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# Conditions for QSR-dissipativity of the interconnection of hybrid systems with the sum of storage functions

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**Abstract**—This note explores the QSR-dissipativity of a class of hybrid dynamical systems. We first revisit and extend some existing definitions of dissipativity for hybrid systems. Subsequently, we explore some results regarding the relations between the QSR-dissipativity property of one open hybrid subsystem and its stability. Next, we shift focus to studying the negative-feedback interconnection of two hybrid subsystems, deriving new results about the existence of storage functions guaranteeing that the interconnection between two individual QSR-dissipative hybrid subsystems remains QSR-dissipative in the presence of external input signals. Analyzing such interconnections of hybrid systems may be intricate due to the possibility of both synchronous and asynchronous jumps occurring among the two subsystems, as previously acknowledged by the existing literature.

## I. INTRODUCTION

For an autonomous dynamical system, roughly speaking isolated from their environment, a Lyapunov function is an abstract energy that is of crucial importance for its stability analysis. This energy approach has been extended to the energy balance of input–state–output systems with their environment by Willems [1], [2], and also by Hill and Moylan [3]. These input–state–output systems are also called *open* in opposition to autonomous/isolated systems and allow exchanges of energy with the environment, by following a vocabulary from Physics. The intuitive idea is that an open dynamical system interacts with its environment through its inputs and outputs: a given function of these variables (the supply rate) represents how a relevant quantity (abstract energy) flows in and out of the system. The supply is split into two parts: a part is stored and an other one is dissipated. The stored amount (storage function) is a function of the state of the system.

Two of the most important features of dissipativity are that, under suitable conditions, a dissipative system without input might be stable (the storage function leading naturally to a potential candidate Lyapunov function) [3], [4] and that dissipativity can be much more easily propagated structurally by connections of subsystems, in contrast to the stability properties. The study of dissipativity theory and its fundamental role in analyzing and controlling dynamical systems is also extremely active. Several interesting and strong contributions were recently published, whereas a two-part

special edition of the IEEE Control Systems Magazine (CSM) was entirely dedicated to the subject in 2022, [5], [6].

A large attention has been dedicated to extend the dissipativity to complex systems such as constrained systems, impulsive systems, switched systems or hybrid systems, with a particular focus on passivity, which is a particular case of dissipativity. Among the large literature, we can cite, as surveys, [7]–[9] in addition to significant papers related to hybrid automatas [10], [11], the dissipativity of impulsive systems from [12]–[14], the passivity of hybrid systems [9], [15]–[17], the dissipativity of switched systems [18]–[20]. Generally the class of supply rates is chosen to be the class of quadratic forms with respect to the extended vector gathering the inputs and the outputs. Such a choice, referred to as *Quadratic Supply Rate (QSR)-dissipativity*, allows to obtain outstanding results dealing with the QSR-dissipativity of the connection between subsystems.

This note aims to revisit QSR-dissipativity for hybrid dynamical systems when using the formalism provided in [21]. Generic and new sufficient conditions for the propagation of the QSR-dissipativity for the negative feedback interconnection are provided. The two cases considering QSR-dissipativity with a prescribed hybrid supply rate or *some* hybrid supply rate are investigated. These conditions consist in the unification, with a coherent formalism, of several results in the literature for various frameworks and various types of systems.

The contributions of this paper with respect to the recent literature are as follows:

- Extension of the result dealing with the passivity in [16], [17] to the framework of dissipativity;
- Extension of the framework of dissipativity for switched systems to the framework of hybrid systems [18];
- Consider the results in [10] with the formalism of [21], in order to cope also with jump-dissipativity and jumps of the state.
- Consider interconnections of hybrid systems with outputs and exogenous inputs, as opposed to the setting obtained as a decomposition of an autonomous system in [4], where supply rates are neither functions of the outputs nor of external inputs.

The paper is organized as follows. Section II presents firstly the framework of hybrid system and (QSR)-dissipativity and recalls secondly the strong link with stability. Section III provides the main unification result gathering the conditions to ensure the QSR-dissipativity of the negative feedback interconnection of two hybrid subsystems. Section IV is dedicated to a collection of comments and corollaries recovering known

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results of the literature. Finally Section V concludes the paper and offers a few perspectives.

**Notation.** The sets  $\mathbb{R}$ ,  $\mathbb{R}_{\geq 0}$ ,  $\mathbb{R}^n$  and  $\mathbb{R}^{n \times m}$  denote the set of real numbers, the set of real non-negative numbers, the  $n$ -dimensional Euclidean space and the set of all real  $n \times m$  matrices, respectively. For a matrix  $Y$ ,  $Y^\top$  means its transpose.  $\mathbb{S}_n$  stands for the set of symmetric matrices of dimension  $n$ . For matrices  $W, Z \in \mathbb{S}_n$ ,  $W \succ Z$  ( $W \succeq Z$ ) means that  $W - Z$  is positive (semi-)definite, and  $W \prec Z$  means that  $W - Z$  is negative definite.  $I_n$  and  $0_{n \times m}$  denote identity matrix of dimension  $n \times n$  and null matrix of dimension  $n \times m$ , with  $0_n = 0_{n \times n}$ . The  $\star$  in the expression of a matrix denotes symmetric blocks. For square matrices  $W$  and  $Z$ ,  $\text{diag}(W, Z)$  corresponds to the block-diagonal matrix. For  $x \in \mathbb{R}^n$ ,  $|x|$  denotes its Euclidean norm and given a closed set  $\mathcal{A} \subset \mathbb{R}^n$ ,  $|x|_{\mathcal{A}} = \min_{z \in \mathcal{A}} |x - z|$  denotes the distance from  $x$  to the set  $\mathcal{A}$ . A function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class- $\mathcal{K}$  if it is continuous, zero at zero and strictly increasing. A function  $\alpha : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0}$  is of class- $\mathcal{K}_\infty$ , if it is of class- $\mathcal{K}$  and if  $\lim_{t \rightarrow +\infty} \alpha(t) = +\infty$ . For a set  $S$ ,  $\bar{S}$  denotes its closure.

## II. PRELIMINARIES

This section introduces the studied setting and recalls the necessary definitions and tools.

### A. Preliminaries on dissipativity of hybrid systems

An open (or input-state-output) hybrid dynamical system may be defined, as in [21], by

$$\tilde{\mathcal{H}} : \begin{cases} \dot{x} \in \tilde{F}(x, u_c), & (x, u_c) \in \tilde{C}, \\ x^+ \in \tilde{G}(x, u_d), & (x, u_d) \in \tilde{D}, \\ y_c = \tilde{h}_c(x), \\ y_d = \tilde{h}_d(x). \end{cases} \quad (1)$$

In  $\tilde{\mathcal{H}}$ , depicted on Fig. 1,  $x \in \mathbb{R}^n$  is the state,  $u_c \in \mathbb{R}^{m_c}$  and  $u_d \in \mathbb{R}^{m_d}$  are the controlled inputs for the flows and jumps, which may be gathered in a concatenated input vector  $u = [u_c^\top \ u_d^\top]^\top \in \mathbb{R}^{m_c+m_d}$ , while  $y_c \in \mathbb{R}^{p_c}$  and  $y_d \in \mathbb{R}^{p_d}$  are the outputs, which may be gathered in a concatenated output  $y = [y_c^\top \ y_d^\top]^\top \in \mathbb{R}^{p_c+p_d}$  and are functions of the state thanks respectively to functions  $\tilde{h}_c : \mathbb{R}^n \rightarrow \mathbb{R}^{p_c}$  and  $\tilde{h}_d : \mathbb{R}^n \rightarrow \mathbb{R}^{p_d}$ . The set  $\tilde{C} \subset \mathbb{R}^n \times \mathbb{R}^{m_c}$  is the flow set and the set-valued mapping  $\tilde{F} : \mathbb{R}^n \times \mathbb{R}^{m_c} \rightrightarrows \mathbb{R}^n$  is the flow map. The set  $\tilde{D} \subset \mathbb{R}^n \times \mathbb{R}^{m_d}$  is the jump set and the set-valued mapping  $\tilde{G} : \mathbb{R}^n \times \mathbb{R}^{m_d} \rightrightarrows \mathbb{R}^n$  is the jump map. In the sequel, we assume the *hybrid basic conditions* [21, Section 6.2] that ensure the nominal well-posedness.

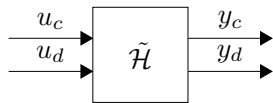


Fig. 1. An open hybrid system  $\tilde{\mathcal{H}}$ .

Dissipativity basically means that the variation of the energy stored in the system is no more than the supplied

energy from outside the system. Besides this intuitive idea, there exists a large variety of points of view for dissipativity leading to several types of formalism and definitions. In his seminal paper [1], Willems presented two main approaches: the definition of dissipativity with a storage function and the related dissipation inequality and also the definition of dissipativity free of a storage function by a fixed lower bound for the integral of supply rate. Under suitable conditions, such as controllability, Dynamics Programming establishes bridges between the two definitions (see [1], [7], [22]). Here we will consider the local dissipation inequalities point of view.

First of all, let us define a candidate storage function related to a compact set.

**Definition 1.** [9, Definition 9.3] Let  $\mathcal{A} \subset \mathbb{R}^n$  be a compact set. The function  $V : \text{dom}V \rightarrow \mathbb{R}_{\geq 0}$  is a candidate storage function with respect to  $\mathcal{A}$  associated to the hybrid system (1) if the following conditions hold:

- $\Pi_c(\tilde{C}) \cup \Pi_d(\tilde{D}) \cup \tilde{G}(\tilde{D}) \subset \text{dom}V$ , where for a set  $S_c \subset \mathbb{R}^n \times \mathbb{R}^{m_c}$ ,  $\Pi_c(S_c) := \{x \in \mathbb{R}^n, \exists u_c \in \mathbb{R}^{m_c}, (x, u_c) \in S_c\}$  and for a set  $S_d \subset \mathbb{R}^n \times \mathbb{R}^{m_d}$ ,  $\Pi_d(S_d) := \{x \in \mathbb{R}^n, \exists u_d \in \mathbb{R}^{m_d}, (x, u_d) \in S_d\}$ ;
- $V$  is *continuously differentiable*<sup>1</sup> on an open set containing  $\Pi_c(\tilde{C})$ ;
- there exists a  $\mathcal{K}$ -function  $\alpha_1$  such that  $\alpha_1(|x|_{\mathcal{A}}) \leq V(x)$ , for all  $x \in \Pi_c(\tilde{C}) \cup \Pi_d(\tilde{D}) \cup \tilde{G}(\tilde{D})$  and  $V(\mathcal{A}) = 0$ . Roughly speaking<sup>2</sup>,  $V$  is positive definite with respect to  $\mathcal{A}$  on  $\Pi_c(\tilde{C}) \cup \Pi_d(\tilde{D}) \cup \tilde{G}(\tilde{D})$ .

By following [8, Chapters 3 and 12] and [4], [16], we have the next definitions.

**Definition 2.** Consider two functions  $r_c : \mathbb{R}^{m_c} \times \mathbb{R}^{p_c} \rightarrow \mathbb{R}$  and  $r_d : \mathbb{R}^{m_d} \times \mathbb{R}^{p_d} \rightarrow \mathbb{R}$ . A hybrid system  $\tilde{\mathcal{H}}$  described by (1) is said to be dissipative with respect to a couple of supply rates  $(r_c, r_d)$  ( $r_c$  and  $r_d$  are then respectively called continuous and discrete supply rates, or flow and jump supply rates), if there exists a candidate storage function  $V : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$ , that satisfies:

$$\begin{cases} \langle \nabla V(x); \xi \rangle \leq r_c(u_c, y_c), \forall (x, u_c) \in \tilde{C}, \forall \xi \in \tilde{F}(x, u_c), \\ V(\eta) - V(x) \leq r_d(u_d, y_d), \forall (x, u_d) \in \tilde{D}, \forall \eta \in \tilde{G}(x, u_d). \end{cases} \quad (2)$$

$V$  is thus called a storage function for  $\tilde{\mathcal{H}}$ . Furthermore, it is said to be “flow-dissipative” (respectively, “jump-dissipative”) if it is dissipative with  $r_d \equiv 0$  (respectively,  $r_c \equiv 0$ ). Finally, the hybrid system  $\tilde{\mathcal{H}}$  is called strictly dissipative if it is dissipative and there exist two positive definite functions with respect to  $\mathcal{A}$ ,  $\rho_c, \rho_d : \mathbb{R}^n \rightarrow \mathbb{R}_{\geq 0}$  such that the storage function satisfies

$$\begin{cases} \langle \nabla V(x); \xi \rangle + \rho_c(x) \leq r_c(u_c, y_c), \\ \quad \forall (x, u_c) \in \tilde{C}, \forall \xi \in \tilde{F}(x, u_c), \\ V(\eta) - V(x) + \rho_d(x) \leq r_d(u_d, y_d), \\ \quad \forall (x, u_d) \in \tilde{D}, \forall \eta \in \tilde{G}(x, u_d). \end{cases} \quad (3)$$

<sup>1</sup>This assumption may be relaxed to  $V$  continuous and local Lipschitz, see [9, Definition 9.3]

<sup>2</sup>See [9, Definition A.26]. Given two nonempty sets  $\mathcal{A} \subset \mathcal{S}$ , a function  $V : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$  is said to be positive definite with respect to  $\mathcal{A}$  on  $\mathcal{S}$ , if  $V(x) > 0$  for all  $x \in \mathcal{S}/\mathcal{A}$  and  $V(\mathcal{A}) = 0$ .

An important special case of dissipativity is passivity. Indeed, passivity has played a crucial role in the field of control systems theory thanks to its structural properties: strong link to the stability of a system; system's performance improved by a feedback control (see for instance [23] for passivity-based control (PBC) techniques). In the case of hybrid systems, the following definition takes place for passivity [17].

**Definition 3.** A hybrid system  $\tilde{\mathcal{H}}$ , with  $m_c = p_c$  and  $m_d = p_d$ , is called passive if it is dissipative with respect to a couple of supply rates  $(r_c, r_d)$  defined by  $r_c(u_c, y_c) = u_c^\top y_c$  and  $r_d(u_d, y_d) = u_d^\top y_d$ . Furthermore, it is called flow-passive (respectively, jump-passive) if it is passive with  $r_d \equiv 0$  (respectively,  $r_c \equiv 0$ ).

**Remark 1.** Passivity is motivated by a large class of practical systems, for which the supply rates correspond to the physical power, as a product of kinetic and potential values (as for instance electrical current and voltage) generally gathered as outputs and inputs of the system. Their dimensions are constrained to  $m_c = p_c$  and  $m_d = p_d$ .

Dissipativity with quadratic supply rates (QSR) offers a less restrictive analysis of dynamical systems as it encompasses passivity and other well-studied characteristics of dynamical systems such as  $l_2$ -gain analysis.

**Definition 4.** A hybrid system  $\tilde{\mathcal{H}}$  is called QSR-dissipative if it is dissipative with respect to a couple of supply rates  $(r_c, r_d)$ , called hybrid supply rate, given by

$$\begin{aligned} r_c(u_c, y_c) &= y_c^\top Q_c y_c + 2y_c^\top S_c u_c + u_c^\top R_c u_c, \\ r_d(u_d, y_d) &= y_d^\top Q_d y_d + 2y_d^\top S_d u_d + u_d^\top R_d u_d, \end{aligned}$$

for matrices  $Q_j \in \mathbb{S}_{p_j}$ ,  $S_j \in \mathbb{R}^{p_j \times m_j}$ ,  $R_j \in \mathbb{S}_{m_j}$ ,  $j \in \{c, d\}$ . Furthermore, it is called QSR-flow-dissipative (respectively, QSR-jump-dissipative) if it is QSR-dissipative with  $r_d \equiv 0$  (respectively,  $r_c \equiv 0$ ).

Note that, by definition, passive systems are also QSR-dissipative, but the converse is false. Besides allowing to study systems with different numbers of inputs and outputs, QSR-dissipativity also allows analyzing systems with unstable internal dynamics<sup>3</sup>, which is not possible in the passivity setting when the storage function is positive definite.

It should be underlined that the notion of dissipativity for hybrid systems is not intuitive. As an illustration, consider the particular case of the passivity of switched systems: Switching between passive systems may render the switched system not passive [24]. Conversely, by switching adequately, it is possible to render passive a switched system with non-passive modes [20]. See also [15], [18] and [25].

### B. Implication of QSR-dissipativity about stability

As mentioned in the work of Willems [1] and also Hill and Moylan [3], the dissipativity is useful for the stability analysis via Lyapunov method. First of all, the candidate storage

<sup>3</sup>i.e., the dynamics of the zero-input system  $u_c \equiv 0, u_d \equiv 0$ .

function with respect to a compact set  $\mathcal{A}$  is a natural choice for a candidate Lyapunov function. Nevertheless, the conditions in Definition 1 are not enough. We need to strengthen these assumptions (see [21, Definition 7.21]) by assuming that there exist  $\mathcal{K}_\infty$ -functions  $\alpha_1$  and  $\alpha_2$  such that the candidate storage function is bounded by:

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \leq \alpha_2(|x|_{\mathcal{A}}). \quad (4)$$

We can summarize the following results.

**Proposition 1.** Suppose that the hybrid system  $\tilde{\mathcal{H}}$  is strictly QSR-dissipative, with respect to a compact set  $\mathcal{A}$ , and that the storage function satisfies (4), with  $Q_c \preceq 0$  and  $Q_d \preceq 0$ , then its zero-input dynamics ( $u_c \equiv 0, u_d \equiv 0$ ) is asymptotically stable, with respect to  $\mathcal{A}$ .

*Proof.* The proof is obtained by rewriting Conditions (3) and by following [21].  $\square$

The strict QSR-dissipativity can be relaxed thanks to a zero-state detectability argument as provided in [3] and extended for hybrid system framework in [4].

**Proposition 2.** Suppose that the hybrid system  $\tilde{\mathcal{H}}$  is QSR-dissipative, that the storage function satisfied (4), and finally that  $\tilde{\mathcal{H}}$  is zero-state detectable. Then its zero-input dynamics ( $u_c \equiv 0, u_d \equiv 0$ ) is asymptotically stable if  $Q_c \prec 0$  and  $Q_d \prec 0$  and stable if  $Q_c \preceq 0$  and  $Q_d \preceq 0$ .

**Remark 2.** Passivity may be a restrictive property. By definition, it imposes  $m_c = p_c$  and  $m_d = p_d$ . Furthermore, with  $Q_c = 0$  and  $Q_d = 0$ , it establishes a strong link with stable systems thanks to the previous propositions. See [17] for detailed results.

**Remark 3.** It is noteworthy that generally the dissipativity is referred to as w.r.t a compact  $\mathcal{A}$ , see for instance [17] and [9, Chapter 9]. This is mainly related to the candidate storage function (Definition 1) and stability analysis purposes but the dissipation inequalities do not involve explicitly  $\mathcal{A}$ .

In addition to the consequences of dissipativity in terms of stability, one other fundamental property of dissipativity is its potential and conditional propagation by interconnecting systems, in contrast to the notion of stability. This statement is more challenging for the interconnection of hybrid systems, mainly due to the fact that they may jump asynchronously. The following section investigates the QSR-dissipativity of the negative feedback interconnection of hybrid systems.

## III. NEGATIVE FEEDBACK INTERCONNECTION OF HYBRID SYSTEMS

Among the most significant interconnections in control theory, it is natural to consider the negative feedback-interconnection of two open-hybrid systems  $\mathcal{H}_i$  of states  $x_i$ ,  $i \in \{1, 2\}$ , each of them described by (1). In order to allow the connections graphically represented in Fig. 2, we impose  $m_{j,1} = p_{j,2}$ ,  $m_{j,2} = p_{j,1}$ , for all  $j \in \{c, d\}$ <sup>4</sup>. Roughly

<sup>4</sup>Note that, differently from passivity, QSR-dissipativity allows for  $m_{j,1} \neq p_{j,1}$  and  $m_{j,2} \neq p_{j,2}$ , for  $j \in \{c, d\}$ , that is, for different number of inputs and outputs in each subsystem.



speaking, we have the connections

$$\begin{cases} u_{j,1} = \tilde{u}_{j,1} - y_{j,2}, \\ u_{j,2} = \tilde{u}_{j,2} + y_{j,1}, \end{cases} \quad \forall j \in \{c, d\}, \quad (5)$$

where  $\tilde{u}_{j,1}$  and  $\tilde{u}_{j,2}$  are additional external inputs.

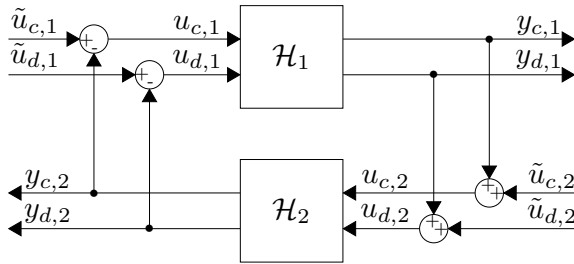


Fig. 2. Negative feedback interconnection  $\mathcal{H}_{12}$  of two hybrid systems  $\mathcal{H}_1$  and  $\mathcal{H}_2$ .

That results in a new augmented open hybrid system  $\mathcal{H}_{12}$  described by (1) by setting,  $\forall j \in \{c, d\}$ :

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad u_j = \begin{bmatrix} u_{j,1} \\ u_{j,2} \end{bmatrix}, \quad \tilde{u}_j = \begin{bmatrix} \tilde{u}_{j,1} \\ \tilde{u}_{j,2} \end{bmatrix}, \quad y_j = \begin{bmatrix} y_{j,1} \\ y_{j,2} \end{bmatrix}$$

$$F(x, \tilde{u}_c) = \begin{bmatrix} \tilde{F}_1(x_1, \tilde{u}_{c,1} - y_{c,2}) \\ \tilde{F}_2(x_2, \tilde{u}_{c,2} + y_{c,1}) \end{bmatrix},$$

$G(x, \tilde{u}_d)$  defined by (11) and flow and jump sets  $C$  and  $D$  defined, respectively, by (12) and (13). Note that the expression of the jump map  $G(x, \tilde{u}_d)$  is more intricate due to the fact that the jumps of the two hybrid subsystems  $\mathcal{H}_1$  and  $\mathcal{H}_2$  may occur simultaneously or asynchronously.

Such interconnection between open hybrid systems plays a crucial role in the study of networks of systems. It may be also viewed as the fundamental negative closed loop between a hybrid system and a controller that is also a hybrid system. Such interconnection should be considered with care, see [26] for a detailed discussion. For instance, we may have the following issues:

- The interconnection between hybrid systems may have (new) Zeno solutions, that do not belong to the set of solutions to every subsystem; and in this case reduce the hybrid time-domain of the solutions;
- The time domain of the solutions of the interconnection and the time domain of the individual subsystems may be of different nature due to possible empty individual jump- or flow-sets.

Several particular frameworks of QSR-dissipative (passivity [17] or small-gain theorems [26]–[28]) have been well studied. Nevertheless, the generic QSR-dissipativity of interconnected hybrid systems is under-examined and needs additional clarifications. We can only cite [10], [11] which consider a specific finite and hybrid automata.

Physically, it is natural to think that the energy of the interconnected system is the sum of the energies of the subsystems composing the interconnection. Here because the storage function is an abstract energy, it is natural to look for weighted sums of the storage functions of the subsystems for the storage function of the interconnection,

that is  $V(x) = \alpha_1 V_1(x_1) + \alpha_2 V_2(x_2)$ , with  $\alpha_1 > 0$  and  $\alpha_2 > 0$ . This class of storage functions may be normalized by setting  $\alpha_1 = 1$  and  $\alpha_2 = \alpha > 0$ .

The following theorem provides results about the QSR-dissipativity of the negative feedback interconnection in Fig. 2 and defined by (5)–(13). Corollaries concerning QSR-flow-dissipativity and QSR-jump-dissipativity will follow in the next section.

**Theorem 1.** Assume that, for each  $i \in \{1, 2\}$ ,  $\mathcal{H}_i$  is QSR-dissipative with  $C^1$ -storage function  $V_i : \mathbb{R}^{n_i} \rightarrow \mathbb{R}_{\geq 0}$ , with respect to a compact set  $\mathcal{A}_i$  and associated to the continuous supply rate  $r_{c,i}$ , defined by matrices  $Q_{c,i} \in \mathbb{S}_{p_{c,i}}$ ,  $S_{c,i} \in \mathbb{R}^{p_{c,i} \times m_{c,i}}$  and  $R_{c,i} \in \mathbb{S}_{m_{c,i}}$ , and to the discrete supply rate  $r_{d,i}$ , defined by matrices  $Q_{d,i} \in \mathbb{S}_{p_{d,i}}$ ,  $S_{d,i} \in \mathbb{R}^{p_{d,i} \times m_{d,i}}$  and  $R_{d,i} \in \mathbb{S}_{m_{d,i}}$ . Denote  $\zeta_d$  and  $\zeta_c$  the extended vectors  $\zeta_j = [y_j^\top \quad \tilde{u}_j^\top]^\top \in \mathbb{R}^{p_{j,1} + p_{j,2} + m_{j,1} + m_{j,2}}$ ,  $\forall j \in \{c, d\}$ . Then the interconnected hybrid system  $\mathcal{H}_{12}$ , depicted by Fig. 2, is also QSR-dissipative with storage function  $V : \mathbb{R}^{n_1 + n_2} \rightarrow \mathbb{R}_{\geq 0}$ ,  $x \mapsto V_1(x_1) + \alpha V_2(x_2)$ , with respect to the compact set  $\mathcal{A}_{12} = \mathcal{A}_1 \times \mathcal{A}_2$ ,  $\alpha > 0$ , related to the continuous supply rate

$$r_{c,12}(\tilde{u}_c, y_c) = \zeta_c^\top M_c \zeta_c, \quad M_c = \begin{bmatrix} Q_{c,12} & S_{c,12} \\ \star & R_{c,12} \end{bmatrix}, \quad (6)$$

with

$$Q_{c,12} = \begin{bmatrix} Q_{c,1} + \alpha R_{c,2} & -S_{c,1} + \alpha S_{c,2}^\top \\ \star & R_{c,1} + \alpha Q_{c,2} \end{bmatrix},$$

$$S_{c,12} = \begin{bmatrix} S_{c,1} & \alpha R_{c,2} \\ -R_{c,1} & \alpha S_{c,2} \end{bmatrix},$$

$$R_{c,12} = \text{diag}(R_{c,1}, \alpha R_{c,2}),$$

and to the discrete supply rate

$$r_{d,12}(\tilde{u}_d, y_d) = \zeta_d^\top M_d \zeta_d, \quad M_d = \begin{bmatrix} Q_{d,12} & S_{d,12} \\ \star & R_{d,12} \end{bmatrix}, \quad (7)$$

where  $Q_{d,12} \in \mathbb{S}_{p_{d,1} + p_{d,2}}$ ,  $S_{d,12} \in \mathbb{R}^{(p_{d,1} + p_{d,2}) \times (m_{d,1} + m_{d,2})}$ , and  $R_{d,12} \in \mathbb{S}_{m_{d,1} + m_{d,2}}$  are matrices fulfilling the three following conditions:

i)

$$\zeta_d^\top (M_1 - M_d) \zeta_d \leq 0, \quad (8)$$

$$\forall (x, \tilde{u}_d) \in \mathbb{R}^{n_1 + n_2} \times \mathbb{R}^{m_{d,1} + m_{d,2}} \text{ such that } (x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \in \tilde{D}_1 \text{ and } (x_2, \tilde{u}_{d,2} + h_{d,1}(x_1)) \notin \tilde{D}_2;$$

ii)

$$\zeta_d^\top (\alpha M_2 - M_d) \zeta_d \leq 0, \quad (9)$$

$$\forall (x, \tilde{u}_d) \in \mathbb{R}^{n_1 + n_2} \times \mathbb{R}^{m_{d,1} + m_{d,2}} \text{ such that } (x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \notin \tilde{D}_1 \text{ and } (x_2, \tilde{u}_{d,2} + h_{d,1}(x_1)) \in \tilde{D}_2;$$

iii)

$$\zeta_d^\top (M_1 + \alpha M_2 - M_d) \zeta_d \leq 0, \quad (10)$$

$$\forall (x, \tilde{u}_d) \in \mathbb{R}^{n_1 + n_2} \times \mathbb{R}^{m_{d,1} + m_{d,2}} \text{ such that } (x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \in \tilde{D}_1 \text{ and } (x_2, \tilde{u}_{d,2} + h_{d,1}(x_1)) \in \tilde{D}_2,$$

where  $M_1$  and  $M_2$  are defined as

$$G(x, \tilde{u}_d) = \begin{cases} \left\{ \begin{bmatrix} \eta_1 \\ x_2 \end{bmatrix}, \forall \eta_1 \in \tilde{G}_1(x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \right\} & \text{if } (x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \in \tilde{D}_1 \text{ and } (x_2, \tilde{u}_{d,2} + h_{d,1}(x_1)) \notin \tilde{D}_2; \\ \left\{ \begin{bmatrix} x_1 \\ \eta_2 \end{bmatrix}, \forall \eta_2 \in \tilde{G}_2(x_2, \tilde{u}_{d,2} + h_{d,1}(x_1)) \right\} & \text{if } (x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \notin \tilde{D}_1 \text{ and } (x_2, \tilde{u}_{d,2} + h_{d,1}(x_1)) \in \tilde{D}_2; \\ \left\{ \begin{bmatrix} \eta_1 \\ \eta_2 \end{bmatrix}, \forall (\eta_1, \eta_2) \in \tilde{G}_1(x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \times \tilde{G}_2(x_2, \tilde{u}_{d,2} + \tilde{h}_{d,1}(x_1)) \right\} & \text{if } (x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \in \tilde{D}_1 \text{ and } (x_2, \tilde{u}_{d,2} + \tilde{h}_{d,1}(x_1)) \in \tilde{D}_2; \end{cases} \quad (11)$$

$$C = \{(x, \tilde{u}_c) \in \mathbb{R}^{n_1+n_2} \times \mathbb{R}^{m_c, 1+m_{c,2}} : (x_1, \tilde{u}_{c,1} - \tilde{h}_{c,2}(x_2)) \in \tilde{C}_1, (x_2, \tilde{u}_{c,2} + \tilde{h}_{c,1}(x_1)) \in \tilde{C}_2\}; \quad (12)$$

$$D = \{(x, \tilde{u}_d) \in \mathbb{R}^{n_1+n_2} \times \mathbb{R}^{m_d, 1+m_{d,2}} : (x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \in \tilde{D}_1\} \\ \cup \{(x, u_d) \in \mathbb{R}^{n_1+n_2+m_{d,1}+m_{d,2}} : (x_2, \tilde{u}_{d,2} + \tilde{h}_{d,1}(x_1)) \in \tilde{D}_2\}. \quad (13)$$

$$M_1 = \begin{bmatrix} Q_{d,1} & -S_{d,1} & S_{d,1} & 0_{p_1 \times m_2} \\ \star & R_{d,1} & -R_{d,1} & 0_{p_2 \times m_2} \\ \star & \star & R_{d,1} & 0_{m_1 \times m_2} \\ \star & \star & \star & 0_{m_2} \end{bmatrix}, \quad (14)$$

$$M_2 = \begin{bmatrix} R_{d,2} & S_{d,2} & 0_{p_1 \times m_1} & R_{d,2} \\ \star & Q_{d,2} & 0_{p_2 \times m_1} & S_{d,2} \\ \star & \star & 0_{m_1} & 0_{m_1 \times m_2} \\ \star & \star & \star & R_{d,2} \end{bmatrix}. \quad (15)$$

□

There are two ways to read Theorem 1:

- If  $M_d$  is given, Conditions (8)–(10) are a test to check the prescribed supply rate induced by  $M_d$ ;
- If  $M_d$  is free, such a matrix can always be found so that  $\mathcal{H}_{12}$  is QSR-dissipative with respect to *some* hybrid supply rate. This is ensured by the feasibility of the linear matrix inequalities  $M_1 - M_d \prec 0$ ,  $\alpha M_2 - M_d \prec 0$  and  $M_1 + \alpha M_2 - M_d \prec 0$  with respect to the symmetric variable  $M_d$ , and by consequence the feasibility of conditions (8)–(10).

*Proof.* The proof consists in two parts. The first one is dedicated to the flow behaviour and the second part to the jump one.

**Flow behaviour:** For all  $(x, \tilde{u}_c) \in C$ , for all  $\xi = [\xi_1^\top \ \xi_2^\top]^\top \in F(x, \tilde{u}_c)$ , it yields:

$$\langle \nabla V(x); \xi \rangle = \langle \nabla V_1(x_1); \xi_1 \rangle + \alpha \langle \nabla V_2(x_2); \xi_2 \rangle \\ \leq r_{c,1}(\tilde{u}_{c,1} - y_{c,2}, y_{c,1}) + \alpha r_{c,2}(\tilde{u}_{c,2} + y_{c,1}, y_{c,2}) \\ = r_{c,12}(\tilde{u}_c, y_c),$$

with  $r_{c,12}$  defined in Equation (6).

**Jump behaviour:** Thanks to the relation of the interconnection we can reformulate:

$$r_{d,1}(\tilde{u}_{d,1} - y_{d,2}, y_{d,1}) = \zeta_d^\top M_1 \zeta_d; \quad (16)$$

$$r_{d,2}(\tilde{u}_{d,2} + y_{d,1}, y_{d,2}) = \zeta_d^\top M_2 \zeta_d. \quad (17)$$

For all  $(x, \tilde{u}_d) \in D$ , for all  $\eta = [\eta_1^\top \ \eta_2^\top]^\top \in G(x, \tilde{u}_d)$ , we have:

$$V(\eta) - V(x) = V_1(\eta_1) - V_1(x_1) + \alpha(V_2(\eta_2) - V_2(x_2)). \quad (18)$$

In order to upper the latter, let us consider  $\eta = [\eta_1^\top \ \eta_2^\top]^\top \in G(x, \tilde{u}_d)$  and the three cases covering  $(x, \tilde{u}_d) \in D$ :

- i) For all  $(x, \tilde{u}_d)$  such that  $(x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \in \tilde{D}_1$  and  $(x_2, \tilde{u}_{d,2} + h_{d,1}(x_1)) \notin \tilde{D}_2$ : we have that  $\eta_2 = x_2$  and it yields, thanks to Condition (8):

$$V(\eta) - V(x) = V_1(\eta_1) - V_1(x_1), \\ \leq r_{d,1}(\tilde{u}_{d,1} - y_{d,2}, y_{d,1}), \\ \leq r_{d,12}(\tilde{u}_d, y_d). \quad (19)$$

- ii) For all  $(x, \tilde{u}_d)$  such that  $(x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \notin \tilde{D}_1$  and  $(x_2, \tilde{u}_{d,2} + h_{d,1}(x_1)) \in \tilde{D}_2$ : we have that  $\eta_1 = x_1$  and it yields, thanks to Condition (9):

$$V(\eta) - V(x) = \alpha(V_2(\eta_2) - V_2(x_2)), \\ \leq \alpha r_{d,2}(\tilde{u}_{d,2} + y_{d,1}, y_{d,2}), \\ \leq r_{d,12}(\tilde{u}_d, y_d). \quad (20)$$

- iii) For all  $(x, \tilde{u}_d)$  such that  $(x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \in \tilde{D}_1$  and  $(x_2, \tilde{u}_{d,2} + h_{d,1}(x_1)) \in \tilde{D}_2$ : it yields, thanks to Condition (10):

$$V(\eta) - V(x) = V_1(\eta_1) - V_1(x_1) + \alpha(V_2(\eta_2) - V_2(x_2)), \\ \leq r_{d,1}(\tilde{u}_{d,1} - y_{d,2}, y_{d,1}) \\ + \alpha r_{d,2}(\tilde{u}_{d,2} + y_{d,1}, y_{d,2}), \\ \leq r_{d,12}(\tilde{u}_d, y_d). \quad (21)$$

In all the cases,  $V(\eta) - V(x) \leq r_{d,12}(\tilde{u}_d, y_d)$ , which concludes the proof. □

Theorem 1 calls a large set of comments and several comparisons and discussions, gathered in the next section.

#### IV. DISCUSSIONS AND COMPARISONS

Derivations of Theorem 1 in particular cases lead to the following corollaries, recalling contributions in the literature.

**Corollary 1.** *The interconnection of flow- (respectively jump) QSR-dissipative hybrid systems is flow- (respectively jump) QSR-dissipative.*

*Proof.* In the flow context, Theorem 1 applies with  $M_1 = M_2 = M_d = 0$  and Conditions (8)–(10) are trivially satisfied. In the jump context, Theorem 1 applies with  $M_c = 0$ .  $\square$

**Corollary 2.** *The interconnection of flow-passive hybrid systems is flow-passive.*

*Proof.* Theorem 1 applies by imposing  $\alpha = 1$  and  $Q_{c,i} = R_{c,i} = 0$   $S_{c,i} = 1/2I$  for  $i \in \{1, 2\}$ .  $\square$

**Corollary 3.** *The interconnection of jump-passive hybrid systems is jump-passive if the two following conditions hold:*

- $-y_{d,2}^\top(y_{d,1} + \tilde{u}_{d,2}) \leq 0, \forall(x, \tilde{u}_d) \in \mathbb{R}^{n_1+n_2} \times \mathbb{R}^{m_{d,1}+m_{d,2}}$  such that  $(x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \in \tilde{D}_1$  and  $(x_2, \tilde{u}_{d,2} + h_{d,1}(x_1)) \notin \tilde{D}_2$ ;
- $y_{d,1}^\top(y_{d,2} - \tilde{u}_{d,1}) \leq 0, \forall(x, \tilde{u}_d) \in \mathbb{R}^{n_1+n_2} \times \mathbb{R}^{m_{d,1}+m_{d,2}}$  such that  $(x_1, \tilde{u}_{d,1} - \tilde{h}_{d,2}(x_2)) \notin \tilde{D}_1$  and  $(x_2, \tilde{u}_{d,2} + h_{d,1}(x_1)) \in \tilde{D}_2$ .

*Proof.* In the jump-passive context, selecting  $\alpha = 1$ ,  $M_d = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}$  in Theorem 1 vanishes condition (10). In addition, in conditions (8)–(9), inequalities  $\zeta_d^\top(M_1 - M_d)\zeta_d \leq 0$  and  $\zeta_d^\top(M_2 - M_d)\zeta_d \leq 0$  respectively read  $-y_{d,2}^\top(y_{d,1} + \tilde{u}_{d,2}) \leq 0$  and  $y_{d,1}^\top(y_{d,2} - \tilde{u}_{d,1}) \leq 0$ .  $\square$

Corollaries 2 and 3 recover respectively [17, Theorem 1] and [17, Theorem 2].

#### V. CONCLUSIONS

In this work, we provided new *unifying* results regarding the negative-feedback interconnection of hybrid QSR-dissipative systems. Theorem 1 can either be used to test whereas the weighted sum of storages is a storage guaranteeing the QSR-dissipativity property of the interconnection for a pre-described supply rate function or to derive simple (and always feasible) LMI tests assuring that the interconnection is QSR-dissipative concerning a new supply rate function to be found. Future work will regard the extension to possibly more general interconnections, as in the setting of [29], and the exploration of relations between QSR-dissipativity and the stabilizability of hybrid subsystems, as recently studied for both nonlinear continuous [30] and discrete-time systems [31].

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