Leveraging the properties of the Christoffel function for anomaly detection in data stream

Louise Travé-Massuyès

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December 15, 2023
Graz, Austria

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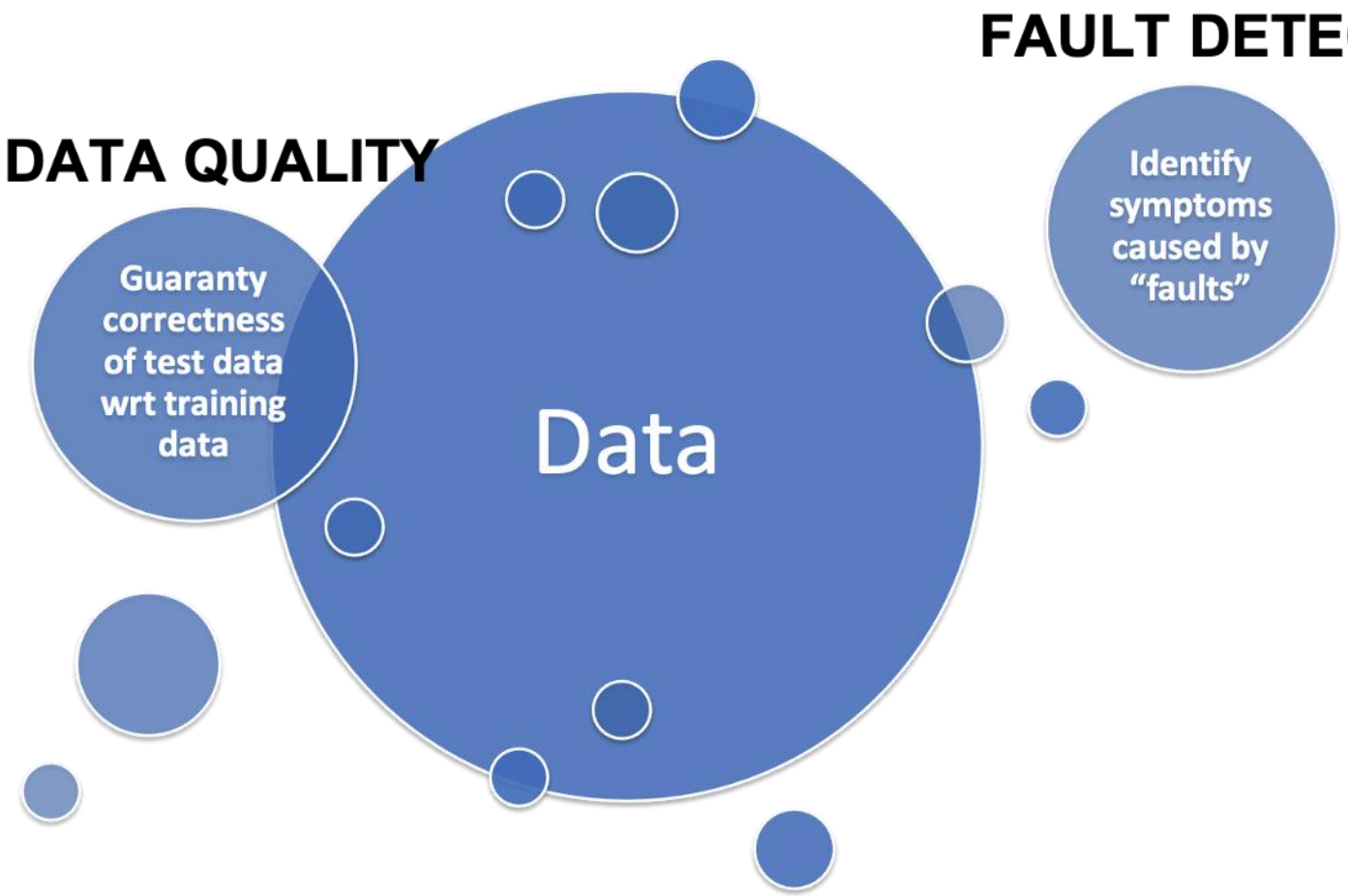
Chair **ADDX** on Anomaly Detection and Diagnnosis

Anomaly detection



Are there intruders ?

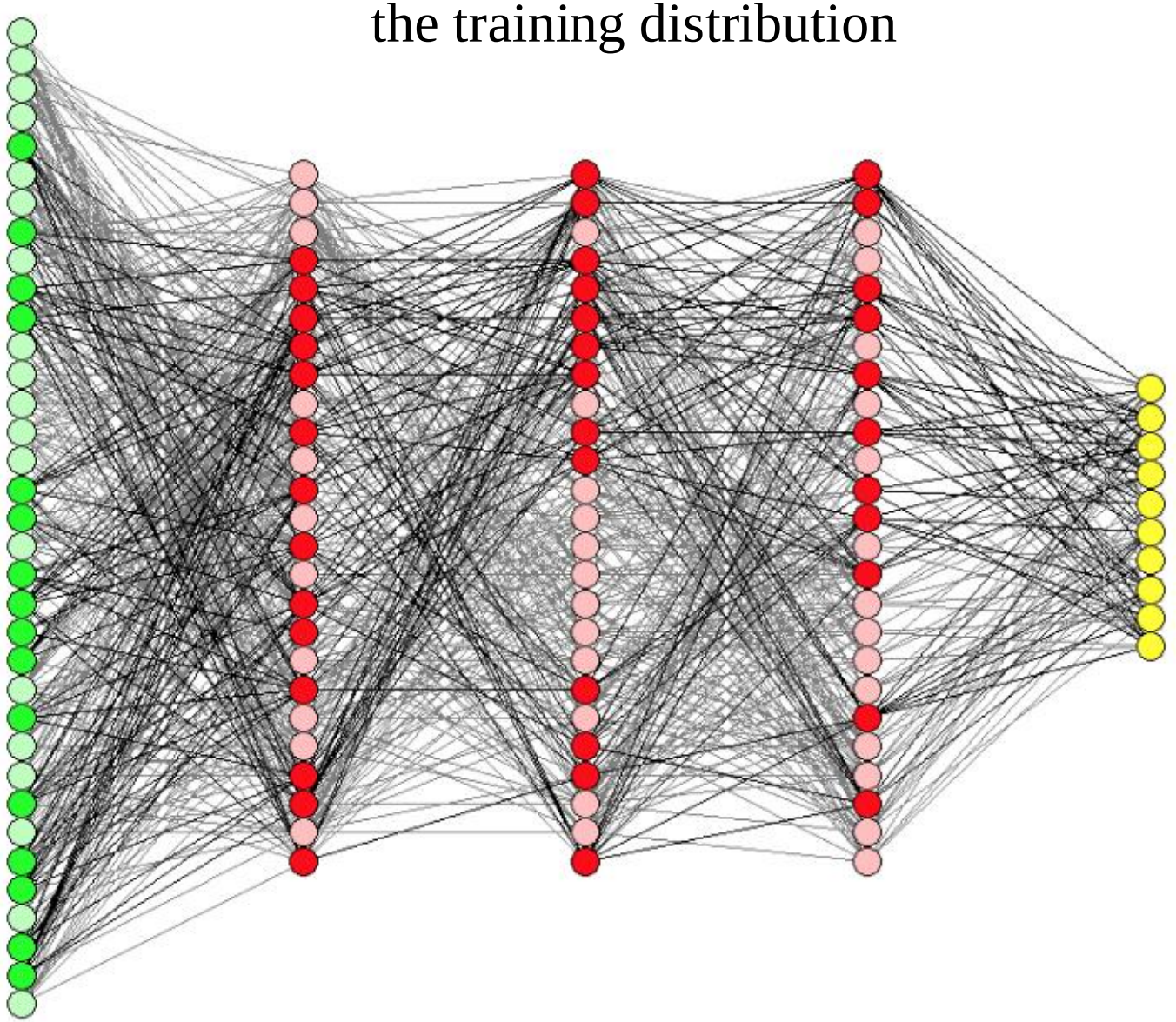
Anomaly

- an instance from a distinct distribution
- a rare or low-probability instance

Data quality

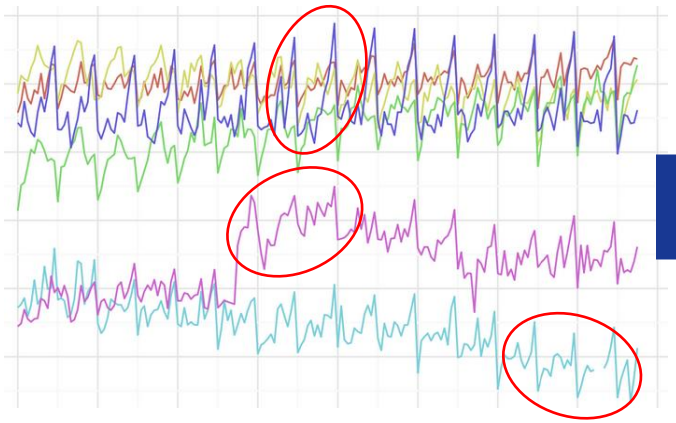
Training data

AI systems reliability is based on inputs lying in the training distribution



Fault detection

Anomalies

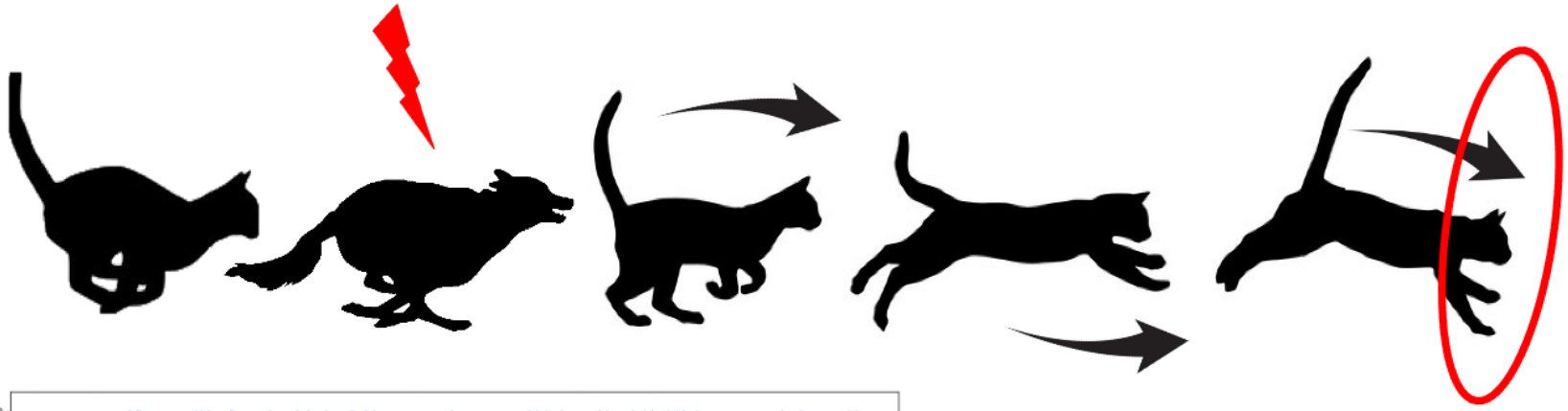


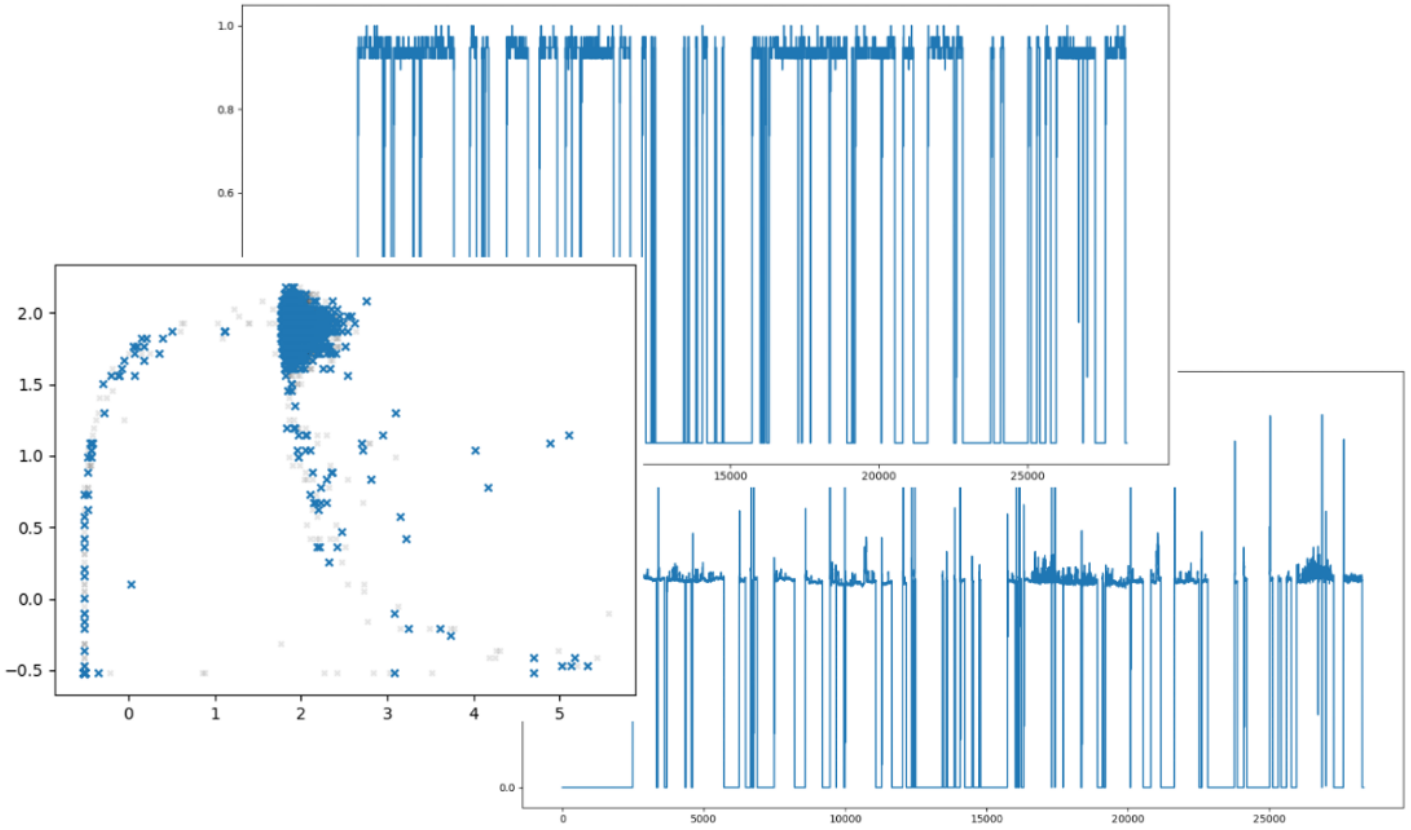
Symptoms

DIAGNOSIS

Root cause

Anomaly detection in data streams





Data stream requirements

- Detection on the fly
- Incremental update → speed
- Low complexity → frugality
- Concept drift → non stationarity

Note that these requirements exclude deep learning methods

State of the Art

- Adaptation of time series methods
  - Prediction models, e.g. based on exponential smoothing
  - Do not account for concept drift
- Dynamic clustering
  - Outliers do not belong to clusters or are in low density clusters
- Methods relying on KNN
  - Based on number of neighbours, e.g., MCOD
  - Based on local density (LOF and variants)
- Statistical methods
  - Parametric methods, e.g., based on GMM (Smartsifter)
  - Non parametric methods, e.g., on line Multiple Kernel Density Estimation (MKDE)

Many methods deal with *transiency*, *concept drift*, *infinity* and *time dependency*, mostly through the use of windows

But no rapid model update and retain no memory of previously acquired knowledge, window size dependency

This work

A **hybrid AI** anomaly detection method for data streams that:

- leverages the Christoffel function
  - related to the Christoffel-Darboux kernel borrowed from the theory of approximation and orthogonal polynomials
  - advocated for data mining by J.-B. Lasserre and E. Pauwels (2019)
- benefits from a clean algebraic framework
- fulfils all data stream requirements
- needs little tuning or no tuning at all

This work

A **hybrid AI** anomaly detection method for data streams that:

- leverages the Christoffel function
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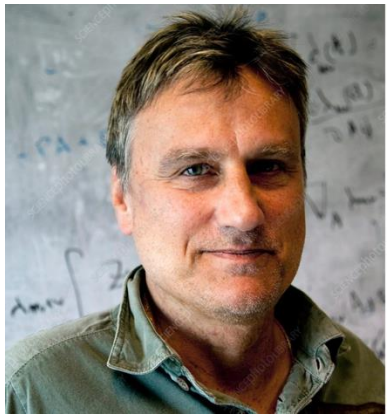
- benefits from a clean algebraic framework
- fulfils all data stream requirements
- needs little tuning

A collaboration between two ANITI chairs:

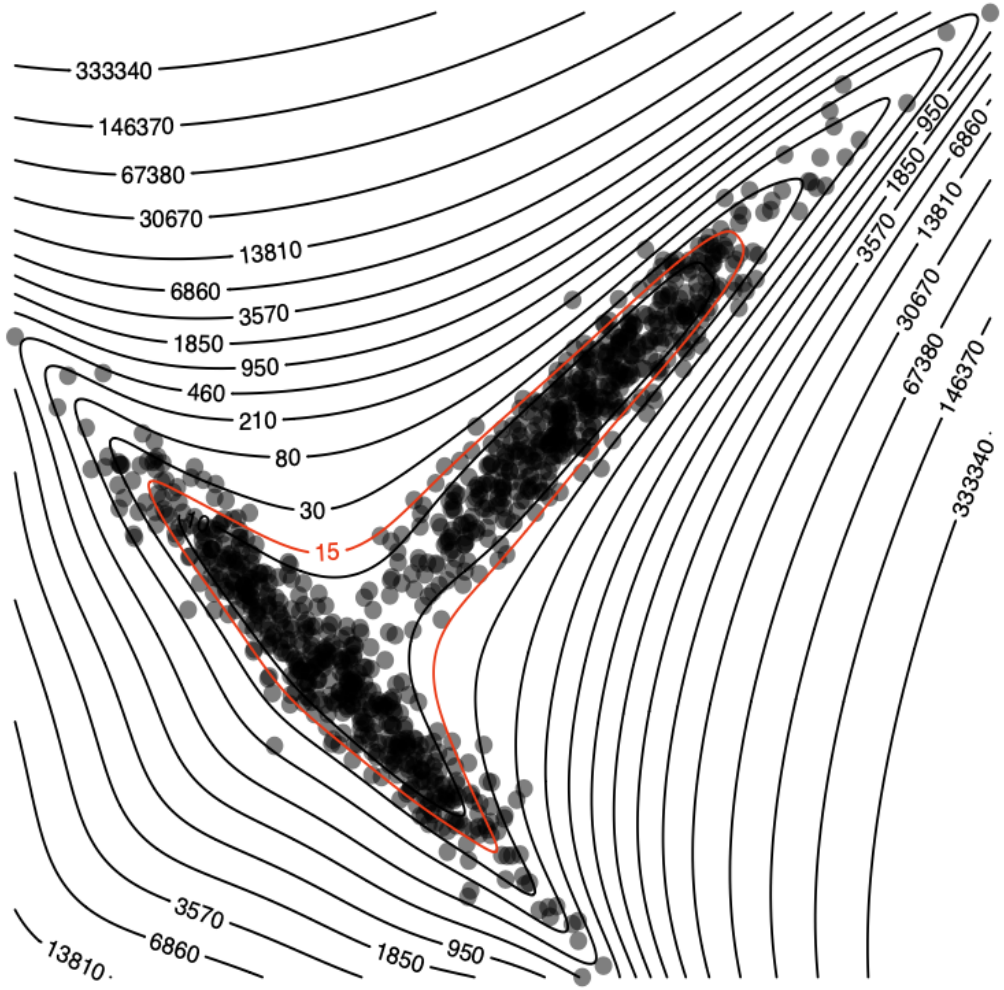
- **Polynomial Optimization** for Machine Learning and Data Analysis (*Jean-Bernard Lasserre*)
- Synergistic Transformations in Model Based and Data Based **Diagnosis** (*Louise Travé-Massuyès*)



PhD thesis of Kévin Ducharlet

Détection d'anomalies dans les flux de données pour une application dans les réseaux de capteurs (in french), PhD thesis, Computer science & Control, INSA, defended on Septembre 28, 2023.

Capturing the shape of a cloud of points



Consider a cloud of data points
$(x(i))_{i\in\mathbb{N}} \subset \mathbb{R}^p$

The red curve is the level set:
$\mathcal{L}_\gamma = \{\mathbf{x} : Q_d(\mathbf{x}) \leq \gamma\}, \gamma \in \mathbb{R}_+$

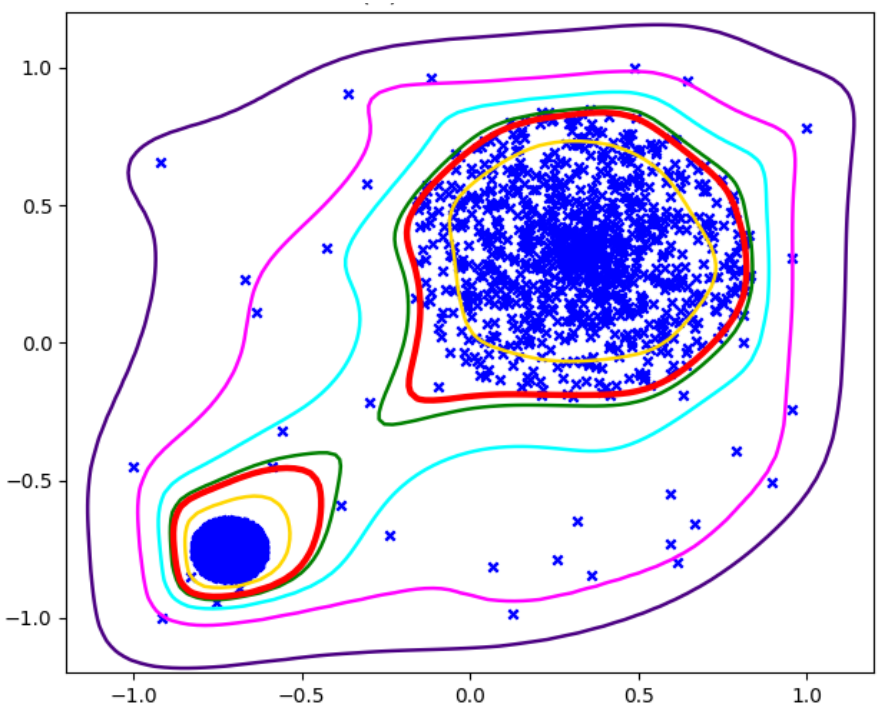
of a certain polynomial $Q_d \in \mathbb{R}[x1, x2]$ of degree $2d$.

Notice that $\mathcal{L}_\gamma$ captures the shape of the cloud.

Capturing the shape of a cloud of points (2)

Level sets obtained for a multi-density two disks dataset

CF level sets (d=6)

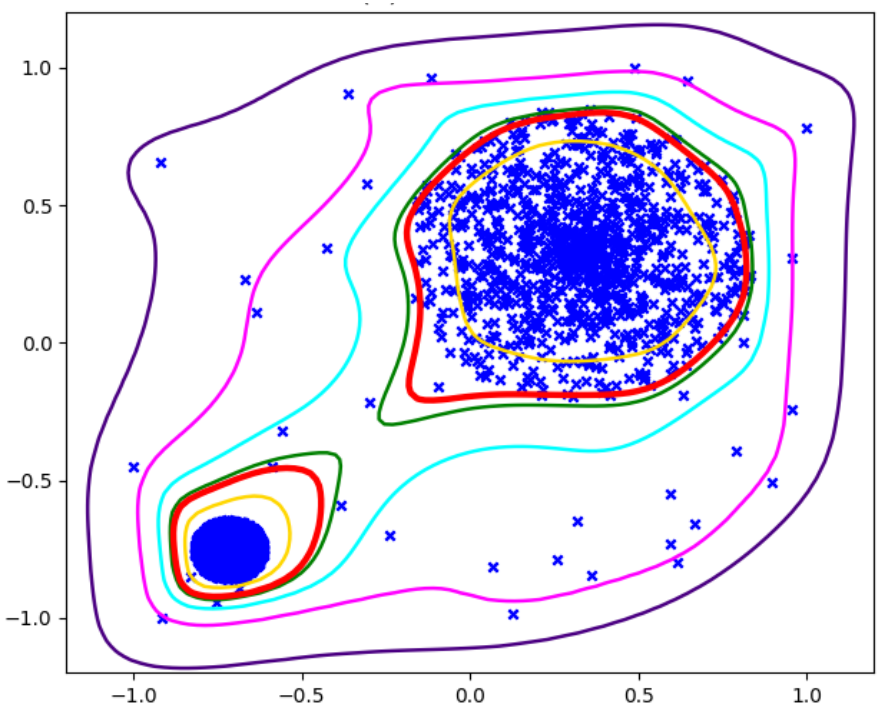


✓ The red level set nicely captures the two clusters

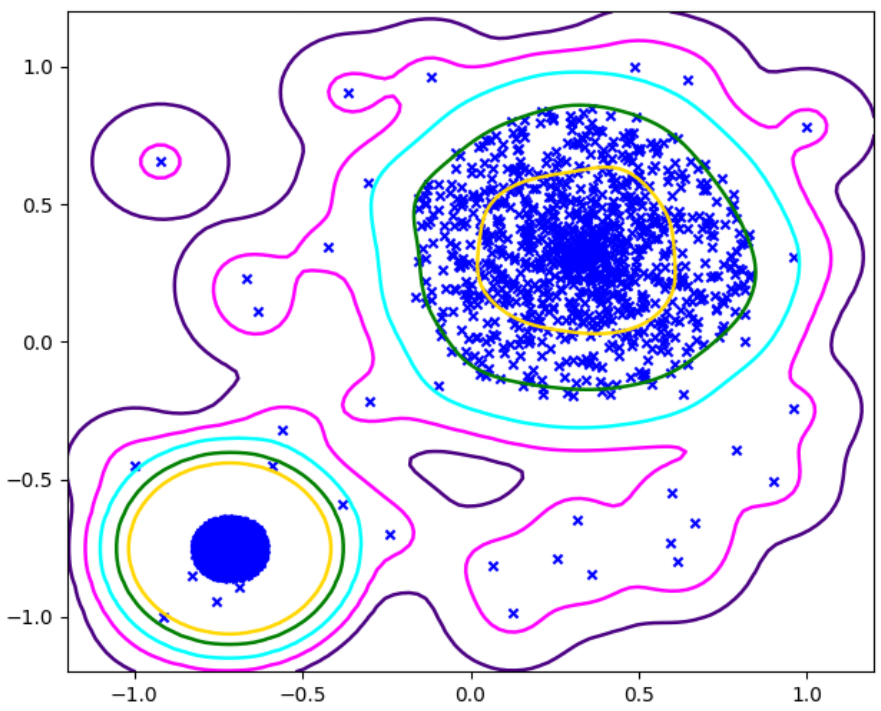
Capturing the shape of a cloud of points (2)

Level sets obtained for a multi-density two disks dataset with CF and MKDE gaussian kernel

CF level sets (d=6)



MKDE level sets



Method	AUROC	AP
CF	0.9745	0.7050
MKDE	0.9645	0.6648

The Christoffel function

- Let μ be a Borel measure on a compact set $\Omega \subset \mathbb{R}^p$ with nonempty interior,
- Form the vector $\mathbf{v}_d(\mathbf{x})$ from a basis of p-variate polynomials of degree at most d:

$$\mathbf{v}_d(\mathbf{x}) = (P_1(\mathbf{x}), \ldots, P_{s(d)}(\mathbf{x}))^T \quad \text{of size } s(d) = \binom{p+d}{p}.$$

$\mathbf{v}_d(\mathbf{x})$ is the vector of all monomials of degree less than or equal to d in the graded lexicographic order

☞ $$Q_d^\mu(\mathbf{x}) = \mathbf{v}_d(\mathbf{x})^T \underbrace{\mathbf{M}_d(\mu)}^{-1} \mathbf{v}_d(\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^p$$

Moment matrix of μ

The **Christoffel function** $\Lambda_d^\mu : \mathbb{R}^p \to \mathbb{R}_+$ is defined by:

$$\Lambda_d^\mu(\mathbf{x})^{-1} = Q_d^\mu(\mathbf{x})$$

Λ_d^μ encodes properties of the underlying measure μ.

Empirical measure

In our case

$$\mu_n := \frac{1}{n}\sum_{i=1}^{n}\delta_{x(i)}$$

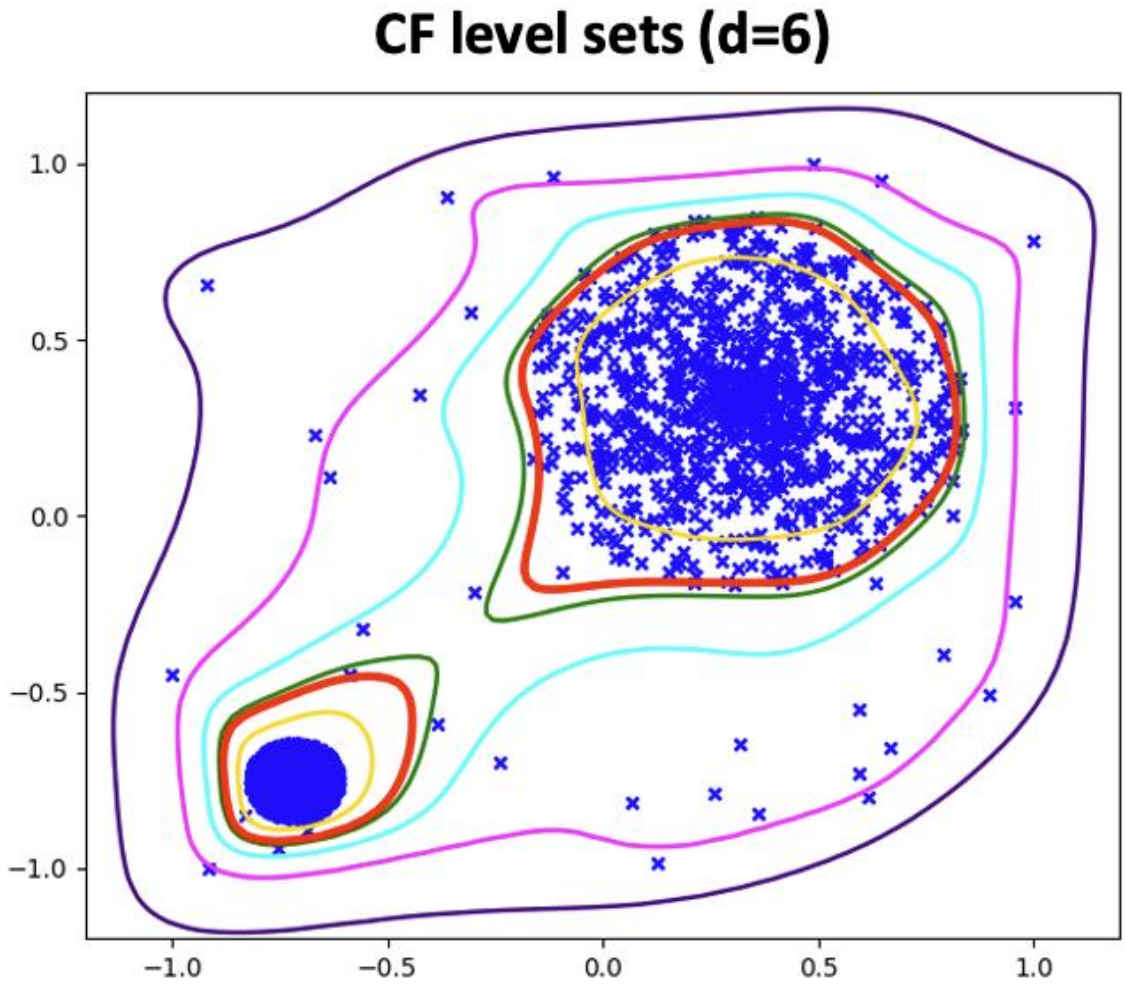
is the EMPIRICAL measure associated with the cloud of data points $(x(i))_{i\leq n}$ sampled from an unknown measure μ on Ω.

Empirical moment matrix of μ_n:

$$\mathbf{M}_d(\mu) = \int_{\mathbb{R}^p} \mathbf{v}_d(\mathbf{x})\mathbf{v}_d(\mathbf{x})^T\, d\mu(\mathbf{x}) \quad \longrightarrow \quad \mathbf{M}_d(\mu_n) = \tfrac{1}{n}\textstyle\sum_{\mathbf{x}\in\mathcal{X}} \mathbf{v}_d(\mathbf{x})\mathbf{v}_d(\mathbf{x})^T$$

Property: The samples belonging to the support Ω of the empirical measure μ_n are confined by a specific level set $\Omega_{\gamma_{d,p}}$, where $\gamma_{d,p} = Cd^{3p/2}$ and C a problem-related constant (cf. (Lasserre *et al.* 2022), Theorem 7.3.3).

Specific level set



The red level set corresponds to the set $\Omega_{\gamma_{d,p}}$ with the threshold $\gamma_{d,p}=d^{3p/2}$ **as dictated by the CF theory** (C=1)

In summary

In our case

$$\mu_n := \frac{1}{n}\sum_{i=1}^{n}\delta_{x(i)}$$

is the EMPIRICAL measure associated with the cloud of data points $(x(i))_{i\leq n}$ sampled from an unknown measure μ on Ω.

☞ ... and quite remarkably

The level sets of $\Lambda_d^{\mu_n}(\mathbf{x})^{-1}$ match the density variations of the cloud of points $(x(i))_{i\leq n}$

$\rightarrow \Lambda_d^{\mu_n}(\mathbf{x})^{-1}$ is a good scoring function for anomaly detection

In particular, the level set

$$\{\mathbf{x}\in\mathbb{R}^p : \Lambda_d^{\mu_n}(\mathbf{x})^{-1} \leq \gamma_{d,p} = Cd^{3p/2}\}$$

identifies the support Ω of μ, even for moderate values of d.

Dealing with data streams: DyCF method

The inverse CF $\Lambda_d^{\mu_n}(\mathbf{x})^{-1}$ does not depend on the number of points *n*, but only on the number of dimensions *p* and the degree d

1) Low memory: $\mathbf{M}_d(\mu_n)^{-1}$ can be seen as en encoding of the whole data set

2) Low computation: incremental update of $\Lambda_d^{\mu_n}(\mathbf{x})^{-1}$ with rank-one update of the inverse $\mathbf{M}_d(\mu_n)^{-1}$

When a point ξ is added to the cloud of *n* points, i.e.,

$$\mu_n \rightarrow \frac{1}{n+1}(n\,\mu_n + \delta_\xi)$$

$\rightarrow$ a new cloud with n+1 points

☞ The Sherman-Morrisson-Woodbury formula allows for a simple RANK-ONE UPDATE of the inverse $\mathbf{M}_d(\mu_n)^{-1}$

Incremental update

Incremental inversion of a matrix A with the Sherman-Morrison-Woodbury formula:

$$(A + uv^T)^{-1} = A^{-1} - \frac{A^{-1}uv^TA^{-1}}{1 + v^TA^{-1}u}$$

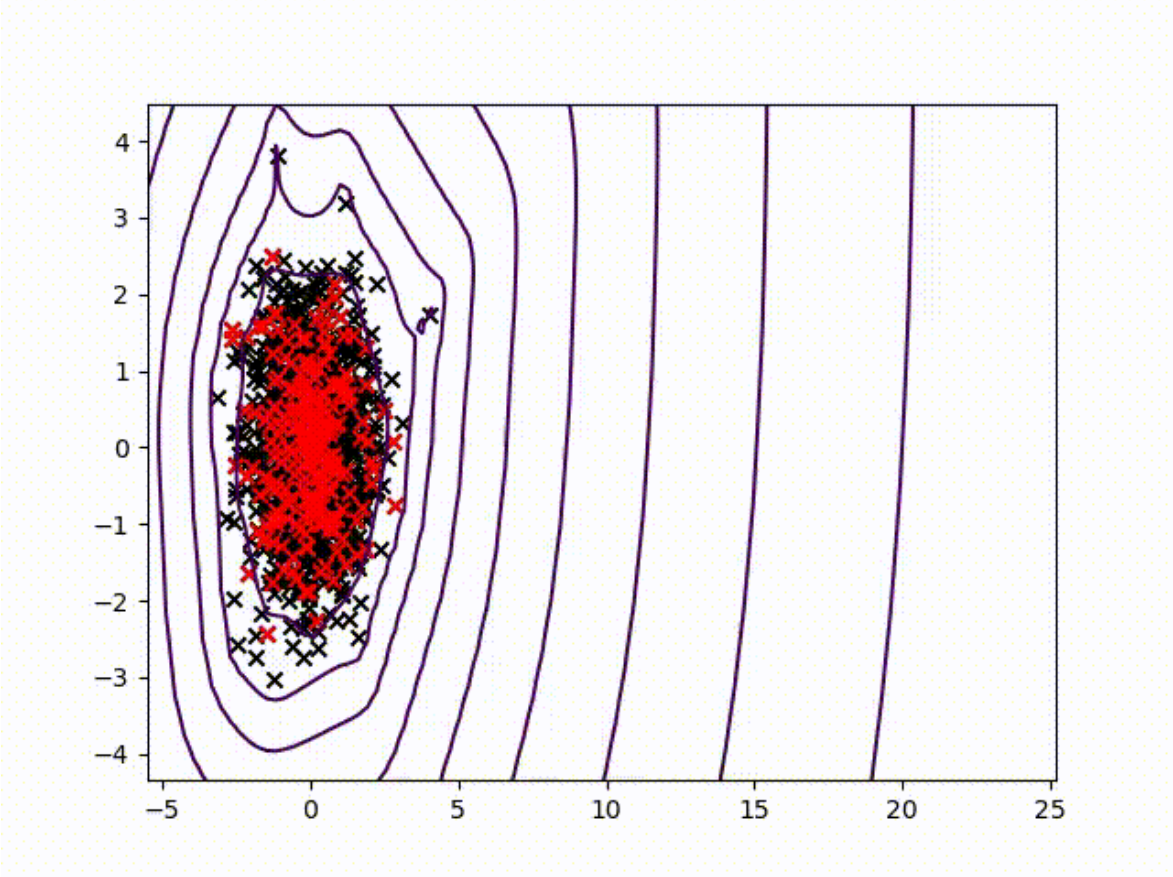
The moment matrix $\mathbf{M}_d(\mu_n)^{-1}$ can be rewritten with the incremental formula:

$$\mathbf{M}_d(\mu_{n+1}) = \frac{1}{n+1}[\underbrace{n\mathbf{M}_d(\mu_n)}_{A} + \underbrace{\mathbf{v}_d(\mathbf{x}_{n+1})}_{u}\underbrace{\mathbf{v}_d(\mathbf{x}_{n+1})^T}_{v^T}]$$

$$\longmapsto \quad ((n+1)\mathbf{M}_d(\mu_{n+1}))^{-1} = (n\mathbf{M}_d(\mu_n))^{-1} - \frac{(n\mathbf{M}_d(\mu_n))^{-1}\mathbf{v}_d(\mathbf{x}_{n+1})\mathbf{v}_d(\mathbf{x}_{n+1})^T(n\mathbf{M}_d(\mu_n))^{-1}}{1 + \mathbf{v}_d(\mathbf{x}_{n+1})^T(n\mathbf{M}_d(\mu_n))^{-1}\mathbf{v}_d(\mathbf{x}_{n+1})}$$

Incremental update

$$\longmapsto \quad ((n+1)\mathbf{M}_d(\mu_{n+1}))^{-1} = (n\mathbf{M}_d(\mu_n))^{-1} - \frac{(n\mathbf{M}_d(\mu_n))^{-1}\mathbf{v}_d(\mathbf{x}_{n+1})\mathbf{v}_d(\mathbf{x}_{n+1})^T(n\mathbf{M}_d(\mu_n))^{-1}}{1+\mathbf{v}_d(\mathbf{x}_{n+1})^T(n\mathbf{M}_d(\mu_n))^{-1}\mathbf{v}_d(\mathbf{x}_{n+1})}$$



Christoffel function score

DyCF requires only one parameter to be fixed: d

The theory dictates to use the level set defined by $\Omega_{\gamma_{d,p}}$, where $\gamma_{d,p} = Cd^{3p/2}$.

Normalized scoring function : $S_{d,p}(\mathbf{x}) = \frac{\Lambda_d^{\mu}(\mathbf{x})^{-1}}{\gamma_{d,p}}$.

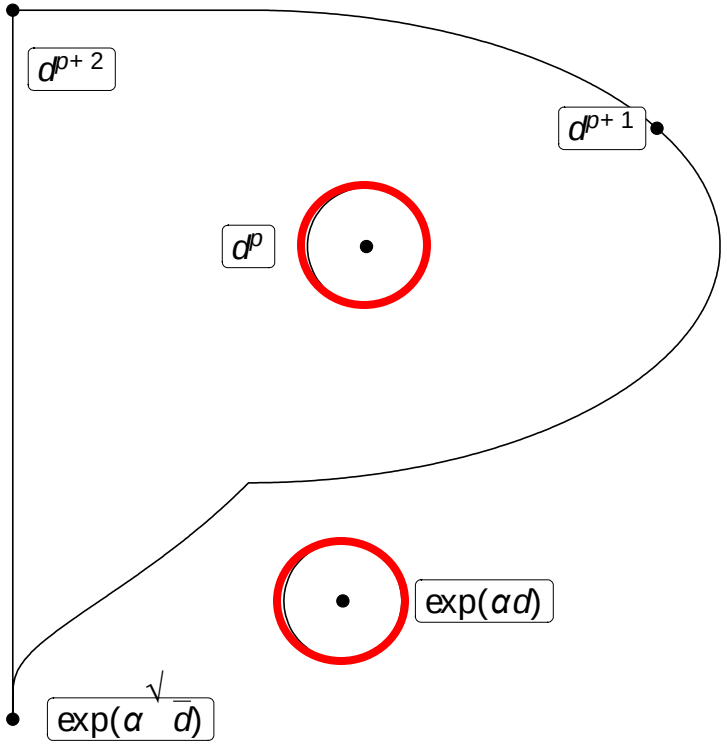
If C=1, a point $\mathbf{x}$ is defined as an outlier if $S_{d,p}(\mathbf{x}) \geq 1$.

Now, can we achieve no tuning at all ?

Leveraging the growth properties of CF

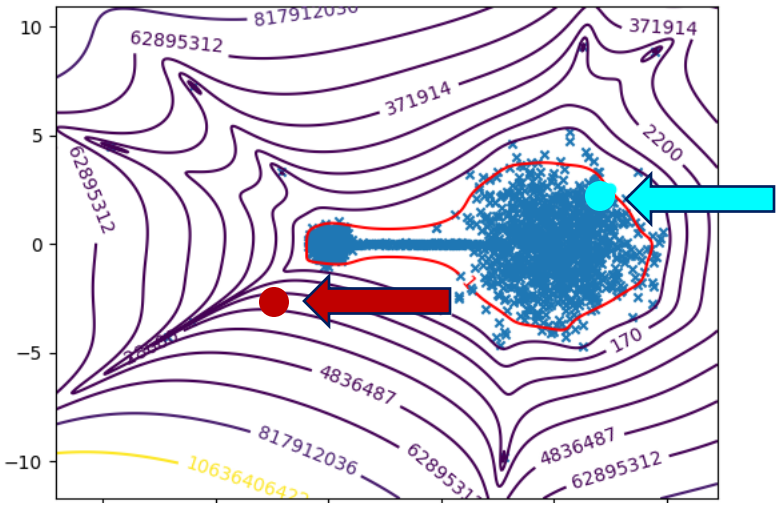
As d grows, $\Lambda_d^{\mu}(\mathbf{x})^{-1}$ has:

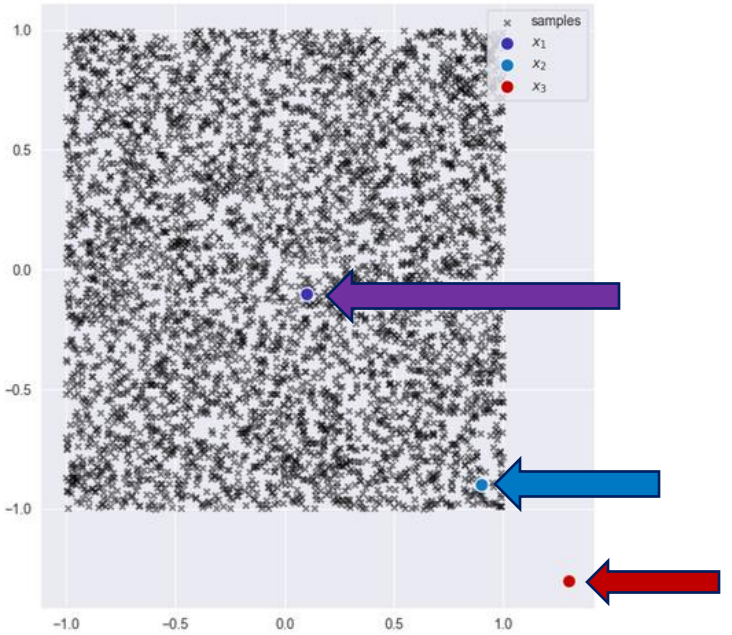
$$\begin{cases} \text{POLYNOMIAL growth} & \text{INSIDE } \Omega \\ \text{EXPONENTIAL growth} & \text{OUTSIDE } \Omega \end{cases}$$

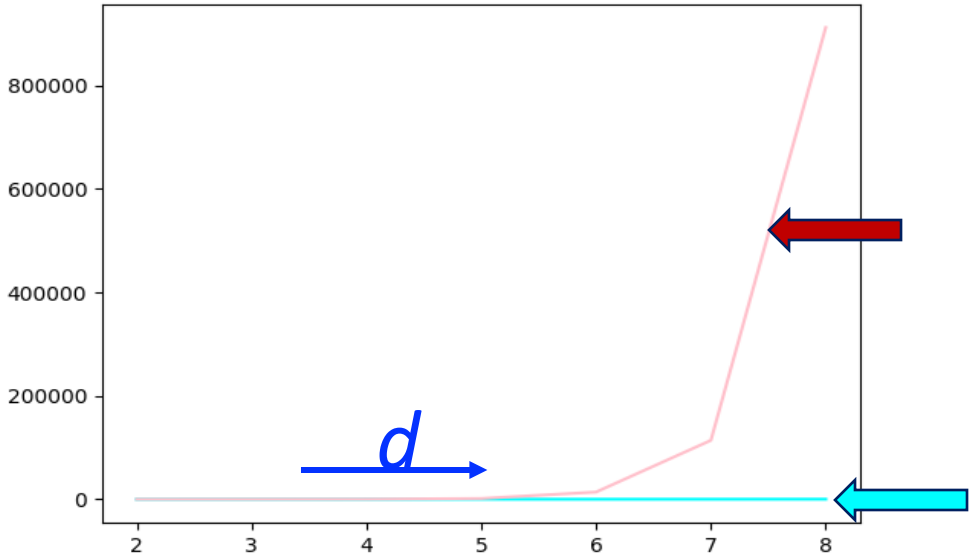


Cf. (Lasserre *et al.* 2022), Lemmas 4.3.1 and 4.3.2)

CF growth property

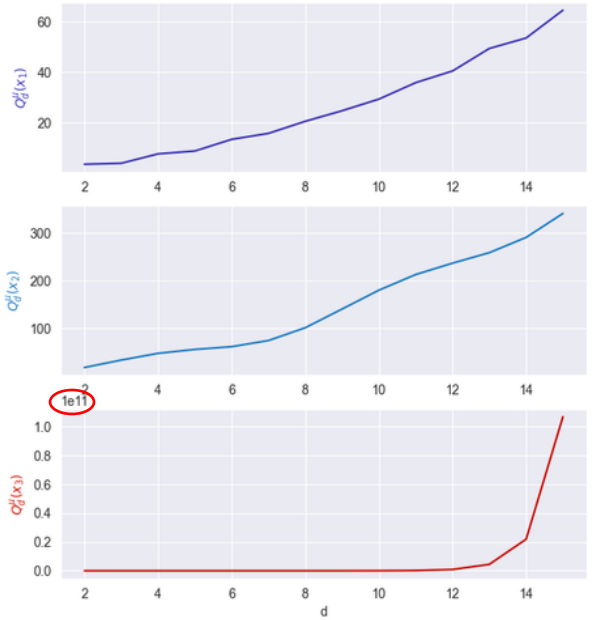


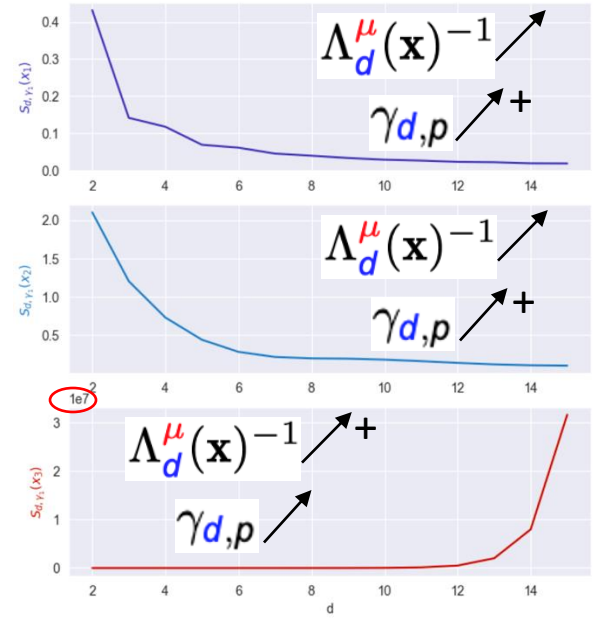




Normalized score:

$$S_{d,p}(\mathbf{x}) = \frac{\Lambda_d^{\mu}(\mathbf{x})^{-1}}{\gamma_{d,p}}$$





DyCG: Dynamic CF Growth method

DyCG : two DyCF models of degrees d_{min} and d_{max}

DyCG scoring function: $S'_{d_{max},d_{min},p}(\mathbf{x}) = \frac{S_{d_{max},p}(\mathbf{x}) - S_{d_{min},p}(\mathbf{x})}{d_{max} - d_{min}}$

Outlierness threshold is 0:

$$\text{Inliers} \longrightarrow S_{d_{max},p}(\mathbf{x}) < S_{d_{min},p}(\mathbf{x}) \longrightarrow S'_{d_{max},d_{min},p}(\mathbf{x}) < 0$$

$$\text{Outliers} \longrightarrow S_{d_{max},p}(\mathbf{x}) \geq S_{d_{min},p}(\mathbf{x}) \longrightarrow S'_{d_{min},d_{max},p}(\mathbf{x}) \geq 0$$

d_{min} and d_{max} are fixed at 2 and 8 once and for all: DyCG is tuning free.

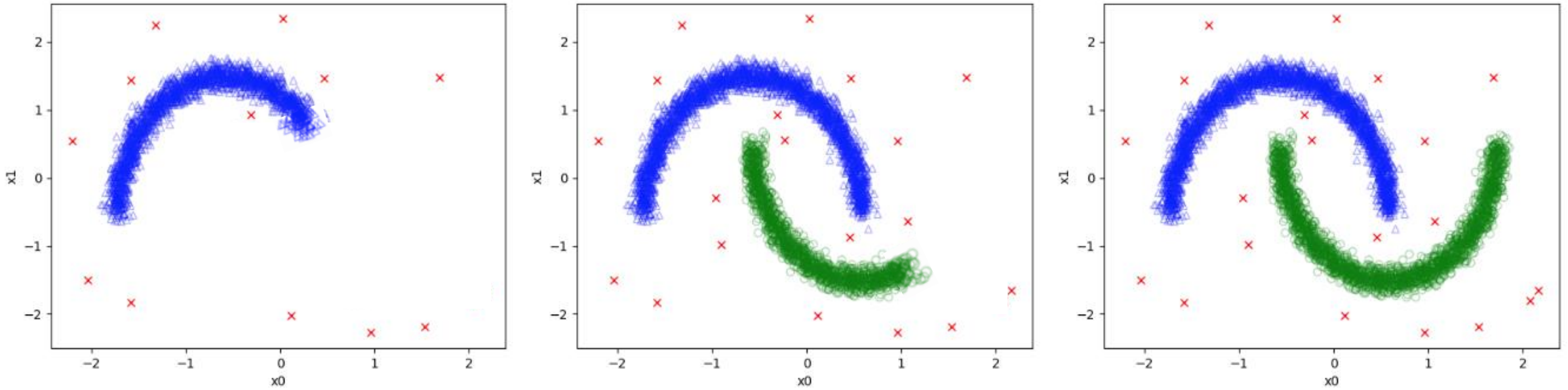
Evaluation

Two moons data stream (tm)

2 x 2500 samples building two moons
(500 for initial training)

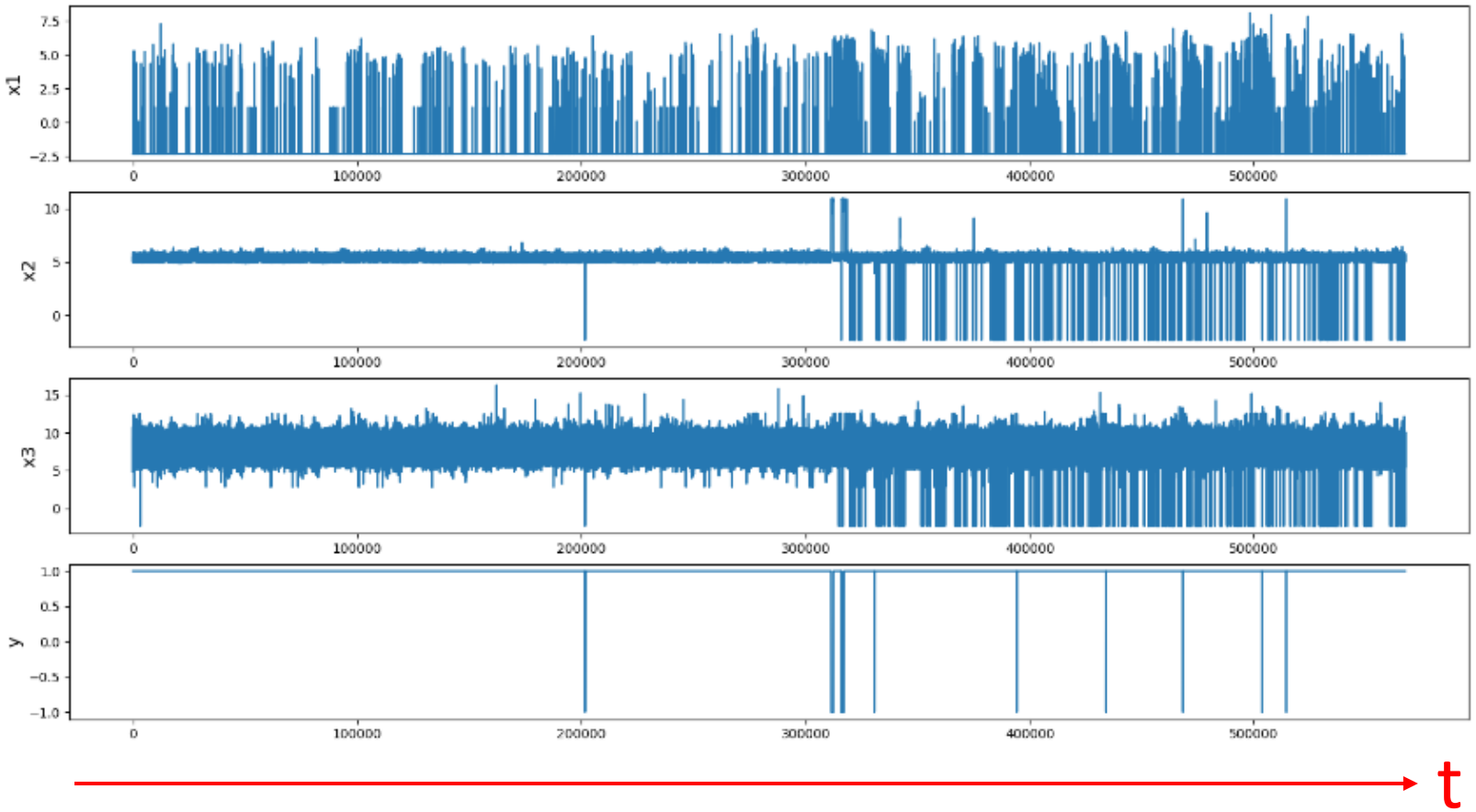
+

20 outliers uniformly distributed



Http data stream (http)

From the Outlier Detection Data Streams ODDS library (https://odds.cs.stonybrook.edu/)
3 variables, 567498 observations (50000 for initial training), 2211 outliers (≃0,4%)

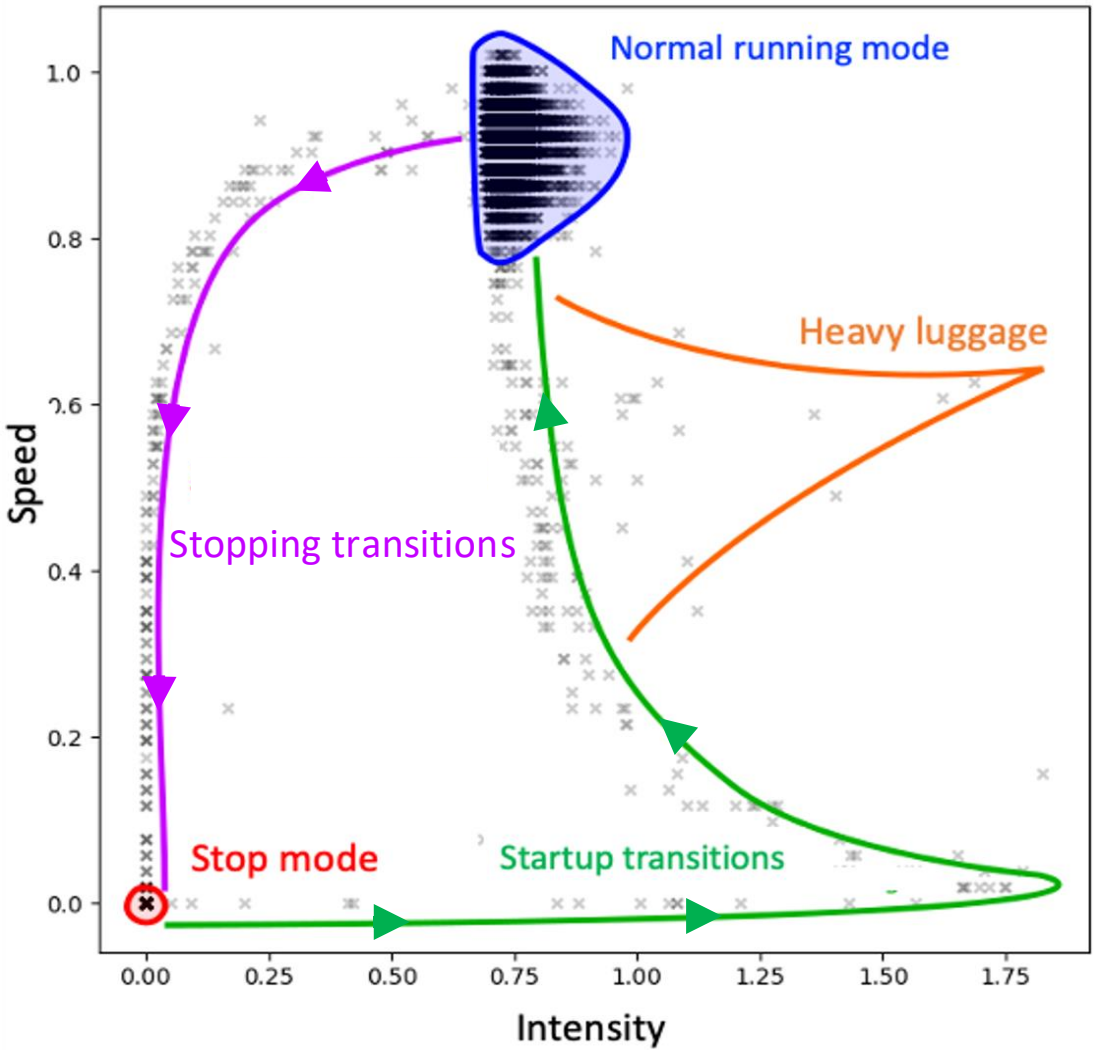


labels

Industrial luggage conveyor data stream (lc)

Carl Berger-Levrault project

Multimode system, 2 variables, 166926 observations (15000 for initial training), 17 introduced outliers



Methods

Sliding window Multivariable Kernel Density Estimation

Based on a Gaussian Mixture Model

Based on the number (parameter *k*) of neighboring points laying at a given distance (parameter *R*)

KNN based method that contrasts sample local density with that of its neighbors

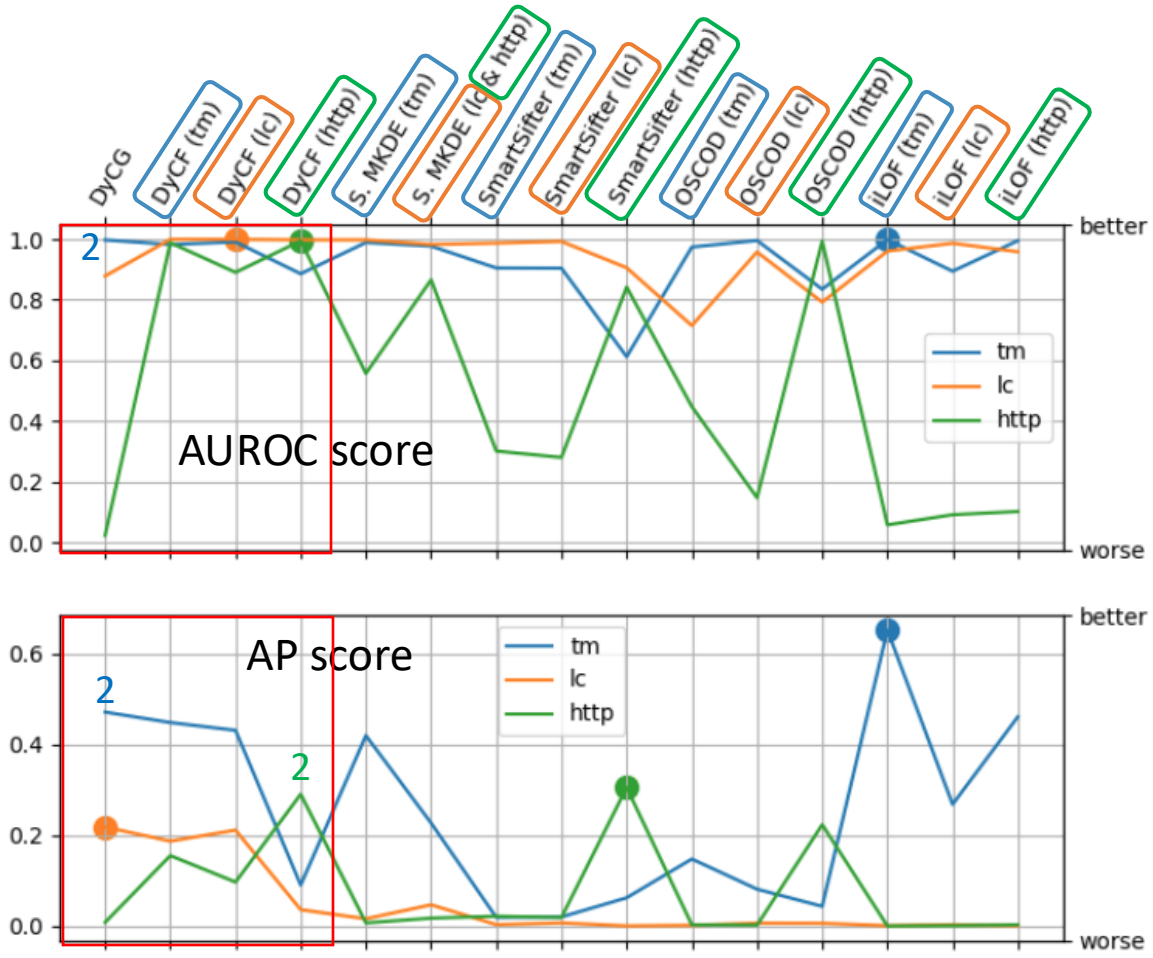
Method	Parameters	Range	ω_{tm}	ω_{lc}	ω_{http}
Sliding MKDE	threshold (not informed)	-	-	-	-
	window size W	$\{100, 200, 500, 1000, 2000, 5000\}$	200	5000	5000
	kernel (fixed)	gaussian	-	-	-
	bandwidth (fixed)	Scott's rule	-	-	-
SmartSifter	threshold (not informed)	-	-	-	-
	nb of gaussians k	$\{2, 5, 10, 15, 20\}$	15	10	2
	disounting param r	$\{0.001, 0.005, 0.01\}$	0.01	0.01	0.001
	stability param α	$\{1, 1.5, 2\}$	1.5	1	2
OSCOD	nb of neighbors k (not informed)	-	-	-	-
	radius R	$\{0.1, 0.2, 0.5, 1, 1.2, 1.5\}$	1.2	0.2	0.5
	window size W	$\{100, 200, 500, 1000, 2000, 5000\}$	200	100	5000
ILOF	threshold (not informed)	-	-	-	-
	nb of neighbors k	$\{5, 10, 15, 20, 25, 30\}$	5	25	30
	window size W	$\{100, 200, 500\}$	200	100	500
DyCF	degree d	$\{2, 4, 6, 8\}$	6	8	2
DyCG	no need	-	-	-	-

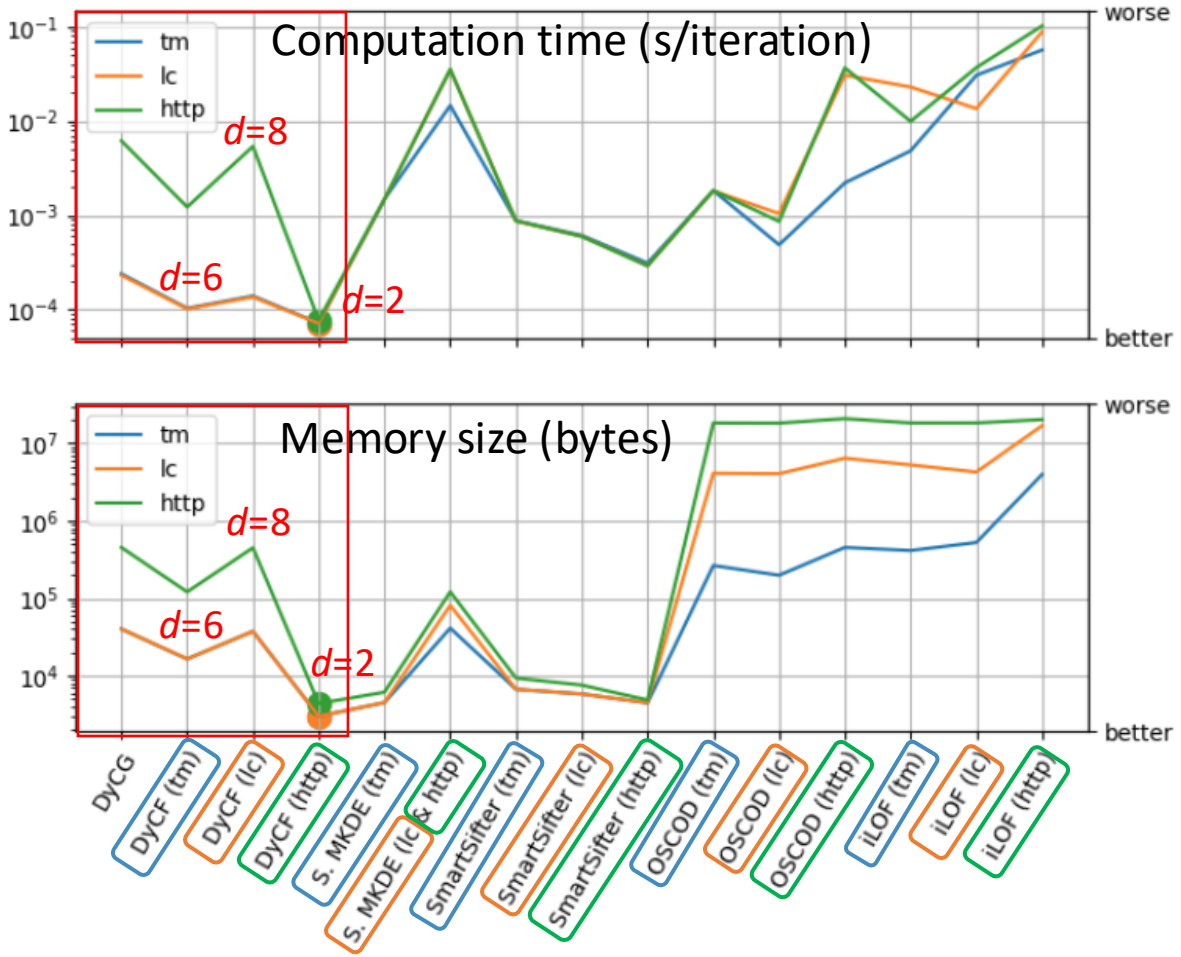
No deep learning method because no frugality, no fast update, no low tuning.

Metrics

- AUROC (Area Under the ROC Sensitivity-Specificity curve)
  - The higher the better (a value of 0.5 is not better than a random classifier)
- AUPRC (Area Under the Precision-Recall Curve) estimated as AP (Average Precision)
  - Higher value indicates better precision-recall performance
  - Relevant for imbalanced data sets
- Computation time for one iteration
- Memory size

Results

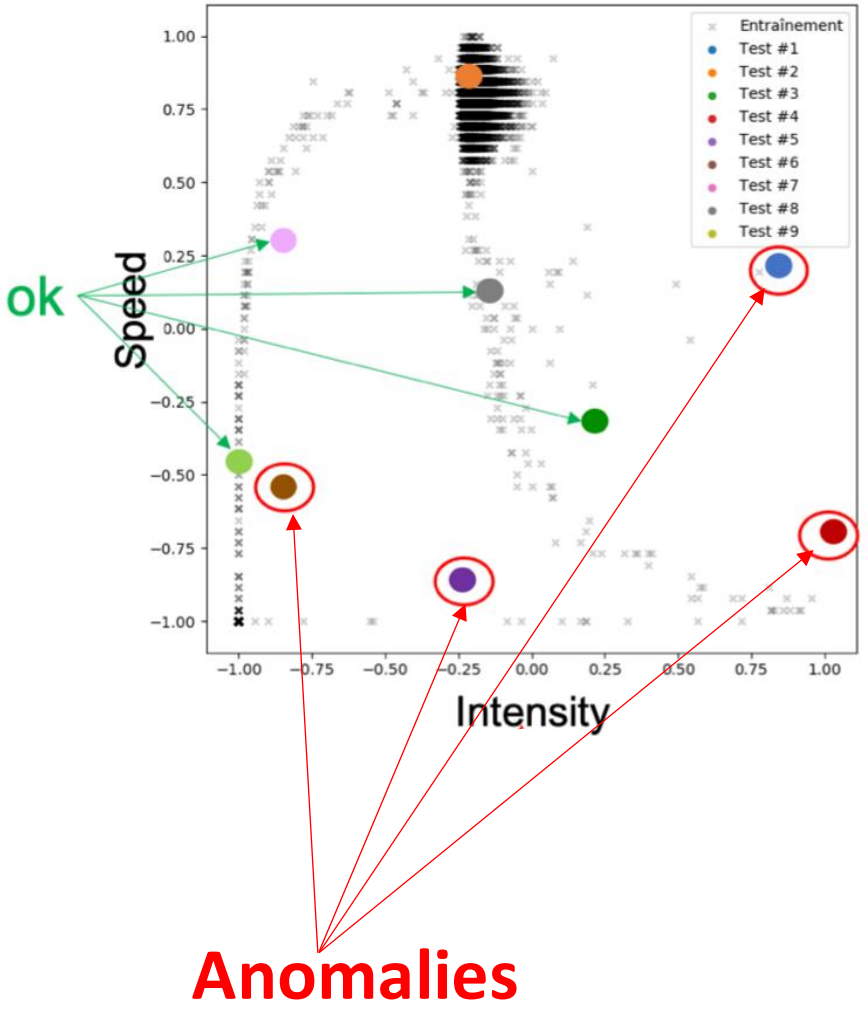




- DyCF and DyCG are globally positioned at the top
- Best scores for the industrial data stream
- DyCG and ILOF are bad for the http data stream
  (p= 3, d=8)

Industrial luggage conveyor data stream (Ic)

Carl Berger-Levrault project



Conclusions

- *Dy-CF* and *Dy-CG* are simple and easy-to-use methods with little or no tuning at all
- They achieve excellent results compared to other more tricky anomaly detection methods
- The Christoffel function provides interesting theoretical foundations
- It nicely deals with data streams thanks to the moment matrix encoding and its incremental update
- Future work:
  - adding forgetting ability
  - scaling up to high dimensions
  - extend to abnormal trajectory detection
  - integrate with other tasks.

(Lasserre *et al.* 2022) Lasserre JB, Pauwels E, Putinar M (2022) The Christoffel–Darboux Kernel for Data Analysis. Cambridge Monographs on Applied and Computational Mathematics, Cambridge University Press, https://doi.org/10.1017/9781108937078

(Ducharlet *et al.* 2022) Ducharlet, K., Travé-Massuyès, L., Lasserre, J. B., Le Lann, M. V., & Miloudi, Y., Leveraging the Christoffel-Darboux kernel for online outlier detection. 2022. URL https://hal.science/hal-03562614.

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