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THE ASYMPTOTIC DISTRIBUTION OF ELKIES PRIMES FOR REDUCTIONS OF ABELIAN VARIETIES IS GAUSSIAN

ALEXANDRE BENOIST AND JEAN KIEFFER

ABSTRACT. We generalize the notion of Elkies primes for elliptic curves to the setting of abelian varieties with real multiplication (RM), and prove the following. Let A be an abelian variety with RM over a number field whose attached Galois representation has large image. Then the number of Elkies primes (in a suitable range) for reductions of A modulo primes converges weakly to a Gaussian distribution around its expected value. This refines and generalizes results obtained by Shparlinski and Sutherland in the case of non-CM elliptic curves, and has implications for the complexity of the SEA point counting algorithm for abelian surfaces over finite fields.

1. INTRODUCTION

1.1. **Setup.** Let E be an elliptic curve over a finite field \mathbb{F}_q . We say that a prime number ℓ is *Elkies* for E if there exists an ℓ -isogeny with domain E defined over \mathbb{F}_q . This terminology stems from the Schoof–Elkies–Atkin (SEA) algorithm for determining $\#E(\mathbb{F}_q)$ [Sch95]; this algorithm is faster if E has many small Elkies primes ℓ , as Elkies’s method can then be applied to determine $\#E(\mathbb{F}_q) \bmod \ell$. In order to assess the overall complexity of the SEA algorithm, Shparlinski and Sutherland proved that there are enough Elkies primes on average, either when considering all elliptic curves over a fixed \mathbb{F}_q [SS14] or when considering reductions of a fixed, non-CM elliptic curve over \mathbb{Q} modulo primes in a large interval [SS15]. For further results in a non-average setting, see [Shp15].

We may also consider Elkies primes for abelian varieties of higher dimensions. Let A be a polarized abelian variety of dimension g over \mathbb{F}_q . We say that a prime ℓ , coprime to q and the degree of the polarization, is Elkies for A if there exists an \mathbb{F}_q -rational subgroup $G \subset A[\ell]$ which is maximal isotropic for the Weil pairing; in that case, the quotient A/G is also equipped with a polarization of the same degree. More generally, if A has real multiplication (RM) by an order \mathcal{O} in a totally real number field, i.e. if A is equipped with a primitive embedding $\mathcal{O} \hookrightarrow \text{End}_{\mathbb{F}_q}(A)$ such that every $x \in \mathcal{O}$ is invariant under the Rosati involution, we say that a prime ideal \mathfrak{l} of \mathcal{O} is Elkies for A if $A[\mathfrak{l}]$ admits a maximal isotropic subgroup G defined over \mathbb{F}_q and stable under \mathcal{O} , or in other words, if there exists an \mathbb{F}_q -rational \mathfrak{l} -isogeny from A , as defined in [BJW17]. This notion of Elkies primes

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is a suitable analogue of the classical definition in the context of the SEA algorithm on principally polarized abelian surfaces with or without RM [Kie22].

1.2. Main results. In this paper, we show that the number of Elkies primes in certain ranges for reductions of a fixed abelian variety A with RM over a number field asymptotically follows a Gaussian distribution, provided that the Galois representation attached to A has a large enough adelic image.

To formulate this last condition precisely, we introduce the following notation. Let F be the field of definition of A and let G_F be its absolute Galois group. If ℓ is a large enough prime, the ℓ -adic Tate module $T_\ell(A)$ of A is a free $\mathcal{O} \otimes \mathbb{Z}_\ell$ -module of rank $2h$ where $h = g/d$, endowed with a nondegenerate alternating form with values in $\mathcal{O} \otimes \mathbb{Z}_\ell$, as we review in Section 2. If n is a sufficiently large integer, we can therefore consider the global Galois representation

$$\widehat{\rho}_n : G_F \rightarrow \mathrm{GSp}_{2h}(\mathcal{O} \otimes \widehat{\mathbb{Z}}_{\geq n}), \quad \text{where } \widehat{\mathbb{Z}}_{\geq n} := \prod_{\ell \text{ prime}, \ell \geq n} \mathbb{Z}_\ell.$$

We say that A has *large Galois image* if $\widehat{\rho}_n(G_F)$ contains $\mathrm{Sp}_{2h}(\mathcal{O} \otimes \widehat{\mathbb{Z}}_{\geq n})$ for large enough n . Assuming that \mathcal{O} is the whole endomorphism ring of A over \mathbb{Q} (a necessary condition), one can sometimes guarantee that A has large Galois image, as in Serre's open image theorem in the case $d = 1$ [Ser85]: we review this theorem and its RM analogues in Section 2.

Our main result on the distribution of Elkies primes is then the following.

Theorem 1.1. *Assume the generalized Riemann hypothesis (GRH). Let \mathcal{O} be an order in a totally real number field K of degree d , and let A be a polarized abelian variety of dimension $g \geq 1$ defined over a number field F with RM by \mathcal{O} with large Galois image.*

For a real number L , denote by $\mathcal{P}_K(L, 2L)$ the set of prime ideals \mathfrak{l} of K such that $N_{K/\mathbb{Q}}(\mathfrak{l}) \in [L, 2L]$, and define $\mathcal{P}_F(P, 2P)$ similarly. For a prime \mathfrak{p} of F of good reduction for A and $L \geq 1$, let $N_e(\mathfrak{p}, L)$ be the number of Elkies primes $\mathfrak{l} \in \mathcal{P}_K(L, 2L)$ for $A_{\mathfrak{p}}$. Further define $\alpha_h \in (0, 1)$ by the formula

$$\alpha_h = \sum_{(d_1, \dots, d_r) \in \Sigma_h} \frac{1}{2^r} \cdot \prod_{i=1}^r \frac{1}{d_i} \cdot \prod_{k=1}^h \frac{1}{\#\{j : d_j = k\}!}$$

where Σ_h denotes the set of unordered partitions of the integer $h = g/d$.

Then, as $L, P \rightarrow \infty$ with $P \gg L^n$ for every positive integer n , the random variable

$$\begin{aligned} X_{P,L} : \mathcal{P}_F(P, 2P) &\longrightarrow \mathbb{R} \\ \mathfrak{p} &\longmapsto \frac{N_e(\mathfrak{p}, L) - \alpha_h \#\mathcal{P}_K(L, 2L)}{\sqrt{\alpha_h(1 - \alpha_h) \#\mathcal{P}_K(L, 2L)}} \end{aligned}$$

converges weakly to the standard Gaussian distribution with mean value 0 and variance 1.

Intuitively, α_h is the probability that \mathfrak{l} will be Elkies for $A_{\mathfrak{p}}$ for random \mathfrak{l} and \mathfrak{p} ; weak convergence to the Gaussian distribution of Theorem 1.1 is what we would obtain from the central limit theorem in the naive probabilistic model where the events “ \mathfrak{l} is Elkies for $A_{\mathfrak{p}}$ ” are all independent. We list the first few values of α_h in Table 1.

h	1	2	3	4	5	6	7	8
α_h (exact value)	$\frac{1}{2}$	$\frac{3}{8}$	$\frac{5}{16}$	$\frac{35}{128}$	$\frac{63}{256}$	$\frac{231}{1024}$	$\frac{429}{2048}$	$\frac{6435}{32768}$
α_h (approximate value)	0.5	0.375	0.3125	0.2734	0.2461	0.2256	0.2095	0.1964

TABLE 1. Values of α_h

We prove Theorem 1.1 by analyzing the moments $\mathbb{E}(X_{P,L}^k)$ of $X_{P,L}$ when $\mathcal{P}_F(P, 2P)$ is equipped with the uniform probability measure. In fact, Theorem 1.1 follows directly from the following result: see for instance [Bil95, Theorem 30.2].

Theorem 1.2. *Assume GRH, and keep notation from Theorem 1.1. Let $k \geq 0$ be any integer, and let m_k be the moment of order k of the standard Gaussian distribution (thus $m_k = 0$ for odd k .) Then $\mathbb{E}(X_{P,L}^k)$ converges to m_k as $P, L \rightarrow \infty$ with $P \gg L^n$ for every positive integer n . More precisely, we have*

$$\mathbb{E}(X_{P,L}^k) = m_k + O_{A,k} \left(\frac{1}{L^{1/2} \log(L)^{1/2}} + \frac{L^{k(2h^2+h+3/2)} \log(P)^2}{\log(L)^{k/2} P^{1/2}} \right).$$

Here the notation $O_{A,k}$ means that the implicit constants in Landau's notation are allowed to depend on A (hence on F , \mathcal{O} , and h) and k .

In the case of elliptic curves, Theorem 1.2 refines [SS15] as we consider moments of all orders and provide an asymptotic equivalent of the even moments rather than an upper bound. To the best of our knowledge, Theorem 1.2 is also the first quantitative result on the distribution of Elkies primes in higher dimensions. In particular, a consequence of this theorem is that there are enough Elkies primes to run the SEA algorithm in dimension 2 on average over reductions of a fixed abelian variety: see [Kie22, Def. 3.7].

The proof of Theorem 1.2 is inspired from [SS15]: we apply an explicit version of the Čebotarev density theorem (which relies on GRH) to number field extensions of F cut out by torsion subgroups of A , and count how many elements in their Galois groups corresponds to \mathfrak{l} being Elkies for $A_{\mathfrak{p}}$. The result then follows from rearranging the summations and from a combinatorial argument to determine the leading term in the moments of $X_{P,L}$.

We also provide numerical experiments on the distribution of Elkies primes in large ranges in the case of elliptic curves: it was actually the very smooth aspect of the data which prompted us to try and prove Theorem 1.1.

One might wonder if this convergence result to a Gaussian distribution also holds when considering all elliptic curves (or more generally abelian varieties) over a fixed \mathbb{F}_q as in [SS14]. To answer this, it seems that one would need careful control on the class numbers appearing in the distribution of traces of Frobenius for elliptic curves over \mathbb{F}_q .

1.3. Organization. In Section 2, we review the properties of Galois representations attached to abelian varieties with RM, characterize Elkies primes both in terms of Frobenius elements in the Galois representation and in terms of the existence of isogenies, and recall results from the literature on large Galois images. In Section 3, we count matrices

in $\mathrm{GSp}_{2h}(\mathcal{O}/\mathfrak{l}\mathcal{O})$ (and related groups) corresponding to Elkies primes, a key input to the Čebotarev density theorem. We prove Theorem 1.2 in Section 4, and report on our numerical experiments in Section 5.

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2. GALOIS REPRESENTATIONS AND ELKIES PRIMES

In this section, we review basic facts on the structure of torsion subgroups of abelian varieties with RM over any field (§2.1). Then we characterize Elkies primes for such abelian varieties in terms of the existence of isogenies (§2.2) and, in the case of finite fields or reductions of abelian varieties over number fields, in terms of the action of Frobenius on torsion subgroups (§2.3). Finally, we review deeper results on large Galois images (§2.4).

2.1. Torsion subgroups of abelian varieties with RM. Throughout, we use the notation listed in Table 2. For the reader's convenience, the table also includes symbols defined later in this section. For now, F is any field, and A is an abelian variety over F with real multiplication by an order \mathcal{O} as in the introduction.

Recall that whenever ℓ is prime to p , the Tate module $T_\ell(A)$ is a free \mathbb{Z}_ℓ -module of rank $2g$ on which the Weil pairing e_ℓ is nondegenerate. If further ℓ is prime to d_A , then e_ℓ also gives a nondegenerate alternating form on $A[\ell]$. We may also view $T_\ell(A)$ as an $\mathcal{O} \otimes \mathbb{Z}_\ell$ -module, using the action of \mathcal{O} as endomorphisms of A .

Lemma 2.1. *Assume that ℓ is coprime to p , d_A and $c_{\mathcal{O}}$, so that $\mathcal{O} \otimes \mathbb{Z}_\ell = \mathcal{O}_K \otimes \mathbb{Z}_\ell$.*

- (1) $T_\ell(A)$ is a free $\mathcal{O} \otimes \mathbb{Z}_\ell$ -module of rank $2h$.
- (2) There exists a unique $\mathcal{O} \otimes \mathbb{Z}_\ell$ -bilinear alternating form $\psi_\ell : T_\ell(A) \times T_\ell(A) \rightarrow \mathcal{O} \otimes \mathbb{Z}_\ell$ with the following property: for every $x, y \in T_\ell(A)$, $e_\ell(x, y) = \mathrm{Tr}_{K/\mathbb{Q}}(\psi_\ell(x, y))$.

Proof. (1) This is [Rib76, Prop. 2.1.1].

- (2) The existence and uniqueness of ψ_ℓ after tensoring with \mathbb{Q}_ℓ is [Chi92, Lemma 1.2.1]. In fact, ψ_ℓ exists at the level of $\mathcal{O} \otimes \mathbb{Z}_\ell$ -modules by [BGK06, Lemma 3.1]. \square

Under the assumptions of Lemma 2.1, we also consider the decomposition of $\mathcal{O}/\ell\mathcal{O}$ as a product of fields:

$$\mathcal{O}/\ell\mathcal{O} = \prod_{\mathfrak{l}|\ell} \mathcal{O}/\mathfrak{l}\mathcal{O}.$$

Then for each $\mathfrak{l}|\ell$, we define the \mathfrak{l} -torsion subgroup $A[\mathfrak{l}] \subset A[\ell]$ as

$$(1) \quad A[\mathfrak{l}] = \bigcap_{f \in \mathfrak{l}} \ker(f) = \{x \in A[\ell] : f(x) = 0 \text{ for every } f \in \mathfrak{l}\}.$$

K	a totally real number field
d	the degree of K over \mathbb{Q}
\mathcal{O}	an order of K
\mathcal{O}_K	the ring of integers in K
$c_{\mathcal{O}}$	the conductor of \mathcal{O} , an ideal supported at primes dividing $[\mathcal{O}_K : \mathcal{O}]$
$N_{K/\mathbb{Q}}$	the norm map for ideals or elements of K/\mathbb{Q}
$\mathrm{Tr}_{K/\mathbb{Q}}$	the trace map for elements of K/\mathbb{Q}
ℓ	a prime number in $\mathbb{Z}_{\geq 1}$
\mathfrak{l}	a prime ideal of \mathcal{O} above ℓ
e	the inertia degree of \mathfrak{l} , i.e. the integer e such that $N_{K/\mathbb{Q}}(\mathfrak{l}) = \ell^e$
F	a field
p	the characteristic of F (either positive or 0)
G_F	the absolute Galois group of F
χ_{ℓ}	the cyclotomic character $G_F \rightarrow \mathbb{Z}_{\ell}^{\times}$
A	a polarized abelian variety over F with RM by \mathcal{O} , i.e. endowed with a primitive embedding $\mathcal{O} \hookrightarrow \mathrm{End}_F(A)$ whose image consists of elements that are invariant under the Rosati involution
g	the dimension of A ; in particular $d g$
h	the integer g/d
d_A	the degree of the polarization of A
π_A	the Frobenius endomorphism of A (if F is finite)
$T_{\ell}(A)$	the ℓ -adic Tate module of A
e_{ℓ}	the Weil pairing on $T_{\ell}(A)$, with values in \mathbb{Z}_{ℓ}
$A[\ell]$	the ℓ -torsion subgroup of A
ψ_{ℓ}	the \mathcal{O} -linear alternating form on $A[\ell]$ defined in Lemma 2.1
$A[\mathfrak{l}]$	the \mathfrak{l} -torsion subgroup of A , as defined in (1) below
ρ_{ℓ}	the ℓ -adic Galois representation with target $\mathrm{GSp}_{2h}(\mathcal{O} \otimes \mathbb{Z}_{\ell})$, cf. (2) below
$\bar{\rho}_{\ell}, \bar{\rho}_{\mathfrak{l}}$	the Galois representations modulo ℓ and \mathfrak{l} as in (3), (4) below
λ	the multiplier character $\mathrm{GSp}_{2h} \rightarrow \mathbb{G}_m$, as in Definition 2.3
$\mathrm{GSp}_{2h}(R; U)$	the subset of $\mathrm{GSp}_{2h}(R)$ given by $\lambda^{-1}(U)$, as in Definition 2.3
$\mathcal{S}_{2h, \mathbb{F}_q}(\lambda_0)$	the split matrices in $\mathrm{GSp}_{2h}(\mathbb{F}_q)$ with multiplier λ_0 , as in Definition 2.8.

TABLE 2. List of notations

Lemma 2.2. *Assume that ℓ is coprime to p , d_A and $c_{\mathcal{O}}$. Then we have a direct sum decomposition*

$$A[\ell] = \bigoplus_{\mathfrak{l}|\ell} A[\mathfrak{l}]$$

where for each $\mathfrak{l}|\ell$, the summand $A[\mathfrak{l}]$ is an $(\mathcal{O}/\mathfrak{l}\mathcal{O})$ -vector space of dimension $2h$. This direct sum is orthogonal with respect to ψ_ℓ , and the restriction of ψ_ℓ to each $A[\mathfrak{l}]$ is nondegenerate.

Proof. The decomposition of $A[\ell]$ as a direct sum is a consequence of Lemma 2.1(1). Let us check that this decomposition is orthogonal with respect to ψ_ℓ . Let $\mathfrak{l} \neq \mathfrak{l}'$ be prime ideals above ℓ , and fix an element $f \in \mathfrak{l}$ which is invertible modulo \mathfrak{l}' . If $x \in A[\mathfrak{l}]$ and $y \in A[\mathfrak{l}']$, then there exists $y' \in A[\mathfrak{l}']$ such that $y = f(y')$. By \mathcal{O} -linearity of ψ_ℓ , we get

$$\psi_\ell(x, y) = \psi_\ell(x, f(y')) = \psi_\ell(f(x), y') = \psi_\ell(0, y') = 0.$$

Finally, each piece is nondegenerate by [BGK06, Lemma 3.2]. \square

We now include the action of the Galois group G_F in the picture. Let ℓ be coprime to p , d_A and $c_{\mathcal{O}}$. By equivariance of the Weil pairing (see for instance [BGK06, Lemma 4.7]), we have for all $\sigma \in G_F$ and $x, y \in T_\ell(A)$:

$$e_\ell(\sigma(x), \sigma(y)) = \chi_\ell(\sigma)e_\ell(x, y).$$

The action of σ on $A[\ell]$ is also \mathcal{O} -linear because the elements of \mathcal{O} , seen as endomorphisms, are defined over F by assumption. By nondegeneracy of $\text{Tr}_{K/\mathbb{Q}}$, we have for all $x, y \in T_\ell(A)$:

$$\psi_\ell(\sigma(x), \sigma(y)) = \chi_\ell(\sigma)\psi_\ell(x, y).$$

In other words, σ preserves ψ_ℓ up to multiplication by the scalar $\chi_\ell(\sigma) \in \mathbb{Z}_\ell^\times$.

In order to identify the action of σ on $A[\ell]$ as an element in a standard symplectic group, we choose once and for all a symplectic basis (v_1, \dots, v_{2h}) of $T_\ell(A)$ as an $\mathcal{O} \otimes \mathbb{Z}_\ell$ -module. This means that the alternating form ψ_ℓ in this basis takes the standard form

$$J_{2h} = \begin{pmatrix} 0 & I_h \\ -I_h & 0 \end{pmatrix},$$

where I_h denotes the $h \times h$ identity matrix. We summarize our notation for the attached symplectic group in the following definition.

Definition 2.3. We denote by GSp_{2h} the general symplectic group with respect to the standard form J_{2h} : for any commutative ring R , we have

$$\text{GSp}_{2h}(R) = \{m \in \text{GL}_{2h}(R) : m^\top J_{2h} m = \lambda(m)J_{2h} \text{ for some } \lambda(m) \in R^\times\}.$$

We call the character $\lambda : \text{GSp} \rightarrow \mathbb{G}_m$ appearing in this equation the *multiplier*. The kernel of λ is Sp_{2h} , the usual symplectic group. If U is a subset of R^\times , we also write

$$\text{GSp}_{2h}(R; U) = \{m \in \text{GSp}_{2h}(R) : \lambda(m) \in U\}.$$

Assuming that ℓ is coprime to d_A and $c_{\mathcal{O}}$, the same vectors v_1, \dots, v_{2h} also form a symplectic basis of $A[\ell]$ as an $\mathcal{O}/\ell\mathcal{O}$ -module, and for every prime $\mathfrak{l}|\ell$ of \mathcal{O} , a symplectic basis of $A[\mathfrak{l}]$ as an $(\mathcal{O}/\mathfrak{l}\mathcal{O})$ -vector space.

Summarizing, we identify the action of $\sigma \in G_F$ on $T_\ell(A)$ with an element of the general symplectic group,

$$(2) \quad \rho_\ell(\sigma) \in \text{GSp}_{2h}(\mathcal{O} \otimes \mathbb{Z}_\ell; \mathbb{Z}_\ell^\times) \subset \text{GSp}_{2h}(\mathcal{O} \otimes \mathbb{Z}_\ell)$$

such that

$$\lambda(\rho_\ell(\sigma)) = \chi_\ell(\sigma) \in \mathbb{Z}_\ell^\times \subset (\mathcal{O} \otimes \mathbb{Z}_\ell)^\times.$$

We identify the action of σ on the ℓ -torsion subgroup $A[\ell]$ with the image of σ under the reduced representation

$$(3) \quad \bar{\rho}_\ell(\sigma) \in \mathrm{GSp}_{2h}(\mathcal{O}/\ell\mathcal{O}; (\mathbb{Z}/\ell\mathbb{Z})^\times).$$

For each prime \mathfrak{l} of \mathcal{O} , we also identify the action of σ on $A[\mathfrak{l}]$ with an element

$$(4) \quad \bar{\rho}_\mathfrak{l}(\sigma) \in \mathrm{GSp}_{2h}(\mathcal{O}/\mathfrak{l}\mathcal{O}).$$

with the same multiplier $\chi_\ell(\sigma)$. We call ρ_ℓ the ℓ -adic Galois representation, and $\bar{\rho}_\ell$ (resp. $\bar{\rho}_\mathfrak{l}$) the Galois representation modulo ℓ (resp \mathfrak{l}), attached to A . By Lemma 2.1(1) and Lemma 2.2, the decompositions

$$A[\ell] = \bigoplus_{\mathfrak{l}|\ell} A[\mathfrak{l}] \quad \text{and} \quad \mathrm{GSp}_{2h}(\mathcal{O}/\ell\mathcal{O}) = \prod_{\mathfrak{l}|\ell} \mathrm{GSp}_{2h}(\mathcal{O}/\mathfrak{l}\mathcal{O})$$

are compatible in the sense that the following diagram commutes:

$$\begin{array}{ccc} G_F & \xrightarrow{\bar{\rho}_\ell} & \mathrm{GSp}_{2h}(\mathcal{O}/\ell\mathcal{O}; (\mathbb{Z}/\ell\mathbb{Z})^\times) & \longrightarrow & \mathrm{GSp}_{2h}(\mathcal{O}/\mathfrak{l}\mathcal{O}). \\ & & \searrow & \xrightarrow{\bar{\rho}_\mathfrak{l}} & \\ & & & & \end{array}$$

In particular, if ℓ splits completely in \mathcal{O} , then $\bar{\rho}_\ell(G_F)$ is a subgroup of $\mathrm{GSp}_{2h}(\mathbb{Z}/\ell\mathbb{Z})^d$ consisting of tuples of matrices (m_1, \dots, m_d) such that $\lambda(m_1) = \dots = \lambda(m_d)$. At the other extreme, if ℓ is inert in \mathcal{O} , then $\bar{\rho}_\ell(G_F)$ is a subgroup of $\mathrm{GSp}_{2h}(\mathbb{F}_{\ell^d}; (\mathbb{Z}/\ell\mathbb{Z})^\times)$.

The representation $\bar{\rho}_\mathfrak{l}$ can be seen as the restriction modulo \mathfrak{l} of the \mathfrak{l} -adic representation considered in [Chi92, §1.1].

2.2. Elkies primes for abelian varieties with RM. Let us restate the definition of Elkies primes given in the introduction. We are mainly interested in finite fields, but for now, our discussion remains valid over any field. We keep the notation from Table 2, and assume throughout that the prime ideals \mathfrak{l} we consider are coprime with p , d_A , and $c_\mathcal{O}$.

Definition 2.4. We say that \mathfrak{l} is *Elkies* for A if there exists an F -rational subgroup of $A[\mathfrak{l}]$ that is maximal isotropic for the Weil pairing e_ℓ and stable under \mathcal{O} . Note that this last condition is automatic when $e = 1$, as $\mathcal{O}/\mathfrak{l}\mathcal{O}$ only consists of scalars.

We can equivalently phrase this definition in terms of isotropic subspaces for ψ_ℓ .

Lemma 2.5. *The prime \mathfrak{l} is Elkies for A if and only if there exists a maximal isotropic sub- $(\mathcal{O}/\mathfrak{l}\mathcal{O})$ -vector space of $A[\mathfrak{l}]$ that is maximal isotropic for ψ_ℓ and F -rational.*

Proof. Suppose \mathfrak{l} is Elkies for A , i.e. there exists an \mathbb{F}_q -rational \mathbb{F}_ℓ -vector space $G \subset A[\mathfrak{l}]$ which is maximal isotropic for the Weil pairing and stable under \mathcal{O} . We may also view $G \subset A[\mathfrak{l}]$ as an F -rational sub- $\mathcal{O}/\ell\mathcal{O}$ -vector space of dimension h . By Lemma 2.1(2), the trace of ψ_ℓ vanishes on $G \times G$, so ψ_ℓ vanishes on $G \times G$ as well as the trace is nondegenerate.

Conversely, if $G \subset A[\mathfrak{l}]$ be a maximal isotropic subspace for ψ_ℓ in $A[\mathfrak{l}]$. Seen as an \mathbb{F}_ℓ -vector space, G has dimension eh while $A[\mathfrak{l}]$ has dimension $2eh$. Moreover G is isotropic for the Weil pairing by Lemma 2.1(2). Therefore, ℓ is Elkies for A . \square

Definition 2.4 is a suitable generalization of the notion of Elkies primes for elliptic curves [SS14], abelian surfaces without RM [Kie22, §3.2], and abelian surfaces with RM in the case of split primes [Kie22, §4.1]. Moreover, there is still a close link between Elkies primes and the existence of F -rational isogenies compatible with the RM structure and the polarization of A . Let us specify this link in more detail, for motivation only, as it will not be used in the rest of the paper.

First we introduce the following notation. The Néron–Severi group $\mathrm{NS}(A)$ of A (the group of line bundles on A up to algebraic equivalence) is related to the endomorphisms of A , as follows. The \mathbb{Q} -algebra $\mathrm{End}^0(A) = \mathrm{End}_{\overline{F}}(A) \otimes \mathbb{Q}$ is endowed with the Rosati involution \dagger coming from our choice of polarization on A . Let $\mathrm{End}^0(A)^\dagger$ denote the sub-vector space of elements invariant under \dagger , and $\mathrm{End}(A)^\dagger = \mathrm{End}^0(A)^\dagger \cap \mathrm{End}(A)$. There is an isomorphism $\mathrm{NS}(A) \otimes \mathbb{Q} \simeq \mathrm{End}^0(A)^\dagger$, which depends on the chosen polarization of A [Mum70, (3) p.190]. Given $\alpha \in \mathrm{End}(A)^\dagger$ and two abelian varieties A, B with RM by \mathcal{O} , we say that an isogeny $\phi : A \rightarrow B$ is an α -isogeny if the RM structures of A and B are compatible via ϕ , and if the pullback of the polarization of B via ϕ (seen as an element of $\mathrm{NS}(A)$) corresponds to α via the previous isomorphism. The element α is then necessarily totally positive [Mum70, (IV) p.209]. Equivalently, we ask that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & A & \longrightarrow & A^\vee \\ & & \downarrow \phi & & \uparrow \phi^\vee \\ & & B & \longrightarrow & B^\vee \end{array}$$

commutes, where \vee denotes duals and the unlabeled arrows are the polarizations. This implies that $\ker(\phi)$ is maximal isotropic in $A[\alpha]$ for its canonical nondegenerate pairing; conversely, if $K \subset A[\alpha]$ is a maximal isotropic subspace, then A/K carries a unique polarization of degree d_A such that the quotient isogeny $A \rightarrow A/K$ is an α -isogeny [Mum70, Cor. p.231]. Recall that our Elkies primes are prime to p , d_A and $c_{\mathcal{O}}$.

- Proposition 2.6.** (1) *The prime \mathfrak{l} is Elkies if and only if there exists an abelian variety B over k with RM by \mathcal{O} and an F -rational \mathfrak{l} -isogeny $\phi : A \rightarrow B$ in the sense of [BJW17, Def. 4.1].*
- (2) *Let $\mathfrak{l}_1, \dots, \mathfrak{l}_r$ be distinct Elkies primes for A , and let $k_1, \dots, k_r \geq 0$ be integers such that $\mathfrak{l}_1^{k_1} \cdots \mathfrak{l}_r^{k_r}$ is trivial in the narrow class group of \mathcal{O} . Let $\alpha \in \mathcal{O}$ be a totally positive generator of this product. Then there exists an abelian variety B over F with RM by \mathcal{O} and endowed with a polarization of degree d_A , and an F -rational α -isogeny $\phi : A \rightarrow B$.*

Proof of Proposition 2.6. (1) directly comes from the definition of \mathfrak{l} -isogenies.

We now prove (2). For $1 \leq i \leq r$, let $K_i \subset A[\mathfrak{l}_i]$ be F -rational, maximal isotropic, and \mathcal{O} -stable subgroups as in Definition 2.4. Define now $K'_i = A[\mathfrak{l}_i^{m_i}]$ if $k_i = 2m_i$ is even, and $K'_i = A[\mathfrak{l}_i^{m_i+1}] \cap \eta^{-1}(K_i)$, where $\eta \in \mathcal{O}$ is any element whose \mathfrak{l}_i -adic valuation is exactly m_i , when $k_i = 2m_i + 1$ is odd. We can check that K'_i is independent of the choice of η , and that it is an F -rational, \mathcal{O} -stable, maximal isotropic subspace in $A[\mathfrak{l}_i^{k_i}]$. By Lemma 2.1 and

the Chinese remainder theorem, we have

$$A[\alpha] = \bigoplus_{i=1}^r A[\mathfrak{l}_i^{k_i}].$$

Moreover, the restriction of the pairing on $A[\alpha]$ to each subgroup $A[\mathfrak{l}_i^{k_i}]$ is precisely the Weil pairing $e_{\ell_i} \pmod{\ell_i^{k_i}}$, where $\ell_i \in \mathbb{Z}$ denotes the prime below \mathfrak{l}_i , and the direct sum is orthogonal, as can be seen from the functorial properties of those pairings [Mum70, p. 228]. Therefore $K = K'_1 \oplus \cdots \oplus K'_r$ is maximal isotropic in $A[\alpha]$, and is the kernel of the isogeny ϕ we are looking for. \square

2.3. Elkies primes and the action of Frobenius. We keep the notation of Table 2, and assume first that $F = \mathbb{F}_q$ is a finite field. Let π_A denote the Frobenius endomorphism of A . We continue to assume that \mathfrak{l} is prime to p , d_A and $c_{\mathcal{O}}$. We can also view the Frobenius map as an element $\pi \in G_F$.

Lemma 2.7. *Let A be an abelian variety over $F = \mathbb{F}_q$ with RM by \mathcal{O} . The prime \mathfrak{l} is Elkies for A if and only if π_A admits a maximal isotropic stable subspace in $A[\mathfrak{l}]$, if and only if $\bar{\rho}_{\mathfrak{l}}(\pi) \in \mathrm{GSp}_{2h}(\mathcal{O}/\mathfrak{l}\mathcal{O})$ admits a maximal isotropic stable subspace in $(\mathcal{O}/\mathfrak{l}\mathcal{O})^{2h}$.*

Proof. This is a restatement of Lemma 2.5, using the fact that a subspace of $A[\mathfrak{l}]$ is \mathbb{F}_q -rational if and only if it is stable under π_A . \square

Lemma 2.7 prompts us to make the following definition.

Definition 2.8. Let k be a finite field. We say that a matrix $m \in \mathrm{GSp}_{2h}(k)$ is *split* if it leaves some maximal isotropic subspace of k^{2h} stable. We denote by $\mathcal{S}_{2h,k} \subset \mathrm{GSp}_{2h}(k)$ the subset of split matrices, and for $\lambda_0 \in k^\times$, we write

$$\mathcal{S}_{2h,k}(\lambda_0) := \{m \in \mathcal{S}_{2h,k} : \lambda(m) = \lambda_0\}.$$

We note that $\mathcal{S}_{2h,k}(\lambda_0)$ is a conjugacy-invariant subset of $\mathrm{GSp}_{2h}(k)$.

Since $\chi_{\ell}(\pi) = q$, another restatement of Lemma 2.5 is the following.

Lemma 2.9. *The prime \mathfrak{l} is Elkies for A if and only if $\bar{\rho}_{\mathfrak{l}}(\pi) \in \mathcal{S}_{2h,\mathcal{O}/\mathfrak{l}\mathcal{O}}(q)$.*

We now switch gears and assume that F is a number field. We fix a polarized abelian variety A over F with RM by \mathcal{O} . For every prime \mathfrak{p} of F with residue field $F_{\mathfrak{p}}$ of good reduction for A , the reduction $A_{\mathfrak{p}}$ of A modulo \mathfrak{p} is a polarized abelian variety of dimension g over $F_{\mathfrak{p}}$ with RM by \mathcal{O} . Indeed, the listed properties can all be formulated in terms of isogenies between abelian varieties and their duals, and such isogenies lift uniquely to Néron models at \mathfrak{p} by [BLR90, §1.4, Prop. 4]. We can characterize Elkies primes for $A_{\mathfrak{p}}$ in terms of the Galois representations $\bar{\rho}_{\mathfrak{l}}$ evaluated at Frobenius elements in G_F .

Proposition 2.10. *Let \mathfrak{p} be a prime of good reduction for A above $p \in \mathbb{Z}$, and let \mathfrak{l} be a prime of \mathcal{O} that is coprime to p, d_A and $c_{\mathcal{O}}$. Then \mathfrak{l} is Elkies for the reduction $A_{\mathfrak{p}}$ if and only if $\bar{\rho}_{\mathfrak{l}}(\sigma_{\mathfrak{p}}) \in \mathcal{S}_{2h,\mathcal{O}/\mathfrak{l}\mathcal{O}}(N_{F/\mathbb{Q}}(\mathfrak{p}))$, where $\sigma_{\mathfrak{p}} \in G_F$ is any Frobenius element at \mathfrak{p} (unique up to conjugation in G_F).*

Proof. Denote by F' the field of definition of $A[\mathfrak{l}]$, i.e. the smallest number field such that the representation $\bar{\rho}_{\mathfrak{l}} : G_F \rightarrow \mathrm{GSp}_{2h}(\mathcal{O}/\mathfrak{l}\mathcal{O})$ factors through $\mathrm{Gal}(F'/F)$. Let \mathfrak{P} be a prime of F' above \mathfrak{p} , and let $\sigma_{\mathfrak{p}} \in G_F$ be a Frobenius element stabilizing \mathfrak{P} ; we can consider $\sigma_{\mathfrak{p}}$ as a (uniquely specified) element of $\mathrm{Gal}(F'/F)$. Reduction modulo \mathfrak{P} defines a bijection $A[\mathfrak{l}] \rightarrow A_{\mathfrak{p}}[\mathfrak{l}]$ by [ST68, §1, Lemma 2], so our choice of fixed symplectic basis of $T_{\ell}(A)$ also fixes a symplectic basis of $A_{\mathfrak{p}}[\mathfrak{l}]$ as an $(\mathcal{O}/\mathfrak{l}\mathcal{O})$ -vector space. By definition, $\sigma_{\mathfrak{p}}$ induces the Frobenius map of the extension of residue fields $F'_{\mathfrak{P}}/F_{\mathfrak{p}}$. Therefore, $\bar{\rho}_{\mathfrak{l}}(\sigma_{\mathfrak{p}})$ is precisely the matrix of the Frobenius endomorphism $\pi_{A_{\mathfrak{p}}}$ in the symplectic basis of $A_{\mathfrak{p}}[\mathfrak{l}]$ specified above. We now apply Lemma 2.9, using the fact that $\chi_{\ell}(\sigma_{\mathfrak{p}}) = N_{F/\mathbb{Q}}(\mathfrak{p}) \bmod \ell$. \square

Proposition 2.10 indicates that the Čebotarev density theorem in F'/F will provide information on how often a fixed prime \mathfrak{l} is Elkies for the reduced abelian varieties $A_{\mathfrak{p}}$ as \mathfrak{p} grows. In order to apply this theorem, we need to know what the Galois group $\mathrm{Gal}(F'/F)$ is: this is the purpose of the “large Galois image” hypothesis in Theorem 1.1.

2.4. Large Galois images. We keep notation from Table 2; here, F is a number field. To formalize the definition of large Galois images used in the introduction, we introduce the following notation. If n is an integer, we write

$$\widehat{\mathbb{Z}}_{\geq n} = \prod_{\ell \text{ prime}, \ell \geq n} \mathbb{Z}_{\ell}.$$

The ℓ -adic Galois representations $\rho_{\ell} : G_F \rightarrow \mathrm{GSp}_{2h}(\mathcal{O} \otimes \mathbb{Z}_{\ell})$ can be combined into a global representation

$$\widehat{\rho}_n : G_F \rightarrow \mathrm{GSp}_{2h}(\mathcal{O} \otimes \widehat{\mathbb{Z}}_{\geq n})$$

Definition 2.11. We say that A has *large Galois image* if for some integer $n \geq 1$, the image of $\widehat{\rho}_n$ contains $\mathrm{Sp}_{2h}(\mathcal{O} \otimes \widehat{\mathbb{Z}}_{\geq n})$. Because the cyclotomic character χ_{ℓ} is surjective for large enough ℓ , an equivalent condition is that for some large enough n ,

$$\widehat{\rho}_n(G_F) = \mathrm{GSp}_{2h}(\mathcal{O} \otimes \widehat{\mathbb{Z}}_{\geq n}; (\widehat{\mathbb{Z}}_{\geq n})^{\times}).$$

In the main results of this paper, Theorems 1.1 and 1.2, we only consider abelian varieties with RM that have large Galois images. In this subsection, we gather some necessary and sufficient conditions for this to happen.

Proposition 2.12. *If A has large Galois image, then $\mathrm{End}_{\overline{\mathbb{Q}}}(A) = \mathcal{O}$. In particular A is simple of type I in Albert’s classification.*

Proof. Since we assumed the RM embedding $\mathcal{O} \hookrightarrow \mathrm{End}(A)$ to be primitive, it is sufficient to prove that $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \otimes \mathbb{Q} = K$. Let F' be a number field over which all endomorphisms of A are defined. Since $G_{F'}$ is an open subgroup of finite index in G_F , there exists a prime ℓ such that $\rho_{\ell}(G_{F'})$ still contains $\mathrm{Sp}_{2h}(\mathcal{O} \otimes \mathbb{Z}_{\ell})$. By Faltings [Fal83], $\mathrm{End}_{F'}(A) \otimes \mathbb{Q}_{\ell}$ is the commutant of $\rho_{\ell}(G_{F'})$ in $\mathrm{End}(T_{\ell}(A) \otimes \mathbb{Q}_{\ell})$.

We claim that the commutant of $\mathrm{Sp}_{2h}(\mathcal{O} \otimes \mathbb{Z}_{\ell})$ in $\mathrm{End}(T_{\ell}(A) \otimes \mathbb{Q}_{\ell})$ is precisely given by the action of elements of $\mathcal{O} \otimes \mathbb{Q}_{\ell}$ on $T_{\ell}(A)$. This would prove that $\mathrm{End}_{F'}(A) \otimes \mathbb{Q}_{\ell}$ is contained in $\mathcal{O} \otimes \mathbb{Q}_{\ell}$, hence $\mathrm{End}_{\overline{\mathbb{Q}}}(A) \otimes \mathbb{Q} = K$ as required.

To show that the claim holds, choose an element $\gamma \in \text{End}(T_\ell(A) \otimes \mathbb{Q}_\ell)$ commuting with $\text{Sp}_{2h}(\mathcal{O} \otimes \mathbb{Z}_\ell)$. In particular, considering scalar matrices in Sp_{2h} , we see that γ acts \mathcal{O} -linearly: we can therefore consider γ as a $2h \times 2h$ matrix with coefficients in $\mathcal{O} \otimes \mathbb{Q}_\ell$. Since $\mathcal{O} \otimes \mathbb{Q}_\ell$ is a product of fields, it is now sufficient to show that for any field k , the commutant of $\text{Sp}_{2h}(k)$ consists of scalar matrices only.

This last fact well-known (the Lie algebra representation of \mathfrak{sp}_{2h} on \mathfrak{sl}_{2h} is irreducible), but for completeness, we include a short proof when k is infinite. Let γ be a $2h \times 2h$ matrix over k commuting with $\text{Sp}_{2h}(k)$. Consider any symplectic basis (v_1, \dots, v_{2h}) of k^{2h} , and let $x_1, \dots, x_h \in k^\times$ be such that $x_1, \dots, x_h, x_1^{-1}, \dots, x_h^{-1}$ are distinct. The endomorphism whose matrix in the basis (v_1, \dots, v_{2h}) is $\text{Diag}(x_1, \dots, x_r, x_1^{-1}, \dots, x_r^{-1})$ is symplectic, so v_1, \dots, v_{2h} are eigenvectors of γ . As each nonzero element of k^{2h} is part of some symplectic basis, we deduce that each nonzero vector is an eigenvector for γ , hence γ is a scalar. \square

Conversely, we have the following theorem, after results of Serre [Ser85], Ribet [Rib76], Chi [Chi92] and Banaszak–Gajda–Krasoń [BGK06].

Theorem 2.13. *Assume that $\text{End}_{\overline{\mathbb{Q}}}(A) = \mathcal{O}$ and either:*

- $d = 1$ and $g \in \{2, 6\}$, or
- $h = g/d$ is odd.

Then A has large Galois image.

Proof. After making a finite extension of F , which only shrinks the image of the Galois representation, we may assume that the Zariski closure \mathcal{G}_ℓ of $\rho_\ell(G_F)$ inside $\text{GSp}_{2h}(\mathbb{Q}_\ell)$ is connected for all ℓ [Ser85, §2.5]. After taking another finite extension of F , we may also assume that the ℓ -adic Galois representations of A are all independent in the sense of [Ser85, §2.1]. The goal is then to prove that $\rho_\ell(G_F)$ contains $\text{Sp}_{2h}(\mathcal{O} \otimes \mathbb{Z}_\ell)$ for large enough ℓ . The case $d = 1$ is Serre’s open image theorem [Ser85, Thm. 3], while [BGK06, Thm. 6.16] covers the cases where h is odd (and can be applied as \mathcal{G}_ℓ is connected.) \square

In particular, if $\text{End}_{\overline{\mathbb{Q}}}(A) = \mathcal{O}$ and A is either an abelian surface or has odd dimension, then A has large Galois image.

3. COUNTING SPLIT MATRICES IN $\text{GSp}_{2h}(\mathbb{F}_q)$

Our goal here is to provide estimates for the cardinality of $\mathcal{S}_{2h, \mathbb{F}_q}(\lambda_0)$ for $\lambda_0 \in \mathbb{F}_q^\times$. In Section 4, we will use them with $\mathbb{F}_q = \mathcal{O}/\mathfrak{I}\mathcal{O}$ when applying the Čebotarev density theorem.

Since $\mathcal{S}_{2h, \mathbb{F}_q}(\lambda_0)$ is conjugacy-invariant, it is a reunion of conjugacy classes of $\text{GSp}_{2h}(\mathbb{F}_q)$, and those have been classified: see for instance [Wil12, Section 6.2]. One key element of the classification is the characteristic polynomial, so we start by studying its link with Elkies primes in §3.1. We use this to count elements in $\mathcal{S}_{2h, \mathbb{F}_q}(\lambda_0)$, up to negligible terms, in §3.2. If $h = 2$, one can actually get an exact count, as we review in §3.3.

3.1. Characteristic polynomials and Elkies primes.

Lemma 3.1. *Let $m \in \mathrm{GSp}_{2h}(\mathbb{F}_q)$. Then m leaves a maximal isotropic subspace of \mathbb{F}_q^{2h} stable if and only if m is conjugate in $\mathrm{GSp}_{2h}(\mathbb{F}_q)$ to a matrix of the form*

$$\begin{pmatrix} M & \star \\ 0 & \lambda(m)M^{-\top} \end{pmatrix}$$

for some $M \in \mathrm{GL}_h(\mathbb{F}_q)$, where $M^{-\top}$ denotes the inverse transpose of M .

Proof. Assume that m admits a maximal isotropic stable subspace $V \subset \mathbb{F}_q^{2h}$. Then we can find a symplectic basis of \mathbb{F}_q^{2h} whose first h vectors generate V , i.e. we can find $Q \in \mathrm{Sp}_{2h}(\mathbb{F}_q)$ such that

$$QmQ^{-1} = \begin{pmatrix} M & \star \\ 0 & M' \end{pmatrix}$$

where $M, M' \in \mathrm{GL}_h(\mathbb{F}_q)$. Because $\lambda(QmQ^{-1}) = \lambda(m)$, we must have $M' = \lambda(m)M^{-\top}$.

Conversely, assume that QmQ^{-1} has the specified form for some $Q \in \mathrm{GSp}_{2h}(\mathbb{F}_q)$. Let V be the span of the first h vectors of the canonical basis of \mathbb{F}_q^{2h} . Then $Q(V)$ is a maximal isotropic subspace of \mathbb{F}_q^{2h} that is stable under m . \square

Definition 3.2. For a monic polynomial $P \in \mathbb{F}_q[X]$ of degree r with constant coefficient $a_0 \in \mathbb{F}_q^\times$ and $\lambda_0 \in \mathbb{F}_q^\times$, we define the λ_0 -reciprocal polynomial of P to be the monic polynomial

$$\tilde{P}^{\lambda_0}(X) = \frac{1}{a_0} X^r P\left(\frac{\lambda_0}{X}\right).$$

Proposition 3.3. *Let $m \in \mathrm{GSp}_{2h}(\mathbb{F}_q)$, let $\lambda_0 = \lambda(m)$, and let χ_m be the characteristic polynomial of m . If m is split, then there exists $P \in \mathbb{F}_q[X]$ such that $\chi_m = P\tilde{P}^{\lambda_0}$.*

Proof. We may assume m is block-triangular as in Lemma 3.1. Let P denote the characteristic polynomial of M . Then the characteristic polynomial of $\lambda(m)M^{-\top}$ is \tilde{P}^{λ_0} . \square

Our next aim is to prove a partial converse to Proposition 3.3 assuming that χ_m is separable, i.e. has only simple roots over an algebraic closure of \mathbb{F}_q . For a monic polynomial $P(X) = X^n + a_{n-1}X^{n-1} + \dots + a_0$ of $\mathbb{F}_q[X]$ of degree n , we denote by c_P its companion matrix:

$$c_P = \begin{pmatrix} 0 & \dots & \dots & 0 & -a_0 \\ 1 & \ddots & & \vdots & -a_1 \\ 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \dots & 0 & 1 & -a_{n-1} \end{pmatrix}.$$

Proposition 3.4. *Let $\chi \in \mathbb{F}_q[X]$ be a monic separable polynomial of degree $2h$ of the form $\chi(X) = P\tilde{P}^{\lambda_0}$ with $P \in \mathbb{F}_q[X]$ and $\lambda_0 \in \mathbb{F}_q^\times$. Factor $P = P_1 \cdots P_r \in \mathbb{F}_q[X]$ into irreducible*

Proposition 3.5. *Let $m \in \mathrm{GSp}_{2h}(\mathbb{F}_q)$, let $\lambda_0 = \lambda(m)$. If the characteristic polynomial χ_m of m is separable and of the form PP^{λ_0} for some $P \in \mathbb{F}_q[X]$, then m is split.*

3.2. Estimating the size of $\mathcal{S}_{2h, \mathbb{F}_q}(\lambda_0)$. We recall that

$$\#\mathrm{GSp}_{2h}(\mathbb{F}_q) = (q-1) \cdot \#\mathrm{Sp}_{2h}(\mathbb{F}_q) = (q-1) \cdot q^{h^2} \cdot \prod_{i=1}^h (q^{2i} - 1) = q^{2h^2+h+1} + O(q^{2h^2+h}).$$

In the following, we will write $f(h) = 2h^2 + h + 1$. As in Theorem 1.1, we set

$$\alpha_h = \sum_{(d_1, \dots, d_r) \in \Sigma_h} \frac{1}{2^r} \cdot \prod_{i=1}^r \frac{1}{d_i} \cdot \prod_{k=1}^h \frac{1}{\#\{j : d_j = k\}}!$$

The main result in this subsection is the following. Recall that the notation O_h means that the implied constants are allowed to depend on h , but not on λ_0 .

Proposition 3.6. *We have $\#\mathcal{S}_{2h, \mathbb{F}_q}(\lambda_0) = \alpha_h q^{f(h)-1} + O_h(q^{f(h)-2})$.*

Proposition 3.5 suggests that split matrices m with separable characteristic polynomial are easier to count. Let $\mathcal{S}_{2h, \mathbb{F}_q}^{sep}(\lambda_0)$ (resp. $\mathcal{S}_{2h, \mathbb{F}_q}^{insep}(\lambda_0)$) be the set of elements $m \in \mathcal{S}_{2h, \mathbb{F}_q}(\lambda_0)$ such that χ_m is separable (resp. inseparable). We obviously have

$$\mathcal{S}_{2h, \mathbb{F}_q}(\lambda_0) = \mathcal{S}_{2h, \mathbb{F}_q}^{sep}(\lambda_0) \sqcup \mathcal{S}_{2h, \mathbb{F}_q}^{insep}(\lambda_0),$$

and we will count elements in each piece, beginning with the inseparable part.

Lemma 3.7. *We have $\#\mathcal{S}_{2h, \mathbb{F}_q}^{insep}(\lambda_0) = O_h(q^{f(h)-2})$.*

Proof. Define $\mathrm{GSp}_{2h}^{insep}(\mathbb{F}_q; \{\lambda_0\})$ as the set of elements of $\mathrm{GSp}_{2h}(\mathbb{F}_q)$ of multiplier λ_0 and whose characteristic polynomial is inseparable. We will in fact prove the stronger claim

$$\#\mathrm{GSp}_{2h}^{insep}(\mathbb{F}_q; \{\lambda_0\}) = O_h(q^{f(h)-2}).$$

To this end, we wish to view $\mathrm{GSp}_{2h}^{insep}(\mathbb{F}_q; \{\lambda_0\})$ as the set of \mathbb{F}_q -points of a certain variety. Let $\Delta : \mathrm{GSp}_{2h} \rightarrow \mathbb{A}^1$ be the morphism which maps m to the discriminant of its characteristic polynomial. The points $m \in \mathrm{GSp}_{2h}$ for which $\Delta(m) = 0$ are precisely the elements whose characteristic polynomial is inseparable. Moreover, the restriction of the morphism $\lambda : \mathrm{GSp}_{2h} \rightarrow \mathbb{G}_m$ to elements m for which $\Delta(m) = 0$ is surjective: indeed, if λ_1 is a point of \mathbb{G}_m , then $\lambda(\sqrt{\lambda_1} \mathrm{Id}_{2h}) = \lambda_1$. Thus, the set of points of GSp_{2h} of multiplier λ_0 and whose characteristic polynomial is inseparable is a subvariety of GSp_{2h} of dimension $\dim(\mathrm{GSp}_{2h}) - 2$, defined by polynomial equations whose degrees are independent of λ_0 .

In [LW54], Lang and Weil proved that the number of points defined over \mathbb{F}_q of a variety of dimension r is $O(q^r)$, where the implicit constant only depends on the dimension and the degree of the variety. The order of $\mathrm{GSp}_{2h}(\mathbb{F}_q)$ is a polynomial expression in q of degree $f(h)$, thus the dimension of GSp_{2h} is $f(h)$ and $\#\mathrm{GSp}_{2h}^{insep}(\mathbb{F}_q; \lambda_0) = O_h(q^{f(h)-2})$. \square

We now estimate the size of $\mathcal{S}_{2h, \mathbb{F}_q}^{sep}(\lambda_0)$. For a partition (d_1, \dots, d_r) of the integer h such that $d_1 \leq \dots \leq d_r$, we denote by $D_{(d_1, \dots, d_r)}(\lambda_0)$ the set of conjugacy classes in $\mathrm{GSp}_{2h}(\mathbb{F}_q)(\lambda_0)$ whose characteristic polynomial is separable and factors as

$$P_1 \cdots P_r \cdot \widetilde{P}_1^{\lambda_0} \cdots \widetilde{P}_r^{\lambda_0}$$

where P_i is irreducible of degree d_i for every $i \in \{1, \dots, r\}$. By Proposition 3.5, there is a one-to-one correspondence between $D_{(d_1, \dots, d_r)}(\lambda_0)$ and a set of characteristic polynomials. Moreover, $\mathcal{S}_{2h, \mathbb{F}_q}^{sep}(\lambda_0)$ is the reunion of all elements of $D_{(d_1, \dots, d_r)}(\lambda_0)$ as (d_1, \dots, d_r) runs through partitions of h .

We first show that the conjugacy classes of $D_{(d_1, \dots, d_r)}(\lambda_0)$ all have the same size. Recall that for a element $m \in \mathrm{GSp}_{2h}(\mathbb{F}_q)$, the number of elements conjugated to m is $\#\mathrm{GSp}_{2h}(\mathbb{F}_q)/\#C(m)$ where $C(m)$ is the centralizer of m .

Lemma 3.8. *Let $P(X) = X^n + a_{n-1}X^{n-1} + \dots + a_0$ be a monic irreducible polynomial of $\mathbb{F}_q[X]$ of degree n . The number of elements $m \in \mathrm{GL}_n(\mathbb{F}_q)$ which commute with the companion matrix c_P is equal to $q^n - 1$.*

Proof. Let u be the endomorphism of \mathbb{F}_q^n associated to the matrix c_P . Then, for every nonzero $x \in \mathbb{F}_q^n$, the family $(x, u(x), \dots, u^{n-1}(x))$ is a basis of \mathbb{F}_q^n because P is irreducible. An element v in $C(u)$ is determined by $v(x)$ since for every $i \in \{0, \dots, n-1\}$, we have $v(u^i(x)) = u^i(v(x))$. If $v(x) \neq 0$, then v maps the basis $(x, u(x), \dots, u^{n-1}(x))$ to the basis $(v(x), u(v(x)), \dots, u^{n-1}(u(x)))$, so v is invertible. Therefore, the elements $m \in \mathrm{GL}_n(\mathbb{F}_q)$ commuting with c_P are in one-to-one correspondence with the nonzero elements of \mathbb{F}_q^n . \square

Lemma 3.9. *With the above notation, the cardinality of each element of $D_{(d_1, \dots, d_r)}(\lambda_0)$ is*

$$\frac{\#\mathrm{GSp}_{2h}(\mathbb{F}_q)}{(q-1) \prod_{i=1}^r (q^{d_i} - 1)}.$$

Proof. By Proposition 3.5, a representative of the class is the block-diagonal matrix

$$m = \mathrm{Diag} \left(c_{P_1}, \dots, c_{P_r}, \lambda_0 c_{P_1}^{-\top}, \dots, \lambda_0 c_{P_r}^{-\top} \right).$$

Matrices in $C(m)$ preserve the invariant subspaces of m , so they are also block-diagonal of the form

$$\mathrm{Diag} \left(N_1, \dots, N_r, \lambda' N_1^{-\top}, \dots, \lambda' N_r^{-\top} \right)$$

where N_i commutes with c_{P_i} for every i . By Lemma 3.8, the number of elements N_i in $\mathrm{GL}_{d_i}(\mathbb{F}_q)$ which commute with c_{P_i} is $q^{d_i} - 1$. Hence,

$$\#C(m) = (q-1) \prod_{i=1}^r (q^{d_i} - 1),$$

where the first factor $(q-1)$ corresponds to the choice of the multiplier λ' . \square

Second, we estimate the size of $D_{(d_1, \dots, d_r)}(\lambda_0)$. We denote by $I_{(d_1, \dots, d_r)}(\lambda_0)$ the set of tuples of irreducible polynomials $(P_1, \dots, P_r) \in \mathbb{F}_q[X]^r$ such that P_i is irreducible of degree d_i for all i and the product $P_1 \cdots P_r \cdot \widetilde{P}_1^{\lambda_0} \cdots \widetilde{P}_r^{\lambda_0}$ is separable. In the next lemma, we identify $D_{(d_1, \dots, d_r)}(\lambda_0)$ with a set of characteristic polynomials.

Lemma 3.10. *Consider the map*

$$H : \begin{cases} I_{(d_1, \dots, d_r)}(\lambda_0) & \rightarrow D_{(d_1, \dots, d_r)}(\lambda_0) \\ (P_1, \dots, P_r) & \mapsto P_1 \cdots P_r \cdot \widetilde{P}_1^{\lambda_0} \cdots \widetilde{P}_r^{\lambda_0}. \end{cases}$$

Then, for every $\chi \in D_{(d_1, \dots, d_r)}(\lambda_0)$, we have

$$\#H^{-1}(\chi) = 2^r \cdot \prod_{k=1}^h \#\{j : d_j = k\}!$$

Proof. Fix an element $(P_1, \dots, P_r) \in H^{-1}(\chi)$. Then, because χ is separable, choosing another element of $H^{-1}(\chi)$ consists in choosing one element of the pair $\{P_i, \widetilde{P}_i^{\lambda_0}\}$ for every $i \in \{1, \dots, r\}$, as well as a permutation of the tuple $(P_{i_{k,1}}, \dots, P_{i_{k,s}})$ where $P_{i_{k,1}}, \dots, P_{i_{k,s}}$ are the polynomials of degree k , for every $k \in \{1, \dots, h\}$. \square

Proof of Proposition 3.6. By Lemma 3.9, the number of elements in $\mathcal{S}_{2h, \mathbb{F}_q}^{sep}(\lambda_0)$ is

$$\#\mathcal{S}_{2h, \mathbb{F}_q}^{sep}(\lambda_0) = \sum_{(d_1, \dots, d_r) \in \Sigma_h} \#D_{(d_1, \dots, d_r)}(\lambda_0) \cdot \frac{\#\mathrm{GSp}_{2h}(\mathbb{F}_q)}{(q-1) \prod_{i=1}^r (q^{d_i} - 1)}.$$

For a partition (d_1, \dots, d_r) of h with $d_1 \leq \dots \leq d_r$, we estimate the size of $D_{(d_1, \dots, d_r)}(\lambda_0)$ by determining the size of $I_{(d_1, \dots, d_r)}(\lambda_0)$ and using Lemma 3.10.

The last coefficients of a monic polynomial P of degree d such that $P = \widetilde{P}^{\lambda_0}$ are determined by the first coefficients, so the number of irreducible polynomials P such that $P = \widetilde{P}^{\lambda_0}$ is $O(q^{d-1})$. For every i , one has to choose P_i such that $P_i \neq \widetilde{P}_i^{\lambda_0}$ and $P_i \neq P_j, \widetilde{P}_j^{\lambda_0}$ for indices $j < i$. Then, according to a formula from Gauss for the number of irreducible polynomials of $\mathbb{F}_q[X]$ of given degree (see [CM11] for a proof), the number of choices for P_i is $\frac{1}{d_i} q^{d_i} + O(q^{d_i-1})$. Hence

$$\#I_{(d_1, \dots, d_r)}(\lambda_0) = \left(\prod_{i=1}^r \frac{1}{d_i} \right) q^h + O_h(q^{h-1}).$$

Therefore,

$$\#D_{(d_1, \dots, d_r)}(\lambda_0) = \left(\frac{1}{2^r} \cdot \prod_{i=1}^r \frac{1}{d_i} \cdot \prod_{k=1}^h \frac{1}{\#\{j : d_j = k\}!} \right) q^h + O_h(q^{h-1})$$

and consequently

$$\#\mathcal{S}_{2h, \mathbb{F}_q}^{sep}(\lambda_0) = \alpha_h q^{f(h)-1} + O(q^{f(h)-2}).$$

Combining this with Lemma 3.7 ends the proof. \square

3.3. The special case $h = 2$. When $h = 2$ and $q = \ell$ is an odd prime, we are able to determine the exact cardinality of $\mathcal{S}_{2h, \mathbb{F}_\ell}$, through the classification of the conjugacy classes of $\mathrm{GSp}_4(\mathbb{F}_\ell)$ and the computation of the cardinality of the classes in [Bre11].

Proposition 3.11. *We have*

$$\#\mathcal{S}_{4, \mathbb{F}_\ell} = \frac{(3\ell^3 + 7\ell^2 + 7\ell + 11)(\ell + 1)(\ell - 1)^3 \ell^4}{8}.$$

Proof. Conjugacy classes of $\mathrm{GSp}_4(\mathbb{F}_\ell)$ have been sorted in different types [Wil12, Section 6.2] according to the factorization of the characteristic polynomial, and the number of classes of each type is known. We also notice that for each class whose characteristic polynomial χ splits as $P\tilde{P}^{\lambda_0}$, there exists a representative of the form

$$m = \begin{pmatrix} M & \star \\ 0 & \lambda(m)M^{-\tau} \end{pmatrix},$$

so the converse of Proposition 3.3 is true in $\mathrm{GSp}_4(\mathbb{F}_\ell)$ even if χ_m is not separable. Thus, we have to count the number of elements which belong to a class whose characteristic polynomial splits as $P\tilde{P}^{\lambda_0}$. The order of each center has been computed in [Bre11, Table 1] (beware that the antisymmetric matrix used to define GSp_{2h} there is not J_{2h} , so notation differs from [Wil12]). Thus we can deduce the size of each conjugacy class. \square

The proof of Proposition 3.11 suggests the following question: does the converse of Proposition 3.5 hold true in $\mathrm{GSp}_{2h}(\mathbb{F}_q)$ for all h , even if χ_m is not separable?

4. THE DISTRIBUTION OF ELKIES PRIMES

In this Section, we prove Theorem 1.2. We introduce the character sum U_k , similar to the sum U in [SS15, eq. (4)], in §4.1. We control the small terms in this sum in §4.2 and we estimate the dominant term in §4.3. Finally, we conclude the proof in §4.4.

4.1. Setup. We keep the notation from Theorem 1.1. We may suppose that P and L are sufficiently large, so that $A_{\mathfrak{p}}$ is well-defined for every $\mathfrak{p} \in \mathcal{P}_F(P, 2P)$, and if $\mathcal{L} = \mathfrak{l}_1 \cdots \mathfrak{l}_r$ is the product of r distinct primes of $\mathcal{P}_K(L, 2L)$, then

$$G_{\mathcal{L}} := \mathrm{Gal}(F(A[\mathcal{L}])/F)$$

is contained between $\mathrm{Sp}_{2h}(\mathcal{O}/\mathcal{L}\mathcal{O})$ and $\mathrm{GSp}_{2h}(\mathcal{O}/\mathcal{L}\mathcal{O})$. This is harmless since we want to establish an asymptotic result.

The Landau prime ideal theorem [Lan03] for the fields K and F asserts that

$$\#\mathcal{P}_K(L, 2L) \sim \frac{L}{\log(L)} \quad \text{and} \quad \#\mathcal{P}_F(P, 2P) \sim \frac{P}{\log(P)}.$$

Let

$$\delta_{\mathfrak{p}, \mathfrak{l}} = \begin{cases} (1 - \alpha_h) & \text{if } \mathfrak{l} \text{ is Elkies for } A_{\mathfrak{p}}, \\ -\alpha_h & \text{otherwise} \end{cases}$$

For a product $\mathfrak{l}_1 \cdots \mathfrak{l}_r$, we define

$$\delta_{\mathfrak{p}, \mathfrak{l}_1 \cdots \mathfrak{l}_r} = \delta_{\mathfrak{p}, \mathfrak{l}_1} \cdots \delta_{\mathfrak{p}, \mathfrak{l}_r}.$$

We further set

$$\mu = \alpha_h \#\mathcal{P}_K(L, 2L) \quad \text{and} \quad \sigma = \sqrt{\alpha_h(1 - \alpha_h)\#\mathcal{P}_K(L, 2L)}.$$

By definition,

$$\begin{aligned} N_e(\mathfrak{p}, L) - \mu &= (1 - \alpha_h)N_e(\mathfrak{p}, L) - \alpha_h(\#\mathcal{P}_K(L, 2L) - N_e(\mathfrak{p}, L)) \\ &= \sum_{\mathfrak{l} \in \mathcal{P}_K(L, 2L)} \delta_{\mathfrak{p}, \mathfrak{l}}. \end{aligned}$$

For any integer $k \geq 1$, the k -th moment of $X_{P, L}$ is

$$\begin{aligned} \mathbb{E}(X_{P, L}^k) &= \frac{1}{\#\mathcal{P}_F(P, 2P)} \sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \left(\frac{N_e(\mathfrak{p}, L) - \mu}{\sigma} \right)^k \\ &= \frac{1}{\#\mathcal{P}_F(P, 2P) \cdot \sigma^k} \sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \left(\sum_{\mathfrak{l} \in \mathcal{P}_K(L, 2L)} \delta_{\mathfrak{p}, \mathfrak{l}} \right)^k \\ &= \frac{1}{\#\mathcal{P}_F(P, 2P) \cdot \sigma^k} \sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \sum_{\substack{\mathfrak{l}_1, \dots, \mathfrak{l}_k \\ \in \mathcal{P}_K(L, 2L)}} \delta_{\mathfrak{p}, \mathfrak{l}_1 \dots \mathfrak{l}_k}. \end{aligned}$$

Hence, we are led to considering the sums

$$U_k := \sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \sum_{\substack{\mathfrak{l}_1, \dots, \mathfrak{l}_k \\ \in \mathcal{P}_K(L, 2L)}} \delta_{\mathfrak{p}, \mathfrak{l}_1 \dots \mathfrak{l}_k}.$$

We expect compensations in the sum U_k when some primes among $\mathfrak{l}_1, \dots, \mathfrak{l}_k$ appear an odd number of times, and we will sort terms according to the number of distinct primes. In the spirit of the proof of [SS15, Theorem 1], for $0 \leq j \leq k$, let $\mathcal{Q}_{k, j}$ be the set of tuples $(\mathfrak{l}_1, \dots, \mathfrak{l}_k)$ of primes in $\mathcal{P}_K(L, 2L)$ such that $\mathfrak{l}_1 \dots \mathfrak{l}_k = \mathfrak{a}^2 \mathfrak{b}$ where \mathfrak{b} is a squarefree product of j prime ideals and \mathfrak{a} is the product of $\frac{k-j}{2}$ prime ideals ($\mathcal{Q}_{k, j}$ is empty if $k-j$ is odd). If $k = 2\nu$ is even, we also define $\mathcal{Q}'_{k, 0}$ to be the set of tuples $(\mathfrak{l}_1, \dots, \mathfrak{l}_k)$ such that the \mathfrak{l}_i 's can be grouped in ν distinct pairs. We will see that the dominant term comes from the contribution of the terms of $\mathcal{Q}'_{k, 0}$. We begin by estimating the other terms.

4.2. Small terms. We want to prove the following result, which generalizes the lemmas 5 and 6 in [SS15]. As in Theorem 1.1, the dependency on A in Landau's notation includes the dependency on F , \mathcal{O} and h .

Proposition 4.1. *Assume GRH. For $P > 2L$ and a product $\mathcal{L} = \mathfrak{l}_1 \dots \mathfrak{l}_r$ of r distinct primes of $\mathcal{P}_K(L, 2L)$, we have*

$$\sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \delta_{\mathfrak{p}, \mathcal{L}} = O_{A, r} \left(\frac{P}{\log(P)L^r} + L^{f(h)r} P^{1/2} \log(P) \right).$$

The proof of this proposition is based on the Čebotarev density theorem in the Galois group $G_{\mathcal{L}}$, which is a subgroup of

$$\mathrm{GSp}_{2h} \left(\prod_{i=1}^r \mathcal{O}/\mathfrak{l}_i \mathcal{O} \right) \cong \prod_{i=1}^r \mathrm{GSp}_{2h}(\mathcal{O}/\mathfrak{l}_i \mathcal{O}).$$

Thus, an element $m \in G_{\mathcal{L}}$ can be identified with an element

$$(m_1, \dots, m_r) \in \prod_{i=1}^r \mathrm{GSp}_{2h}(\mathcal{O}/\mathfrak{l}_i \mathcal{O})$$

and its multiplier $\lambda(m)$ with an element

$$(\lambda_1, \dots, \lambda_r) \in \prod_{i=1}^r (\mathcal{O}/\mathfrak{l}_i \mathcal{O})^\times.$$

For $(\lambda_1, \dots, \lambda_r) \in \lambda(G_{\mathcal{L}})$, we define

$$G_{\mathcal{L}}(\lambda_1, \dots, \lambda_r) := \{m \in G_{\mathcal{L}} : \lambda(m) = (\lambda_1, \dots, \lambda_r)\}$$

which can be identified with

$$\prod_{i=1}^r \mathrm{GSp}_{2h}(\mathcal{O}/\mathfrak{l}_i \mathcal{O}, \{\lambda_i\}).$$

In particular, by the large Galois image assumption,

$$\#G_{\mathcal{L}}(\lambda_1, \dots, \lambda_r) = \prod_{i=1}^r \#\mathrm{Sp}_{2h}(\mathcal{O}/\mathfrak{l}_i \mathcal{O}) \quad \text{and} \quad \#G_{\mathcal{L}} = \#\lambda(G_{\mathcal{L}}) \cdot \prod_{i=1}^r \#\mathrm{Sp}_{2h}(\mathcal{O}/\mathfrak{l}_i \mathcal{O}).$$

Let us now construct the conjugacy classes in $G_{\mathcal{L}}$ we are interested in. Given a tuple $(\varepsilon_1, \dots, \varepsilon_r) \in \{\pm 1\}^r$, we denote by

$$\mathcal{C}_{\mathfrak{l}_1, \dots, \mathfrak{l}_r}(\varepsilon_1, \dots, \varepsilon_r) \subset G_{\mathcal{L}}$$

the set of elements $m = (m_1, \dots, m_r)$ of $G_{\mathcal{L}}$ such that $m_i \in \mathcal{S}_{2h, \mathcal{O}/\mathfrak{l}_i \mathcal{O}}$ if $\varepsilon_i = 1$ and $m_i \notin \mathcal{S}_{2h, \mathcal{O}/\mathfrak{l}_i \mathcal{O}}$ if $\varepsilon_i = -1$. This set is stable by conjugation in $G_{\mathcal{L}}$. For a given prime \mathfrak{p} of good reduction for A , if we set

$$\varepsilon_{\mathfrak{p}, \mathfrak{l}_i} = \begin{cases} 1 & \text{if } \mathfrak{l}_i \text{ is Elkies for } A_{\mathfrak{p}}, \\ -1 & \text{otherwise.} \end{cases}$$

then by Proposition 2.10, the Frobenius element $\sigma_{\mathfrak{p}}$ at \mathfrak{p} in $G_{\mathcal{L}}$ satisfies

$$(\bar{\rho}_{\mathfrak{l}_1}(\sigma_{\mathfrak{p}}), \dots, \bar{\rho}_{\mathfrak{l}_r}(\sigma_{\mathfrak{p}})) \in \mathcal{C}_{\mathfrak{l}_1, \dots, \mathfrak{l}_r}(\varepsilon_1, \dots, \varepsilon_r)$$

if and only if $\varepsilon_{\mathfrak{p}, \mathfrak{l}_i} = \varepsilon_i$ for all i .

We also need to determine the size of this conjugacy class. For $\lambda_i \in (\mathcal{O}/\mathfrak{l}_i \mathcal{O})^\times$, we define

$$\begin{cases} C_{\mathfrak{l}_i}^1(\lambda_i) & = \#\mathcal{S}_{2h, \mathcal{O}/\mathfrak{l}_i \mathcal{O}}(\lambda_i), \\ C_{\mathfrak{l}_i}^{-1}(\lambda_i) & = \#(\mathrm{GSp}_{2h}(\mathcal{O}/\mathfrak{l}_i \mathcal{O}; \{\lambda_i\}) - \mathcal{S}_{2h, \mathcal{O}/\mathfrak{l}_i \mathcal{O}}(\lambda_i)). \end{cases}$$

Considering the preimage of each multiplier $(\lambda_1, \dots, \lambda_r) \in \lambda(G_{\mathcal{L}})$ separately, we immediately obtain

$$\#\mathcal{C}_{\mathfrak{l}_1, \dots, \mathfrak{l}_r}(\varepsilon_1, \dots, \varepsilon_r) = \sum_{(\lambda_1, \dots, \lambda_r) \in \lambda(G_{\mathcal{L}})} \prod_{i=1}^r C_{\mathfrak{l}_i}^{\varepsilon_i}(\lambda_i).$$

Lemma 4.2. *Let $\mathcal{L} = \mathfrak{l}_1 \cdots \mathfrak{l}_r$ be a product of distinct primes of $\mathcal{P}_K(L, 2L)$ and $(\varepsilon_1, \dots, \varepsilon_r)$ be an element of $\{\pm 1\}^r$. Then, for $x > 2L$, we have*

$$\begin{aligned} \#\{\mathfrak{p} : N(\mathfrak{p}) \leq x \text{ and } \varepsilon_{\mathfrak{p}, \mathfrak{l}_i} = \varepsilon_i \text{ for all } i\} &= \frac{\sum_{(\lambda_1, \dots, \lambda_r) \in \lambda(G_{\mathcal{L}})} \prod_{i=1}^r C_{\mathfrak{l}_i}^{\varepsilon_i}(\lambda_i)}{\#G_{\mathcal{L}}} \frac{x}{\log(x)} \\ &+ O_{A,r} \left(N(\mathfrak{l}_1)^{f(h)} \cdots N(\mathfrak{l}_r)^{f(h)} x^{1/2} \log(x) \right). \end{aligned}$$

Proof. This follows from an effective version of the Čebotarev density theorem in $G_{\mathcal{L}}$ [Ser81, §2, Equation (20_R)] for the set $\mathcal{C}_{\mathfrak{l}_1, \dots, \mathfrak{l}_r}(\varepsilon_1, \dots, \varepsilon_r)$. In the left-hand side of (20_R), an upper bound on $\#\mathcal{C}_{\mathfrak{l}_1, \dots, \mathfrak{l}_r}(\varepsilon_1, \dots, \varepsilon_r)$ is the order of $G_{\mathcal{L}}$, which is $O_r(N(\mathfrak{l}_1)^{f(h)} \cdots N(\mathfrak{l}_r)^{f(h)})$. The degree n of the extension $F(A[\mathcal{L}])/F$ is equal to the order of $G_{\mathcal{L}}$. Since $x > 2L$, we have $\log(n) = O_{r,h}(\log(x))$. For every $i \in \{1, \dots, r\}$, let ℓ_i be the prime number below \mathfrak{l}_i . The ramified primes in the extension $F(A[\mathcal{L}])/F$ lie among the divisors of ℓ_1, \dots, ℓ_r and the primes of bad reduction of A (this follows from the Néron-Ogg-Shafarevich criterion), so

$$\log \left(\prod_{i=1}^r \ell_i \right) = O_r(\log(x))$$

under the assumption $x > 2L$. □

Proof of Proposition 4.1. We have

$$\sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \delta_{\mathfrak{p}, \mathcal{L}} = \sum_{\substack{(\gamma_1, \dots, \gamma_r) \\ \in \{1 - \alpha_h, -\alpha_h\}^r}} \gamma_1 \cdots \gamma_r \cdot \#\{\mathfrak{p} : N(\mathfrak{p}) \leq x \text{ and } \varepsilon_{\mathfrak{p}, \mathfrak{l}_i} = \varepsilon_i \text{ for all } i\}$$

where $\varepsilon_i = 1$ if $\gamma_i = 1 - \alpha_h$ and $\varepsilon_i = -1$ if $\gamma_i = -\alpha_h$. Write

$$S_{(\lambda_1, \dots, \lambda_r)}(\mathfrak{l}_1, \dots, \mathfrak{l}_r) = \sum_{\substack{(\gamma_1, \dots, \gamma_r) \\ \in \{1 - \alpha_h, -\alpha_h\}^r}} \gamma_1 \cdots \gamma_r \cdot \prod_{i=1}^r C_{\mathfrak{l}_i}^{\varepsilon_i}(\lambda_i).$$

By the previous lemma,

$$\begin{aligned} (5) \quad \sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \delta_{\mathfrak{p}, \mathcal{L}} &= \frac{\sum_{(\lambda_1, \dots, \lambda_r) \in \lambda(G_{\mathcal{L}})} S_{(\lambda_1, \dots, \lambda_r)}(\mathfrak{l}_1, \dots, \mathfrak{l}_r)}{\#G_{\mathcal{L}}} \cdot \frac{P}{\log(P)} \\ &+ O_{A,r} \left(N(\mathfrak{l}_1)^{f(h)} \cdots N(\mathfrak{l}_r)^{f(h)} P^{1/2} \log(P) \right). \end{aligned}$$

Fix $(\lambda_1, \dots, \lambda_r) \in \lambda(G_{\mathcal{L}})$. Then,

$$S_{(\lambda_1, \dots, \lambda_r)}(\mathfrak{l}_1, \dots, \mathfrak{l}_r) = \prod_{i=1}^r S_{\lambda_i}(\mathfrak{l}_i)$$

where $S_{\lambda_i}(\mathfrak{l}_i) := (1 - \alpha_h)C_{\mathfrak{l}_i}^1(\lambda_i) - \alpha_h C_{\mathfrak{l}_i}^{-1}(\lambda_i)$. We have $S_{\lambda_i}(\mathfrak{l}_i) = O_h(N(\mathfrak{l}_i)^{f(h)-2})$ according to Proposition 3.6. Thus,

$$S_{(\lambda_1, \dots, \lambda_r)}(\mathfrak{l}_1, \dots, \mathfrak{l}_r) = O_{A,r} \left(N(\mathfrak{l}_1)^{f(h)-2} \dots N(\mathfrak{l}_r)^{f(h)-2} \right).$$

In this equality, the implicit constant is independent of $(\lambda_1, \dots, \lambda_r)$. Therefore,

$$\sum_{(\lambda_1, \dots, \lambda_r) \in \lambda(G_{\mathcal{L}})} S_{(\lambda_1, \dots, \lambda_r)}(\mathfrak{l}_1, \dots, \mathfrak{l}_r) = O_{A,r} \left(\#\lambda(G_{\mathcal{L}}) \cdot N(\mathfrak{l}_1)^{f(h)-2} \dots N(\mathfrak{l}_r)^{f(h)-2} \right).$$

Consequently,

$$\frac{\sum_{(\lambda_1, \dots, \lambda_r) \in \lambda(G_{\mathcal{L}})} S_{(\lambda_1, \dots, \lambda_r)}(\mathfrak{l}_1, \dots, \mathfrak{l}_r)}{\#G_{\mathcal{L}}} = O_{A,r} \left(\frac{1}{L^r} \right).$$

Inserting this upper bound and writing $N(\mathfrak{l}_i) \leq 2L$ in (5) ends the proof. \square

4.3. The dominant term. When k is even, the dominant term of U_k corresponds to the contribution of elements of $\mathcal{Q}'_{k,0}$. We begin by estimating the size of this set. For a positive integer ν , we recall that $m_{2\nu}$ is the moment of order 2ν of the standard Gaussian distribution. Its value is

$$m_{2\nu} = (2\nu - 1) \cdot (2\nu - 3) \dots 3 \cdot 1.$$

Lemma 4.3. *Let ν be a positive integer. Then,*

$$\begin{aligned} \#\mathcal{Q}'_{2\nu,0} &= m_{2\nu} \frac{L^\nu}{\log(L)^\nu} + O_\nu \left(\frac{L^{\nu-1}}{\log(L)^{\nu-1}} \right), \quad \text{and} \\ \#(\mathcal{Q}_{2\nu,0} - \mathcal{Q}'_{2\nu,0}) &= O_\nu \left(\frac{L^{\nu-1}}{\log(L)^{\nu-1}} \right). \end{aligned}$$

Proof. For $n \in \{1, \dots, \nu\}$, let \mathcal{A}_n be the set of tuples (A_1, \dots, A_n) of disjoint subsets of $\{1, \dots, 2\nu\}$ such that:

- for every $i \in \{1, \dots, n\}$, $A_i \neq \emptyset$,
- for every $(i, j) \in \{1, \dots, n\}^2$, $A_i \cap A_j = \emptyset$,
- for every $i \in \{1, \dots, n\}$, $\#A_i$ is even,
- $\bigsqcup_{i=1}^n A_i = \{1, \dots, 2\nu\}$.

We equip $\mathcal{P}_K(L, 2L)$ with an arbitrary total order $<$. We also define \mathcal{B}_L^n to be the set of ordered n -tuples of distinct prime ideals of $\mathcal{P}_K(L, 2L)$. Let $s = (\mathfrak{l}_1, \dots, \mathfrak{l}_{2\nu})$ be an element of $\mathcal{Q}_{2\nu,0}$ such that $\text{lcm}(\mathfrak{l}_1 \dots \mathfrak{l}_{2\nu})$ has n distinct prime factors, and $\mathfrak{l}'_1 < \dots < \mathfrak{l}'_n$ be the primes such that

$$\{\mathfrak{l}_1, \dots, \mathfrak{l}_{2\nu}\} = \{\mathfrak{l}'_1, \dots, \mathfrak{l}'_n\}.$$

Then, we define $b_s = (\iota'_1, \dots, \iota'_n)$. For $j \in \{1, \dots, n\}$, we set

$$A_j^s = \{i \in \{1, \dots, 2\nu\} : \iota_i = \iota'_j\}$$

and $a_s = (A_1^s, \dots, A_n^s)$.

With this notation, the set $\mathcal{Q}_{2\nu,0}$ is in one-to-one correspondence with

$$\bigsqcup_{1 \leq n \leq \nu} \mathcal{A}_n \times \mathcal{B}_L^n$$

via $s \mapsto (a_s, b_s)$, and $\mathcal{Q}'_{2\nu,0}$ is in one-to-one correspondence with $\mathcal{A}_\nu \times \mathcal{B}'_L$.

If n is fixed, we have

$$\#\mathcal{B}_L^n = \binom{\#\mathcal{P}_K(L, 2L)}{n} \sim \frac{L^n}{n! \log(L)^n}$$

as L goes to infinity. For $n = \nu$, we have

$$\#\mathcal{A}_\nu = \binom{2\nu}{2} \cdot \binom{2\nu-2}{2} \cdots \binom{2}{2} = \nu! \cdot m_{2\nu},$$

so $\#(\mathcal{A}_\nu \times \mathcal{B}'_L) \sim m_{2\nu} \frac{L^\nu}{\log(L)^\nu}$. On the other hand, for $n \leq \nu - 1$, we have

$$\#\mathcal{B}_L^n = O\left(\frac{L^{\nu-1}}{\log(L)^{\nu-1}}\right).$$

Since $\#\mathcal{A}_n$ is a constant independent of L , we obtain

$$\sum_{n=1}^{\nu-1} \#\mathcal{A}_n \cdot \#\mathcal{B}_L^n = O_\nu\left(\frac{L^{\nu-1}}{\log(L)^{\nu-1}}\right). \quad \square$$

We are able to prove a more precise statement than Proposition 4.1 for elements of $\mathcal{Q}'_{2\nu,0}$.

Proposition 4.4. *Let $(\iota_1, \dots, \iota_{2\nu}) \in \mathcal{Q}'_{2\nu,0}$. Then,*

$$\sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \delta_{\mathfrak{p}, \iota_1 \dots \iota_{2\nu}} = (\alpha_h (1 - \alpha_h))^\nu \frac{P}{\log(P)} + O_{A, \nu} \left(\frac{P}{\log(P)L} + L^{f(h)\nu} P^{1/2} \log(P) \right).$$

Proof. Assume that $\{\iota_1, \dots, \iota_{2\nu}\} = \{\iota'_1, \dots, \iota'_\nu\}$ where $\iota'_1 < \dots < \iota'_\nu$. Given $(\gamma_1, \dots, \gamma_\nu)$ in $\{1 - \alpha_h, -\alpha_h\}^\nu$, denote by $\mathcal{D}'_{\iota'_1, \dots, \iota'_\nu}(\gamma_1, \dots, \gamma_\nu)$ the set of primes $\mathfrak{p} \in \mathcal{P}_F(P, 2P)$ such that for every i , the prime ι'_i is Elkies for $A_{\mathfrak{p}}$ if $\gamma_i = 1 - \alpha_h$, and ι'_i is not Elkies for $A_{\mathfrak{p}}$ if $\gamma_i = -\alpha_h$. As in Lemma 4.2, the Čebotarev density theorem yields:

$$\begin{aligned} \#\mathcal{D}'_{\iota'_1, \dots, \iota'_\nu}(\gamma_1, \dots, \gamma_\nu) &= \frac{\sum_{(\lambda_1, \dots, \lambda_\nu) \in \lambda(G_{\iota'_1, \dots, \iota'_\nu})} \prod_{i=1}^{\nu} C_{\iota'_i}^{\varepsilon_i}(\lambda_i)}{\#G_{\iota'_1, \dots, \iota'_\nu}} \frac{P}{\log(P)} + O_{A, \nu} \left(L^{\nu f(h)} P^{1/2} \log(P) \right) \\ &= (\alpha_h)^{k_1} (1 - \alpha_h)^{\nu - k_1} \frac{P}{\log(P)} + O_{A, \nu} \left(\frac{P}{\log(P)L} + L^{\nu f(h)} P^{1/2} \log(P) \right) \end{aligned}$$

where k_1 is the number of entries γ_i equal to $1 - \alpha_h$. Then,

$$\begin{aligned}
\sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \delta_{\mathfrak{p}, l_1 \dots l_{2\nu}} &= \sum_{\substack{(\gamma_1, \dots, \gamma_\nu) \\ \in \{1 - \alpha_h, -\alpha_h\}^\nu}} (1 - \alpha_h)^{2k_1} (\alpha_h)^{2(\nu - k_1)} \cdot \#D_{l_1, \dots, l_\nu}(\gamma_1, \dots, \gamma_\nu) \\
&= \sum_{k_1=0}^{\nu} \binom{\nu}{k_1} (1 - \alpha_h)^{2k_1} (\alpha_h)^{2(\nu - k_1)} (\alpha_h)^{k_1} (1 - \alpha_h)^{\nu - k_1} \frac{P}{\log(P)} \\
&\quad + O_{A, \nu} \left(\frac{P}{\log(P) \cdot L} + L^{f(h)\nu} P^{1/2} \log(P) \right) \\
&= (\alpha_h (1 - \alpha_h))^\nu \frac{P}{\log(P)} + O_{A, \nu} \left(\frac{P}{\log(P) \cdot L} + L^{f(h)\nu} P^{1/2} \log(P) \right). \quad \square
\end{aligned}$$

4.4. Conclusion of the proof. We go back to estimating the moments of $X_{P, L}$. First, assume that k is odd, and write $k = 2\nu + 1$. Then

$$U_k = \sum_{j=0}^{\nu} \sum_{\substack{(l_1, \dots, l_{2\nu+1}) \\ \in \mathcal{Q}_{2\nu+1, 2j+1}}} \sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \delta_{\mathfrak{p}, l_1 \dots l_{2\nu+1}}.$$

For $j \in \{0, \dots, \nu\}$, we have

$$\#\mathcal{Q}_{2\nu+1, 2j+1} = O_\nu \left(\frac{L^{\nu+j+1}}{\log(L)^{\nu+j+1}} \right),$$

so by Proposition 4.1,

$$\begin{aligned}
&\sum_{\substack{(l_1, \dots, l_{2\nu+1}) \\ \in \mathcal{Q}_{2\nu+1, 2j+1}}} \sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \delta_{\mathfrak{p}, l_1 \dots l_{2\nu+1}} \\
&= O_{A, \nu} \left(\frac{L^{\nu+j+1}}{\log(L)^{\nu+j+1}} \left(\frac{P}{L^{2j+1} \log(P)} + L^{f(h)(2j+1)} P^{1/2} \log(P) \right) \right).
\end{aligned}$$

The dominant terms occur for $j = 0$ and $j = \nu$. By getting rid of the non-dominant terms, we obtain

$$U_k = O_{A, k} \left(\frac{L^\nu P}{\log(L)^{\nu+1} \log(P)} + \frac{L^{(2\nu+1)(f(h)+1)} P^{1/2} \log(P)}{\log(L)^{2\nu+1}} \right).$$

We finally plug this upper bound into the expression for $\mathbb{E}(X_{P, L}^k)$ in §4.1. We have

$$\sigma^k \cdot \#\mathcal{P}_F(P, 2P) \underset{P, L \rightarrow +\infty}{\sim} (\alpha_h (1 - \alpha_h))^{\nu+1/2} \frac{P}{\log(P)} \cdot \frac{L^{\nu+1/2}}{\log(L)^{\nu+1/2}}.$$

Hence,

$$\begin{aligned}\mathbb{E}(X_{P,L}^k) &= \frac{U_k}{\#\mathcal{P}_F(P, 2P)\sigma^k} \\ &= O_{A,k} \left(\frac{1}{L^{1/2} \log(L)^{1/2}} + \frac{L^{(2\nu+1)(f(h)+1)-\nu-1/2} \log(P)^2}{\log(L)^{\nu+1/2} P^{1/2}} \right),\end{aligned}$$

proving Theorem 1.2 for odd k .

Second, assume that $k = 2\nu$ is even. We also write

$$U_{2\nu} = \sum_{j=0}^{\nu} \sum_{(l_1, \dots, l_{2\nu}) \in \mathcal{Q}_{2\nu, 2j}} \sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \delta_{\mathfrak{p}, l_1 \dots l_{2\nu}}.$$

For $j \in \{1, \dots, \nu\}$, we obtain as above

$$\sum_{(l_1, \dots, l_{2\nu}) \in \mathcal{Q}_{2\nu, 2j}} \sum_{\mathfrak{p} \in \mathcal{P}_F(P, 2P)} \delta_{\mathfrak{p}, l_1 \dots l_{2\nu}} = O_{A,\nu} \left(\frac{L^{\nu+j}}{\log(L)^{\nu+j}} \left(\frac{P}{L^{2j} \log(P)} + L^{2f(h)j} P^{1/2} \log(P) \right) \right).$$

Now assume that $j = 0$. By Lemma 4.3 and Proposition 4.4, the contribution of elements of $\mathcal{Q}'_{2\nu, 0}$ to U_k is

$$\begin{aligned}m_{2\nu} \frac{L^\nu}{\log(L)^\nu} (\alpha_h \cdot (1 - \alpha_h))^\nu \frac{P}{\log(P)} \\ + O_{A,\nu} \left(\frac{P}{\log(P)L} + \frac{PL^{\nu-1}}{\log(P) \log(L)^{\nu-1}} + \frac{L^{(f(h)+1)\nu} P^{1/2} \log(P)}{\log(L)^\nu} \right).\end{aligned}$$

while the contribution of elements from $\mathcal{Q}_{2\nu, 0} - \mathcal{Q}'_{2\nu, 0}$ is

$$O_\nu \left(\frac{P \log(L)^{\nu-1}}{\log(P) \log(L)^{\nu-1}} \right).$$

The dominant terms in the above upper bounds occur for $j = 0$ and $j = \nu$, and we have

$$\begin{aligned}U_{2\nu} &= m_{2\nu} \frac{L^\nu}{\log(L)^\nu} (\alpha_h \cdot (1 - \alpha_h))^\nu \frac{P}{\log(P)} \\ &\quad + O_{A,\nu} \left(\frac{P}{\log(P)} \frac{L^{\nu-1}}{\log(L)^{\nu-1}} + \frac{L^{(2f(h)+2)\nu} P^{1/2} \log(P)}{\log(L)^{2\nu}} \right).\end{aligned}$$

Therefore,

$$\mathbb{E}(X_{P,L}^k) = m_k + O_{A,k} \left(\frac{\log(L)}{L} + \frac{L^{(2f(h)+1)\nu} \log(P)^2}{\log(L)^\nu P^{1/2}} \right).$$

This concludes the proof of Theorem 1.2; Theorem 1.1 is a consequence of this theorem and [Bil95, Theorem 30.2].

5. NUMERICAL EXPERIMENTS

At the beginning of this project, we performed numerical experiments with *SageMath* [Sag24] in order to confirm experimentally the estimate of [SS15, Theorem 1]. All the experiments presented in this section were made with the non-CM elliptic curve E given by the Weierstrass equation $y^2 + y = x^3 - x^2$ defined over \mathbb{Q} (Cremona label 11a3). We began by computing some values of the left-hand side of [SS15, Theorem 1] by fixing one of the variable L or P and by letting the other one vary. Fig. 1 shows the evolution of the left-hand side for $\nu = 1$ for three values of L (namely 25, 100 and 250) and P varying between 10^3 and $5 \cdot 10^6$.

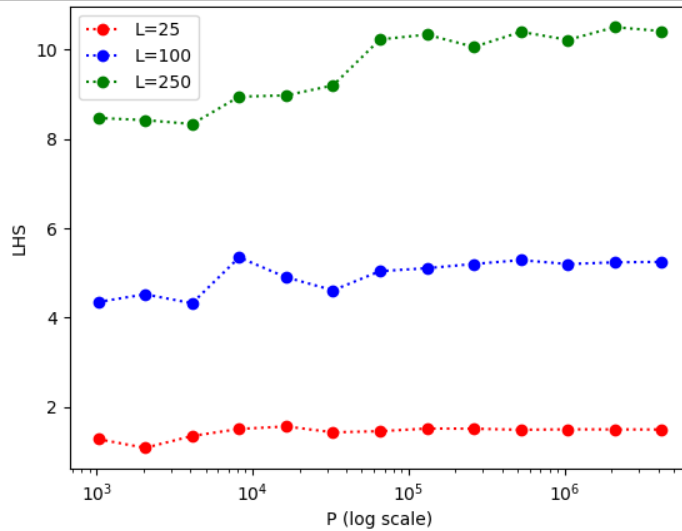


FIGURE 1. Moment of order 2 for $P \in [10^3, 5 \cdot 10^6]$

This graph suggests that the left-hand side has a finite limit (which depends on L) as P goes to infinity. To go further, we analyzed the distribution of $N_e(p, L)$, i.e. the number of primes $p \in \mathcal{P}_{\mathbb{Q}}(P, 2P)$ such that $N_e(p, L) = n$ as n varies between 0 and $\pi(2L) - \pi(L) + 1$. We observed that this distribution has a Gaussian shape when P is much larger than L as in Fig. 2.

We then tried to predict the mean value and the standard deviation as a function of L through a naive probabilistic model, relying on the fact that the standard hypothesis that 50% of prime numbers are Elkies is correct. In other words, for every $p \in \mathcal{P}_{\mathbb{Q}}(P, 2P)$, a prime $\ell \in [L, 2L]$ has a probability 1/2 to be Elkies for the reduced curve E_p , and those events are independent. Then, the number of Elkies primes for E_p in $[L, 2L]$ follows a

binomial distribution $\mathcal{B}(\pi(2L) - \pi(L), 1/2)$, whose expected value μ and deviation σ are

$$\mu = \frac{\pi(2L) - \pi(L)}{2} \quad \text{and} \quad \sigma = \frac{\sqrt{\pi(2L) - \pi(L)}}{2}.$$

Therefore, when P is much larger than L , we expect the actual distribution of Elkies primes to look like a Gaussian function with those parameters. In Fig. 2, we plot the distribution for $L = 250$ and $P = 10^7$ in blue and the associated Gaussian red; we see that the naive model fits very well with the reality.

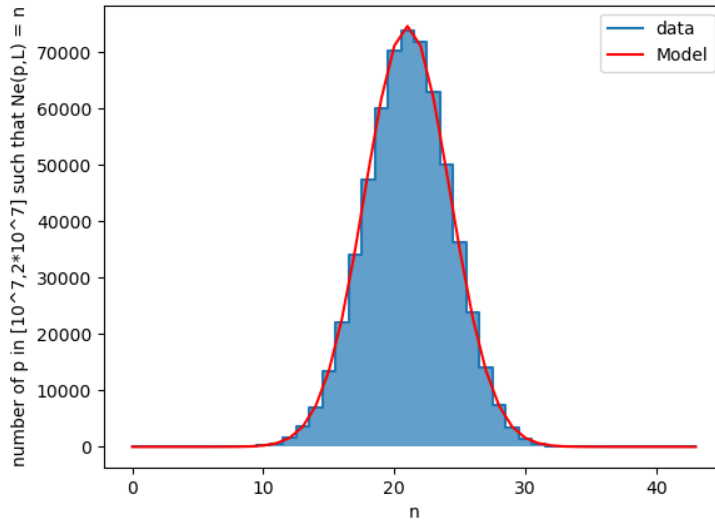


FIGURE 2. Distribution with $L = 250$ and $P = 10^7$

The predicted value of the left-hand side of [SS15, Theorem 1] for $\nu = 1$ is the moment of order 2 of the binomial distribution $\mathcal{B}(\pi(2L) - \pi(L), 1/2)$, which is $(\pi(2L) - \pi(L))/4$. On Fig. 3, we fix $P = 10^5$ and we let L vary in $[20, 500]$. We plot the evolution of the left-hand side for in blue and the predicted value in red. We see that the model is accurate for small values of L , but when L is larger than \sqrt{P} , a gap between the model and the reality starts appearing.

All in all, these numerical experiments gave us the idea that the distribution of Elkies primes converges to a Gaussian function when P and L go to infinity with P growing quickly compared with L . The naive model allowed us to predict the parameters of this Gaussian function in the setting of elliptic curves, setting us on the path towards Theorem 1.1.

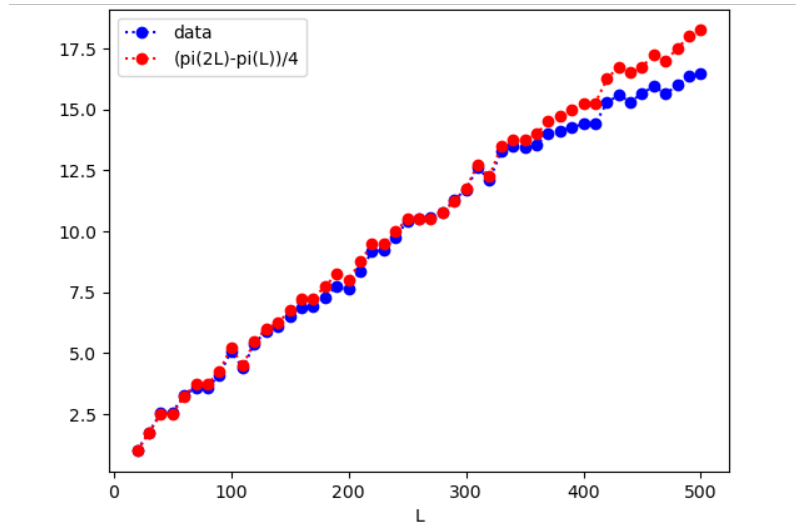


FIGURE 3. Evolution of the moment of order 2, with $P = 10^5$ and $L \in [20, 500]$

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(Alexandre Benoist) INRIA NANCY GRAND EST
 ORCID: 0009-0002-3942-0961
 Email address: alexandre.benoist@polytechnique.edu

(Jean Kieffer) LORIA (LABORATOIRE LORRAIN DE RECHERCHE EN INFORMATIQUE ET SES APPLICATIONS) AND CNRS (CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE)
 ORCID: 0000-0002-9953-0137
 Email address: jean.kieffer@loria.fr