



HAL
open science

High-order conditions for an analytic function to be locally positive definite

Lionel Rosier

► **To cite this version:**

Lionel Rosier. High-order conditions for an analytic function to be locally positive definite. 2024. hal-04796631

HAL Id: hal-04796631

<https://hal.science/hal-04796631v1>

Preprint submitted on 21 Nov 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

High-order conditions for an analytic function to be locally positive definite

Lionel Rosier

Université du Littoral Côte d'Opale,
Laboratoire de Mathématiques Pures et Appliquées J. Liouville,
B.P. 699, F-62228 Calais, France
`Lionel.Rosier@univ-littoral.fr`

Abstract. We consider an analytic function from \mathbb{R}^n to \mathbb{R} and provide some sufficient conditions involving homogeneous polynomials with respect to some family of dilations ensuring that the function presents a strict local minimum at some point. For a polynomial function, we show how to use Tarski-Seidenberg theorem or Sturm theorem to investigate the same issue.

Keywords: analytic function; extrema; locally positive definite function; weighted homogeneity; high-order conditions; polynomial function; Tarski-Seidenberg theorem; Sturm theorem

1 Introduction

Optimization in spaces of finite dimension is a classical part of a course in real analysis, which is taught because of its importance in many applications, e.g. in Engineering, Automatic Control, Neural Networks, etc. In Automatic Control, a Lyapunov function has often to be searched for to ensure that a closed-loop system is stabilized at the origin (see e.g. [1]). A given function is a *Lyapunov function* for the system if and only if it is *positive definite* and its Lie derivative is *negative definite*. Thus, there is a real need for having efficient tests to decide whether a given nonlinear smooth function is positive definite. We shall restrict our study to analytic functions, even if analytic Lyapunov functions may fail to exist for asymptotically stable systems with analytic vector fields (see e.g. [1]).

Most of the attention is usually paid to the first order condition (null gradient to select critical points) and to the second order condition (positive definite Hessian matrix for the existence of a strict minimum) (see e.g. [15,20]). Of course, those conditions do not cover all the situations, and it could happen that terms of order greater than 2 play an important role in the positive definiteness of the function.

The aim of this article is to revisit the classical issue of the search of extrema for a smooth function of several variables by using high-order conditions based upon homogeneous polynomials associated with a family of dilations. The use of weighted homogeneity is now classical in Automatic Control (see e.g. [1]).

Assuming that all the critical points are known, we want to find sufficient conditions ensuring that a given critical point corresponds to a local extremum or a saddle point. It is an easy exercise to prove that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ which is analytic in a neighborhood of 0 and vanishes at 0 presents a strict minimum at 0 if and only if it can be expanded as $f(x) = a_{2k}x^{2k} + \dots$ with $k \in \mathbb{N}^*$ and $a_{2k} > 0$. The situation is much more delicate in dimension $n \geq 2$. If f is a polynomial function, then using Tarski-Seidenberg theorem (see e.g. [3,5,6,11,17,18]), it is possible to decide whether the function is locally or globally positive definite. We refer the reader to [2,9,16] and the references therein for a description of the algorithms used to decide whether the polynomial function f is positive definite.

The paper is outlined as follows. In Section 2, we provide necessary conditions and sufficient conditions for an analytic function to be positive definite by using expansions with respect to certain families of dilations. We provide a lot of examples suggesting that a necessary and sufficient condition is difficult to guess. In Section 3, we restrict ourselves to polynomial functions and show how Tarski-Seidenberg theorem can be used to decide whether a polynomial function is positive definite. Rather than using a general (involved and costly) algorithm to mimic Tarski-Seidenberg theorem, we describe a method based on Sturm sequences and apply it on a few examples of functions $f \in \mathbb{R}[x_1, x_2]$. The paper ends with some concluding remarks.

2 High-order conditions for the positive definiteness of an analytic function

To fix the framework, using translations if needed, we can assume that the critical point under investigation is $x = 0$ and that the value of the function at 0 is merely 0. We consider a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ ($n \in \mathbb{N}^*$) as above which is analytic in a neighborhood of 0 and is expanded as

$$f(x) = \sum_{e \in \mathbb{N}^n} a_e x^e \quad \text{for } |x_i| \leq b_i, \ 1 \leq i \leq n. \quad (1)$$

In (1), $(b_1, \dots, b_n) \in (0, +\infty)^n$, $x^e = \prod_{i=1}^n x_i^{e_i}$ for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$, and $e = (e_1, \dots, e_n) \in \mathbb{N}^n$. From the above assumptions,

$$a_{(0, \dots, 0)} = a_{(1, 0, \dots, 0)} = \dots = a_{(0, \dots, 0, 1)} = 0.$$

If $M := \max_{|x_i|=b_i} |f(x)|$, then by [10, Chapter 5, Lemma 1.1],

$$|a_e| \leq \frac{M}{b^e} \quad \forall e \in \mathbb{N}^n. \quad (2)$$

We shall say that the function f is (*locally*) *positive definite*, if there exists $\varepsilon > 0$ such that

$$\forall x \in \mathbb{R}^n \ (0 < \|x\| < \varepsilon \Rightarrow f(x) > 0). \quad (3)$$

We shall denote $f > 0$ locally, in short. We are mainly concerned with the issue of providing conditions about the coefficients a_e ensuring that f is locally positive definite; that is, 0 is a strict local minimum of f .

We denote $|x| = x_1 + \cdots + x_n$ and $\|x\| = (x_1^2 + \cdots + x_n^2)^{\frac{1}{2}}$ for $x \in \mathbb{R}^n$, and $P_k(x) := \sum_{|e|=k} a_e x^e$ the homogeneous polynomial of degree k in the expansion of f , so that $f(x) = \sum_{k=0}^{\infty} P_k(x)$.

It is well-known that if $f(0) = 0$, and 0 is a critical point (i.e. $\frac{\partial f}{\partial x_i}(0) = 0 \quad \forall i \in \{1, \dots, n\}$) at which the Hessian matrix $H = \left(\frac{\partial^2 f}{\partial x_j \partial x_i}(0) \right)_{1 \leq i, j \leq n}$ is positive definite, then f is locally positive definite at 0. In other words,

$$(P_0 = P_1 = 0 \text{ and } P_2 > 0) \Rightarrow f > 0 \text{ locally.} \quad (4)$$

Of course, the converse is only *partially* true. If f is locally positive definite, then $P_0 = P_1 = 0$ and $P_2(x) = \frac{1}{2}x^t H x \geq 0$ for all $x \in \mathbb{R}^n$, but H is not necessarily positive definite. For instance the function $f(x_1, x_2) = x_1^2 + x_2^4$ is (locally or globally) positive definite, but the matrix $H = \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix}$ is not positive definite.

When the quadratic form $P_2(x) = \frac{1}{2}x^t H x$ is degenerate, we have to look at terms of higher order (here x_2^4) to decide whether the function f is definite positive or not. That example shows also that the variables x_i should be given different weights, which amounts to expanding f as a series of polynomials homogeneous with respect to certain families of dilations (see below).

Let us give first necessary conditions for f to be locally positive definite.

Proposition 1. *Assume that f be analytic and locally positive definite. Then we have the following properties.*

1. *There exists $k \in \mathbb{N}^*$ such that $P_1 = \cdots = P_{2k-1} = 0$, $P_{2k} \geq 0$ and $P_{2k} \not\equiv 0$.*
2. *For each $i \in \{1, \dots, n\}$, there exists $k_i \in \mathbb{N}^*$ such that $\partial_{x_i}^l f(0) = 0$ for $0 \leq l < 2k_i$ and $\partial_{x_i}^{2k_i} f(0) > 0$.*
3. *Let $\underline{b} := \min(b_1, \dots, b_n)$. Then there exists $\rho_0 \in (0, \underline{b})$ such that for any $\rho \in (0, \rho_0)$ there exists $N_0 \in \mathbb{N}^*$ such that for any x with $\|x\| = \rho$ there is some $k \in \{2, \dots, N_0\}$ with $P_1(x) = \cdots = P_{k-1}(x) = 0$ and $P_k(x) \neq 0$ (actually $P_k(x) > 0$).*

Proof.

1. We need the following lemma.

Lemma 1. *Let $\delta \in (0, 1)$. Then there exists $C > 0$ such that*

$$\forall x \in \mathbb{R} \quad (|x| \leq \delta \underline{b} \Rightarrow \forall k \in \mathbb{N} \quad \sum_{j=k+1}^{\infty} |P_j(x)| \leq C \left(\frac{|x|}{\underline{b}} \right)^{k+1}). \quad (5)$$

Proof of Lemma 1: By (2), for $|x| \leq \delta \underline{b}$

$$|P_j(x)| \leq M \sum_{|e|=j} \prod_{i=1}^n \left(\frac{|x_i|}{b_i} \right)^{e_i} \leq M \left(\sum_{i=1}^n \frac{|x_i|}{b_i} \right)^j \leq M \left(\frac{|x|}{\underline{b}} \right)^j,$$

so that

$$\sum_{j=k+1}^{\infty} |P_j(x)| \leq M \sum_{j=k+1}^{\infty} \left(\frac{|x|}{\underline{b}}\right)^j = \frac{M}{1 - \frac{|x|}{\underline{b}}} \left(\frac{|x|}{\underline{b}}\right)^{k+1} \leq C \left(\frac{|x|}{\underline{b}}\right)^{k+1}$$

with $C = \frac{M}{1-\delta}$. \square

Let $l \in \mathbb{N}^*$ be the first index for which $P_l \neq 0$. Note that $P_l(tx) = t^l P_l(x)$ for $(t, x) \in \mathbb{R} \times \mathbb{R}^n$. By Lemma 1, for any $x \in S^{n-1} := \{x \in \mathbb{R}^n; \|x\| = 1\}$,

$$f(tx) = t^l P_l(x) + O(t^{l+1}) \quad \text{as } t \rightarrow 0. \quad (6)$$

If $l \in 1+2\mathbb{N}$, P_l is odd, so there exists $x \in S^{n-1}$ with $P_l(x) < 0$ and $f(tx) < 0$ for $0 < t \ll 1$, contradicting the fact that f be locally positive definite. We obtain the same conclusion if $l \in 2\mathbb{N}^*$ and $P_l(x) < 0$ for some $x \in S^{n-1}$.

2. Pick any $i \in \{1, \dots, n\}$. The function $x_i \rightarrow f(0, \dots, x_i, \dots, 0)$ is analytic and locally positive definite. Therefore, it may be expanded as $f(0, \dots, x_i, \dots, 0) = c_{2k_i} x_i^{2k_i} + \dots$ with $c_{2k_i} > 0$. It follows that

$$\partial_{x_i}^l f(0) = 0 \text{ for } 0 \leq l < 2k_i \text{ and } \partial_{x_i}^{2k_i} f(0) > 0.$$

3. Pick $\rho_0 \in (0, \underline{b})$ such that

$$\forall x \in \mathbb{R}^n \quad (0 < \|x\| < \rho_0 \Rightarrow f(x) > 0).$$

If the conclusion is false, one may find some number $\rho \in (0, \rho_0)$, a sequence (x_j) in ρS^{n-1} , and a sequence $N_j \nearrow +\infty$ such that

$$P_1(x_j) = \dots = P_{N_j}(x_j) = 0. \quad (7)$$

Extracting a subsequence if needed, we can assume that $x_j \rightarrow x$ in ρS^{n-1} . But from (7) and the fact that $N_j \nearrow +\infty$, we infer that $P_i(x) = 0$ for all $i \geq 1$, so that $f(x) = 0$, which is a contradiction. The fact that $P_k(x) > 0$ comes from (6) for $l = k$ and $t \rightarrow 0^+$. \square

Remark 1. 1. The converse of the third item in Lemma 1 is not true. Consider

$$f(x_1, x_2) = x_1^2 + x_2^4 - 4x_1x_2^2 = (x_1 - 2x_2^2)^2 - 3x_2^4.$$

Then for $x = (x_1, x_2) \neq (0, 0)$, $P_1(x) = 0$ and $P_2(x) > 0$ if $x_1 \neq 0$, while $P_1(x) = P_2(x) = P_3(x) = 0$ and $P_4(x) > 0$ if $x_1 = 0$. Thus 3. holds with $N_0 = 4$. However f is not locally positive definite, for $f(x_2^2, x_2) = -2x_2^4$. Note also that there does not exist any number $\delta > 0$ such that

$$(0 < t < \delta \text{ and } \|x - (0, 1)\| < \delta) \Rightarrow P_1(tx) + P_2(tx) + P_3(tx) \geq 0.$$

Indeed $P_1(s^2, s) + P_2(s^2, s) + P_3(s^2, s) = -3s^4 < 0$ for $0 < s < 1$.

Let us recall the classical definition of weighted homogeneity (see e.g. [1]). For a given sequence $r = (r_1, \dots, r_n) \in (0, +\infty)^n$, we define the one-parameter family of *dilations* $(\delta_t^r)_{t>0}$ associated with r by

$$\delta_t^r(x) := (t^{r_1}x_1, \dots, t^{r_n}x_n) \quad \forall x = (x_1, \dots, x_n) \in \mathbb{R}^n, \forall t > 0.$$

Note that the set $\{\delta_t^r(x); x \in S^{n-1}, 0 < t < t_0\} \cup \{0\}$ is a neighborhood of 0 for each $t_0 > 0$.

A function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ is said to be δ^r -homogeneous of degree $p \in \mathbb{R}^+$ if

$$h(\delta_t^r(x)) = t^p h(x) \quad \forall x \in \mathbb{R}^n, \forall t > 0.$$

From now on, we assume that $r = (r_1, \dots, r_n) \in (\mathbb{N}^*)^n$. Then, from $f(x) = \sum_{e \in \mathbb{N}^n} a_e x^e$, we obtain that

$$f(\delta_t^r(x)) = \sum_{e \in \mathbb{N}^n} a_e \prod_{i=1}^n (t^{r_i} x_i)^{e_i} = \sum_{p=0}^{\infty} t^p \sum_{\substack{e \in \mathbb{N}^n \\ r \cdot e = p}} a_e x^e.$$

Thus

$$f(x) = \sum_{p=0}^{\infty} \sum_{\substack{e \in \mathbb{N}^n \\ r \cdot e = p}} a_e x^e =: \sum_{p=0}^{\infty} H_p(x) \quad (8)$$

where the polynomial H_p is δ^r -homogeneous of degree p . (Note that $H_p = 0$ if $\{e \in \mathbb{N}^n; r \cdot e = p\} = \emptyset$.) We note also that the relation $H_p(\delta_t^r(x)) = t^p H_p(x)$ is valid for all $(x, t) \in \mathbb{R}^{n+1}$.

The following lemma extends Lemma 1 to the framework of the weighted homogeneity.

Lemma 2. *Let f be an analytic function expanded as in (1), let $r = (r_1, \dots, r_n) \in (\mathbb{N}^*)^n$, and let the sequence $(H_p)_{p \geq 0}$ be as in (8).*

1. *For any $p \in \mathbb{N}$, let $m_p := \min_{x \in S^{n-1}} H_p(x)$ and $M_p := \max_{x \in S^{n-1}} H_p(x)$. Then*

$$m_p t^p \leq H_p(\delta_t^r(x)) \leq M_p t^p \quad \forall x \in S^{n-1}, \forall t > 0. \quad (9)$$

2. *For any $p \in \mathbb{N}$, there exist $t_0 > 0$ and $C_p > 0$ such that*

$$\sum_{q \geq p+1} |H_q(\delta_t^r(x))| \leq C_p t^{p+1} \quad \forall x \in S^{n-1}, \forall t \in (0, t_0). \quad (10)$$

Proof.

1. From $m_p \leq H_p(x) \leq M_p$ for $x \in S^{n-1}$, we infer that

$$m_p t^p \leq H_p(\delta_t^r(x)) = t^p H_p(x) \leq M_p t^p \quad \forall x \in S^{n-1}, \forall t > 0.$$

2. Note that $H_0 = 0$. Pick any $\delta \in (0, 1)$ and let $C > 0$ be as in Lemma 1. Then for $p \in \mathbb{N}^*$ and $x \in \mathbb{R}^n$ with $|x| \leq \delta \underline{b}$, doing the same computations as in the proof of Lemma 1, we obtain

$$|H_p(x)| \leq \sum_{\substack{e \in \mathbb{N}^n \\ r \cdot e = p}} |a_e x^e| \leq \sum_{\substack{e \in \mathbb{N}^n \\ e \neq 0}} |a_e x^e| \leq C \frac{|x|}{\underline{b}} \leq C \delta.$$

Pick $t_0 > 0$ such that

$$\forall x \in S^{n-1} \quad |\delta_{2t_0}^r(x)| \leq \delta \underline{b}.$$

Then for $x \in S^{n-1}$ and $t \in (0, t_0)$, noticing that $\delta_t^r = \delta_{\frac{t}{2t_0}}^r \circ \delta_{2t_0}^r$, we obtain for any $p \in \mathbb{N}$

$$\begin{aligned} \sum_{q \geq p+1} |H_q(\delta_t^r(x))| &= \sum_{q \geq p+1} \left(\frac{t}{2t_0}\right)^q |H_q(\delta_{2t_0}^r(x))| \\ &\leq C\delta \left(\frac{t}{2t_0}\right)^{p+1} \left(1 - \frac{t}{2t_0}\right)^{-1} \\ &\leq C_p t^{p+1} \end{aligned}$$

with $C_p := 2C\delta(2t_0)^{-(p+1)}$. \square

The following result extends Proposition 1 to the framework of the weighted homogeneity. Its proof, being similar to those of Proposition 1, is omitted.

Proposition 2. *Let f be an analytic function expanded as in (1), let $r = (r_1, \dots, r_n) \in (\mathbb{N}^*)^n$, and let the sequence $(H_p)_{p \geq 0}$ be as in (8). Then the following properties hold:*

1. *There exists $k \in \mathbb{N}^*$ such that $H_1 = \dots = H_{2k-1} = 0$, $H_{2k} \geq 0$ and $H_{2k} \not\equiv 0$.*
2. *There exists $\rho_0 \in (0, \underline{b})$ such that for any $\rho \in (0, \rho_0)$ there exists $N_0 \in \mathbb{N}^*$ such that for any x with $\|x\| = \rho$ there is some $k \in \{2, \dots, N_0\}$ with $H_1(x) = \dots = H_{k-1}(x) = 0$ and $H_k(x) > 0$.*

Remark 2. Proposition 2 can be used to show that some given function is not positive definite. For instance, consider

$$f(x_1, x_2) = x_1^2 x_2^3 + x_1^4 + x_2^{10}.$$

Pick $r = (r_1, r_2) = (5, 2)$. Then $f(x) = H_{16}(x) + H_{20}(x)$ with $H_{16}(x) = x_1^2 x_2^3$ and $H_{20}(x) = x_1^4 + x_2^{10}$. As for $x_1 \neq 0$ and $x_2 < 0$ we have $H_{16}(x_1, x_2) < 0$, the conclusion of the first item in Proposition 2 fails, so that f is not positive definite. Note that with the usual dilation $r = (r_1, r_2) = (1, 1)$, we have $P_4(x) = x_1^4 \geq 0$ for all $x \in \mathbb{R}^2$.

Theorem 1. *Let f be analytic around 0 and expanded as in (1). Pick $(r_1, \dots, r_n) \in (\mathbb{N}^*)^n$ and $p_0 \in \mathbb{N}^*$, and let $(H_p)_{p \geq 0}$ be the sequence of δ^r -homogeneous polynomials H_p as in (8). Then the two following assertions are equivalent:*

- (i) $\exists C > 0, \exists t_1 > 0 \quad f(\delta_t^r(x)) \geq Ct^{p_0} \quad \forall x \in S^{n-1}, \forall t \in (0, t_1);$
- (ii) $\exists \tilde{C} > 0, \exists \tilde{t}_1 > 0 \quad \sum_{p \leq p_0} H_p(\delta_t^r(x)) \geq \tilde{C}t^{p_0} \quad \forall x \in S^{n-1}, \forall t \in (0, \tilde{t}_1).$

Proof. We infer from (10) applied with $p = p_0$ that $\sum_{p \geq p_0+1} |H_p(\delta_t^r(x))| \leq C_{p_0} t^{p_0+1}$. The result follows at once, since

$$f(\delta_t^r(x)) = \sum_{p \leq p_0} H_p(\delta_t^r(x)) + \sum_{p \geq p_0+1} H_p(\delta_t^r(x)).$$

\square

Corollary 1. *Let f be analytic around 0 and expanded as $f(x) = \sum_{p \in \mathbb{N}} P_p(x)$, where the polynomial P_p is homogeneous of degree p for each $p \in \mathbb{N}$. Then the following assertions are equivalent:*

- (i) *the function f is locally positive definite;*
- (ii) $\exists p_0 \in \mathbb{N}, \exists C > 0, \exists \epsilon > 0 \quad f(x) \geq C\|x\|^{p_0} \quad \forall x \in \mathbb{R}^n \text{ with } \|x\| < \epsilon;$
- (iii) $\exists p_0 \in \mathbb{N}, \exists C > 0, \exists \epsilon > 0 \quad \sum_{p \leq p_0} P_p(x) \geq C\|x\|^{p_0} \quad \forall x \in \mathbb{R}^n \text{ with } \|x\| < \epsilon.$

Proof. If f is positive definite, then by Lojasiewicz inequality (see [4,8,12,13]) there are some strictly positive constants C, ϵ, α such that $f(x) \geq C\|x\|^\alpha$ for all $x \in \mathbb{R}^n$ with $\|x\| < \epsilon$. Picking any $p_0 \in \mathbb{N}$ with $p_0 \geq \alpha$ and changing the constants C, ϵ if needed, we see that (i) and (ii) are equivalent. The equivalence of (ii) and (iii) follows from Theorem 1 with $\delta_t^r(x) = tx$. \square

Remark 3. 1. In practice, for any given $p_0 \in \mathbb{N}$, the condition in (iii) can be tested by using Tarski-Seidenberg theorem (see below). Indeed, the set $E := \{(C, \epsilon) \in \mathbb{R}^2; \forall x \in \mathbb{R}^n, (x, C, \epsilon) \in F\}$ where

$$F := \{(x, C, \epsilon) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}; \quad C > 0 \text{ and } \epsilon > 0 \text{ and} \\ (\|x\|^2 \geq \epsilon^2 \text{ or } x = 0 \text{ or } (\sum_{p \leq p_0} P_p(x))^2 \geq C^2 \|x\|^{2p_0})\}$$

is semi-algebraic, so that we can know if it is empty or not. This yields an algorithm to check whether the function f is positive definite, even if we don't know when to stop (there is no bound for p_0).

2. The condition in (iii) cannot be relaxed into $\exists p_0, p_1 \in \mathbb{N}, \exists C > 0, \exists \epsilon > 0$

$$\sum_{p \leq p_0} P_p(x) \geq C\|x\|^{p_1} \quad \forall x \in \mathbb{R}^n \text{ with } \|x\| < \epsilon. \quad (11)$$

Note first that explicit Lojasiewicz exponents for positive definite polynomial functions may be found in [8,12]. If $P \in \mathbb{R}[z_1, \dots, z_n]$ is positive definite, then by [8, Theorem 1.5]

$$P(x) \geq C\|x\|^{(\deg P - 1)^n + 1} \quad (12)$$

in a neighborhood of 0 for some constant $C > 0$. The example $P(x_1, x_2) := x_1^{2k} + (x_1 - x_2^k)^2$ for $k \in \mathbb{N}^*$ from [12] shows that the Lojasiewicz exponent $(\deg P - 1)^n + 1$ is (almost) sharp. Indeed, setting $y = \delta_t^r(x) := (t^k x_1, t x_2)$ for $0 < t < 1$ and $x \in S^1$, we have that $t^{2k} \leq t^{2k} x_1^2 + t^2 x_2^2 = \|y\|^2 \leq t^2$ and

$$f(y) = f(\delta_t^r(x)) = t^{2k^2} x_1^{2k} + t^{2k} (x_1 - x_2^k)^2 \geq C t^{2k^2} \geq C \|y\|^{2k^2}.$$

The Lojasiewicz exponent is then $2k^2$. If $f(x) = P(x) + x_2^{2k+1}$, then $f(t^k, t) = t^{2k^2} + t^{2k+1} < 0$ for $t \in (-1, 0)$ and $k \geq 2$. Therefore f is not locally positive definite, and Corollary 1 is not valid if we take in (11) $p_1 = 2k^2 > p_0 = 2k$.

Theorem 1 also yields a high-order sufficient condition for the function f to be locally positive definite.

Corollary 2. *Let f be an analytic function in a neighborhood of 0 expanded as in (1) and such that $f(0) = 0$. Assume the existence of a sequence of weights $(r_1, \dots, r_n) \in (\mathbb{N}^*)^n$, of an integer $p_0 \in \mathbb{N}^*$ and of a number $\mu > 0$ such that, with the δ^r -homogeneous polynomials H_p as in (8), we have*

$$H_{p_0}(x) > 0 \quad \forall x \in S^{n-1}, \quad (13)$$

$$\sum_{1 \leq p < p_0} H_p(x) \geq 0 \quad \forall x \in \mathbb{R}^n \text{ with } \|x\| < \mu. \quad (14)$$

Then f is locally positive definite. Furthermore, there exist $t_1 > 0$ and $C > 0$ such that

$$f(\delta_t^r(x)) \geq Ct^{p_0} \quad \forall x \in S^{n-1}, \quad \forall t \in (0, t_1). \quad (15)$$

Let $k \in \mathbb{N}^$ be as in the first item of Proposition 2 ($2k \leq p_0$). Then there exist $C' > 0$ and $t_2 \in (0, t_1)$ such that*

$$f(\delta_t^r(x)) \leq C't^{2k} \quad \forall x \in S^{n-1}, \quad \forall t \in (0, t_2). \quad (16)$$

Proof. Let $m_{p_0} := \min_{x \in S^{n-1}} H_{p_0}(x)$. Then $m_{p_0} > 0$ by (13). Pick $t_0 > 0$ such that $\|\delta_t^r(x)\| < \mu$ for $0 < t < t_0$ and $\|x\| = 1$. We infer from (9) and (14) that

$$\sum_{p \leq p_0} H_p(\delta_t^r(x)) \geq H_{p_0}(\delta_t^r(x)) \geq m_{p_0} t^{p_0} \quad \forall x \in S^{n-1}, \quad \forall t \in (0, t_0).$$

Then (15) follows from Theorem 1. On the other hand, (16) follows from Proposition 2 and (10). \square

Example 1. Let

$$f(x_1, x_2) = x_1^2 x_2^2 - 5x_1^3 x_2^3 + x_1^6 - x_1^3 x_2^4 - x_1^5 x_2^2 + x_2^8. \quad (17)$$

Looking at the monomials x_1^6 and x_2^8 , we select $r = (r_1, r_2) = (4, 3)$, so that

$$f(x) = H_{14}(x) + H_{21}(x) + H_{24}(x) + H_{26}(x)$$

with

$$H_{14}(x) := x_1^2 x_2^2, \quad H_{21}(x) := -5x_1^3 x_2^3, \quad H_{24}(x) := x_1^6 - x_1^3 x_2^4 + x_2^8, \quad \text{and} \quad H_{26}(x) := -x_1^5 x_2^2.$$

Pick $p_0 = 24$. From $|x_1^3 x_2^4| \leq (x_1^6 + x_2^8)/2$, we infer that $H_{24}(x) \geq (x_1^6 + x_2^8)/2 > 0$ for $x \neq 0$. On the other hand

$$H_{14}(x) + H_{21}(x) = x_1^2 x_2^2 (1 - 5x_1 x_2) \geq 0 \quad \text{for } \|x\| \ll 1.$$

It follows from Corollary 2 that f is locally positive definite.

Some remarks are in order.

Remark 4. 1. Corollary 2 provides a high-order extension of the classical test for the existence of a strict minimum based upon the Hessian matrix.

2. The conclusion of Corollary 2 is certainly valid with less regularity about f (assuming merely that $f \in C^p(\mathbb{R}^n)$ for some $p \in \mathbb{N}^*$) by using Taylor expansion at some order, but we will not provide such a statement.
3. Since H_{p_0} cannot be odd by (13), p_0 has to be taken in $2\mathbb{N}^*$.
4. By Proposition 1, some monomials $x_1^{2k_1}, \dots, x_n^{2k_n}$ should appear with positive coefficients in the expansion of f . A possible choice of the sequence of weights $r = (r_1, \dots, r_n)$ is then $r_i = p/k_i$ for $1 \leq i \leq n$, where p is the lowest common multiple of k_1, \dots, k_n . However, doing such a choice with the *least values of* k_1, \dots, k_n and $p_0 = 2p$ may not work. Consider e.g.

$$f(x_1, x_2) := x_1^2 - 2x_1x_2 + x_2^2 + x_1^4 + x_2^4. \quad (18)$$

The least value of k_1 (resp. k_2) is 1 (resp. 1). Picking $r = (1, 1)$ and $p_0 = 2$, we see that $H_2(x_1, x_2) = P_2(x_1, x_2) = (x_1 - x_2)^2$, and H_2 is not positive definite. One may nevertheless apply Corollary 2 with $r = (1, 1)$ by taking $p_0 = 4$.

5. Condition (13) is *not a necessary condition* for f to be a locally positive definite function. Consider

$$f(x_1, x_2) := x_1^2x_2^2 + (x_1^8 - 2x_1^4x_2^4 + \frac{1}{2}x_2^8) =: P_4(x) + P_8(x). \quad (19)$$

Since $f(x_1, x_2) = x_1^2x_2^2(1 - 2x_1^2x_2^2) + x_1^8 + \frac{1}{2}x_2^8$ and $1 - 2x_1^2x_2^2 > 0$ for $\|x\| \ll 1$, f is locally positive definite. If we pick e.g. $r = (r_1, r_2) = (1, 1)$, then $H_4(0, x_2) = P_4(0, x_2) = 0$, H_4 is not positive definite, while $H_8(x_1, x_1) = P_8(x_1, x_1) = -x_1^8/2$, and hence H_8 is not positive definite. More generally, it is easy to see that for any choice $r = (r_1, r_2) \in (\mathbb{N}^*)^2$, there is no $p_0 \in \mathbb{N}^*$ such that (13) holds. Thus (13) is never satisfied.

6. Condition (14) is *not a necessary condition* for f to be a locally positive definite function. Consider

$$f(x_1, x_2) := x_1^2x_2^2 - x_1x_2^4 + x_1^6 + x_2^6. \quad (20)$$

If we pick $r = (r_1, r_2) \in (\mathbb{N}^*)^2$ with $r_1 \neq r_2$, then it is easy to see that there is no $p_0 \in \mathbb{N}^*$ such that (13) holds. Take now $r = (r_1, r_2) = (1, 1)$ and $p_0 = 6$. Then $H_6(x_1, x_2) = x_1^6 + x_2^6$ is positive definite so that (13) holds. From $|x_1x_2^4| \leq (x_1^2x_2^2 + x_2^6)/2$, we infer that

$$f(x_1, x_2) \geq \frac{1}{2}x_1^2x_2^2 + x_1^6 + \frac{1}{2}x_2^6 > 0 \text{ for } x \neq 0,$$

i.e. f is (globally) positive definite. However

$$\sum_{n < 6} H_n(x_1, x_1^{\frac{1}{4}}) = x_1^{\frac{5}{4}} - x_1^2 < 0 \text{ for } 0 < x_1 < 1.$$

The above remarks show that the converse of Corollary 2, namely

“ f locally positive definite $\Rightarrow \exists (r_1, \dots, r_n, p_0) \in (\mathbb{N}^*)^{n+1}$ such that (13) and (14) hold”

is not true. It appears that $\sum_{n \leq p_0} H_n$ can be locally positive definite by combining the strict positivity of H_{p_0} at some points, and the strict positivity of $\sum_{n < p_0} H_n$ at the other points.

The following result provides sufficient conditions for the existence of a high-order saddle point.

Proposition 3. *Let f be an analytic function around 0 expanded as in (1) with $f(0) = 0$. Assume the existence of a sequence of weights $(r_1, \dots, r_n) \in (\mathbb{N}^*)^n$ and of an integer $p_0 \in \mathbb{N}^*$ such that*

$$H_n = 0 \quad \forall n < p_0 \quad (21)$$

$$\exists x_1, x_2 \in S^{n-1} \quad H_{p_0}(x_1) > 0 \text{ and } H_{p_0}(x_2) < 0. \quad (22)$$

Then there exists $t_0 > 0$ such that

$$f(\delta_t^r(x_1)) > 0 \quad \forall t \in (0, t_0), \quad (23)$$

$$f(\delta_t^r(x_2)) < 0 \quad \forall t \in (0, t_0). \quad (24)$$

In particular, 0 is neither a local minimum nor a local maximum of f around $x = 0$.

Proof. This is a direct consequence of

$$f(\delta_t^r(x_j)) = t^{p_0} H_{p_0}(x_j) + \sum_{n > p_0} H_n(\delta_t^r(x_j)) = t^{p_0} H_{p_0}(x_j) + O(t^{p_0+1}) \text{ as } t \rightarrow 0$$

for $j = 1, 2$. □

3 Positive definiteness of a polynomial function

3.1 Tarski-Seidenberg theorem

In this section, we assume that $f \in \mathbb{R}[x_1, \dots, x_n]$, $n \geq 1$ and that $f(0) = 0$. We can expand f as

$$f(x) = \sum_{e \in \mathbb{N}^n, 0 < |e| \leq d} a_e x^e \quad \text{for } x \in \mathbb{R}^n,$$

where d denotes the degree of f . The polynomial function f is (locally) positive definite around the origin if and only if

$$\exists \varepsilon > 0 \quad \{x \in \mathbb{R}^n; 0 < \|x\|^2 < \varepsilon^2 \text{ and } f(x) \leq 0\} = \emptyset.$$

It turns out that the set $\{x \in \mathbb{R}^n; 0 < \|x\|^2 < \varepsilon^2 \text{ and } f(x) \leq 0\}$ is semi-algebraic, so that the classical results in real algebraic geometry can be used to decide whether it is empty for some $\varepsilon > 0$.

Let us give first a few definitions and results. (We follow closely [11].)

A set in \mathbb{R}^n is said to be *semi-algebraic* if it is a finite union of finite intersections of sets defined by a polynomial equation (i.e. $\{x \in \mathbb{R}^n; P(x) = 0\}$) or inequality (i.e. $\{x \in \mathbb{R}^n; P(x) > 0\}$ or $\{x \in \mathbb{R}^n; P(x) \geq 0\}$).

It is easy to see that a finite union (resp. a finite intersection) of semi-algebraic sets is also semi-algebraic, and that the complement of a semi-algebraic set is also semi-algebraic. For $n, m \in \mathbb{N}^*$, the linear map $\pi : (x, y) \in \mathbb{R}^{n+m} = \mathbb{R}^n \times \mathbb{R}^m \rightarrow y \in \mathbb{R}^m$ is a projection. It turns out that projections of semi-algebraic sets are also semi-algebraic, according to Tarski-Seidenberg theorem (see e.g. [5,6,11,17,18]):

Theorem 2. (*Tarski-Seidenberg*) *If E is a semi-algebraic set in \mathbb{R}^{n+m} and if π denotes the projection $\pi : (x, y) \in \mathbb{R}^{n+m} \rightarrow y \in \mathbb{R}^m$, then $\pi(E)$ is a semi-algebraic set in \mathbb{R}^m .*

We notice that the above result can be stated as follows: the set

$$\{y \in \mathbb{R}^m; \exists x \in \mathbb{R}^n, (x, y) \in E\}$$

is semi-algebraic. Furthermore, we can replace \exists by \forall , for the complement of a semi-algebraic set is semi-algebraic. Thus eliminating finitely many symbols among \exists, \forall at the beginning of any formula involving a semi-algebraic set yields a semi-algebraic set.

For our problem, we infer from Tarski-Seidenberg theorem that the set

$$F := \{\varepsilon \in \mathbb{R}; \forall x \in \mathbb{R}^n, (x, \varepsilon) \in E\}$$

is semi-algebraic, where we take

$$E := \{(x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}; \varepsilon > 0 \text{ and } (\|x\|^2 \geq \varepsilon^2 \text{ or } x = 0 \text{ or } f(x) > 0)\}.$$

Using algorithms as those in [7,9,14,16], one can obtain the polynomial functions of the variable $\varepsilon \in \mathbb{R}$ defining the set F . Note that, F being a semi-algebraic set in \mathbb{R} , it is a finite union of intervals. It remains to determine those intervals using for instance Sturm theorem (see below) to localize roughly the roots of the polynomial functions involved in the definition of F , and Newton's method to find the roots with more accuracy. This allows us to answer the question of whether F is nonempty (i.e. the function f is locally positive definite).

Alternatively, one can avoid determining precisely the set F by replacing a numerical coefficient in the expansion of f , say a_{e_0} with $0 < |e_0| \leq d$, by a variable denoted by $y \in \mathbb{R}$. Let $f_y(x) := \sum_{0 < |e| \leq d, e \neq e_0} a_e x^e + y x^{e_0}$ and

$$G := \{(x, \varepsilon, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}; \varepsilon > 0 \text{ and } (\|x\|^2 \geq \varepsilon^2 \text{ or } x = 0 \text{ or } f_y(x) > 0)\}.$$

The set G is clearly semi-algebraic. It follows that the set

$$H := \{y \in \mathbb{R}; \exists \varepsilon \in \mathbb{R}, \forall x \in \mathbb{R}^n, (x, \varepsilon, y) \in G\}$$

is also semi-algebraic. After having determined the polynomial functions involved in the definition of H , it remains to test whether $a_{e_0} \in H$ or not (which is easy).

Example 2. Let us consider the quadratic function $f(x) = \|x\|^2$. Let us denote $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Pick $e_0 = (0, \dots, 2)$. Then one readily sees that

$$\begin{aligned} E &= \{(x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}; \varepsilon > 0 \text{ and } (\|x\|^2 \geq \varepsilon^2 \text{ or } x = 0 \text{ or } \|x\|^2 > 0)\} \\ &= \{(x, \varepsilon) \in \mathbb{R}^n \times \mathbb{R}; \varepsilon > 0\}, \\ F &= (0, +\infty), \\ G &= \{(x, \varepsilon, y) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}; \varepsilon > 0 \text{ and } (\|x\|^2 \geq \varepsilon^2 \text{ or } x = 0 \text{ or } \|x'\|^2 + yx_n^2 > 0)\}, \\ H &= (0, +\infty). \end{aligned}$$

We stress that the sets F and H are expected to be derived by using algorithms inspired by Tarski-Seidenberg theorem, but here they can be obtained directly. We notice that F is nonempty, and that $1 \in H$, so that both approaches give that f is locally positive definite.

3.2 Sturm theorem

The proofs of Tarski-Seidenberg theorem in [5,6,11] rest on the analysis of the changes of signs of sequences of polynomial functions of one variable, which is inspired by Sturm theorem. It turns out that we can use directly Sturm theorem to test whether the polynomial function is positive definite, without using algorithms to describe the projections of semi-algebraic sets.

The Sturm sequence associated with a polynomial function $P \in \mathbb{R}[x]$ is defined inductively as follows. First, we take $P_0 := P$ and $P_1 := P'$. If P_{i-2} and P_{i-1} are defined with $P_{i-1} \neq 0$, we pick $P_i := -R$, where R is obtained by doing the Euclidean division of P_{i-2} by P_{i-1} , namely

$$P_{i-2} = P_{i-1}Q + R, \quad Q, R \in \mathbb{R}[x], \quad \deg(R) < \deg(P_{i-1}).$$

Thus the Sturm sequence reads (P_0, \dots, P_k) with $P_0 = P$, $\deg(P_0) > \deg(P_1) > \dots > \deg(P_k)$, $P_k \neq 0$, and $P_k | P_{k-1}$.

For $a \in \mathbb{R}$ with $P(a) \neq 0$ we denote by $N(a)$ the number of changes of signs in the sequence $(P_0(a), \dots, P_k(a))$, i.e.

$$\begin{aligned} N(a) &= \#\{i \in [[0, k-1]]; \exists j \in [[i+1, k]], P_i(a)P_j(a) < 0 \text{ and} \\ &\quad \forall l \in [[i+1, j-1]] P_l(a) = 0\}. \end{aligned}$$

Then Sturm theorem (see e.g. [5]) can be stated as follows.

Theorem 3. (*Sturm*) Let $P \in \mathbb{R}[x]$ with no multiple root and let $a, b \in \mathbb{R}$ with $P(a)P(b) \neq 0$. Then the number of real roots of P in $[a, b]$ is $N(a) - N(b)$.

Remark 5. 1. Saying that P has no multiple root is equivalent to saying that P and P' are relatively prime, or that $P_k \in \mathbb{R}^*$.
2. If P has multiple roots, then $P_k \notin \mathbb{R}^*$ is (up to the sign) the greatest common divisor of P and P' . It is then sufficient to apply Sturm theorem to $\tilde{P} := P/P_k$. Indeed, the roots of \tilde{P} are simple and are the same as those of P .

The main advantage of that result is that, after computing formally the Sturm sequence, one can compute easily the *number* of roots in some interval $[a, b]$ without having to compute the roots (which may be a hard task).

In our problem, one roughly has to check that there is no root in some interval $[a, b]$. More precisely, for given $\varepsilon > 0$, we set $C_\varepsilon := [-\varepsilon, \varepsilon]^n$. We want to test whether

$$\forall x \in C_\varepsilon \setminus \{0\} \quad f(x) > 0. \quad (25)$$

If $f(x) > 0$ for all $x \in \partial C_\varepsilon$, then (25) is equivalent to

$$\forall x' \in (-\varepsilon, \varepsilon)^{n-1}, \quad \#\{x_n \in [-\varepsilon, \varepsilon]; f(x', x_n) = 0\} = \begin{cases} 0 & \text{if } x' \neq 0; \\ 1 & \text{if } x' = 0. \end{cases} \quad (26)$$

The condition in (26) can be checked by using Sturm theorem for the variable $x_n \in [-\varepsilon, \varepsilon]$.

In practice, we check that $f(x) > 0$ for any extremal point $x \in \{-\varepsilon, \varepsilon\}^n$, next for x in the segment between two extremal points by using Sturm theorem, etc. until proving that $f(x) > 0$ for all $x \in \partial C_\varepsilon$, or merely for $x \in [-\varepsilon, \varepsilon]^{n-1} \times \{-\varepsilon, \varepsilon\}$. The last step is the verification of (26).

Let us consider two examples.

Example 3. Consider first the polynomial function

$$f(x_1, x_2) = x_1^4 + x_1 x_2^3 + x_2^2.$$

Pick $\varepsilon = 1$, so that $C_\varepsilon = [-1, 1]^2$. Then

$$f(1, 1) = 3, \quad f(1, -1) = 1, \quad f(-1, 1) = 1, \quad f(-1, -1) = 3.$$

Let us compute the values of f on ∂C_ε . We find

$$\begin{aligned} f(x_1, \pm 1) &= x_1^4 + (1 \pm x_1) > 0 \quad \forall x_1 \in [-1, 1], \\ f(\pm 1, x_2) &= (1 \pm x_2^3) + x_2^2 > 0 \quad \forall x_2 \in [-1, 1]. \end{aligned}$$

Thus $f(x) > 0$ for all $x \in \partial C_\varepsilon$.

Let us compute the Sturm sequence for the variable x_1 . We pick $P_0(x_1) = x_1^4 + x_1 x_2^3 + x_2^2$ and $P_1(x_1) = \partial P_0 / \partial x_1 = 4x_1^3 + x_2^3$. The Euclidean division of P_0 by P_1 reads

$$x_1^4 + x_1 x_2^3 + x_2^2 = (4x_1^3 + x_2^3) \left(\frac{1}{4} x_1 \right) + \frac{3}{4} x_1 x_2^3 + x_2^2,$$

We obtain $R(x_1) = \frac{3}{4} x_1 x_2^3 + x_2^2$. We have two cases to consider.

(i) if $x_2 = 0$, then $R = 0$, so that $P_1 | P_0$. We notice that the map $x_1 \rightarrow f(x_1, 0) = x_1^4$ has one root in $[-1, 1]$.

(ii) if $x_2 \neq 0$, then $R \neq 0$ and $P_2(x_1) = -\frac{3}{4} x_1 x_2^3 - x_2^2$. The Euclidean division of P_1 by P_2 reads

$$4x_1^3 + x_2^3 = \left(-\frac{3}{4} x_1 x_2^3 - x_2^2 \right) \left(-\frac{16}{3} \frac{x_1^2}{x_2^3} + \frac{64}{9} \frac{x_1}{x_2^4} - \frac{256}{27} \frac{1}{x_2^5} \right) + x_2^3 - \frac{256}{27 x_2^3}.$$

Note that $x_2^3 - \frac{256}{27x_2^3} < 0$ for all $x_2 \in [-1, 1]$.

Therefore $P_3(x_1) = -x_2^3 + \frac{256}{27x_2^3} \in \mathbb{R}^*$ and we do not have to compute P_4 . We find

$$P_0(-1) = 1 - x_2^3 + x_2^2 > 0, \quad P_1(-1) = -4 + x_2^3 < 0, \quad P_2(-1) = \frac{3}{4}x_2^3 - x_2^2 < 0$$

and $P_3(-1) = -x_2^3 + \frac{256}{27x_2^3} > 0$, so that $N(-1) = 2$. On the other hand,

$$P_0(1) = 1 + x_2^3 + x_2^2 > 0, \quad P_1(1) = 4 + x_2^3 > 0, \quad P_2(1) = -\frac{3}{4}x_2^3 - x_2^2 < 0, \quad P_3(1) > 0,$$

so that $N(1) = 2$. Since $N(-1) - N(1) = 0$, the function $x_1 \rightarrow f(x_1, x_2)$ has no root in $[-1, 1]$ for $x_2 \in [-1, 1] \setminus \{0\}$. This shows that the function f is locally positive definite.

Example 4. Consider again the polynomial function $f(x_1, x_2) = x_1^2 - 4x_1x_2^2 + x_2^4$ from Remark 1. If $0 < \varepsilon \ll 1$ and $x_1 = \pm\varepsilon$, we have that $|x_2^4 - 4x_1x_2^2| \leq C\varepsilon^3$ and $f(\pm\varepsilon, x_2) > 0$ for $x_2 \in [-\varepsilon, \varepsilon]$. Pick $P_0(x_1) = x_1^2 - 4x_1x_2^2 + x_2^4$, $P_1(x_1) = \partial P_0 / \partial x_1 = 2x_1 - 4x_2^2$. The Euclidean division of P_0 by P_1 reads

$$x_1^2 - 4x_1x_2^2 + x_2^4 = (2x_1 - 4x_2^2)\left(\frac{1}{2}x_1 - x_2^2\right) - 3x_2^4,$$

and hence $P_2(x_1) = 3x_2^4 \in \mathbb{R}^*$ if $x_2 \neq 0$.

If $x_2 = 0$, the map $x_1 \rightarrow f(x_1, 0) = x_1^2$ has one root in $[-\varepsilon, \varepsilon]$. If $x_2 \in [-\varepsilon, \varepsilon] \setminus \{0\}$, we have that

$$P_0(-\varepsilon) = \varepsilon^2 + 4\varepsilon x_2^2 + x_2^4 > 0, \quad P_1(-\varepsilon) = -2\varepsilon - 4x_2^2 < 0, \quad P_2(-\varepsilon) = 3x_2^4 > 0$$

so that $N(-\varepsilon) = 2$, and that for $\varepsilon \ll 1$,

$$P_0(\varepsilon) = \varepsilon^2 - 4\varepsilon x_2^2 + x_2^4 > 0, \quad P_1(\varepsilon) = 2\varepsilon - 4x_2^2 > 0, \quad P_2(\varepsilon) = 3x_2^4 > 0$$

so that $N(\varepsilon) = 0$. It follows that the map $x_1 \rightarrow f(x_1, x_2)$ has two roots in $[-\varepsilon, \varepsilon]$, and hence the function f is not positive definite.

4 Conclusion

In this paper, we provided high-order conditions for an analytic function to be locally positive definite at the origin. We provided necessary conditions and sufficient conditions for the local positive definiteness, but a necessary and sufficient condition seems hard to find. Nevertheless, in the subclass of polynomial functions, the theorem of Tarski-Seidenberg and those of Sturm can be used to decide whether a given polynomial function is locally positive definite. However, the corresponding computations using iterative projections on subspaces are involved and they do not use the structure of the function. Furthermore, they do not provide any bound from below of the function.

References

1. A. Bacciotti, L. Rosier, *Liapunov Functions and Stability in Control Theory*, 2nd Edition, Springer-Verlag Berlin Heidelberg 2005.
2. S. Basu, R. Pollack, M.-F. Roy. *Algorithms in Real Algebraic Geometry*. Springer-Verlag, 2003.
3. R. Benedetti, J.-J. Risler. *Real Algebraic and Semi-algebraic Sets*. Hermann, 1990.
4. E. Bierstone, P. D. Milman, *Semianalytic and Subanalytic Sets*, Publ. Math. I.H.E.S. **67** (1988), 5–42.
5. J. Bochnak, M. Coste, M.-F. Roy. *Real Algebraic Geometry*. Ergebnisse der Mathematik und ihrer Grenzgebiete. Springer-Verlag 1988.
6. P. Cohen, *A simple proof of Tarski's theorem on elementary algebra*. Mimeographed manuscript, Stanford University 1967, 6 pp.
7. D. Grigoriev, N. Vorobjov, *Solving systems of polynomial inequalities in subexponential time*, Journal of Symbolic Computation 2006.
8. J. Gwoździewicz, *The Lojasiewicz exponent of an analytic function at an isolated zero*, Comment. Math. Helv. **74** (1999), no. 3, 364–375.
9. J. Heintz, M.-F. Roy, P. Solernò, *Sur la complexité du principe de Tarski-Seidenberg*, Bull. Soc. Math. France **118** (1990), no. 1, 101–126.
10. G. Hochschild. *The structure of Lie groups*. Holden-Day, Inc., San Francisco London Amsterdam 1965.
11. L. Hörmander. *The Analysis of Linear Partial Operators II*. Springer-Verlag Berlin Heidelberg 1983.
12. J. Kollár, *An effective Lojasiewicz inequality for real polynomials*, Period. Math. Hungar. **38** (1999), no. 3, 213–221.
13. S. Lojasiewicz, *Ensembles semi-analytiques*, preprint IHES, Bures-sur-Yvette, 1965.
14. J. Renegar, *On the computational complexity and geometry of the first order theory of the reals*, Technical Report 856, Cornell University Ithaca (1988).
15. L. Rosier. *Calcul Différentiel et Applications*. ISTE editions, London 2024.
16. M. Safey El Din, *Algorithmes efficaces en géométrie algébrique réelle*, Journées Nationales du Calcul Formel 2007.
17. A. Seidenberg, *A new decision method for elementary algebra*. Ann. of Math. **60**, 365–374 (1954).
18. A. Tarski, *A decision method for elementary algebra and geometry*, Manuscript, Berkeley, 63 pp. (1951).
19. B. Teissier, *Variétés polaires. I. Invariants polaires des singularités d'hypersurfaces*, Invent. Math. **40** (1977), no. 3, 267–292.
20. J. Vauthier, J.-J. Prat. *Cours d'analyse mathématique de l'agrégation*. Masson Paris 1992.