilon \rightarrow 0^+ \\ (t,x) \rightarrow (1,s)}} f_u \left( \frac{t}{\epsilon}, \frac{x}{\epsilon}, 0 \right).$$

Moreover,  $R$  is locally monotone in  $\mathbb{R} \setminus \{c_i\}_{i=1}^n$ , i.e. for each  $s_0 \in \mathbb{R} \setminus \{c_i\}_{i=1}^n$ ,

$$\text{either } \liminf_{\delta \rightarrow 0} \inf_{s_0 - \delta < s < s' < s_0 + \delta} R(s) - R(s') \geq 0 \quad \text{or} \quad \limsup_{\delta \rightarrow 0} \sup_{s_0 - \delta < s < s' < s_0 + \delta} R(s) - R(s') \leq 0.$$

(H4') For each  $(t, x) \in \mathbb{R}_+ \times \mathbb{R}$ ,  $f(t, x, u)/u$  is non-increasing in  $u > 0$ .

(H5') There exists  $M > 0$  such that  $f(t, x, u) < 0$  for  $(t, x, u) \in \mathbb{R}_+ \times \mathbb{R} \times [M, \infty)$ .

(H6') For each  $i = 1, \dots, n$ , let  $\Lambda_1^{(i)}$  be the principal eigenvalue given by

$$\Lambda_1^{(i)} = \inf \{ \Lambda \in \mathbb{R} : \exists \phi \in C_{loc}^2(\mathbb{R}), \phi > 0, \phi'' + g_i(y)\phi \leq \Lambda \phi \}. \quad (2.19)$$

**Theorem 2.9.** *Given  $\{c_i\}_{i=1}^n$ , and  $f(t, x, u)$  satisfying (H1') – (H6'). If  $u_0$  is compactly supported and nontrivial, then the population spreads to the right at speed  $c_*$ , where*

$$c_* = \sup \{ s \geq 0 : \rho^\dagger(s) = 0 \},$$

where  $\rho^\dagger$  is the unique FL-solution of

$$\begin{cases} \min\{\rho, \rho + H(s, \rho)\} = 0 & \text{for } s \in (0, \infty) \setminus \{c_i\}, \\ \min\{\rho(c_i), \rho(c_i) + F^{(i)}(\rho'(c_i+), \rho'(c_i-))\} = 0 & \text{for each } i, \\ \rho(0) = 0 \quad \text{and} \quad \lim_{s \rightarrow +\infty} \rho(s)/s = +\infty. \end{cases} \quad (2.20)$$

where

$$H(s, p) = -sp + p^2 + R(s),$$

$$F^{(i)}(\tilde{p}_+, \tilde{p}_-) = \max \left\{ \Lambda_1^{(i)} - \frac{c_i^2}{4}, H^-(c_i+, \tilde{p}_+), H^+(c_i-, \tilde{p}_-) \right\},$$

such that  $H^\mp(c_i\pm, \cdot)$  are the decreasing/increasing parts of  $H(c_i\pm, \cdot)$  given by

$$H^-(c_i+, p) = H(c_i+, \min\{p, c_i/2\}), \quad H^+(c_i-, p) = H(c_i-, \max\{p, c_i/2\}),$$

and  $\Lambda_1^{(i)}$  is the principal eigenvalue given by (2.19).

## 2.5 Related optimal control formulations

It is well known that the viscosity solutions of Hamilton-Jacobi equations correspond naturally to the value functions of certain optimal control problems, whereas the viscosity solutions of variational inequalities correspond to the value functions of two-player, zero-sum deterministic differential games with stopping time [21]. In fact, following the control formulation for Hamilton-Jacobi equations stated in [36], we conjecture that the following formulation should hold:

$$\rho(s) = \inf_{\substack{z \in X \\ z(0)=s}} \left[ \sup_{T \in [0, \infty]} \int_0^T e^{-s'} \ell(z(s'), z(s') - \dot{z}(s')) ds' \right], \quad (2.21)$$

where  $X = H^1([0, \infty); [0, \infty))$ ,  $T \in [0, \infty]$  is any constant and the cost function  $\ell$  is given by

$$\ell(s, \alpha) := \frac{\alpha^2}{4} - R(s) \text{ if } s \in \mathbb{R} \setminus \{c_i\}_{i=1}^n, \quad \ell(s, \alpha) := \frac{\alpha^2}{4} - \Lambda^{(i)} \text{ if } s = c_i.$$

By a change of variables  $\tau = t(1 - e^{-s'})$ , one can show that  $w(t, x) = t\rho(x/t)$  satisfies

$$w(t, x) = \inf_{\substack{\gamma(0)=x \\ \gamma(t)=0}} \left[ \sup_{\theta \in [0, t]} \int_0^{t \wedge \theta} \ell \left( \frac{\gamma(\tau)}{t - \tau}, -\dot{\gamma}(\tau) \right) d\tau \right]. \quad (2.22)$$

By applying the arguments in [26, Lemma 2.4], one can check that the above is consistent with the known max-min formulas involving stopping times when the running cost  $\ell$  is a continuous function [21, 11]. When the minimum with  $\rho$  is not taken in the problem (2.20), then the unique viscosity solution can be characterized via the optimal control formulation (with  $T$  and  $\theta$  taken to be  $+\infty$  in (2.21) and (2.22) respectively); See [36, 3].

## 2.6 An earlier result: Viscosity solution in the sense of Ishii

First, let  $\tilde{H}$  be the truncated version of the Hamiltonian  $H$ , given by

$$\tilde{H}(s, p) = -sp + p^2 + g(-\infty) \quad \text{for } s < c_1, \quad \tilde{H}(s, p) = -sp + p^2 + g(+\infty) \quad \text{for } s > c_1 \quad (2.23)$$

and  $\tilde{H}(c_1, p) = -c_1p + p^2 + g(-\infty) \vee g(+\infty)$ . Note that  $\tilde{H}$  uses only the information  $g(\pm\infty)$  but does not depend on the specific form of the profile of  $g$ . The following left and right limits of  $H(s, p)$  at  $s = c_1$ , as functions of  $p$ , will be used later.

$$\tilde{H}(c_1-, p) = -c_1p + p^2 + g(-\infty) \quad \text{and} \quad \tilde{H}(c_1+, p) = -c_1p + p^2 + g(+\infty). \quad (2.24)$$

In a previous paper [40], we studied equation (1.1) in the case

$$\sup_{y \in \mathbb{R}} g(y) \leq \max\{g(+\infty), g(-\infty)\}, \quad (2.25)$$

which is connected to the unique  $\hat{\rho}_{base}$  such that (2.12) holds and satisfies, in viscosity sense,

$$\min\{\rho, \rho + \tilde{H}(s, \rho')\} = 0 \quad \text{for } s > 0. \quad (2.26)$$

Since the Hamiltonian function  $\tilde{H}$  is discontinuous at  $c_1$ , the viscosity solution needs to be interpreted in a relaxed sense introduced by Ishii [38]. As this definition is well known, we skip it here and refer the reader to Definition D.1 in the appendix.

We will show in Section 5.2 that  $\rho$  is an Ishii solution to the equation (2.26) if and only if it is a FL-solution of (2.9) with  $A = A_0 := \max\{\min_{\mathbb{R}} H(c_1+, \cdot), \min_{\mathbb{R}} H(c_1-, \cdot)\}$ . See Proposition 5.8.

In [40], we showed that the rate function of problem (1.1) selects  $\hat{\rho}_{base}$  provided that (2.25) holds. This means that the spreading speed  $s_{base}$  is as predicted by the the equations (2.26)-(2.12) (with solution in the sense of Ishii).

**Theorem 2.10.** *For each  $c_1 > 0$ , there exists a unique  $\hat{\rho}_{base}$  which satisfies (2.26) in the sense of Ishii and the boundary conditions (2.12). Furthermore, if (2.25) is valid, then  $\bar{c}_* = \underline{c}_* = s_{base}$ , where  $s_{base} = \sup\{s \geq 0 : \hat{\rho}_{base}(s) = 0\}$ .*

*Proof.* See [40, Proposition 1.7, Theorem 1 and Remark 1.6]. □

For the case when  $g(y)$  does not satisfy (2.25), we may compare the solution  $u$  of (1.1) with the solution  $\tilde{u}$  of the same problem with  $g$  replaced by the truncation  $\min\{g, \max\{g(\pm\infty)\}\}$ , to deduce that the spreading speed is always bounded from below by  $s_{base}$ . However, we will show in Corollary 4.3 below that if  $\Lambda_1 > \max\{g(\pm\infty)\}$ , then this lower bound is not optimal.

**Corollary 2.11.** *Suppose  $\sup g > \max\{g(\pm\infty)\}$ . For each  $c_1 > 0$ ,*

$$\underline{c}_* \geq s_{base},$$

where  $\underline{c}_*$  is the minimal spreading speed and  $s_{base}$  be given in Theorem 2.10.

*Remark 2.12.* In case (2.25) holds, the spreading speed can then be determined as soon as explicit solution  $\hat{\rho}_{base}$  of (2.26)-(2.12) (in the sense of Ishii) can be constructed. This gives an alternative verification of the formula (3.1) (in case  $g(-\infty) \geq g(+\infty)$ ) and formula (3.2) (in case  $g(+\infty) \geq g(-\infty)$ ) based on the viscosity solution in sense of Ishii [40].

### 3 Applications and Explicit Formulas

As applications, we apply Theorem 1 to treat (1.1), which concerns the case when there is a single environmental shifting speed  $c_1$ . We will derive explicit formulas for the spreading speed in terms of  $c_1, g(+\infty), g(-\infty)$  and  $\Lambda_1$ , where  $\Lambda_1$  is the principal eigenvalue of (2.4) defined by (2.5). To simplify the notations, we denote for the remainder of this section

$$r_- = g(-\infty), \quad \text{and} \quad r_+ = g(+\infty).$$

Thanks to standard properties of the principal eigenvalue  $\Lambda_1$  (see Proposition 4.2(a)) we have

$$\Lambda_1 \in [\max\{r_-, r_+\}, \infty).$$

*Remark 3.1.* For  $g(y) = r_- \chi_{\{y < 0\}} + r_m \chi_{\{0 \leq y < L\}} + r_+ \chi_{\{y \geq L\}}$  with  $r_m > \max\{r_-, r_+\}$  that there exists  $\underline{L} \geq 0$  such that

$$\Lambda_1 > \max\{r_-, r_+\} \quad \text{if and only if} \quad L > \underline{L}$$

where  $\underline{L} = 0$  when  $r_- = r_+$  and

$$\underline{L} = \frac{1}{\sqrt{r_m - \max\{r_-, r_+\}}} \operatorname{arccot} \left( \sqrt{\frac{r_m - \max\{r_-, r_+\}}{|r_- - r_+|}} \right) > 0 \quad \text{when } r_- \neq r_+.$$

See [29, Theorem 2.1] for the precise statement.

The following theorem says that the spreading speed  $c_*$  is enhanced according to the specific profile of  $g(\cdot)$ .

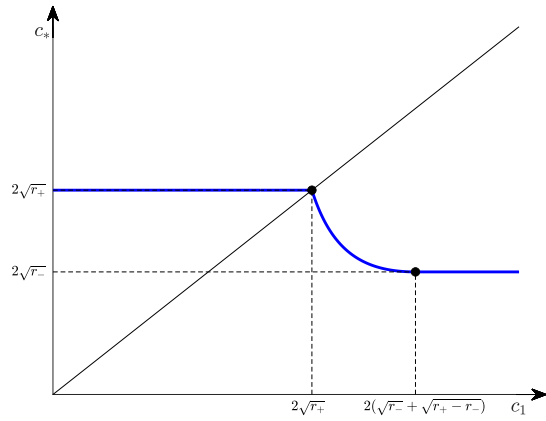
**Theorem 3.2.** *Let  $u$  be a solution of (1.1) with compactly supported, nonnegative, nontrivial initial data, then the rightward spreading speed*

$$c_* = \bar{c}_* = \underline{c}_* = \begin{cases} 2\sqrt{r_+} & \text{if } c_1 \leq 2\sqrt{r_+}, \\ c_1 & \text{if } 2\sqrt{r_+} < c_1 \leq 2\sqrt{\Lambda_1}, \\ \frac{c_1}{2} - \sqrt{\Lambda_1 - r_-} + \frac{r_-}{\frac{c_1}{2} - \sqrt{\Lambda_1 - r_-}} & \text{if } 2\sqrt{\Lambda_1} < c_1 \leq 2(\sqrt{r_-} + \sqrt{\Lambda_1 - r_-}), \\ 2\sqrt{r_-} & \text{if } c_1 > 2(\sqrt{r_-} + \sqrt{\Lambda_1 - r_-}). \end{cases}$$

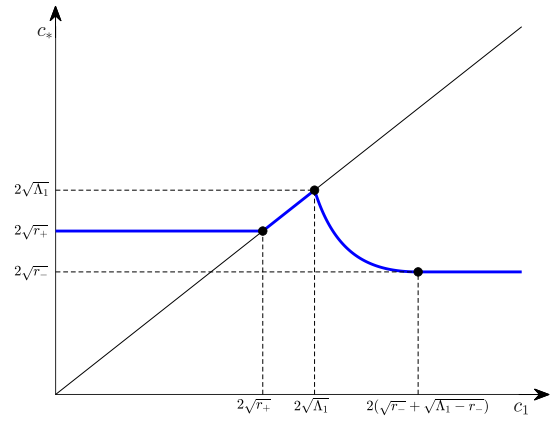
*In particular, the mapping  $c_1 \mapsto c_*$  is in general non-monotone, see panels (b), (d) and (f) of Figure 1 for illustration.*

*Remark 3.3.* The case  $c_1 \in (2\sqrt{r_+}, 2\sqrt{\Lambda_1})$  is contained in [8]. In fact, under this assumption, they proved the existence of a family of forced waves. Moreover, it is proved that solutions with sufficiently fast decaying initial data (including those that are compactly supported) converge locally uniformly to the unique minimal forced wave.

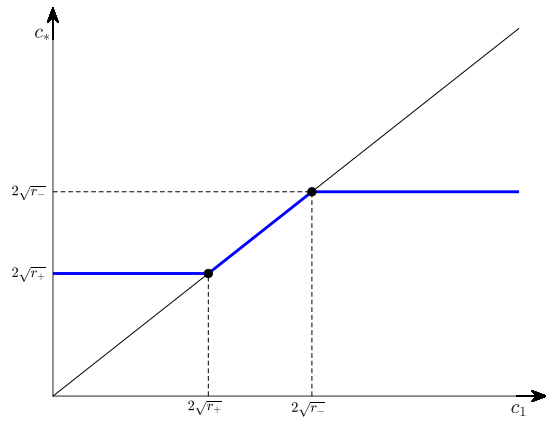
*Remark 3.4.* In case that  $\Lambda_1 = r_- > r_+$ , then case (iii) in Theorem 3.2 is eliminated. In case that  $\Lambda_1 = r_+ > r_-$ , then case (ii) in Theorem 3.2 is eliminated. In case that  $\Lambda_1 = r_- = r_+$ , then cases (ii) and (iii) in Theorem 3.2 are eliminated. In our former paper [40], we showed that  $c_* = s_{base}$  when  $\sup g \leq \max\{r_{\pm}\}$  with three cases:  $r_+ > r_-$ ,  $r_+ < r_-$  and  $r_+ = r_-$ . Next corollaries extend the validity of  $c_* = s_{base}$  to all  $g$  such that  $\Lambda_1 = \max\{r_{\pm}\}$ . See Figure 1.



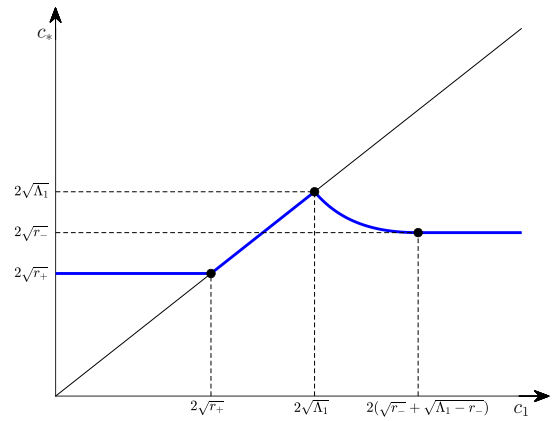
(a)  $\Lambda_1 = r_+ > r_- > 0$



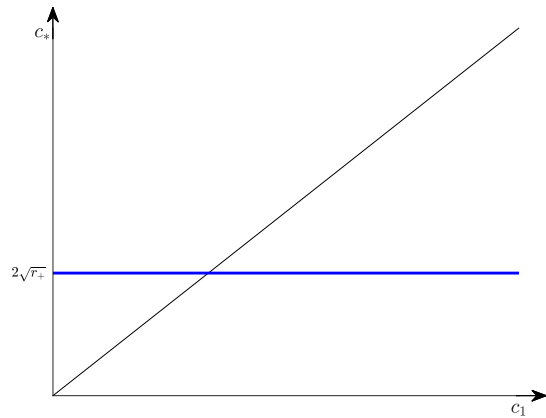
(b)  $\Lambda_1 > r_+ > r_- > 0$



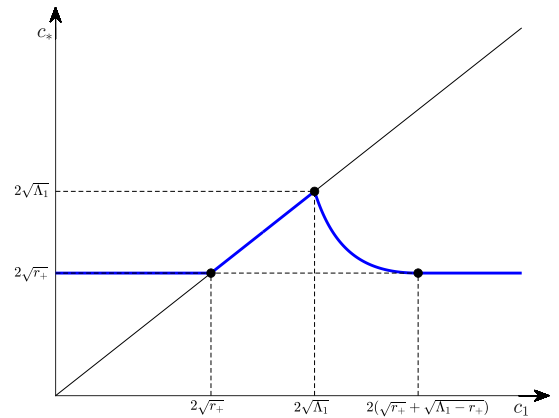
(c)  $\Lambda_1 = r_- > r_+ > 0$



(d)  $\Lambda_1 > r_- > r_+ > 0$



(e)  $\Lambda_1 = r_- = r_+ > 0$



(f)  $\Lambda_1 > r_- = r_+ > 0$

Figure 1: The dependence of spreading speed  $c_*$  on  $c_1$ . Here  $r_{\pm} = g(\pm\infty)$ . The case  $c_* = c_1$  is also indicated in [8]. Nonlocal pulling is illustrated by the curved part of the blue lines in panels (a), (b), (d) and (f). The part where  $c_*$  coincides either with the KPP speed of the limiting system at  $\pm\infty$  is indicated by the horizontal part of the blue lines in (a)-(f).

**Corollary 3.5.** *Suppose  $\Lambda_1 = r_+ > r_-$  and let  $u$  be a solution of (1.1) with compactly supported, nonnegative, nontrivial initial data, then*

$$c_* = \bar{c}_* = \underline{c}_* = \begin{cases} 2\sqrt{r_+} & \text{if } c_1 \leq 2\sqrt{r_+}, \\ \frac{c_1}{2} - \sqrt{r_+ - r_-} + \frac{r_-}{\frac{c_1}{2} - \sqrt{r_+ - r_-}} & \text{if } 2\sqrt{r_+} < c_1 \leq 2(\sqrt{r_-} + \sqrt{r_+ - r_-}) \\ 2\sqrt{r_-} & \text{if } c_1 > 2(\sqrt{r_-} + \sqrt{r_+ - r_-}) \end{cases} \quad (3.1)$$

See Figure 1(a) for the dependence of  $c_*$  on the shifting speed  $c_1$ .

**Corollary 3.6.** *Suppose  $\Lambda_1 = r_- > r_+$  and  $u$  be a solution of (1.1) with compactly supported, nonnegative, nontrivial initial data, then*

$$c_* = \bar{c}_* = \underline{c}_* = \begin{cases} 2\sqrt{r_+} & \text{if } c_1 \leq 2\sqrt{r_+}, \\ c_1 & \text{if } 2\sqrt{r_+} < c_1 \leq 2\sqrt{r_-}, \\ 2\sqrt{r_-} & \text{if } c_1 > 2\sqrt{r_-}. \end{cases} \quad (3.2)$$

See Figure 1(c) for the dependence of  $c_*$  on the shifting speed  $c_1$ .

**Corollary 3.7.** *Suppose  $\Lambda_1 = r_- = r_+$  and  $u$  be a solution of (1.1) with compactly supported, nonnegative, nontrivial initial data, then*

$$c_* = \bar{c}_* = \underline{c}_* = 2\sqrt{\Lambda_1}. \quad (3.3)$$

See Figure 1(e) for the dependence of  $c_*$  on the shifting speed  $c_1$ .

*Remark 3.8.* If  $c_1 \leq 0$ , then  $c_* = 2\sqrt{r_+}$  (as  $R(s) \equiv r_+$  for  $s > 0$ ), see also [35].

Finally, it is easy to obtain the following result on the leftward spreading speed by considering the rightward spreading speed of  $v(t, x) := u(t, -x)$  and apply Theorem 3.2 and Corollaries 3.5–3.7.

**Theorem 3.9.** *Let  $u$  be a solution of (1.1) with compactly supported, nonnegative, nontrivial initial data, then the leftward spreading speed*

$$c^* = \bar{c}^* = \underline{c}^* = \begin{cases} 2\sqrt{r_-} & \text{if } c_1 \geq -2\sqrt{r_-}, \\ c_1 & \text{if } -2\sqrt{r_-} > c_1 \geq -2\sqrt{\Lambda_1}, \\ \frac{c_1}{2} - \sqrt{\Lambda_1 - r_+} + \frac{r_+}{\frac{c_1}{2} - \sqrt{\Lambda_1 - r_+}} & \text{if } -2\sqrt{\Lambda_1} > c_1 \geq -2(\sqrt{r_+} + \sqrt{\Lambda_1 - r_+}), \\ 2\sqrt{r_+} & \text{if } c_1 < -2(\sqrt{r_+} + \sqrt{\Lambda_1 - r_+}). \end{cases}$$

where

$$\begin{cases} \bar{c}^* = \inf \{c > 0 \mid \limsup_{t \rightarrow \infty} \sup_{x < -ct} u(t, x) = 0\}, \\ \underline{c}^* = \sup \{c > 0 \mid \liminf_{t \rightarrow \infty} \inf_{-ct < x < -ct+1} u(t, x) > 0\}. \end{cases} \quad (3.4)$$

*Proof of Theorem 3.2.* We will give an explicit formula for the unique FL-solution of (2.9) satisfying (2.12).

Case (i):  $0 < c_1 \leq 2\sqrt{r_+}$ . Define

$$\rho_1(s) := \max\{s^2/4 - r_+, 0\}.$$

It is easy to see that  $\rho_1(s)$  is nonnegative and satisfies the first equation of (2.9) in a classical sense. Clearly,  $\rho_1$  is automatically a FL-subsolution as  $\rho_1(c_1) = 0$ . Note that  $A = \Lambda_1 - \frac{c_1^2}{4} \geq r_+ - \frac{c_1^2}{4} \geq 0$ . Therefore, for any  $\psi \in C_{pw}^1$ , we have  $F_A(\psi'_+(c_1+), \psi'_-(c_1-)) \geq A \geq 0$ . This

implies that  $\rho_1$  is also a FL-supersolution, and hence,  $\rho_1(s)$  is a FL-solution of (2.9). Since  $\rho_1$  also satisfies (2.12),  $\hat{\rho}_A = \rho_1$  by uniqueness, It then follows from the definition of  $c_*$  in (2.15) that  $c_* = 2\sqrt{r_+}$ .

Case (ii):  $2\sqrt{r_+} < c_1 \leq 2\sqrt{\Lambda_1}$ .

Define

$$\rho_2(s) := \begin{cases} s^2/4 - r_+, & s \geq c_1 + \sqrt{c_1^2 - 4r_+}, \\ \max\left\{\frac{c_1 + \sqrt{c_1^2 - 4r_+}}{2}(s - c_1), 0\right\} & 0 \leq s \leq c_1 + \sqrt{c_1^2 - 4r_+}. \end{cases} \quad (3.5)$$

One could directly check that  $\rho_2 \in C^1((0, c_1)) \cap C^1((c_1, +\infty))$  and satisfies the first equation of (2.9). Since  $\rho_2(c_1) = 0$  and  $A \geq 0$ , we infer that  $\rho_2$  is a FL-solution of (2.9) due to the same reason as in Case (i). Therefore, by locating the free boundary point in (3.5), we get  $c_* = c_1$ .

Case (iii):  $2\sqrt{\Lambda_1} < c_1 \leq 2(\sqrt{r_-} + \sqrt{\Lambda_1 - r_-})$ . Set

$$\mu_+ = \frac{c_1}{2} + \sqrt{\Lambda_1 - r_+}, \quad \mu_- = \frac{c_1}{2} - \sqrt{\Lambda_1 - r_-} > 0. \quad (3.6)$$

Define

$$\rho_3(s) := \begin{cases} s^2/4 - r_+, & s \geq 2\mu_+, \\ \mu_+s - (\mu_+^2 + r_+), & c_1 \leq s \leq 2\mu_+, \\ \mu_-s - (\mu_-^2 + r_-), & c_* \leq s \leq c_1, \\ 0, & 0 \leq s \leq c_*. \end{cases}$$

where  $\mu_+$  and  $\mu_-$  are as in (3.6). Noting that

$$\rho_3(c_1) = -A = \frac{c_1^2}{4} - \Lambda_1 > 0.$$

It is easy to check that  $\rho_3$  satisfies (2.9) in the classical sense when  $s \neq c_1$  or  $c_* = \mu_- + r_-/\mu_-$ . The FL-subsolution property at  $s = c_*$  holds since  $\rho_3(c_*) = 0$ . To show it is indeed a FL-subsolution, it suffices to consider the case that  $\rho_3 - \psi$  attains a global maximum at  $s_0 = c_1$  for some test function  $\psi \in C_{pw}^1$ . It then follows from  $\rho(s) - \psi(s) \leq \rho(c_1) - \psi(c_1)$  for any  $s$  close to  $c_1$  that  $\psi'(c_1+) \geq \rho'_3(c_1+) = \mu_+ \geq \frac{c_1}{2}$ , and  $\psi'(c_1-) \leq \rho'_3(c_1-) = \mu_- \leq \frac{c_1}{2}$ . This implies

$$\rho_3(c_1) + H^-(c_1+, \psi'(c_1+)) = \rho_3(c_1) + H(c_1+, \frac{c_1}{2}) = \frac{c_1^2}{4} - \Lambda_1 - \frac{c_1^2}{4} + r_+ = r_+ - \Lambda_1 \leq 0,$$

and

$$\rho_3(c_1) + H^+(c_1-, \psi'(c_1-)) = \rho_3(c_1) + H(c_1-, \frac{c_1}{2}) = \frac{c_1^2}{4} - \Lambda_1 - \frac{c_1^2}{4} + r_- = r_- - \Lambda_1 \leq 0,$$

where we used  $\rho_3(c_1) = -A = \frac{c_1^2}{4} - \Lambda_1$ . Also, we obtain that

$$\rho_3(c_1) + F_A(\psi'(c_1+), \psi'(c_1-)) \leq 0.$$

This yields  $\rho_3$  is a FL-subsolution of (2.9).

Next, we verify that  $\rho_3$  is a FL-supersolution. Note that  $\rho_3$  is a classical solution (including at the point  $s = 2\mu_+$ ) except for two non-differentiable points  $s_1 = c_1$  and  $s_2 = c_* = \frac{r_-}{\mu_-} + \mu_-$ . Also, observe from  $c_1 \leq 2(\sqrt{r_-} + \sqrt{\Lambda_1 - r_-})$  that  $\mu_- \leq \sqrt{r_-}$ , and hence

$$s_2 = \mu_- + \frac{r_-}{\mu_-} \geq 2\mu_-. \quad (3.7)$$

Suppose first that a test function  $\psi \in C_{pw}^1$  touches  $\rho_3$  from below at  $c_1$ , then

$$\rho_3(c_1) + F_A(\psi'(c_1+), \psi'(c_1-)) \geq \rho_3(c_1) + A = 0.$$

Suppose next that a test function  $\psi \in C_{pw}^1$  touches  $\rho_3$  from below at  $s_2 = \frac{r_-}{\mu_-} + \mu_-$ , it then follows that  $0 \leq \psi'(s_2) \leq \mu_-$  (note that  $\psi \in C^1$  near  $s_2$ ).

$$\rho_3(s_2) + H(s_2, \psi'(s_2)) = -s_2\psi'(s_2) + [\psi'(s_2)]^2 + r_- \geq -c_*\mu_- + \mu_-^2 + r_- = 0,$$

where the first inequality is due to  $\psi'(s_2) \in [0, \mu_-]$ , and that  $p \mapsto -s_2p + p^2 + r_+$  is monotone decreasing in  $[0, \mu_-]$  (thanks to (3.7)). We can then conclude that  $\rho_3(s)$  is a FL-solution of (2.9) in  $(0, \infty)$ , and hence,  $c_* = s_2 = \frac{r_-}{\mu_-} + \mu_-$ .

Case (iv):  $c_1 > 2(\sqrt{r_-} + \sqrt{\Lambda_1 - r_-})$ . Define

$$\rho_4(s) := \begin{cases} s^2/4 - r_+, & s \geq 2\mu_+, \\ \mu_+s - (\mu_+^2 + r_+), & c_1 \leq s \leq 2\mu_+, \\ \mu_-s - (\mu_-^2 + r_-), & 2\mu_- \leq s \leq c_1, \\ \max\{\frac{s^2}{4} - r_-, 0\}, & 0 \leq s \leq 2\mu_-. \end{cases}$$

Noting that  $\rho_4$  is a classical solution except at two points  $c_1$  and  $s_3 = 2\sqrt{r_-}$ . For  $s = c_1$ , we could argue similarly to that in the case (iii) to obtain that the junction condition for super- and subsolution hold true. For  $s_3 = 2\sqrt{r_-}$ ,  $\rho_4(s_3) = 0$  implies that the junction condition for subsolution hold at  $s = s_3$ . Now suppose that  $\rho_4 - \psi$  attains a global minimum at  $s_3$  for some test function  $\psi \in C_{pw}^1$ , then we infer that  $0 \leq \psi'(s_3) \leq \mu_-$  and

$$\rho_4(s_3) + H(s_3, \psi'(s_3)) = -s_3\psi'(s_3) + [\psi'(s_3)]^2 + r_- \geq r_- - \frac{s_3^2}{4} = 0,$$

where we used  $\min_p(-s_3p + p^2) = -s_3^2/4$  and  $s_3 = 2\sqrt{r_-}$ . As a consequence,  $\rho_4$  is a FL-solution and  $c_* = s_3 = 2\sqrt{r_-}$ .  $\square$

*Proof of Corollaries 3.4-3.6.* By part (b) of Theorem 1 implies that  $c_* = s_{base}$ . Then it is a direct consequence of [40, Theorem 6 (iv) and Theorem 7].  $\square$

## 4 Preliminary Results

We will give some preliminary results in this section in preparation of the proof of the main result (Theorem 1) in the next section.

To study the behavior of  $u$  at the leading edge, we consider the rate function

$$w^\epsilon(t, x) = -\epsilon \log u^\epsilon(t, x). \quad (4.1)$$

We first observe that  $\hat{w}(t, x) := \lim_{\epsilon \rightarrow 0^+} w^\epsilon(t, x)$ , if exists, is 1-homogeneous.

**Lemma 4.1.** *Suppose  $w^\epsilon \rightarrow \hat{w}$  in  $C_{loc}((0, \infty) \times [0, \infty))$ , as  $\epsilon \rightarrow 0^+$ , then  $\hat{w}(t, x) = t\hat{\rho}(x/t)$  for some function  $\hat{\rho}$ .*

*Proof.* Fix a constant  $A > 0$ , then

$$\hat{w}(At, Ax) = \lim_{\epsilon \rightarrow 0^+} -\epsilon \log u \left( \frac{At}{\epsilon}, \frac{Ax}{\epsilon} \right) = A \lim_{(\epsilon/A) \rightarrow 0^+} -(\epsilon/A) \log u \left( \frac{t}{\epsilon/A}, \frac{x}{\epsilon/A} \right) = A\hat{w}(t, x).$$

And the lemma follows if we take  $\hat{\rho}(s) := \hat{w}(1, s)$ .  $\square$

Suppose the limit function  $\hat{w}(t, x) = t\hat{\rho}(x/t)$  exists, and define

$$\hat{s} = \sup\{s \geq 0 : \hat{\rho}(s) = 0\}, \quad (4.2)$$

then we immediately have

$$u^\epsilon(t, x) = \exp\left(-\frac{w^\epsilon(t, x)}{\epsilon}\right) = \exp\left(-\frac{\hat{w}(t, x) + o(1)}{\epsilon}\right) = \exp\left(-\frac{t\hat{\rho}(x/t) + o(1)}{\epsilon}\right) \rightarrow 0$$

whenever  $x/t > \hat{s}$ . This gives the first part of (2.3).

Furthermore, it can be shown (e.g. [47, Lemma 3.1] or [21, Sect. 4]) that  $u^\epsilon(t, x)$  is bounded away from zero in the interior of  $\{(t, x) : \hat{w}(t, x) = 0\}$ , i.e. the second part of (2.3) holds. Hence, the study of the spreading speed  $c_*$  reduces to the determination of the free boundary point  $\hat{s}$  of  $\hat{\rho}$ , given in (4.2). Next, we collect the properties of the eigenvalue problem (2.4) as well as a few technical results for Hamilton-Jacobi equations with general Hamiltonians  $H(s, p)$ .

#### 4.1 The eigenvalue problem associated with $g(y)$

Note that  $s_{base}$  depends only on the values of  $g(\pm\infty)$ . The next questions are if and when the invasion is enhanced by the specific profile of  $g$ . The answer is completely determined by the eigenvalue  $\Lambda_1$  given by (2.5). This and several other notions of principal eigenvalues are analyzed in [14]. Here, we recall some basic properties of  $\Lambda_1$  and the associated positive eigenfunction.

**Proposition 4.2.** *Let  $\Lambda_1$  be given by (2.5).*

- (a) *Then  $\Lambda_1 \geq \max\{g(-\infty), g(+\infty)\}$  and the eigenvalue problem (2.4) has a positive solution in  $C_{loc}^2(\mathbb{R})$  if and only if  $\Lambda \in [\Lambda_1, \infty)$ .*
- (b) *If, in addition,  $\Lambda_1 > \max\{g(-\infty), g(+\infty)\}$ , then  $\Lambda_1$  is a simple eigenvalue of (2.4) and the following statements hold.*

- (i) *Let  $\lambda_\pm := \sqrt{\Lambda_1 - g(\pm\infty)}$  and  $\Phi_1(y)$  be the positive eigenfunction corresponding to  $\Lambda = \Lambda_1$ , then*

$$\Phi_1(y) = \exp(-\lambda_\pm y + o(y)) \quad \text{as } y \rightarrow \pm\infty,$$

*i.e. for any sufficiently small  $\eta > 0$ , there exist positive numbers  $\bar{C}_\eta, \underline{C}_\eta$ , such that*

$$\begin{cases} \underline{C}_\eta e^{-(\lambda_+ + \eta)y} \leq \Phi_1(y) \leq \bar{C}_\eta e^{-(\lambda_+ - \eta)y} & \text{if } y \geq 0, \\ \underline{C}_\eta e^{(\lambda_- + \eta)y} \leq \Phi_1(y) \leq \bar{C}_\eta e^{(\lambda_- - \eta)y} & \text{if } y \leq 0. \end{cases} \quad (4.3)$$

- (ii) *Suppose (2.4) has a positive eigenfunction  $\tilde{\Phi} \in C_{loc}^2(\mathbb{R})$  for some  $\tilde{\Lambda} \in \mathbb{R}$  such that  $\tilde{\Phi} \rightarrow 0$  as  $|y| \rightarrow \infty$ , then  $\tilde{\Lambda} = \Lambda_1$  and  $\tilde{\Phi} \in \text{span}\{\Phi_1\}$ .*

- (c) *If  $\Lambda_1 = \max\{g(-\infty), g(+\infty)\}$ . Then for any  $\eta > 0$ , there exists  $g_\eta : \mathbb{R} \rightarrow \mathbb{R}$ , such that*

$$\begin{cases} g_\eta(x) = g(x) & \text{for all } |x| \gg 1, \\ g(x) \leq g_\eta(x) \leq g(x) + \eta & \text{for all } x \in \mathbb{R}, \\ \Lambda_1^\eta := \Lambda_1(g_\eta) & \text{satisfies } \Lambda_1^\eta > \max\{g(\pm\infty)\}. \end{cases}$$

For the convenience of the reader, we provide the proof of the above result in Appendix A. The following result describes the effect of  $\Lambda_1$  in enhancing the spreading speed  $c_*$ .

**Corollary 4.3.** *Let  $\Lambda_1$  be given by (2.5) and let  $s_{base}$  be given by Section 2.6.*

- (a) *If  $\Lambda_1 = \max\{g(\pm\infty)\}$ , then for each  $c_1 > 0$ , we have  $c_* = s_{base}$ .*
- (b) *If  $\Lambda_1 > \max\{g(\pm\infty)\}$ , then for some  $c_1 > 0$ , we have  $c_* > s_{base}$ .*

*Proof.* Statement (a) follows from Theorem 1. Statement (b) is a direct consequence of Theorem 3.2.  $\square$



## 4.2 The continuity of subsolutions

We discuss the weak continuity condition for sub-solutions, which are half-relaxed limits of solutions to reaction-diffusion equations. This property first appeared in [6].

**Lemma 4.4.** *Suppose  $\underline{\rho}$  is nonnegative and satisfies  $\underline{\rho}(0) = 0$ , and satisfies*

$$\min\{\rho, \rho + H(s, \rho')\} \leq 0 \quad \text{in } [0, \infty)$$

*in viscosity sense (of Ishii).*

- (a) *If  $H(s, 0) > 0$  for each  $s > 0$ , then  $\underline{\rho}$  is nondecreasing.*
- (b) *If  $\lim_{|p| \rightarrow \infty} \inf_{s \in K} H(s, p) \rightarrow \infty$  for each compact set  $K \subset [0, \infty)$ , then  $\underline{\rho} \in \text{Lip}_{loc}([0, \infty))$ . In particular, it satisfies the weak continuity condition:*

$$\underline{\rho}(c_1) = \limsup_{s \rightarrow c_1+} \underline{\rho}(s) \quad \text{and} \quad \underline{\rho}(c_1) = \limsup_{s \rightarrow c_1-} \underline{\rho}(s). \quad (4.4)$$

*Proof.* Part (a) is due to [40, Lemma 2.9]. For Part (b), fix a bounded interval  $K = [0, \bar{s}]$  with  $\bar{s} > 0$ , and let  $M > 0$  be given such that

$$H(s, p) > 0 \quad \text{for all } s \in [0, \bar{s} + 1], |p| \geq M.$$

Fix any point  $s_0 \in [0, \bar{s}]$ , we claim that

$$\underline{\rho}(s) - \underline{\rho}(s_0) \leq M|s - s_0| \quad \text{for all } s \in [0, \bar{s}]. \quad (4.5)$$

For this purpose, define

$$\Psi(s) = \begin{cases} \alpha + M|s - s_0| & \text{for } s \in [0, \bar{s}], \\ \alpha + M \left[ |\bar{s} - s_0| + \frac{1}{\bar{s}+1-s} - 1 \right] & \text{for } s \in [\bar{s}, \bar{s} + 1], \end{cases}$$

for any  $\alpha > 0$ , then  $\Psi$  is continuously differentiable except at  $s_0$ , such that

$$|\Psi'(s)| \geq M \quad \text{for } s \in [0, \bar{s} + 1] \setminus \{s_0\}. \quad (4.6)$$

Next, take the minimal  $\alpha$  such that  $\Psi$  touches  $\underline{\rho}$  from above at some point  $s_1 \in [0, \bar{s} + 1]$ , which is possible since  $\underline{\rho}$  is upper semicontinuous, and thus bounded on  $[0, \bar{s} + 1]$ .

If  $s_1 = s_0$ , then  $\alpha = \rho(s_0)$  and we obtain (4.5). Suppose to the contrary that  $s_1 \neq s_0$ . We first observe that  $\underline{\rho}(s_1) > 0$ , which follows from  $\Psi(s_0) < \Psi(s)$  for  $s \in [0, \bar{s} + 1]$  and

$$\underline{\rho}(s_1) = \Psi(s_1) > \Psi(s_0) \geq \underline{\rho}(s_0) \geq 0.$$

Now  $\underline{\rho}(s_1) > 0$ , so the definition of viscosity subsolution implies that

$$0 \geq \underline{\rho}(s_1) + H(s_1, \Psi'(s_1)).$$

But the right hand side is strictly positive thanks to (4.6) and the choice of  $M$ . This is a contradiction. Therefore, (4.5) is proved.

Since  $M$  depends only on  $K = [0, \bar{s}]$  but not on  $s_0$ , we can reverse the role of  $s$  and  $s_0$  in (4.5) to conclude that

$$|\underline{\rho}(s) - \underline{\rho}(s_0)| \leq M|s - s_0| \quad \text{for all } s, s_0 \in [0, \bar{s}].$$

This proves the Lipschitz continuity of  $\underline{\rho}$  in any compact subset of  $[0, \infty)$ .  $\square$

### 4.3 Critical slope lemmas

Let  $U$  be an open interval in  $\mathbb{R}$  containing  $c_1$ , and recall that

$$C_{pw}^1(U) = C(U) \cap C^1(U \cap (-\infty, c_1]) \cap C^1(U \cap [c_1, \infty)).$$

**(HH)** Assume  $p \mapsto H(s, p)$  is convex and coercive, and  $H(c_1+, p)$  and  $H(c_1-, p)$  exist.

**Lemma 4.5.** *Assume that **(HH)** holds. Let  $\underline{\rho} : U \rightarrow [0, \infty)$  satisfy the following:*

(i)  $\underline{\rho}$  is a viscosity subsolution of

$$\min\{\rho, \rho + H(s, \rho')\} = 0 \quad \text{in } \{s \in U : s > c_1\}.$$

(ii)  $\underline{\rho}$  satisfies the weak continuity condition (4.4).

(iii)  $\underline{\rho}(c_1) > 0$ .

Suppose there is a test function  $\varphi \in C_{pw}^1(U)$  that touches  $\underline{\rho}$  from above only at  $c_1$ . Let  $p_+ = \varphi'(c_1+)$ , and

$$\bar{p}_+ := \inf \{ \bar{p} \in \mathbb{R} : \exists r > 0, \varphi(s) + \bar{p}(s - c_1) \geq \underline{\rho}(s) \text{ for } 0 \leq s - c_1 < r \}. \quad (4.7)$$

Then  $-\infty < \bar{p}_+ \leq 0$  and

$$\underline{\rho}(c_1) + H(c_1+, p_+ + \bar{p}_+) \leq 0.$$

The proof is a modification of that in [36, Lemmas 2.9 and 2.10], where we use the weak continuity condition. For the convenience of the reader, we provide the proof in Section B.

*Remark 4.6.* Suppose, in addition, that  $\underline{\rho}$  is a viscosity subsolution of  $\min\{\rho, \rho + H(s, \rho')\} = 0$  in  $\{s \in U : s < c_1\}$ . Then

$$\underline{\rho}(c_1) + H(c_1-, -p_- - \bar{p}_-) \leq 0,$$

where  $-p_- = \varphi'(c_1-)$  and  $\bar{p}_- \in (-\infty, 0]$  is given by

$$\bar{p}_- := \inf \{ \bar{p} \in \mathbb{R} : \exists r > 0, \varphi(s) - \bar{p}(s - c_1) \geq \underline{\rho}(s) \text{ for } -r < s - c_1 \leq 0 \}. \quad (4.8)$$

**Lemma 4.7.** *Assume that **(HH)** is valid. Suppose  $\bar{p}$  is a viscosity supersolution of*

$$\rho + H(s, \rho') = 0 \quad \text{in } \{s \in U : s > c_1\}.$$

and there is a test function  $\varphi \in C_{pw}^1(U)$  that touches  $\bar{p}$  from below only at  $c_1$ . Let  $p_+ = \varphi'(c_1+)$ , and

$$\underline{p}_+ := \sup \{ \underline{p} \in \mathbb{R} : \exists r > 0, \varphi(s) + \underline{p}(s - c_1) \leq \bar{p}(s) \text{ for } 0 \leq s - c_1 < r \}. \quad (4.9)$$

If  $\underline{p}_+ < \infty$ , then

$$\bar{p}(c_1) + H(c_1+, p_+ + \underline{p}_+) \geq 0, \quad \text{with } \underline{p}_+ \geq 0.$$

Note that no weak continuity condition is needed as we do not claim the finiteness of  $\underline{p}_+$ . The proof of Lemma 4.7 is also included in Appendix B.

*Remark 4.8.* Suppose, in addition,  $\bar{p}$  is a viscosity supersolution of  $\min\{\rho, \rho + H_-(s, \rho')\} = 0$  in  $\{s \in U : s < c_1\}$ . Let  $-p_- = \varphi'(c_1-)$  and

$$\underline{p}_- := \sup \{ \underline{p} \in \mathbb{R} : \exists r > 0, \varphi(s) - \underline{p}(s - c_1) \leq \bar{p}(s) \text{ for } -r < s - c_1 \leq 0 \}. \quad (4.10)$$

If  $\underline{p}_- < +\infty$ , then

$$\bar{p}(c_1) + H(c_1-, -p_- - \underline{p}_-) \geq 0 \quad \text{and} \quad \underline{p}_- \geq 0.$$

## 5 Proof of Main Results

**Lemma 5.1.** *Let  $u$  be a nonnegative, nontrivial solution of (1.1) and assume  $\inf g > \delta_0 > 0$ . Then for any  $\eta \in (0, 2\sqrt{\delta_0})$*

$$\liminf_{t \rightarrow \infty} \inf_{|x| < (2\sqrt{\delta_0} - \eta)t} u(t, x) \geq \delta_0. \quad (5.1)$$

*Proof.* Choose  $\delta_1 \in (\delta_0, \inf g)$ , then  $u$  is a subsolution to

$$\tilde{u}_t - \tilde{u}_{xx} = \tilde{u}(\delta_1 - \tilde{u}) \quad \text{in } (0, \infty) \times \mathbb{R}$$

with compactly supported initial data  $u_0$ . It is a classical result that (5.1) holds for  $\tilde{u}$ . It follows by comparison principle that (5.1) holds also for  $u$ .  $\square$

Fix a solution  $u(t, x)$  of (1.1), and let

$$u^\epsilon(t, x) = u\left(\frac{t}{\epsilon}, \frac{x}{\epsilon}\right), \quad \text{and} \quad w^\epsilon(t, x) = -\epsilon \log u^\epsilon(t, x),$$

then  $w^\epsilon$  satisfies

$$\begin{cases} w_t^\epsilon - \epsilon w_{xx}^\epsilon + |w_x^\epsilon|^2 + g\left(\frac{x - c_1 t}{\epsilon}\right) - e^{-w^\epsilon/\epsilon} = 0 & \text{for } (t, x) \in (0, \infty) \times \mathbb{R}, \\ w^\epsilon(0, x) = \begin{cases} -\epsilon \log u_0(x/\epsilon) & \text{if } x/\epsilon \in \text{Int}(\text{supp } u_0), \\ +\infty & \text{otherwise.} \end{cases} \end{cases} \quad (5.2)$$

Consider the half-relaxed limits [6]:

$$w^*(t, x) = \limsup_{\substack{\epsilon \rightarrow 0 \\ (t', x') \rightarrow (t, x)}} w^\epsilon(t, x) \quad \text{and} \quad w_*(t, x) = \liminf_{\substack{\epsilon \rightarrow 0 \\ (t', x') \rightarrow (t, x)}} w^\epsilon(t, x) \quad (5.3)$$

**Lemma 5.2.** *Let  $w^*$  and  $w_*$  be given as above. Then  $w^*(t, x) = t\rho^*(x/t)$  and  $w_*(t, x) = t\rho_*(x/t)$ , for some upper semicontinuous function  $\rho^*$  and lower semicontinuous function  $\rho_*$ .*

*Proof.* The existence of  $\rho^*$  and  $\rho_*$  is similar to Lemma 4.1 and is omitted. The semicontinuity are due to the half-relaxed limits in the definition of  $w^*, w_*$ .  $\square$

**Lemma 5.3.**  *$\rho^*(s) \geq \rho_*(s) \geq 0$  for all  $s \geq 0$  and  $\rho^*(0) = 0$ . Moreover,  $\rho_*(s)/s \rightarrow +\infty$  as  $s \rightarrow +\infty$ .*

*Proof.* By maximum principle, one can establish uniform upper bound of  $u$ , i.e.  $u(t, x) \leq M_0 := \max\{\sup |u_0|, \sup g\}$ , so that  $w^\epsilon(t, x) \geq -\epsilon \log M_0$ . This implies  $w^* \geq w_* \geq 0$  and hence  $\rho^* \geq \rho_* \geq 0$ .

To show  $\rho^*(0) = 0$ , it suffices to prove  $w^*(t, 0) = 0$  for all  $t > 0$ . By Lemma 5.1, we have

$$w^*(t, 0) = \limsup_{\substack{\epsilon \rightarrow 0 \\ (t', x') \rightarrow (t, 0)}} w^\epsilon(t', x') \leq \limsup_{\substack{\epsilon \rightarrow 0 \\ (t', x') \rightarrow (t, 0)}} -\epsilon \log u\left(\frac{t'}{\epsilon}, \frac{x'}{\epsilon}\right) \leq -\lim_{\epsilon \rightarrow 0} \epsilon \log \delta_0 = 0.$$

Finally, by similar argument to that in [40, Lemma B.3], we have  $w_*(0, x) = +\infty$  for all  $x > 0$ . It then follows from lower semicontinuity of  $w_*$  that

$$\liminf_{s \rightarrow +\infty} \frac{\rho_*(s)}{s} = \liminf_{s \rightarrow +\infty} w_*\left(\frac{1}{s}, 1\right) \geq w_*(0, 1) = +\infty.$$

This completes the proof.  $\square$

## 5.1 Verification of flux-limited solutions property

The main result of this subsection is the following.

**Proposition 5.4.** *Let  $\rho^*$  and  $\rho_*$  be given in Lemma 5.2. Then  $\rho^*$  (resp.  $\rho_*$ ) is a FL-subsolution (resp. FL-supersolution) of (2.9) with flux limiter given by (2.13), that is,  $A = \Lambda_1 - \frac{|c_1|^2}{4}$ .*

We divide the proof of Proposition 5.4 into the verification of FL-subsolution and supersolution.

**Lemma 5.5.** *Let  $\rho^*$  be given by Lemma 5.2. Then*

- (a)  $\rho^* \in \text{Lip}_{loc}([0, \infty))$ ;
- (b)  $\rho^*$  satisfies the weak continuity condition (4.4);
- (c)  $\rho^*$  is a FL-subsolution of (2.9) with  $A = \Lambda_1 - \frac{c_1^2}{4}$ .

*Proof.* By construction  $\rho^* : [0, \infty) \rightarrow [0, \infty)$  is upper semicontinuous. It is standard to show that  $w^*(t, x) = t\rho^*(x/t)$  is a viscosity subsolution to

$$\begin{cases} \min\{w^*, w_t^* + |w_x^*|^2 + g(-\infty)\} = 0 & \text{in } \{(t, x) : 0 < x < c_1 t\}, \\ \min\{w^*, w_t^* + |w_x^*|^2 + g(+\infty)\} = 0 & \text{in } \{(t, x) : x > c_1 t > 0\}, \\ \min\{w^*, w_t^* + |w_x^*|^2 + \inf_{y \in \mathbb{R}} g(y)\} = 0 & \text{in } (0, +\infty) \times (0, +\infty). \end{cases} \quad (5.4)$$

From the third equation, we deduce as in [46, Lemma 2.3] that, in viscosity sense,

$$\min\{\rho^*, \rho^* - s(\rho^*)' + |(\rho^*)'|^2 + \inf_{\mathbb{R}} g\} \leq 0 \quad \text{in } (0, \infty). \quad (5.5)$$

Since also  $\rho^*(0) = 0$  (thanks to Lemma 5.3), we infer from Lemma 4.4 that  $\rho^* \in \text{Lip}_{loc}([0, \infty))$ . This proves assertion (a). Assertion (a) implies (b).

The proof of (c) is inspired by [31]. From the first two equations of (5.4), we deduce that, in viscosity sense,

$$\min\{\rho^*, \rho^* + H(s, (\rho^*)')\} \leq 0 \quad \text{for } s \in (0, \infty) \setminus c_1,$$

with  $H(s, p)$  given in (2.23). It remains to show that  $\rho^*$  is a subsolution to the second equation of (2.9). For this purpose, let  $\psi \in C_{pw}^1$  and suppose  $\rho^* - \psi$  has a strict global maximum point<sup>1</sup> at  $c_1$ , and that  $\psi(c_1) = \rho^*(c_1) > 0$ . Denote

$$A = \Lambda_1 - \frac{c_1^2}{4}, \quad \lambda = -\rho^*(c_1) = -\psi(c_1), \quad p_+ = \psi'(c_1+), \quad p_- = -\psi'(c_1-).$$

(Note the negative sign in the definition of  $p_-$ .) We want to show

$$-\lambda + \max\{A, H^-(c_1+, p_+), H^+(c_1-, -p_-)\} \leq 0. \quad (5.6)$$

(Observe that if  $A \geq 0$ , then any nonnegative (sub)solution  $\rho^*(s)$  satisfying the junction condition  $\min\{\rho, \rho + F_A(\rho'(c_1+), \rho'(c_1-))\} \leq 0$  must vanish at the point  $c_1$ , i.e. the case  $\rho^*(c_1) > 0$  is null.) We first claim that

$$H^-(c_1+, p_+) \leq \lambda, \quad \text{and} \quad H^+(c_1-, -p_-) \leq \lambda. \quad (5.7)$$

By Lemma 4.5,

$$H(c_1+, p_+ + \bar{p}_+) \leq \lambda \quad \text{for some } \bar{p}_+ \in (-\infty, 0]. \quad (5.8)$$

<sup>1</sup>See [2, Proposition 3.1] for several equivalent definitions of viscosity solution.

Hence, using the fact that  $H^-$  is decreasing in  $p$  and  $H^- \leq H$ ,

$$H^-(c_1+, p_+) \leq H^-(c_1+, p_+ + \bar{p}_+) \leq H(c_1+, p_+ + \bar{p}_+) \leq \lambda.$$

Arguing similarly, we also have  $H^+(c_1-, -p_-) \leq \lambda$ . This proves (5.7).

It remains to show  $A \leq \lambda$ , where  $A = \Lambda_1 - \frac{c_1^2}{4}$ .

Suppose for contradiction  $A > \lambda$ . Then by (5.7), we have

$$A > \max\{\min_p H(c_1\pm, p)\} = \max\{g(\pm\infty)\} - \frac{c_1^2}{4}, \quad (5.9)$$

In particular,  $\Lambda_1 > \max\{g(\pm\infty)\}$ . Define

$$\phi_0(x) = \mu_+ \max\{x, 0\} - \mu_- \min\{x, 0\}, \quad (5.10)$$

where

$$\mu_+ = \frac{c_1}{2} + \sqrt{\Lambda_1 - g(+\infty)}, \quad \text{and} \quad \mu_- = \frac{-c_1}{2} + \sqrt{\Lambda_1 - g(-\infty)}. \quad (5.11)$$

Note that  $\mu_\pm$  are also determined uniquely (thanks to (5.9)) by

$$\begin{cases} H(c_1+, \mu_+) = A & \text{and} & \mu_+ \geq \operatorname{argmin} H(c_1+, \cdot), \\ H(c_1-, -\mu_-) = A & \text{and} & -\mu_- \leq \operatorname{argmin} H(c_1-, \cdot). \end{cases} \quad (5.12)$$

By (5.8) and that  $\lambda < A = H(c_1+, \mu_+)$ , we have  $H(c_1+, p_+ + \bar{p}_+) < H(c_1+, \mu_+)$  and thus  $p_+ + \bar{p}_+ < \mu_+$  (here we have used the fact that  $\mu_+$  is the larger root of  $H(c_1+, p) = A$ ). By definition of  $\bar{p}_+$  in (4.7), we deduce that there exists a small neighborhood  $(c_1 - r, c_1 + r)$  of  $c_1$  (with  $0 < r < \min\{1, c_1\}$ ) such that (by Lipschitz continuity of  $\rho^*$ )

$$\frac{\rho^*(c_1)}{2} < \rho^*(s) \leq -\lambda + \phi_0(s - c_1) \quad \text{for } c_1 \leq s < c_1 + r,$$

with the second inequality being an equality iff  $s = c_1$ . By arguing similarly, along with the definition of  $\bar{p}_-$  in (4.8), we have

$$\frac{\rho^*(c_1)}{2} < \rho^*(s) \leq -\lambda + \phi_0(s - c_1) \quad \text{for } c_1 - r < s \leq c_1,$$

with the second equality holds iff  $s = c_1$ . In other words,  $-\lambda + \phi_0(s - c_1)$  is also a test function touching  $\rho^*$  from above at  $c_1$  in  $(c_1 - r, c_1 + r)$ . Hence, letting

$$Q_r := \{(t, x) : x/t \in [c_1 - r, c_1 + r], |t - 1| < r\}, \quad (5.13)$$

we get

$$w^*(t, x) = t\rho^*(x/t) \leq \varphi^0(t, x) := \frac{A - \lambda}{4}(t - 1)^2 + t \left( -\lambda + \phi_0\left(\frac{x - c_1 t}{t}\right) \right) \quad \text{in } Q_r$$

with equality iff  $(t, x) = (1, c_1)$ . We can then choose  $\delta \in (0, (1 - r)\rho^*(c_1))$  small such that

$$w^*(t, x) + \delta \leq \varphi^0(t, x) \quad \text{on } \partial Q_r. \quad (5.14)$$

Next, define

$$\varphi^\epsilon(t, x) := \frac{A - \lambda}{4}(t - 1)^2 - t\lambda - \epsilon \log \Psi\left(\frac{x - c_1 t}{\epsilon}\right) \quad \text{for } (t, x) \in Q_r,$$

where  $\Psi(y) = e^{-\frac{c_1 y}{2}} \Phi_1(y)$  and  $\Phi_1$  is the positive eigenfunction given in Proposition 4.2. Thanks to Proposition 4.2(b)(i), one has  $\varphi^\epsilon \rightarrow \varphi^0$  in  $C_{loc}$  since

$$-\epsilon \log \left[ e^{-\frac{c_1}{2} \frac{x-c_1 t}{\epsilon}} \Phi_1 \left( \frac{x-c_1 t}{\epsilon} \right) \right] \rightarrow \phi_0(x-c_1 t) = t \phi_0 \left( \frac{x-c_1 t}{t} \right) \quad \text{locally uniformly.}$$

Hence, we deduce from (5.14) that

$$w^\epsilon(t, x) + \delta/2 \leq \varphi^\epsilon(t, x) \quad \text{on } \partial Q_r, \quad (5.15)$$

for sufficiently small  $\epsilon$ . Next, we observe that  $\varphi^\epsilon - \delta/2$  satisfies

$$\varphi^\epsilon_t - \epsilon \varphi^\epsilon_{xx} + |\varphi^\epsilon_x|^2 + g \left( \frac{x-c_1 t}{\epsilon} \right) - e^{-\frac{(2\varphi^\epsilon - \delta)}{2\epsilon}} = \frac{A-\lambda}{2}(t-1) - \lambda + A + o(1) > 0,$$

where we used

$$\begin{aligned} \varphi^\epsilon(t, x) - \frac{\delta}{2} &\geq t \left( -\lambda + \phi_0 \left( \frac{x}{t} - c_1 \right) \right) - \frac{\delta}{2} + o(1) \\ &\geq (1-r) \rho^* \left( \frac{x}{t} \right) - \frac{\delta}{2} + o(1) \geq \frac{(1-r) \rho^*(c_1)}{2} + o(1) > 0 \quad \text{for } (t, x) \in Q_r \end{aligned}$$

to deduce  $e^{-\frac{2\varphi^\epsilon - \delta}{2\epsilon}} = o(1)$  in the last equality, and  $|t-1| \leq r < 1$  in the strict inequality. Hence  $\varphi^\epsilon - \delta/2$  is a supersolution to the equation (5.2) of  $w^\epsilon$ . In view of the boundary condition (5.15), the comparison principle yields

$$w^\epsilon(t, x) + \delta/2 \leq \varphi^\epsilon(t, x) = \varphi^0(t, x) + o(1) \quad \text{in } Q_r. \quad (5.16)$$

By definition of  $w^*(1, c_1) = \rho^*(c_1) = -\lambda$  (recall that  $\psi$  touches  $\rho^*$  from above), there exists  $(t^\epsilon, x^\epsilon) \rightarrow (1, c_1)$  such that  $w^\epsilon(t^\epsilon, x^\epsilon) \rightarrow -\lambda$ . Substituting  $(t, x) = (t^\epsilon, x^\epsilon)$  into (5.16) and letting  $\epsilon \rightarrow 0$ , we have

$$-\lambda + \delta/2 \leq \varphi^0(1, c_1) = -\lambda \quad \text{for some } \delta > 0,$$

which leads to a contradiction. Therefore,  $A \leq \lambda$ . This concludes the proof.  $\square$

Next, we show the FL-supersolution property of  $\rho_*$ .

**Lemma 5.6.** *The lower limit  $\rho_*$  is a FL-supersolution of (2.9) with  $A = \Lambda_1 - \frac{c_1^2}{4}$ .*

*Proof.* Again, it is standard to check that  $w_*$  is a viscosity supersolution to the first two equations of (5.4) in the viscosity sense. This implies again, by [46, Lemma 2.3] that  $\rho_*$  is the viscosity supersolution of the first equation of (2.9).

It remains to verify the remaining junction condition of (2.9). Suppose there is a test function  $\psi \in C_{pw}^1$  that touches  $\rho_*$  from below only at  $s = c_1$ , and denote

$$\lambda = -\rho_*(c_1) = -\psi(c_1), \quad p_+ = \psi'(c_1+), \quad p_- = -\psi'(c_1-).$$

By way of contradiction, we further assume that

$$\max\{A, H^-(c_1+, p_+), H^+(c_1-, -p_-)\} < \lambda. \quad (5.17)$$

Let  $A_0 = \max\{g(\pm\infty)\} - |c_1|^2/4$ ,  $\mu_\pm$  be given in terms of  $A \in [A_0, \infty)$  as in (5.12), and  $\phi_0$  be defined as in (5.10). By Proposition 4.2(c), for any  $\eta > 0$ , there exists  $g_\eta \in C(\mathbb{R}, \mathbb{R})$  such that  $\Lambda_1^\eta > \max\{g(\pm\infty)\}$  and  $\|g - g_\eta\|_\infty \leq \eta$ . In particular, if  $A > A_0$  (or say  $\Lambda_1 > \max\{g(\pm\infty)\}$ ), we could set  $\eta = 0$  and  $g_0 \equiv g$ . Let  $\Psi^\eta(y) = e^{-\frac{c_1}{2} y} \Phi_1^\eta(y)$  where  $\Phi_1^\eta(y)$  is a positive and bounded eigenfunction associated with  $\Lambda_1^\eta$ . For any  $\eta \geq 0$ , we could also define  $\mu_+^\eta$  and  $\mu_-^\eta$  with  $\Lambda_1$  replace by  $\Lambda_1^\eta$  in (5.11). Accordingly,  $\phi_0^\eta$  would be defined in a similar manner. We first prove the following claim.

**Claim 5.7.** *There exists  $\eta_0 > 0$ , such that  $-\lambda + \phi_0^\eta(s - c_1)$  touches  $\rho_*$  from below strictly at  $c_1$  for any  $\eta \in [0, \eta_0)$ .*

Let  $\underline{p}_\pm$  be given in (4.9) and (4.10). In the case that  $0 \leq \underline{p}_+ < +\infty$ , we have

$$H^-(c_1+, p_+) < \lambda \leq H(c_1+, p_+ + \underline{p}_+),$$

which implies  $p_+ + \underline{p}_+ \geq \operatorname{argmin} H(c_1+, \cdot)$ . Together with

$$A < \lambda, \quad H(c_1+, \mu_+) = A$$

with  $\mu_+$  being the larger root of  $p \mapsto H(c_1+, p) - A$ , we deduce that  $p_+ + \underline{p}_+ > \mu_+$ . Note that  $\mu_+^\eta \rightarrow \mu_+$  as  $\eta \rightarrow 0^+$ . Therefore,  $p_+ + \underline{p}_+ > \mu_+^\eta$  for any sufficiently small  $\eta \geq 0$ . This yields  $\rho_*(s) > -\lambda + \phi_0^\eta(s - c_1)$  in a right neighborhood of  $c_1$  (which depends on  $\eta$ ). Moreover, this last statement is obviously valid if  $\underline{p}_+ = +\infty$ .

In the case that  $\underline{p}_- < +\infty$ , we could argue similarly to get for any sufficiently small  $\eta \geq 0$ ,  $\rho_*(s) > -\lambda + \phi_0^\eta(s - c_1)$  in a left neighborhood of  $c_1$ , which is clearly true in the case that  $\underline{p}_- = +\infty$ . As a consequence, the Claim 5.7 is proved.

Now for fixed  $\eta \in (0, \min\{\eta_0, \frac{\lambda-A}{4}\})$ , there exists  $r \in (0, 1)$ , such that  $-\lambda + \phi_0^\eta(s - c_1)$  touching  $\rho_*$  from below strictly at  $c_1$  in  $(c_1 - r, c_1 + r)$ . Letting  $Q_r$  be given in (5.13), we get

$$w_*(t, x) = t\rho_*(x/t) \geq \varphi^{0,\eta}(t, x) := \frac{A - \lambda}{4}(t - 1)^2 + t \left( -\lambda + \phi_0^\eta \left( \frac{x - c_1 t}{t} \right) \right) \quad \text{in } Q_r$$

with equality holds iff  $(t, x) = (1, c_1)$ . Then there exists  $\delta(\eta) > 0$  such that

$$w_*(t, x) \geq \varphi^{0,\eta}(t, x) + \delta \quad \text{on } \partial Q_r.$$

Define

$$\varphi^{\epsilon,\eta}(t, x) = \frac{A - \lambda}{4}(t - 1)^2 - t\lambda - \epsilon \log \left[ e^{-\frac{c_1}{2} \cdot \frac{x - c_1 t}{\epsilon}} \Phi^\eta \left( \frac{x - c_1 t}{\epsilon} \right) \right], \quad \text{for } (t, x) \in Q_r.$$

Clearly,  $\varphi^{\epsilon,\eta} \rightarrow \varphi^{0,\eta}$  in  $C_{loc}$  since

$$-\epsilon \log \left[ e^{-\frac{c_1}{2} \cdot \frac{x - c_1 t}{\epsilon}} \Phi^\eta \left( \frac{x - c_1 t}{\epsilon} \right) \right] \rightarrow \phi_0^\eta(x - c_1 t) = t\phi_0^\eta \left( \frac{x - c_1 t}{t} \right) \quad \text{locally uniformly.}$$

Therefore,

$$w^\epsilon(t, x) \geq \varphi^{\epsilon,\eta}(t, x) + \delta/2 \quad \text{on } \partial Q_r.$$

for sufficiently small  $\epsilon$ . Now we verify  $\varphi^{\epsilon,\eta} + \delta/2$  is a subsolution of (5.2). Indeed,

$$\begin{aligned} & \varphi_t^{\epsilon,\eta} - \epsilon \varphi_{xx}^{\epsilon,\eta} + |\varphi_x^{\epsilon,\eta}|^2 + g \left( \frac{x - c_1 t}{\epsilon} \right) - e^{-\frac{(2\varphi^{\epsilon,\eta} + \delta)}{2\epsilon}} \\ & \leq \frac{A - \lambda}{2}(t - 1) - \lambda + \Lambda_1^\eta - \frac{c_1^2}{4} + \|g - g_\eta\|_\infty, \\ & \leq \frac{A - \lambda}{2}(t - 1) - \lambda + A + \Lambda_1^\eta - \Lambda_1 + \|g - g_\eta\|_\infty \\ & \leq \frac{A - \lambda}{2}(t - 1) - \lambda + A + 2\eta \\ & \leq \frac{-\lambda + A}{2} + 2\eta < 0. \end{aligned}$$

It then follows from the maximum principle that, for all small  $\epsilon$ ,

$$w^\epsilon(t, x) - \delta/2 \geq \varphi^{\epsilon,\eta}(t, x) = \frac{A - \lambda}{4}(t - 1)^2 - t\lambda + \phi_0^\eta(x - c_1 t) + o(1) \quad \text{for } (t, x) \in Q_r. \quad (5.18)$$

Choose  $(t^\epsilon, x^\epsilon) \rightarrow (1, c_1)$  such that  $w^\epsilon(t^\epsilon, x^\epsilon) \rightarrow w_*(1, c_1) = \rho_*(c_1) = -\lambda$ . Evaluating (5.18) at  $(t^\epsilon, x^\epsilon)$  and then letting  $\epsilon \rightarrow 0$ , we again deduce that  $-\lambda - \delta/2 \geq \varphi^{0,\eta}(1, c_1) = -\lambda$ , which is a contradiction. This concludes the proof.  $\square$

*Proof of Proposition 5.4.* It is a direct consequence of Lemmas 5.5 and 5.6.  $\square$

## 5.2 Equivalence between Ishii solution and FL solution with $A = A_0$

This section is a special case of [36, Section 7] with the general Hamilton  $H(s, p)$  being discontinuous at  $c_1$ .

**Proposition 5.8.** *Let  $H(s, p) : [(0, \infty) \setminus \{c_1\}] \times \mathbb{R}$  be convex in  $p$  and such that  $H(c_1 \pm, p)$  are well-defined and coercive, and  $\operatorname{argmin} H(c_1+, \cdot) = \operatorname{argmin} H(c_1-, \cdot)$ .*

Define

$$\tilde{H}(s, p) = H(s, p) \quad \text{if } s \neq c_1, \quad \text{and} \quad \tilde{H}(c_1, p) := \max\{H(c_1-, p), H(c_1+, p)\}.$$

Then for any given nonnegative function  $\rho$ , it is a FL-supersolution (resp. FL-subsolution) to

$$\begin{cases} \min\{\rho, \rho + H(s, \rho')\} = 0 & \text{in } (0, \infty) \setminus \{c_1\}, \\ \min\{\rho(c_1), \rho(c_1) + \max\{A_0, H^-(c_1+, \rho'(c_1+)), H^+(c_1-, \rho'(c_1-))\}\} = 0, \end{cases} \quad (5.19)$$

with  $A_0 := \max\{\min_{\mathbb{R}} H(c_1+, \cdot), \min_{\mathbb{R}} H(c_1-, \cdot)\}$ , if and only if it is a viscosity supersolution (resp. subsolution) in the sense of Ishii to

$$\min\{\rho, \rho + \tilde{H}(s, \rho')\} = 0 \quad \text{in } (0, \infty), \quad (5.20)$$

where the definition of viscosity sub/supersolutions of (5.20) in sense of Ishii is given in Definition D.1.

Note also that FL-supersolution (resp. subsolution) with  $A \leq A_0$  is equivalent to the case  $A = A_0$ . For the particular Hamiltonians satisfying

$$H(c_1-, p) = -c_1 p + p^2 + g(-\infty) \quad \text{and} \quad H(c_1+, p) = -c_1 p + p^2 + g(+\infty). \quad (5.21)$$

that we consider in this paper, one has  $A_0 = \max\{g(-\infty), g(+\infty)\} - c_1^2/4$ .

*Proof of Proposition 5.8.* Denote  $\tilde{p}$  be the common value of  $\operatorname{argmin} H(c_1 \pm, \cdot)$ . First, we show sufficiency, i.e. super/subsolution in sense of Ishii implies FL-super/subsolution.

Let  $\rho$  be a viscosity supersolution of (5.20) in the sense of Ishii. Then  $\rho \geq 0$  for all  $s$ .

Let  $\psi(s)$  be a vertex test function touching  $\rho$  from below at  $s = c_1$ . Denote

$$\lambda = -\rho(c_1), \quad p_+ = \psi'(c_1+), \quad p_- = -\psi'(c_1-). \quad (5.22)$$

We need to show  $F_{A_0}(p_+, -p_-) \geq \lambda$ , where

$$F_{A_0}(p_+, -p_-) = \max\{A_0, H^-(c_1+, p_+), H^+(c_1-, -p_-)\}.$$

By the critical slope results (Lemma 4.7) and Remark 4.8), there exist  $\underline{p}_{\pm} \geq 0$  such that

$$H(c_1+, p_+ + \underline{p}_+) \geq \lambda \quad \text{and} \quad H(c_1-, -p_- - \underline{p}_-) \geq \lambda. \quad (5.23)$$

(These  $\underline{p}_{\pm}$  are given in (4.9)-(4.10). If any of them is infinite, then simply take a large enough positive number satisfying (5.23).)

If  $A_0 \geq \lambda$ , then  $F_{A_0}(p_+, -p_-) \geq A_0 \geq \lambda$ , and we are done.

If  $p_+ + \underline{p}_+ \leq \tilde{p}$  (resp.  $-p_- - \underline{p}_- \geq \tilde{p}$ ), then we are done, since

$$H^-(c_1+, p_+) \geq H^-(c_1+, p_+ + \underline{p}_+) = H(c_1+, p_+ + \underline{p}_+) \geq \lambda \quad (\text{resp. } H^+(c_1-, -p_-) \geq \lambda).$$



Henceforth, we assume

$$\begin{cases} A_0 < \lambda \leq \min\{H(c_1+, p_+ + \underline{p}_+), H(c_1-, -p_- - \underline{p}_-)\}, \\ -\infty < -p_- - \underline{p}_- < \tilde{p} < p_+ + \underline{p}_+ < +\infty. \end{cases} \quad (5.24)$$

By the definition of the critical slopes, the second line in (5.24) means that  $\rho - \tilde{\psi}(s)$  has a strict local minimum at  $s = c_1$ , where  $\tilde{\psi} \in C^1$  is the special smooth test function

$$\tilde{\psi}(s) = \psi(c_1) + \tilde{p}(s - c_1).$$

By solution property in the Ishii sense (see Definition D.1(b)), we have

$$A_0 = \max\{\min H(c_1\pm, \cdot)\} = \max\{H(c_1\pm, \tilde{p})\} \geq \lambda.$$

This is a contradiction with (5.24), and shows that  $\rho$  is FL-supersolution with flux limiter  $A_0$ .

Next, we show subsolution in sense of Ishii implies FL-subsolution.

Let  $\rho(c_1) > 0$  and let  $\psi \in C_{pw}^1$  be a vertex test function touching  $\rho$  from above at  $s = c_1$ . We need to show

$$\max\{A_0, H^-(c_1+, p_+), H^+(c_1-, -p_-)\} \leq \lambda, \quad (5.25)$$

where  $\lambda, p_+, p_-$  are as in (5.22), and  $H^-(s, \cdot)$  and  $H^+(s, \cdot)$  denotes the decreasing and increasing part of  $H(s, \cdot)$ , respectively.

By critical slope results in Lemma 4.5 and Remark 4.6 ( $\rho$  enjoys weak continuity property thanks to Lemma 4.4), there exist finite real numbers  $\bar{p}_\pm \leq 0$  (given by (4.7)-(4.8)) such that

$$H(c_1+, p_+ + \bar{p}_+) \leq \lambda \quad \text{and} \quad H(c_1-, -p_- - \bar{p}_-) \leq \lambda. \quad (5.26)$$

In particular, we deduce that

$$A_0 \leq \lambda. \quad (5.27)$$

Moreover, (5.26) also implies that at  $(t_0, 0)$ ,

$$\begin{aligned} H^-(c_1+, p_+) &\leq H^-(c_1+, p_+ + \bar{p}_+) \quad \text{since } H^-(c_1+, \cdot) \text{ is nonincreasing,} \\ &\leq H(c_1+, p_+ + \bar{p}_+) \leq \lambda. \end{aligned} \quad (5.28)$$

Similarly, we also obtain  $H^+(c_1-, -p_-) \leq \lambda$ . Combining with (5.27) and (5.28), we obtain (5.25). This proves that  $w$  is a FL-subsolution with  $A = A_0$ .

Next, we show the converse statement, i.e. FL-super/subsolution implies super/subsolution in sense of Ishii.

Let  $\rho$  be a FL-supersolution of (5.19), and let  $\psi$  be a  $C^1$  test function touching  $\rho$  from below at  $s = c_1$ . Then we have  $\rho \geq 0$  for all  $s$ , and

$$\max\{H(c_1+, \psi'(c_1)), H(c_1-, \psi'(c_1))\} \geq \max\{H^-(c_1+, \psi'(c_1)), H^+(c_1-, \psi'(c_1))\} \geq -\psi(c_1).$$

This proves that  $\rho$  is viscosity supersolution of (5.20) in the Ishii sense.

Finally, let  $\rho$  be a FL-subsolution of (5.19) with  $A = A_0$ , and  $\psi$  be a  $C^1$  test function touching  $\rho$  from below at  $s = c_1$ . Then we have

$$\tilde{H}(c_1, \psi'(c_1)) = \max\{H^-(c_1+, \psi'(c_1)), H^+(c_1-, \psi'(c_1))\} \leq -\rho(c_1).$$

Now, since  $\tilde{p} = \operatorname{argmin} H(c_1+, \cdot) = \operatorname{argmin} H(c_1-, \cdot)$ , we either have  $\psi'(c_1) \geq \tilde{p}$  or  $\psi'(c_1) < \tilde{p}$ . In the former case, we have  $H(c_1-, \psi'(c_1)) = H^+(c_1-, \psi'(c_1)) \leq -\rho(c_1)$ . In the latter case, we have  $H(c_1+, \psi'(c_1)) = H^-(c_1+, \psi'(c_1)) \leq -\rho(c_1)$ . This implies that

$$\tilde{H}_*(c_1, \psi'(c_1)) = \min\{H(c_1-, \psi'(c_1)), H(c_1+, \psi'(c_1))\} \leq -\rho(c_1).$$

i.e.  $\rho$  is a viscosity subsolution of (5.20) in the sense of Ishii.  $\square$

Next, we specialize to the class of Hamiltonian defined in (2.23), and prove the first part of Theorem 1.

*Proof of Theorem 1, first part.* We establish Theorem 1 in case  $\Lambda_1 \leq \max\{g(\pm\infty)\}$ . Then  $\Lambda_1 = \max\{g(\pm\infty)\}$  (thanks to Proposition 4.2(a)). By Proposition 5.4,  $\rho^*$  (resp.  $\rho_*$ ) is a FL-subsolution (resp. FL-supersolution) of (2.9) with

$$A = A_0 := \max\{g(\pm\infty)\} - \frac{c_1^2}{4}.$$

Thanks to Proposition 5.8,  $\rho^*$  and  $\rho_*$  are viscosity sub- and supersolution of (2.26) in the Ishii sense. Moreover, it follows from Lemma 5.3 that

$$\rho^*(0) = 0 \leq \rho_*(0), \quad \text{and} \quad \rho_*(s)/s \rightarrow +\infty \quad \text{as } s \rightarrow +\infty. \quad (5.29)$$

Hence, we may apply the comparison principle [40, Proposition 2.11] for viscosity solutions in the Ishii sense to deduce that

$$\rho^*(s) \leq \rho_*(s) \quad \text{for all } s \geq 0.$$

Since also  $\rho^* \geq \rho_*$  by construction (see (5.3)), we conclude that  $\rho^* \equiv \rho_*$ . We define  $\hat{\rho}_{A_0}$  to be the common value. This proves the existence and uniqueness of  $\hat{\rho}_{A_0}$  stated in Proposition 2.3. (Note that this also settles the case  $A \leq A_0$ , as they yield the same equation (2.9).)

Furthermore,  $w^\epsilon(t, x) \rightarrow t\hat{\rho}_{A_0}(x/t)$  in  $C_{loc}((0, \infty) \times (0, \infty))$ . Let  $s_{base} = \sup\{s \geq 0 : \hat{\rho}_{A_0}(s) = 0\}$ , then  $\hat{\rho}_{A_0}(s) > 0$  for  $s > s_{base}$ . This gives

$$u^\epsilon(t, x) = e^{-\frac{w^\epsilon(t, x)}{\epsilon}} \rightarrow 0 \quad \text{locally uniformly for } \{(t, x) : t > 0, x > s_{base}t\},$$

i.e.  $\bar{c}^* \leq s_{base}$ , where  $\bar{c}^*$  is the maximal spreading speed given in (2.1).

Next, we observe that  $\hat{\rho}_{A_0}$  is monotone increasing (Lemma 4.4(a)), so that  $\hat{\rho}_{A_0}(s) = 0$  for  $s \in [0, s_{base}]$  and hence

$$w^\epsilon(t, x) \rightarrow 0 \quad \text{in } C_{loc}(\{(t, x) : t > 0, 0 \leq x < s_{base}t\}).$$

It then follows as in [47, Lemma 3.1] that

$$\liminf_{\epsilon \rightarrow 0} \inf_K u^\epsilon(t, x) \geq \inf g > 0$$

for each compact subset  $K \subset \{(t, x) : t > 0, 0 \leq x < s_{base}t\}$ . For each  $\eta > 0$ , we may take  $K = \{(1, s) : \inf g/2 \leq s \leq s_{base} - \eta\}$ , and deduce

$$\liminf_{t \rightarrow \infty} \inf_{\frac{\inf g}{2}t \leq x \leq (s_{base} - \eta)t} u(t, x) = \liminf_{\epsilon \rightarrow 0} \inf_K u^\epsilon(t, x) > 0 \quad \text{for any } \eta > 0.$$

Since  $\eta > 0$  is arbitrary, this implies  $\underline{c}_* \geq s_{base}$ . Combining with  $\bar{c}_* \leq s_{base}$ , we obtain  $\bar{c}_* = \underline{c}_* = s_{base}$ . This concludes the proof of Theorem 1 in the case  $\Lambda_1 = \max\{g(\pm\infty)\}$ .  $\square$

Having verified that  $w^*$  and  $w_*$  are FL-subsolution and FL-supersolution of (2.9), one may apply the arguments in [36] to obtain a comparison principle. Here, however, we will follow the arguments due to Lions and Souganidis [44] to show that they are in fact viscosity sub- and supersolutions of certain Kirchhoff junction conditions, and establish the more general comparison principle (see Appendix C).

*Remark 5.9.* The concept of FL-sub/supersolutions was originally introduced in [36, 37], in which the authors established the comparison principle based on the construction of certain ‘‘vertex test functions’’.

### 5.3 Verification of Kirchhoff junction conditions

Let  $B \in \mathbb{R}$  be given. We consider the Hamilton-Jacobi equation with Kirchhoff junction condition:

$$\begin{cases} \min\{\rho, \rho + H(s, \rho')\} = 0 & \text{for } s \neq c_1, \\ \min\{\rho(c_1), \min\{\rho(c_1) + H(c_1 \pm, \rho'(c_1 \pm)), \rho'(c_1-) - \rho'(c_1+) - B\}\} \leq 0, \\ \min\{\rho(c_1), \max\{\rho(c_1) + H(c_1 \pm, \rho'(c_1 \pm)), \rho'(c_1-) - \rho'(c_1+) - B\}\} \geq 0. \end{cases} \quad (5.30)$$

The definition of viscosity solution to the above problem also involves the use of piecewise  $C^1$  test functions.

**Definition 5.10.** (a) We say that  $\underline{\rho}$  is a viscosity subsolution of (5.30) provided (i)  $\underline{\rho}$  is upper semicontinuous, and (ii) if  $\underline{\rho} - \psi$  has a local maximum point at some  $s_0$  such that  $\psi \in C_{pw}^1$  and  $\underline{\rho}(s_0) > 0$ , then

$$\begin{aligned} \underline{\rho}(s_0) + H(s_0, \psi'(s_0)) &\leq 0 && \text{in case } s_0 \neq c_1, \\ \min\{\underline{\rho}(c_1) + H(c_1 \pm, \psi'(c_1 \pm)), \psi'(c_1-) - \psi'(c_1+) - B\} &\leq 0 && \text{in case } s_0 = c_1. \end{aligned}$$

(b) We say that  $\bar{\rho}$  is a viscosity supersolution of (5.30) provided (i)  $\bar{\rho}$  is lower semicontinuous, (ii)  $\bar{\rho} \geq 0$  for all  $s$ , and (iii) if  $\bar{\rho} - \psi$  has a local minimum point at some  $s_0$  such that  $\psi \in C_{pw}^1$  then

$$\begin{aligned} \bar{\rho}(s_0) + H(s_0, \psi'(s_0)) &\geq 0 && \text{in case } s_0 \neq c_1, \\ \max\{\bar{\rho}(c_1) + H(c_1 \pm, \psi'(c_1 \pm)), \psi'(c_1-) - \psi'(c_1+) - B\} &\geq 0 && \text{in case } s_0 = c_1. \end{aligned}$$

(c) We say that  $\rho$  is a viscosity solution of (5.30) provided it is a viscosity subsolution and supersolution of (5.30).

Next, for each flux-limiter  $A \geq A_0$ , where  $A_0 = \max\{g(\pm\infty)\} - \frac{c_1^2}{4}$ , we associate a Kirchhoff junction condition parameter  $B$  as follows:

$$B = -\mu_+ - \mu_-,$$

where  $\mu_+, \mu_-$  are uniquely determined in terms of  $A$  by

$$H^+(c_1+, \mu_+) = A \quad \text{and} \quad H^-(c_1-, -\mu_-) = A. \quad (5.31)$$

By recalling the definition of  $H^+(c_1 \pm, p)$  and  $H^-(c_1 \pm, p)$  in (2.7) and (2.8), we deduce

$$\mu_+ = \frac{1}{2}(c_1 + \sqrt{c_1^2 + 4(A - g(+\infty))}), \quad \mu_- = \frac{1}{2}(-c_1 + \sqrt{c_1^2 + 4(A - g(-\infty))})$$

**Lemma 5.11.** *Let  $A \geq A_0 := \max\{g(\pm\infty)\} - \frac{c_1^2}{4}$ , and define  $\mu_{\pm}$  in terms of  $A$  as in (5.31). If  $\underline{\rho}$  is a FL-subsolution to (2.9), then it is a viscosity subsolution of the problem (5.30) with Kirchhoff junction condition with parameter  $B = -\mu_+ - \mu_-$ .*

*Proof.* It remains to show that  $\underline{\rho}$  is a subsolution to the second equation of (5.30). For this purpose, let  $\psi \in C_{pw}^1$  and suppose  $\underline{\rho} - \psi$  has a strict global maximum point at  $c_1$ , and that  $\psi(c_1) = \underline{\rho}(c_1) > 0$ . Denote

$$\lambda = -\underline{\rho}(c_1) = -\psi(c_1), \quad p_+ = \psi'(c_1+), \quad p_- = -\psi'(c_1-).$$

Suppose

$$H(c_1+, p_+) > \lambda \quad \text{and} \quad H(c_1-, p_-) > \lambda, \quad (5.32)$$

we need to show that

$$-p_- - p_+ + \mu_+ + \mu_- \leq 0. \quad (5.33)$$

Thanks to the critical slope lemma (Lemma 4.5),  $H(c_1+, p_+ + \bar{p}_+) \leq \lambda$  for some  $\bar{p}_+ < 0$ , it follows that  $p_+ \geq \operatorname{argmin} H(c_1+, \cdot)$ . Similarly, we have  $-p_- \leq \operatorname{argmin} H(c_1-, \cdot)$ .

By the definition of FL-subsolution (see Definition 2.2), it follows that  $\lambda \geq A$ . This, together with the fact that  $\mu_+$  (resp.  $-\mu_-$ ) is the larger (resp. smaller) root of  $p \mapsto H(c_1\pm, p) - A$ , implies

$$p_+ \geq \mu_+ \quad \text{and} \quad -p_- \leq -\mu_-. \quad (5.34)$$

Therefore, we obtain  $-p_+ - p_- + \mu_+ + \mu_- \leq 0$ .  $\square$

**Lemma 5.12.** *Let  $A \geq A_0$ . If  $\bar{\rho}$  is a FL-supersolution of (2.9), then it is a viscosity supersolution of the problem (5.30) with Kirchhoff condition with parameter  $B = -\mu_+ - \mu_-$ .*

*Proof.* It remains to verify the second condition of (5.30). Suppose there is a test function  $\psi \in C_{pw}^1$  that touches  $\bar{\rho}$  from below only at  $s = c_1$ .

Denote

$$\lambda = -\bar{\rho}(c_1), \quad p_+ = \psi'(c_1+), \quad p_- = -\psi'(c_1-).$$

Suppose

$$H(c_1+, p_+) < \lambda \quad \text{and} \quad H(c_1-, -p_-) < \lambda, \quad (5.35)$$

we need to show

$$-p_- - p_+ + \mu_+ + \mu_- \geq 0. \quad (5.36)$$

Since  $\bar{\rho}$  is a FL-supersolution, we have  $A \geq \lambda$ . Upon considering (5.35), and also  $\mu_+$  (resp.  $-\mu_-$ ) being the larger (resp. smaller) root of  $H^+(c_1+, p) = A$  (resp.  $H^-(c_1-, p) = A$ ), we deduce<sup>2</sup>

$$p_+ < \mu_+ \quad \text{and} \quad -p_- > -\mu_-. \quad (5.37)$$

This implies (5.36).  $\square$

**Corollary 5.13.**  $\rho^*$  (resp.  $\rho_*$ ) are viscosity subsolution (resp. supersolution) of (5.30) with  $B = -\mu_+ - \mu_-$  where  $\mu_\pm$  are associated with  $A = \Lambda_1 - \frac{c_1^2}{4}$  via (5.12).

*Proof.* Fix  $A = \Lambda_1 - \frac{c_1^2}{4} > \max\{g(\pm\infty)\} - \frac{c_1^2}{4}$ . Define  $\mu_+, \mu_-$  by (5.12).

By Proposition 5.4,  $\rho^*$  and  $\rho_*$  are FL-subsolution and supersolution of (2.9) with  $A = \Lambda_1 - \frac{c_1^2}{4}$ , respectively. By Lemmas 5.11 and 5.12, they are viscosity sub- and supersolutions of (5.30) with  $B = -\mu_+ - \mu_-$ .  $\square$

## 5.4 Proof of main results

*Proof of Proposition 2.3.* Recall that  $A_0 = \max\{g(\pm\infty)\} - \frac{c_1^2}{4}$ . Let  $\underline{\rho}$  and  $\bar{\rho}$  be a pair of FL-subsolution and FL-supersolution of (2.9) for some  $A \geq A_0$ , such that (2.11) holds.

If  $A \leq A_0$  holds, then by Proposition 5.8,  $\underline{\rho}, \bar{\rho}$  is a pair of viscosity sub- and supersolution of (2.26) in the sense of Ishii. The comparison principle follows from [40, Proposition 2.11].

Henceforth, we assume  $A > A_0$ . Then, it follows from Lemmas 5.11 and 5.12 that  $\underline{\rho}, \bar{\rho}$  are a pair of sub- and supersolutions of (5.30). The comparison principle then follows from Theorem C.5.  $\square$

---

<sup>2</sup>Note that we do not need  $\bar{p}_+ < +\infty$  here, comparing with the proof of the previous verification for junction subsolution. This asymmetry in the arguments of super and subsolutions is due to the fact that  $H$  is quasiconvex and coercive.

*Proof of Corollary 2.4 and Theorem 1.* For given function  $g$ , denote  $A = \Lambda_1 - \frac{c_*^2}{4}$ , where  $\Lambda_1$  is given in (2.5). By Proposition 4.2(a),  $A \geq A_0 := \max\{g(\pm\infty)\} - \frac{c_*^2}{4}$ .

For the case  $A = A_0$ , in view of Proposition 5.8, the problem is equivalent to (2.26) and (2.12). The existence and uniqueness of  $\hat{\rho}_{A_0}$  follows from [40, Proposition 1.7(b)]. The convergence  $w^\epsilon(t, x) \rightarrow t\hat{\rho}(x/t)$  in  $C_{loc}$  and determination of spreading speed is given in the first part of the proof of Theorem 1 (see Subsection 5.2).

For the case  $A > A_0$ , it follows that  $\rho^*$  and  $\rho_*$  are viscosity sub- and supersolutions of (5.30) and satisfies  $\rho^*(0) = \rho_*(0) = 0$  and  $\rho_*(s)/s \rightarrow +\infty$  as  $s \rightarrow +\infty$  (see Lemma 5.3). Hence, Theorem C.5 implies  $\rho^* \leq \rho_*$ . Arguing similarly as in the proof of the first part Theorem 1 (in Subsection 5.2), we conclude that (i) (2.9)-(2.12) has a unique FL-solution  $\hat{\rho}_A(s)$ ; (ii)  $w^\epsilon(t, x) \rightarrow t\hat{\rho}_A(x/t)$  in  $C_{loc}$ ; (iii) the spreading speed is given by  $c_* = \hat{s}_A = \sup\{s \geq 0 : \hat{\rho}_A(s) = 0\}$ .  $\square$

## A Proof of Proposition 4.2

*Proof of Proposition 4.2.* For (a), observe that if  $\phi'' + g(y)\phi \leq \Lambda\phi$  in  $\mathbb{R}$  for some positive  $\phi \in C_{loc}^2(\mathbb{R})$  and  $\Lambda < g(+\infty)$ , then  $\phi'' < 0$  for  $y \gg 1$ . Since  $\phi > 0$  in  $\mathbb{R}$ , we deduce that  $\phi' > 0$  for  $y \gg 1$ , and hence  $\phi'(+\infty) \in [0, +\infty)$  and  $\phi(+\infty) \in (0, +\infty]$  both exist. However, this means  $\limsup_{y \rightarrow \infty} \phi''(y) \leq (\Lambda - g(+\infty))\phi(+\infty) < 0$ , which contradicts  $\phi'(+\infty) \geq 0$ . This proves  $\Lambda_1 \geq g(+\infty)$ . Similarly, we can show that  $\Lambda_1 \geq g(-\infty)$ . Next, we apply [14, Theorem 1.4] to infer that (2.4) has a positive solution if and only if  $\Lambda \in [\Lambda_1, \infty)$ . (Note that  $\Lambda_1 = -\lambda_1(L, \mathbb{R})$  in the notation of [14].) This proves (a).

For (b)(i), we first apply [14, Proposition 1.11(ii)] to deduce that if  $\Lambda_1 > \max\{g(\pm\infty)\}$ , then  $\Lambda_1$  is a simple eigenvalue and  $\Phi_1$  converges exponentially to zero as  $|y| \rightarrow \infty$ . To establish (4.3), it suffices to prove the estimation of accurate decay rate of  $\Phi_1$  at  $+\infty$ . Since  $\Lambda_1 > g(+\infty)$ , there exists  $a_0 > 0$  such that  $\Lambda_1 > g(y)$  for any  $y \geq a_0$ . Now we define

$$\bar{\lambda}(a) := \sup_{y \geq a} \sqrt{\Lambda_1 - g(y)}, \quad \underline{\lambda}(a) := \inf_{y \geq a} \sqrt{\Lambda_1 - g(y)}, \quad \forall y \geq a_0.$$

Fix  $a \geq a_0$ . Then for any  $\epsilon > 0$  and  $M > 0$ , it is easy to check that  $\bar{\Phi}_\epsilon^M(y) = Me^{-\underline{\lambda}(a)y} + \epsilon$  is a supersolution of (2.4) on  $[a, +\infty)$ , that is,

$$(\bar{\Phi}_\epsilon^M)'' + (g(y) - \Lambda_1)\bar{\Phi}_\epsilon^M \leq 0 \text{ on } [a, +\infty).$$

For any given small  $\epsilon_0 > 0$ , there exists  $M_1 > e^{\underline{\lambda}(a)a}\Phi_1(a)$  such that  $\bar{\Phi}_{\epsilon_0}^{M_1}(y) > \Phi_1(y)$  on  $[a, +\infty)$ . Using the sliding argument or strong maximum principle, we infer that  $\bar{\Phi}_\epsilon^{M_1}(y) > \Phi_1(y)$  on  $[a, +\infty)$  for any  $\epsilon \in (0, \epsilon_0]$ . Letting  $\epsilon \rightarrow 0^+$ , we have  $M_1 e^{-\underline{\lambda}(a)y} \geq \Phi_1(y)$  on  $[a, +\infty)$ .

Similarly, for any  $M > 0$  and  $\epsilon \in (0, e^{-a\bar{\lambda}(a)})$ ,  $\underline{\Phi}_\epsilon^M(y) := \max\{0, M(e^{-\bar{\lambda}(a)y} - \epsilon)\}$  is a subsolution of (2.4) in  $(a, +\infty)$ , that is,

$$(\underline{\Phi}_\epsilon^M)'' + (g(y) - \Lambda_1)\underline{\Phi}_\epsilon^M \geq 0 \text{ on } [a, +\infty).$$

For any given small  $\epsilon_0 > 0$ , there exists  $M_2 > 0$  such that  $\underline{\Phi}_{\epsilon_0}^{M_2}(y) < \Phi_1(y)$  on  $[a, +\infty)$ . By the sliding argument again, it follows that  $\underline{\Phi}_\epsilon^{M_2}(y) < \Phi_1(y)$  on  $[a, +\infty)$  for any  $\epsilon \in (0, \epsilon_0]$ . Letting  $\epsilon \rightarrow 0^+$ , we have  $M_2 e^{-\bar{\lambda}(a)y} \leq \Phi_1(y)$  on  $[a, +\infty)$ . Consequently, there exist  $\underline{C} > 0, \bar{C} > 0$  (dependent on  $a$ ), such that

$$\underline{C}e^{-\bar{\lambda}(a)y} \leq \Phi_1(y) \leq \bar{C}e^{-\underline{\lambda}(a)y}, \quad \forall y \geq 0.$$

Noting that  $\bar{\lambda}(a)$  and  $\underline{\lambda}(a)$  are continuous on  $[a_0, +\infty)$  with  $\bar{\lambda}(+\infty) = \underline{\lambda}(+\infty) = \lambda_+$ . Therefore, for any sufficiently small  $\eta > 0$ , there exists  $a > a_0$  such that  $\lambda_+ - \eta \leq \underline{\lambda}(a) \leq \bar{\lambda}(a) \leq \lambda_+ + \eta$ . This implies the first inequality in (4.3) is valid.

For (b)(ii), suppose  $(\tilde{\Lambda}, \tilde{\Phi})$  is an eigenpair of (2.4). By the first assertion of the Proposition,  $\tilde{\Lambda} \in [\Lambda_1, \infty)$ . On the one hand, if  $\tilde{\Lambda} = \Lambda_1$ , then we can immediately conclude by the fact that  $\Lambda_1$  is simple (by [14, Proposition 1.1(ii)]).

On the other hand, if  $\tilde{\Lambda} > \Lambda_1$ , then one can prove that  $\tilde{\Phi}(y) \sim e^{-\sqrt{\tilde{\Lambda}-g(+\infty)y+o(y)}}$  as  $y \rightarrow +\infty$ . It then follows that  $\tilde{\Phi}(y)/\Phi_1(y) \rightarrow 0$  as  $y \rightarrow +\infty$ . By repeating the argument, we also obtain  $\tilde{\Phi}(y)/\Phi_1(y) \rightarrow 0$  as  $y \rightarrow -\infty$ . We can then touch  $\Phi_1(y)$  from below with  $k\tilde{\Phi}(y)$  to obtain, from the strong maximum principle, that  $\Phi_1(x) = k\tilde{\Phi}(x)$  for some  $k > 0$ . In this case  $\tilde{\Lambda} = \Lambda_1$ , a contradiction.

For (c), fix an arbitrary  $\eta > 0$  we choose, for each  $k \in \mathbb{N}$ , a continuous function such that

$$\begin{cases} g_\eta^k(x) = g(x), & |x| \geq k+1, \\ g(x) \leq g_\eta^k(x) \leq g(x) + \eta, & l \leq |x| \leq k+1, \\ g_\eta^k(x) = g(x) + \eta, & |x| \leq k. \end{cases}$$

If  $\Lambda_1(g_\eta^k) > \Lambda_1(g)$  for some  $k$ , we are done. Suppose to the contrary that  $\Lambda_1(g_\eta^k) \equiv \Lambda_1(g)$  for all  $k$ , and let  $\Phi_k \in C_{loc}^2$  be a positive eigenfunction of  $\Lambda_1(g_\eta^k)$ . By Harnack inequality, there exists a positive function  $C(R)$  independent of  $k$  such that

$$\frac{1}{C(R)} \leq \frac{\Phi_k(x)}{\Phi_k(0)} \leq C(R) \quad \text{for } k \in \mathbb{N}, |x| \leq R.$$

Normalizing by  $\Phi_k(0) = 1$ , we see that  $\{\Phi_k\}$  is bounded in  $C^2([-R, R])$  for each  $R$ . It follows that (up to a subsequence)  $\Phi_k$  converges in  $C_{loc}^1(\mathbb{R})$  to a positive eigenfunction  $\tilde{\Phi} \in C_{loc}^2(\mathbb{R})$  satisfying

$$\tilde{\Phi}'' + (g(x) + \eta)\tilde{\Phi} = \Lambda_1(g)\tilde{\Phi} \quad \text{in } \mathbb{R}.$$

By assertion (a), it follows that  $\Lambda_1(g) \geq \Lambda_1(g + \eta)$ , which is impossible since  $\Lambda_1(g + \eta) = \Lambda_1(g) + \eta$ .  $\square$

## B Proof of Lemmas 4.5 and 4.7

Following the same procedure in [36, Lemmas 2.9 and 2.10], it suffices to prove Lemma 4.7 without weak continuity condition and then show the finiteness of  $\bar{p}_+$  in Lemma 4.5 with weak continuity condition.

*Proof of Lemma 4.7.* By the definition of  $\underline{p}_+$ , we see that  $\underline{p}_+ \geq 0$ . For any sufficiently small  $\epsilon > 0$ , there exists  $r_\epsilon \in (0, \epsilon)$  such that

$$\bar{\rho}(s) \geq \varphi(s) + (\underline{p}_+ - \epsilon)(s - c_1) \quad \text{for } 0 \leq s - c_1 \leq r_\epsilon$$

and there exists  $s_\epsilon \in (c_1, c_1 + \frac{r_\epsilon}{2})$  such that

$$\bar{\rho}(s_\epsilon) < \varphi(s_\epsilon) + (\underline{p}_+ + \epsilon)(s_\epsilon - c_1).$$

Now construct a smooth function  $\Psi : \mathbb{R} \rightarrow [-1, 0]$  such that

$$\Psi(s) = \begin{cases} 0 & \text{for } s \in (-1/2, 1/2), \\ -1 & \text{for } s \notin (-1, +1) \end{cases}$$

and define

$$\Phi(s) = \varphi(s) + 2\epsilon\Psi_{r_\epsilon}(s) + \begin{cases} (\underline{p}_+ + \epsilon)(s - c_1) & \text{if } s \in U \cap (c_1, +\infty) \\ 0 & \text{if } s \in U \cap (-\infty, c_1] \end{cases}$$

with  $\Psi_{r_\epsilon}(s) = r_\epsilon \Psi((s - c_1)/r_\epsilon)$ . It then follows that  $\Phi(c_1) = \varphi(c_1) = \bar{\rho}(c_1)$  and

$$\begin{cases} \Phi(c_1 + r_\epsilon) = \varphi(c_1 + r_\epsilon) + (\underline{p}_+ - \epsilon)r_\epsilon \leq \bar{\rho}(c_1 + r_\epsilon), \\ \Phi(s_\epsilon) = \varphi(s_\epsilon) + (\underline{p}_+ + \epsilon)(s_\epsilon - c_1) > \bar{\rho}(s_\epsilon). \end{cases}$$

This implies that there exists a point  $\bar{s}_\epsilon \in (c_1, c_1 + r_\epsilon)$  such that  $\bar{\rho} - \Phi$  attains a local minimum at  $\bar{s}_\epsilon$ . Therefore, by the definition of viscosity supersolution and  $H(\cdot, p)$  is convex in  $p$ , we obtain

$$\Phi(\bar{s}_\epsilon) + \sup_{s \in (c_1, c_1 + r_\epsilon)} H(s, \Phi'(\bar{s}_\epsilon)) \geq \bar{\rho}(\bar{s}_\epsilon) + H^*(\bar{s}_\epsilon, \Phi'(\bar{s}_\epsilon)) \geq 0,$$

which yields

$$\Phi(\bar{s}_\epsilon) + \sup_{s \in (c_1, c_1 + r_\epsilon)} H(s, \varphi'(\bar{s}_\epsilon) + 2\epsilon\Psi'_{r_\epsilon}(\bar{s}_\epsilon) + \underline{p}_+ + \epsilon) \geq 0.$$

Letting  $\epsilon \rightarrow 0^+$ , we reach

$$\varphi(c_1) + H(c_1+, \varphi'(c_1) + \underline{p}_+) \geq 0.$$

Now the conclusion follows immediately from the fact that  $\bar{\rho}(c_1) = \varphi(c_1)$ .  $\square$

*Proof of Lemma 4.5.* We only show that  $\bar{p}_+ > -\infty$ . Without loss of generality, we might assume that  $\varphi(c_1) = \underline{\rho}(c_1) > 0$ . Suppose by contradiction that  $\bar{p}_+ = -\infty$ , then there exists  $p_n \rightarrow -\infty$  and  $r_n > 0$  such that

$$\varphi(s) + p_n(s - c_1) \geq \underline{\rho}(s) \text{ for } 0 \leq s - c_1 < r_n.$$

Modifying  $\varphi$  if necessary (e.g.,  $\varphi + (s - c_1)^2$ ), we could further assume that

$$\varphi(s) + p_n(s - c_1) > \underline{\rho}(s) \text{ for } 0 < s - c_1 \leq r_n. \quad (\text{B.1})$$

For fixed  $n$ , since  $\underline{\rho}$  satisfies the weak continuity condition (4.4), it then follows that there exists  $s_m \in (c_1, c_1 + r_n)$  such that  $s_m \rightarrow c_1$  and  $\underline{\rho}(s_m) \rightarrow \underline{\rho}(c_1)$  as  $m \rightarrow +\infty$ . Define

$$\Psi_m(s) = \varphi(s) + p_n(s - c_1) + \frac{(s_m - c_1)^2}{s - c_1}, \quad s > c_1.$$

For each  $m \in \mathbb{N}$ , there exists  $\hat{s}_m \in (c_1, c_1 + r_n]$  such that  $\Psi_m - \underline{\rho}$  attains the minimum at  $\hat{s}_m$ . Then

$$0(1) = \Psi_m(s_m) - \underline{\rho}(s_m) \geq \Psi_m(\hat{s}_m) - \underline{\rho}(\hat{s}_m) \geq \varphi(\hat{s}_m) + p_n(\hat{s}_m - c_1) - \underline{\rho}(\hat{s}_m). \quad (\text{B.2})$$

Suppose  $\hat{s}_m \rightarrow s_0 \neq c_1$  (up to a subsequence) as  $m \rightarrow \infty$ . Then letting  $m \rightarrow +\infty$  in (B.2), we have

$$0 \geq \varphi(s_0) + p_n(s_0 - c_1) - \liminf_{m \rightarrow \infty} \underline{\rho}(\hat{s}_m) > 0,$$

where the second inequality follows from (B.1) and the upper semicontinuity of  $\underline{\rho}$ . This is a contradiction. Therefore, we conclude that  $\hat{s}_m \rightarrow c_1$  and  $\underline{\rho}(\hat{s}_m) \rightarrow \underline{\rho}(c_1) > 0$  as  $m \rightarrow \infty$ . Now we might assume  $\underline{\rho}(\hat{s}_m) > 0$  for each  $m \in \mathbb{N}$ , by the definition of viscosity subsolution, we obtain

$$\underline{\rho}(\hat{s}_m) + H_* \left( \hat{s}_m, \varphi'(\hat{s}_m) + p_n - \frac{(s_m - c_1)^2}{(\hat{s}_m - c_1)^2} \right) \leq 0.$$

Note that  $\inf_{s \in (c_1, \hat{s}_m]} H(s, p) \leq H_*(\hat{s}_m, p)$ . Then we pass to the limit as  $m \rightarrow +\infty$  in the inequality above and get

$$\underline{\rho}(c_1) + H(c_1+, \varphi'(c_1+) + p_n^0) \leq 0,$$

where  $p_n^0 = p_n - \limsup_{m \rightarrow \infty} \frac{(s_m - c_1)^2}{(\hat{s}_m - c_1)^2} \in [-\infty, 0]$ . It then follows from  $\liminf_{p \rightarrow -\infty} H(c_1+, p) \geq 0$  that  $p_n^0 > -\infty$  and  $p_n^0$  is bounded from below by a constant  $C$  which only depends on  $H(c_1+, p)$ ,  $\underline{\rho}(c_1)$  and  $\varphi'(c_1+)$ . But this also implies  $p_n \geq C$ , which leads to a contradiction. This completes the proof of the finiteness of  $\bar{p}_+$ .  $\square$

## C Comparison Principle for problem with Kirchhoff condition

The comparison principle for FL-solutions was first proved by Imbert and Monneau [36, 37]. Subsequently, Lions and Souganidis gave an alternative proof by connecting it to the Kirchhoff junction condition [45, 44]. We combine the arguments of the latter and of [40] to prove a comparison result that allows for solutions that grows superlinearly.

Let  $\mathcal{P} = \{c_i\}_{i=1}^n$  for some  $0 < c_1 < \dots < c_n$  and  $B_i \in \mathbb{R}$  for all  $i$ . We establish a comparison principle for viscosity sub- and supersolutions of the Hamilton-Jacobi equation

$$\min\{\rho, \rho + H(s, \rho')\} = 0 \quad \text{in } (0, \infty) \setminus \{c_i\} \quad (\text{C.1})$$

with the following Kirchhoff junction condition at  $c_i$

$$\rho'(c_i-) - \rho'(c_i+) - B_i = 0 \quad \text{for } i = 1, \dots, n, \quad (\text{C.2})$$

and boundary conditions  $\rho(0) = 0$  and  $\rho(s)/s \rightarrow +\infty$  as  $s \rightarrow +\infty$ . Here, we assume the following for the Hamiltonian function  $H(s, p)$ .

(A1) For any given  $L > 0$  and each  $s_0 \in (\frac{1}{L}, L) \setminus \{c_i\}_{i=1}^n$ , there exists  $\delta_0 = \delta_0(L)$  and  $h_0 \in \{\pm 1\}$  such that

$$H^*(s, p) - H_*(s', p) \leq \omega_L(|s - s'|)(1 + |p|)$$

for all  $s, s'$  such that  $0 < |s - s_0| + |s' - s_0| < \delta_0$  and  $(s' - s)h_0 < 0$ . Here  $\omega_L : [0, \infty) \rightarrow [0, \infty)$  is a modulus of continuity for each  $L > 0$ , i.e. it is continuous with  $\omega_L(0) = 0$ , and  $H^*$  (resp.,  $H_*$ ) is the upper (resp., lower) semicontinuous envelope of  $H$  with respect to the first variable, that is,

$$H^*(s, p) = \limsup_{s' \rightarrow s} H(s', p) \quad (\text{resp., } H_*(s, p) = \liminf_{s' \rightarrow s} H(s', p)). \quad (\text{C.3})$$

(A2)  $p \mapsto H(s, p)$  is convex,  $\inf_{s > 0} H(s, 0) > 0$  and  $\lim_{|p| \rightarrow \infty} \left[ \inf_{s \in K} H(s, p) \right] \rightarrow \infty$  for each compact set  $K \subset (0, \infty)$ .

(A3) The one-sided limits  $H(c_i \pm, p)$  are well defined.

(A4) There exists  $\bar{s} > \max\{c_i\}_{i=1}^n$  such that  $H(s, p) = -sp + \tilde{H}(s, p) + R(s)$  for  $s \geq \bar{s}$  and such that  $\tilde{H}(s, p)$  is non-increasing in  $s \in [\bar{s}, \infty)$ . Moreover,  $R \in L^\infty$  is locally monotone in  $s \in [\bar{s}, \infty)$ .

*Remark C.1.* The definition of local monotonicity of  $R(s)$  is stated in (H3') of Subsection 2.4; see also [15]. Note also that (A1) implies that  $H(s, p)$  is locally monotone as a function of two variables.

*Remark C.2.* For our purpose, we will take  $H(s, p) = -sp + p^2 + R(s)$ , where  $R$  equals to positive constants on  $(0, c_1)$  and on  $(c_1, \infty)$ . It is obvious that (A1)-(A4) hold.

To define what it means by a viscosity solution to (C.1)–(C.2), we recall the space of piecewise  $C^1$  test functions

$$C_{pw}^1 = C_{loc}^0((0, \infty)) \cap C^1((0, c_1]) \cap C^1([c_1, c_2]) \cap \dots \cap C^1([c_{n-1}, c_n]) \cap C_{loc}^1([c_n, \infty)).$$

and use the notations in (C.3).

**Definition C.3.** (a) We say that  $\underline{\rho}$  is a viscosity subsolution to (C.1)–(C.2) if  $\underline{\rho}$  is upper semicontinuous on  $(0, +\infty)$  and it satisfies, in the viscosity sense,

$$\begin{cases} \min\{\rho, \rho + H(s, \rho')\} \leq 0 & \text{for } s \in (0, \infty) \setminus \mathcal{P}, \\ \min\{\rho(c_i), \min\{\rho'(c_i-) - \rho'(c_i+) - B_i, \rho(c_i) + H(c_i \pm, \rho'(c_i \pm))\}\} \leq 0 & \text{for each } 1 \leq i \leq n. \end{cases} \quad (\text{C.4})$$



i.e. whenever  $\underline{\rho} - \psi$  has a strict local maximum point at  $s_0$  for some  $\psi \in C_{pw}^1$  and  $\underline{\rho}(s_0) > 0$ , we have

$$\underline{\rho}(s_0) + H_*(s_0, \psi'(s_0)) \leq 0 \quad \text{if } s_0 \in (0, \infty) \setminus \mathcal{P},$$

$$\min\{\psi'(c_i-) - \psi'(c_i+) - B_i, \underline{\rho}(c_i) + H(c_i \pm, \psi'(c_i \pm))\} \leq 0 \quad \text{if } s_0 = c_i.$$

(b) We say that  $\bar{\rho}$  is a viscosity supersolution to (C.1)–(C.2) if  $\bar{\rho}$  is lower semicontinuous on  $(0, +\infty)$  and it satisfies, in the viscosity sense,

$$\begin{cases} \min\{\rho, \rho + H(s, \rho')\} \geq 0 & \text{for } s \in (0, \infty) \setminus \mathcal{P}, \\ \min\{\rho(c_i), \max\{\rho'(c_i-) - \rho'(c_i+) - B_i, \rho(c_i) + H(c_i \pm, \rho'(c_i \pm))\}\} \geq 0 & \text{for each } i. \end{cases} \quad (\text{C.5})$$

i.e.  $\bar{\rho} \geq 0$  for all  $s > 0$  and whenever  $\underline{\rho} - \psi$  has a strict local minimum point at  $s_0$  for some  $\psi \in C_{pw}^1$ , we have

$$\bar{\rho}(s_0) + H^*(s_0, \psi'(s_0)) \geq 0 \quad \text{if } s_0 \in (0, \infty) \setminus \mathcal{P},$$

$$\max\{\psi'(c_i-) - \psi'(c_i+) - B_i, \bar{\rho}(c_i) + H(c_i \pm, \psi'(c_i \pm))\} \geq 0 \quad \text{if } s_0 = c_i.$$

(c) We say that  $\bar{\rho}$  is a viscosity solution to (C.1)–(C.2), if it is both a viscosity subsolution and viscosity supersolution.

*Remark C.4.* The above setting includes the case with general  $r(x, t)$  with infinitely many shifts (in that case  $\tilde{H}_i(p) = p^2$  and  $R$  is locally monotone except possibly at  $c_i$ ), as well as the case when there is finitely many shifts, but periodic homogenization in between (in that case  $R \equiv 0$ ).

**Theorem C.5.** *Let  $\underline{\rho}$  and  $\bar{\rho}$  be, respectively, viscosity sub- and supersolutions of (C.1)–(C.2), such that*

$$\underline{\rho}(0) \leq 0 \leq \bar{\rho}(0), \quad \underline{\rho}(s) < \infty \quad \text{for all } s \geq 0, \quad \text{and} \quad \frac{\bar{\rho}(s)}{s} \rightarrow +\infty \quad \text{as } s \rightarrow +\infty.$$

*Then  $\underline{\rho}(s) \leq \bar{\rho}(s)$  in  $[0, +\infty)$ .*

*Remark C.6.* The above theorem directly implies the corresponding comparison principle for KL-super/subsolutions (see Proposition 2.3). This is because  $\bar{\rho}$  and  $\underline{\rho}$  being a pair FL-super/subsolution (for an arbitrarily given  $A \in \mathbb{R}$ ) implies the corresponding super/subsolution property with Kirchhoff junction condition (for some  $B = B(A)$ ); See Subsection 5.3.

*Proof of Theorem C.5.* First of all, we may assume without loss that  $\underline{\rho}$  is nonnegative,  $\underline{\rho}(0) = 0$  and  $\underline{\rho} \in \text{Lip}_{loc}([0, \infty))$ . This can be achieved by replacing  $\underline{\rho}$  by  $\max\{0, \underline{\rho}\}$ . Since  $0$  and  $\underline{\rho}$  (and hence also  $\max\{\underline{\rho}, 0\}$  is automatically a viscosity subsolution (in Ishii sense) to  $\min\{\rho, \rho + H(s, \rho')\} = 0$ , Lemma 4.5 is applicable and it is therefore locally Lipschitz continuous.

Suppose to the contrary that

$$\sup_{s \geq 0} [\underline{\rho}(s) - \bar{\rho}(s)] > 0. \quad (\text{C.6})$$

**Step #1.** We may assume without loss of generality that  $\underline{\rho}(s) - \bar{\rho}(s) \rightarrow -\infty$  as  $s \rightarrow +\infty$ .

If  $\limsup_{s \rightarrow \infty} \frac{\underline{\rho}(s)}{s} < \infty$ , then we are done. Otherwise, we proceed as in [40, Proposition 2.11]. First, observe that  $\underline{\rho} \in \text{Lip}_{loc}((0, \infty))$ . By Rademacher's theorem, it is differentiable in some  $[0, \infty) \setminus \mathcal{S}$  where  $\mathcal{S}$  has zero Lebesgue measure. Hence, we may choose  $s_k \in [\bar{s}, \infty)$ ,  $s_k \rightarrow \infty$  such that  $\underline{\rho}$  is differentiable at  $s_k$ , and

$$\inf_k \underline{\rho}(s_k) > 0 \quad \text{and} \quad R(s_k) \rightarrow \limsup_{s \rightarrow \infty} R(s).$$

Note that the latter is a consequence of local monotonicity of  $R$  thanks to (A4). Next, define

$$\underline{\rho}_k(s) := \begin{cases} \underline{\rho}(s) - \nu_k & \text{for } 0 \leq s \leq s_k, \\ \underline{\rho}(s_k) - \nu_k + \underline{\rho}'(s_k)(s - s_k) & \text{for } s > s_k, \end{cases}$$

where  $\nu_k = \sup_{[s_k, \infty)} R - R(s_k)$  (note that  $\nu_k \rightarrow 0$ ). Observe that  $\underline{\rho}_k$  is a viscosity subsolution in  $[0, \infty)$  with linear growth as  $s \rightarrow \infty$ . Indeed,  $\underline{\rho}_k$  is viscosity subsolution of (C.1) in  $[0, s_k)$  on the one hand, and classical subsolution of (C.1) in  $[s_k, \infty)$  on the other hand, since

$$\begin{aligned} \underline{\rho}_k + H(s, \underline{\rho}'_k) &= \underline{\rho}_k - s\underline{\rho}'_k + \tilde{H}(s, \underline{\rho}'_k) + R(s) \\ &= [a_k - \nu_k + b_k(s - s_k)] - sb_k + \tilde{H}(s, b_k) + R(s) \\ &\leq a_k - \nu_k - s_k b_k + \tilde{H}(s_k, b_k) + R(s) \\ &= (a_k - s_k b_k + \tilde{H}(s_k, b_k) + R(s_k)) + (R(s) - R(s_k) - \nu_k) \\ &\leq \underline{\rho}(s_k) + H(s_k, \underline{\rho}'(s_k)) \leq 0, \end{aligned}$$

where we adopted the notation  $a_k = \underline{\rho}(s_k)$ ,  $b_k = \underline{\rho}'(s_k)$ .

We can then replace  $\underline{\rho}$  by  $\underline{\rho}_k$ , if necessary. Note that (C.6) still holds provided  $k$  is sufficiently large, since  $\nu_k \rightarrow 0$  and  $s_k \rightarrow +\infty$ .

In the rest of the proof, we will show the comparison result is valid, i.e.

$$\max\{\underline{\rho}_k(s), 0\} \leq \bar{\rho}(s) \quad \text{in } [0, \infty), \quad \text{for all sufficiently large } k.$$

Granted, then we can take  $k \rightarrow \infty$  to deduce that  $\underline{\rho} \leq \bar{\rho}$ .

**Step #2.** For  $\lambda \in (0, 1)$ , define  $W(s) = \lambda \underline{\rho}(s) - \bar{\rho}(s)$ . Then choose  $\lambda \nearrow 1$ ,  $0 < s_0 < \bar{s}_0$  such that

$$\eta_0 := W(s_0) = \max_{s \in [0, \infty)} W(s) > \max \left\{ \sup_{[\bar{s}_0 - 1, \infty)} W, (1 - \lambda) \sup_{\substack{s \in [0, \bar{s}_0] \\ |p| \leq 2|s_0|}} |H(s, p)| \right\} \quad (\text{C.7})$$

For given  $1 \leq i \leq n$ , we consider two cases. Either (i) there is a sequence  $\lambda_j \nearrow 1$  such that  $s_0 \neq c_i$  for all  $j$ , or (ii) there exists a sequence  $\lambda_j \nearrow 1$  such that  $s_0 = c_i$  for all  $j$ . We first consider case (i),  $s_0 \neq c_i$ .

**Step #3.** Next, define

$$\Psi_\alpha(s, t) = \lambda \underline{\rho}(s + \alpha^{-1/2} h_0) - \bar{\rho}(t) - \frac{\alpha}{2} |s - t|^2 - \frac{1 - \lambda}{2} |s - s_0|^2, \quad (\text{C.8})$$

where  $h_0$  is given in (A1).

**Claim C.7.** *There exists  $\bar{\alpha} > 0$  such that if  $\alpha > \bar{\alpha}$ , then the following statements hold.*

- (i)  $\Psi_\alpha$  has an interior local maximum  $(s_1, t_1)$  in  $(\frac{s_0}{2}, \frac{\bar{s}_0 + s_0}{2}) \times (0, \bar{s}_0)$ .
- (ii)  $\Psi_\alpha(s_1, t_1) \geq \Psi_\alpha(s_0, s_0) = \eta_0 + o(1) > 0$ .
- (iii)  $\alpha |s_1 - t_1|^2 \rightarrow 0$  as  $\alpha \rightarrow \infty$ .
- (iv)  $(s_1, t_1) \rightarrow (s_0, s_0)$  as  $\alpha \rightarrow \infty$ .

Denote  $Q = [\frac{s_0}{2}, \frac{\bar{s}_0 + s_0}{2}] \times [0, \bar{s}_0]$ . Clearly,  $\Psi_\alpha$  is upper semicontinuous in  $Q$ , therefore, it attains the maximum at  $(s_1, t_1) \in Q$ . By (C.7) and  $\underline{\rho} \in \text{Lip}_{loc}((0, \infty))$ , it follows that

$$\sup_Q \Psi_\alpha \geq \Psi_\alpha(s_0, s_0) = \eta_0 + \lambda(\underline{\rho}(s_0 + \alpha^{-1/2} h_0) - \underline{\rho}(s_0)) = \eta_0 + o(1)$$

where  $o(1)$  is considered with respect to  $\alpha \rightarrow +\infty$ . This proves (ii). For (iii), first observe that  $\alpha|s_1 - t_1|^2 = O(1)$  which is a direct consequence of statement (ii). In particular,  $(s_1, t_1) \rightarrow (\hat{s}, \hat{s})$  for some  $\hat{s} \in [\frac{s_0}{2}, \frac{\bar{s}_0 + s_0}{2}]$ . By (ii), we can write

$$\frac{\alpha}{2}|s_1 - t_1|^2 \leq -W(s_0) + [W(s_1) + \bar{\rho}(s_1)] - \bar{\rho}(t_1) - \frac{1-\lambda}{2}|s_1 - s_0|^2 + o(1).$$

Since  $s \mapsto W(s) + \bar{\rho}(s)$  and  $t \mapsto -\bar{\rho}(t)$  are both upper semicontinuous, we can take  $\alpha \rightarrow \infty$  to obtain

$$0 \leq \limsup_{\alpha \rightarrow \infty} \frac{\alpha}{2}|s_1 - t_1|^2 \leq -W(s_0) + W(\hat{s}) - \frac{1-\lambda}{2}|\hat{s} - s_0|^2 \leq 0,$$

where the last inequality follows from the fact that  $W$  attains global maximum at  $s_0$ . This proves (iii) and (iv). Finally, statement (iv) yields  $(s_1, t_1)$  must be an interior point of  $Q$ . This proves (i).

**Step #4.** Fixing  $t = t_1$ , observe that for  $\alpha > \bar{\alpha}$ ,  $\underline{\rho}(s_1 + \alpha^{-1/2}h_0) \geq \eta_0 + \bar{\rho}(t_1) + o(1) > 0$  since  $\bar{\rho} \geq 0$ , and  $s \mapsto \underline{\rho}(s) - \varphi(s)$  attains a local maximum at  $s = \hat{s}_1 := s_1 + \alpha^{-1/2}h_0$ , where

$$\varphi(s) = \frac{1}{\lambda} \left[ \bar{\rho}(t_1) + \frac{\alpha}{2}|s - \alpha^{-1/2}h_0 - t_1|^2 + \frac{1-\lambda}{2}|s - \alpha^{-1/2}h_0 - s_0|^2 \right].$$

By definition of viscosity subsolution, we have

$$\underline{\rho}(\hat{s}_1) + H_* \left( \hat{s}_1, \frac{\alpha(s_1 - t_1)}{\lambda} + \frac{1-\lambda}{\lambda}(s_1 - s_0) \right) \leq 0. \quad (\text{C.9})$$

Using the convexity of  $H_*$ , we have

$$\lambda H_* \left( \hat{s}_1, \frac{\alpha(s_1 - t_1)}{\lambda} + \frac{1-\lambda}{\lambda}(s_1 - s_0) \right) + (1-\lambda) H_* (\hat{s}_1, -(s_1 - s_0)) \geq H_* (\hat{s}_1, \alpha(s_1 - t_1)).$$

Substitute into (C.9)

$$\lambda \underline{\rho}(\hat{s}_1) + H_* (\hat{s}_1, \alpha(s_1 - t_1)) \leq (1-\lambda) H_* (\hat{s}_1, -(s_1 - s_0)) \leq (1-\lambda) \sup_{\substack{s \in [0, \bar{s}_0] \\ |p| \leq 2|\bar{s}_0|}} H(s, p). \quad (\text{C.10})$$

Next, we fix  $s = s_1$  and observe that  $t \mapsto \bar{\rho}(t) - \psi(t)$  has an interior local minimum at  $t = t_1$ , where

$$\psi(t) = \lambda \underline{\rho}(s_1) - \frac{\alpha}{2}|t - s_1|^2 - \frac{1-\lambda}{2}|s_1 - s_0|^2.$$

Hence,

$$\bar{\rho}(t_1) + H^*(t_1, \alpha(s_1 - t_1)) \geq 0 \quad (\text{C.11})$$

since  $s_0 \notin \mathcal{P}$  and thus  $t_1 \notin \mathcal{P}$ . Combining (C.10) and (C.11), we have

$$\lambda \underline{\rho}(\hat{s}_1) - \bar{\rho}(t_1) \leq H^*(t_1, \alpha(s_1 - t_1)) - H_* (\hat{s}_1, \alpha(s_1 - t_1)) + (1-\lambda) \sup_{\substack{s \in [0, \bar{s}_0] \\ |p| \leq 2|\bar{s}_0|}} H(s, p) \quad (\text{C.12})$$

**Step #5.** Observe that  $\lambda \underline{\rho}(\hat{s}_1) - \bar{\rho}(t_1) \geq \Psi_\alpha(s_1, t_1) \geq \eta_0 + o(1)$ . Using also  $|h_0| = 1$  (by (A1)) and  $\alpha|s_1 - t_1|^2 = o(1)$  (by Claim C.7(iii)), we have

$$(t_1 - \hat{s}_1)h_0 = [t_1 - s_1 - \alpha^{-1/2}h_0]h_0 \leq |t_1 - s_1| - \alpha^{-1/2} < 0.$$

We can then apply (A1) to (C.12) to get

$$o(1) + \eta_0 \leq \omega_L(|t_1 - s_1 - \alpha^{-1/2}h_0|(\alpha|t_1 - s_1| + 1)) + (1-\lambda) \sup_{s \in [0, \bar{s}_0], |p| \leq 2|\bar{s}_0|} H(s, p). \quad (\text{C.13})$$

Using  $\alpha|s_1 - t_1|^2 = o(1)$  and

$$|t_1 - s_1 - \alpha^{-1/2}h_0|(\alpha|t_1 - s_1| + 1) \leq (\alpha^{1/2}|t_1 - s_1| + 1)(\alpha^{1/2}|t_1 - s_1| + \alpha^{-1/2}),$$

we let  $\alpha \rightarrow \infty$  in (C.13) to obtain

$$\eta_0 \leq (1 - \lambda) \sup_{s \in [0, \bar{s}_0], |p| \leq 2|\bar{s}_0|} H(s, p),$$

which is a contradiction. This concludes the proof when  $W(s) = \lambda \underline{\rho}(s) - \bar{\rho}(s)$  attains a local maximum at  $s_0$ , such that  $s_0 \neq c_i$  for all  $i$ .

It remains to consider the case (ii) (see Step #2), when there is  $\lambda_j \nearrow 1$  such that  $\lambda_j \underline{\rho} - \bar{\rho}$  has a global maximum at  $s_0 = c_i$  for some  $i$ . Since this holds for a sequence of  $\lambda_j \nearrow 1$ , we reduce to the case that  $\underline{\rho} - \bar{\rho}$  attains the global maximum at  $c_i$  for some  $i$ . For convenience, let's assume  $s_0 = c_1$ . Next, define

$$a = \underline{\rho}(c_1) \quad \text{and} \quad b = \bar{\rho}(c_1)$$

and assume to the contrary that  $a > b \geq 0$ . (We will show that  $a \leq b$  so there is a contradiction.)

**Step #6.** We claim that the critical slopes of  $\underline{\rho}$ , given as follows, are finite.

$$p_- = \sup\{\bar{p} \in \mathbb{R} : \exists r > 0, \underline{\rho}(c_1) + \bar{p}(s - c_1) \geq \underline{\rho}(s) \text{ for } -r < s - c_1 \leq 0\} \quad (\text{C.14})$$

$$p_+ = \sup\{\bar{p} \in \mathbb{R} : \exists r > 0, \underline{\rho}(c_1) - \bar{p}(s - c_1) \geq \underline{\rho}(s) \text{ for } 0 \leq s - c_1 < r\}. \quad (\text{C.15})$$

Indeed, they are finite because  $\underline{\rho}$  is locally Lipschitz continuous. Moreover, we have

$$\begin{cases} a + H(c_1-, p_-) \leq 0, & a + H(c_1+, -p_+) \leq 0, & \text{and} \\ \min\{p'_- + p'_+ - B_1, a + H(c_1\pm, \mp p'_\pm)\} \leq 0 & \text{for } (p'_-, p'_+) \in (-\infty, p_-] \times (-\infty, p_+], \end{cases} \quad (\text{C.16})$$

where the former is due to Lemma 4.5 and Remark 4.6 (note that  $\underline{\rho} \in \text{Lip}_{loc}$ , so there exists at least one test function in  $C_{pw}^1$ ), while the latter holds by considering the test function  $\psi(s) = \underline{\rho}(c_1) + p'_- \min\{s - c_1, 0\} - p'_+ \max\{s - c_1, 0\}$ , which touches  $\underline{\rho}$  from above at  $c_1$ .

Next, define

$$p_-^* = \limsup_{s \rightarrow c_1-} \frac{\underline{\rho}(s) - \underline{\rho}(c_1)}{s - c_1} \quad \text{and} \quad p_+^* = -\liminf_{s \rightarrow c_1+} \frac{\underline{\rho}(s) - \underline{\rho}(c_1)}{s - c_1}. \quad (\text{C.17})$$

Note that

$$p_-^* \geq p_- \quad \text{and} \quad p_+^* \geq p_+, \quad (\text{C.18})$$

$$\text{since } p_- = \liminf_{s \rightarrow c_1-} \frac{\underline{\rho}(s) - \underline{\rho}(c_1)}{s - c_1} \quad \text{and} \quad p_+ = -\limsup_{s \rightarrow c_1+} \frac{\underline{\rho}(s) - \underline{\rho}(c_1)}{s - c_1}.$$

**Step #7.** We improve (C.16) to

$$\begin{cases} a + H(c_1-, p_-^*) \leq 0, & a + H(c_1+, -p_+^*) \leq 0, & \text{and} \\ \min\{p'_- + p'_+ - B_1, a + H(c_1\pm, \mp p'_\pm)\} \leq 0 & \text{for } (p'_-, p'_+) \in (-\infty, p_-^*] \times (-\infty, p_+^*]. \end{cases} \quad (\text{C.19})$$

Indeed, since  $\underline{\rho}$  is locally Lipschitz,  $\underline{\rho}'$  exists a.e. and  $\underline{\rho}(s) - \underline{\rho}(c_1) = \int_{c_1}^s \underline{\rho}'(t) dt$ , the definition of  $p_+^*$  implies that, for each  $\delta > 0$ , the set  $\{s \in (c_1, c_1 + \delta) : \underline{\rho}'(s) < -p_+^* + \delta\}$  has positive measure. This implies that there is a sequence  $s_k \searrow c_1$  such that  $\underline{\rho}$  is differentiable at  $s_k$  and also that  $\limsup_{k \rightarrow \infty} \underline{\rho}'(s_k) \leq -p_+^*$ . Hence, letting  $k \rightarrow \infty$  in  $\underline{\rho}(s_k) + H(s_k, \underline{\rho}'(s_k)) \leq 0$ , we obtain

$$a + H(c_1+, \liminf_{k \rightarrow \infty} \underline{\rho}'(s_k)) \leq 0.$$

Noting that  $H$  is convex in  $p$  variable,  $\liminf_{k \rightarrow \infty} \underline{\rho}'(s_k) \leq -p_+^* \leq -p_+$ , and using the first part of (C.16), we deduce

$$a + H(c_1+, -p'_+) \leq 0 \quad \text{for all } p'_+ \in [p_+, p_+^*]. \quad (\text{C.20})$$

By a completely similar argument, we also have

$$a + H(c_1-, p'_-) \leq 0 \quad \text{for all } p'_- \in [p_-, p_-^*]. \quad (\text{C.21})$$

Combining (C.20) and (C.21) into (C.16), we obtain (C.19).

**Step #8.** We claim that the critical slopes of  $\bar{\rho}$ , defined as follows, are well-defined but possibly equals  $-\infty$ .

$$q_- = \inf\{\bar{q} \in \mathbb{R} : \exists r > 0, \bar{\rho}(c_1) + \bar{q}(s - c_1) \leq \bar{\rho}(s) \text{ for } -r < s - c_1 \leq 0\} \quad (\text{C.22})$$

$$q_+ = \inf\{\bar{q} \in \mathbb{R} : \exists r > 0, \bar{\rho}(c_1) - \bar{q}(s - c_1) \leq \bar{\rho}(s) \text{ for } 0 \leq s - c_1 < r\}. \quad (\text{C.23})$$

Indeed,

$$\underline{\rho}(s) - \bar{\rho}(s) \leq \underline{\rho}(c_1) - \bar{\rho}(c_1) \quad \text{for all } s, \quad \text{with equality holds at } s = c_1, \quad (\text{C.24})$$

i.e. the locally Lipschitz function  $\underline{\rho}(s) - \underline{\rho}(c_1) + \bar{\rho}(c_1)$  touches  $\bar{\rho}(s)$  from below at  $s = c_1$ . This shows that  $q_-$  and  $q_+$  are well-defined in  $\overline{\mathbb{R}} \cup \{-\infty\}$ .

Next, we observe that

$$q_- \leq p_-^* \quad \text{and} \quad q_+ \leq p_+^*, \quad (\text{C.25})$$

which is due to (C.17), (C.24), and

$$q_- = \limsup_{s \rightarrow c_1^-} \frac{\bar{\rho}(s) - \bar{\rho}(c_1)}{s - c_1} \quad \text{and} \quad q_+ = -\liminf_{s \rightarrow c_1^+} \frac{\bar{\rho}(s) - \bar{\rho}(c_1)}{s - c_1}.$$

**Step #9.** Suppose  $q_{\pm} > -\infty$ , then we have

$$\begin{cases} b + H(c_1-, q_-) \geq 0, & b + H(c_1+, -q_+) \geq 0 \quad \text{and} \\ \max\{q'_- + q'_+ - B_1, b + H(c_1\pm, \mp q'_{\pm})\} \geq 0, & \text{for } (q'_-, q'_+) \in [q_-, \infty) \times [q_+, \infty), \end{cases} \quad (\text{C.26a})$$

$$\quad (\text{C.26b})$$

where the former holds by virtue of the critical slope lemma (Lemma 4.7 and Remark 4.8), and the latter holds by considering the  $C_{pw}^1$  test function  $\psi(s) = \bar{\rho}(c_1) + q'_- \min\{s - c_1, 0\} - q'_+ \max\{s - c_1, 0\}$ .

If  $q_- = -\infty$  (resp.  $q_+ = -\infty$ ) then take  $q_-$  (resp.  $q_+$ ) large and negative enough (but finite) to satisfy both (C.25) and (C.26a). Then, for any  $(q'_-, q'_+) \in [q_-, \infty) \times [q_+, \infty)$ , the test function  $\psi(s) = \bar{\rho}(c_1) + q'_- \min\{s - c_1, 0\} - q'_+ \max\{s - c_1, 0\}$  touches  $\bar{\rho}$  at  $s = c_1$ . Then it follows that (C.26b) holds.

**Step #10.**

In view of (C.19), (C.25), and (C.26), we may apply the Lemma C.8 with  $(H_1(p) = H(c_1-, p)$  and  $H_2(p) = H(c_1+, -p)$ ,  $p \in \mathbb{R}$ ,  $p_1 = p_-^*$ ,  $p_2 = p_+^*$ ,  $q_1 = q_-$  and  $q_2 = q_+$ ) to conclude that  $a \leq b$ . This is a contradiction to the assumption that  $a > b$ . The proof is now complete.  $\square$

The following key lemma is due to Lions and Souganidis [44].

**Lemma C.8.** *Assume that  $H_1, \dots, H_m \in C(\mathbb{R})$ ,  $p_1, \dots, p_m, q_1, \dots, q_m \in \mathbb{R}$ , and  $a, b \in \mathbb{R}$  are such that, for all  $i = 1, \dots, m$ ,*

1.  $p_i \geq q_i$ ,  $a + H_i(p_i) \leq 0 \leq b + H_i(q_i)$  for all  $i$ ,
2.  $\min(\sum_i p'_i - B, \min_i(a + H_i(p'_i))) \leq 0$  for each  $p'_i \in (-\infty, p_i]$ ,

3.  $\max(\sum_i q'_i - B, \max_i(b + H_i(q'_i))) \leq 0$  for each  $q'_i \in [q_i, \infty)$ .

Then  $a \leq b$ .

*Remark C.9.* By replacing  $p_1$  by  $p_1 + B$  and  $q_1$  by  $q_1 + B$ , and redefining

$$H_1(\cdot) \quad \text{to be} \quad H_1(\cdot + B),$$

one can reduce Lemma C.8 to the case  $B = 0$ , which is exactly [44, Lemma 3.1].

## D Definition of viscosity solution in sense of Ishii

**Definition D.1.** Let  $\hat{\rho} : (0, \infty) \rightarrow \mathbb{R}$ .

(a) We say that  $\hat{\rho}$  is a viscosity subsolution of

$$\min\{\rho, \rho + \tilde{H}(s, \rho')\} = 0 \quad \text{for } s \geq 0 \quad (\text{D.1})$$

in the sense of Ishii provided (i)  $\hat{\rho}$  is upper semicontinuous, (ii) if  $\hat{\rho} - \psi$  has a local maximum point at  $s_0 > 0$  such that  $\psi \in C^1$  and  $\hat{\rho}(s_0) > 0$ , then

$$\hat{\rho}(s_0) + \tilde{H}_*(s, \psi'(s_0)) \leq 0,$$

where  $\tilde{H}_*(s, p)$  is the lower envelope of  $\tilde{H}(s, p)$ , i.e.

$$\tilde{H}_*(s, p) = \begin{cases} -sp + p^2 + g(-\infty) & \text{for } s < c_1, \\ \tilde{H}(c_1-, p) \wedge \tilde{H}(c_1+, p) = -c_1p + p^2 + \min\{g(-\infty), g(+\infty)\} & \text{for } s = c_1, \\ -sp + p^2 + g(+\infty) & \text{for } s > c_1. \end{cases}$$

(b) We say that  $\hat{\rho}$  is a viscosity supersolution of (D.1) in the sense of Ishii provided (i)  $\hat{\rho}$  is lower semicontinuous, (ii)  $\hat{\rho} \geq 0$  for all  $s > 0$ , (iii) if  $\hat{\rho} - \psi$  has a local minimum point at  $s_0 > 0$  such that  $\psi \in C^1$ , then

$$\hat{\rho}(s_0) + \tilde{H}^*(s, \psi'(s_0)) \geq 0,$$

where  $\tilde{H}^*(s, p)$  is the upper envelope of  $\tilde{H}(s, p)$ , i.e.

$$\tilde{H}^*(s, p) = \begin{cases} -sp + p^2 + g(-\infty) & \text{for } s < c_1, \\ \tilde{H}(c_1-, p) \vee \tilde{H}(c_1+, p) = -c_1p + p^2 + \max\{g(-\infty), g(+\infty)\} & \text{for } s = c_1, \\ -sp + p^2 + g(+\infty) & \text{for } s > c_1. \end{cases}$$

(c) We say that  $\hat{\rho}$  is a viscosity solution of (D.1) in the sense of Ishii if it is both subsolution and supersolution of (2.26) in the sense of Ishii.

## References

- [1] D. G. ARONSON AND H. F. WEINBERGER, *Multidimensional nonlinear diffusion arising in population genetics*, Adv. in Math., 30 (1978), pp. 33–76.
- [2] G. BARLES, *An introduction to the theory of viscosity solutions for first-order Hamilton-Jacobi equations and applications*, in Hamilton-Jacobi equations: approximations, numerical analysis and applications, vol. 2074 of Lecture Notes in Math., Springer, Heidelberg, 2013, pp. 49–109.
- [3] G. BARLES AND E. CHASSEIGNE, *On Modern Approaches of Hamilton-Jacobi Equations and Control Problems with Discontinuities: A Guide to Theory, Applications, and Some Open Problems*, vol. 104, Springer Nature, 2023.

- [4] G. BARLES AND B. PERTHAME, *Discontinuous solutions of deterministic optimal stopping time problems*, RAIRO Modél. Math. Anal. Numér., 21 (1987), pp. 557–579.
- [5] ———, *Exit time problems in optimal control and vanishing viscosity method*, SIAM J. Control Optim., 26 (1988), pp. 1133–1148.
- [6] ———, *Comparison principle for Dirichlet-type Hamilton-Jacobi equations and singular perturbations of degenerated elliptic equations*, Appl. Math. Optim., 21 (1990), pp. 21–44.
- [7] H. BERESTYCKI, O. DIEKMANN, C. J. NAGELKERKE, AND P. A. ZEGELING, *Can a species keep pace with a shifting climate?*, Bull. Math. Biol., 71 (2009), pp. 399–429.
- [8] H. BERESTYCKI AND J. FANG, *Forced waves of the Fisher-KPP equation in a shifting environment*, J. Differential Equations, 264 (2018), pp. 2157–2183.
- [9] H. BERESTYCKI, F. HAMEL, AND G. NADIN, *Asymptotic spreading in heterogeneous diffusive excitable media*, J. Funct. Anal., 255 (2008), pp. 2146–2189.
- [10] H. BERESTYCKI AND G. NADIN, *Spreading speeds for one-dimensional monostable reaction-diffusion equations*, J. Math. Phys., 53 (2012), pp. 115619, 23.
- [11] H. BERESTYCKI AND G. NADIN, *Asymptotic spreading for general heterogeneous fisher-kpp type equations*, Mem. Amer. Math. Soc., (2022).
- [12] H. BERESTYCKI AND L. ROSSI, *Reaction-diffusion equations for population dynamics with forced speed. I. The case of the whole space*, Discrete Contin. Dyn. Syst., 21 (2008), pp. 41–67.
- [13] ———, *Reaction-diffusion equations for population dynamics with forced speed. II. Cylindrical-type domains*, Discrete Contin. Dyn. Syst., 25 (2009), pp. 19–61.
- [14] ———, *Generalizations and properties of the principal eigenvalue of elliptic operators in unbounded domains*, Comm. Pure Appl. Math., 68 (2015), pp. 1014–1065.
- [15] X. CHEN AND B. HU, *Viscosity solutions of discontinuous Hamilton-Jacobi equations*, Interfaces Free Bound., 10 (2008), pp. 339–359.
- [16] F.-D. DONG, J. SHANG, W. FAGAN, AND B. LI, *Persistence and spread of solutions in a two-species Lotka-Volterra competition-diffusion model with a shifting habitat*, SIAM J. Appl. Math., 81 (2021), pp. 1600–1622.
- [17] Y. DU, Y. HU, AND X. LIANG, *A climate shift model with free boundary: enhanced invasion*, J. Dynam. Differential Equations, 35 (2023), pp. 771–809.
- [18] Y. DU, L. WEI, AND L. ZHOU, *Spreading in a shifting environment modeled by the diffusive logistic equation with a free boundary*, J. Dynam. Differential Equations, 30 (2018), pp. 1389–1426.
- [19] A. DUCROT, T. GILETTI, J.-S. GUO, AND M. SHIMOJO, *Asymptotic spreading speeds for a predator-prey system with two predators and one prey*, Nonlinearity, 34 (2021), pp. 669–704.
- [20] L. C. EVANS, *Periodic homogenisation of certain fully nonlinear partial differential equations*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics, 120 (1992), pp. 245–265.
- [21] L. C. EVANS AND P. E. SOUGANIDIS, *A PDE approach to geometric optics for certain semilinear parabolic equations*, Indiana Univ. Math. J., 38 (1989), pp. 141–172.
- [22] J. FANG, Y. LOU, AND J. WU, *Can pathogen spread keep pace with its host invasion?*, SIAM J. Appl. Math., 76 (2016), pp. 1633–1657.
- [23] J. FANG, X. YU, AND X.-Q. ZHAO, *Traveling waves and spreading speeds for time-space periodic monotone systems*, J. Funct. Anal., 272 (2017), pp. 4222–4262.
- [24] G. FAYE, T. GILETTI, AND M. HOLZER, *Asymptotic spreading for Fisher-KPP reaction-diffusion equations with heterogeneous shifting diffusivity*, Discrete Contin. Dyn. Syst. Ser. S, 15 (2022), pp. 2467–2496.

- [25] M. FREIDLIN, *Limit theorems for large deviations and reaction-diffusion equations*, Ann. Probab., 13 (1985), pp. 639–675.
- [26] M. I. FREIDLIN AND T.-Y. LEE, *Wave front propagation and large deviations for diffusion-transmutation process*, Probab. Theory Related Fields, 106 (1996), pp. 39–70.
- [27] J. GARNIER, T. GILETTI, AND G. NADIN, *Maximal and minimal spreading speeds for reaction diffusion equations in nonperiodic slowly varying media*, J. Dynam. Differential Equations, 24 (2012), pp. 521–538.
- [28] Y. GIGA AND N. HAMAMUKI, *Hamilton-Jacobi equations with discontinuous source terms*, Comm. Partial Differential Equations, 38 (2013), pp. 199–243.
- [29] L. GIRARDIN, T. GILETTI, AND H. MATANO, *Spreading properties of the fisher-kpp equation when the intrinsic growth rate is maximal in a moving patch of bounded size*, 2024.
- [30] L. GIRARDIN AND K.-Y. LAM, *Invasion of open space by two competitors: spreading properties of monostable two-species competition-diffusion systems*, Proc. Lond. Math. Soc. (3), 119 (2019), pp. 1279–1335.
- [31] J. GUERAND, *Effective nonlinear Neumann boundary conditions for 1D nonconvex Hamilton-Jacobi equations*, J. Differential Equations, 263 (2017), pp. 2812–2850.
- [32] F. HAMEL, *Reaction-diffusion problems in cylinders with no invariance by translation. II. Monotone perturbations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 14 (1997), pp. 555–596.
- [33] F. HAMEL AND G. NADIN, *Spreading properties and complex dynamics for monostable reaction-diffusion equations*, Comm. Partial Differential Equations, 37 (2012), pp. 511–537.
- [34] M. HOLZER AND A. SCHEEL, *Accelerated fronts in a two-stage invasion process*, SIAM J. Math. Anal., 46 (2014), pp. 397–427.
- [35] C. HU, J. SHANG, AND B. LI, *Spreading speeds for reaction-diffusion equations with a shifting habitat*, J. Dynam. Differential Equations, 32 (2020), pp. 1941–1964.
- [36] C. IMBERT AND R. MONNEAU, *Flux-limited solutions for quasi-convex Hamilton-Jacobi equations on networks*, Ann. Sci. Éc. Norm. Supér. (4), 50 (2017), pp. 357–448.
- [37] ———, *Quasi-convex Hamilton-Jacobi equations posed on junctions: the multi-dimensional case*, Discrete Contin. Dyn. Syst., 37 (2017), pp. 6405–6435.
- [38] H. ISHII, *Hamilton-Jacobi equations with discontinuous Hamiltonians on arbitrary open sets*, Bull. Fac. Sci. Engrg. Chuo Univ., 28 (1985), pp. 33–77.
- [39] A. N. KOLMOGOROV, I. G. PETROVSKII, AND N. S. PISKUNOV, *Etude de l'équation de diffusion avec accroissement de la quantité de matière, et son application à un problème biologique*, Bjul. Moskowskogo Gos. Univ., 17 (1937), pp. 1–26.
- [40] K.-Y. LAM AND X. YU, *Asymptotic spreading of KPP reactive fronts in heterogeneous shifting environments*, J. Math. Pures Appl. (9), 167 (2022), pp. 1–47.
- [41] B. LI, S. BEWICK, J. SHANG, AND W. F. FAGAN, *Persistence and spread of a species with a shifting habitat edge*, SIAM J. Appl. Math., 74 (2014), pp. 1397–1417.
- [42] X. LIANG AND X.-Q. ZHAO, *Asymptotic speeds of spread and traveling waves for monotone semi-flows with applications*, Comm. Pure Appl. Math., 60 (2007), pp. 1–40.
- [43] P.-L. LIONS, G. PAPANICOLAOU, AND S. R. VARADHAN, *Homogenization of hamilton-jacobi equations*, Unpublished preprint, (1987).
- [44] P.-L. LIONS AND P. SOUGANIDIS, *Well-posedness for multi-dimensional junction problems with Kirchoff-type conditions*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 28 (2017), pp. 807–816.



- [45] P.-L. LIONS AND P. E. SOUGANIDIS, *Viscosity solutions for junctions: well posedness and stability*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl., 27 (2016), pp. 535–545.
- [46] Q. LIU, S. LIU, AND K.-Y. LAM, *Stacked invasion waves in a competition-diffusion model with three species*, J. Differential Equations, 271 (2021), pp. 665–718.
- [47] S. LIU, Q. LIU, AND K.-Y. LAM, *Asymptotic spreading of interacting species with multiple fronts II: Exponentially decaying initial data*, J. Differential Equations, 303 (2021), pp. 407–455.
- [48] A. B. POTAPOV AND M. A. LEWIS, *Climate and competition: the effect of moving range boundaries on habitat invasibility*, Bull. Math. Biol., 66 (2004), pp. 975–1008.
- [49] W. SHEN, *Variational principle for spreading speeds and generalized propagating speeds in time almost periodic and space periodic KPP models*, Trans. Amer. Math. Soc., 362 (2010), pp. 5125–5168.
- [50] J.-B. WANG, W.-T. LI, F.-D. DONG, AND S.-X. QIAO, *Recent developments on spatial propagation for diffusion equations in shifting environments*, Discrete Contin. Dyn. Syst. Ser. B, 27 (2022), pp. 5101–5127.
- [51] H. F. WEINBERGER, *Long-time behavior of a class of biological models*, SIAM J. Math. Anal., 13 (1982), pp. 353–396.
- [52] T. YI AND X.-Q. ZHAO, *Propagation dynamics for monotone evolution systems without spatial translation invariance*, J. Funct. Anal., 279 (2020), pp. 108722, 50.
- [53] ———, *Global dynamics of evolution systems with asymptotic annihilation*, Discrete Contin. Dyn. Syst., 43 (2023), pp. 2693–2720.

This figure "image001.png" is available in "png" format from:

<http://arxiv.org/ps/2410.14007v1>