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Solving non-causal schemes for anisotropic eikonal equations, with quasi-linear complexity

Jean-Marie Mirebeau*, Rawane Mansour†

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Abstract

We introduce a numerical algorithm for solving anisotropic eikonal equations, whose complexity is quasi-linear $\mathcal{O}(N \ln^2(N/\varepsilon))$ with respect to the number N of discretization points, and logarithmic w.r.t. the numerical tolerance $\varepsilon > 0$, with explicit constants depending on the metric defining the PDE geometry. In contrast with the fast-marching method, our algorithm does *not* rely on the causality property, and for this reason it can be applied to a variety of discretization schemes: semi-Lagrangian, Eulerian, or based on a Lax-Friedrichs relaxation of the eikonal PDE. Our method uses a narrow band to compute the eikonal front propagation, whose width is tuned depending on the properties of the discretization scheme and of the metric. Numerical experiments, involving anisotropic metrics arising in seismology, image segmentation and motion planning, illustrate the efficiency of the method.

1 Introduction

The eikonal equation is a first order non-linear PDE, characterizing the distance map defined by a local metric. It is a simple model of front propagation, whose numerical solution plays a fundamental role in numerous applications such as motion planning, geometry processing, or seismic traveltime tomography, see e.g. [PPKC10]. Discretizations of the eikonal equation can be split into two categories, depending on whether they do or do not satisfy the *causality property*, see (9) below. *Causal* schemes can be solved using the Fast Marching Method (FMM) which is a generalization of Dijkstra’s algorithms on graphs, with complexity $\mathcal{O}(N \ln N)$ where N denotes the number of discretization points¹, following the seminal work [Tsi95, Set96]. Unfortunately, the design of a causal numerical scheme becomes complex when the metric is (strongly) anisotropic [KS98, SV03, Mir14a, Mir14a], i.e. when the front propagates faster in some directions than others, and one cannot avoid using a wide discretization stencil [Mir16] which is a serious drawback. *Non-causal* schemes can on the other hand be solved up to an arbitrary tolerance $\varepsilon > 0$ using a variety of iterative methods [Zha05, BR06, CCF11, JW08], for a cost $\text{Cost}(N, \varepsilon, \mathcal{F})$ - which strikingly is *not known* for any of these algorithms - depending on the number of points N , the error tolerance ε on the residue of the numerical scheme, and the properties of the metric henceforth denoted \mathcal{F} describing the geometry of the PDE. In this paper, we introduce a narrow band method for solving non-causal discretizations of the eikonal equation, in the spirit of [CCF11], see algorithm 1, and perform a thorough analysis of its numerical complexity $\text{Cost}(N, \varepsilon, \mathcal{F})$ which is

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¹A even stricter condition, referred to as strict causality, allows solving the scheme in $\mathcal{O}(N)$ time using a variant of Dial’s algorithm [Tsi95].

quasi-linear $\mathcal{O}(N \ln^2(N/\varepsilon))$ with explicit constants depending on the metric \mathcal{F} , see theorem 1.3. The proof applies to semi-Lagrangian, Eulerian, or Lax-Friedrichs discretizations of the eikonal equation, associated with Riemannian or Finslerian anisotropic metrics, and relies on a new geometrical concept referred to as α -acuteness.

In order to describe our results in more detail, we need to introduce some notations. Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain with Lipschitz boundary, equipped with a Finslerian metric² $\mathcal{F} : \bar{\Omega} \times \mathbb{R}^d \rightarrow \mathbb{R}$ denoted $(x, v) \mapsto \mathcal{F}_x(v)$. Then there exists a unique viscosity solution [BCD97] denoted $\mathcal{U} : \bar{\Omega} \rightarrow \mathbb{R}$ to the anisotropic eikonal PDE:

$$\forall x \in \Omega, \mathcal{F}_x^*(-\nabla \mathcal{U}(x)) = 1, \quad \text{where } \mathcal{F}_x^*(\eta) := \max_{\mathcal{F}_x(\xi) \leq 1} \langle \eta, \xi \rangle \quad (1)$$

denotes the dual metric, with null Dirichlet boundary conditions $\mathcal{U} = 0$ on $\partial\Omega$ for simplicity. In fact, this PDE characterizes the Finslerian distance to the boundary:

$$\mathcal{U}(x) = \inf_{\gamma} \int_0^1 \mathcal{F}_{\gamma(t)}(\gamma'(t)) dt, \quad \text{subject to } \gamma \in \text{Lip}([0, 1], \bar{\Omega}), \gamma(0) = x, \gamma(1) \in \partial\Omega. \quad (2)$$

In other words, $\mathcal{U}(x)$ is the minimal time needed to reach $\partial\Omega$ when starting from $x \in \Omega$. In this introduction, we discretize the eikonal equation using a *semi-Lagrangian* approach similar to [Tsi95, BR06], but let us emphasize that our results also apply to *Eulerian* discretizations such as [Set96, Mir19], and to a *Lax-Friedrichs* relaxation inspired by [KOQ04], see appendices A and B. Let $\mathring{X} \subset \bar{\Omega}$ and $\partial X \subset \mathbb{R}^d$ be disjoint finite sets, referred to as the discrete interior and boundary, and let $X := \mathring{X} \sqcup \partial X$. For all $x \in \mathring{X}$ we consider a stencil $\mathcal{V}(x)$ which is a finite and non-empty collection of tuples $(v_1, \dots, v_d) \in (\mathbb{R}^d)^d$ such that $x + v_i \in X$ for all $1 \leq i \leq d$. The *semi-lagrangian update operator* $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$ is defined as

$$\Lambda u(x) := \min_{(v_1, \dots, v_d) \in \mathcal{V}(x)} \min_{(\xi_1, \dots, \xi_d) \in \Xi_d} \mathcal{F}_x\left(\sum_{1 \leq i \leq d} \xi_i v_i\right) + \sum_{1 \leq i \leq d} \xi_i u(x + v_i), \quad (3)$$

for any discrete map $u \in \mathbb{R}^X$ and any interior vertex $x \in \mathring{X}$. By convention $\Lambda u(x) = 0$ for all $x \in \partial X$. We have $\xi_1, \dots, \xi_d \geq 0$ and $\xi_1 + \dots + \xi_d = 1$ in (3). Indeed, Ξ_d henceforth denotes the collection of d -plets of barycentric coordinates, defined as

$$\Xi_d := \{\xi \in [0, \infty[^d \mid \langle \xi, \mathbb{1} \rangle = 1\}, \quad \text{where } \mathbb{1} := (1, \dots, 1) \in \mathbb{R}^d. \quad (4)$$

Under suitable assumptions, the operator Λ admits a fixed point, which approximates the solution to the eikonal equation (1), see remark 1.7. The update operator (3) has a clear geometrical interpretation (5) when the stencils arise from a triangulation, yet in practice many other constructions have been proposed [Tsi95, KS98, Set96, Mir14b, Mir14a], and for our purposes we only require assumption 1.2 below.

Remark 1.1 (Stencil arising from a triangulation). *Assume that \mathring{X} and ∂X denote the interior and boundary vertices, respectively, of a triangulation \mathcal{T} of Ω . For any interior vertex $x \in \mathring{X}$ define $\mathcal{V}(x)$ as the collection of all tuples $(v_1, \dots, v_d) \in (\mathbb{R}^d)^d$ (up to permutation) such that the simplex $[x, x + v_1, \dots, x + v_d] \in \mathcal{T}$, following [BR06]. Denote by $\mathbf{V}(x)$ the union of these simplices, which is a neighborhood of x . Then (3) can be reformulated as*

$$\Lambda u(x) = \min_{y \in \partial \mathbf{V}(x)} \mathcal{F}_x(y - x) + \mathbf{I}_{\mathcal{T}} u(y), \quad (5)$$

²I.e. $(x, v) \mapsto \mathcal{F}_x(v)$ is jointly continuous w.r.t. the position $x \in \bar{\Omega}$ and the velocity $v \in \mathbb{R}^d$, and \mathcal{F}_x is a (possibly asymmetric) norm w.r.t. the velocity v for any fixed position $x \in \bar{\Omega}$.

where $I_{\mathcal{T}}$ denotes piecewise linear interpolation on the triangulation \mathcal{T} . The fixed point equation $\Lambda u = u$ is thus a discrete counterpart of Bellman's optimality principle $\mathcal{U}(x) = \min_{y \in \partial \mathcal{V}(x)} d_{\mathcal{F}}(x, y) + \mathcal{U}(y)$ which is satisfied by the distance map \mathcal{U} to the boundary (2), where $d_{\mathcal{F}}$ denotes the travel-time (or path-length) distance associated with the metric \mathcal{F} .

Let us introduce the collection of outward offsets from an interior vertex $x \in \overset{\circ}{X}$

$$\mathcal{E}(x) := \{v_i \mid (v_1, \dots, v_d) \in \mathcal{V}(x), 1 \leq i \leq d\}, \quad (6)$$

and denote by $d_{\mathcal{E}} : X \times X \rightarrow [0, \infty]$ the (asymmetric and possibly infinite) distance on the positively weighted oriented graph with an edge $(x, x + v)$ traversed in time $\mathcal{F}_x(v) > 0$, for all $x \in \overset{\circ}{X}$ and all $v \in \mathcal{E}(x)$.

In algorithm 1 below, the narrow band B_{n+1} is defined in terms of the distance $d_{\mathcal{E}}$ to a set of points denoted Y_n . In practice, this distance is computed using Dijkstra's algorithm on the graph defined by \mathcal{E} with null boundary conditions on the set Y_n , see algorithm 2 page 17.

Assumption 1.2. (*Connectedness to the boundary*) For all $x \in \overset{\circ}{X}$, there exists $y \in \partial X$ such that $d_{\mathcal{E}}(x, y) < \infty$. (*Non-degeneracy of simplices*) For all $x \in \overset{\circ}{X}$ and all $(v_1, \dots, v_d) \in \mathcal{V}(x)$ one has $\det(v_1, \dots, v_d) \neq 0$.

Algorithm 1 Narrow band solver of $\Lambda u = u$, with guaranteed complexity

Inputs: $X = \overset{\circ}{X} \sqcup \partial X$ the discretized domain, Λ the update operator, parameters τ, r, T, K .

Initialization: Initialize a mutable $u : X \rightarrow [0, \infty]$ to the value ∞ identically.

For $n = 0, 1, \dots$ **until** $u \leq (n - r)\tau$ identically on X **do**

Construct $Y_n := \{y \in X \mid u(y) \leq (n - 1)\tau\} \cup \partial X$

Construct $B_{n+1} := \{x \in X \mid u(x) \geq (n - r + 1)\tau \text{ and } d_{\mathcal{E}}(x, Y_n) \leq T\}$

For $k = 1, \dots, K$ **do**

For all $x \in B_{n+1}$ **do** $u(x) \leftarrow \Lambda u(x)$ **in parallel**

The main result of this paper is a complete convergence and complexity analysis of algorithm 1, which is a numerical solver of the fixed point problem $\Lambda u = u$ discretizing the eikonal equation, based on a carefully designed narrow band B along which the values of the computed solution are repeatedly updated. In the related literature [Tsi95, CCF11], points within the narrow band are usually referred to as *Active*, points which leave the narrow band are tagged as *Frozen* afterwards, and other points are *Far*. The complexity of algorithm 1 is dominated by the update operations $u(x) \leftarrow \Lambda u(x)$ in the last line, which are regarded as elementary operations. The narrow band B itself is obtained using Dijkstra's method, see algorithm 2, which has a non-dominant cost. In practice, the minimization over Ξ_d in (3) has a closed form in the case of a Riemannian metric [Tsi95, SV03], but for more complicated metrics it needs to be addressed using iterative convex optimization methods [DCC⁺21].

We denote equivalently $V \in \mathcal{V}(x)$ or $(x, V) \in \mathcal{V}$, and we regard the stencil element $V = (v_1, \dots, v_d) \in (\mathbb{R}^d)^d$ as a matrix defined column-wise. In the next result we let $\ln^+ := \max\{1, \ln\}$, and for any point $x \in \bar{\Omega}$, vector $\xi \in \mathbb{R}^d$, and asymmetric norm F , we denote

$$\mathcal{F}_x^V(\xi) := \mathcal{F}_x(V\xi) \quad F_{\max} := \max_{\|\xi\|=1} F(\xi), \quad F_{\min} := \min_{\|\xi\|=1} F(\xi). \quad (7)$$

Theorem 1.3. *Under assumption 1.2, there exists a unique $u_* \in \mathbb{R}^X$ such that $\Lambda u_* = u_*$. Given a tolerance $\varepsilon > 0$, and using the parameters τ, r, T, K of proposition 2.14 and lemma 3.7,*

algorithm 1 produces an approximate solution $u \in \mathbb{R}^X$ such that $u - \varepsilon \leq u_* \leq u$. It uses $\mathcal{O}(N\kappa^2 \ln^+(Nh\kappa^2/\varepsilon)(\kappa + \ln^+(Nh/\varepsilon)))$ elementary solution updates (of the form $u(x) \leftarrow \Lambda u(x)$), where the constant only depends on the space dimension d , with $N := \#(X)$ and

$$h := \max_{(x,V) \in \mathcal{V}} (\mathcal{F}_x^V)_{\max}, \quad h_{\min} := \min_{(x,V) \in \mathcal{V}} (\mathcal{F}_x^V)_{\min}, \quad \kappa := h/h_{\min}. \quad (8)$$

Algorithm 1 has purposely a simple structure, which makes it amenable to the complexity analysis of theorem 1.3. In the numerical experiments section 4, we show that the parameter K is in practice advantageously replaced with an adaptive stopping criterion in the corresponding loop. We also provide simple and efficient heuristics for setting the parameters τ , r and T - indeed, the parameters used in the proof of theorem 1.3 are not sharp enough to provide good performance.

For comparison, the FMM can solve the fixed point system $\Lambda u_* = u_*$ exactly (up to machine precision) using at most $M := \sum_{x \in \hat{X}} \#(\mathcal{E}(x))$ elementary solution updates, each coming with a logarithmic overhead cost for maintaining a heap structure, resulting in the overall cost $\mathcal{O}(M \ln N)$ which is similarly quasi-linear. However, the FMM needs the update operator Λ to obey the following *causality* property [Mir19, Appendix A]: for all $u, v \in \mathbb{R}^X$, and all $t \in \mathbb{R}$

$$u^{<t} = v^{<t} \Rightarrow (\Lambda u)^{\leq t} = (\Lambda v)^{\leq t}, \quad \text{where } u^{<t}(x) := \begin{cases} u(x) & \text{if } u(x) < t, \\ \infty & \text{otherwise,} \end{cases} \quad (9)$$

and likewise for $u^{\leq t}$. In other words, any update value $\Lambda u(x)$ can be expressed in terms of strictly smaller values of u (compare with (37) page 22). The celebrated ‘‘acuteness implies causality’’ principle [Tsi95, KS98, SV03, GV23] states that (9), in the case of the semi-Lagrangian update operator (3), is equivalent to assuming that the norm \mathcal{F}_x^V satisfies the classical acuteness condition of definition 1.4 below, for each $x \in \hat{X}$ and $V \in \mathcal{V}(x)$. A line of works has been devoted to the construction of stencils $\mathcal{V}(x)$, depending on the local metric \mathcal{F}_x , such that this condition is satisfied [KS98, SV03, Mir14b, Mir14a]; however, when the anisotropy of \mathcal{F} increases the obtained stencils \mathcal{V} inevitably become wide [Mir16] and may contain many elements, which negatively affects accuracy and complexity.

We recall below the classical acuteness condition and introduce, as announced, a relaxation referred to as α -acuteness where $\alpha > 0$ is regarded as a timescale, which is sufficient for the proof of theorem 1.3. The α -acuteness condition is *always satisfied* for sufficiently large α , see proposition 1.5 below, in contrast with the classical acuteness condition which puts severe constraints on the norm \mathcal{F}_x^V and thus on the scheme design.

Definition 1.4. Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be a differentiable asymmetric norm, and let $\alpha > 0$. The α -acuteness condition (resp. classical acuteness condition) requires that for all $\xi \in \Xi_d$

$$\langle \xi, \exp(-\frac{1}{\alpha} \nabla F(\xi)) \rangle < 1, \quad (\text{resp. } \nabla F(\xi) \in [0, \infty]^d), \quad (10)$$

where \exp is applied componentwise. We denote $\rho_F(\alpha) := \max\{\langle \xi, \exp(-\frac{1}{\alpha} \nabla F(\xi)) \rangle \mid \xi \in \Xi_d\}$.

In the special case where $F = \mathcal{F}_x^V$ and \mathcal{F}_x is proportional to the Euclidean norm, the classical acuteness condition is equivalent to $\langle v_i, v_j \rangle \geq 0$ for all $1 \leq i < j \leq d$, where $V = (v_1, \dots, v_d)$. Hence the name, see [KS98, SV03, Mir14b, Mir14a, GV23] a proof and various generalizations. Definition 1.4 has a straightforward extension to non-differentiable norms, see proposition 2.23. Propositions 1.5 and 1.6 below are key ingredients of the proof of theorem 1.3, and are established in section 2.3.

Proposition 1.5. *Let $F : \mathbb{R}^d \rightarrow \mathbb{R}$ be an asymmetric norm. If $\alpha > F_{\max}^2/F_{\min}$ (resp. if $\alpha > 0$ and F satisfies the classical acuteness condition) then $\rho_F(\alpha) < 1$, and more precisely*

$$\rho_F(\alpha) \leq 1 - \frac{F_{\min}}{\alpha\sqrt{d}} \left(1 - \frac{F_{\max}^2}{\alpha F_{\min}}\right), \quad \left[\text{resp. } \rho_F(\alpha) \leq 1 - \frac{F_{\min}}{F_{\max}\sqrt{d}} \left(1 - \exp\left(-\frac{F_{\max}}{\alpha}\right)\right) \right]. \quad (11)$$

The eikonal equation, and the corresponding numerical schemes, are often reformulated using an exponential change of coordinates - known as the Kruřkov³ transform - which introduces a timescale $\alpha > 0$ in the problem. The next result implies that the exponentially transformed update operator is contracting iff the α -acuteness condition is satisfied.

Proposition 1.6. *Let F be an asymmetric norm on \mathbb{R}^d , and let $\alpha > 0$. Define for any $u \in \mathbb{R}^d$*

$$\lambda_F(u) := \min_{\xi \in \Xi_d} F(\xi) + \langle \xi, u \rangle, \quad \phi_F := E_\alpha \lambda_F E_\alpha^{-1}, \quad \text{with } E_\alpha(u) := -\exp(-u/\alpha). \quad (12)$$

If F is uniformly convex and continuously differentiable on $\mathbb{R}^d \setminus \{0\}$, and $\xi \in \Xi_d$ minimizes (12, left), then

$$\nabla \lambda_F(u) = \xi, \quad \left[\frac{d}{dt} \phi_F(v+t) \right]_{t=0} = \langle \xi, \exp(-\frac{1}{\alpha} \nabla F(\xi)) \rangle,$$

where $v := E_\alpha u$. As a result one has $\phi_F(v+t) \leq \phi_F(v) + \rho_F(\alpha)t$, for all $v \in \mathbb{R}^d$ and $t \geq 0$.

Comparison with the state of the art. A variety of numerical methods already allow solving non-causal discretizations of the eikonal equation, known as Fast Sweeping (FS), Adaptive Gauss-Siedel Iteration (AGSI), Fast Iterative Method (FIM), or Banded Fast Marching (BFM) [Zha05, BR06, JW08, CCF11]. These methods usually work well in applications, and their numerical complexity has been estimated to $\mathcal{O}(N)$, $\mathcal{O}(N \ln N)$, or $\mathcal{O}(N^{1+1/d})$ based on heuristic considerations and empirical observations, see [Zha05, CCF11, BR06] and section 4. However, no complete complexity analysis has been published to the knowledge of the authors. In contrast with our result theorem 1.3, these empirical estimates also do not take into account the fixed point tolerance $\varepsilon > 0$, or the anisotropy of the metric \mathcal{F} .

Limitations. A large body of work has been devoted to the numerical solution of the eikonal equation. Important topics of interest include parallelization and GPU acceleration [JW08], linear formulations using Varadhan’s asymptotic expansion of the heat kernel, methods with high order of accuracy, inverse problems known as traveltime tomography, avoiding the curse of dimensionality in high dimensional problems, and many more. They all are outside the scope of this paper, see the survey article [CLPQ20] for some discussion.

Contributions. We establish a quasi-linear complexity guarantee for a numerical method solving anisotropic eikonal equations, see theorem 1.3, which is based on a carefully designed narrow band and applies to semi-Lagrangian, Eulerian or Lax-Friedrichs discretizations of the PDE. The proof relies on a new geometrical property referred to as α -acuteness, see definition 1.4.

Outline. We study in section 2 a simplified algorithm, in order to illustrate the main geometrical concepts while avoiding some technical difficulties, in which the narrow band is defined using the exact solution and provided by an oracle. Our main result theorem 1.3 is established in section 3. It is illustrated by numerics in section 4, and extended to Eulerian and Lax-Friedrichs schemes in appendices A and B.

³Kruřkov also shifts the solution by 1, but this was not convenient for our purposes.

Remark 1.7 (Convergence to the viscosity solution). *Standard techniques for the convergence analysis of monotone discretization schemes show that the fixed point u_h of the update operator (3) with discretization scale (8, left) denoted h , converges uniformly $u_h \rightarrow \mathcal{U}$ as $h \rightarrow 0$ to the viscosity solution \mathcal{U} of (1), under mild assumptions. See for instance [BR06, Mir14a] in the semi-Lagrangian setting, [Mir19] in the Eulerian setting, [KOQ04] in the Lax-Friedrichs setting, or [AL24] for optimal convergence rates. Such convergence results to the continuum limit are however not within the scope of this paper.*

Notations: The space \mathbb{R}^d is equipped with the Euclidean norm $\|\cdot\|$ and scalar product $\langle \cdot, \cdot \rangle$. The operator norm of a matrix A is also denoted $\|A\| := \max\{\|Ax\| \mid \|x\| \leq 1\}$. The collection of $d \times d$ matrices which are invertible (resp. symmetric, resp. symmetric positive definite) is denoted $\text{GL}(\mathbb{R}^d)$ (resp. \mathcal{S}_d , resp. \mathcal{S}_d^{++}).

The letter F always refers to an asymmetric norm, i.e. a mapping $F : \mathbb{R}^d \rightarrow \mathbb{R}_+$ satisfying $F(\lambda x) = \lambda F(x)$ for all $\lambda > 0$, $F(x+y) \leq F(x) + F(y)$ and $(F(x) = 0 \Leftrightarrow x = 0)$, for all $x, y \in \mathbb{R}^d$; for readability, the word *asymmetric* is often dropped in the following.

The notation $X = \overset{\circ}{Y} \sqcup \partial Y$ means that the set X is the union of two *disjoint* sets $\overset{\circ}{Y}$ and ∂Y . By \mathbb{R}^X we denote the collection of all maps $u : X \rightarrow \mathbb{R}$. When the set X is clear from context, we denote $\{u \leq t\} := \{x \in X \mid u(x) \leq t\}$, for any $t \in \mathbb{R}$. We denote by $\max(u) := \max\{u(x) \mid x \in X\}$ the largest value attained by $u \in \mathbb{R}^X$, and by $\max\{u, t\} \in \mathbb{R}^X$ the map defined pointwise by $\max\{u(x), t\}$ for all $x \in X$, where $t \in \mathbb{R}$.

2 Proof in a simplified setting

In this section, we establish a variant of the algorithmic complexity result theorem 1.3 in a simplified setting, where the narrow band needs to be provided by an oracle, since its definition involves the exact solution of the problem interest. We present in section 2.1 some basic results on discrete monotone operators, used in section 2.2 to evaluate the accuracy and complexity of the simplified algorithm in an abstract axiomatic setting. We study in section 2.3 the α -acuteness geometrical property, proving propositions 1.5 and 1.6 from the introduction, and we show that the semi-Lagrangian scheme (3) satisfies the required axioms.

2.1 Discrete monotone operators

Throughout this section we fix a finite set X , and denote by \mathbb{R}^X the collection of maps $u : X \rightarrow \mathbb{R}$. The set \mathbb{R}^X is equipped with the following partial order, and strict partial order

$$u \preceq u' \stackrel{\text{def}}{\Leftrightarrow} (\forall x \in X, u(x) \leq u'(x)), \quad u \prec u' \stackrel{\text{def}}{\Leftrightarrow} (\forall x \in X, u(x) < u'(x)).$$

A (discrete) operator on the set X is an arbitrary mapping $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$. The notations Λu and $\Lambda(u)$ are used interchangeably, with the aim to improve readability; for instance we usually write $\Lambda u(x)$ rather than $\Lambda(u)(x)$, for all $u \in \mathbb{R}^X$ and $x \in X$. Operators which are monotone⁴, i.e. non-decreasing w.r.t. the partial order \preceq , are a classical tool in the study the of discretization schemes for degenerate elliptic equations in general [BBM23], and eikonal equations in particular [BR06]. We gather below some basic results needed for our analysis, see [BR06, BBM23] and references therein for more discussion. In order to alleviate notations, we often identify a real $t \in \mathbb{R}$ with a constant mapping in \mathbb{R}^X .

⁴Note to be confused with monotone operators in Hilbert spaces, which is an unrelated concept.

Definition 2.1. An operator $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$ is said:

- monotone if $\forall u, u' \in \mathbb{R}^X, u \preceq u' \Rightarrow \Lambda u \preceq \Lambda u'$.
- ρ -sub-additive if $\forall u \in \mathbb{R}^X, \forall t \in [0, \infty[, \Lambda(u + t) \preceq \Lambda u + \rho t$.
- super-multiplicative if $\forall u \in \mathbb{R}^X, \forall \gamma \geq 1, \Lambda(\gamma u) \succeq \gamma \Lambda u$.

We usually assume $\rho \in [0, 1]$. By convention, a 1-sub-additive operator is said sub-additive.

Lemma 2.2. (a) If Λ is ρ -sub-additive, then $\forall u \in \mathbb{R}^X, \forall t \in [0, \infty[, \Lambda(u - t) \succeq \Lambda u - \rho t$.

(b) If Λ is super-multiplicative, then $\forall u \in \mathbb{R}^X, \forall \gamma \in [0, 1], \Lambda(\gamma u) \preceq \gamma \Lambda u$. (Thus $\Lambda(0) \preceq 0$.)

Proof. (a) Apply the ρ -sub-additivity axiom to $u' := u - t$ and $t' := t$. (b) If $\gamma \in]0, 1]$, then apply the super-multiplicativity axiom to $u' := \gamma u$ and $\gamma' := 1/\gamma$. In the special case $\gamma = 0$, note that $\Lambda(2 \times 0) \succeq 2\Lambda(0)$, hence $\Lambda(0) \preceq 0$ as announced, and therefore $\Lambda(0 \times u) \preceq 0 \times \Lambda(u)$. \square

In order to avoid any confusion, let us immediately mention that the semi-Lagrangian update operator (3) is neither ρ -sub-additive for any $\rho < 1$, nor super-multiplicative; these properties are instead satisfied by the exponentially transformed operator, see proposition 2.24 below.

As an illustration, two of these axioms imply a contraction property in the L^∞ norm.

Corollary 2.3. Let $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$ be a monotone and ρ -sub-additive operator. Then $\|\Lambda u - \Lambda u'\|_\infty \leq \rho \|u - u'\|_\infty$ for all $u, u' \in \mathbb{R}^X$. If $\rho < 1$ then Λ admits a unique fixed point $u_* \in \mathbb{R}^X$, and $\|\Lambda^n u - u_*\|_\infty \leq \rho^n \|\Lambda u - u\|_\infty / (1 - \rho)$ for all $u \in \mathbb{R}^X$.

Proof. Observe that $u - t \preceq u' \preceq u + t$ with $t := \|u - u'\|_\infty$, and thus $\Lambda u - \rho t \preceq \Lambda u' \preceq \Lambda u + \rho t$ by monotony and ρ -sub-additivity, as announced. The second part directly follows from the Picard-Lindelöf fixed point theorem. \square

Proposition 2.4. Consider a family of operators $(\Lambda_i)_{i \in I}$ on the set X , and assume that $\Lambda u(x) := \inf_{i \in I} \Lambda_i u(x)$ (resp. $\Lambda u(x) := \sup_{i \in I} \Lambda_i u(x)$) is finite for all $u \in \mathbb{R}^X, x \in X$. If Λ_i is monotone, or ρ -sub-additive, or super-multiplicative, for each $i \in I$, then so is Λ .

Proof. Assume that Λ_i is monotone for all $i \in I$, and let $x \in X$ be arbitrary. If $u, u' \in \mathbb{R}^X$ are such that $u \preceq u'$, then $\Lambda_i u(x) \leq \Lambda_i u'(x)$ for all $i \in I$, and thus $\inf_{i \in I} \Lambda_i u(x) \leq \inf_{i \in I} \Lambda_i u'(x)$, which proves that Λ is monotone as announced. Proceed likewise for the sup, and other properties. \square

Sub- and super-solutions. To any operator $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$ is associated a fixed point problem, already considered in corollary 2.3 : find $u_* \in \mathbb{R}^d$ such that $\Lambda u_* = u_*$. Sub- and super-solutions of Λ (the operator name is omitted when it is clear from context), which obey a one-sided inequality instead, play a fundamental for role in the theory of monotone operators.

Definition 2.5. Let $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$ be an operator. A super-solution is any map $\underline{u} \in \mathbb{R}^X$ such that $\Lambda \underline{u} \preceq \underline{u}$; it is said strict if $\Lambda \underline{u} \prec \underline{u}$. A sub-solution is any map $\bar{u} \in \mathbb{R}^X$ such that $\Lambda \bar{u} \succeq \bar{u}$; likewise it is said strict if $\Lambda \bar{u} \succ \bar{u}$.

Proposition 2.6. Let $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$ be a ρ -sub-additive operator, for some $\rho < 1$. Let $u \in \mathbb{R}^X$ and $\bar{c}, \underline{c} \geq 0$ be such that $u - \bar{c} \preceq \Lambda u \preceq u + \underline{c}$. Then $\bar{u} := u - \bar{c}/(1 - \rho)$ and $\underline{u} := u + \underline{c}/(1 - \rho)$ are a sub-solution and a super-solution respectively (both strict if $u - \bar{c} \prec \Lambda u \prec u + \underline{c}$).

Proof. For any $t \geq 0$ one has $\Lambda(u + t) \preceq \Lambda u + \rho t \preceq u + \underline{c} + \rho t = (u + t) + \underline{c} - (1 - \rho)t$, hence $\underline{u} := u + t$ is a super-solution provided $t \geq \underline{c}/(1 - \rho)$, as announced, and likewise for \bar{u} . \square

Proposition 2.7. Let $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$ be a monotone operator on a finite set $X = \overset{\circ}{Y} \sqcup \partial Y$.

- (Weak comparison principle) If Λ is sub-additive, then for any $u, u' \in \mathbb{R}^X$

$$(u \prec u' \text{ on } \partial Y, \text{ and } u - \Lambda u \prec u' - \Lambda u' \text{ on } \overset{\circ}{Y}) \Rightarrow u \prec u'.$$

- (Comparison principle) If Λ is ρ -sub-additive, for some $\rho < 1$, then for any $u, u' \in \mathbb{R}^X$

$$(u \preceq u' \text{ on } \partial Y, \text{ and } u - \Lambda u \preceq u' - \Lambda u' \text{ on } \overset{\circ}{Y}) \Rightarrow u \preceq u'.$$

Proof. Assume 1-sub-additivity of Λ , $u \prec u'$ on ∂Y and $u - \Lambda u \prec u' - \Lambda u'$ on $\overset{\circ}{Y}$. Let $x \in X$ be such that $t := u(x) - u'(x)$ is maximal, and assume for contradiction that $t \geq 0$. By assumption this implies $x \in \overset{\circ}{Y}$. Then $t = u(x) - u'(x) < \Lambda u(x) - \Lambda u'(x) \leq \Lambda(u' + t)(x) - \Lambda u'(x) \leq t$, using successively the assumption, monotony, and sub-additivity, which is the desired contradiction.

Assume now ρ -sub-additivity for some $\rho < 1$, $u \preceq u'$ on ∂Y and $u - \Lambda u \preceq u' - \Lambda u'$ on $\overset{\circ}{Y}$. Then for any $\varepsilon > 0$ one has $u \prec u' + \varepsilon$ on ∂Y , and $u' + \varepsilon - \Lambda(u' + \varepsilon) \succeq u' - \Lambda u' + (1 - \rho)\varepsilon \succ u' - \Lambda u' \succeq u - \Lambda u$ on $\overset{\circ}{Y}$. Thus $u \prec u' + \varepsilon$ by the previous argument, $\forall \varepsilon > 0$, and therefore $u \preceq u'$ as announced. \square

Exponential change of variables. This transformation of the unknowns is common in the study of eikonal equations and minimal time optimal control problems [BCD97], and attributed to Kruřkov. In the continuous setting, it turns the degenerate elliptic operator $\|\nabla u\| - 1$ into the elliptic operator $u/\alpha + \|\nabla u\|$, where $\alpha > 0$. In the discrete setting, in a similar spirit, the sub-additive semi-Lagrangian update operator becomes ρ -sub-additive in the exponential coordinates, for some $\rho < 1$, see proposition 1.6.

In the following, we denote by $\mathbb{R}_\infty :=] - \infty, \infty]$ the real line extended to the right only, and equipped with the topology of $]0, 1]$. We fix a parameter $\alpha > 0$, and define $E_\alpha : \mathbb{R}_\infty \rightarrow \mathbb{R}$ and $E_\alpha^{-1} : \mathbb{R} \rightarrow \mathbb{R}_\infty$ by

$$E_\alpha(u) := \begin{cases} -\exp(-u/\alpha), & \text{if } u \in \mathbb{R}, \\ 0 & \text{if } u = \infty, \end{cases} \quad E_\alpha^{-1}(v) := \begin{cases} -\alpha \ln(-v) & \text{if } v \in] - \infty, 0[, \\ \infty & \text{if } v \geq 0. \end{cases} \quad (13)$$

Note that $E_\alpha^{-1}(E_\alpha(u)) = u$ and $E_\alpha(E_\alpha^{-1}(v)) = \min\{v, 0\}$. By convention, the maps E_α and E_α^{-1} are applied pointwise to elements of \mathbb{R}_∞^X and \mathbb{R}^X respectively.

Proposition 2.8. Let $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$ be an operator which extends to a continuous function $\Lambda : \mathbb{R}_\infty^X \rightarrow \mathbb{R}_\infty^X$ (with the same notation). Let $\alpha > 0$ and $\Phi := E_\alpha \Lambda E_\alpha^{-1} : \mathbb{R}^X \rightarrow \mathbb{R}^X$. The following holds: (a) Φ is a continuous operator, (b) if Λ is monotone, then Φ is monotone, (c) if Λ is sub-additive, then Φ is super-multiplicative.

Proof. (a) The continuity of Φ follows by composition from the continuity of E_α , Λ and E_α^{-1} . (b) Likewise, the monotony of Φ follows by composition from the monotony of E_α , Λ and E_α^{-1} . (c) Assume that Λ is ρ -sub-additive, for some $\rho \in [0, 1]$. Let $\gamma \geq 1$ and $v \in \mathbb{R}^X$. Then

$$\Phi[\gamma v] = -\exp(-\alpha^{-1} \Lambda(u - \alpha \ln \gamma)) \succeq -\exp(-\alpha^{-1} (\Lambda u - \rho \alpha \ln \gamma)) = \gamma^\rho \Phi(v),$$

with $u = E_\alpha^{-1}(v)$. We used successively (i) $E_\alpha^{-1}(\gamma v) = E_\alpha^{-1}(v) - \alpha \ln \gamma$, and (ii) the ρ -sub-additivity of Λ as in lemma 2.2 (a), and the monotony of E_α . Choosing $\rho = 1$ we conclude the proof. \square

Partial application of monotone operators. Given an operator $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$, and a subset $B \subset X$, we denote for all $u \in \mathbb{R}^X$

$$\Lambda_B u(x) := \begin{cases} \Lambda u(x) & \text{if } x \in B, \\ u(x) & \text{else.} \end{cases}$$

Such a restricted operator implicitly appears in the last line of algorithm 1, choosing B as the narrow band.

Proposition 2.9 (Partial application preserves sub- and super-solutions). *Let $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$ be a monotone operator, and let $B \subset X$ be a subset. Let $u \in \mathbb{R}^X$ and $u' := \Lambda_B u$. If $\Lambda u \preceq u$ then $\Lambda u' \preceq u'$. Likewise if $\Lambda u \succeq u$ then $\Lambda u' \succeq u'$.*

Proof. Consider the case $\Lambda u \preceq u$ of a super-solution. Then $\Lambda u \preceq u' \preceq u$ by assumption, thus $\Lambda u' \preceq \Lambda u$ by monotony, and therefore $\Lambda u' \preceq u'$ as announced. The case $\Lambda u \succeq u$ is similar. \square

Lemma 2.10 (ρ -sub-additivity in a subdomain). *Let $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$ be a monotone and ρ -sub-additive operator, let $u \in \mathbb{R}^X$ and $t \geq 0$. If $u - \Lambda_B u \preceq t$, then $\Lambda_B^k u - \Lambda_B^{k+1} u \preceq \rho^k t$, $\forall k \geq 0$.*

Proof. By an immediate induction argument, it suffices to prove the case $k = 1$. Denoting $u' := \Lambda_B u$ we have $u \preceq u' + t$, hence $\Lambda u \preceq \Lambda(u' + t) \preceq \Lambda u' + \rho t$ by monotony and ρ -sub-additivity. Recalling that $u = u'$ outside B , we obtain $\Lambda_B u \preceq \Lambda_B u' + \rho t$, which concludes. \square

2.2 Convergence analysis, when the narrow band is provided by an oracle

We estimate the complexity and the numerical error of a simplified and hypothetical variant of algorithm 1, whose narrow band is defined in terms of the exact problem solution (hence the need for an *oracle* providing this data). The analysis builds on the results of section 2.1, and requires the following assumptions.

Assumption 2.11. *We denote by X a finite set, and by $\Phi : \mathbb{R}^X \rightarrow \mathbb{R}^X$ an operator which is monotone, ρ -sub-additive for some $\rho < 1$, and super-multiplicative. We denote by $v_* \in \mathbb{R}^X$ the fixed point of Φ , see corollary 2.3, and assume that $-1 \preceq v_* \prec 0$ on X .*

Finally and most importantly, we assume that the following level sets of v_* are known, and use them in the construction of our simplified algorithm, see proposition 2.13 below. For all $n \in \mathbb{Z}$

$$X_n^- := \{x \in X \mid v_*(x) \leq -2^{-n}\}, \quad X_n^+ := \{x \in X \mid v_*(x) \geq -2^{-n}\}.$$

Lemma 2.12 (A family of super-solutions). *Under assumption 2.11. Fix an integer $r > 0$, define for all $0 \leq n \leq 2^{r-1}$*

$$\underline{v}_n := (1 - n2^{-r+1})v_* + 2^{-n}. \quad (14)$$

The following holds, restricting to $n > 0$ for (c)

$$(a) \Phi \underline{v}_n \preceq \underline{v}_n - (1 - \rho)2^{-n}, \quad (b) \underline{v}_n \succeq 0 \text{ on } X_n^+, \quad (c) \underline{v}_n \succeq \underline{v}_{n-1} + 2^{-n} \text{ on } X_{n-r}^-.$$

Choose $0 < \delta \leq 1/2$, and assume that $2^{-n} \leq \delta |\max(v_)|$ and $0 \leq n \leq \delta 2^r$. Then*

$$(d) v_* \preceq \underline{v}_n \preceq (1 - 3\delta)v_*, \quad (e) u_* \preceq \underline{u}_n \preceq u_* - \alpha \ln(1 - 3\delta),$$

where $u_ := E_\alpha^{-1}v_*$ and $\underline{u}_n := E_\alpha^{-1}\underline{v}_n$, for some $\alpha > 0$, see the exponential transformation (13).*

Proof. (a) Follows from the super-multiplicativity, used as in lemma 2.2 (b), and the ρ -sub-additivity of Φ . (b) Holds by definition of X_n^+ . (c) One has $\underline{v}_n - \underline{v}_{n-1} = -2^{-r+1}v_* - 2^{-n} \succeq 2^{-n}$ on X_{n-r}^- . (d) One has $0 \preceq \underline{v}_n - v_* \preceq -2\delta v_* + 2^{-n} \leq 3\delta|v_*|$, since $v_* \prec 0$, as announced. (e) Immediate in view of the logarithmic expression of E_α^{-1} on $] -\infty, 0[$. \square

Proposition 2.13 (Narrow band iterative algorithm, with oracle). *Under assumption 2.11 and with the notation (14). Define $v_0 := 0$, and for all $1 \leq n \leq 2^{r-1}$ let*

$$v_n := \Phi_{B_n^*}^K(v_{n-1}), \quad \text{where } B_n^* := X_{n-r}^+ \cap X_n^-, \quad (15)$$

where K is s.t. $\rho^K \leq (1-\rho)2^{-r}$. Then v_n is a super-solution, and $v_* \preceq v_n \preceq \underline{v}_n$, $0 \leq n \leq 2^{r-1}$.

Proof. One has $\Phi(0) \preceq 0$ by super-multiplicativity, see lemma 2.2 (b), hence v_0 is a super-solution. Since v_n is obtained by repeated partial applications of the monotone operator Φ to v_0 , it is a super-solution by proposition 2.9, as announced. By the comparison principle proposition 2.7, we obtain that $v_n \succeq v_*$. The monotony of Φ also implies that $v_n \leq v_{n-1}$ for all $0 < n \leq 2^{r-1}$.

One has $v_0 := 0 \preceq 1 + v_* = \underline{v}_0$, recalling that $v_* \succeq -1$ by assumption 2.11. We then proceed by induction and assume that $v_{n-1} \preceq \underline{v}_{n-1}$ for some $n \geq 1$. Recall that v_{n-1} is a super-solution, hence so is $\Phi_{B_n^*}(v_{n-1})$ by proposition 2.9, and therefore $v_* \preceq \Phi_{B_n^*}(v_{n-1})$ by the comparison principle proposition 2.7. We thus have on the set B_n^*

$$v_{n-1} - \Phi_{B_n^*}(v_{n-1}) \preceq \underline{v}_{n-1} - v_* = (n-1)2^{-r+1}|v_*| + 2^{-n+1} \preceq n2^{-n+1}.$$

Therefore, successively using lemma 2.10, noting that $2n\rho^K \leq 1 - \rho$, and using lemma 2.12 (a), we obtain on the set B_n^*

$$v_n - \Phi_{B_n^*}(v_n) = \Phi_{B_n^*}^K(v_{n-1}) - \Phi_{B_n^*}^{K+1}(v_{n-1}) \preceq \rho^K n 2^{-n+1} \leq (1-\rho)2^{-n} \preceq \underline{v}_n - \Phi_{B_n^*}(\underline{v}_n). \quad (16)$$

In addition, we have $v_n \preceq v_0 = 0 \preceq \underline{v}_n$ on X_n^+ , and $v_n = v_{n-1} \preceq \underline{v}_{n-1} \preceq \underline{v}_n$ on X_{n-r}^- . By the comparison principle proposition 2.7, applied to $\dot{Y} := B_n^*$ and $\partial Y := X \setminus B_n^* \subset X_n^+ \cup X_{n-r}^-$, we obtain $v_n \preceq \underline{v}_n$, which concludes. \square

The algorithm (15) is not practical, because the sets $(B_n^*)_{0 \leq n \leq 2^{r-1}}$ depend on the exact fixed point v_* , and thus need to be provided by an oracle. Nevertheless, let us examine the complexity of this hypothetical method, which is a straightforward consequence of the preceding two results. We denote by $\lceil s \rceil \in \mathbb{Z}$ the ceiling of a real number $s \in \mathbb{R}$.

Proposition 2.14. *Under assumption 2.11 and using the notations of lemma 2.12 and proposition 2.13. Let $\varepsilon > 0$, let v_n be defined by (15) with the parameters n, r, K below, and let $u_n := E_\alpha^{-1}v_n$ and $u_* := E_\alpha^{-1}v_*$. Then $u_* \preceq u_n \preceq u_* + \varepsilon$, and at most $NK(r+1)$ elementary solution updates are used in the iterations of (15), where $N := \#(X)$ is the discrete domain cardinality. The parameters are defined in terms of $\mathcal{N} := \max(u_*)/\alpha$ and $p = 1 - \rho > 0$ as*

$$\begin{aligned} \delta &:= \frac{1 - \exp(-\varepsilon/\alpha)}{3} \geq \frac{\min\{\varepsilon/\alpha, 1\}}{6} & n &:= \left\lceil \frac{|\ln \delta| + \max(u_*)/\alpha}{\ln 2} \right\rceil = \mathcal{O}(\mathcal{N} + |\ln \delta|), \\ r &:= \left\lceil \frac{\ln(n/\delta)}{\ln 2} \right\rceil = \mathcal{O}(\ln \mathcal{N} + |\ln \delta|), & K &:= \left\lceil \frac{r \ln 2 + |\ln(1-\rho)|}{|\ln \rho|} \right\rceil = \mathcal{O}\left(\frac{\ln \mathcal{N} + |\ln \delta| + |\ln p|}{p}\right). \end{aligned}$$

Proof. By construction, $0 \leq \delta \leq 1/3$ and $\varepsilon = -\alpha \ln(1 - 3\delta)$, $2^{-n} \leq \delta |\max(v_*)|$, $n2^{-r} \leq \delta$, and $\rho^K \leq (1-\rho)2^{-r}$, as required by lemma 2.12 and proposition 2.13. (The lower bound for δ follows from the inequality $1 - \exp(-x) \geq \min\{1, x\}/2$, for all $x \geq 0$.) Thus $u_* \preceq u_n \preceq \underline{u}_n \preceq u_* + \varepsilon$, as announced. Finally, for any given $x \in X$, one has $x \in B_n^*$ iff $-2^{n-r} \leq v_*(x) \leq -2^{-n}$ iff $n_*(x) \leq n \leq n_*(x) + r$ where $n_*(x) := -\ln(-v_*(x))/\ln 2$, thus the elementary solution update “ $v(x) \leftarrow \Phi v(x)$ ” is used at most $K(r+1)$ times overall in (15). \square

2.3 Properties of an elementary update

We study in this subsection the convex optimization problem (17), which is the elementary building block of the semi-Lagrangian update operator (1). The α -acuteness condition of definition 1.4 plays a central role, and propositions 1.5 and 1.6 from the introduction are established as a consequence of propositions 2.19 and 2.21 to 2.23. For any $u, v \in \mathbb{R}^I$ define

$$\lambda_F(u) := \min_{\xi \in \Xi_I} F(\xi) + \langle u, \xi \rangle, \quad \phi_F(v) := E_\alpha \lambda_F E_\alpha^{-1} v. \quad (17)$$

Assumption 2.15. *We fix throughout this subsection a timescale $\alpha > 0$ and a norm $F : \mathbb{R}^I \rightarrow \mathbb{R}_+$ (always possibly asymmetric, by convention), where $I > 0$ is the space dimension. We assume that F is continuously differentiable on $\mathbb{R}^I \setminus \{0\}$, and uniformly convex (i.e. $F - \varepsilon \|\cdot\|$ is also a norm for some $\varepsilon > 0$). This simplifying assumption is eventually removed in proposition 2.23.*

Our estimates depend on upper and lower bounds for F , and often involve the dual norm F^* . We introduce the corresponding notations below, and gather classical elementary relations between these quantities in lemma 2.16.

$$F_{\max} := \max_{\|\xi\|=1} F(\xi), \quad F_{\min} := \min_{\|\xi\|=1} F(\xi), \quad F^*(\eta) := \max_{F(\xi) \leq 1} \langle \eta, \xi \rangle. \quad (18)$$

Recall that any norm is its own bi-dual: $F^{**} = F$. Also, since F is positively 1-homogeneous, its gradient is positively 0-homogeneous, and it satisfies Euler's identity: $\forall \xi \in \mathbb{R}^I \setminus \{0\}$,

$$\langle \nabla F(\xi), \xi \rangle = F(\xi). \quad (19)$$

Lemma 2.16. *For any norm F , any $\xi, \eta \in \mathbb{R}^I$ in (20, ii), and $\xi \neq 0$ in (20, iv), one has*

$$\max_{\xi \neq 0} \|\nabla F(\xi)\| = F_{\max}, \quad \langle \eta, \xi \rangle \leq F(\xi)F^*(\eta), \quad F_{\max}^* = F_{\min}^{-1}, \quad F^*(\nabla F(\xi)) = 1. \quad (20)$$

Proof of (20, i): One has $F_{\max} \leq \max_{\|\xi\|=1} \|\nabla F(\xi)\| = \max_{\xi \neq 0} \|\nabla F(\xi)\|$ using successively (19) and the 0-homogeneity of ∇F ; conversely choose ξ such that $\|\xi\| = 1$ and $F(\xi) = F_{\max}$ is maximal, note that ξ is proportional with $\nabla F(\xi)$ by Lagrange's optimality conditions, and thus $F(\xi) = \langle \xi, \nabla F(\xi) \rangle = \|\nabla F(\xi)\|$ using successively Euler's identity and the equality case in Cauchy-Schwartz's inequality. *Proof of (20, ii):* Clear if $\xi = 0$, and otherwise note that $F^*(\eta) \geq \langle \eta, \xi/F(\xi) \rangle$ by (18, iii). *Proof of (20, iii):* Choosing ξ such that $\|\xi\| = 1$ and $F(\xi) = F_{\min}$ is minimal, and letting $\eta := \xi$ in (20, ii) we obtain $1 = \langle \xi, \xi \rangle \leq F(\xi)F^*(\xi) \leq F_{\min}F_{\max}^*$. Conversely, choose η such that $\|\eta\| = 1$ and $F^*(\eta) = F_{\max}^*$ is maximal; note that $\xi := \nabla F^*(\eta)$ satisfies $F(\xi) \leq 1$, by (18, iii) and the envelope theorem, which is also discussed in the next point. Thus $F_{\max}^* = F(\eta) = \langle \eta, \nabla F^*(\eta) \rangle = \langle \eta, \xi \rangle \leq \|\xi\| \leq F(\xi)/F_{\min} \leq 1/F_{\min}$. *Proof of (20, iv):* We establish that $F(\nabla F^*(\eta)) = 1$ for any $\eta \neq 0$, which is equivalent to the announced property up to exchanging the roles of F and F^* . Denote $B := \{\xi \in \mathbb{R}^I \mid F(\xi) \leq 1\}$, which is compact and convex. For any $\eta \neq 0$, the maximum (18, iii) defining $F^*(\eta)$ is attained at some $\xi \in B$. By positive homogeneity and uniform convexity of F , this maximizer is unique and satisfies $F(\xi) = 1$. By the envelope theorem, one has $\nabla F^*(\eta) = \xi$, and therefore $F(\nabla F^*(\eta)) = 1$ as announced. \square

The function $\lambda_F : \mathbb{R}^I \rightarrow \mathbb{R}$ is not an operator in the sense of section 2.1, but it nevertheless satisfies properties that are clearly counterparts of those of definition 2.1.

Proposition 2.17. • (Monotonicity) $\forall u, u' \in \mathbb{R}^I, u \preceq u' \Rightarrow \lambda_F(u) \leq \lambda_F(u')$.

- (Sub-additivity) $\forall u \in \mathbb{R}^I, \forall t \geq 0, \lambda_F(u + t) \leq \lambda_F(u) + t$.

Proof. These properties are clear for any linearized mapping $\lambda_F^\xi(u) := F(\xi) + \langle \xi, u \rangle$, where $\xi \in \Xi_I$ is fixed, hence they hold for the minimal envelope λ_F similarly to proposition 2.4. \square

The last key property for our analysis, the ρ -sub-additivity of ϕ_F , cannot be proved by linearization in contrast with proposition 2.17. Instead, a more careful analysis is needed, using the optimality conditions of the optimization problem (17, left), and the α -acuteness condition. We recall the notation $\mathbb{1} := (1, \dots, 1) \in \mathbb{R}^I$.

Lemma 2.18 (Karush-Kuhn-Tucker relation). *For any $u \in \mathbb{R}^I$, the convex optimization problem (17, left) admits a unique minimizer $\xi \in \Xi_I$, and one has the optimality relation:*

$$\nabla F(\xi) + u = \lambda_F(u)\mathbb{1} + \nu, \quad \text{for some } \nu \in [0, \infty]^I \text{ satisfying } \langle \nu, \xi \rangle = 0. \quad (21)$$

Proof. Since F is a uniformly convex norm, and since the convex set Ξ_I does not contain any pair of collinear points, the restriction $F|_{\Xi_I}$ is a strictly convex function. Therefore (17, left) is the minimization of a strictly convex functional over a compact convex and non-empty set, and thus admits a unique minimizer $\xi \in \Xi_I$.

In view of the state constraints $\langle \xi, \mathbb{1} \rangle = 1$ and $\xi \succeq 0$ defining Ξ_I , the KKT relations for the minimization problem (17, left) state that $\nabla F(\xi) + u = \lambda\mathbb{1} + \nu$ for some $\lambda \in \mathbb{R}$, and some ν obeying (21, right). Taking the inner product with ξ we obtain $\lambda = \langle \lambda\mathbb{1} + \nu, \xi \rangle = \langle \nabla F(\xi) + u, \xi \rangle = F(\xi) + \langle u, \xi \rangle = \lambda_F(\xi)$, using Euler's identity (19), which concludes. \square

The exponential and the inverse function are applied componentwise to vectors in the following, similarly to definition 1.4, e.g. $1/v := (1/v_i)_{1 \leq i \leq I}$ for any $v = (v_i)_{1 \leq i \leq I} \in (\mathbb{R} \setminus \{0\})^I$.

Proposition 2.19. *The mapping λ_F is concave and continuously differentiable, and ϕ_F is differentiable on $\mathbb{R}^I \setminus \{0\}$. Furthermore for any $u \in \mathbb{R}^I$, denoting by $\xi \in \Xi_I$ the minimizer of (17, left), one has*

$$\nabla \lambda_F(u) = \xi, \quad \left[\frac{d}{dt} \phi_F(v + t) \right]_{t=0} = \langle \xi, \exp(-\nabla F(\xi)/\alpha) \rangle, \quad (22)$$

with $v := E_\alpha u$. As a result, $\phi_F(v + t) \leq \phi_F(v) + \rho_F(\alpha)t$, for all $v \in \mathbb{R}^X$ and $t \geq 0$.

Proof. The function $\lambda_F : \mathbb{R}^I \rightarrow \mathbb{R}$ is concave, since it is defined as the minimum of the family of (affine hence) concave functions $\lambda_F^\xi(u) := F(\xi) + \langle u, \xi \rangle$, $\xi \in \Xi_I$. Note that $\nabla \lambda_F^\xi(u) = \xi$, for any $u \in \mathbb{R}^I$. By the envelope theorem [Car01, Theorem 6.1], one has $\nabla \lambda_F(u) = \nabla \lambda_F^\xi(u) = \xi$ where $\xi \in \Xi_I$ is the unique minimizer in (17, left), see lemma 2.18, which establishes (22, left). Then

$$\begin{aligned} \left[\frac{d}{dt} \phi_F(v + t) \right]_{t=0} &= E'_\alpha(\lambda_F(u)) \langle \nabla \lambda_F(u), (E_\alpha^{-1})'(v) \rangle = -\frac{1}{\alpha} \exp\left(-\frac{\lambda_F(u)}{\alpha}\right) \langle \xi, \frac{\alpha}{v} \rangle = \phi_F(v) \langle \xi, \frac{1}{v} \rangle, \\ -\langle \xi, \frac{1}{v} \rangle &= \langle \xi, \exp\left(\frac{u}{\alpha}\right) \rangle = \langle \xi, \exp\left(\frac{-\nabla F(\xi) + \lambda_F(u) + \nu}{\alpha}\right) \rangle = \frac{-1}{\phi_F(v)} \langle \xi, \exp\left(-\frac{\nabla F(\xi)}{\alpha}\right) \rangle \end{aligned} \quad (23)$$

using the composition rule for derivatives in the first line, and the KKT relation (21) in the second line. Together these two lines yield (22, right). The KKT multiplier ν disappears from (23, r.h.s.) thanks to the complementary condition (21, right) (componentwise, one has $\xi_i = 0$ whenever $\nu_i \neq 0$, for each $1 \leq i \leq I$).

In order to establish the final statement, we need to generalize (22) to the case where $v = (v_1, \dots, v_I) \in \mathbb{R}^X$ is arbitrary, rather than negative as previously assumed. Denote $J := \{1 \leq$

$i \leq I \mid v_i \geq 0\}$, so that $u_i := E_\alpha^{-1}v_i = \infty$ for all $i \in J$. In the degenerate case where $J = \{1, \dots, I\}$, one has $u = (\infty)_{1 \leq i \leq I}$, thus $\lambda_F(u) = \infty$ and $\phi_F(v+t) = 0$ for all $t \geq 0$. Otherwise, the optimization problem (17) defining $\lambda_F(u)$ remains well posed with the convention $0 \times \infty = 0$ in the inner product $\langle \xi, u \rangle$. It has a finite value $\lambda_F(u) < \infty$ and a minimizer $\xi \in \Xi_I$ such that $\xi_j = 0$ for all $j \in J$. This effectively amounts to consider a similar optimization problem posed on $\Xi_{d-\#(J)}$ where the variables $(\xi_i)_{i \in J}$ are eliminated, and thus (22) still holds. The estimate of $\phi_F(v+t)$ follows by integration, recalling that $\langle \xi, \exp(-\nabla F(\xi)/\alpha) \rangle \leq \rho_F(\alpha)$ by definition 1.4. \square

The next two propositions together establish proposition 1.5, announced in the introduction, under the assumption 2.15 which is then removed in proposition 2.23. For sharpness, we write the obtained estimates of $\rho_F(\alpha)$ in terms of

$$F_{\Xi_{\min}} := \min_{\xi \in \Xi_I} F(\xi), \quad F_{\Xi_{\infty}} := \max_{\xi \in \Xi_I} \|\nabla F(\xi)\|. \quad (24)$$

Observing that $F_{\Xi_{\min}} \geq F_{\min}/\sqrt{I}$ (since $\|\xi\| \geq 1/\sqrt{I}$ for all $\xi \in \Xi_I$) and $F_{\Xi_{\infty}} \leq F_{\max}$ (by (20, i)), we obtain (11, left and right) from propositions 2.21 and 2.22 respectively.

Lemma 2.20. *The function $r(x) := \exp(x) - 1 - x$ satisfies $0 \leq r(x) \leq x^2$ for all $x \leq 1$.*

Proof. Differentiating twice we find that the function r is convex on \mathbb{R} , and $r(0) = r'(0) = 0$, hence $r \geq 0$. Likewise the function $s(x) := x^2 - r(x)$ is convex on $]-\infty, \ln 2]$, and $s(0) = s'(0) = 0$, hence $s \geq 0$ on this interval. Finally, s is concave on $[\ln 2, 1]$ and takes positive values at the endpoints of this interval, hence it is positive on this interval. \square

Proposition 2.21. *One has $\rho_F(\alpha) \leq 1 - \frac{F_{\Xi_{\min}}}{\alpha} (1 - \frac{F_{\Xi_{\infty}}^2}{\alpha F_{\min}})$, for all $\alpha \geq F_{\Xi_{\infty}}^2/F_{\min}$.*

Proof. Denote $\eta := -\nabla F(\xi)/\alpha$, for some $\xi \in \Xi_I$, and observe that $\|\eta\| \leq \|\nabla F(\xi)\|/\alpha \leq F_{\Xi_{\infty}}/\alpha \leq F_{\min}/F_{\Xi_{\infty}} \leq 1$. We have the estimates

$$\begin{aligned} \langle \xi, \exp(\eta) \rangle &= \langle \xi, \mathbb{1} + \eta + r(\eta) \rangle = 1 + \langle \eta, \xi \rangle + \langle \xi, r(\eta) \rangle \leq 1 - F(\xi)/\alpha + F(\xi)F^*(r(\eta)), \\ F^*(r(\eta)) &\leq F_{\max}^* \|r(\eta)\| \leq \|\eta\|^2/F_{\min} = \alpha^{-2} \|\nabla F(\xi)\|^2/F_{\min} \leq \alpha^{-2} F_{\Xi_{\infty}}^2/F_{\min}, \end{aligned}$$

using Euler's identity and (20, ii) in the first line, and (20, iii) in the second line. Combining the two lines we obtain $\langle \xi, \exp(-\nabla F(\xi)/\alpha) \rangle \leq 1 - \alpha^{-1} F(\xi) (1 - \alpha^{-1} F_{\Xi_{\infty}}^2/F_{\min})$. The result follows by definition 1.4 of $\rho_F(\alpha)$, and definition (24) of $F_{\Xi_{\min}}$. \square

Proposition 2.22. *Assume the classical acuteness condition, see definition 1.4, namely that $\nabla F(\xi) \succeq 0$ for all $\xi \in \Xi_I$. Then $\rho_F(\alpha) \leq 1 - \frac{F_{\Xi_{\min}}}{F_{\Xi_{\infty}}} (1 - \exp(-\frac{F_{\Xi_{\infty}}}{\alpha}))$, for all $\alpha > 0$.*

Proof. By convexity, one has $\exp(-t) \leq (1 - t/M) + (t/M) \exp(-M)$ for all $0 \leq t \leq M$. We choose $M := F_{\Xi_{\infty}}/\alpha$, let $\eta := \nabla F(\xi)/\alpha$ for some $\xi \in \Xi_I$, and obtain using Euler's identity (19)

$$\langle \xi, \exp(-\eta) \rangle \leq \langle \xi, (\mathbb{1} - \eta/M) + (\eta/M) \exp(-M) \rangle = 1 - \alpha^{-1} F(\xi) (1 - \exp(-M))/M. \quad \square$$

Proposition 2.23 (Extension to non-uniformly convex and non-differentiable norms). *Let F be an arbitrary asymmetric norm on \mathbb{R}^I (removing assumption 2.15). Define the subgradient set*

$$\begin{aligned} \partial F(\xi) &:= \{\eta \in \mathbb{R}^I \mid \forall \dot{\xi} \in \mathbb{R}^I, F(\xi + \dot{\xi}) \geq F(\xi) + \langle \eta, \dot{\xi} \rangle\}, \\ \rho_F(\alpha) &:= \max\{\langle \xi, \exp(-\eta/\alpha) \rangle \mid \xi \in \Xi_I, \eta \in \partial F(\xi)\}. \end{aligned}$$

Likewise the classical acuteness condition becomes $\partial F(\xi) \subset [0, \infty]^I$ for all $\xi \in \Xi_I$, and $F_{\Xi_{\infty}} := \max\{\|\eta\| \mid \xi \in \Xi_I, \eta \in \partial F(\xi)\}$. The estimates of propositions 2.21 and 2.22 still hold, and $\phi_F(v+t) \leq \phi_F(v) + \rho_F(\alpha)t$ and for all $v \in \mathbb{R}^I$ and $t \geq 0$.

Proof. The proofs of propositions 2.21 and 2.22 do not rely on continuous differentiability or uniform convexity, and thus immediately extend to this setting by replacing $\nabla F(\xi)$ with an arbitrary element of $\partial F(\xi)$.

Define $F_n := (F^* + F_0/n)^* + F_0/n$ for all $n \in \mathbb{N}^*$, where F_0 denotes the Euclidean norm. Since $F^* + F_0/n$ is a uniformly convex norm, its dual norm is continuously differentiable on $\mathbb{R}^I \setminus \{0\}$, and therefore F_n is both continuously differentiable and uniformly convex, thus fitting assumption 2.15. In addition, F_n converges uniformly to F on bounded subsets of \mathbb{R}^I as $n \rightarrow \infty$, hence $\lambda_{F_n} \rightarrow \lambda_F$ uniformly, and thus $\phi_{F_n} \rightarrow \phi_F$ locally uniformly. Finally, since all cluster points of $\nabla F_n(\xi)$ as $n \rightarrow \infty$ belong to $\partial F(\xi)$, for any $\xi \in \mathbb{R}^d \setminus \{0\}$, we obtain $\limsup_{n \rightarrow \infty} \rho_{F_n}(\alpha) \leq \rho_F(\alpha)$. We conclude by passing $\phi_{F_n}(v+t) \leq \phi_{F_n}(v) + \rho_{F_n}(\alpha)t$ to the limit as $n \rightarrow \infty$. \square

Axioms verification for the semi-Lagrangian scheme We show in the conclusion of this subsection that the semi-Lagrangian update operator (3), denoted Λ and acting on $X = \overset{\circ}{X} \sqcup \partial X$, obeys the axioms of the simplified convergence analysis presented in section 2.2.

Proposition 2.24. *For any $u \in \mathbb{R}^X$, and any interior vertex $x \in \overset{\circ}{X}$, one has*

$$\Lambda u(x) = \min_{V \in \mathcal{V}(x)} \lambda_{\mathcal{F}_x^V}(u_x^V) \quad (25)$$

where for $V = (v_1, \dots, v_d) \in \mathcal{V}(x)$ we denoted $u_x^V := (u(x+v_1), \dots, u(x+v_d)) \in \mathbb{R}^d$, and $\mathcal{F}_x^V(\xi) := \mathcal{F}_x(V\xi)$. As a result Λ is monotone and sub-additive. The operator $\Phi := E_\alpha \Lambda^\top E_\alpha^{-1}$ is monotone, super-multiplicative, and ρ -sub-additive, with the following parameters

$$\alpha := \max_{(x,V) \in \mathcal{V}} \frac{2(\mathcal{F}_x^V)_{\max}^2}{(\mathcal{F}_x^V)_{\min}}, \quad \rho := 1 - \frac{\sigma}{2\alpha}, \quad \text{where } \sigma := \min_{(x,V) \in \mathcal{V}} (\mathcal{F}_x^V)_{\Xi \min}. \quad (26)$$

Thus $\sigma \geq h/(\kappa\sqrt{d})$, $2h \leq \alpha \leq 2\kappa h$, and $\rho \leq 1 - (4\kappa^2\sqrt{d})^{-1}$ using the notations (8).

Proof. The expression of Λ immediately follows from its definition (3), and from the definition (17) of an elementary update λ_F . The monotonicity and sub-additivity of Λ follow from the counterparts of these properties for λ_F established in proposition 2.17, and from proposition 2.4 on the properties of the minimum of several schemes. The monotony and super-multiplicativity of Φ follow from proposition 2.8 on the exponential change of coordinates, and the ρ -sub-additivity from propositions 2.19 and 2.21. The final estimates of σ and α immediately follow from the definitions (8) of h and κ , and imply the estimate of ρ . \square

The parameter σ , defined in (26, right) can be interpreted geometrically, in the setting of remark 1.1, as the minimal distance from any interior point $x \in \overset{\circ}{X}$ to the boundary of its stencil $\mathbf{V}(x)$. The parameters (α, ρ) of proposition 2.24 can be replaced with any other pair such that $\alpha > 0$ and $1 > \rho \geq \max_{V \in \mathcal{V}(x)} \rho_{\mathcal{F}_x^V}(\alpha)$, as in the following example.

Remark 2.25. *Assume that \mathcal{F}_x^V satisfies the classical acuteness condition, for all $x \in \overset{\circ}{X}$ and all $V \in \mathcal{V}(x)$. Then $\Phi := E_\alpha \Lambda E_\alpha^{-1}$ is ρ -sub-additive for any $\alpha > 0$, with*

$$\rho := 1 - \min_{(x,V) \in \mathcal{V}} \frac{(F_x^V)_{\Xi \min}}{(F_x^V)_{\Xi \infty}} \left(1 - \exp\left(-\frac{(F_x^V)_{\Xi \infty}}{\alpha}\right) \right). \quad (27)$$

Choosing a very small $\alpha > 0$ yields a thin narrow band $B_n^* = \{x \in X \mid (n-r)\tau \leq u_*(x) \leq n\tau\}$ in the simplified algorithm of proposition 2.13, where u_* is the fixed point of Λ and $\tau := \alpha \ln 2$. Thus B_n^* contains few points, possibly a single one. This is reminiscent of the FMM solver which accepts points one by one, ordered by increasing values of the exact solution u_* . The FMM indeed also applies under these assumptions, see the discussion surrounding the causality property (9).

3 Proof of the main result

This section is devoted to the convergence and complexity analysis of algorithm 1, including the construction of the narrow band using an oriented graph distance, which was avoided in section 2. We study this construction in section 3.1 in an axiomatic setting, and use the obtained properties in section 3.2 for the convergence analysis of algorithm 1. Finally, we verify in section 3.3 that the operators of interest obey the required axioms, thus completing the proof of theorem 1.3.

3.1 Comparison of sublevel sets

The main result of this section, proposition 3.5, compares the sublevel sets of the fixed points of two operators; it is eventually applied in section 3.3 to the semi-Lagrangian update operator (3) discretizing the eikonal equation, and to a graph based approximation (33) whose fixed point is trivially computed using the classical Dijkstra algorithm 2. For that purpose, we introduce new axioms, which are more specific to our setting than those of definition 2.1, and distinguish between interior and boundary points.

Definition 3.1. *An operator $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$ on a set $X = \overset{\circ}{X} \sqcup \partial X$ satisfies the property:*

- Additive invariance if $\forall u \in \mathbb{R}^X, \forall t \in \mathbb{R}, \Lambda(u + t) = \Lambda u + t$ on $\overset{\circ}{X}$.
- σ -sub-multiplicativity if $\forall u \in \mathbb{R}^X, \forall t \in [0, \infty[, \Lambda[(1 + t)u] \preceq (1 + t)\Lambda u - \sigma t$ on $\overset{\circ}{X}$.
- C_{past} -memory if $\forall u \in \mathbb{R}^X, \forall x \in \overset{\circ}{X}$, one has

$$\Lambda \max\{u, t\}(x) = \Lambda u(x), \quad \text{for any } t \leq \Lambda u(x) - C_{\text{past}}.$$

We usually assume $\sigma > 0$ and $C_{\text{past}} \geq 0$.

The axiom of σ -sub-multiplicativity, for some $\sigma > 0$, allows perturbing weak sub- and super-solutions of Λ into *strict* ones in proposition 3.4, similarly to ρ -sub-additivity in proposition 2.6. The axiom of C_{past} -memory means, in the context of (3), that the arrival time at a given point only depends on recent arrival times, which eases the construction of the narrow band in our method. We prove below a reflected axiom, the stability of the properties of interest upon taking the maximum or minimum of a family of operators, and a version of the comparison principle.

Lemma 3.2. *If Λ is σ -sub-multiplicative then $\forall t \in [0, 1], \Lambda[(1 - t)u] \succeq (1 - t)\Lambda u + \sigma t$ on $\overset{\circ}{X}$.*

Proof. If $0 \leq t < 1$, then apply the σ -sub-multiplicativity axiom to $u' := (1 - t)u$ and $t' := t/(1 - t)$. If $t = 1$, then observe that $\Lambda(2 \times 0) \leq 2\Lambda(0) - \sigma$, using $u' = 0$ and $t' = 1$. \square

Lemma 3.3. *Consider a family of operators $(\Lambda_i)_{i \in I}$ on a set $X = \overset{\circ}{X} \sqcup \partial X$, and assume that $\Lambda u(x) := \inf_{i \in I} \Lambda_i u(x)$ (resp. $\Lambda u(x) := \sup_{i \in I} \Lambda_i u(x)$) is finite for all $u \in \mathbb{R}^X, x \in X$. If Λ_i is additively invariant, or σ -sub-multiplicative, or has C_{past} -memory (resp. assuming also monotony and sub-additivity), for each $i \in I$, then so is Λ .*

Proof. We only prove the case of C_{past} -memory in the sup case, where the additional monotony and sub-additivity axioms are needed as announced. The other announced properties are proved similarly to proposition 2.4, and are essentially trivial. In the following, we fix $x \in \overset{\circ}{X}$ and assume that $t \leq \Lambda u(x) - C_{\text{past}}$. Then $\Lambda \max\{u, t\}(x) \geq \Lambda u(x)$ since Λ is monotone, see proposition 2.4. On the other hand, one has for any $i \in I$ denoting $\delta := \Lambda u(x) - \Lambda_i u(x)$

$$\Lambda_i \max\{u, t\}(x) \leq \Lambda_i[\max\{u, t - \delta\} + \delta](x) \leq \Lambda_i \max\{u, t - \delta\}(x) + \delta = \Lambda_i u(x) + \delta = \Lambda u(x)$$

using successively (i) monotony, (ii) sub-additivity, and (iii) C_{past} -memory, which concludes. \square

Proposition 3.4. *Let $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$ be an operator on a finite set $X = \overset{\circ}{X} \sqcup \partial X$ which is monotone, sub-additive, and σ -sub-multiplicative for some $\sigma > 0$. Consider a partition $X = \overset{\circ}{Y} \sqcup \partial Y$ such that $\partial X \subset \partial Y$. For any $\bar{u}, \underline{u} \in \mathbb{R}^X$,*

$$\text{if } \bar{u} \preceq \underline{u} \text{ on } \partial Y, \quad \text{and } \max\{0, \bar{u} - \Lambda \bar{u}\} \preceq \underline{u} - \Lambda \underline{u} \text{ on } \overset{\circ}{Y}, \quad \text{then } \bar{u} \preceq \underline{u}.$$

Proof. Let $C \geq 0$ be such that $C + \min_{\partial Y} \underline{u} > 0$, and let $u_\varepsilon := (1 + \varepsilon)\underline{u} + C\varepsilon$ for some $\varepsilon > 0$. Then $u_\varepsilon \succ \underline{u}$ on ∂Y by choice of C . In addition, one has $\Lambda u_\varepsilon \preceq (1 + \varepsilon)\Lambda \underline{u} + C\varepsilon - \sigma\varepsilon$ on $\overset{\circ}{Y}$ by sub-additivity and σ -sub-multiplicativity, which shows that $u_\varepsilon - \Lambda u_\varepsilon \succeq (1 + \varepsilon)(\underline{u} - \Lambda \underline{u}) + \sigma\varepsilon \succ \bar{u} - \Lambda \bar{u}$ on $\overset{\circ}{Y}$. By the weak comparison principle proposition 2.7, we obtain that $\bar{u} \prec u_\varepsilon$ for any $\varepsilon > 0$, and thus $\bar{u} \preceq \underline{u}$ as announced. \square

As announced, the next result compares the sub-levelsets of the fixed points of two operators. Recall that we use the shorthand $\{u \leq t\} := \{x \in X \mid u(x) \leq t\}$, for any $u : X \rightarrow \mathbb{R}$ and $t \in \mathbb{R}$.

Proposition 3.5. *Let $\Lambda, \Gamma : \mathbb{R}^X \rightarrow \mathbb{R}^X$ be two operators on a finite set $X = \overset{\circ}{X} \sqcup \partial X$, such⁵ that $\Gamma - C_{\text{comp}} \leq \Lambda \leq \Gamma$ for some constant $C_{\text{comp}} \geq 0$. Assume that Λ is monotone, additively invariant, σ -sub-multiplicative for some $\sigma > 0$, and has C_{past} -memory. Consider $u_*, \hat{u} \in \mathbb{R}^X$, a partition $X = \overset{\circ}{Y} \sqcup \partial Y$ satisfying $\partial X \subset \partial Y$, and thresholds $t \leq t'$, such that the following holds*

$$\Lambda u_* = u_*, \quad \{u_* \leq t\} \subset \partial Y \subset \{u_* \leq t'\}, \quad \hat{u} = 0 \text{ on } \partial Y, \quad \Gamma \hat{u} = \hat{u} \text{ on } \overset{\circ}{Y}. \quad (28)$$

Then $\bar{u} := s\hat{u} + t - C_{\text{past}}$ and $\underline{u} := \hat{u} + t'$ satisfy $\bar{u} \preceq u_ \preceq \underline{u}$ on $\overset{\circ}{Y}$, where $s := \sigma / (C_{\text{comp}} + \sigma)$. Thus, $\forall T \geq 0$*

$$\{u_* \leq sT + t - C_{\text{past}}\} \subset \{\hat{u} \leq T\} \subset \{u_* \leq T + t'\}. \quad (29)$$

Proof. One has $\Lambda \underline{u} = \Lambda \hat{u} + t' \preceq \Gamma \hat{u} + t' = \underline{u}$ on $\overset{\circ}{Y}$, using successively (i) the additive invariance of Λ , (ii) the comparison assumption $\Lambda \leq \Gamma$, and (iii) the fixed point assumption (28, iv). In addition $\underline{u} = t' \succeq u_*$ on ∂Y , using (28, ii and iii), hence $\underline{u} \succeq u_*$ by the comparison principle proposition 3.4, as announced. Denoting $t_* := t - C_{\text{past}}$, we have on $\overset{\circ}{Y}$

$$\Lambda \bar{u} = \Lambda(s\hat{u} + t_*) = \Lambda(s\hat{u}) + t_* \succeq s\Lambda \hat{u} + (1 - s)\sigma + t_* \succeq s(\Gamma \hat{u} - C_{\text{comp}}) + (1 - s)\sigma + t_* = \bar{u},$$

using successively (i) additive invariance, (ii) σ -sub-multiplicativity as in lemma 3.2 since $s \in]0, 1]$, (iii) the assumption $\Gamma - C_{\text{comp}} \leq \Lambda$, and (iv) the definition of s and (28, iv). In addition $\bar{u} = t_* \leq \max\{u_*, t_*\}$ on ∂Y , and one has on $\overset{\circ}{Y}$

$$\max\{u_*, t_*\} = u_* = \Lambda u_* = \Lambda \max\{u_*, t_*\}$$

using successively (i) the observation that $t_* \leq t \preceq u_*$ on $\overset{\circ}{Y}$ by (28, ii), (ii) the fixed point assumption (28, i), and (iii) the fact that Λ has C_{past} -memory and $\Lambda u_* = u_* \succeq t = t_* + C_{\text{past}}$ on $\overset{\circ}{Y}$. Using again the comparison principle proposition 3.4 we obtain $\bar{u} \preceq \max\{u_*, t_*\}$ on X , thus $\bar{u} \preceq u_*$ on $\overset{\circ}{Y}$, as announced. We have established that $\bar{u} \leq u_* \leq \underline{u}$ on $\overset{\circ}{Y}$, equivalently $u_* - t' \preceq \hat{u} \preceq (u_* - t_*)/s$ on $\overset{\circ}{Y}$, which implies $E \cap \overset{\circ}{Y} \subset E' \cap \overset{\circ}{Y} \subset E'' \cap \overset{\circ}{Y}$ where E, E', E'' denote the sublevel sets appearing in (29). We conclude the proof of (29) noting that $\partial Y \subset E'$ and $\partial Y \subset E''$ by (28, ii and iii). \square

⁵i.e. $\Gamma u - C_{\text{comp}} \preceq \Lambda u \preceq \Gamma u$ for all $u \in \mathbb{R}^X$

3.2 The complete algorithm and its complexity

We replace in this subsection the ideal but non-computable narrow band used in proposition 2.13, with an approximation based on proposition 3.5. This completes the analysis of algorithm 1, under the assumptions listed below, which are checked in section 3.3 and appendices A and B for the numerical schemes of interest.

Assumption 3.6. *We denote by $X = \overset{\circ}{X} \sqcup \partial X$ a finite set, and by $\Lambda, \Phi, \Gamma : \mathbb{R}^X \rightarrow \mathbb{R}^X$ three operators. The operator Φ is subject to assumption 2.11: it is monotone, ρ -sub-additive, super-multiplicative, and its fixed point satisfies $-1 \preceq v_* \prec 0$. We assume in addition that $v_* = -1$ on ∂X . The operator Λ is additively invariant, σ -sub-multiplicative, has C_{past} -memory, and $\Phi = E_\alpha \Lambda E_\alpha^{-1}$. The operator Γ satisfies $\Gamma - C_{\text{comp}} \leq \Lambda \leq \Gamma$, and for any partition $X = \overset{\circ}{Y} \sqcup \partial Y$ there exists a solution $\hat{u} \in \mathbb{R}^X$ to $\Gamma \hat{u} = \hat{u}$ on $\overset{\circ}{Y}$ and $\hat{u} = 0$ on ∂Y . The constants satisfy $\rho, \alpha, \sigma > 0$ and $C_{\text{past}}, C_{\text{comp}} \geq 0$.*

We use the following notations from lemma 2.12 and proposition 2.13 of section 2.2:

$$\underline{v}_n := (1 - n2^{-r+1})v_* + 2^{-n}, \quad X_n^+ := \{v_* \geq -2^{-n}\}, \quad X_n^- := \{v_* \leq -2^{-n}\}, \quad B_n^* := X_{n-r}^+ \cap X_n^-,$$

where $r > 0$ and $n \geq 0$ are integers. The following lemma constructs an approximation B_n of the ideal narrow band B_{n+1}^* , from an approximation of the explicit super-solution \underline{v}_n . The definition of B_n matches the one given in algorithm 1 when Γ is the graph based approximation of Λ described in section 3.3.

Lemma 3.7. *Under assumption 3.6. Let $v_n \in \mathbb{R}^X$ be such that $v_* \preceq v_n \preceq \underline{v}_n$, with $0 \leq n \leq 2^{r-3}$ and $r \geq 3$. Then*

$$X_{n-r+1}^+ \subset \{v_n \geq -2^{-(n-r+1)}\} \subset X_{n-r}^+. \quad (30)$$

Let $\hat{u}_n \in \mathbb{R}^X$ satisfy $\hat{u}_n = 0$ on $Y_n := \{v_n \leq -2^{-(n-1)}\} \cup \partial X$, and $\Gamma \hat{u}_n = \hat{u}_n$ on $X \setminus Y_n$. Then

$$X_{n-1}^- \subset Y_n \subset X_n^-, \quad X_{n+1}^- \subset \{\hat{u}_n \leq T\} \subset X_{n+\hat{r}}^-, \quad (31)$$

with $T := (2\tau + C_{\text{past}})/s$ and $\hat{r} := \lceil T/\tau \rceil$, where $\tau := \alpha \ln 2$ and $s := \sigma/(C_{\text{comp}} + \sigma)$. Finally

$$B_{n+1}^* := X_{n-r+1}^+ \cap X_{n+1}^- \subset B_{n+1} := \{v_n \geq -2^{-(n-r+1)} \text{ and } \hat{u}_n \leq T\} \subset X_{n-r}^+ \cap X_{n+\hat{r}}^-. \quad (32)$$

Proof. Recall that $v_* < 0$ by assumption. At any point of X , one has the sequence of implications

$$v_* \geq -\varepsilon \Rightarrow v_n \geq -\varepsilon \Rightarrow \underline{v}_n \geq -\varepsilon \Leftrightarrow v_* \geq -\frac{\varepsilon + 2^{-n}}{1 - n2^{-r+1}} \Rightarrow v_* \geq -2\varepsilon,$$

choosing $\varepsilon = 2^{-(n-r+1)}$ and using that $n2^{-r+1} \leq 1/4$ in the last inequality; the inclusions (30) follow. Similarly, at any point of X we have the sequence of reversed implications

$$v_* \leq -\varepsilon \Leftarrow v_n \leq -\varepsilon \Leftarrow \underline{v}_n \leq -\varepsilon \Leftrightarrow v_* \leq -\frac{\varepsilon + 2^{-n}}{1 - n2^{-r+1}} \Leftarrow v_* \leq -2\varepsilon,$$

choosing $\varepsilon := 2^{-(n-1)}$ and using that $n2^{-r+1} \leq 1/4$ in the last inequality; the inclusions (31, left) follow, noting that $\partial X \subset \{v_* = -1\} \subset X_n^-$. Let $u_* := E_\alpha^{-1}v_*$, in such way that $\Lambda u_* = u_*$ and $X_n^- = \{u_* \leq n\tau\}$. Applying proposition 3.5 with $t := (n-1)\tau$ and $t' := n\tau$, the sets $\partial Y := Y_n$ and $\overset{\circ}{Y} := X \setminus Y_n$, and observing that $sT + t - C_{\text{past}} = (n+1)\tau$, we obtain (31, right) which concludes the proof. Finally, the set intersection of (30) and (31, right) yields (32), which concludes. \square

Corollary 3.8. *Under assumption 3.6, choose $\varepsilon > 0$ and define the parameters n, r, K as in proposition 2.14. Define $v_0 := 0$ and let $v_n := \Phi_{B_n}^K v_{n-1}$ for all $0 < n \leq n$, where the narrow band B_n is defined by (32). Then v_n is a super-solution and $v_* \preceq v_n \preceq \underline{v}_n$, for all $0 < n \leq n$. Finally we have $u_* \preceq u_n := E_\alpha^{-1} v_n \preceq u_* + \varepsilon$, and at most $NK(r + \hat{r} + 1)$ elementary solution updates are used in the iterations defining v_n .*

Proof. Denote by v_n^* the sequence of proposition 2.13, which is the simplified algorithm with oracle, and recall that $v_n^* \preceq \underline{v}_n$ for all $0 \leq n \leq n$. We claim that $v_n \preceq v_n^*$ for all $0 \leq n \leq n$: indeed $v_0 = v_0^* = 0$, and $v_n = \Phi_{B_n}^K v_{n-1} \preceq \Phi_{B_n}^K v_{n-1}^* \preceq \Phi_{B_n^*}^K v_{n-1}^* = v_n^*$ for all $0 < n \leq n$, using (i) the induction hypothesis $v_{n-1} \preceq v_{n-1}^*$ and the monotony of the operator Φ , and (ii) the inclusion $B_n^* \subset B_n$ see (32) and the fact that v_{n-1}^* is a sub-solution. We have shown that $v_n \preceq v_n^* \preceq \underline{v}_n$ for all $0 \leq n \leq n$, and therefore $u_n \preceq u_n^* := E_\alpha^{-1} v_n^* \preceq u_* + \varepsilon$. In addition, v_n is a super-solution of Φ , since $v_0 = v_0^*$ is a supersolution and using proposition 2.9 for induction, hence $v_n \succeq v_*$ by the comparison principle proposition 2.7, and therefore $u_n \geq u_*$ by monotony of E_α^{-1} . Finally, for any given $x \in X$, the membership $x \in B_n$ implies $-2^{-(n-r-1)} \leq v_*(x) \leq -2^{-(n+\hat{r}-1)}$ by (32), equivalently $n_*(x) - \hat{r} \leq n \leq n_*(x) + r$ where $n_*(x) := 1 - \ln(-v_*(x)) / \ln 2$, thus the elementary solution update “ $v(x) \leftarrow \Phi v(x)$ ” is used at most $K(r + \hat{r} + 1)$ times overall along the iterations, which concludes. \square

3.3 Verification of the axioms for the semi-Lagrangian scheme

We construct in this section a graph based approximation Γ of the semi-Lagrangian update operator Λ defined in (3). We check that it satisfies assumption 3.6, and conclude the proof of the complexity estimate theorem 1.3. For that purpose, we define for all $u \in \mathbb{R}^I$

$$\lambda_F(u) := \min_{\xi \in \Xi_I} F(\xi) + \langle \xi, u \rangle, \quad \gamma_F(u) := \min_{1 \leq i \leq I} F(b_i) + u_i, \quad F_{\Xi_{\max}} := \max_{1 \leq i \leq I} F(b_i).$$

Here and in the next proposition, I is a positive integer, F is a norm on \mathbb{R}^I (always possibly asymmetric, by convention) and $(b_i)_{1 \leq i \leq I}$ denotes the canonical basis of \mathbb{R}^I .

Proposition 3.9. *For any norm F on \mathbb{R}^I , one has for all $u \in \mathbb{R}^I$*

- (additive invariance) $\forall t \in \mathbb{R}, \lambda_F(u + t) = \lambda_F(u) + t$.
- (σ -sub-multiplicativity) $\forall t \geq 0, \lambda_F[(1 + t)u] \leq (1 + t)\lambda_F(u) - \sigma t$ with $\sigma := F_{\Xi_{\min}}$, see (24).
- ($F_{\Xi_{\max}}$ -memory) $\lambda_F(\max\{u, t\}) = \lambda_F(u)$ for all $t \leq \lambda_F(u) - F_{\Xi_{\max}}$.
- (Comparison) $\gamma_F(u) - F_{\Xi_{\max}} \leq \lambda_F(u) \leq \gamma_F(u)$.

Proof. Similarly to proposition 2.17, additive invariance and σ -sub-multiplicativity hold for any affine mapping $\lambda_F^\xi(u) := F(\xi) + \langle u, \xi \rangle$, where $\xi \in \Xi^I$ is fixed, hence for their minimal envelope.

Proof of comparison: one has $\gamma_F(u) - F_{\Xi_{\max}} \leq \min_{1 \leq i \leq I} u_i = \min_{\xi \in \Xi_i} \langle \xi, u \rangle \leq \lambda_F(u)$, and $\lambda_F(u) \leq \gamma_F(u)$ since the canonical basis is contained in Ξ_I , as announced. *Proof of $F_{\Xi_{\max}}$ -memory:* one has $\lambda_F(u) \leq \gamma_F(u) \leq F_{\Xi_{\max}} + \min_{1 \leq i \leq I} u_i$. From $t \leq \lambda_F(u) - F_{\max}$ we thus obtain $t \leq \min_{1 \leq i \leq I} u_i$, hence $\max\{u, t\} = u$, and therefore $\lambda_F(\max\{u, t\}) = \lambda_F(u)$, as announced. \square

We introduce a graph based operator Γ , defined for all $x \in \mathring{X}$ as follows (recalling (6))

$$\Gamma u(x) := \min_{v \in \mathcal{E}(x)} \mathcal{F}_x(v) + u(x + v), \quad \text{where } \mathcal{E}(x) := \{v_j \mid (v_1, \dots, v_d) \in \mathcal{V}(x), 1 \leq j \leq d\}, \quad (33)$$

and $\Gamma u(x) := 0$ for all $x \in \partial X$. Clearly, if $X = \overset{\circ}{Y} \sqcup \partial Y$ with $\partial Y \subset \partial X$, then the graph distance map to ∂Y can be characterized as the unique fixed point of Γ with null boundary conditions on ∂Y , as follows

$$(\Gamma \hat{u} = \hat{u} \text{ on } \overset{\circ}{Y}, \text{ and } \hat{u} = 0 \text{ on } \partial Y) \Leftrightarrow \forall x \in X, \hat{u}(x) = d_{\mathcal{E}}(x, \partial Y) \left(:= \min_{y \in \partial Y} d_{\mathcal{E}}(x, y) \right). \quad (34)$$

With this choice of Γ , the narrow bands B_{n+1} defined in algorithm 1 and (32) are thus identical.

Corollary 3.10. *Under assumption 1.2 (connected stencil), and assuming that $\Phi := E_{\alpha} \Lambda E_{\alpha}^{-1}$ is ρ -sub-additive for some $\alpha > 0$ and $\rho < 1$, as in proposition 2.24. The operator Λ on $X := \overset{\circ}{X} \sqcup \partial X$ defined in (3) is additively invariant, σ -sub-multiplicative, has C_{past} -memory, and satisfies $\Gamma - C_{\text{comp}} \leq \Lambda \leq \Gamma$. The fixed point u_* of Λ satisfies $0 \preceq u_* \preceq N h_{\mathcal{E}}$. We denoted $N := \#(X)$, the constant $\sigma > 0$ is given in (26, right), and one has $C_{\text{comp}} = C_{\text{past}} = h_{\mathcal{E}} = \max_{(x,v) \in \mathcal{E}} \mathcal{F}_x(v)$ (thus $h_{\mathcal{E}} \leq h$, see (8)).*

Proof. For any interior discretization point $x \in \overset{\circ}{X}$ and stencil element $V = (v_1, \dots, v_d) \in \mathcal{V}(x)$, one has $(\mathcal{F}_x^V)_{\Xi_{\max}} = \max_{1 \leq i \leq d} \mathcal{F}_x(v_i)$ using the notation (7). Note also that $\Gamma u(x) = \min\{\gamma_{\mathcal{F}_x^V}(u_x^V) \mid V \in \mathcal{V}(x)\}$, for all $x \in \overset{\circ}{X}$. The additive invariance, σ -sub-multiplicativity, C_{past} -memory, and comparison with Γ thus follow from the counterparts of these properties for λ_F and γ_F established in proposition 3.9, and from lemma 3.3 on the minimum of several schemes.

The map $\hat{u}(x) := d_{\mathcal{E}}(x, \partial X)$ satisfies $\hat{u} \preceq N h_{\mathcal{E}}$ by the connectedness assumption 1.2 and since the oriented graph has N vertices with edgelengths bounded by $h_{\mathcal{E}}$. One has $\Gamma \hat{u} = \hat{u}$ in view of (34), thus $\Lambda \hat{u} \preceq \hat{u}$ and therefore $\Phi \hat{v} \preceq \hat{v}$ where $\hat{v} := E_{\alpha} \hat{u}$ and $\Phi := E_{\alpha} \Lambda E_{\alpha}^{-1}$. By corollary 2.3 there exists a unique fixed point v_* of Φ , therefore $v_* \preceq \hat{v} < 0$ by the comparison principle proposition 2.7, and thus $u_* := E_{\alpha} v_* \preceq \hat{u} \preceq N h_{\mathcal{E}}$ as announced. \square

Final complexity estimate. We conclude the proof of theorem 1.3. By corollary 3.8, the computation of the approximate fixed point u_n requires at most $NK(r + \hat{r} + 1)$ elementary updates, with the parameters $N := \#(X)$, K and r from proposition 2.14, and $\hat{r} := \lceil (2 + \frac{C_{\text{past}}}{\alpha \ln 2})(1 + \frac{C_{\text{comp}}}{\sigma}) \rceil$ from lemma 3.7. Combining these estimates we obtain the complexity

$$\mathcal{O}\left(\frac{N}{p} \ln^+\left(\frac{\max(u_*)}{p\varepsilon}\right) \left(\frac{C_{\text{past}} C_{\text{comp}}}{\alpha \sigma} + \ln^+\left(\frac{\max(u_*)}{\varepsilon}\right)\right)\right) \quad (35)$$

$$= \mathcal{O}\left(N \kappa^2 \sqrt{d} \ln^+\left(\frac{N h \kappa^2 \sqrt{d}}{\varepsilon}\right) (\kappa \sqrt{d} + \ln^+\left(\frac{N h}{\varepsilon}\right))\right) \quad (36)$$

where $p := 1 - \rho$ and $\ln^+ := \max\{1, \ln\}$. We assumed for simplicity in (35) that $\varepsilon \leq \alpha$, thus $\varepsilon/(6\alpha) \leq \delta \leq 1/3$, and that $\sigma \leq C_{\text{comp}}$, up to considering $\varepsilon' := \min\{\varepsilon, \alpha\}$ and $\sigma' := \min\{\sigma, C_{\text{comp}}\}$. We inserted in (36) the estimates of $\rho, \alpha, \sigma, C_{\text{past}}, C_{\text{comp}}$ and $\max(u_*)$ established in proposition 2.24 and corollary 3.10, which are specific of the semi-Lagrangian scheme (3).

Complexity of the narrow bands construction. For completeness, we briefly recall in algorithm 2 how a variant of Dijkstra's algorithm can be used to construct the narrow bands of algorithm 1. The insertion in the penultimate line of algorithm 2 is conditioned to the inequality $\hat{u}(x) < T$, in such way that the graph distance $d_{\mathcal{E}}(x, Y_n)$ is correctly computed up to the value T which is required for the construction of the narrow band B_{n+1} , and is overestimated elsewhere. As a result of this conditional insertion, any given point $x \in X$ which is inserted in L at step n satisfies $x \in B_{n+1}$. Each such point triggers at most $e(x)$ updates of \hat{u} , where $e(x) := \#\{y \in X \mid x = y + v \text{ for some } v \in \mathcal{E}(y)\}$, thus $E := \sum_{x \in X} e(x) = \sum_{y \in \overset{\circ}{X}} \#\mathcal{E}(y)$ is

the total number of edges in the graph. In addition, as already observed in corollary 3.8, there are at most $r + \hat{r} + 1$ values of n such that $x \in B_{n+1}$. The overall cost of algorithm 2 is thus $\mathcal{O}((r + \hat{r} + 1)E \ln N)$, accounting for the logarithmic overhead of maintaining the heap structure, which is clearly not dominant.

Algorithm 2 Dijkstra-based narrow band computation

Inputs: An increasing sequence $Y_0 \subset Y_1 \subset \dots \subset Y_n \subset X$ of finite sets, $Y_{-1} := \emptyset$. A positively weighted graph structure on X , here defined by the edges $\mathcal{E}(x)$ and corresponding costs $c(x, x + v) := \mathcal{F}_x(v)$ for each $v \in \mathcal{E}(x)$, $x \in \overset{\circ}{X}$.

Variables: An array $\hat{u} : X \rightarrow [0, \infty]$ initialized to $+\infty$. A list L initially empty and containing points of X sorted by the corresponding values of \hat{u} (implemented using a heap for efficiency).

For $n = 0, \dots, n$ **do**

Set $\hat{u}(x) = 0$ for all $x \in Y_n \setminus Y_{n-1}$, and insert these points in L (previously empty).

While L is non-empty **do**

Remove from L an element y which minimizes $\hat{u}(y)$.

For all $x \in X$ **such that** $y = x + v$ **for some** $v \in \mathcal{E}(x)$ **do**

If $c(x, y) + \hat{u}(y) < \hat{u}(x)$ **then**

$\hat{u}(x) \leftarrow c(x, y) + \hat{u}(y)$

If $\hat{u}(x) < T$ **then** insert x in L if not present, or update its position otherwise.

Yield $\hat{u}_n := \hat{u}$, satisfying $\hat{u}_n(x) \geq d_{\mathcal{E}}(x, Y_n)$ for all $x \in X$, with equality if $d_{\mathcal{E}}(x, Y_n) \leq T$.

4 Numerical experiments

We illustrate the proposed Narrow Band (NB) method algorithm 1 with a toy one-dimensional numerical experiment, a large two-dimensional test case from seismology featuring anelliptic anisotropy, and two synthetic test cases featuring strong Riemannian anisotropy and asymmetric anisotropy respectively. Our objective is to show that the parameters of the NB method are easy to set in practice, and to illustrate the quasi-linear complexity established in theorem 1.3.

For comparison⁶, we implemented the Global Iteration (GI), Fast Sweeping (FS) [Zha05], Adaptive Gauss Siedel Iteration (AGSI) [BR06], and Fast Iterative Method (FIM) [JW08], see algorithms 3 to 5 and [JW08, Algorithm 4.2]. At termination, the approximate numerical solution produced by each method satisfies $u - \varepsilon \preceq \Lambda u \preceq u$ by construction⁷, where $\varepsilon > 0$ is a tolerance parameter and Λ denotes the update operator of the scheme.

We report for each method the number of evaluations of the update operator Λ until termination. We do not report computation time⁸, which was not particularly optimized. In particular, we did not parallelize the numerical implementations, despite a clear opportunity to do so outlined in the last line of algorithm 1.

⁶The Banded Fast Marching method [CCF11] is close in spirit to the proposed NB method, despite the lack of complexity guarantee. A comparison with this method would have been interesting in principle. However, it is more complex and less formally specified than the AGSI, FIM and FS, hence it was not implemented.

⁷This implies $u_* \preceq u \preceq (1 + \varepsilon/\sigma)u_*$ where u_* is the exact fixed point of Λ , using the σ -sub-additivity of the update operator as in lemma 3.2 and the comparison principle, where the constant $\sigma > 0$ is given in corollaries 3.10, A.7 and B.6 for the Semi-Lagrangian, Lax-Friedrichs, or Eulerian scheme respectively.

⁸For reference, the anelliptic test case BP 2007 TTI of size 12596×1801 completed in 25min on a single core of a Macbook pro laptop equipped with a M1 max processor and 32GB ram, using the Lax-Friedrichs scheme and the proposed Narrow Band method.

The three discretizations of the eikonal equation considered in this paper have different strengths and weaknesses, about which we share here our subjective and empirical experience [Mir14b, Mir14a, Mir19]. The semi-Lagrangian scheme usually has the best accuracy among first order methods, especially if one uses a sufficiently refined stencil with e.g. 8 points in two dimensions; on the negative side, the update operator (3) is defined by an optimization problem which can be non-trivial to implement for anelliptic metric structures [DCC⁺21]. The Eulerian scheme has a rather simple and fast to compute update operator (52), by solving univariate quadratic equations; on the negative side, it only applies to specific families of metrics, namely Randers or Riemannian. The Lax-Friedrichs scheme has the simplest update operator (43), which is easily implemented for any metric structure; on the negative side, it is highly diffusive and never satisfies the causality property (9).

The complexity of the FS, AGSI, and FIM methods is empirically close to $\mathcal{O}(N)$ on “easy” problems where e.g. the metric $\mathcal{F}_x(v) = c(x)\|v\|$ is proportional to the identity matrix, the cost function $c > 0$ is smooth, and the Eulerian scheme is used, where N denotes the number of discretization points. In this numerical section, we purposely focus on “difficult” test cases, involving the strongly non-causal Lax-Friedrichs scheme in the first two numerical experiments, and either strong anisotropy or strong asymmetry in the other two experiments, so as to illustrate the limitations of the FS, AGSI and FIM in difficult situations. Under these conditions, their average number of iterations per point grows linearly with the largest grid dimension, see fig. 1, resulting in superlinear complexity $\mathcal{O}(N^{1+\frac{1}{d}})$ (for a square domain with $N^{\frac{1}{d}}$ points on each side).

Algorithm 3 Global Iteration

Inputs : An update operator $\Lambda : \mathbb{R}^X \rightarrow \mathbb{R}^X$, a tolerance $\varepsilon > 0$.

Variables : An array $u : X \rightarrow]0, \infty]$, initialized to ∞ .

Until $\Lambda u \succeq u - \varepsilon$ **do**

For all $x \in X$ **do** $u(x) \leftarrow \Lambda u(x)$ **in parallel.**

Algorithm 4 Fast Sweeping [Zha05].

Same inputs and variables as GI.

Until $\Lambda u \succeq u - \varepsilon$ **do**

For $i = 1, \dots, d$ **do**

For all $x = (x_1, \dots, x_d) \in X$ **in order** of increasing x_i **do** $u(x) \leftarrow \Lambda u(x)$.

For all $x = (x_1, \dots, x_d) \in X$ **in order** of decreasing x_i **do** $u(x) \leftarrow \Lambda u(x)$.

Algorithm 5 Adaptive Gauss Siedel Iteration [BR06].

Same inputs and variables as GI.

Additional variable : A FIFO queue L , initialized with the points of ∂X .

While L is non-empty **do**

Pop $x \in X$ from the front of L .

If $\Lambda u(x) < u(x) - \varepsilon$ **then**

$u(x) \leftarrow \Lambda u(x)$

Append to L all the neighbors of x which are not already in L .

Discussion of the timescale α . The analysis of the proposed numerical method is based, above all, on two ingredients which are the monotony of the update operator Λ , and the ρ -sub-additivity of the update operator $\Phi := E_\alpha \Lambda E_\alpha^{-1}$ written in Kruřkov’s exponential coordinates

(13), with $\rho < 1$. Assuming differentiability of Λ , for simplicity, these assumptions read

$$\frac{\partial \Lambda u(x)}{\partial u(y)} \geq 0, \quad \sum_{y \in X} \frac{\partial \Lambda u(x)}{\partial u(y)} \exp\left(\frac{u(y) - \Lambda u(x)}{\alpha}\right) \leq \rho, \quad (37)$$

for all $x, y \in X$ and $u \in \mathbb{R}^X$. The timescale $\alpha > 0$ thus heuristically measures the admissible amount of *non-causality*: the arrival time $\Lambda u(x)$ may depend on ulterior values $u(y) > \Lambda u(x)$, as opposed to (9), but the dependency $\frac{\partial \Lambda u(x)}{\partial u(y)} \leq \rho \exp(\frac{\Lambda u(x) - u(y)}{\alpha})$ decays exponentially fast as a function of the non-causal differences $u(y) - \Lambda u(x) \geq 0$, with the timescale α .

As announced in the introduction we replace the fixed number of iterations K within the narrow band B_n in algorithm 1 with a stopping criterion, so as to both reduce complexity and eliminate this parameter. More precisely, at each inner iteration, we perform the update $u(x) \leftarrow \Lambda u(x)$ at each point $x \in B_n$ which does not satisfy

$$\exp\left(\frac{u_n - \Lambda u(x)}{\alpha}\right) - \exp\left(\frac{u_n - u(x)}{\alpha}\right) \leq \varepsilon_* \quad (38)$$

where u_n is the lower bound for the values in the narrow band B_n , and $\varepsilon_* > 0$ is a fixed threshold.

The choice of timescale $\alpha > 0$ can be guided by proposition 2.24 for the Semi-Lagrangian scheme, proposition B.4 for the Eulerian scheme, and proposition A.6 for the Lax-Friedrichs scheme. Unfortunately, the theoretical value is often excessively large for practical use, and likewise the other narrow band parameters are often unusable, even in the toy one-dimensional example of section 4.1. For this reason, we present simple heuristics for parameter setting.

Heuristic choice of narrow band parameters. *Timescale α .* Consider the eikonal equation associated with a metric \mathcal{F} on a bounded domain $\Omega \subset \mathbb{R}^d$, discretized on the grid $\bar{\Omega} \cap h\mathbb{Z}^d$ with scale $h > 0$. Denote by $V_{\min} := 1/\max\{\mathcal{F}_x(v) \mid x \in \bar{\Omega}, \|v\| = 1\}$ the minimal propagation speed. We use $\alpha \in [2h/V_{\min}, 5h/V_{\min}]$ in the following experiments. An excessively large value of α increases computational cost (linearly w.r.t. α empirically), and an excessively small value increases the residue $\|u - \Lambda u\|_\infty$ of the numerical solution produced by the NB method (this quantity was always computed and checked to validate the experiments).

Threshold $\varepsilon_ > 0$.* The condition (38) implies $\Lambda u(x) \geq u(x) - e\alpha\varepsilon_*$ for all $x \in B_n$ such that $u(x) \leq \alpha$ (using that $\exp'(s) \geq 1/e, \forall s \geq -1$), i.e. at points whose value is to be frozen soon. We use $\varepsilon_* := \frac{\varepsilon}{e\alpha}$ in the following experiments, for consistency with the stopping criterion of the GI, AGSI, FS and FIM.

Time step $\tau > 0$. Lemma 3.7 suggests $\tau = \alpha \ln 2$. We use $\tau = \alpha/2$ in the following experiments, which slightly improves accuracy.

Narrow band width $r\tau$. A significant step of the convergence analysis is the estimate (16) which similar to (38) but with the tolerance $\varepsilon_* = (1 - \rho)2^{-r}$, corresponding to $r\tau = \alpha \ln((1 - \rho)e\alpha/\varepsilon)$. We use $r\tau = \alpha \ln(\alpha/\varepsilon)$ in the following experiments, since the contraction factor ρ is often unknown, hard to estimate, or excessively pessimistic.

Narrow band extension parameter T . Lemma 3.7 suggests $T = (2\tau + C_{\text{past}})(\sigma + C_{\text{comp}})/\sigma$; however σ, C_{past} and C_{comp} may again be unknown, hard to estimate, or excessively pessimistic. We use $T = 5\tau$ in the following experiments, which is sufficiently large to ensure that the back of the narrow band never catches up with the front.

Finally, let us mention that the test cases feature point sources $u(x_0) = 0$ for some $x_0 \in \Omega$, and outflow boundary conditions $u = \infty$ on $\partial\Omega$. This is consistent with the intended applications, but differs slightly from the setting of theorem 1.3.

4.1 One dimensional toy experiment, Lax-Friedrichs scheme

We solve the toy problem $|u'| = 1$ on $\Omega :=]0, 1[$, with the Dirichlet boundary condition $u(0) = 0$, and the outflow boundary condition $u(1) = \infty$. The exact viscosity solution of this eikonal PDE is $u_*(x) = x$ for all $x \in \Omega$. The Lax-Friedrichs update operator (43) reads

$$\Lambda u(x) := \min\{hC_0 + \min\{u(x+h), u(x-h)\}, c_1(h - \frac{1}{2}|u(x+h) - u(x-h)|) + \frac{1}{2}(u(x+h) + u(x-h))\},$$

for all $x \in \mathring{X} :=]0, 1[\cap h\mathbb{Z}$, where the grid scale $h > 0$ is chosen as the inverse of an integer. The scheme parameters need to obey $0 < c_1 < 1 \leq C_0$; we choose arbitrarily $c_1 = 1/3$ and $C_0 = 2$. Observing that the solution u is non-decreasing we obtain the simplified but equivalent scheme

$$\Lambda u(x) = \min\left\{2h + u(x-h), \frac{h + 2u(x-h) + u(x+h)}{3}\right\}. \quad (39)$$

Narrow band parameters. The parameters arising from the convergence analysis are not reasonable for numerical applications, even in this simple example. Specifically, corollary A.7 suggests using $\alpha = 216h$, to achieve a contraction factor $\rho = 1 - 1/1296$. Proposition 2.14, with $h = 1/1000$ and target accuracy $\varepsilon = 10^{-4}h$, suggests using $K = 33979$ inner iterations, so as to satisfy (38) with $\varepsilon_* \approx 4.1 \times 10^{-12}$, and to use a narrow band of width $r\tau \approx 4117h$. Lemma 3.7 suggests using $T \approx 2096h$ for the narrow band extension. In practice, the narrow band would be larger than the domain, the threshold ε_* would be comparable with machine precision, and the number K of inner loops would be greater than what GI uses overall.

On the other hand, instantiating (37, right) with the scheme (39) and the known eikonal PDE solution $u_*(x) = x$, yields $\frac{2}{3} \exp(-h/\alpha) + \frac{1}{3} \exp(h/\alpha) \leq \rho$. Letting $\alpha = h/\ln(3/2) \approx 2.46h$ thus yields the contraction factor $\rho = 1 - 1/18$; however, this only holds in the vicinity of u_* .

In practice, we use the middle ground $\alpha = 5h$ for the timescale, and use the tolerance $\varepsilon = 10^{-4}h$; thus following our heuristics $\tau = \alpha/2 = 2.5h$, $r\tau = \alpha \ln(\alpha/\varepsilon) \approx 54h$, $T = 5\tau = 12.5h$ and $\varepsilon_* = \frac{\varepsilon}{e\alpha} \approx 7.4 \times 10^{-6}$. Figure 2 illustrates the narrow band, which is rather thin and allows concentrating the numerical computations in the region where they are most useful. In contrast, the AGSI queue quickly grows and encompasses most of the domain, resulting in poor efficiency similarly to GI.

The NB method uses in average 220 updates per point with the specified parameters, independently of the problem size (this relatively high number, for such a simple problem, is due to the high diffusivity and non-causality of the Lax-Friedrichs scheme). In contrast the AGSI uses 203, 684, 2849 updates per point for the problem sizes 200, 1000 and 5000 respectively, with the same tolerance $\varepsilon = 10^{-4}h$, and the FIM, FS and GI are even more costly (in that order), see fig. 1 (i).

4.2 Anelliptic anisotropy, Lax-Friedrichs scheme

The BP 2007 TTI test case⁹ is a large two-dimensional dataset describing the underground properties below the seafloor using Thomsen's elastic parameters [Tho86], which is commonly used in seismic tomography benchmarks. The discretization grid has shape 12596×1801 , with grid scale $h = 6.25m$. Thomsen's parameters describe a part of the Hooke elasticity tensor, which is sufficient to compute the arrival time of the seismic pressure waves by solving the eikonal equation associated with the metric (40) below. The speed profile is tilted, anisotropic and anelliptic, with sharp velocity contrasts at the boundary of the different media (rock, salt domes, water).

⁹Created by Hemang Shah and provided courtesy of BP Exploration Operation Company Limited.

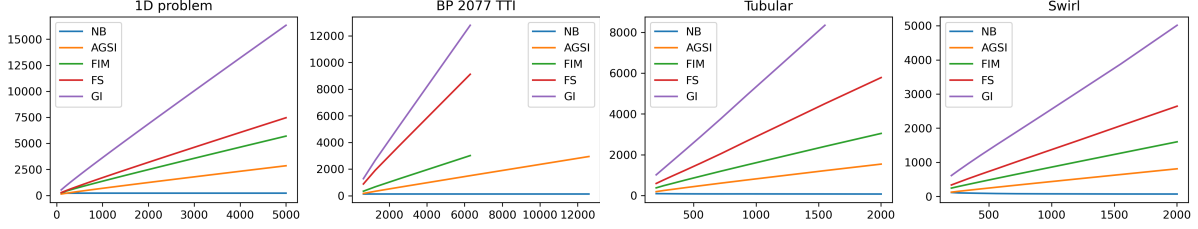


Figure 1: Average number of updates per point N_{up} , as a function of the largest grid dimension N_x , in the four considered test cases. Observe that N_{up} grows linearly with N_x when using the AGSI, FIM, FS and GI methods, but remains constant with the proposed NB method.

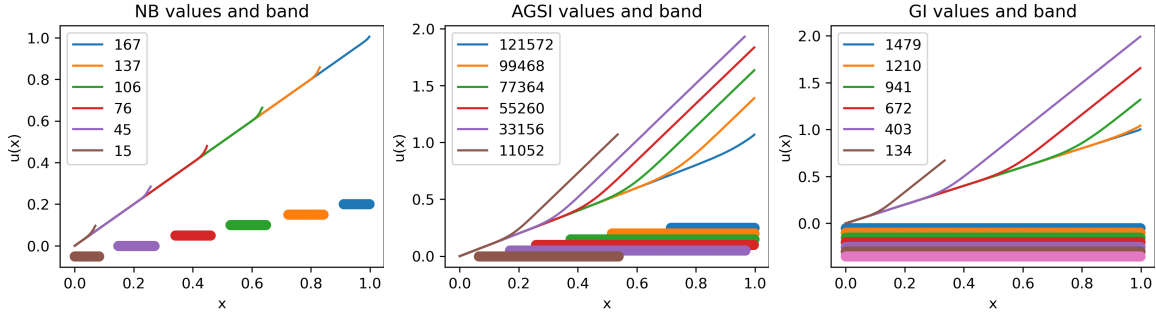


Figure 2: Solution values (thin lines) and effective band of active points (thick lines; i.e. the narrow band, queue, or whole domain), at the specified iteration counts of the main loop of the NB, AGSI, and GI methods. Problem size $N_x = 400$, see text for the method parameters.

Formally, consider a bounded domain $\Omega \subset \mathbb{R}^d$ equipped with $c \in C^0(\bar{\Omega}, \mathbb{R}^{d \times d \times d \times d})$, the reduced Hooke tensor, obeying the usual assumptions of symmetry ($c_{ijkl} = c_{klij} = c_{jikl}$) and positivity ($\sum_{ijkl} c_{ijkl}(x) m_{ij} m_{kl} > 0$ for any $m \in \mathcal{S}_d$), where the indices i, j, k, l always range over $\{1, \dots, d\}$. Define for any point $x \in \bar{\Omega}$ and any (co-)vector $\eta \in \mathbb{R}^d$ the (dual-)metric

$$\mathcal{F}_x^*(\eta) := \sqrt{\|c(x; \eta)\|}, \quad \text{where } c(x; \eta)_{jk} := \sum_{i,l} c_{ijkl}(x) \eta_i \eta_l. \quad (40)$$

We denote by $\|\cdot\|$ the operator norm of a $d \times d$ matrix, which in this case is also the largest eigenvalue since $c(x; \eta) \in \mathcal{S}_d^{++}$ for any $\eta \neq 0$ in view of the assumptions on c . As announced the norm \mathcal{F}^* is anelliptic (non-Riemannian), and has a complex structure depending in full generality on the numerous¹⁰ independent coefficients of the reduced Hooke tensor c , which complicates the implementation of semi-Lagrangian [DCC⁺21] or Eulerian schemes. In contrast, the Lax-Friedrichs scheme implementation (43) is trivial since it only requires evaluating $\mathcal{F}_x^*(\eta)$. We solve $\mathcal{F}^*(\nabla u) = 1$ on $\Omega \setminus \{x_0\}$ with the point source boundary condition $u(x_0) = 0$ where x_0 is an interior point close to the top-right corner, and outflow boundary conditions on $\partial\Omega$.

The Lax-Friedrichs scheme parameters $C_0(x)$ and $c_1(x)$ are defined pointwise in this experiment, for best efficiency, and are adjusted numerically so as to satisfy $C_0(x)^{-1} \|\xi\|_\infty \leq \mathcal{F}_x^*(\xi) \leq c_1(x)^{-1} \|\xi\|_1$ sharply, by sampling over a set of 20 directions ξ . We normalized the data so that the maximum pressure wave velocity is $V_{\text{max}} = 1$, and we use the timescale $\alpha = 2h/V_{\text{min}} \approx 6h$,

¹⁰There are 6 independent coefficients if $d = 2$, and 15 if $d = 3$. The two-dimensional BP 2007 TTI test case only provides four: Thomsen's parameters V_p , ε , δ , and the tilt angle θ , but they are sufficient to approximate well (40) and thus the pressure wave speed profile.

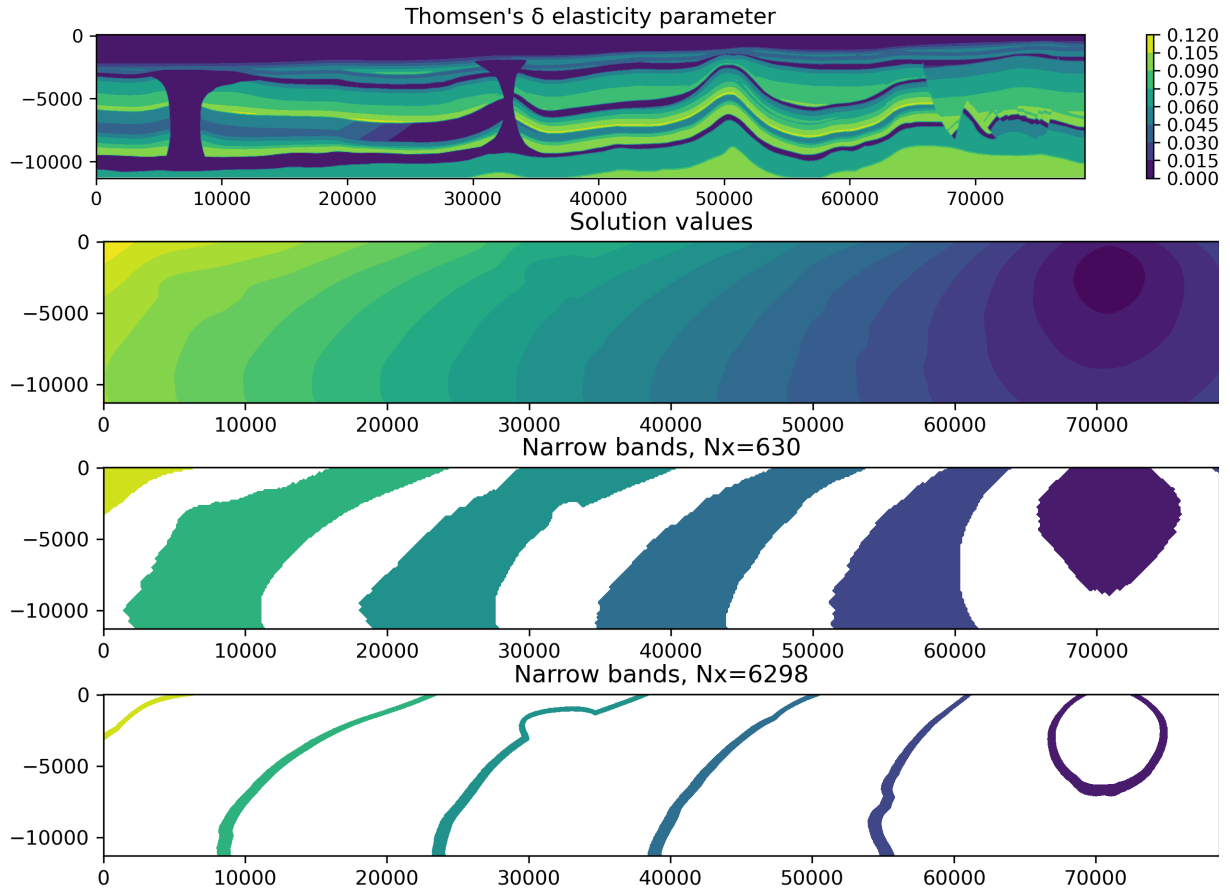


Figure 3: (i) Geometry of the BP 2077 TTI test case. (ii) Contour plot of the numerical solution of the eikonal equation. (iii, iv) Narrow band of active points, at several steps of the NB method, for the horizontal problem sizes $N_x = 630$ and $N_x = 6298$, see text for parameters.

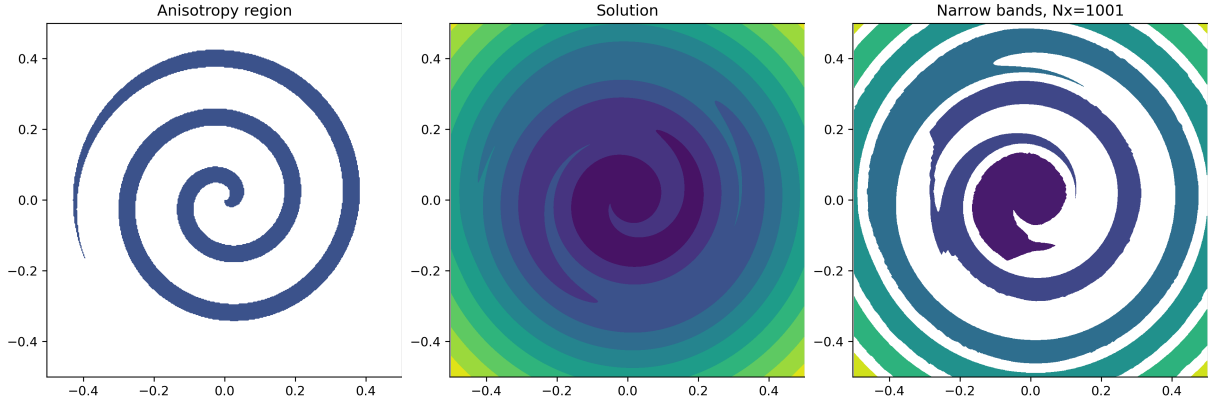


Figure 4: Tubular test case. (i) The Riemannian metric has eigenvalues 1 and $1/20^2$ in the spiral blue region, with an eigenvector tangential to the spiral curve; and equals Id elsewhere. (ii) Contour plot of the numerical solution. (iii) Narrow band of active points, at several steps of the NB method, problem size $N_x = 1001$, see text for other parameters.

tolerance $\varepsilon = 10^{-4}h$, and following our heuristic $\tau = \alpha/2$, $r\tau = \alpha \ln(\alpha/\varepsilon)$, $T = 5\tau$. Some narrow bands produced during the computation are illustrated on fig. 3.

The proposed NB method uses in average 130 updates per point, independently of the problem size. In contrast, the AGSI uses 172, 635 and 2946 updates per point for the problem sizes 629×90 , 2519×360 and 12596×1801 (model subsampled by a factor 20, 5, and original size respectively), and the FIM, FS and GI are even more costly (in that order).

4.3 Strongly anisotropy Riemannian test case, Semi-Lagrangian scheme.

The following test case, introduced in [BC11], is inspired by applications to tubular structure segmentation in medical images. We use a semi-Lagrangian discretization, based on the four-point two-dimensional diamond stencil on the Cartesian grid; this numerical scheme similar to the seminal work [Tsi95], except that the Riemannian anisotropy destroys the causality property (9), and makes the FMM inapplicable. The front propagation is fast and strongly anisotropic in the neighborhood of a given curve, and slow and isotropic elsewhere, which leads to slow convergence of the AGSI as observed in [BC11]. This test case motivated the design of adaptive numerical schemes, using wide stencils depending on the local value of the Riemannian metric and designed using a tool from algorithmic geometry known as Selling's matrix decomposition, in order to apply the FMM and solve the problem in a single pass [Mir14a, Mir19]. We show here that the original problem discretization can also be solved efficiently, provided the AGSI is replaced with the NB method.

Formally, the metric has a Riemannian structure $\mathcal{F}_x(v) := \sqrt{\langle v, \mathcal{M}(x)v \rangle}$ on the domain $\Omega :=]-1/2, 1/2[^2$. It satisfies $\mathcal{M}(x) = \text{Id}$, except when $x \in \Omega$ lies in a band of width $1/20$ along the spiral curve $\Gamma : r \in [0, R] \mapsto r(\cos \omega r, \sin \omega r)$ with $\omega := 12\pi$ and $R = 0.43$, in which case $\mathcal{M}(x)$ has eigenvalues 1 and $1/20$, with the second eigenvector tangential to Γ . We solve $\langle \nabla u, \mathcal{M}^{-1} \nabla u \rangle = 1$ on $\Omega \setminus \{x_0\}$ with the point source boundary condition $u(x_0) = 0$ at the domain center $x_0 := (0, 0)$, and outflow boundary conditions on $\partial\Omega$.

We use the timescale $\alpha = 5h$, tolerance $\varepsilon = 10^{-4}h$, where $h > 0$ denotes the grid scale, and following our heuristic $\tau = \alpha/2$, $r\tau = \alpha \ln(\alpha/\varepsilon)$, and $T = 5\tau$. The number of updates per point of the NB method slightly decreases from 97 to 78 as the problem size increases from 201^2 to 2001^2 . In contrast, the AGSI uses 196, 389, 770 and 1544 updates per point for the problem

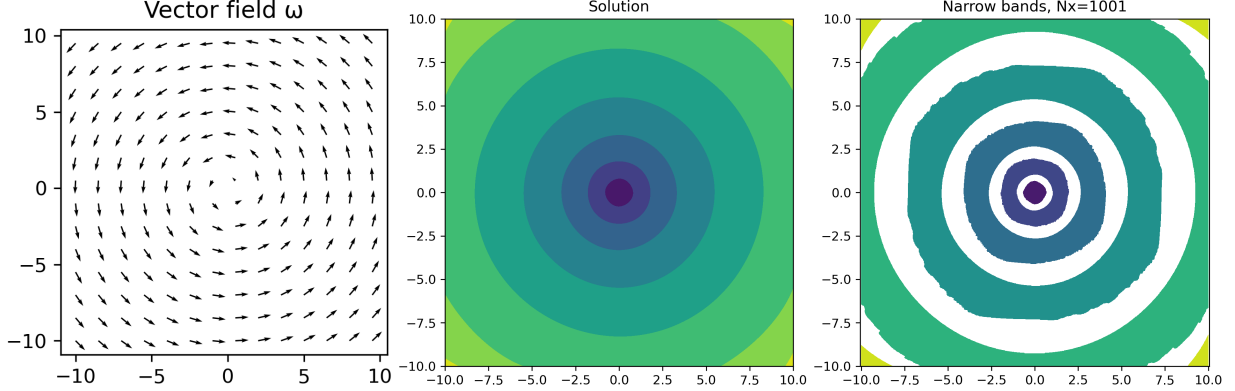


Figure 5: Swirl test case (i) Vector field ω , defining the asymmetric part of the metric. (ii) Solution, which is radial up to boundary effects. (iii) Some narrow bands, $N_x = 1001$, $\alpha = 3h$.

sizes 201^2 , 433^2 , 931^2 and 2001^2 , and the FIM, FS and GI are even more costly (in that order), see fig. 1 (iii).

4.4 Strongly asymmetric Randers test case, Eulerian scheme.

The following test case is inspired by Zermelo’s problem of a boat navigating a whirlpool of water [BRS04]. It involves a Randers metric with a strong asymmetric term defined by a rotating vector field, whose norm almost saturates Randers local controllability criterion (48). We use a four-point Eulerian scheme, similar to the seminal work [RT92, Set96], except that the Randers asymmetric bias term destroys the causality property (9), and makes the FMM inapplicable. A test case with a similar structure is considered in [Mir14b], where the AGSI is shown to be inefficient; in that work, a semi-Lagrangian scheme is used in combination with adaptive stencils depending on the local value of the metric and designed using a tool from arithmetic known as the Stern-Brocot tree, which is specific to the two-dimensional setting, so as to restore causality and apply the single pass FMM. We show below that the original simple Eulerian problem discretization can also be solved efficiently, provided the AGSI is replaced with the NB method.

Formally, the domain $\Omega :=]-10, 10[^2$ equipped with the Randers metric $\mathcal{F}_x(v) := \|x\| + \langle \omega(x), v \rangle$, where $\omega(x) := \rho \frac{\|x\|^2}{1+\|x\|^2} \frac{x^\perp}{\|x\|}$ with $\rho := 0.98 < 1$ so as to obey the Randers compatibility condition, see definition B.1. We solve $\|\nabla u - \omega\| = 1$ on $\Omega \setminus \{x_0\}$ with the point source boundary condition $u(x_0) = 0$ at the domain center $x_0 := (0, 0)$, and outflow boundary conditions on $\partial\Omega$.

We use the timescale $\alpha = 5h$, tolerance $\varepsilon = 10^{-4}h$, where $h > 0$ denotes the grid scale, and following our heuristic $\tau = \alpha/2$, $r\tau = \alpha \ln(\alpha/\varepsilon)$ and $T = 5\tau$. The number of updates per point of the NB method slightly decreases from 115 to 70 as the problem size increases from 201^2 to 2001^2 . In contrast, the AGSI uses 125, 219, 410 and 810 updates per point for the problem sizes 201^2 , 433^2 , 931^2 and 2001^2 , and the FIM, FS and GI are even more costly (in that order), see fig. 1 (iv).

Conclusion

We establish in this paper, to our knowledge, the first quasi-linear complexity result for a numerical solver of anisotropic eikonal equations which *does not* rely on the causality property

(9). The method concentrates the computation on a narrow band, whose width (a range of solution values) is essentially proportional to a characteristic time $\alpha > 0$ which quantifies the *non-causality* discretization scheme, and depends logarithmically on the numerical tolerance $\varepsilon > 0$. The method applies to semi-Lagrangian, Eulerian, and Lax-Friedrichs schemes.

The proposed algorithm does work in practice, provided the excessively pessimistic parameters derived from the theoretical analysis are replaced with suitable heuristics. It shines particularly well, in comparison with other iterative solvers of eikonal equations, in test cases which are large along at least one axis, and which either (i) are addressed using the highly non-causal Lax-Friedrichs relaxation scheme, or (ii) feature strong anisotropy or asymmetry. Avenues for future work include addressing more complex front propagation models than the eikonal PDE, establishing sharper theoretical bounds leading to method parameters usable in practice, and developing a high performance parallel numerical implementation.

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A Lax-Friedrichs scheme

We consider in this appendix a discretization of anisotropic eikonal PDEs based on a Lax-Friedrichs relaxation similar to [KOQ04], which is combined with a graph based limiter. We prove that this scheme can be solved using the proposed narrow band algorithm 1 with quasi-linear complexity, see corollary A.7. For that purpose, we need to introduce some notations.

Let $\Omega \subset \mathbb{R}^d$ be a bounded open domain with Lipschitz boundary, equipped with $\mathcal{F}^* \in C^0(\bar{\Omega} \times \mathbb{R}^d)$ a Finslerian (dual-)metric. Thus $\xi \in \mathbb{R}^d \mapsto \mathcal{F}_x^*(\xi)$ is a norm, possibly asymmetric, and satisfying

$$C_0^{-1} \|\xi\|_\infty \leq \mathcal{F}_x^*(\xi) \leq C_1 \|\xi\|_1, \quad (41)$$

for all $x \in \bar{\Omega}$, $\xi \in \mathbb{R}^d$, where $C_1, C_0 > 0$ are constants, and $\|\xi\|_1 := \sum_{i=1}^d |\xi_i|$ and $\|\xi\|_\infty := \max_{i=1}^d |\xi_i|$. Let $h > 0$ be a fixed gridscale, and let $X := \mathring{X} \sqcup \partial X$ be the discretization domain with interior $\mathring{X} := \Omega \cap h\mathbb{Z}^d$ and boundary $\partial X := \{x + \tau hb_i \mid x \in \mathring{X}, 1 \leq i \leq d, \tau = \pm 1\} \setminus X$, where $(b_i)_{i=1}^d$ denotes the canonical basis of \mathbb{R}^d . We consider the numerical scheme

$$\max \left\{ \max_{\substack{1 \leq i \leq d \\ \tau \in \{-1, 1\}}} \frac{u(x) - u(x - \tau hb_i)}{C_0 h}, \mathcal{F}_x^*(\nabla_h u(x)) - \frac{h}{2c_1} \Delta_h u(x) \right\} = 1, \quad (42)$$

for all $x \in \mathring{X}$, where $0 < c_1 < C_1^{-1}$ is a constant. This scheme is defined in terms of the standard centered finite differences discretizations of the gradient and Laplacian differential operators:

$$\nabla_h u(x) := \sum_{1 \leq i \leq d} \frac{u(x + hb_i) - u(x - hb_i)}{2h} b_i, \quad \Delta_h u(x) := \sum_{1 \leq i \leq d} \frac{u(x + hb_i) - 2u(x) + u(x - hb_i)}{h^2}.$$

Choosing a smooth u and letting $h \rightarrow 0$ we find that (42) is consistent with $\max\{\|\nabla u(x)\|_\infty / C_0, \mathcal{F}_x^*(\nabla u(x))\} = 1$, up to a first order truncation error, hence with the anisotropic eikonal PDE $\mathcal{F}_x^*(\nabla u(x)) = 1$ in view of (41). (For readability, we changed the sign convention w.r.t. (1).)

Solving for $u(x)$ in (42), we obtain the update operator $\Lambda u(x) := \min\{\Lambda_0 u(x), \Lambda_1 u(x)\}$ where

$$\Lambda_0 u(x) := hC_0 + \min_{\substack{1 \leq i \leq d \\ \tau \in \{-1, 1\}}} u(x + \tau hb_i), \quad \Lambda_1 u(x) := \frac{c_1}{d} (h - \mathcal{F}_x(h \nabla_h u(x))) + \sum_{\substack{1 \leq i \leq d \\ \tau \in \{-1, 1\}}} \frac{u(x + \tau hb_i)}{2d},$$

for any $u \in \mathbb{R}^X$ and any interior point $x \in \mathring{X}$. By convention $\Lambda u(x) = 0$ for all $x \in \partial X$. For convenience, we study the elementary operator $\lambda(u) := \min\{\lambda_0(u), \lambda_1(u)\}$, $u \in \mathbb{R}^{2d}$, where

$$\lambda_0(u) := C_0 + \min(u), \quad \lambda_1(u) := \frac{c_1}{d} (1 - F^*(Du)) + \frac{\langle u, \mathbb{1}_{2d} \rangle}{2d}, \quad Du := \left(\frac{u_{+i} - u_{-i}}{2} \right)_{1 \leq i \leq d} \quad (43)$$

and where C_0 and $c_1 > 0$ are constants, and F^* is a norm, always possibly asymmetric. Here and below, the components of $u \in \mathbb{R}^{2d}$ are denoted by $u_{\tau i}$, $\tau \in \{-1, 1\}$, $1 \leq i \leq d$. The following assumption is used throughout propositions A.2 and A.6 and lemmas A.3 to A.5.

Assumption A.1. F^* is a norm on \mathbb{R}^d , satisfying $C_0^{-1}\|\xi\|_\infty \leq F^*(\xi) \leq C_1\|\xi\|_1$ for all $\xi \in \mathbb{R}^d$. The constants $C_0, C_1, c_1 > 0$ and $\delta := (1 - C_1c_1)/(2d) > 0$ are all positive.

We recall that $\min(u), \max(u) \in \mathbb{R}$ denote the smallest and largest element of $u \in \mathbb{R}^{2d}$, whereas $\max\{u, t\} \in \mathbb{R}^{2d}$ denotes the elementwise maximum with $t \in \mathbb{R}$, see notations.

Proposition A.2. *The following properties hold for all $u \in \mathbb{R}^{2d}$:*

- (Monotony) $\forall u' \in \mathbb{R}^{2d}, u \preceq u' \Rightarrow \lambda(u) \leq \lambda(u')$, and in addition $\lambda_1(u') - \lambda_1(u) \geq \delta\|u' - u\|_1$.
- (Translation invariance) $\forall t \in \mathbb{R}, \lambda(u + t) = \lambda(u) + t$.
- (Comparison) $\lambda_0(u) - C_{\text{comp}} \leq \lambda(u) \leq \lambda_0(u)$, with $C_{\text{comp}} := C_0$.
- (C_{past} -memory) $\lambda(u) = \lambda(\max\{u, t\})$ for all $t \leq \lambda(u) - C_{\text{past}}$, with $C_{\text{past}} := C_0$.
- (σ -sub-multiplicativity) $\forall t \geq 0, \lambda((1 + t)u) \leq (1 + t)\lambda(u) - t\sigma$, with $\sigma := c_1/d$.

Proof. *Proof of monotony:* one has $F^*(Du) - F^*(Du') \leq F^*(Du - Du') \leq C_1\|Du - Du'\|_1 \leq C_1\|u - u'\|_1/2$ using successively the triangular inequality for the asymmetric norm F^* , assumption A.1, and the finite differences expression (43, right). The announced inequality for λ_1 follows remarking that $\langle u' - u, \mathbb{1}_{2d} \rangle = \|u' - u\|_1$ since $u' \succeq u$. Finally, λ is monotone since it is defined as the maximum of λ_1 which is monotone by the above argument, and of λ_0 which is clearly monotone. *Translation invariance* is clear. *Comparison* follows from the observation that $\lambda_1(u) \geq \lambda_1(\min(u)\mathbb{1}_{2d}) = c_1/d + \min(u) \geq \min(u) = \lambda_0(u) - C_{\text{comp}}$ using monotony for the first inequality. *Proof of C_{past} -dependency:* one has $t \leq \lambda(u) - C_0 \leq \lambda_0(u) - C_0 = \min(u)$, hence $\max\{u, t\} = u$. *Proof of σ -sub-multiplicativity:* λ_0 (resp. λ_1) is the sum of the 1-positively homogeneous min function (resp. the function $u \mapsto \frac{1}{2d}(\langle u, \mathbb{1}_{2d} \rangle - 2c_1F^*(Du))$) and of the constant C_0 (resp. c_1/d), hence is C_0 -sub-multiplicative (resp. c_1/d -). The result follows for $\lambda = \min\{\lambda_0, \lambda_1\}$ by lemma 3.3 on the properties of the minimum of two operators, and since $C_0 \geq C_1^{-1} \geq c_1 \geq c_1/d$ by assumption A.1 (use that $C_0^{-1} \leq F^*(b_1) \leq C_1$, and $1 - C_1c_1 > 0$). \square

Define $\phi_0 := E_\alpha \lambda_0 E_\alpha^{-1}$, where $\alpha > 0$, see (13). One easily checks that for all $v \in \mathbb{R}^d$

$$\phi_0(v) = \exp(-C_0/\alpha) \min\{0, \min(v)\}, \quad (44)$$

and therefore ϕ_0 is ρ_0 -sub-additive, with $\rho_0 := \exp(-C_0/\alpha)$. A similar result is established for the complete update operator in proposition A.6, preceded with some technical lemmas.

Lemma A.3. *Let $\lambda : \mathbb{R}^I \rightarrow \mathbb{R}$ be an arbitrary function which is differentiable at $u \in \mathbb{R}^I$. Let $\phi := E_\alpha \lambda E_\alpha^{-1}$ and $v := E_\alpha u$, where $\alpha > 0$. Then*

$$\left[\frac{d}{dt} \phi(v + t) \right]_{t=0} = \rho(\alpha; \lambda, u), \quad \text{where } \rho(\alpha; \lambda, u) := \langle \exp([u - \lambda(u)]/\alpha), \nabla \lambda(u) \rangle.$$

Proof. By the differentiation rule for compositions, using the explicit expression (13) of E_α

$$\left[\frac{d}{dt} \phi(v + t) \right]_{t=0} = E'_\alpha(\lambda(u)) \langle E_\alpha^{-1}(v), \nabla \lambda(u) \rangle = \frac{1}{\alpha} \exp\left(-\frac{\lambda(u)}{\alpha}\right) \left\langle -\frac{\alpha}{v}, \nabla \lambda(u) \right\rangle. \quad \square$$

Lemma A.4. *If $\lambda_1(u) \leq \lambda_0(u)$, for some $u \in \mathbb{R}^{2d}$, then $\max(u) \leq \min(u) + C_0/\delta$.*

Proof. One has $\lambda_1(u) \geq \lambda_1(\min(u)\mathbb{1}_{2d}) + \delta\|u - \min(u)\|_1 \geq \min(u) + \delta(\max(u) - \min(u))$ by proposition A.2 (monotony). We conclude recalling the expression (43) of λ_0 . \square

Lemma A.5. For any $u \in \mathbb{R}^{2d}$, such that F^* is differentiable at $Du \in \mathbb{R}^d$, one has

$$\nabla \lambda_1(u) = \frac{1}{2d} (1 - c_1 \tau \partial_i F^*(Du))_{1 \leq i \leq d}^{\tau \in \{-1, 1\}} \in \mathbb{R}^{2d}. \quad (45)$$

If $\lambda_1(u) \leq \lambda_0(u)$ and $\alpha \geq C_0/\delta$, then $\rho_1(\alpha; \lambda_1, u) \leq 1 - \frac{c_1}{\alpha d} + \frac{C_0^2}{\alpha^2 \delta^2}$.

Proof. The expression (45), where $\partial_i F^*$ denotes the partial derivative of $F^* : \mathbb{R}^d \rightarrow \mathbb{R}$ w.r.t. the i -th coordinate, is obtained by differentiating (43, center). Let us note the identities

$$\|\nabla \lambda_1(u)\|_1 = \langle \mathbb{1}_{2d}, \nabla \lambda_1(u) \rangle = 1, \quad \langle u, \nabla \lambda_1(u) \rangle = \lambda_1(u) - c_1/d, \quad (46)$$

for any $u \in \mathbb{R}^{2d}$, using (left) proposition A.2 (monotony) which implies that $\nabla \lambda_1(u) \in [0, \infty]^{2d}$, and (right) Euler's identity for the 1-homogeneous function $u \mapsto \lambda_1(u) - c_1/d$. We assume $\lambda_1(u) = 0$ in the following, w.l.o.g. thanks to the additive invariance of λ_1 , and obtain denoting $r(x) := \exp(x) - x - 1$ and inserting (46) in the last estimate

$$\rho(\alpha; \lambda_1, u) = \langle \exp\left(\frac{u}{\alpha}\right), \nabla \lambda_1(u) \rangle = \langle \mathbb{1}_{2d} + \frac{u}{\alpha} + r\left(\frac{u}{\alpha}\right), \nabla \lambda_1(u) \rangle \leq 1 - \frac{c_1}{\alpha d} + \left\| r\left(\frac{u}{\alpha}\right) \right\|_\infty. \quad (47)$$

One has $\min(u) + c_1/d = \lambda_1(\min(u) \mathbb{1}_{2d}) \leq \lambda_1(u) = 0 \leq \lambda_0(u) = \min(u) + C_0$ by assumption and monotony of λ_1 . Thus $-C_0 \leq \min(u) \leq 0$, therefore $\max(u) \leq C_0/\delta$ by lemma A.4, hence $\|u/\alpha\|_\infty \leq 1$ by assumption. Recalling that $0 \leq r(t) \leq t^2$ for all $t \leq 1$, see lemma 2.20, and inserting this estimate in (47), we conclude the proof. \square

Proposition A.6. One has $\phi(v+t) \leq \phi(v) + \rho(\alpha)t$ for all $v \in \mathbb{R}^{2d}$, $t \geq 0$, with $\phi := E_\alpha \lambda E_\alpha^{-1}$ and

$$\rho(\alpha) := \max \left\{ \exp\left(-\frac{C_0}{\alpha}\right), 1 - \frac{c_1}{\alpha d} + \frac{C_0^2}{\alpha^2 \delta^2} \right\} < 1, \quad \text{assuming } \alpha > \alpha_0 := \frac{dC_0^2}{c_1 \delta^2}$$

Proof. Let us first check that one has a continuous extension $\lambda : \mathbb{R}_\infty^{2d} \rightarrow \mathbb{R}_\infty$, where $\mathbb{R}_\infty :=]-\infty, \infty]$ is equipped with the topology of $]0, 1]$, in such way that ϕ is continuously defined on \mathbb{R}^d . Consider a converging sequence $u_n \rightarrow u$ as $n \rightarrow \infty$, with $u_n \in \mathbb{R}^{2d}$ and $u \in \mathbb{R}_\infty^{2d}$. If $\max(u) < \infty$, then $\lambda(u_n) \rightarrow \lambda(u)$ by continuity of λ on \mathbb{R}^d . If $\min(u) = \infty$, then $\lambda(u_n) \geq \min(u_n) \rightarrow \infty$ as $n \rightarrow \infty$, which proves continuity. Finally, if $\min(u) < \infty$ and $\max(u) = \infty$, then $\lambda(u_n) = \lambda_0(u_n) = \min(u_n) + C_0$ for sufficiently large n by lemma A.4, and we conclude noting that λ_0 is continuous on \mathbb{R}_∞^{2d} . (In contrast, λ_1 is ill-defined on \mathbb{R}_∞^{2d} due to the indeterminate form $\infty - \infty$ in (43, right).)

Now let $v_0 \in \mathbb{R}^d$, and $u(t) := E_\alpha^{-1}(v_0 + t) \in \mathbb{R}_\infty^{2d}$ for all $t \in \mathbb{R}$. Assume that $\min(v_0) < \max(v_0)$, otherwise the result is clear, and let $t_0 := (\max(v_0) - R \min(v_0))/(R-1)$, with $R := \exp(\alpha C_0/\delta)$. By construction $[\max(u(t)) < \min(u(t)) + C_0 \delta < \infty]$ iff $t < t_0$. For all $t \geq t_0$ one has $\phi(v_0 + t) = \phi_0(v_0 + t)$ by lemma A.4, which is $\exp(-\alpha C_0)$ -Lipschitz by (44) as a function of t . On the other hand for all $t < t_0$ one has $\phi(v_0 + t) = \min\{\phi_0(v_0 + t), \phi_1(v_1 + t)\}$, which is $\rho(\alpha)$ -Lipschitz w.r.t. t in view of lemma A.5, for any $\alpha \geq C_0/\delta$ (note that $\alpha_0 \geq C_0/\delta$ since $C_0 \geq c_1$, $\delta \leq 1$, $d \geq 1$). Finally, the choice of α_0 ensures that $\rho(\alpha) < 1$. \square

Corollary A.7. The operator Λ is monotone, translation invariant, satisfies $\Gamma - C'_{\text{comp}} \leq \Lambda \leq \Gamma$ where the graph based approximation is defined as $\Gamma := \Lambda_0$, has C'_{past} -memory, is σ' -sub-multiplicative, and $\Phi := E_{\alpha'} \Lambda E_{\alpha'}^{-1}$ is ρ' -sub-additive. The fixed point u_* of Λ satisfies $0 \preceq u_* \preceq N h_\mathcal{E}$ where $N := \#(X)$. The constants are defined in terms of those of propositions A.2 and A.6 as follows: $C'_{\text{comp}} = C'_{\text{past}} = h_\mathcal{E} := h C_0$, $\sigma' := h \sigma > 0$, $\alpha' := 2h \alpha_0$, and $\rho' := 1 - \frac{1}{2} \left(\frac{c_1 \delta}{C_0 d}\right)^2 < 1$.

Proof. The properties follow from propositions A.2 and A.6 and the identity $\Lambda u(x) = h \lambda(\tilde{u}/h)$, for any $x \in X$ and $u \in \mathbb{R}^X$, by choosing $\tilde{u}_{\tau i} := u(x + \tau h b_i)$ and $F^* := \mathcal{F}_x^*$, but with the constant $\rho(2\alpha_0)$ instead of ρ' . Denoting $\alpha := 2\alpha_0$, we have $\exp(-\frac{C_0}{\alpha}) \leq 1 - \frac{C_0}{2\alpha} \leq 1 - \frac{c_1}{2\alpha} = 1 - \frac{c_1}{\alpha d} + \frac{C_0^2}{\alpha^2 \delta^2} = \rho'$, using successively that $\exp(-x) \leq 1 - x/2$ for all $0 \leq x \leq 1$ for the first inequality, basic inequalities $\delta \leq 1$, $c_1 \leq C_0$, $d \geq 1$ which follow from the assumptions, and the expression of α_0 ; thus $\rho' = \rho(2\alpha_0)$ as required. The existence, uniqueness and estimate of u_* is similar to corollary 3.10. \square

We have gathered all the ingredients necessary to apply algorithm 1, and establish a counterpart of theorem 1.3 for the Lax-Friedrichs scheme. Specializing the generic complexity estimate (35) of algorithm 1 with the values of corollary A.7 we obtain

$$\mathcal{O}\left(\frac{N}{p} \ln^+\left(\frac{NhC_0}{p\varepsilon}\right) \ln^+\left(\frac{NhC_0}{\varepsilon}\right)\right) \quad \text{with } p := 1 - \rho' = \frac{1}{2}\left(\frac{c_1\delta}{C_0d}\right)^2.$$

B Eulerian scheme

We establish in this appendix the counterpart (61) of our main complexity result theorem 1.3, for an Eulerian discretization of the eikonal equation first considered in [RT92, Set96] and generalized to Riemann and Randers type anisotropy in [Mir19]. The key ingredient of the proof is that the Eulerian scheme (52) can be regarded as an instance of the semi-Lagrangian scheme (3), using a diagonal metric in a higher dimensional space, see lemma B.2. In particular, the classical Eulerian [RT92, Set96] and semi-Lagrangian [Tsi95] (using the four point stencil) discretizations of the isotropic eikonal equation are mathematically equivalent. This coincidence has likely been noticed before, but curiously no proof has been published to the author's knowledge. In order to proceed, we need to introduce some notations.

Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain with Lipschitz boundary, and let $\mathcal{D} \in C^0(\bar{\Omega}, \mathcal{S}_d^{++})$ and $\omega \in C^0(\bar{\Omega}, \mathbb{R}^d)$ be subject to the following compatibility condition: for all $x \in \bar{\Omega}$

$$\|\omega(x)\|_{\mathcal{D}(x)} \leq \omega_{\max} < 1, \text{ on } \bar{\Omega}, \quad \text{where } \|v\|_{\mathcal{D}} := \sqrt{\langle v, \mathcal{D}v \rangle}. \quad (48)$$

Our objective is to numerically approximate the unique viscosity solution $\mathcal{U} : \bar{\Omega} \rightarrow \mathbb{R}$ of

$$\|\nabla \mathcal{U} + \omega\|_{\mathcal{D}} = 1, \text{ in } \bar{\Omega}, \quad \mathcal{U} = 0 \text{ on } \partial\Omega. \quad (49)$$

The solution $\mathcal{U}(x)$ of this variant of the eikonal PDE (1) is the traveltime (or path-length) distance (2) from a point $x \in \Omega$ to the boundary $\partial\Omega$, with respect to the Randers metric $\mathcal{F}_x(\xi) := \|\xi\|_{\mathcal{D}(x)-1} + \langle \omega(x), \xi \rangle$, see [Mir19]. Randers metrics arise, for instance, in Zermelo's navigation problem describing a vehicle is subject to drift, and the condition (48) ensures the local controllability of the system, see [BRS04]. The special case where $\omega = 0$ identically on Ω corresponds to a Riemannian eikonal PDE, without drift. In order to discretize (49), we consider a finite and non-empty set of offsets $E \subset \mathbb{Z}^d \setminus \{0\}$, and coefficients $\mu \in C^0(\bar{\Omega} \times E,]0, \infty[)$ such that

$$\mathcal{D}(x) := \sum_{e \in E} \mu(x, e) e e^\top, \quad 0 < \mu_{\min} \leq \mu(x, e) \leq \mu_{\max}, \quad (50)$$

for all $x \in \bar{\Omega}$, $e \in E$. In the special case of a diagonal matrix, which is denoted $\mathcal{D}(x) = \text{Diag}(\mu_1(x), \dots, \mu_d(x))$, we recover the classical scheme [RT92, Set96] by choosing $E = \{b_1, \dots, b_d\}$

as the canonical basis of \mathbb{R}^d and letting $\mu(x, b_i) := \mu_i(x)$ for all $1 \leq i \leq d$. Non-diagonal anisotropy, introduced in [Mir19] for this class of schemes, is discussed in remark B.7.

Let $h > 0$ be a fixed grid scale, and let

$$\mathring{X} := \Omega \cap h\mathbb{Z}^d, \quad \partial X := \{x + \tau h e \mid x \in \mathring{X}, e \in E, \tau = \pm 1\} \setminus \mathring{X}, \quad (51)$$

denote the discretized domain interior and boundary. The Eulerian update value $\lambda = \Lambda u(x)$, where $u \in \mathbb{R}^X$ and $x \in \mathring{X}$, is defined as the unique solution of the finite differences equation

$$\sum_{e \in E} \mu(x, e) \max \{0, \lambda - u(x - e) + h\langle \omega(x), e \rangle, \lambda - u(x + e) - h\langle \omega(x), e \rangle\}^2 = h^2. \quad (52)$$

The exact solution λ can be numerically computed by solving at most $\#(E)$ univariate quadratic equations, see [Set96] or [MGB⁺23, Algorithm 3]. Replacing λ with $u(x)$ in (52), and using (50, left) and a Taylor expansion, we find that this scheme is first order consistent with (49), see [Mir19] for more discussion and a convergence analysis. By convention $\Lambda u(x) = 0$ for all $x \in \partial X$.

Definition B.1 (Randers norm). *For any $M \in \mathcal{S}_I^{++}$ and $\omega \in \mathbb{R}^I$ obeying the compatibility condition $\|\omega\|_{M^{-1}} < 1$, we define the Randers norm $F(\xi) := \|\xi\|_M + \langle \omega, \xi \rangle$, $\xi \in \mathbb{R}^I$. This norm is asymmetric whenever $\omega \neq 0$.*

A Randers norm is clearly convex and 1-positively homogeneous, and for all $\xi \in \mathbb{R}^I \setminus \{0\}$

$$0 < (1 - \|\omega\|_{M^{-1}})\|\xi\|_M \leq F(\xi) \leq (1 + \|\omega\|_{M^{-1}})\|\xi\|_M \leq 2\|\xi\|_M, \quad (53)$$

using the Cauchy-Schwartz inequality $\langle \omega, \xi \rangle \leq \|\omega\|_{M^{-1}}\|\xi\|_M$ and the compatibility condition. As announced, we draw in lemma B.2 a connection between the Eulerian update operator (52) and the semi-Lagrangian update λ_F defined in proposition 1.6. It is used in proposition B.4 and corollary B.6 to establish the properties of the Eulerian scheme allowing to apply algorithm 1 and prove a counterpart of theorem 1.3 in this setting.

Lemma B.2. *Let $\mu_1, \dots, \mu_I > 0$, and $\omega := (\omega_1, \dots, \omega_I) \in \mathbb{R}^I$ be such that $\sum_{i=1}^I \mu_i \omega_i^2 < 1$. Let F be the Randers norm of parameters $M = \text{Diag}(\mu_1^{-1}, \dots, \mu_I^{-1})$ and ω . Then $\lambda := \lambda_F(u)$ is the unique solution to*

$$\sum_{1 \leq i \leq I} \mu_i \max\{0, \lambda - u_i - \omega_i\}_+^2 = 1. \quad (54)$$

Proof. Denote $f(\lambda) := \sum_{i=1}^I \mu_i \max\{0, \lambda - u_i - \omega_i\}_+^2$ for all $\lambda \in \mathbb{R}$. Observe that f is continuous, vanishes on $] -\infty, \lambda_0]$ where $\lambda_0 := \min\{u_i + \omega_i \mid 1 \leq i \leq I\}$, is strictly increasing on $[\lambda_0, \infty[$, and that $f(\lambda) \rightarrow \infty$ as $\lambda \rightarrow \infty$. Thus there exists a unique solution to $f(\lambda) = 1$, as announced.

In the following, we let $\lambda := \lambda_F(u)$, and denote by $\xi \in \Xi_I$ the minimizer of the convex optimization problem (12) defining this value. By the KKT relations lemma 2.18, there exists $\nu \in [0, \infty[^I$ obeying the complementary condition $\langle \xi, \nu \rangle = 0$ and

$$\frac{M\xi}{\|\xi\|_M} + \omega + u = \lambda \mathbb{1} + \nu, \quad \text{hence } \frac{M\xi}{\|\xi\|_M} = \max\{0, \lambda \mathbb{1} - u - \omega\}. \quad (55)$$

The identity (55, right) is proved by distinguishing two cases, given $1 \leq i \leq I$: (i) if $\xi_i > 0$, then $\nu_i = 0$ by the complementarity condition, and thus $0 < \mu_i^{-1} \xi_i / \|\xi\|_M = \lambda - u_i - \omega_i + \nu_i = \lambda - u_i - \omega_i$, (ii) if $\xi_i = 0$, then recalling that $\nu_i \geq 0$ we obtain that $0 = \mu_i^{-1} \xi_i / \|\xi\|_M = \lambda - u_i - \omega_i + \nu_i \geq \lambda - u_i - \omega_i$. Finally, applying $\|\cdot\|_{M^{-1}}$ to (55, right) we obtain $1 = \|M\xi\|_{M^{-1}} / \|\xi\|_M = \|\max\{0, \lambda \mathbb{1} - u - \omega\}\|_{M^{-1}}$, which establishes (54) and concludes the proof. \square

Assumption B.3. The domain $\Omega \subset \mathbb{R}^d$ is open and bounded. The Randers eikonal equation parameters $\mathcal{D} \in C^0(\bar{\Omega}, \mathcal{S}_d^{++})$ and $\omega \in C^0(\bar{\Omega}, \mathbb{R}^d)$ obey the compatibility condition (48). The finite difference stencil $E \subset \mathbb{Z}^d \setminus \{0\}$ and coefficients $\mu \in C^0(\bar{\Omega} \times E,]0, \infty[)$ are positive and bounded (50). The discrete domain $X = \overset{\circ}{X} \sqcup \partial X$ is defined by (51) in terms of the grid scale $h > 0$. The update operator $\lambda = \Lambda u(x)$ solves (52) for all $x \in \overset{\circ}{X}$, and $\Lambda u(x) = 0$ for all $x \in \partial X$.

Proposition B.4. Under assumption B.3. For any interior vertex $x \in \overset{\circ}{X}$ one has

$$\Lambda u(x) = \min_{\tau \in \{-1, 1\}^E} \lambda_{hF_x^\tau}(u_x^\tau), \quad (56)$$

where the Randers norm F_x^τ on \mathbb{R}^E , and the collection of values $u_x^\tau \in \mathbb{R}^E$, are defined by

$$F_x^\tau(\xi) := \sqrt{\sum_{e \in E} \frac{\xi(e)^2}{\mu(x, e)}} + \sum_{e \in E} \tau(e) \langle \omega(x), e \rangle \xi(e), \quad u_x^\tau := (u(x + h\tau(e)e))_{e \in E}, \quad (57)$$

for all $\tau \in \{-1, 1\}^E$ and $\xi \in \mathbb{R}^E$. As a result, Λ is monotone and sub-additive. The operator $\Phi := E_\alpha \Lambda E_\alpha^{-1}$ is monotone, super-multiplicative, and ρ -sub-additive, where $\sigma > 0$, $\alpha > 0$ and $\rho < 1$ are defined as

$$\sigma := h \frac{1 - \omega_{\max}}{\sqrt{\mu_{\max} \#(E)}}, \quad \alpha := \frac{8h\sqrt{\mu_{\max}}}{(1 - \omega_{\max})\mu_{\min}}, \quad \rho := 1 - \frac{\sigma}{2\alpha}. \quad (58)$$

Proof. The function F_x^τ has the structure of a Randers norm, with parameters $M_x = \text{Diag}(\mu(x, e)^{-1}, e \in E)$ and $\omega_x^\tau = (\tau(e) \langle \omega(x), e \rangle)_{e \in E}$. The Randers compatibility condition is satisfied since

$$\langle \omega_x^\tau, M_x^{-1} \omega_x^\tau \rangle = \sum_{e \in E} \mu(x, e) \langle \omega(x), e \rangle^2 = \langle \omega(x), D(x)\omega(x) \rangle \leq \omega_{\max} < 1$$

using the successively (i) the diagonal structure of M_x , (ii) the consistency property (50, left), and (iii) the corresponding assumption (49, right) on $\bar{\Omega}$. We therefore obtain using (53)

$$(1 - \omega_{\max}) \|\xi\|_{M_x} \leq F_x^\tau(\xi) \leq 2 \|\xi\|_{M_x}, \quad \text{hence } (F_x^\tau)_{\min} \geq \frac{1 - \omega_{\max}}{\sqrt{\mu_{\max}}}, \quad (F_x^\tau)_{\max} \leq \frac{2}{\sqrt{\mu_{\min}}}, \quad (59)$$

and $(F_x^\tau)_{\Xi_{\min}} \geq (F_x^\tau)_{\min} / \sqrt{\#(E)}$ as observed in (24). Noting the analogy between (56) and (25), we obtain (58) as in proposition 2.24, and inserting (59) we conclude the proof. \square

Remark B.5. Consider the Riemannian case, where the (co-)vector field ω inducing the Randers asymmetry vanishes identically. Then the operator $\Phi := E_\alpha \Lambda E_\alpha^{-1}$ is ρ -sub-additive for any $\alpha > 0$, where $\rho := 1 - (1 - \omega_{\max}) \sqrt{\mu_{\min} / (2\mu_{\max} \#(E))} \exp(-2/(\alpha \sqrt{\mu_{\min}}))$. Indeed, the norm F_x^τ satisfies the classical acuteness condition definition 1.4 thanks to its diagonal structure, hence remark 2.25 applies, and this estimate follows from (27) and (59). Note that in this special case, the scheme (52) may also be solved using the FMM [Mir19].

We define a graph-based operator $\Gamma : \mathbb{R}^X \rightarrow \mathbb{R}^X$ as follows: for all $u \in \mathbb{R}^X$ and $x \in \overset{\circ}{X}$

$$\Gamma u(x) := \min_{e \in E} \min_{\tau \in \{-1, 1\}} h(\mu(x, e)^{-\frac{1}{2}} + \tau \langle \omega(x), e \rangle) + u(x + \tau e), \quad (60)$$

with $\Gamma u(x) = 0$ for all $x \in \partial X$ by convention. Similarly to (34), this operator characterizes the distance $d_\mathcal{E}$ on the positively weighted oriented graph with an edge $(x, x + \tau e)$ for any $x \in \overset{\circ}{X}$, $\tau \in \{-1, 1\}$ and $e \in E$, traversed in time $hF_x^\tau(b_e) = h(\mu(x, e)^{-\frac{1}{2}} + \tau \langle \omega(x), e \rangle)$, where $(b_e)_{e \in E}$ denotes the canonical basis of \mathbb{R}^E .

Corollary B.6. *Under assumption B.3, and assuming that $\Phi := E_\alpha \Lambda E_\alpha^{-1}$ is ρ -sub-additive for some $\alpha > 0$ and $\rho < 1$, as in proposition B.4. The operator Λ is additively invariant, σ -sub-multiplicative, has C_{past} -memory, and satisfies $\Gamma - C_{\text{comp}} \leq \Lambda \leq \Gamma$. The fixed point u_* of Λ satisfies $0 \preceq u_* \preceq Nh\varepsilon$. We denote $N := \#(X)$, the constant $\sigma > 0$ is given in (58), and $C_{\text{comp}} = C_{\text{past}} = h\varepsilon = h \max_{x \in \mathring{X}}^{e \in E} (\mu(x, e)^{-\frac{1}{2}} + |\langle \omega(x), e \rangle|)$ (thus $h\varepsilon \leq 2h/\sqrt{\mu_{\min}}$, see (59)).*

Proof. For any $x \in \mathring{X}$ and $e \in E$, there exists an integer $n \geq 0$ such that $x, x+e, \dots, x+ne \in \mathring{X}$ and $x + (n+1)e \in \partial X$, by construction (51) and since Ω is bounded. This establishes a counterpart of the connectedness assumption 1.2, and from this point the proof is identical to corollary 3.10. \square

We have gathered all the ingredients necessary to apply algorithm 1, and establish a counterpart of theorem 1.3 for the Eulerian scheme. Specializing the generic complexity estimate (35) with the values and bounds on $\alpha, \rho, \sigma, C_{\text{past}}, C_{\text{comp}}$ and $\max(u_*)$ established in proposition B.4 and corollary B.6 we obtain

$$\mathcal{O}\left(\frac{N}{p} \ln^+\left(\frac{Nh}{p\varepsilon\sqrt{\mu_{\min}}}\right)\left(\sqrt{\#(E)} + \ln^+\left(\frac{Nh}{\varepsilon\sqrt{\mu_{\min}}}\right)\right)\right), \quad \text{with } p := 1 - \rho = \frac{(1 - \omega_{\max})^2 \mu_{\min}}{16\mu_{\max}\sqrt{\#(E)}}. \quad (61)$$

Again, this estimate is quasi-linear w.r.t. N , and logarithmic w.r.t. the tolerance ε , with constants depending on the scheme structure. Finally, we discuss the matrix decomposition (50), and a shortcoming of our analysis which is the dependency of (61) on the minimal coefficient value μ_{\min} (rather than e.g. $\mathcal{D}_{\min} > 0$ such that $\mathcal{D} \geq \mathcal{D}_{\min} \text{Id}$).

Remark B.7 (Construction of the matrix decomposition coefficients). *The Selling decomposition of a symmetric positive definite matrix $D \in \mathcal{S}_d^{++}$, $d \in \{2, 3\}$, is efficiently computable and yields coefficients $\mu(D, e) \geq 0$, for all $e \in \mathbb{Z}^d \setminus \{0\}$, which are continuous (piecewise affine) w.r.t. D and satisfy*

$$D = \sum_{e \in \mathbb{Z}^d \setminus \{0\}} \mu(D, e) ee^\top, \quad \text{with } \mu(D, e) = 0 \text{ if } \|e\| > C(d)\sqrt{\|D\|\|D^{-1}\|}, \quad (62)$$

where the constant $C(d)$ only depends on the dimension [BBM23]. It is suggested in [Mir19] to choose the scheme coefficients $\hat{\mu}(x, e) = \mu(\mathcal{D}(x), e)$. One strength of this approach is that Selling's decomposition is adaptive, and avoids using large offsets unless anisotropy is strong, see (62, right). On the negative side, these non-negative coefficients typically vanish on part of $\bar{\Omega}$ (unless \mathcal{D} is diagonal), thus fail (50) and make (61) inapplicable.

Assume that $0 < c_{\min} \text{Id} \preceq \mathcal{D}(x) \preceq c_{\max} \text{Id}$ for all $x \in \bar{\Omega}$, define $E := \{e \in \mathbb{Z}^d \mid C(d)\sqrt{2c_{\max}/c_{\min}}\}$, and define $c_0 > 0$ by $\text{Id}/c_0 := \sum_{e \in E} ee^\top$ (this matrix is proportional to the identity by symmetry of the set E). Then set for all $e \in E$ and $x \in \bar{\Omega}$

$$\mu(x, e) := \mu(\mathcal{D}(x) - c_{\min} \text{Id} / 2, e) + c_{\min} c_0 / 2.$$

These coefficients are positively bounded below as required by (50), hence the quasi-linear complexity estimate (61) holds. However, we see the dependency of (61) on μ_{\min} as an artifact of our proof technique, rather than a defect of Selling's decomposition, and would recommend using the unmodified coefficients $\hat{\mu}$ in applications, see [BBM23, Appendix B] for a related discussion.