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On the set of bad primes in the study of the Casas–Alvero conjecture

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On the set of bad primes in the study of the Casas–Alvero conjecture

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Dedicated to Professor Osamu Saeki on the occasion of his sixtieth birthday

Abstract

The Casas–Alvero conjecture predicts that every univariate polynomial over a field of characteristic zero having a common factor with each of its derivatives $H_i(f)$ is a power of a linear polynomial. One approach to proving the conjecture is to first prove it for polynomials of some small degree d , compile a list of bad primes for that degree (namely, those primes p for which the conjecture fails in degree d and characteristic p) and then deduce the conjecture for all degrees of the form dp^ℓ , $\ell \in \mathbb{N}$, where p is a good prime for d . In this paper we calculate certain distinguished monomials appearing in the resultant $R(f, H_i(f))$ and obtain a (non-exhaustive) list of bad primes for every degree $d \in \mathbb{N} \setminus \{0\}$.

1 Introduction

In the year 2001 Eduardo Casas–Alvero published a paper on higher order polar germs of plane curve singularities [1]. His work on polar germs inspired him to make the following conjecture (according to the testimony of José Manuel Aroca, E. Casas communicated the problem orally well before 2001).

Let K be a field, d a strictly positive integer and $f = x^d + a_1x^{d-1} + \dots + a_{d-1}x + a_d$ a monic univariate polynomial of degree d over K . Let

$$H_i(f) = \binom{d}{i}x^{d-i} + \binom{d-1}{i}a_1x^{d-i-1} + \dots + \binom{i}{i}a_{d-i}$$

be the i -th Hasse derivative of f .

Definition 1 *The polynomial f is said to be a **Casas–Alvero polynomial** if for each $i \in \{1, \dots, d-1\}$ it has a non-constant common factor with its i -th Hasse derivative $H_i(f)$.*

Note that, by definition, a Casas-Alvero polynomial f has a common root with $H_{d-1}(f)$. In particular, if $\text{char } K = 0$, it has at least one root $b \in K$, regardless of whether or not K is algebraically closed. Making the change of variables $x \rightsquigarrow x - b$, we may assume that 0 is a root of f , in other words, $a_d = 0$. In the sequel, we will always make this assumption without mentioning it explicitly.

Conjecture 1 (Casas–Alvero) *Assume that $\text{char } K = 0$. If $f \in K[x]$ is a Casas-Alvero polynomial of degree d with $a_d = 0$, then $f(x) = x^d$.*

For $i \in \{1, \dots, d-1\}$, let $R_i = R(f, H_i(f)) \in K[a_1, \dots, a_{d-1}]$ be the resultant of f and $H_i(f)$. The polynomials f and $H_i(f)$ have a common factor if and only if $R_i = 0$. Thus f is Casas–Alvero if and only if the point $(a_1, \dots, a_{d-1}) \in V(R_1, \dots, R_{d-1}) \subset K^{d-1}$. In those terms the Conjecture can be reformulated as follows:

Conjecture 2 *Let $V = V(R_1, \dots, R_{d-1}) \subset K^{d-1}$. Then $V = \{0\}$. In other words,*

$$\sqrt{(R_1, \dots, R_{d-1})} = (a_1, \dots, a_{d-1}) \quad (1)$$

or, equivalently,

$$a_i^N \in (R_1, \dots, R_{d-1}) \text{ for all } i \in \{1, \dots, d-1\} \text{ and some } N \in \mathbb{N}. \quad (2)$$

If $\text{char } K = p > 0$, the Conjecture is false in general. The simplest counterexample is the polynomial $f(x) = x^{p+1} - x^p$.

Remark 2 *Let $K \subset K'$ be a field extension. The induced extension*

$$K[a_1, \dots, a_{d-1}] \subset K'[a_1, \dots, a_{d-1}]$$

is faithfully flat. Since the polynomials R_1, \dots, R_{d-1} have coefficients in \mathbb{Z} , (2) holds in $K[a_1, \dots, a_{d-1}]$ if and only if it holds in $K'[a_1, \dots, a_{d-1}]$. Hence the truth of the conjecture depends only on the characteristic of K but not on the choice of the field K itself.

Remark 3 *Formulae (1) and (2) can be interpreted in terms of Gröbner bases. Namely, (1) and (2) are equivalent to saying that for any choice of monomial ordering and Gröbner basis (f_1, \dots, f_s) of (R_1, \dots, R_{d-1}) , after renumbering the f_j , the leading monomial of f_j is a power of a_j .*

We will write $\text{CA}_{d,p}$ for the statement “The Casas-Alvero conjecture holds for polynomials of degree d over fields of characteristic p ”.

The following equivalences are known for each d ([5], [6]) :

$\text{CA}_{d,0}$ holds \iff $\text{CA}_{d,p}$ holds for some prime number $p \iff$ $\text{CA}_{d,p}$ holds for all but finitely many primes p .

Definition 4 *A prime number p is said to be a **bad prime for d** if $\text{CA}_{d,p}$ is false. If p is not a bad prime for d , we will say that p is a **good prime for d** .*

Proposition 5 ([6], Propositions 2.2 and 2.6) *Take a strictly positive integer d , a prime number p and a non-negative integer ℓ . Assume that $\text{CA}_{d,p}$ holds. Then so do $\text{CA}_{dp^\ell, p}$ and $\text{CA}_{dp^\ell, 0}$.*

This result suggests the following general approach to the problem :

- (1) prove the conjecture for a small number d ;
- (2) compile lists of good and bad primes for d ;
- (3) conclude that $CA_{dp^\ell,0}$ holds for all the primes p that are known to be good for d .

In particular, this shows the importance of knowing which primes are good or bad for a given degree d .

The above approach has been carried out up to $d \leq 7$ ([2], [3], [4], [5], [6]). Some integers cannot be written in the form dp^ℓ where p is a good prime for d . For example,

$$12 = 2^2 \cdot 3, \quad 20 = 2^2 \cdot 5, \quad 24 = 2^3 \cdot 3, \quad 28 = 2^2 \cdot 7, \quad 30 = 2 \cdot 3 \cdot 5, \quad 36 = 2^2 \cdot 3^2, \quad 40 = 2^3 \cdot 5, \dots$$

$CA_{12,0}$ has been proved by [2] with the aid of a computer, by using a very clever strategy to cut down the computation of resultants and Gröbner basis. Thus the smallest degree d for which $CA_{d,0}$ is not known is $d = 20$.

In this paper we show that for each $i \in \{1, \dots, d-1\}$, the monomials $\left(1 - \binom{d}{i}\right)^{d-i} a_{d-i}^d$ and $(-1)^{(d-1)(d-i)} \binom{d}{i}^{d-1} a_{d-1}^{d-i} a_{d-i}$ appear in the resultant R_i (unless $i = 1$ in which case the two monomials are the same and the coefficient is $(1-d)^{d-1}$). Moreover, the monomials a_{d-i}^d are the only pure powers appearing in any of the R_i . We then use these facts to compile a (non-exhaustive) list of bad primes for each $d \in \mathbb{N}_{>0}$, namely all the primes p for which there exists $i \in \{1, \dots, d-1\}$ such that $p \mid \binom{d}{i} - 1$.

Acknowledgements. The fact that the monomial $\left(1 - \binom{d}{i}\right)^{d-i} a_{d-i}^d$ appears in R_i and is the only pure power appearing there was first proved by Rosa de Frutos' in her Ph.D. thesis [3], Proposition 2.2.1, page 17.

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2 A list of bad primes

Unless otherwise specified, from now till the end of this paper we shall regard the R_i as elements of the polynomial ring $\mathbb{Z}[a_1, \dots, a_{d-1}]$.

Theorem 6 ([3], Proposition 2.2.1) *For each $i \in \{1, \dots, d-1\}$, the monomial $(-1)^{d-i} \left(\binom{d}{i} - 1\right)^{d-i} a_{d-i}^d$ appears in the resultant R_i . Moreover, the monomials a_{d-i}^d are the only pure powers appearing in any of the R_i .*

Proof: The polynomial R_i is the resultant of

$$f = x^d + a_{d-1}x^{d-1} + \dots + a_{d-1}x$$

and

$$H_i(f) = \binom{d}{i} x^{d-i} + \binom{d-i-1}{i} a_1 x^{d-i-1} + \dots + \binom{i+1}{i} a_{d-i-1} x + \binom{i}{i} a_{d-i}.$$

Notation. For $i, j \in \{1, \dots, d-1\}$, we denote by \tilde{a}_{ij} the element $\binom{d-j}{i} a_j$.

Note that for all $i \in \{1, \dots, d-1\}$, $\tilde{a}_{i,d-i} = a_{d-i}$.

The resultant R_i equals the determinant $D(d, i)$ of the following matrix $M(d, i)$:

$$M(d, i) = \begin{pmatrix} \overbrace{\begin{matrix} 1 & a_1 & a_2 & \cdots & a_{d-i-1} \\ 0 & 1 & a_1 & \cdots & a_{d-i-2} \\ \vdots & \vdots & & & \\ 0 & \cdots & 0 & \cdots & 1 \end{matrix}}^{d-i} & \overbrace{\begin{matrix} a_{d-i} & \cdots & a_{d-1} & 0 & \cdots & \cdots & 0 \\ a_{d-i-1} & a_{d-i} & \cdots & a_{d-1} & 0 & \cdots & 0 \\ & & & & & \vdots & \vdots \\ a_1 & a_2 & \cdots & a_{d-i} & \cdots & a_{d-1} & 0 \end{matrix}}^d \\ \hline \binom{d}{i} & \tilde{a}_{i,1} & \tilde{a}_{i,2} & \cdots & \tilde{a}_{i,d-i-1} & & \\ 0 & \binom{d}{i} & \tilde{a}_{i,1} & \cdots & \tilde{a}_{i,d-i-2} & & \\ \vdots & \vdots & & & & & \\ \vdots & \vdots & & & & & \\ \vdots & \vdots & & & & & \\ \vdots & \vdots & & & & & \\ \vdots & \vdots & & & & & \\ 0 & 0 & \cdots & \cdots & 0 & & \\ & & & \cdots & 0 & \binom{d}{i} \tilde{a}_{i,1} & \tilde{a}_{i,2} & \cdots & a_{d-i} \end{pmatrix}$$

By definition, the determinant $D(d, i)$ of the $(2d-i) \times (2d-i)$ matrix $M(d, i) = (m_{\ell j})$ is

$$\Delta = \sum_{\sigma \in \Sigma_{2d-i}} (-1)^{\epsilon(\sigma)} m_{\sigma(1),1} m_{\sigma(2),2} \cdots m_{\sigma(2d-i),2d-i}, \quad (3)$$

where Σ_{2d-i} is the group of permutations of $\{1, \dots, 2d-i\}$ and

$$\begin{aligned} \epsilon(\sigma) &= 0 && \text{if } \sigma \text{ is even} \\ &= 1 && \text{if } \sigma \text{ is odd.} \end{aligned}$$

First of all, note that the last column of $M(d, i)$ has only one non-zero entry that equals a_{d-i} . Hence $a_{d-i} \mid D(d, i)$. In particular no pure power of a_j can appear in $D(d, i)$ for $j \neq d-i$.

Remark 7 *The entry a_{d-i} appears only in the last d columns of $M(d, i)$: exactly once in each of the last i columns and exactly twice in each of the columns numbered $d-i+1, d-i+2, \dots, 2d-2i$.*

By inspection of the matrix $M(d, i)$, we see that

- (1) a monomial ω appearing in $D(d, i)$ cannot be divisible by a_{d-i}^{d+1}
- (2) if $a_{d-i}^d \nmid \omega$, then ω is not a pure power of a_{d-i}
- (3) if $a_{d-i}^d \mid \omega$, then in the notation of formula (3), $\omega = (-1)^{\epsilon(\sigma)} m_{\sigma(1),1} \cdots m_{\sigma(2d-i),2d-i}$ with $\sigma(j) = j$ for $j \in \{2d-2i+1, \dots, 2d-1\}$ and $\sigma(j) \in \{j, j-d+i\}$ for $j \in \{d-i+1, \dots, 2d-2i\}$.

The term in (3) corresponding to $\sigma = \text{Id}$ is the product of the elements on the main diagonal of $M(d, i)$; this product is equal to a_{d-i}^d . There are other choices of $\sigma \in \Sigma_d$ for which the corresponding summand in (3) is of the form ca_{d-i}^d , where $c \in \mathbb{Z}$. Let us group all of these terms together and calculate the total coefficient of a_{d-i}^d in (3)

Now, the coefficient of a_{d-i}^d in $D(d, i)$ is given by the coefficient of a_{d-i}^{d-i} in the expansion of the determinant $\tilde{D}(d, i)$ of the $(2d-2i) \times (2d-2i)$ -matrix $\tilde{M}(d, i)$ obtained by deleting the last i lines and the last i columns of $M(d, i)$.

The matrix $\tilde{M}(d, i)$ is

$$\begin{pmatrix} 1 & a_1 & a_2 & \dots & a_{d-i-1} & a_{d-i} & \dots & a_{d-1} & 0 \\ 0 & 1 & a_1 & \dots & a_{d-i-2} & a_{d-i-1} & a_{d-i} & \dots & a_{d-1} \\ \vdots & \vdots & & & \vdots & & & \vdots & \vdots \\ 0 & \dots & 0 & \dots & 1 & a_1 & a_2 & \dots & a_{d-i} \\ \binom{d}{i} & \tilde{a}_{i,1} & \tilde{a}_{i,2} & \dots & \tilde{a}_{i,d-i-1} & a_{d-i} & 0 & \dots & 0 \\ 0 & \binom{d}{i} & \tilde{a}_{i,1} & \dots & \tilde{a}_{i,d-i-2} & \tilde{a}_{i,d-i-1} & a_{d-i} & 0 & \dots \\ \vdots & \vdots & & & \vdots & & & \vdots & \vdots \\ 0 & \dots & 0 & \binom{d}{i} & \dots & \dots & \tilde{a}_{i,d-i-1} & a_{d-i} & 0 \\ 0 & 0 & \dots & \dots & \binom{d}{i} & \dots & \dots & \tilde{a}_{i,d-i+1} & a_{d-i} \end{pmatrix}$$

In this determinant, the first $d-i$ columns do not contain any a_{d-i} and in the last $d-i$ columns, each a_{d-i} appears two times, once in the first $d-i$ rows, once in the last $d-i$ rows.

In each of the last $d-i$ columns we have to choose one of the two a_{d-i} and delete the rest of the line and the rest of the column to which it belongs. Fix one such choice. The corresponding monomial $(-1)^{\epsilon(\sigma)} m_{\sigma(1),1} m_{\sigma(2),2} \dots m_{\sigma(2d-2i),2d-2i}$, $\sigma \in \Sigma_{2d-2i}$, satisfies $\sigma(j) \in \{j, j-d+i\}$ for all $j \in \{d-i+1, \dots, 2d-2i\}$.

Let

$$J = \{\sigma(j) \mid j \in \{d-i+1, \dots, 2d-2i\}\} \quad (4)$$

and $J^c = \{1, \dots, 2d-2i\} \setminus J$. Write $J = \{j_1, \dots, j_{d-i}\} \subset \{1, \dots, 2(d-i)\}$.

$$\text{For all } q, \ell \in \{1, \dots, d-i\} \text{ we have } j_q - j_\ell \neq d-i. \quad (5)$$

The set J^c has the same property. Note that, conversely, every set

$$J = \{j_1, \dots, j_{d-i}\} \subset \{1, \dots, 2(d-i)\}$$

satisfying (5) has the form (4) for a suitable $\sigma \in \Sigma_{2d-2i}$.

The coefficient of the term a_{d-i}^{d-i} in the expansion of $\tilde{D}(d, i)$ corresponding to a given choice of J is the determinant of the matrix $N(d, i, J)$ obtained from the first $(d-i)$ columns of $\tilde{M}(d, i)$ by deleting the rows numbered $j_1, \dots, j_{d-i-1}, j_{d-i}$.

Let

$$k = \#(J^c \cap \{d-i+1, d-i+2, \dots, 2(d-i)\}). \quad (6)$$

There exists a permutation of the rows of $N(d, i, J)$ such that the resulting matrix is an upper triangular matrix with only 1 and $\binom{d}{i}$ on the main diagonal, where 1 appears $(d-i-k)$ times and $\binom{d}{i}$ appears k times. Thus the permutation $\sigma \in \Sigma_{2d-2i}$ is uniquely determined by $J \cap \{d-i+1, \dots, 2d-2i\}$.

We have

$$\det N(d, i, J) = \pm \binom{d}{i}^k. \quad (7)$$

Now, σ is the composition of k transpositions $(j, j-d+i)$ for $j \in J^c \cap \{d-i+1, \dots, 2d-2i\}$. Thus

$$\epsilon(\sigma) \equiv k \pmod{2}. \quad (8)$$

Example To illustrate the process, let us take

$$J = \{d-i+1, d-i+2, \dots, 2(d-i)-3, d-i-2, 2(d-i)-1, 2(d-i)\},$$

1 which means that we chose all the occurrences of a_{d-i} lying on the main diagonal in the last
2 $(d-i)$ rows of $\tilde{M}(d, i)$ except in the column number $2(d-i) - 2$ in which case we chose the
3 occurrence of a_{d-i} at the place $(d-i-2, 2d-2i-2)$. We have $k = 1$.

4 The resulting matrix $N(d, i, J)$ looks like

$$5 \begin{pmatrix} 6 & 1 & a_1 & a_2 & \dots & \dots & \dots & a_{d-i-1} \\ 7 & 0 & 1 & a_1 & \dots & \dots & \dots & a_{d-i-2} \\ 8 & \vdots & \vdots & \vdots & & & & \vdots \\ 9 & 0 & \dots & 0 & 1 & a_1 & a_2 & a_3 \\ 10 & 0 & \dots & \dots & 0 & 0 & 1 & a_1 \\ 11 & 0 & \dots & \dots & \dots & 0 & 0 & 1 \\ 12 & 0 & \dots & \dots & 0 & \binom{d}{i} & \tilde{a}_{i,1} & \tilde{a}_{i,2} \end{pmatrix}$$

13 To obtain an upper triangular matrix, we have to apply a cyclic permutation to the rows $d-i$,
14 $d-i-1$ and $d-i-2$ and we obtain that the desired determinant is $\pm \binom{d}{i}$.

15 Coming back to the proof of the Theorem, for each $k \in \{0, \dots, d-i\}$, there are $\binom{d-i}{k}$ choices
16 of J satisfying (6). Combining this with (7) and (8) and summing over all $k \in \{0, \dots, d-i\}$, we
17 get that the coefficient of a_{d-i}^d in R_i is

$$18 \sum_{k=0}^{d-i} (-1)^k \binom{d-i}{k} \binom{d}{i}^k = (-1)^{d-i} \left(\binom{d}{i} - 1 \right)^{d-i} \quad (9)$$

19 \square

20 **Corollary 8** Take a prime number p such that there exists $i \in \{1, \dots, d-1\}$ for which $p \mid \binom{d}{i} - 1$.
21 Then $CA_{d,p}$ is false.

22 Proof. Assume that $\text{char}(K) = p$. By Theorem 6, no pure power of any of the a_i appears in any
23 of the $R_j \pmod{p}$. Hence the point of K^{d-1} whose i -th coordinate is 1 and all of whose other
24 coordinates are zero belongs to $V(R_1, \dots, R_{d-1})$. \square

25 Using similar arguments, we obtain the following Theorem.

26 **Theorem 9** For $i \in \{2, \dots, d-1\}$, the monomial $(-1)^{(d-1)(d-i)} \binom{d}{i}^{d-1} a_{d-1}^{d-i} a_{d-i}$ appears in the
27 resultant R_i . The term $(-1)^{(d-1)(d-i)} \binom{d}{i}^{d-1} a_{d-1}^{d-i} a_{d-i}$ is the unique monomial in (3) of degree
28 $d-i+1$; all the other monomials appearing in (3) have degree strictly greater than $d-i+1$.

29 Proof. By inspection of the matrix $M(d, i)$, we see that the monomial $(-1)^{(d-1)(d-i)} \binom{d}{i}^{d-1} a_{d-1}^{d-i} a_{d-i}$
30 appears in the resultant R_i : it is the monomial with

$$31 \begin{aligned} 32 \sigma(j) &= d-i+j && \text{for } \{1, \dots, d-i\} \\ 33 &= j-(d-i) && \text{for } j \in \{d-i+1, \dots, 2d-i-1\} \\ 34 &= 2d-i && \text{for } j = 2d-i. \end{aligned}$$

35 Moreover, it is the unique monomial ω of R_i such that $a_{d-1}^{d-i} \mid \omega$.

36 Let us prove the second statement of the Theorem. Let $M^\bullet(d, i)$ be the matrix obtained
37 by deleting the last row and the last column of $M(d, i)$. Let $D^\bullet(d, i) = \det M^\bullet(d, i)$. We need
38 to show that all the monomials appearing in $D^\bullet(d, i)$ have order at least $d-i$ and a_{d-1}^{d-i} is the
39 only one of order exactly $d-i$.

Remark 10 For $\ell, j \in \{1, \dots, 2d - i - 1\}$, we have $m_{\ell j} \in \mathbb{N} \setminus \{0\}$ if and only if one of the following conditions holds:

- (1) $j \in \{d - i + 1, \dots, d - 1\}$ and $\ell = j + d - i$
- (2) $j \in \{1, \dots, d - i\}$ and $\ell \in \{j, j + d - i\}$.

By Remark 10, the last $d - i$ columns of $M^\bullet(d, i)$ do not involve any non-zero constant entries. Hence every monomial $\omega = (-1)^{e(\sigma)} m_{\sigma(1), 1} \cdots m_{\sigma(2d-i-1), 2d-i-1}$ appearing in $D^\bullet(d, i)$ has degree at least $d - i$. Moreover, assume that $\deg \omega = d - i$. Moreover, for $j \in \{1, \dots, d - 1\}$ one of the conditions (1) or (2) of Remark 10 holds with $\ell = \sigma(j)$. Let

$$j(\omega) = \min\{j \in \{1, \dots, d - 1\} \mid \sigma(j) = j + d - i\}.$$

Lemma 11 We have $j(\omega) = 1$.

Proof of Lemma. Assume that $j(\omega) > 1$, aiming for contradiction. By Remark 10,

$$\text{if } j \in \{d - i + 1, \dots, d - 1\}, \text{ then } \sigma(j) = j + d - i. \quad (10)$$

Hence $j(\omega) \leq d - i + 1$.

Take a $j \in \{j(\omega) + d - 2, \dots, 2d - i - 1\}$. Then $j > d - 1$. From (10) we obtain $\sigma(j) \notin \{2d - i - 1, \dots, 2d - i - 1\}$. By inspection of the matrix $M^\bullet(d, i)$, it follows that

$$\sigma(j) \in \{j(\omega) - 1, \dots, d - i\}.$$

By descending induction on j , we obtain

$$\sigma(j) = j - d + 1 \text{ whenever } j \in \{j(\omega) + d - 2, \dots, 2d - i - 1\}. \quad (11)$$

By equation (11), we have $\sigma(j(\omega) + d - 2) = j(\omega) - 1$, so $\sigma(j(\omega) - 1) \neq j(\omega) - 1$. By Remark 10, $\sigma(j(\omega) - 1) = j(\omega) - 1 + d - i$, contradicting the definition of $j(\omega)$. The Lemma is proved. \square

The Theorem follows from the Lemma by inspection of the matrix $M^\bullet(d, i)$. \square

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