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Axiomatic Foundations of the Core for Games in Effectiveness Form*

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Abstract

We provide an axiomatic characterization of the core of games in effectiveness form. We point out that the core, whenever it applies to appropriate classes of these games, coincides with a wide variety of prominent stability concepts in social choice and game theory, such as the Condorcet winner, the Nash equilibrium, pairwise stability, and stable matchings, among others. Our characterization of the core invokes the axioms of weak nonemptiness, coalitional unanimity, and Maskin monotonicity together with a principle of independence of irrelevant states, and uses in its proof a holdover property echoing the conventional ancestor property. Taking special cases of this general characterization of the core, we derive new characterizations of the previously mentioned stability concepts.

Keywords: Effectiveness function, core, axiomatization, holdover property, consistency principle

JEL Classification: C70, C71

1 Introduction

Many theorists in economics and political science have been occupied in studying a wide variety of stability concepts in social choice and game theory for a century or more. Generally speaking, these stability concepts are mainly founded on the idea that given some prevailing state, individuals possess some blocking power to oppose that state and exercise it when they have an interest to do so. A stable state is understood to be a state for which no individual or group of individuals has the power to change the status quo by choosing a more desirable situation. This arises, for example, in a general equilibrium of markets where economic agents on both the demand and supply sides do not have any incentive to alter their consumption or production decisions at the given market price. In the same vein, elections in political systems rely on voting rules (quorum, majority, etc.) that allow some coalitions of voters to impose their chosen candidate on the entire society. In like manner, equilibrium concepts for non-cooperative games (Nash equilibrium, subgame perfect equilibrium, etc.) recommend a state robust to deviations in strategy in both static and dynamic settings. Likewise, many solution concepts for coalitional games (core, stable set, etc.) stress cooperative agreements on utility allocation that no coalition would contest.

In this article, we consider the general framework of games in effectiveness form (henceforth e-form games), first introduced by Rosenthal (1972), which encompasses a vast range of contexts, including voting problems, normal form games, network problems, and matching models, among others. The canonical e-form game has the following features. A set N of players is equipped with preferences over a set A of states. Players are mutually aware of each other's preferences, can form coalitions, and sign binding agreements to oppose a given state. In addition, the blocking power distribution among coalitions is described by an

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“effectiveness function”; given a prevailing state a in A , coalition S is effective for B at a if it can force all players to move from state a to some state in B . Such a function specifies for every coalition S of players and subset B of states of A whether or not S is effective for B . Without going into details, this way of defining the effectivity of coalitions is similar to the “inducement correspondence” introduced by Greenberg (1990) and is more general than the notions of “effectivity function”, “effectiveness relation” and “effectivity correspondence” respectively proposed by Moulin and Peleg (1982), Chwe (1994) and Demuyne et al. (2019). The effectiveness function also corresponds to a special case of the “local effectivity function” suggested by Abdou and Keiding (2003). The literature on effectivity functions generally separates the social states to which the coalitions have a right from the means authorized to achieve them. The first is similar to a notion of rights system (or constitution), that describes to which social states or sets of social states groups of individuals are entitled, while the second describes the rules that delimit the actions permissible to individuals (see for instance Peleg and Peters (2010)). In the standard approach, the effectiveness function describes the rights system allocated to coalitions, while the actions available to individuals are modeled through a game form. The possible outcomes of the game form are then required to be consistent with the effectivity function. More precisely, the game form must provide each group in society with possibilities that are consistent with those provided by the rights system. The standard approach further assumes that the agents are all consequentialists: Once the game form has been defined, the preferences relate only to the outcome of individual acts, not to the nature of the act. In this article, we maintain that e-form games describe, for each coalition, the states to which they can claim, but we allow these states to be identified with authorized behaviors or actions. One of the advantages, especially when we model games in strategic form (see section 3.2), is that agents are allowed to be non consequentialist, as no assumptions are made on the object of individual preferences (acts or consequences of acts).

Since players can behave cooperatively to oppose a given state, the solution concept we consider here is a version of the core of e-games (see Rosenthal, 1972). A state a is core-stable if there are no coalition S of players and a subset B of states for which S is effective for B at a and in which every player in S strictly prefers every state in B to a . The most remarkable feature of the core is the fact that a wide variety of prominent stability concepts in social choice and game theory, such as the Condorcet winner, the Nash equilibrium, pairwise stability, and stable matchings, among others, coincide with the core applied to some classes of e-form games by means of an appropriate effectiveness function (Propositions 3.1, 3.2, 3.3, and 6.8). More precisely, by fixing what constitutes the blocking power of coalitions, we can express these stability concepts in terms of the core for a suitable class of e-form games. The core of e-form games is the counterpart of the core for social environments analyzed by Demuyne et al. (2019). It is also equivalent to the Γ -equilibrium for rights structures introduced by Koray and Yildiz (2018). Recently, Korpela et al. (2020) provide a full characterization of the implementation in Γ -equilibrium via rights structures .

Despite the diversity of existing stability concepts such as those just mentioned, very little is known about the properties that unify them. To address this issue, we propose to axiomatically characterize the core on a vast range of classes of e-form games. Broadly speaking, the axiomatic method is adopted in the design of solutions for decision problems. It begins with the formulation of the properties of solutions, called axioms, and addresses the question of existence of solutions satisfying some combinations of the axioms. A characterization theorem identifies a unique solution, or a class of solutions, satisfying the axioms which are viewed as building blocks in their construction. The study of the axiomatic foundations of the core across different classes of e-form games appears to be of primary importance since it permits to establish that a wide variety of solution concepts that admit a core representation on a specific class of e-form games are constructed from the same set of axioms. Formally, the core is a correspondence that associates a (possibly empty) subset of core-stable states with each e-form game. Perhaps unexpectedly, the core is characterized on a wide range of classes of e-form games by a set of four axioms which are reasonably weak and intuitive (Corollary 4.6). “Weak nonemptiness” requires that when the core is nonempty, a solution to contain at least one state. “Coalitional unanimity” establishes that if a state a is selected for an e-form game, then a must belong to any top set of states B for players in some coalition S effective for B at a . If a state a is selected for an e-form game, then “Maskin monotonicity” asserts that it is also selected in an e-form game where a has (weakly) improved in the preference rankings of all players. “Independence of irrelevant states” specifies that if a state is selected for an e-form game, then it is still selected when non-selected states are removed from the game. This latter axiom is in line with other principles of independence widely used in characterizations of game-theoretic solutions (see Nash, 1950a; Arrow, 1950; Chernoff, 1954; Sen, 1969, 1993). For some classes of e-form games such as, for example, those derived from network problems,

states cannot be removed without withdrawing players associated with them. Our principle of independence permits withdrawing such players when necessary.

We first show that if a solution is coalitionally unanimous and Maskin monotonic, then it is a subsolution of the core (Proposition 4.2). Then, we prove that if a subsolution of the core is nonempty and satisfies independence of irrelevant states, then it is the core (Theorem 4.5), provided the class of e-form games satisfies a new property, called the holdover property, which plays a key role in the proof of this statement. This property echoes the conventional “ancestor property” and specifies that, given a state a in the core of an e-form game, it is always possible to introduce additional states (and their associated players when necessary) in such a way that the core of the new augmented e-form game only contains state a . This methodology constitutes an alternative to the use of the so-called bracing lemma, which is a typical consistency result for many game-theoretic models (see Thomson, 2011). The complementarity of these two approaches is highlighted in Subsection 6.2.

Using the building blocks leading up to our axiomatic characterization of the core (Corollary 4.6), we provide new axiomatic characterizations of the Condorcet winner correspondence (Corollary 5.2), the Nash equilibrium correspondence (Proposition 5.5), and the pairwise stability correspondence (Proposition 5.7). This mainly consists in reformulating our general axioms for specific classes of e-form games underlying these stability concepts and showing that these classes satisfy the holdover property. The Condorcet winner has recently been characterized by Horan et al. (2019) with axioms different from ours. As far as we know, the pairwise stability correspondence has never been characterized axiomatically before. Our characterization of the Nash equilibrium correspondence is compared with the existing ones proposed by Peleg and Tijs (1996) and Ray (2000), allowing two other axiomatic characterizations to be established (Proposition 6.4). By invoking a consistency principle instead of an independence principle and applying a bracing lemma in the framework of e-form games, we provide a second axiomatic characterization of the core (Theorem 6.7) and apply it in the context of the stable matchings (Proposition 6.10).

The rest of the article is organized as follows. Section 2 introduces the framework of e-form games and the related concept of the core. Specific classes of e-form games for which the core coincides with existing stability concepts are constructed in Section 3. Section 4 presents the main axiomatic characterization of the core. Section 5 contains the specific characterizations of the Condorcet winner correspondence, the Nash equilibrium correspondence and the pairwise stability correspondence. The consistency principle for e-form games is discussed in Section 6 and used to characterize the Nash equilibrium correspondence and the stable matchings correspondence. All proofs appear in the Appendix.

2 Preliminaries

2.1 Games in effectiveness form

For a set X , $\mathcal{P}(X)$ denotes the set of all subsets of X , and $\mathcal{P}_0(X)$ denotes the set of all nonempty subsets of X . Let \mathcal{N} (the universe of players) and \mathcal{A} (the universe of states) be two sets. A *coalition* is a finite nonempty subset of \mathcal{N} . For each coalition N and each set of states $A \subseteq \mathcal{A}$, an *effectiveness function* on (N, A) is a family $E := (E_a, a \in A)$, where for all $a \in A$, $E_a : \mathcal{P}(N) \rightarrow \mathcal{P}(\mathcal{P}_0(A))$ is such that $E_a(\emptyset) = \emptyset$.¹ The statement $B \in E_a(S)$ means that when the current state is $a \in A$, coalition S can force the outcome of the e-form game to be in B . Each player i has a complete and transitive *preference relation* \succeq_i over the set A of states. We sometimes assume that the preference relation is also antisymmetric, depending on the context in which we work in, such as voting problems. We respectively denote \succ_i and \sim_i as the *asymmetric and symmetric parts of \succeq_i* . For $a, b \in A$, $a \succ_i b$ means that player i strictly prefers a to b , and $a \sim_i b$ means that player i is indifferent between a and b . Let $\succeq = (\succeq_i)_{i \in N}$ denote the *preference profile* of all players over A .

Definition 2.1. A *game in effectiveness form* (*e-form game*) is a tuple $\Gamma = (N, A, E, \succeq)$ consisting of a coalition N of players, a set A of states, an effectiveness function $E := (E_a, a \in A)$, and a preference profile \succeq over A .

¹Rosenthal (1972) proposed a closely related definition of an effectiveness function which, for every coalition $S \in \mathcal{P}(N)$, maps, independently of N , every state $a \in A$ into a collection of subsets of A . Our definition is inspired by that of the effectivity function proposed by Moulin and Peleg (1982), which assigns, independently of the current state, a collection of subsets of A to every coalition $S \in \mathcal{P}(N)$. Thus, if for all $a, a' \in A$ and every $S \in \mathcal{P}(N)$ it holds that $E_a(S) = E_{a'}(S)$, our definition coincides with that proposed by Moulin and Peleg (1982).

2.2 The core

The core of an e-form game is defined to be the set of outcomes against which there exists no objections. The power, for a coalition, to propose an objection will depend on the outcome it wishes to object to. This distinguishes our notion of the core from the notions of the of a cooperative game or an effectivity function, where this power is invariant. In order for a coalition to make an objection against a outcome, it is also necessary that the members of the coalition unanimously agree that the outcomes to which the objection may lead will always be preferable to the outcome that the coalition is objecting. This last point is in line with other concepts of the core found in the literature. Observe, in particular, that our notion of objection is myopic in the sense that coalitions do not anticipate how their decision to deviate from the current state to the other ones will lead to other changes by other coalitions. This distinguishes our approach from other approaches in the literature that focus on farsightedness.

We now formally introduce the appropriate notion of objection which we use to define the core. Given an e-form game $\Gamma = (N, A, E, \succeq)$, an *objection* against a state $a \in A$ is a pair $(S, B) \in \mathcal{P}_0(N) \times \mathcal{P}_0(A)$ such that:

- (i) $B \in E_a(S)$;
- (ii) for all $b \in B$ and all $i \in S$, $b \succ_i a$ holds.

Definition 2.2. The *core* of $\Gamma = (N, A, E, \succeq)$, denoted by $C(\Gamma)$, is the correspondence that assigns the associated set of states against which there exists no objection:

$$C(\Gamma) = \{a \in A : \text{there is no objection } (S, B) \text{ against } a\}.$$

As an example, let us consider a society N of citizens who have preferences over a set A of three government types, namely dictatorship and democracy with either unanimity or majority principles denoted by D , U and M , respectively. Suppose further that, while dictatorship cannot be overthrown, society can fluctuate across the two democratic systems, provided that a unanimity or a majority of citizens vote in favor of changing the status quo. Formally, the effectiveness function describing this situation is given for all $S \in \mathcal{P}_0(N)$ by:

$$E_D(S) = \emptyset,$$

$$E_U(S) = \begin{cases} \{\{a\}, a \in A\} & \text{if } S = N; \\ \emptyset & \text{otherwise,} \end{cases}$$

and

$$E_M(S) = \begin{cases} \{\{a\}, a \in A\} & \text{if } |S| \geq |N|/2; \\ \emptyset & \text{otherwise.} \end{cases}$$

For any preference profile \succeq over A , we observe that $D \in C(\Gamma)$ where $\Gamma = (N, A, E, \succeq)$. Furthermore, we have $U \notin C(\Gamma)$ if and only if U is Pareto dominated either by D or by M . Finally, $M \in C(\Gamma)$ if and only if M is preferred over each other government type by a majority of citizens.

3 Core representations of typical stability concepts

The interpretation of the core may differ from one class of e-form games to another. In this section, we express some prominent stability concepts in terms of the core in a suitable class of e-form games. The proofs of Propositions 3.1, 3.2, and 3.3 are fairly straightforward and thus will be omitted.

3.1 Voting theory, effectivity functions, and the Condorcet winner

The first class of e-form games we consider concerns the selection of a committee in the theory of voting. A *voting problem* is a tuple (N, A, \succeq) , where $N \subseteq \mathcal{N}$ is, in this context, a finite *set of voters* while $A \subseteq \mathcal{A}$ is a finite *set of candidates*. The profile $\succeq = (\succeq_i)_{i \in N}$ is a *profile of linear orders*, i.e. for each voter i , \succeq_i is a complete, transitive, and antisymmetric preference relation over A . For two different candidates a and b , let $P(a, b)$ denote the number of voters who strictly prefer a to b , so that $P(a, b) + P(b, a) = |N|$. Candidate a is said to *beat* candidate b if $P(a, b) > P(b, a)$, or equivalently $P(a, b) > |N|/2$. For a voting problem (N, A, \succeq) , the *Condorcet winner* is the unique candidate who beats any other candidate in a head-to-head competition whenever it exists. The Condorcet winner is a concept that is frequently viewed as a natural desideratum for voting problems.

In the theory of voting, effectivity functions (Moulin and Peleg, 1982) describe the allocation of decision power among various coalitions of voters. In this framework, coalitional power is independent of the chosen candidate, i.e. for all $a, b \in A$ and all $S \in \mathcal{P}(N)$, $E_a(S) = E_b(S)$.

Specifically, the Condorcet winner allows any coalition with a majority to veto any candidate whenever its members agree to do so. As a consequence, independently of the chosen candidate, the associated effectiveness function $E^{CW} := (E_a^{CW}, a \in A)$ gives veto power to a coalition of size at least $|N|/2$:

$$\forall a \in A, \forall S \in \mathcal{P}_0(N), E_a^{CW}(S) = \begin{cases} \{\{b\}, b \in A\} & \text{if } |S| \geq |N|/2; \\ \emptyset & \text{otherwise.} \end{cases}$$

Let $\mathcal{K}^{CW}(\mathcal{N}, \mathcal{A})$ be the class of e-form games $\Gamma = (N, A, E^{CW}, \succeq)$, where N is a finite subset of \mathcal{N} and A is a finite subset of \mathcal{A} .

Proposition 3.1. *The Condorcet winner² of (N, A, \succeq) is $C(N, A, E^{CW}, \succeq)$.*

The *Condorcet winner correspondence* associates $C(\Gamma) \in \mathcal{P}_0(A)$ with each e-form game $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}^{CW}(\mathcal{N}, \mathcal{A})$.

3.2 Normal form games and Nash equilibrium

As a second class of e-form games, we consider normal form games. In this context, \mathcal{N} and \mathcal{A} represent the universe of players and a universe of strategy profiles, respectively. A *normal form game* is a tuple $(N, (\Sigma_i)_{i \in N}, \succeq)$ where N is a finite *set of players*, $(\Sigma_i)_{i \in N}$ are the *set of strategy profiles* and $\succeq = (\succeq_i)_{i \in N}$ are complete and transitive *preference relations* over *strategy profiles* $\Sigma_N = \times_{i \in N} \Sigma_i$. For each $S \in \mathcal{P}_0(N)$, we define the cartesian product $\Sigma_S = \times_{i \in S} \Sigma_i$ and for $\sigma \in \Sigma_N$, we denote the restriction of σ to Σ_S by σ_S . We recall that $\sigma^* \in \Sigma_N$ is a *Nash equilibrium* (Nash, 1950b) of $(N, (\Sigma_i)_{i \in N}, \succeq)$ if:

$$\forall i \in N, \forall \sigma_i \in \Sigma_i, \sigma^* \succeq_i (\sigma_i, \sigma_{N \setminus \{i\}}^*).$$

In the framework of e-form games, the set Σ_N of strategy profiles can be identified with the set A of states. To define the associated effectiveness function $E^{NE} := (E_\sigma^{NE}, \sigma \in \Sigma_N)$, only individual players can oppose a state by a deviation in strategy:

$$\forall \sigma \in \Sigma_N, \forall S \in \mathcal{P}_0(N), E_\sigma^{NE}(S) = \begin{cases} \{(\sigma'_i, \sigma_{N \setminus \{i\}}) : \sigma'_i \in \Sigma_i\} & \text{if } S = \{i\}; \\ \emptyset & \text{otherwise.} \end{cases}$$

Throughout sections 3.2, 5.2 and 6.1, we suppose that \mathcal{A} is such that for all $A \subseteq \mathcal{A}$, there exist a finite set $N \subseteq \mathcal{N}$ and a set of strategy profiles $(\Sigma_i)_{i \in N}$ such that $A = \times_{i \in N} \Sigma_i$.

We denote $\mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ the class of e-form games $\Gamma = (N, \Sigma_N, E^{NE}, \succeq)$ defined as above.

Proposition 3.2. *The set of Nash equilibria of $(N, (\Sigma_i)_{i \in N}, \succeq)$ coincides with $C(N, \Sigma_N, E^{NE}, \succeq)$.*

²We slightly abuse definition and write Condorcet winner of (N, A, \succeq) for the mapping which associates to each voting problem (N, A, \succeq) the singleton consisting of the Condorcet winner if it exists, and the empty set if there is no Condorcet winner.

The *Nash equilibrium correspondence* associates $C(\Gamma) \in \mathcal{P}_0(\Sigma_N)$ with each e-form game $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$.³

The above result, as well as other similar results on the representation of concepts that arise in the following subsections, should not be confused with the classical question of implementation that appears in the literature on effectivity functions. Implementation theory studies conditions under which an effectivity function naturally identifies a game whose equilibrium outcomes (for a given notion of equilibrium) correspond, for each preference profiles, to the outcomes of the core. In this article, we encompass a game form in an e-form game in such a way that the core outcomes of the associated e-form game coincide to the equilibrium outcomes. This type of results make it clear that several stability concepts can be studied simultaneously thanks to the notion of core of an e-form game.

3.3 Network formation and pairwise stability

The third class of e-form games concerns network formation (Jackson and Wolinsky, 1996). As before, we denote by \mathcal{N} the universe of players. A network problem is a tuple (N, A, \succeq) , where $N \subseteq \mathcal{N}$ is a coalition, (N, g^N) denotes the complete simple graph with vertex set N , $A = \{g : g \subseteq g^N\}$, and $\succeq = (\succeq_i)_{i \in N}$ is a preference profile on A . Precisely, given a coalition N of players, a network on N is defined as a set of edges g that describes pairwise relations among these players. Two players i and j are directly connected in g if and only if $\{i, j\} \in g$. Furthermore, for each player $i \in N$, \succeq_i is a preference relation over the set A describing the affinity of player i with its potential partners. Let (N, A, \succeq) be a network problem. Say that $g \in A$ is pairwise stable if (i) for all $\{i, j\} \in g$, $g \succ_i g \setminus \{\{i, j\}\}$ and (ii) for all $\{i, j\} \in g^N \setminus g$, $g \succeq_k g \cup \{\{i, j\}\}$ for some $k \in \{i, j\}$.⁴ Moreover, define, for all $S \subseteq N$,

$$E_g^{PS}(S) = \begin{cases} \{g \setminus \{\{i, j\}\} : \{i, j\} \in g\}, & \text{if } S = \{i\} \text{ for some } i \in N, \\ \{g \cup \{i, j\}\}, & \text{if } S = \{i, j\} \text{ for some } \{i, j\} \in g^N \setminus g, \\ \emptyset, & \text{otherwise.} \end{cases}$$

We denote $\mathcal{K}^{PS}(\mathcal{N})$ the class of e-form games $\Gamma = (N, A, E^{PS}, \succeq)$.

Proposition 3.3. The set of pairwise stable $a \in A$ of a network problem (N, A, \succeq) coincides with $C(N, A, E^{PS}, \succeq)$.

The *pairwise stability correspondence* associates $C(\Gamma) \in \mathcal{P}_0(A)$ with each e-form game $\Gamma = (N, A, E^{PS}, \succeq) \in \mathcal{K}^{PS}(\mathcal{N})$. Other models of network formation allowing the creation of more than one link at a time by coalitions of arbitrary size (see Dutta and Mutuswami, 1997; Jackson and van den Nouweland, 2005) can be represented by means of appropriate effectiveness functions.

4 Axiomatic characterizations of the core of e-form games

Let \mathcal{K} be a class of e-form games. Throughout this article, we assume that if $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ for some preference profile \succeq over A then $\Gamma' = (N, A, E, \succeq') \in \mathcal{K}$ for all preference profiles \succeq' over A . The (nonempty) *subset of e-form games* over the class \mathcal{K} with a nonempty core is denoted by \mathcal{K}_C .

Definition 4.1. A *solution* on \mathcal{K} is a correspondence φ that associates a (possibly empty) subset $\varphi(\Gamma)$ of A with each e-form game $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$. A *subsolution* of φ on \mathcal{K} is a correspondence ψ associating a subset $\psi(\Gamma)$ of states in $\varphi(\Gamma)$ with each e-form game $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$. A *proper* subsolution ψ of φ on \mathcal{K} is a subsolution of φ on \mathcal{K} such that $\psi \neq \varphi$.

Let \mathcal{K} be a class of e-form games and φ be a solution on \mathcal{K} . The first axiom is the axiom of weak nonemptiness, largely used in the literature by, for example, Peleg (1985), Peleg and Sudhölter (1997),

³The core of e-form games can also coincide with the strong Nash equilibrium concept (Aumann, 1959) by modifying the effectiveness function to allow any coalition to move from some states to other ones.

⁴Jackson and Wolinsky (1996) proposed a stronger version of pairwise stability requiring that adding a link makes one deviating player strictly better off and the other one weakly better off. The notion introduced here, largely used in the literature, was also discussed by Jackson and Wolinsky (1996) and has a simple core representation for e-form games.

Voorneveld and van den Nouweland (1998) and more recently Horan et al. (2019). It requires that the solution contains at least one state whenever the core is nonempty. When $\mathcal{K} = \mathcal{K}_C$, the axiom of weak nonemptiness can be replaced by the stronger axiom of nonemptiness (see Section 6.2.2)

Weak nonemptiness. A solution φ on \mathcal{K} satisfies *weak nonemptiness* if for all $\Gamma \in \mathcal{K}_C$, it holds that $\varphi(\Gamma) \neq \emptyset$.

Maskin monotonicity (Maskin, 1999) is one of the key principles in implementation theory but is also desirable in and of itself (see, for example, Kojima and Manea, 2010; Karakaya and Klaus, 2017). Roughly speaking, this axiom, sometimes called Maskin invariance,⁵ requires that if a state is chosen in an e-form game, then it is also chosen in an e-form game in which the state has (weakly) improved in the preference rankings of all players. We recall that a preference profile \succeq' is a *Maskin monotonic transformation* of a preference profile \succeq at $a \in A$ if $a \succeq'_i b$ for all $b \in A$ where $a \succeq_i b$. In words, any state that is ranked below or at the same level as a under \succeq_i is also ranked below or at the same level as a under \succeq'_i .

Maskin monotonicity. A solution φ on \mathcal{K} satisfies *Maskin monotonicity* if for all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$, all $\Gamma' = (N, A, E, \succeq') \in \mathcal{K}$, and all $a \in \varphi(\Gamma)$ such that \succeq' is a Maskin monotonic transformation of \succeq at a , it holds that $a \in \varphi(\Gamma')$.

The next axiom generalizes the well-known unanimity condition that is customarily imposed in many settings (see, for example, Crès et al., 2011). Roughly speaking, this condition means that if everyone prefers a particular state over all other states, then this particular state must be selected. We say that $B \in \mathcal{P}_0(A)$ is a *top set for $S \in \mathcal{P}_0(N)$ with respect to \succeq* if for all $b \in B$, all $c \in A \setminus B$, and all $i \in S$, it holds that $b \succ_i c$.

Coalitional unanimity. A solution φ on \mathcal{K} satisfies *coalitional unanimity* if for all $\Gamma \in \mathcal{K}$ and all $a \in \varphi(\Gamma)$, if there exists (S, B) such that $B \in E_a(S)$ and B is a top set for S with respect to \succeq , then $a \in \varphi(\Gamma) \cap B$.

Coalitional unanimity ensures that a state a in the solution cannot be outside a set B which is a top set for a coalition S that is effective for B at a .

Theorem 4.2. *If φ is a solution on a class \mathcal{K} of e-form games that satisfies coalitional unanimity and Maskin monotonicity, then φ is a subsolution of the core.*

The statement of Theorem 4.2 implies that if an e-form game Γ has an empty core, then it is not possible to find another nonempty solution on a class containing Γ which satisfies both coalitional unanimity and Maskin monotonicity. The following result can be immediately deduced from Theorem 4.2.

Corollary 4.3. *If φ is a solution on a class \mathcal{K} of e-form games that satisfies weak nonemptiness, coalitional unanimity, and Maskin monotonicity then $\varphi(\Gamma) = C(\Gamma)$ for all $\Gamma \in \mathcal{K}$ with $|C(\Gamma)| = 1$.*

This result is relevant to characterize the core in specific environments such as the Condorcet winner in voting problems under strict preferences (see Section 5.1). It can also be applied to characterize the core over any domain admitting a unique core element: in that case, weak nonemptiness can be replaced by the standard axiom of nonemptiness. For example, over the domain of normal form linear oligopoly games, it is straightforward to see that the Nash equilibrium correspondence is the unique solution satisfying nonemptiness, coalitional unanimity, and Maskin monotonicity.

Loosely speaking, our next axiom establishes that if a state is selected, then this state remains selected even when some non-selected (irrelevant) states are removed. For $N' \in \mathcal{P}_0(N)$ and $A' \in \mathcal{P}_0(A)$, let $E'_a := (E'_a)_{a \in A'}$ be the effectiveness function on (N', A') defined for all $a \in A'$ and all $S \in \mathcal{P}_0(N')$ by $E'_a(S) = \{B \in E_a(S) : B \subseteq A'\}$. Further, the preference profile $\succeq' = (\succeq'_i)_{i \in N'}$ is defined for all $a, b \in A'$ and all $i \in N'$ by $a \succeq'_i b$ if and only if $a \succeq_i b$. The *e-form subgame of Γ on (N', A')* is denoted by $\Gamma' = (N', A', E', \succeq')$.

Independence of irrelevant states. A solution φ on \mathcal{K} satisfies *independence of irrelevant states* if for all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ and all $a \in \varphi(\Gamma)$, if $\Gamma' = (N', A', E', \succeq')$ is an e-form subgame of Γ such that $\Gamma' \in \mathcal{K}$

⁵We refer to Thomson (2018) for a detailed discussion about the usage of “Maskin monotonicity” and “Maskin invariance”.

and $a \in A'$, then $a \in \varphi(\Gamma')$.

We point out that independence of irrelevant states requires that the e-form subgame Γ' belongs to \mathcal{K} . On the one hand, if N is fixed for all e-form games in \mathcal{K} , independence of irrelevant states closely resembles the seminal axiom of independence of irrelevant alternatives like Nash (1950a).⁶ On the other hand, since it is not always possible to remove some states without withdrawing some players for some classes of e-form games, the subsets of states and players in the e-form subgame may be closely connected. To see this, consider the class of e-form games representing network problems defined in section 3.3. To define the concept of pairwise stability in a standard way, we require that any pair of players can be connected. Therefore, for an e-form game (N, A, E^{PS}, \succeq) to belong to $\mathcal{K}^{PS}(\mathcal{N})$, it is necessary for A to be the set of all networks included in the complete network g^N . Consider two players i, j in N connected in a network $g \supseteq \{i, j\}$. Since $A \setminus \{g\}$ is no longer the set of all subsets of g^N , it follows that $(N, A \setminus \{g\}, E^{PS}, \succeq)$ does not belong to $\mathcal{K}^{PS}(\mathcal{N})$. One way, allowed by our axiom, to remove g from this game while remaining in the class $\mathcal{K}^{PS}(\mathcal{N})$, is to remove from N all players connected in g .

Unlike coalitional unanimity and Maskin monotonicity, we will apply weak nonemptiness and independence of irrelevant states on classes of e-form games satisfying a new property that echoes the ancestor property defined in Norde et al. (1996).⁷

Definition 4.4. Holdover property. A class \mathcal{K} of e-form games satisfies the *holdover property* if for all $\Gamma \in \mathcal{K}$ and all $\bar{a} \in C(\Gamma)$, there exists $\bar{\Gamma} \in \mathcal{K}$ such that:

- (i) Γ is an e-form subgame of $\bar{\Gamma}$;
- (ii) $C(\bar{\Gamma}) = \{\bar{a}\}$.

When the holdover property holds, it is possible to establish a result that allows independence of irrelevant states to be applied in a way similar to the so-called bracing lemma, a typical result based on a principle of consistency to characterize solution concepts in game-theoretic and economic models (see Thomson, 2011 and Section 6.2 for a detailed discussion).

Theorem 4.5. *Let \mathcal{K} be a class of e-form games satisfying the holdover property. No proper subsolution of the core satisfies weak nonemptiness and independence of irrelevant states on \mathcal{K} .*

Combining Theorems 4.2 and 4.5, we obtain⁸ our first main characterization of the core of e-form games.

Corollary 4.6. *Let \mathcal{K} be a class of e-form games satisfying the holdover property. The core is the unique solution satisfying weak nonemptiness, coalitional unanimity, Maskin monotonicity, and independence of irrelevant states on \mathcal{K} .*

The logical independence of the axioms is discussed in the Appendix.

5 Applications

5.1 Axiomatic characterization of the Condorcet Winner

We propose adapting the axioms of Section 4 to the class of e-form games $\mathcal{K}^{CW}(\mathcal{N}, \mathcal{A})$ defined in Subsection 3.1 representing voting problems. In this context, coalitional unanimity becomes the very standard axiom of majority property stating that if one candidate is preferred by a majority of voters, then that candidate must be the unique chosen candidate. Let $\mathcal{K} \subseteq \mathcal{K}^{CW}(\mathcal{N}, \mathcal{A})$ be a class of e-form games.

Majority property. A solution φ on \mathcal{K} , satisfies the *majority property* if for all $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}$, if there exists $b \in A$ and $S \in \mathcal{P}_0(N)$ such that $|S| \geq |N|/2$; and further, for all $c \in A \setminus \{b\}$ and all $i \in S$, $b \succ_i c$, then $\varphi(\Gamma) \subseteq \{b\}$.

⁶See also Arrow (1950), Chernoff (1954), and Sen (1969, 1993).

⁷See also Peleg et al. (1996) and Peleg and Sudhölter (1997).

⁸The proof that the core satisfies the four axioms in Corollary 4.6 is left to the reader.

Lemma 5.1. *Majority property is equivalent to coalitional unanimity on $\mathcal{K}^{CW}(\mathcal{N}, \mathcal{A})$.*

Combining Corollary 4.3 with Lemma 5.1, we obtain the following result.

Corollary 5.2. *The Condorcet winner correspondence is the unique solution that satisfies weak nonemptiness, majority property, and Maskin monotonicity on $\mathcal{K}^{CW}(\mathcal{N}, \mathcal{A})$.*

Thus, on the full domain of voting problems there does not exist a nonempty-valued solution that selects the Condorcet winner whenever it exists, and which satisfies both majority property and Maskin monotonicity. Moreover, this result constitutes a new characterization of the Condorcet winner that echoes the one recently established by Horan et al. (2019).

5.2 Axiomatic characterization of the Nash equilibrium

Following the same line of reasoning, we express the axioms of Section 4 in the context of the class of e-form games $\mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ defined in Subsection 3.2 representing normal form games. As a consequence, coalitional unanimity becomes a weak version of the axiom of individual rationality specifying that if the strategic choices of the players permit to reach the favorite strategy profile of a player, then this strategy profile must be selected. Let $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ be a class of e-form games.

Weak individual rationality. A solution φ on \mathcal{K} satisfies *weak individual rationality* if for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$, and all $\sigma \in \varphi(\Gamma)$, if there exist $i \in N$ and $\sigma'_i \in \Sigma_i$ such that for all $\sigma'' \in \Sigma_N$ with $\sigma'' \neq (\sigma'_i, \sigma_{-i})$, $(\sigma'_i, \sigma_{-i}) \succ_i \sigma''$ holds, then $\sigma'_i = \sigma_i$.

Note that weak individual rationality corresponds to coalitional unanimity for any $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$. In Section 6, we establish that weak individual rationality is implied by the two standard axioms of one-person rationality and weak consistency.

The next axiom was first introduced in the context of normal form games by Peleg and Tijs (1996). Let $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$. For $i \in N$, let $\Sigma'_i \subseteq \Sigma_i$ and $E'^{NE} := (E'^{NE})_{\sigma \in \Sigma'_N}$ be the effectiveness function on (N, Σ'_N) defined for all $\sigma \in \Sigma'_N$ and all $S \in \mathcal{P}_0(N)$ by:

$$E'^{NE}_{\sigma}(S) = \begin{cases} \{(\sigma'_i, \sigma_{N \setminus \{i\}}) : \sigma'_i \in \Sigma'_i\} & \text{if } S = \{i\}; \\ \emptyset & \text{otherwise.} \end{cases}$$

Independence of irrelevant strategies. A solution φ on \mathcal{K} satisfies *independence of irrelevant strategies* if for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$, and all $\sigma \in \varphi(\Gamma)$, if $\Gamma' = (N, \Sigma'_N, E'^{NE}, \succeq')$ is an e-form subgame of Γ such that $\Gamma' \in \mathcal{K}$ and $\sigma_i \in \Sigma'_i$ for all $i \in N$, then $\sigma \in \varphi(\Gamma')$.

It is immediate that independence of irrelevant states implies independence of irrelevant strategies over any subclass of $\mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ and that for a fixed set of players N , both axioms are equivalent over any subclass of $\mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$.

The following definition gives a technical condition on a class of e-form games. We say that a game class is extendable if any e-form game of this class can be seen as the “subgame” of another element of this class. Precisely, a class is extendable if any game that belongs to it remains in this class by adding a strategy to one of its players.

Definition 5.3. A class $\mathcal{K} \subsetneq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ is *extendable* if for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$ and all $i \in N$, there exists a set $\tilde{\Sigma}_i$ such that $\Sigma_i \subsetneq \tilde{\Sigma}_i$; and further, if for all preference profiles $\tilde{\succeq}$ over the set $\tilde{\Sigma}_i \times \Sigma_{N \setminus \{i\}}$, $\tilde{\Gamma} = (N, \tilde{\Sigma}_i \times \Sigma_{N \setminus \{i\}}, \tilde{E}^{NE}, \tilde{\succeq}) \in \mathcal{K}$, where \tilde{E}^{NE} is the effectiveness function that extends E^{NE} on $\tilde{\Sigma}_i \times \Sigma_{N \setminus \{i\}}$.

A wide variety of classes of e-form games in $\mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ are extendable. Next example shows that e-form games representing normal form games with compact sets of strategies are extendable. Same type of argument can be given to show, for instance, that so is the class of matrix games.

Example 5.4. Let y be an arbitrary element of $[0, +\infty]$, N a finite set of players and \mathcal{A} , the universe of strategy profiles, such that for each $i \in N$ and $x_i \in [0, y[$, $\times_{i \in N} [0, x_i] \subsetneq \mathcal{A}$. We define the class \mathcal{K}_y as the set of e-form games $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ such that for each $i \in N$ there exists $x_i \in [0, y[$ such that $\Sigma_i = [0, x_i]$. The class \mathcal{K}_y is clearly extendable. Indeed, by keeping the notations of the definition of an extendable class, it suffices to consider a game $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}_y$ such that $\Sigma_i = [0, x_i]$ for an arbitrary $i \in N$ and to define $\tilde{\Sigma}_i = [0, x'_i]$ with $x_i < x'_i < y$.

We now have the material to provide a new axiomatic characterization of the Nash equilibrium correspondence using Theorems 4.2 and 4.5.

Proposition 5.5. *Let $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ be an extendable class of e-form games such that for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $|N| \geq 2$. The Nash equilibrium correspondence is the unique solution that satisfies weak nonemptiness, weak individual rationality, Maskin monotonicity, and independence of irrelevant strategies on \mathcal{K} .*

5.3 Axiomatic characterization of pairwise stable networks

As in the two previous subsections, we write the axioms of Section 4 for the class of e-form games $\mathcal{K}^{PS}(\mathcal{N})$ defined in Subsection 3.3 representing network problems. In this setting, coalitional unanimity becomes an axiom on the (un)desirability of links in a network specifying that if a player is undesirable for another one, then the selected network must not contain the link between them; otherwise, if two players are desirable for each other, then the selected network must link them together.

(Un)desirable Link. A solution φ on \mathcal{K} satisfies *(un)desirable link* if for all $\Gamma = (N, A, E^{PS}, \succeq) \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{PS}(\mathcal{N})$ and all $g \in \varphi(\Gamma)$, if there exists $i \in N$ such that $g \setminus \{\{i, j\}\} \succ_i g'$ for some $j \in N \setminus \{i\}$ and all $g' \subseteq g^N$ with $g' \neq g \setminus \{\{i, j\}\}$, then $\{i, j\} \notin g$; otherwise if there exist $i, j \in N$ such that $g \cup \{\{i, j\}\} \succ_k g'$ for all $k \in \{i, j\}$ and all $g' \subseteq g^N$ with $g' \neq g \cup \{\{i, j\}\}$, then $\{i, j\} \in g$.

Proposition 5.6. *(Un)desirable link is equivalent to coalitional unanimity on $\mathcal{K}^{PS}(\mathcal{N})$.*

Combining Theorem 4.2 with Theorem 4.5, we obtain the following result.

Proposition 5.7. *The pairwise stability correspondence is the unique solution that satisfies weak nonemptiness, (un)desirable link, Maskin monotonicity, and independence of irrelevant states on $\mathcal{K}^{PS}(\mathcal{N})$.*

To the best of our knowledge, this is the first axiomatic characterization of the pairwise stability correspondence.

6 The consistency principle for e-form games

The axiom of consistency (Sobolev, 1975; Peleg, 1986) we consider in this section is of fundamental interest in axiomatic theory and has been examined from numerous angles.⁹

Definition 6.1. Let $\Gamma \in \mathcal{K}$ be an e-form game and $\bar{a} \in A$. The *reduced e-form game of Γ with respect to S and \bar{a}* is the e-form game $\Gamma^{S, \bar{a}} = (S, A_S, E^{S, \bar{a}}, \succeq')$ $\in \mathcal{K}$, where S is a proper coalition of N , and the effectiveness function $E^{S, \bar{a}}$ is defined by setting, for all $a_S \in A_S$ and all $T \in \mathcal{P}_0(S)$,

$$E_{a_S}^{S, \bar{a}}(T) = \{B_S : B_S \times \{\bar{a}_{N \setminus S}\} \in E_{(a_S, \bar{a}_{N \setminus S})}(T)\};$$

and further, where the preference profile $\succeq' = (\succeq'_i)_{i \in S}$ over the set A_S is defined by setting, for all $a_S, a'_S \in A_S$ and all $i \in S$, $a_S \succeq'_i a'_S$ if and only if $(a_S, \bar{a}_{N \setminus S}) \succeq_i (a'_S, \bar{a}_{N \setminus S})$.

The reduced e-form game $\Gamma^{S, \bar{a}} = (S, A_S, E^{S, \bar{a}}, \succeq')$ describes the situation where the members of $N \setminus S$ continue to cooperate with the members of S while the set of possible states is constrained by $\bar{a}_{N \setminus S}$. Then, for every $T \in \mathcal{P}_0(S)$ the effectiveness function $E_{a_S}^{S, \bar{a}}(T)$ corresponds to the effectivity power of T which is consistent with its effectivity power in Γ .

⁹For comprehensive surveys on consistency and its applications, the reader is referred to Thomson (1990, 2011).

Definition 6.2. A class \mathcal{K} is a *reduction-closed class*¹⁰ if for all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$, all $\bar{a} \in A$, and all $i \in N$, it holds that $\Gamma^{\{i\}, \bar{a}} \in \mathcal{K}$.

We now have the material to write consistency in the context of e-form games.

Consistency. A solution φ on \mathcal{K} satisfies *consistency* if for all $\Gamma \in \mathcal{K}$, all $\bar{a} \in \varphi(\Gamma)$, and all $S \subsetneq N$ such that $\Gamma^{S, \bar{a}} \in \mathcal{K}$, it holds that $a_S \in \varphi(\Gamma^{S, \bar{a}})$.

6.1 Consistency and Nash equilibrium

In this subsection, we discuss the possible links between Proposition 5.5 and existing axiomatic characterizations of the Nash equilibrium correspondence using consistency. In the context of normal form games, consistency states that if $\sigma \in \Sigma_N$ belongs to the solution of a game, then σ_S belongs to the solution of the game restricted to S , obtained by fixing the strategies of players outside S at $\sigma_{N \setminus S}$. Given an e-form game $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ and a strategy profile $\bar{\sigma} \in \Sigma_N$, the *reduced e-form game of Γ with respect to S and $\bar{\sigma}$* is the e-form game $\Gamma^{S, \bar{\sigma}} = (S, \Sigma_S, E^{NE, S, \bar{\sigma}}, \succeq') \in \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$, whereby $S \subsetneq N$, the effectiveness function $E^{NE, S, \bar{\sigma}}$ is derived from Definition 6.1 when $E = E^{NE}$ and further, the preference profile $\succeq' = (\succeq'_i)_{i \in S}$ over the set Σ_S is defined for all $\sigma_S, \sigma'_S \in \Sigma_S$ and all $i \in S$ by $\sigma_S \succeq'_i \sigma'_S$ if and only if $(\sigma_S, \bar{\sigma}_{N \setminus S}) \succeq_i (\sigma'_S, \bar{\sigma}_{N \setminus S})$. In the following, we assume that $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ is a reduction-closed class. We propose to weaken consistency on \mathcal{K} by an axiom which requires consistency for only one-person reduced games. Formally, this axiom, called weak consistency is defined as follows.

Weak consistency. A solution φ on \mathcal{K} satisfies *weak consistency* if for all $\Gamma \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$, all $\bar{\sigma} \in \varphi(\Gamma)$, and all $i \in N$ such that $\Gamma^{\{i\}, \bar{\sigma}} \in \mathcal{K}$, it holds that $\sigma_i \in \varphi(\Gamma^{\{i\}, \bar{\sigma}})$.

Since the pioneering work of Peleg and Tijs (1996), several axiomatic characterizations of the Nash equilibrium correspondence or closely related solution concepts for normal form games have been proposed in the literature.¹¹ Most of them use consistency together with one-person rationality that we adapt for e-form games as follows.

One-person rationality. A solution φ on \mathcal{K} satisfies *one-person rationality* if for all one-person e-form games $\Gamma = (\{i\}, \Sigma_i, E^{NE}, \succeq_i) \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$, it holds that $\varphi(\Gamma) = \{\sigma_i \in \Sigma_i : \sigma_i \succeq_i \sigma'_i \text{ for all } \sigma'_i \in \Sigma_i\}$.

The following proposition establishes that one-person rationality together with weak consistency imply weak individual rationality.

Proposition 6.3. *Let $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ be a reduction-closed class, and φ be a solution on \mathcal{K} . If φ satisfies one-person rationality and weak consistency on \mathcal{K} , then φ also satisfies weak individual rationality on \mathcal{K} .*

To introduce the next axiom, defined by Peleg and Tijs (1996), we say that player $d \in N$ is *dummy* in $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ if $|\Sigma_d| = 1$.

Dummy. A solution φ on \mathcal{K} satisfies *dummy* if for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$, if player $d \in N$ is dummy, then $\varphi(\Gamma) = \Sigma_d \times \varphi(\Gamma^{N \setminus \{d\}, \bar{\sigma}})$, where $\bar{\sigma} \in \Sigma_N$.

Ray (2000) proposed a weaker version of the dummy axiom that we can adapt too.

Weak dummy. A solution φ on \mathcal{K} satisfies *weak dummy* if for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$, if player $d \in N$ is dummy, then $\varphi(\Gamma) \subseteq \Sigma_d \times \varphi(\Gamma^{N \setminus \{d\}, \bar{\sigma}})$, where $\bar{\sigma} \in \Sigma_N$.

We establish two other axiomatic characterizations of the Nash equilibrium correspondence which only use standard properties.

¹⁰This property is closely related to the property used by Peleg (1989) called “weak reduced game property” which holds for reduced games with at most two players.

¹¹See, for example, Peleg et al. (1996), Peleg and Sudhölter (1997), Norde et al. (1996) and van Heumen et al. (1996).

Proposition 6.4. *Let $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ be a class of e-form games that is both extendable and reduction-closed, and φ be a solution on \mathcal{K} . The following three properties are equivalent:*

- (i) φ is the Nash equilibrium correspondence;
- (ii) φ satisfies weak nonemptiness, one-person rationality, independence of irrelevant strategies, and weak consistency on \mathcal{K} ;
- (iii) φ satisfies weak nonemptiness, one-person rationality, independence of irrelevant strategies, and weak dummy on \mathcal{K} .

6.2 Ancestor Property and axiomatic characterization of stable matchings

In this subsection, we discuss a standard method to axiomatize game-theoretic solutions with the axiom of consistency. This method is called the “bracing lemma” by Thomson (2011). We show that this method can be adapted to the framework of e-form games and discuss new axiomatizations in matching theory by using the ancestor property together with Theorem 4.2.

6.2.1 Ancestor property and bracing lemma

The following property concerns the nature of the classes of e-form games and adapted the ancestor property defined by Norde et al. (1996) to the framework of e-form games.

Definition 6.5. Ancestor property. A class \mathcal{K} of e-form games satisfies the *ancestor property* if for all $\Gamma = (N, A_N, E, \succeq) \in \mathcal{K}$ and all $a^* \in C(\Gamma)$, there exists $\Gamma' = (N', A'_{N'}, E', \succeq') \in \mathcal{K}$ with $a' \in A'_{N'}$, such that:

- (i) Γ is the reduced e-form game of Γ' with respect to N and a' .
- (ii) $C(\Gamma') = \{a'\}$, where $a'_{N'} = a^*$;

When the ancestor property holds, it is possible to establish a result almost identical to the so-called “bracing lemma” (see Thomson, 2011).

Lemma 6.6. *Let \mathcal{K} be a class of e-form games satisfying the ancestor property. No proper subsolution of the core satisfies weak nonemptiness and consistency on \mathcal{K} .*

Combining Theorem 4.2 and Lemma 6.6, we obtain our second main characterization of the core.

Theorem 6.7. *Let \mathcal{K} be a class of e-form games satisfying the ancestor property. The core is the unique solution satisfying weak nonemptiness, coalitional unanimity, Maskin monotonicity, and consistency on \mathcal{K} .*

6.2.2 A new characterization of the core of matching models

As an application of Theorem 6.7, we propose a new axiomatization of the core of the two-sided one-to-one matching model introduced by Gale and Shapley (1962).¹² The stable matchings correspondence has been axiomatized in the context of one-to-one matching by Sasaki and Toda (1992), Nizamogullari and Özkal-Sanver (2014) and Klaus (2017). Using the universe \mathcal{N} of players, we consider a *matching model* consisting of a finite set $N \subseteq \mathcal{N}$ of individuals, partitioned in two subgroups with equal numbers of *men* M and *women* W . Each man m (resp. woman w) has a complete, transitive, and antisymmetric strict preference relation P_m (resp. P_w) over the set W (resp. M). A *matching* is a one-to-one mapping $\mu : M \rightarrow W$. A *matching problem* is a tuple $(M, W, (P_m)_{m \in M}, (P_w)_{w \in W})$. A matching μ is *stable* in $(M, W, (P_m)_{m \in M}, (P_w)_{w \in W})$ if there is no pair $(m, w) \in M \times W$ such that $w P_m \mu(m)$ and $m P_w \mu^{-1}(w)$.

In the setting of e-form games, the set A of states consists of all possible matchings μ . The preferences of the players $(\succeq_i)_{i \in M \cup W}$ over the set A are induced by their preferences $((P_m)_{m \in M}, (P_w)_{w \in W})$ over their partners as follows: for every $m \in M$, $\mu \succ_m \mu'$ if and only if $\mu(m) P_m \mu'(m)$, and for every $w \in W$, $\mu \succ_w \mu'$ if and only if $\mu^{-1}(w) P_w \mu'^{-1}(w)$. Several effectiveness functions lead to a core representation of the stable

¹²We emphasize that Corollary 4.6 does not apply in the context of matching models because the holdover property may not be satisfied.

matchings. We propose the following definition which will turn out to be essential for our axiomatic result. The requirement of stability here is that any man and woman that are currently not partners can make themselves better off by creating a link between themselves:

$$\forall \mu \in A, \forall (m, w) \in M \times W, E_{\mu}^{SM}(\{m, w\}) = \{\{\mu' : \mu'(m) = w\}\},$$

and for $S \in \mathcal{P}_0(N)$ such that $|S| \neq 2$, $E_{\mu}^{SM}(S) = \emptyset$ for all $\mu \in A$. Unlike other effectivity functions previously defined, observe that $E_{\mu}^{SM}(\{m, w\})$ contains sets which are not singletons. We denote $\mathcal{K}^{SM}(\mathcal{N})$ the class of e-form games $\Gamma = (N, A, E^{SM}, \succeq)$.

Proposition 6.8. *The set of stable matchings of $(M, W, (P_m)_{m \in M}, (P_w)_{w \in W})$ coincides with $C(\Gamma)$, where $\Gamma = (N, A, E^{SM}, \succeq) \in \mathcal{K}^{SM}(\mathcal{N})$.*

The proof of this result is fairly straightforward and thus omitted.¹³ The *stable matchings correspondence* associates $C(\Gamma) \in \mathcal{P}_0(A)$ with each e-form game $\Gamma = (N, A, E^{SM}, \succeq) \in \mathcal{K}^{SM}(\mathcal{N})$. The main advantage of this core representation is to have both an adaptation of coalitional unanimity as well as an equivalence between consistency as defined in Section 6 and the one as usually defined in matching theory (see, for example, Sasaki and Toda, 1992). Indeed, for all $\Gamma \in \mathcal{K}^{SM}(\mathcal{N})$, a reduced e-form game $\Gamma^{S, \bar{\mu}}$ belonging to $\mathcal{K}^{SM}(\mathcal{N})$ has necessarily equal numbers of men and women, which means that there exist $M' \subseteq M$ and $W' \subseteq W$ such that $S = M' \cup W'$ and $|M'| = |W'|$. Following the definition of a reduced e-form game, it is immediate that the associated effectiveness function is defined for all matching $\mu|_{M'} : M' \rightarrow W'$ and all $T \in \mathcal{P}_0(M' \cup W')$ by setting $E_{\mu|_{M'}}^{SM, S, \bar{\mu}}(T) = \{\{\mu'|_{M'} : \mu'|_{M'}(m') = w'\}\}$ if $T = \{m', w'\}$, $(m', w') \in M' \times W'$, and $E_{\mu|_{M'}}^{SM, S, \bar{\mu}}(T) = \emptyset$ otherwise.

In the context of matchings, coalitional unanimity becomes a natural axiom that states that soulmates must be married together, where soulmates are a man and a woman who would prefer to be with each other than any other individual (Leo et al., 2017).

Soulmates. A solution φ on $\mathcal{K}^{SM}(\mathcal{N})$ satisfies *soulmates* if for all $\Gamma = (N, A, E^{SM}, \succeq) \in \mathcal{K}^{SM}(\mathcal{N})$ and all $\mu \in \varphi(\Gamma)$, if there exists B such that for all $\mu' \in B$, $\mu'(m) = w$ and B is a top set for $\{m, w\}$, then $\mu(m) = w$.

Proposition 6.9. *Soulmates is equivalent to coalitional unanimity on $\mathcal{K}^{SM}(\mathcal{N})$.*

The proof of this result is fairly straightforward and thus omitted. Applying Theorem 4.2 and Lemma 6.6, we obtain the following result.

Proposition 6.10. *The stable matchings correspondence is the unique solution that satisfies nonemptiness, soulmates, Maskin monotonicity, and consistency on $\mathcal{K}^{SM}(\mathcal{N})$.*

Since a stable matching always exists (Gale and Shapley, 1962), i.e. $\mathcal{K}^{SM}(\mathcal{N}) = \mathcal{K}_C^{SM}(\mathcal{N})$, the axiom of weak nonemptiness can be replaced by the axiom of nonemptiness in Proposition 6.10.

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¹³It is also possible to define a class of e-form games for which the core coincides with the core of the housing markets (Shapley and Scarf, 1974; Roth and Postlewaite, 1977).

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Appendix

Proof of Theorem 4.2. Let φ be a solution that satisfies coalitional unanimity and Maskin monotonicity on \mathcal{K} . Assume for the sake of contradiction that $\varphi(\Gamma) \not\subseteq C(\Gamma)$ for some $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$. Hence, there is a state $a \in \varphi(\Gamma)$ such that $a \notin C(\Gamma)$, i.e. there exists an objection (S, B) against a in Γ . Let \succeq' be the preference profile defined as follows:

- (i) for all $i \in S$, we first move, in any order, the set of states B to the top of player i 's preferences (if not already the case), i.e. for all $b \in B$ and all $c \in A \setminus B$, it holds that $b \succ'_i c$. Second, we move state a just below the worst state in B according to player i 's preferences, i.e. for all $c \in A \setminus (B \cup \{a\})$, $a \succeq'_i c$. Third, for all $c, d \in A \setminus (B \cup \{a\})$, we assume that $c \succeq'_i d$ if and only if $c \succeq_i d$;
- (ii) for all $i \in N \setminus S$, we leave preferences as they are, i.e. for all $i \in N \setminus S$, $\succeq'_i = \succeq_i$.

Observe that \succeq' is a Maskin monotonic transformation of \succeq at a . Let $\Gamma' = (N, A, E, \succeq') \in \mathcal{K}$. It follows from Maskin monotonicity that $a \in \varphi(\Gamma')$. Furthermore, by the definition of \succeq' , it holds that B is a top set for all members of S . Thus, it follows from coalitional unanimity that $a \in B$, a contradiction. \blacksquare

Proof of Theorem 4.5. Let \mathcal{K} be a class satisfying the holdover property and φ be a proper subsolution of the core, which is defined on \mathcal{K} and satisfies weak nonemptiness and independence of irrelevant states. Since φ is a proper subsolution of the core, there exists $\Gamma \in \mathcal{K}$ and $\bar{a} \in C(\Gamma)$ such that $\bar{a} \notin \varphi(\Gamma)$. By the holdover property, there exists $\bar{\Gamma} \in \mathcal{K}$ such that Γ is an e-form subgame of $\bar{\Gamma}$, where $C(\bar{\Gamma}) = \{\bar{a}\}$. Since φ is a proper subsolution of the core satisfying weak nonemptiness, we deduce that $\varphi(\bar{\Gamma}) = \{\bar{a}\}$. Since φ satisfies independence of irrelevant states, it follows that $\bar{a} \in \varphi(\Gamma)$, a contradiction. \blacksquare

Proof of Lemma 5.1. Let φ be a solution on $\mathcal{K}^{CW}(\mathcal{N}, \mathcal{A})$.

- (i) First, assume that φ satisfies coalitional unanimity on $\mathcal{K}^{CW}(\mathcal{N}, \mathcal{A})$. Let $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}^{CW}(\mathcal{N}, \mathcal{A})$ and assume that there exists $b \in A$ and $S \in \mathcal{P}_0(N)$ with $|S| \geq |N|/2$, such that for all $c \in A \setminus \{b\}$ and all $i \in S$, $b \succ_i c$. Then $\{b\}$ is a top set for all members of S and for all $a \in A$, $\{b\} \in E_a^{CW}(S)$. If $\varphi(\Gamma) = \emptyset$, then there is nothing to prove. If $\varphi(\Gamma) \neq \emptyset$, then there exists $a \in \varphi(\Gamma)$, and by coalitional unanimity, $a \in \varphi(\Gamma) \cap \{b\}$. We deduce that $\varphi(\Gamma) = \{b\}$.
- (ii) Second, assume that φ satisfies the majority property on $\mathcal{K}^{CW}(\mathcal{N}, \mathcal{A})$. Let $\Gamma = (N, A, E^{CW}, \succeq) \in \mathcal{K}^{CW}(\mathcal{N}, \mathcal{A})$ and assume that there exist $a \in \varphi(\Gamma)$ and (S, B) such that $B \in E_a^{CW}(S)$ and B is a top set for all members of S . By the definition of E^{CW} , we deduce that $|S| \geq |N|/2$ and there exists $b \in A$ such that $B = \{b\}$. As a consequence, for all $i \in S$ and all $c \in A \setminus \{b\}$, we have $b \succ_i c$. By the majority property, $\varphi(\Gamma) \subseteq \{b\}$, and we deduce that $a = b$, so $a \in \varphi(\Gamma) \cap \{b\}$.

This concludes the proof. \blacksquare

Proof of Proposition 5.5. Let $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ be an extendable class such that for all $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$, $|N| \geq 2$. By Theorems 4.2 and 4.5, it is sufficient to prove that \mathcal{K} restricted to games with at least two players satisfies the holdover property. Let $\Gamma = (N, \Sigma_N, E^{NE}, \succeq) \in \mathcal{K}$ be an e-form game and i and j denote two distinct players belonging to N . We have seen in Section 3.2 that the set of Nash equilibria of (N, Σ_N, \succeq) is equal to $C(\Gamma)$. Assume that $|C(\Gamma)| \geq 2$ and let σ^* be a member of $C(\Gamma)$. Since \mathcal{K} is extendable, for all $k \in \{i, j\}$, there exist a set $\tilde{\Sigma}_k \supseteq \Sigma_k$ and $s_k, t_k \in \tilde{\Sigma}_k \setminus \Sigma_k$ such that for all weak order $\tilde{\succeq}$ on $\tilde{\Sigma}_N$, $\tilde{\Gamma} = (N, \tilde{\Sigma}_N, \tilde{E}^{NE}, \tilde{\succeq}) \in \mathcal{K}$, where:

$$\tilde{\Sigma}_N = \tilde{\Sigma}_{\{ij\}} \times \Sigma_{N \setminus \{i, j\}}.$$

Let $(\tilde{\succeq}_\ell)_{\ell \in N}$ be the preference profile on $\tilde{\Sigma}_N$ defined as follows:

Rule 1. $\forall \ell \in N, \forall k, k' \in \{i, j\}, \forall \sigma_k \in \Sigma_k, \forall a_k \in \{s_k, t_k\}, \forall a_{k'} \in \{s_{k'}, t_{k'}\}, \forall \tilde{\sigma} \in \tilde{\Sigma}_N$ such that $\forall k \in \{i, j\}, \tilde{\sigma}_k \notin \{s_k, t_k\} \cup \Sigma_k$, we have:

- (i) $(\sigma_k, \tilde{\sigma}_{N \setminus \{k\}}) \sim_\ell (a_k, \tilde{\sigma}_{N \setminus \{k\}}) \sim_\ell (a_{k'}, \tilde{\sigma}_{N \setminus \{k'\}}) \tilde{\succ}_\ell \tilde{\sigma}$,
- (ii) $(a_k, \sigma_{N \setminus \{k\}}^*) \sim_\ell (a_{k'}, \sigma_{N \setminus \{k'\}}^*) \tilde{\succ}_\ell (a_k, \tilde{\sigma}_{N \setminus \{k\}})$.

Rule 2. $\forall \ell \in N, \forall k \in \{i, j\}, \forall a_k \in \{s_k, t_k\}, \forall \sigma \in \Sigma_N$, we have $\sigma \tilde{\succ}_\ell (a_k, \sigma_{N \setminus \{k\}}^*)$.

Rule 3. $\forall \ell \in N, \forall \sigma, \sigma' \in \Sigma_N, \sigma \succeq_\ell \sigma' \Leftrightarrow \sigma \tilde{\succeq}_\ell \sigma'$.

Rule 4. $\forall \ell \in N, \forall k \in \{i, j\}, \forall a_k \in \{s_k, t_k\}, \forall \sigma \in \Sigma_N$ such that $\sigma_{N \setminus \{k\}} \neq \sigma_{N \setminus \{k\}}^*$, we have $(a_k, \sigma_{N \setminus \{k\}}) \succ_{\ell} \sigma$.

Rule 5. $\forall \ell \in N, \forall k \in \{i, j\}, \forall a_k \in \{s_k, t_k\}, \forall a_i \in \{s_i, t_i\}, \forall a_j \in \{s_j, t_j\}, \forall \sigma \in \Sigma_N$ we have $(a_i, a_j, \sigma_{N \setminus \{i, j\}}) \succ_{\ell} (a_k, \sigma_{N \setminus \{k\}})$.

Rule 6. $\forall \ell \in N \setminus \{i, j\}, \forall \sigma \in \Sigma_N, \forall k \in \{i, j\}, \forall a_k \in \{s_k, t_k\}$, we have:

- (i) $(s_i, s_j, \sigma_{N \setminus \{ij\}}) \sim_i (t_i, t_j, \sigma_{N \setminus \{ij\}}) \succ_i (s_i, t_j, \sigma_{N \setminus \{ij\}}) \sim_i (t_i, s_j, \sigma_{N \setminus \{ij\}})$,
- (ii) $(s_i, t_j, \sigma_{N \setminus \{ij\}}) \sim_j (t_i, s_j, \sigma_{N \setminus \{ij\}}) \succ_j (s_i, s_j, \sigma_{N \setminus \{ij\}}) \sim_j (t_i, t_j, \sigma_{N \setminus \{ij\}})$,
- (iii) $(s_i, t_j, \sigma_{N \setminus \{ij\}}) \sim_{\ell} (t_i, s_j, \sigma_{N \setminus \{ij\}}) \sim_{\ell} (s_i, s_j, \sigma_{N \setminus \{ij\}}) \sim_{\ell} (t_i, t_j, \sigma_{N \setminus \{ij\}})$.

Rule 7. $\forall \ell \in N, \succ_{\ell}$ is transitive.

Observe that *Rules* 1 through 7 rank strategy profiles in increasing order according to players' preferences. They ensure that for all $\ell \in N, \succ_{\ell}$ is a weak order of $\tilde{\Sigma}_N$. The proof that $(N, \Sigma_N, E^{NE}, \succeq)$ is an e-form subgame of $(N, \tilde{\Sigma}_N, \tilde{E}^{NE}, \tilde{\succeq})$ is an immediate consequence of *Rule* 3. Observe that *Rules* 2 and 4 ensure that individual deviations of i and j from σ^* to $\tilde{\Sigma}_N \setminus \Sigma_N$ make the situation worse for all players, while individual deviations of i or j from $\Sigma_N \setminus \{\sigma^*\}$ to $\tilde{\Sigma}_N \setminus \Sigma_N$ improve the situation of all players. We want to prove that $C(\tilde{\Gamma}) = \{\sigma^*\}$. Let $\bar{\sigma} \in \tilde{\Sigma}_N \setminus \{\sigma^*\}$. Several cases arise:

- (i) If $\bar{\sigma} \in \Sigma_N \setminus \{\sigma^*\}$, then two subcases can occur:
 - (a) If $\bar{\sigma}_{N \setminus \{i\}} \neq \sigma_{N \setminus \{i\}}^*$, then by *Rule* 4, we have $(s_i, \bar{\sigma}_{N \setminus \{i\}}) \succ_i \bar{\sigma}$;
 - (b) If $\bar{\sigma}_{N \setminus \{i\}} = \sigma_{N \setminus \{i\}}^*$, then $\bar{\sigma}_{N \setminus \{j\}} \neq \sigma_{N \setminus \{j\}}^*$ because $\bar{\sigma} \neq \sigma^*$. Hence, it follows from *Rule* 4 that $(s_j, \bar{\sigma}_{N \setminus \{j\}}) \succ_j \bar{\sigma}$.

In both subcases, we conclude that $\bar{\sigma}$ does not belong to $C(\tilde{\Gamma})$.

- (ii) If $\bar{\sigma} \in \tilde{\Sigma}_N \setminus \Sigma_N$, then two subcases can occur:
 - (a) If there does not exist $k \in \{i, j\}$ such that $\bar{\sigma}_k \in \{s_k, t_k\} \cup \Sigma_k$, then by *Rule* 1, we have $(s_i, \bar{\sigma}_{N \setminus \{i\}}) \succ_i \bar{\sigma}$;
 - (b) If there exists $k \in \{i, j\}$ such that $\bar{\sigma}_k \in \{s_k, t_k\} \cup \Sigma_k$, then three cases arise. In each of these cases, we denote by k' the unique element in $\{i, j\} \setminus \{k\}$;
 - (b₁) If $\bar{\sigma}_k \in \{s_k, t_k\}$ and $\bar{\sigma}_{k'} \in \{s_{k'}, t_{k'}\}$, then by (i) and (ii) of *Rule* 6, there exists an improving deviation of i or j ;
 - (b₂) If $\bar{\sigma}_k \in \{s_k, t_k\}$ and $\bar{\sigma}_{k'} \in \Sigma_{k'}$, then by *Rule* 5, there exists an improving deviation of k' (the case where $\bar{\sigma}_k \in \Sigma_k$ and $\bar{\sigma}_{k'} \in \{s_{k'}, t_{k'}\}$ is similar);
 - (b₃) If $\bar{\sigma}_k \in \{s_k, t_k\} \cup \Sigma_k$ and $\bar{\sigma}_{k'} \notin \{s_{k'}, t_{k'}\} \cup \Sigma_{k'}$, then for all $\sigma_{k'} \in \Sigma_{k'}$, $(\sigma_{k'}, \bar{\sigma}_{N \setminus \{k'\}})$ is an improving deviation of k' from $\bar{\sigma}$. On the one hand, if $(\sigma_{k'}, \sigma_{N \setminus \{k, k'\}}) = \sigma_{N \setminus \{k\}}^*$, then it follows from (ii) of *Rule* 1 that $(\sigma_{k'}, \bar{\sigma}_{N \setminus \{k'\}})$ is an improving deviation of k' from $\bar{\sigma}$. On the other hand, if $(\sigma_{k'}, \sigma_{N \setminus \{k, k'\}}) \neq \sigma_{N \setminus \{k\}}^*$, then it follows from *Rules* 1, 4 and 7 that $(\sigma_{k'}, \bar{\sigma}_{N \setminus \{k'\}})$ is an improving deviation of k' from $\bar{\sigma}$.

In all cases, we conclude that $\bar{\sigma}$ does not belong to $C(\tilde{\Gamma})$.

It remains to prove that σ^* belongs to $C(\tilde{\Gamma})$. First, since σ^* belongs to $C(\Gamma)$, it follows from *Rule* 3 that there is no possible deviation from σ^* to Σ_N . Second, it follows from *Rule* 2 that there does not exist improving deviation from σ^* to $\tilde{\Sigma}_N \setminus \Sigma_N$. Thus, we conclude that $C(\tilde{\Gamma}) = \{\sigma^*\}$. \blacksquare

Proof of Proposition 5.7. By Theorems 4.2 and 4.5, it is sufficient to prove that $\mathcal{K}^{PS}(\mathcal{N})$ satisfies the holdover property. Let $\Gamma = (N, A, E^{PS}, \succeq) \in \mathcal{K}^{PS}(\mathcal{N})$ and take any $\bar{g} \in C(\Gamma)$. If $|N| = 1$, then there is nothing to prove. Assume that $|N| \geq 2$ and let $\bar{\Gamma} = (N', A', E'^{PS}, \succeq') \in \mathcal{K}^{PS}(\mathcal{N})$ be an e-form game where

$N' = N \cup \{\bar{i}\}$. Consider any player $\bar{j} \in N$ and define players' preferences in the following way:

Rule 1. For player $\bar{i} \in N'$, we assume that for all $g, g' \in A' \setminus A$, $g \succ'_{\bar{i}} g'$ iff $|g| > |g'|$; for all $g, g' \in A \setminus \{\bar{g}\}$, $g \sim'_{\bar{i}} g'$; for all $g \in A \setminus \{\bar{g}\}$ and all $g' \in A' \setminus (A \setminus \{\bar{g}\})$, $g \succ'_{\bar{i}} g'$; for all $g \in A' \setminus (A \cup \{\bar{g} \cup \{\{\bar{i}, \bar{j}\}\})$, $g \succeq'_{\bar{i}} \bar{g}$ and $\bar{g} \cup \{\{\bar{i}, \bar{j}\}\} \sim'_{\bar{i}} \bar{g}$.

Rule 2. For player $\bar{j} \in N$, we assume that for all $g, g' \in A$, $g \succeq'_{\bar{j}} g'$ iff $g \succeq_{\bar{j}} g'$; for all $g \in A$ and all $g' \in A' \setminus A$, $g \succ'_{\bar{j}} g'$; and for all $g, g' \in A' \setminus A$, $g \succ'_{\bar{j}} g'$ iff $|g| < |g'|$.

Rule 3. For all players $k \in N \setminus \{\bar{j}\}$, we assume that for all $g, g' \in A$, $g \succeq'_k g'$ iff $g \succeq_k g'$; for all $g \in A$ and all $g' \in A' \setminus A$, $g \succ'_k g'$; for all $g, g' \in A' \setminus A$ such that $\{\bar{i}, \bar{j}\} \in g$ and $\{\bar{i}, \bar{j}\} \in g'$, $g \succ'_k g'$ iff $|g| > |g'|$; for all $g, g' \in A' \setminus A$ such that $\{\bar{i}, \bar{j}\} \notin g$ and $\{\bar{i}, \bar{j}\} \notin g'$, $g \succ'_k g'$ iff $|g| < |g'|$; for all $g, g' \in A' \setminus A$ such that $\{\bar{i}, \bar{j}\} \in g$ and $\{\bar{i}, \bar{j}\} \notin g'$, $g \succ'_k g'$.

First, observe that Γ is an e-form subgame of $\bar{\Gamma}$. It remains to prove that $C(\bar{\Gamma}) = \{\bar{g}\}$. By construction, it must be clear that \bar{g} is pairwise stable in $\bar{\Gamma}$. Then assume that there exists a pairwise stable network \tilde{g} in $\bar{\Gamma}$ such that $\tilde{g} \neq \bar{g}$. We distinguish two cases:

- (i) First, if $\tilde{g} \in A$, then $\tilde{g} \cup \{\{\bar{i}, \bar{j}\}\} \succ'_l \tilde{g}$ for all $l \in \{\bar{i}, \bar{j}\}$. So, \tilde{g} is not pairwise stable;
- (ii) Second, if $\tilde{g} \in A' \setminus A$, we distinguish two subcases:
 - (a) If $\{\bar{i}, \bar{j}\} \in \tilde{g}$ then two cases can occur:
 - i. There exists $k \in N \setminus \{\bar{j}\}$ such that $\{k, \bar{i}\} \notin \tilde{g}$. In this case, we have $\tilde{g} \cup \{\{k, \bar{i}\}\} \succ'_l \tilde{g}$ for all $l \in \{k, \bar{i}\}$. So, \tilde{g} is not pairwise stable;
 - ii. For all $k \in N \setminus \{\bar{j}\}$, $\{k, \bar{i}\} \in \tilde{g}$. In this case, we have $\tilde{g} \setminus \{\{\bar{i}, \bar{j}\}\} \succ'_j \tilde{g}$. So, \tilde{g} is not pairwise stable.
 - (b) If $\{\bar{i}, \bar{j}\} \notin \tilde{g}$, then there exists $k \in N \setminus \{\bar{j}\}$ such that $\{k, \bar{i}\} \in \tilde{g}$ since $\tilde{g} \in A' \setminus A$. In this case, we have $\tilde{g} \setminus \{\{k, \bar{i}\}\} \succ'_k \tilde{g}$. So, \tilde{g} is not pairwise stable, a contradiction.

Thus, we conclude that $C(\bar{\Gamma}) = \{\bar{g}\}$. ■

Proof of Proposition 6.3. Let φ be a solution on $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ where \mathcal{K} is a reduction-closed class. Assume that φ satisfies one-person rationality and weak consistency on \mathcal{K} . Let σ be an element of $\varphi(\Gamma)$. Assume that there exist $i \in N$ and $\sigma'_i \in \Sigma_i$ such that for all $\sigma'' \neq (\sigma'_i, \sigma_{-i})$, $(\sigma'_i, \sigma_{-i}) \succ_i \sigma''$. Since $\sigma \in \varphi(\Gamma)$ and φ satisfies weak consistency, we have $\sigma_i \in \varphi(\Gamma^{\{i\}, \sigma})$. Since φ satisfies one-person rationality, we have $\{\sigma'_i\} = \varphi(\Gamma^{\{i\}, \sigma})$. Hence, we obtain $\sigma_i = \sigma'_i$, which means that φ satisfies weak individual rationality. ■

Proof of Proposition 6.4. Let $\mathcal{K} \subseteq \mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$ be a class of e-form games, which is both extendable and reduction-closed, and φ be a solution on \mathcal{K} . We omit the proof that the Nash equilibrium correspondence satisfies the axioms. Assume that φ satisfies weak nonemptiness, one-person rationality, independence of irrelevant strategies, and weak consistency on \mathcal{K} . Since φ satisfies one-person rationality and weak consistency, then by a similar argument to Peleg and Tijs (1996)[Proposition 2.8], we obtain that φ is a subsolution of the Nash equilibrium correspondence. We conclude with Theorem 4.5 that φ is the Nash equilibrium correspondence. Now, assume that φ satisfies weak nonemptiness, one-person rationality, independence of irrelevant strategies, and weak dummy. Since φ satisfies weak dummy and independence of irrelevant strategies, then by Ray (2000)[Proposition 2], we obtain that φ satisfies weak consistency and conclude with the previous argument that φ is the Nash equilibrium correspondence. ■

Proof of Lemma 6.6. Let \mathcal{K} be a class satisfying the ancestor property and φ be a proper subsolution of the core, defined on \mathcal{K} and which satisfies weak nonemptiness and consistency. Since φ is a proper subsolution of the core, there exist $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ and $a^* \in C(\Gamma)$ such that $a^* \notin \varphi(\Gamma)$. By the ancestor property, there exists $\Gamma' = (N', A'_{N'}, E', \succeq') \in \mathcal{K}$ with $N' \supseteq N$ and $a' \in A'$ such that Γ is the reduced e-form game of Γ' with respect to N and a' with $a'_N = a^*$ and $C(\Gamma') = \{a'\}$. Since φ is a proper subsolution of the core satisfying weak nonemptiness, $\varphi(\Gamma') = \{a'\}$. Since φ satisfies consistency, it follows that $a'_N = a^* \in \varphi(\Gamma)$,

which is a contradiction. ■

Proof of Theorem 6.7. Let \mathcal{K} be a class of e-form games satisfying the ancestor property. From Theorem 4.2 and Lemma 6.6, it suffices to prove that the core defined on \mathcal{K} satisfies weak nonemptiness, coalitional unanimity, Maskin monotonicity, and consistency on \mathcal{K} . By Corollary 4.6, it remains to prove that the core satisfies consistency on \mathcal{K} . Let $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$ and $\bar{a} \in C(\Gamma)$. Let $S \subsetneq N$ be a proper subcoalition of N and $\Gamma^{S, \bar{a}} = (S, A_S, E^{S, \bar{a}}, \succeq')$ be the reduced e-form game of Γ with respect to S and \bar{a} . Assume for the sake of contradiction that $\bar{a}_S \notin C(\Gamma^{S, \bar{a}})$. Then, there exists an objection (T, B_S) , with $T \in \mathcal{P}_0(S)$ and $B_S \in E_{\bar{a}_S}^{S, \bar{a}}(T)$, against \bar{a}_S in $\Gamma^{S, \bar{a}}$. It follows that there exists $B \in E_{\bar{a}}(T)$ with $B = B_S \times \{\bar{a}_{N \setminus S}\}$, and for all $b \in B$ and all $i \in T$, $b \succ_i \bar{a}$, contradicting $\bar{a} \in C(\Gamma)$. ■

Proof of Proposition 6.10. To prove that any solution φ satisfying soulmates and Maskin monotonicity on \mathcal{K}^{SM} is a subsolution of the stable matchings correspondence, it suffices to adapt the proof of Theorem 4.2. The proof that no proper subsolution of the stable matchings correspondence satisfies nonemptiness and consistency on \mathcal{K}^{SM} is a direct consequence of Sasaki and Toda (1992)[Lemma 1, p. 222] and Lemma 6.6. ■

Independence of the axioms

The table below shows that weak nonemptiness, coalitional unanimity, and Maskin monotonicity are independent (the formal proof is left to the reader).¹⁴ To introduce it, we need to define some additional solution concepts. Let \mathcal{K} be an arbitrary class of e-form games. Denote by φ_1 the solution on \mathcal{K} defined by $\varphi_1(\Gamma) = \emptyset$ for all $\Gamma \in \mathcal{K}$. Next, let φ_2 be the solution on \mathcal{K} such that $\varphi_2(\Gamma) = A$ for all $\Gamma = (N, A, E, \succeq) \in \mathcal{K}$. Finally, we say that a state a is top dominated if there exists $S \in \mathcal{P}_0(N)$ and a top set $B \in E_a(S)$ of S such that $a \notin B$. We define φ_3 as the set of states that are not top dominated.

	Weak Non.	Coal. un.	Maskin mon.
φ_1	no	yes	yes
φ_2	yes	no	yes
φ_3	yes	yes	no

Due to the generality of e-form games, the independence of these three axioms with independence of irrelevant states cannot be established for every class \mathcal{K} of e-form games. In particular, independence fails for some classes of e-form games, but it holds in others. To see that independence may fail, observe that Maskin monotonicity can be replaced by independence of irrelevant states in the proof of Theorem 4.2 for each class \mathcal{K} such that every subgame of $\Gamma \in \mathcal{K}$ belongs to \mathcal{K} . To prove this point, assume by contradiction that $a \in \varphi(\Gamma)$ and $a \notin C(\Gamma)$, i.e. there exists an objection (S, B) against a in Γ . At this step, we must assume that the e-form subgame $\Gamma' = (N, B \cup \{a\}, E', \succeq')$ of Γ belongs to \mathcal{K} . Then, it follows from independence of irrelevant states that $a \in \varphi(\Gamma')$. However, B is a top set for S with respect to \succeq' and $B \in E'_a(S)$. By coalitional unanimity, it holds that $a \in B$, a contradiction. As a consequence, the axioms of weak nonemptiness, coalitional unanimity, and independence of irrelevant states imply Maskin monotonicity in specific environments. Moreover, when the core contains at most one state (this is the case, for example, with the class $\mathcal{K}^{CW}(\mathcal{N}, \mathcal{A})$), it is easy to see that independence of irrelevant states is implied by the three other axioms.

On the other hand, there are some classes for which the axioms of Corollary 4.6 are independent. We illustrate such a case on the class \mathcal{K}^{NE} . To do this, let us observe that φ_1 and φ_2 always satisfy independence of irrelevant states. Let \mathcal{K} be a class for which there exists Γ such that $C(\Gamma)$ contains at least two elements. Let $\rho : \mathcal{P}_0(\mathcal{N}) \rightarrow \mathcal{N}$ be a mapping which associates to each $N \in \mathcal{P}_0(\mathcal{N})$ an element of N . Let φ_4 be the solution on \mathcal{K} such that for each $\Gamma \in \mathcal{K}$, $\Gamma = (N, A, E, \succeq)$, $\varphi_4(\Gamma) = \{a \in C(\Gamma) : \forall b \in C(\Gamma), a \succeq_{\rho(N)} b\}$. It is clear that φ_4 satisfies weak nonemptiness, coalitional unanimity, and Maskin monotonicity, but does not satisfy independence of irrelevant states. We conclude by noting that φ_3 satisfies independence of irrelevant states in the specific classes of $\mathcal{K}^{NE}(\mathcal{N}, \mathcal{A})$. In particular, the axioms which appear in proposition 5.5 are independent.

¹⁴In particular, the table shows the independence of the axioms of Corollary 5.2.