



HAL
open science

Uniform ultimate boundedness analysis for linear systems with asymmetric input backlash and dead-zone: A piecewise quadratic Lyapunov function approach

Aurélien Pierron, Jérémie Kreiss, Marc Jungers, Gilles Millérioux, Jérémy Dupont, Martial Martig

► To cite this version:

Aurélien Pierron, Jérémie Kreiss, Marc Jungers, Gilles Millérioux, Jérémy Dupont, et al.. Uniform ultimate boundedness analysis for linear systems with asymmetric input backlash and dead-zone: A piecewise quadratic Lyapunov function approach. *European Journal of Control*, 2024, 80 (part A), pp.101059. <10.1016/j.ejcon.2024.101059>. <hal-04795335>

HAL Id: hal-04795335

<https://hal.science/hal-04795335v1>

Submitted on 21 Nov 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons CC BY 4.0 - Attribution - International License

Uniform Ultimate Boundedness Analysis for Linear Systems with Asymmetric Input Backlash and Dead-zone: A Piecewise Quadratic Lyapunov Function Approach

A. Pierron^{1,2} J. Kreiss¹ M. Jungers¹ G. Millérioux¹ J. Dupont² M. Martig²

Abstract—This paper deals with the interconnection between a linear system and a nonlinear operator consisting of asymmetric input backlash and asymmetric dead-zone. The uniform ultimate boundedness of the system is studied. A piecewise quadratic Lyapunov function, suitable with the polyhedral description of the nonlinear operator is proposed. The conservatism of existing results is therefore reduced. The effectiveness and improvement of the results are assessed using a numerical example.

Index Terms—Backlash, dead-zone, stability analysis, piecewise quadratic Lyapunov function, uniform ultimate boundedness.

I. INTRODUCTION

The system investigated in this paper is a linear plant whose input acts via a nonlinear operator exhibiting asymmetric backlash and asymmetric deadzone. The presence of a backlash operator (labelled also as a play operator [1]) is motivated by a large range of nonlinear system applications (see for instance [2]) and in particular when a cylinder is used. Its intrinsic presence in the well known Hysteresis phenomena and its relation to nonlinear memory effects (see Prandtl-Ishlinskii models [3], [4] for instance) makes it essential in many applications. The deadzone is widespread in mechanics to model solid/dry frictions for instance and is closely associated with the saturation nonlinearity. These two types of nonlinearities are nevertheless rarely studied together, even if they can be coupled in real systems. A main difficulty is that the backlash operator is only piecewise differentiable. In addition and because of the memory effect, such a phenomenon requires to be described by its time derivative, when it exists. Various solutions propose to compensate the backlash operator by an inversion. For instance, [5] mimics the backlash by a hybrid system and proposes a controller as an other hybrid system to compensate the backlash operator. An other example is in [4], where its inverse is constructed analytically and used as a feedforward compensator. However, this requires an exact knowledge of the nonlinearity.

Due to the structure of the studied system which may fail to be stable, we are interested in providing guarantees of

the Uniform Ultimately Boundedness (UUB) property, by following [6]. The analysis is performed thanks to a UUB-Lyapunov function which present the following properties: (i) outside the unit level set (i.e. when its value is greater than 1), it decreases along the trajectory of the system and (ii) after reaching this level-set, the trajectory remains inside it. The aim, here, is to provide tractable sufficient conditions as parameterized Linear Matrix Inequalities (LMIs) and an optimization problem, as a semidefinite program, that minimizes (an estimate of the size) of the unit level-set of this UUB-Lyapunov function.

In order to reduce the conservatism of considering only common quadratic UUB-Lyapunov function to ensure the UUB property [7], we propose here to investigate a piecewise quadratic Lyapunov function. This kind of Lyapunov functions has been introduced in the middle of the 90's to deal with piecewise affine systems, which are defined as affine dynamics over regions corresponding to a polyhedral partition of the state-space (see for instance [8] and references therein). Such a tool has been used successfully for specific hybrid systems [9], [10] or systems with saturation or dead-zone [11], [12], and even more complex nonlinearities [13]. To the best of our knowledge, such a tool has not been considered for complex nonlinear operator with backlash, as the one investigated here.

The paper is organized as follows. Section II introduces the problem by describing the nonlinear operator and recalling the definition of Uniform Ultimate Boundedness. Section III details the preliminaries dealing with the structure of the piecewise quadratic UUB-Lyapunov functions and the characterization of the nonlinear operator. These preliminaries are combined in the main result in Section IV to provide sufficient conditions that ensure the UUB property. These conditions are expressed as parameterized LMIs and an optimization problem to optimize the size of the resulting UUB-set is provided. Section V demonstrates, on an example, the effectiveness of our method and the providing improvement with respect to the literature. We finally conclude in Section VI.

Notation: I_n stands for the identity matrix of dimension n and $0_{n \times m}$ stands for the null matrix of dimensions $n \times m$, with $0_n = 0_{n \times n}$. For a matrix $M \in \mathbb{R}^{n \times m}$, M^\top denotes its transpose. For a square matrix $M \in \mathbb{R}^{n \times n}$, $\text{He}(M) = M + M^\top$ and $M \succ 0_n$ ($M \succeq 0_n$) means that M is positive (semi-)definite. For two symmetric matrices M_1, M_2 of same dimension, $M_1 \succ M_2$ ($M_1 \succeq M_2$) means $M_1 - M_2 \succ 0$

¹ Aurélien Pierron, Jérémie Kreiss, Marc Jungers and Gilles Millérioux are with Université de Lorraine, CNRS, CRAN, F-54000 Nancy, France. {jeremie.kreiss, marc.jungers, gilles.millieroux}@univ-lorraine.fr

² Aurélien Pierron, Jeremy Dupont and Martial Martig are with SPIE Industrie, F-57000 Metz, France. {aurelien.pierron, jeremy.dupont, martial.martig}@spie.com

($M_1 - M_2 \succeq 0$). The same symbol is used for vectors: for a vector $z \in \mathbb{R}^n$, $z \succeq 0$ means that all entries of the vector z are non-negative. For a matrix M , $M_{(j)}$ $M_{(i,j)}$ denote respectively its j -th row and its (i,j) -th entry. The symbol \star denotes a symmetric block in symmetric matrices. For square matrices W and Z , $\text{diag}(W, Z)$ corresponds to the block-diagonal matrix.

II. PROBLEM FORMULATION

A. System description

We consider in this paper the continuous-time system of the form:

$$\begin{aligned} \dot{x}(t) &= Ax(t) + B\Phi[f](t), \\ f(t) &= Kx(t), \end{aligned} \quad (1)$$

where we denote the state $x \in \mathbb{R}^n$, with initial state $x(0) = x_0 \in \mathbb{R}^n$, and the input $f(t) \in \mathbb{R}$ which is scalar for the sake of simplicity. However, the same developments may be applied to extend the study to the multiple input case, by assuming a componentwise definition of the operator Φ . The matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$ and $K \in \mathbb{R}^{1 \times n}$ are constant and given. The nonlinear operator Φ gathers asymmetric backlash and asymmetric dead-zone. It is the main source of difficulty and its definition requires a special attention.

The nonlinear operator $\Phi[f](\cdot)$ exhibits an hysteresis memory and is defined for continuous and piecewise differentiable function $f \in C_{\text{pw}}^1([0, +\infty); \mathbb{R})$. If the unbounded sequence of times $\{t_j\}_{j \in \mathbb{N}}$ is such that f is differentiable over $]t_j, t_{j+1}[$, then we have the evolution of the nonlinearity output given for all time $t \in]t_j, t_{j+1}[$, $\forall j \in \mathbb{N}$:

$$\Phi[f](t) = \begin{cases} l_a \dot{f}(t) & \text{if } \Phi[f](t) \geq 0 \text{ and } \left((\dot{f} \geq 0 \text{ and } \right. \\ & \left. \Phi[f](t) = l_a(f(t) - \rho_a - h)) \text{ or } \right. \\ & \left. (\dot{f}(t) \leq 0 \text{ and } \Phi[f](t) = l_a(f(t) - \rho_a)) \right) \\ l_b \dot{f}(t) & \text{if } \Phi[f](t) \leq 0 \text{ and } \left((\dot{f} \leq 0 \text{ and } \right. \\ & \left. \Phi[f](t) = l_b(f(t) + \rho_b + h)) \text{ or } \right. \\ & \left. (\dot{f}(t) \geq 0 \text{ and } \Phi[f](t) = l_b(f(t) + \rho_b)) \right) \\ 0 & \text{otherwise} \end{cases} \quad (2)$$

Remark 1: Equation (2) defines the three possible derivatives of the nonlinear operator, depending on distinct and asymmetric branches on Figure 1. That follows the definition of the backlash in [6] but adapted to the backlash combines with the asymmetric deadzone and the asymmetry related to the sign of its input.

The parameters of the nonlinearity Φ are $h, l_a, l_b, \rho_a, \rho_b \in \mathbb{R}_{>0}$ and are respectively called the backlash width, the inclination and the threshold when f is positive and negative. The characteristic of the Φ is depicted on Figure 1.

B. Activation of the non-linearity

The function $t \mapsto \Phi[f](t)$ is continuous. The evolution of the output $\Phi[f](t)$ is well defined by relation (2). Nevertheless, the initial condition $\Phi[f](0)$ is not uniquely defined

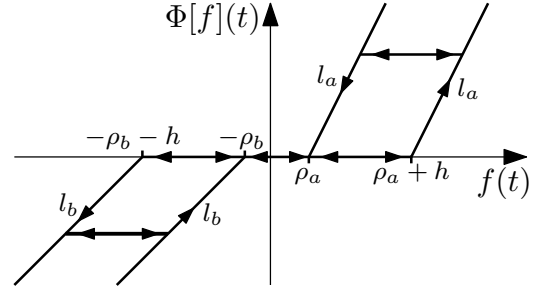


Fig. 1. Characteristic of the non-linear operator Φ .

by the knowledge of the function $f \in C_{\text{pw}}^1([0, +\infty); \mathbb{R})$ but belongs to an interval depending on the initial value $f(0)$:

$$\Phi[f](0) \in \begin{cases} [l_b(f(0) + \rho_b); \min(0, l_b(f(0) + \rho_b + h))], \\ \quad \text{if } f(0) \leq -\rho_b; \\ \{0\}, & \text{if } \rho_b \leq f(0) \leq \rho_a; \\ [\max(0, l_a(f(0) - \rho_a - h)); l_a(f(0) - \rho_a)], \\ \quad \text{if } \rho_a \leq f(0). \end{cases} \quad (3)$$

It can be interpreted as a memory effect.

The following sets Σ_i , $i \in \mathcal{I} = \{1, \dots, 4\}$, are introduced for the formal developments and proofs given in the sequel.

$$\begin{aligned} \Sigma_1 &= \{(x, \phi) \in \mathbb{R}^{n+1}, \phi \geq 0, \\ & \quad \phi + l_a \rho_a \leq l_a Kx \leq \phi + l_a(\rho_a + h)\}, \\ \Sigma_2 &= \{(x, \phi) \in \mathbb{R}^{n+1}, 0 \leq Kx \leq \rho_a, \phi = 0\}, \\ \Sigma_3 &= \{(x, \phi) \in \mathbb{R}^{n+1}, -\rho_b \leq Kx \leq 0, \phi = 0\}, \\ \Sigma_4 &= \{(x, \phi) \in \mathbb{R}^{n+1}, \phi \leq 0, \\ & \quad \phi - l_b(\rho_b + h) \leq l_b Kx \leq \phi - l_b \rho_b\}, \\ \Sigma &= \cup_{i \in \mathcal{I}} \Sigma_i. \end{aligned}$$

We have the following lemma, whose proof is developed in [7] and inspired by [14].

Lemma 1: For any function $f \in C_{\text{pw}}^1([0, +\infty); \mathbb{R})$, let consider an initial condition for the output $\Phi[f](0)$ such that $(f(0), \Phi[f](0)) \in \Sigma$. Then, the function $t \mapsto \Phi[f](t)$ is uniquely defined and $(f(t), \Phi[f](t)) \in \Sigma$, $\forall t \in \mathbb{R}_{\geq 0}$.

Hereafter, we will consider the following assumption.

Assumption 1: The initial condition $\Phi[Kx](0)$ is chosen such that $(Kx(0), \Phi[Kx](0)) \in \Sigma$. \square

C. Uniform Ultimate Boundedness property

The major concern of this study is in the neighborhood of the origin. Indeed, because of the dead zone behavior of Φ around zero, one can easily see that $\Phi[Kx](t) = 0$ and the system (1) is in open-loop. The open-loop system might present instability (since no assumptions are made on matrix A) for the origin, which prevent us to study stability of System (1). Moreover, (1) may exhibit infinite number of equilibrium points (this is the case when zero is an eigenvalue of A).

For all these reasons, we are interested in a weaker notion than stability which is known as Uniform Ultimate Boundedness (UUB) for system (1), whose definition is as follows.

Definition 1 (UUB): [15, Definition 4.6] The trajectory of system (1) is *uniformly ultimately bounded* with ultimate bound b if there exist positive constants b and c , independent of $t_0 > 0$, and for every $a \in (0, c)$, there is $T = T(a, b) \geq 0$, independent of t_0 , such that $\|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b, \forall t \geq t_0 + T$. A set of the state-space implying that $\|x\| \leq b$ is called a UUB-set. \square

The nonlinear operator having affine branches with respect to $f(t) = Kx(t)$, we introduce the next assumption.

Assumption 2: The triplet (A, B, K) is such that the matrices $(A - Bl_k K)$, $k \in \{a, b\}$, are Hurwitz. \square

Assumption 2 allows to cope with possibly not Hurwitz matrix A and will be a necessary condition for the feasibility of the LMIs in the main result (see Theorem 1 in Section IV and also [6]).

Based on Definition 1, we investigate the following problem.

Problem 1 (Uniform Ultimate Boundedness analysis):

Given the system (1) and a gain K such that Assumptions 1 and 2 hold, minimize the size of the UUB-set of System (1), for which the UUB property is guaranteed. \square

III. PRELIMINARIES AND TECHNICAL RESULTS

Our approach to solve Problem 1 is based on the idea presented in [15, Theorem 4.18] using a Lyapunov-like function (called in the following UUB-Lyapunov function) that decreases along the trajectory outside its unit level set and remains less than 1 inside.

Since the system is defined over a polyhedral partition (due to the non-linear operator), and in order to fit as much as possible the UUB-set, we will consider piecewise Lyapunov functions over this partition. This is the main improvement with respect to the result in [7].

In this case, we need to ensure continuous property of the overall UUB-Lyapunov function on the borders of each region. In the following, we will ensure, over each intersection between a region of the partition and the complementary of the unit level set of the UUB-Lyapunov function, that the regional Lyapunov function is decreasing. The associated conditions will involve the state x , the nonlinearity Φ and its time-derivative $\dot{\Phi}$.

Finally, since in [7], it is shown that the non-linearity Φ satisfies *quadratic sector conditions* w.r.t. the state, Φ and $\dot{\Phi}$, and since the sets Σ_i are compatible with intersection of hyperplans depending only on the state, we focus on Lyapunov functions that are quadratic with respect to x only.

Therefore, we are looking in this paper at *continuous piecewise quadratic UUB-Lyapunov function*. This section will provide the tools and notations to formalize Problem 1, before presenting the main result in Section IV. Subsection III-A treats the structure of continuous piecewise quadratic UUB-Lyapunov function. Subsection III-B offers a background for the nonlinear operator.

A. Structure of a continuous piecewise quadratic UUB-Lyapunov function

From the definitions of Σ_i , $i \in \mathcal{I}$, we can divide the state space \mathbb{R}^n in 4 regions bounded by the three parallel

hyperplans $f(t) = Kx(t) \in \{-\rho_b; 0; \rho_a\}$. Figure 1 shows the relevance of such a decomposition. In order to highlight the polyhedral partition, we introduce the notation $\bar{x} = (x^\top \ 1)^\top \in \mathbb{R}^{n+1}$ and the following polyhedral partition (see [8, Definition A.8]) of the state-space \mathbb{R}^n related to the definition of sets Σ_i , $i \in \mathcal{I}$. Let us define $\mathcal{X}_i = \{x \in \mathbb{R}^n, X_i \bar{x} \succeq 0\}$, $i \in \mathcal{I}$, with $X_i \in \mathbb{R}^{n_i \times (n+1)}$,

$$\begin{aligned} X_1 &= \begin{bmatrix} K & -\rho_a \end{bmatrix}, & X_4 &= \begin{bmatrix} -K & -\rho_b \end{bmatrix}, \\ X_2 &= \begin{bmatrix} -K & \rho_a \\ K & 0 \end{bmatrix}, & X_3 &= \begin{bmatrix} K & \rho_b \\ -K & 0 \end{bmatrix}. \end{aligned}$$

We can check that $\cup_{i \in \mathcal{I}} \mathcal{X}_i = \mathbb{R}^n$ and that only the intersections between consecutive sets are not empty, such that only $\mathcal{X}_i \cap \mathcal{X}_{i+1} \neq \emptyset$, $i \in \{1, 2, 3\}$. In addition the sets \mathcal{X}_i are closed and the intersections of their interiors are empty. The origin belongs only to \mathcal{X}_2 and \mathcal{X}_3 . We introduce $\mathcal{I}_0 = \{2, 3\} \subset \mathcal{I}$, the set of indices i such that $0 \in \mathcal{X}_i$. The projection of Σ_i on its n -first components belongs to \mathcal{X}_i , $i \in \mathcal{I}$.

By following the method detailed in [8, Appendix A], we can build the continuity matrices $\bar{F}_i \in \mathbb{R}^{(n+3) \times (n+1)}$, $i \in \mathcal{I}$, based on the three halfplanes $\{x \in \mathbb{R}^n, Kx - \rho_a \succeq 0\}$, $\{x \in \mathbb{R}^n, Kx + 0 \succeq 0\}$ and $\{x \in \mathbb{R}^n, -Kx - \rho_b \succeq 0\}$ as

$$\begin{aligned} \bar{F}_1 &= \begin{bmatrix} K & -\rho_a \\ K & 0 \\ 0_{1 \times n} & 0 \\ I_n & 0_{n \times 1} \end{bmatrix}, & \bar{F}_2 &= \begin{bmatrix} 0_{1 \times n} & 0 \\ K & 0 \\ 0_{1 \times n} & 0 \\ I_n & 0_{n \times 1} \end{bmatrix}, \\ \bar{F}_3 &= \begin{bmatrix} 0_{1 \times n} & 0 \\ 0_{1 \times n} & 0 \\ 0_{1 \times n} & 0 \\ I_n & 0_{n \times 1} \end{bmatrix}, & \bar{F}_4 &= \begin{bmatrix} 0_{1 \times n} & 0 \\ 0_{1 \times n} & 0 \\ -K & -\rho_b \\ I_n & 0_{n \times 1} \end{bmatrix}. \end{aligned}$$

By construction, the origin belonging in \mathcal{X}_2 and \mathcal{X}_3 , the last column of \bar{F}_2 and \bar{F}_3 are trivial. For $i \in \mathcal{I}_0$, we denote $F_i \in \mathbb{R}^{(n+3) \times n}$, the extraction of the first n columns of \bar{F}_i . This choice allows to impose

$$\bar{F}_i \bar{x} = \bar{F}_{i+1} \bar{x}, \forall x \in \mathcal{X}_i \cap \mathcal{X}_{i+1}, i \in \{1, 2, 3\}. \quad (4)$$

The last block rows in \bar{F}_i , i.e. $[I_n \ 0_{n \times 1}]$, translates the continuity of the state in \mathbb{R}^n .

This polyhedral partition $\{\mathcal{X}_i\}_{i \in \mathcal{I}}$ of \mathbb{R}^n , compatible with the sets Σ_i , $i \in \mathcal{I}$, depends only on the state x . For this reason, we focus on regional UUB-Lyapunov functions that are quadratic forms with respect to the state. As a consequence, we can look for a UUB-Lyapunov function in the class of continuous piecewise quadratic functions, defined by

$$V(x) = V_i(x), \quad x \in \mathcal{X}_i, \quad (5)$$

where the regional quadratic functions $V_i(x)$ are given by

$$V_i(x) = \begin{cases} \bar{x}^\top \bar{P}_i \bar{x} = \bar{x}^\top \bar{F}_i^\top T \bar{F}_i \bar{x}, & \forall i \in \mathcal{I}/\mathcal{I}_0, \\ \bar{x}^\top \bar{P}_i \bar{x} = x^\top P_i x = x^\top F_i^\top T F_i x, & \forall i \in \mathcal{I}_0, \end{cases} \quad (6)$$

with $T = T^\top \in \mathbb{R}^{(n+3) \times (n+3)}$, a weighting matrix.

Lemma 2 ([9]): The function $V(\cdot)$ defined by (5) and (6) is continuous over \mathbb{R}^n , irrespectively to the weighting matrix T . \square

Proof: The regional quadratic function (6) are continuous. To prove the continuity property, we have to check the continuity of V on the non-empty intersections of the regions \mathcal{X}_i . Thanks to the relation (4), we have $\forall x \in \mathcal{X}_i \cap \mathcal{X}_{i+1}$, $i \in \{1, \dots, 3\}$

$$V_i(x) = \bar{x}^\top \bar{F}_i^\top T \bar{F}_i \bar{x} = \bar{x}^\top \bar{F}_{i+1}^\top T \bar{F}_{i+1} \bar{x} = V_{i+1}(x).$$

The piecewise quadratic function V is thus continuous over \mathbb{R}^n , without constraint about the weighting matrix T . ■

In order to obtain other required properties of V via Lemmas 3 and 4, let us describe the building of the polyhedral cell bounding related to the polyhedral partition of \mathbb{R}^n [8, Algorithm A.1]:

- If $i \in \mathcal{I}_0$, \bar{E}_i is obtained by deleting all rows of X_i whose last entry is non-zero. E_i is then obtained by extracting the n first columns of the resulting \bar{E}_i .
- If $i \in \mathcal{I} \setminus \mathcal{I}_0$, \mathcal{X}_i is unbounded and \bar{E}_i is obtained by augmenting X_i with the row $[0_{1 \times n} \ 1]$.

This procedure leads to

$$\begin{aligned} \bar{E}_1 &= \begin{bmatrix} K & -\rho_a \\ 0_{1 \times n} & 1 \end{bmatrix}, & \bar{E}_4 &= \begin{bmatrix} -K & -\rho_b \\ 0_{1 \times n} & 1 \end{bmatrix}, \\ E_2 &= K, & E_3 &= -K, \end{aligned}$$

and ensures the implications

$$\forall i \in \mathcal{I}_0, \quad x \in \mathcal{X}_i \Rightarrow E_i x \succeq 0, \quad (7)$$

$$\forall i \in \mathcal{I} \setminus \mathcal{I}_0, \quad x \in \mathcal{X}_i \Rightarrow \bar{E}_i \bar{x} \succeq 0. \quad (8)$$

Lemma 3: Assume that V obeys Equations (6), then there exists a scalar $\beta > 0$ such that $V(x) \leq \beta \|x\|^2$, $\forall x \in \mathbb{R}^n$. □

Proof: The idea of the proof can be found in the appendix of [9]. The main argument is that in an open neighborhood of the origin, V is piecewise quadratic with respect to the state x (instead of \bar{x} , there is no affine term in x or constant term). For $i \in \mathcal{I}_0$, there exists $\beta_i > 0$ such that $V_i(x) \leq \beta_i \|x\|^2$. By construction, there exists $\epsilon > 0$ small enough such that $\|x\|^2 \leq \epsilon$ implies that $x \in \cup_{i \in \mathcal{I}_0} \mathcal{X}_i$. Contraposing this statement implies, $x \in \cup_{i \in \mathcal{I} \setminus \mathcal{I}_0} \mathcal{X}_i$ induces $\|x\|^2 / \epsilon \geq 1$. For each $i \in \mathcal{I} \setminus \mathcal{I}_0$, there exists $\tilde{\beta}_i > 0$ such that $V_i(x) \leq \tilde{\beta}_i \|\bar{x}\|^2 \leq \tilde{\beta}_i (1 + 1/\epsilon) \|x\|^2 = \beta_i \|x\|^2$, with $\beta_i = \tilde{\beta}_i (1 + 1/\epsilon)$. The proof ends by selecting $\beta = \max_{i \in \mathcal{I}} \beta_i$. ■

Lemma 4: If there exist a symmetric matrix $T \in \mathbb{R}^{(n+3) \times (n+3)}$ and matrices $U_i, i \in \mathcal{I}$ of adequate dimensions with nonnegative entries, such that the LMIs

$$F_i^\top T F_i - E_i^\top U_i E_i \succ 0_n, \quad i \in \mathcal{I}_0 \quad (9)$$

$$\bar{F}_i^\top T \bar{F}_i - \bar{E}_i^\top U_i \bar{E}_i \succ 0_{n+1}, \quad i \in \mathcal{I} \setminus \mathcal{I}_0, \quad (10)$$

hold, then there exists $\alpha > 0$ such that $V(x) \geq \alpha \|x\|^2$. □

Proof: The strict inequalities (9) allow the existence of $\alpha_i > 0$ small enough such that 0_n can be replaced in (9) by $\alpha_i I_n$. Multiplying left and right the resulting inequalities by x^\top and x respectively leads to $i \in \mathcal{I}_0$

$$V_i(x) - \alpha_i \|x\|^2 \geq \sum_{j,k} U_{i,(j,k)} (E_{i,(j)} x) (E_{i,(k)} x) \geq 0.$$

The latter inequality comes from implication (7). We proceed in the same way for inequalities (10) by replacing 0_{n+1} by $\text{diag}(\alpha_i I_n, 0)$ and multiplying the result by \bar{x}^\top and \bar{x} , $i \in \mathcal{I} \setminus \mathcal{I}_0$:

$$V_i(x) - \alpha_i \|x\|^2 \geq \sum_{j,k} U_{i,(j,k)} (\bar{E}_{i,(j)} \bar{x}) (\bar{E}_{i,(k)} \bar{x}) \geq 0.$$

The latter inequality is a consequence of implication (8). Selecting $\alpha = \min_{i \in \mathcal{I}} \alpha_i$ completes the proof. ■

Due to Lemmas 2 3 and 4, V can act as a candidate continuous UUB-Lyapunov function.

B. Characteristics of the nonlinear operator

This subsection aims at providing sufficient conditions ensuring that the UUB-Lyapunov function V decreases outside its unit level set, (that is on $\{x \in \mathbb{R}^n, V(x) \geq 1\}$), by taking advantage of the characteristics of the nonlinear operator Φ .

To use Assumption 2, we consider an adaptation of the dual nonlinear operator Ψ related to the nonlinear operator Φ , (see [6] and [7]) by:

$$\Psi[Kx](t) = \Phi[Kx](t) - l_i Kx(t), \quad \forall x \in \mathcal{X}_i, \quad (11)$$

with $l_1 = l_2 = l_a$ and $l_3 = l_4 = l_b$. The characteristic of the nonlinear operator Ψ is depicted in Figure 2. The interest is that for Ψ is bounded, in particular when $x \in \Sigma_1 \cup \Sigma_4$.

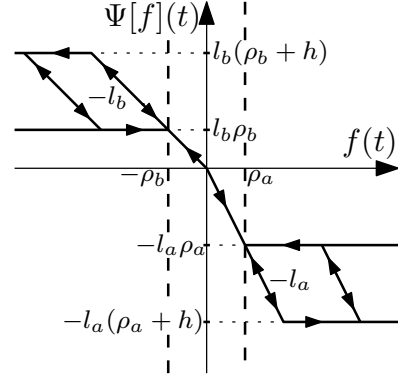


Fig. 2. Characteristic of the dual nonlinear operator Ψ .

In order to express compactly the dynamics and the quadratic sector conditions related to Φ , we consider the augmented vectors

$$z = (x^\top \ \Psi \ \dot{\Psi} \ 1)^\top \in \mathbb{R}^{n+3},$$

$$y = (x^\top \ \Phi \ \dot{\Phi} \ 1)^\top \in \mathbb{R}^{n+3},$$

that are related one another by $y = N_i z$ over $x \in \mathcal{X}_i$, where

$$N_i = \begin{pmatrix} I_n & 0 & 0 & 0 \\ l_i K & 1 & 0 & 0 \\ l_i K(A + B l_i K) & l_i K B & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad i \in \mathcal{I}.$$

In addition, we have $\bar{x} = N_0 z$, with

$$N_0 = \begin{pmatrix} I_n & 0_{n \times 1} & 0_{n \times 1} & 0_{n \times 1} \\ 0_{1 \times n} & 0 & 0 & 1 \end{pmatrix}.$$

It is thus possible to rewrite the dynamics (1) as:

$$\dot{\hat{x}}(t) = \Gamma_i z(t), \quad x \in \mathcal{X}_i, \quad (12)$$

where, $i \in \mathcal{I}$,

$$\Gamma_i = \begin{pmatrix} A + Bl_i K & B & 0_{n \times 1} & 0_{n \times 1} \\ 0_{1 \times n} & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{(n+1) \times (n+3)}.$$

Let us now focus on the characteristics of the nonlinear operator. We can distinguish two kinds of conditions:

- Several conditions can be expressed as polyhedral constraints related to x , Ψ and also $\dot{\Psi}$. They correspond to a reformulation of the sets $\Sigma_i \in \mathbb{R}^{n+1}$, $i \in \mathcal{I}$, and furthermore to the fact that in Σ_2 and Σ_3 , we have $\dot{\Phi} = 0$. It results to the polyhedral constraints $\{z \in \mathbb{R}^{n+3}, \bar{G}_i z \succeq 0\}$, with

$$\bar{G}_1 = \begin{bmatrix} l_1 K & 1 & 0 & 0 \\ 0_{1 \times n} & -1 & 0 & -l_1 \rho_a \\ 0_{1 \times n} & 1 & 0 & l_1(\rho_a + h) \end{bmatrix},$$

$$\bar{G}_4 = \begin{bmatrix} -l_4 K & -1 & 0 & 0 \\ 0_{1 \times n} & -1 & 0 & l_4(\rho_b + h) \\ 0_{1 \times n} & 1 & 0 & -l_4 \rho_b \end{bmatrix},$$

and for $i \in \mathcal{I}_0$,

$$\bar{G}_i = \begin{bmatrix} K & 0 & 0 & (i-2)\rho_b \\ -K & 0 & 0 & (3-i)\rho_a \\ l_i K & 1 & 0 & 0 \\ -l_i K & -1 & 0 & 0 \\ l_i K(A + Bl_i K) & l_i KB & 1 & 0 \\ -l_i K(A + Bl_i K) & -l_i KB & -1 & 0 \end{bmatrix}.$$

- Equation (2) allows sector conditions in Σ_1 and Σ_4 . They are gathered in Lemma 5 which is proven in [7].

Lemma 5: In Σ_1 , we have the following quadratic sector conditions, for any $\alpha_1 \geq 1$:

$$\dot{\Phi}(\Psi + l_1(\rho_a + h/2)) \leq 0, \quad (13)$$

$$\dot{\Phi}(\dot{\Phi} - \alpha_1 l_1 K \dot{x}) \leq 0, \quad (14)$$

which imply that the quadratic forms in z satisfy respectively $z^\top M_{1,1} z \leq 0$ and $z^\top M_{2,1} z \leq 0$, with

$$M_{1,1} = N_1^\top \begin{pmatrix} 0 & 0 & -K^\top l_1 & 0 \\ \star & 0 & 1 & 0 \\ \star & \star & 0 & l_1(\rho_a + \frac{h}{2}) \\ \star & \star & \star & 0 \end{pmatrix} N_1,$$

$$M_{2,1} = N_1^\top \begin{pmatrix} 0 & 0 & -(\alpha_1 l_1 KA)^\top & 0 \\ \star & 0 & -(\alpha_1 l_1 KB)^\top & 0 \\ \star & \star & 2 & 0 \\ \star & \star & \star & 0 \end{pmatrix} N_1.$$

In Σ_4 , we have the following quadratic sector conditions, for any $\alpha_4 \geq 1$:

$$\dot{\Phi}(\Psi - l_4(\rho_b + h/2)) \leq 0, \quad (15)$$

$$\dot{\Phi}(\dot{\Phi} - \alpha_4 l_4 K \dot{x}) \leq 0, \quad (16)$$

which imply that the quadratic forms in z satisfy respectively $z^\top M_{1,4} z \leq 0$ and $z^\top M_{2,4} z \leq 0$, with

$$M_{1,4} = N_4^\top \begin{pmatrix} 0 & 0 & -K^\top l_2 & 0 \\ \star & 0 & 1 & 0 \\ \star & \star & 0 & -l_2(\rho_b + \frac{h}{2}) \\ \star & \star & \star & 0 \end{pmatrix} N_4,$$

$$M_{2,4} = N_4^\top \begin{pmatrix} 0 & 0 & -(\alpha_4 l_4 KA)^\top & 0 \\ \star & 0 & -(\alpha_4 l_4 KB)^\top & 0 \\ \star & \star & 2 & 0 \\ \star & \star & \star & 0 \end{pmatrix} N_4.$$

□

Proof: The proof can be found in the Lemmas in [7]. ■

IV. MAIN RESULT

This section provides sufficient conditions, expressed in terms of parameterized Linear Matrix Inequalities, in order to obtain a solution to Problem 1. Theorem 1 gathers these sufficient conditions and finally an optimization problem is presented to select a solution optimizing the size of the UUB-set.

Theorem 1: Consider the system (1) that verifies Assumption 1. Assume that there exist a symmetric matrix $T \in \mathbb{R}^{(n+3) \times (n+3)}$, matrices U_i and W_i , $i \in \mathcal{I}$ with suitable dimensions and nonnegative components, positive scalars $\tau_{0,i}$, $\tau_{1,1}$, $\tau_{2,1}$, $\tau_{1,4}$, $\tau_{2,4}$, such that the inequalities (9), (10) and

$$\Pi_i + \bar{G}_i^\top W_i \bar{G}_i \prec 0, \quad i \in \mathcal{I}_0, \quad (17)$$

$$\Pi_i - \tau_{1,i} M_{1,i} - \tau_{2,i} M_{2,i} + \bar{G}_i^\top W_i \bar{G}_i \prec 0, \quad i \in \mathcal{I} \setminus \mathcal{I}_0, \quad (18)$$

where

$$\Pi_i = \text{He}(\Gamma_i^\top \bar{P}_i N_0)$$

$$- \tau_{0,i} (\text{diag}(0_n, 0, 0, 1) - N_0^\top \bar{P}_i N_0), \quad i \in \mathcal{I},$$

hold. Then, for any initial choice of initial conditions $(Kx(0), \Phi[Kx](0))$ satisfying Assumption 2, the system (1) is UUB with the UUB-Lyapunov function V given by (5)–(6) and the UUB-set

$$L_V = \{x \in \mathbb{R}^n, V(x) \leq 1\}. \quad (19)$$

□

Proof: Let us consider the function V having the structure detailed in (6). V is continuous over \mathbb{R}^n . The unit level set L_V , defined by (19) is then closed. The inequalities (9) and (10) being satisfied, Lemmas 3 and 4 apply and thus V is a UUB-Lyapunov function candidate. Consider the structure of matrices Π_i , we have

$$\begin{aligned} z^\top \Pi_i z &= 2\bar{x}^\top \bar{P}_i \bar{x} - \tau_{0,i}(1 - \bar{x}^\top \bar{P}_i \bar{x}), \\ &= \dot{V}(x) - \tau_{0,i}(1 - V(x)), \quad \forall i \in \mathcal{I}. \end{aligned}$$

For $i \in \mathcal{I}_0$, thanks to the strict inequality (17), there exists $\delta_i > 0$ such that

$$\begin{aligned} \dot{V}(x) - \tau_{0,i}(1 - V(x)) + \delta_i \|x\|^2 \\ \leq - \sum_{j,k} W_{i,(j,k)} (\bar{G}_{i,(j)} z) (\bar{G}_{i,(k)} z), \end{aligned}$$

leading to $\dot{V}(x) \leq -\frac{\delta_i}{\alpha}V(x)$ in $\mathcal{X}_i \setminus L_V$, $i \in \mathcal{I}_0$, due to the S -procedure.

For $i \in \mathcal{I} \setminus \mathcal{I}_0$, the strict inequality (18) implies that there exists $\delta_i > 0$ such that

$$\begin{aligned} \dot{V}(x) - \tau_{0,i}(1 - V(x)) - \tau_{1,i}z^\top M_{1,i}z - \tau_{2,i}z^\top M_{2,i}z \\ + \delta_i \|x\|^2 \leq - \sum_{j,k} W_{i,(j,k)}(\bar{G}_{i,(j)}z)(\bar{G}_{i,(k)}z), \end{aligned}$$

leading to $\dot{V}(x) \leq -\frac{\delta_i}{\alpha}V(x)$ in $\mathcal{X}_i \setminus L_V$, $i \in \mathcal{I} \setminus \mathcal{I}_0$. Finally denoting $\delta = \min_{i \in \mathcal{I}} \delta_i$, we have

$$\dot{V}(x) \leq -\frac{\delta}{\alpha}V(x), \quad \forall x \notin L_V.$$

The latter inequality guarantees that, outside L_V , the function V decreases exponentially to reach (in finite time) the value 1 and moreover remains below the value 1 for all future time (see [15]). In other words, the set L_V is positively invariant by the dynamics (1). ■

Remark 2: It is noteworthy that the use of the S -procedure with matrices W_i in (17) and (18) is not the only way to consider the polyhedral constraints. Other structures are proposed in [16] for instance. One may also use a more general cone sector conditions in Σ_2 and Σ_3 to take into account the case when $\Phi = 0$ and $\dot{\Phi} = 0$, as in [7]. Finally it should be underlined that the nonlinear operator is not the unique one satisfying the sector bounded conditions in zones 1 and 4.

Among all the solutions of Theorem 1, we would like to select the one minimizing the size of the level set L_V (19). The difficulty is to have a cost function of an optimization problem that is related to the size of a level set of a piecewise quadratic positive function and also that depends only in the variables of the constraints in Theorem 1. By following the discussion in [8, Section 7.1], the level set of a piecewise quadratic positive function consists in a collection of ellipsoids parts. We suggest to consider the average of the size of these ellipsoids as an approximation of the size of the level set L_V . In other words, we would like to minimize $\text{Tr}(Q)$, where $Q = Q^\top$ such that

$$\begin{bmatrix} Q & I_n & \cdots & I_n \\ I_n & P_1 & & 0_n \\ \vdots & & \ddots & \\ I_n & 0_n & & P_4 \end{bmatrix} \succ 0, \quad (20)$$

which is equivalent, due to a Schur complement, to $Q \succ \sum_{i \in \mathcal{I}} P_i^{-1}$.

Remark 3: Other choices may be possible to approximate the size of L_V . Furthermore, it is possible to weight the matrices P_i in (20) to take into account the ratio of the fragments of ellipsoids composing L_V . The size of each ellipsoid is related to P_i and not \bar{P}_i , that contains also a shift of the ellipsoid center.

Let us provide the optimization problem that offers a solution to Problem 1.

$$\min_{T, Q, U_i, W_i, \tau_{0,i}, \tau_{1,1}, \tau_{2,1}, \tau_{1,4}, \tau_{2,4}} \text{Tr}(Q),$$

under (9), (10), (18), (17).

Notice that the constraints in the optimization problem are LMIs if the parameters $\tau_{0,i}$ are given.

V. NUMERICAL ILLUSTRATION

In order to illustrate our result and compare it with the available results in the literature, we consider the example coming from [7]. The linear system is a double integrator and the gain K satisfies Assumption 2.

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad K = \begin{pmatrix} -2 & -3 \end{pmatrix}.$$

$$l_a = 1, \quad l_b = 1.2, \quad \rho_a = 0.1, \quad \rho_b = 0.2, \quad h = 0.2.$$

We impose $\tau_{0,i} = \tau_0$, $\forall i \in \mathcal{I}$, to have a single parameter in the line search. The optimal value of $\text{Tr}(Q)$ depends on the parameter τ_0 . This dependency is depicted in Figure 3. In order to numerically validate the fact that $\text{Tr}(Q)$ approaches the area of L_V , we compute *a posteriori* the area of L_V after solving the optimization problem. We can see on Figure 3, that the two curves have the same shape. We can explain the curve as follows: for $\tau_{0,i}$ equal to zero, our conditions are not feasible because it is not possible with A not Hurwitz to obtain stability of the origin. We need to take into account the fact that $V(t)$ decreases only outside L_V . In addition, $\tau_{0,i}$ should be chosen such that $(A - Bl_k K + \frac{\tau_0}{2})$ is Hurwitz, $k \in \{a, b\}$, that is $\tau_{0,i} < \max_{k \in \{a, b\}} -2\text{Re}(A - Bl_k K) = 1.76$. In other cases, the conditions are not feasible. We consider in the following $\tau_{0,i} = 0.6$, which corresponds to the argument of the minimum in Figure 3.

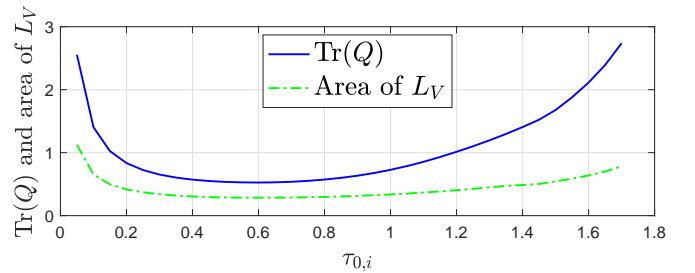


Fig. 3. Optimal value of $\text{Tr}(Q)$ and numerical area of L_V , depending on $\tau_{0,i}$.

We consider the initial condition $x_0 = \begin{pmatrix} 0.8 & 0.4 \end{pmatrix}^\top$ and we choose $\Phi[Kx](0) = -2.5$, which verifies the constraint (3).

We depict the time trajectory of the state $x(t)$ on Figure 4. It is shown that $V(t)$ decreases piecewise exponentially (faster than linearly in the semilog scale) to reach 1 at $t = 2.7$ s and after remains below 1.

The value of the nonlinearity $\Phi[Kx](t)$ follows the characteristic of the nonlinear operator as depicted in Figure 5.

Finally, we plot the state trajectory in the state plan $x(t) \in \mathbb{R}^2$ in Figure 6.

It is clear on Figure 6, that L_V (in red solid line) is a smaller set than the ellipsoid (in black solid line) generated by the conditions in [7] and consequently provides a less

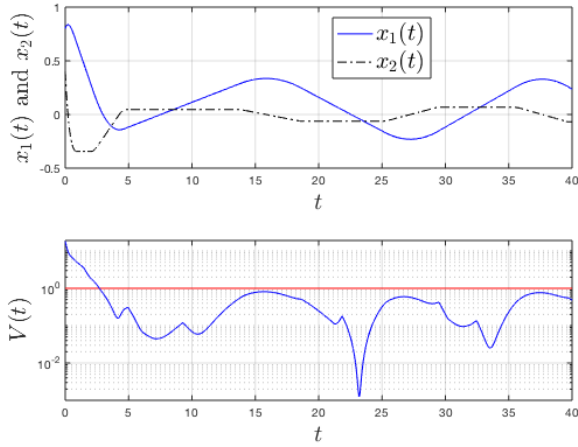


Fig. 4. Time-trajectory of the state $x(t)$ and of the UUB-Lyapunov function $V(t)$.

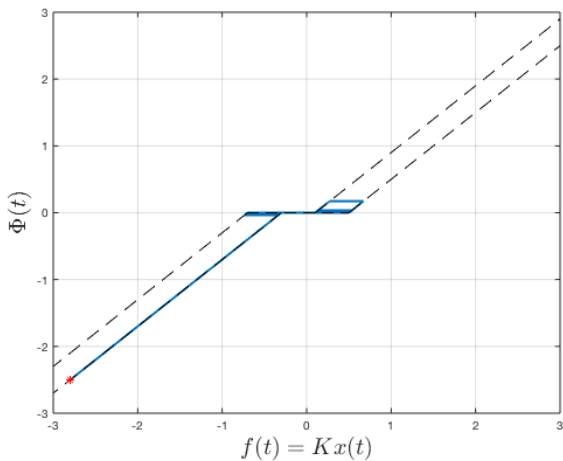


Fig. 5. Characteristic of the nonlinear operator. The starting point corresponds to $(Kx_0, \Phi[Kx](0)) = (-2.8, -2.5)$, depicted with a red cross. The trajectory $(Kx(t), \Phi[Kx](t))$ is in blue. The characteristic of Figure 1 is recalled in back dashes.

conservative solution to Problem 1. L_V is composed of arcs of ellipsoids in the sets \mathcal{X}_i , $i \in \mathcal{I}$ as expected.

By looking at the state trajectories on Figure 6, we can exhibit several behaviors of the system inside the UUB-set: in particular, we can show limit cycles and unstable equilibrium points. For the initial condition $x_0 = (0.8 \ 0.4)^\top$, the state-trajectory tends to the set L_V and converges to a limit cycle that lies into L_V . Furthermore, zero is an eigenvalue of A with associated eigenvector $v_0 = (1 \ 0)^\top$. The horizontal segment

$$\begin{aligned} \mathcal{S} &= \{\mu v_0 \in \mathbb{R}^n, -\rho_b \leq \mu K v_0 \leq \rho_a\}, \\ &= \{(\mu \ 0)^\top, \mu \in [-0.05; 0.15]\} \end{aligned}$$

is a not empty continuum set of equilibrium points.

In Figure 6, two other trajectories are depicted starting from $x_0 = (0.3 \ -0.5)^\top$ and $x_0 = (-0.1 \ 0.5)^\top$.

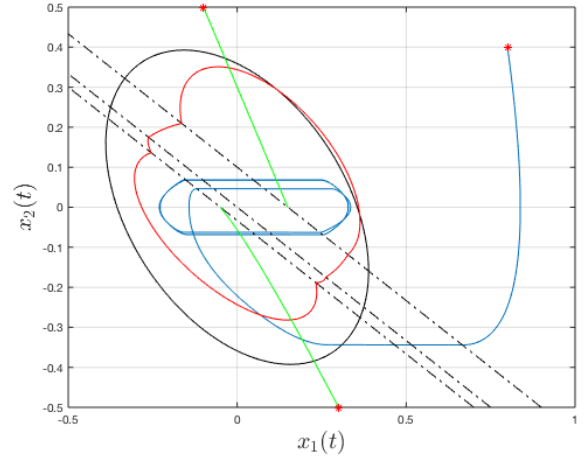


Fig. 6. Phase portrait of the system. The bounds of the polyhedral partition \mathcal{X}_i are depicted with dashed-dot lines. The black ellipsoid is the UUB-set provided in [7] and the red curve depicted L_V from Theorem 1. Several trajectories are depicted starting from $x_0 = (0.8 \ 0.4)^\top$ (in blue solid lines) and from $x_0 = (0.3 \ -0.5)^\top$ and $x_0 = (-0.1 \ 0.5)^\top$ (in green solid lines).

They respectively belong to \mathcal{X}_4 and \mathcal{X}_1 and reach the end points of the segment \mathcal{S} , which are only attractive over \mathcal{X}_4 and \mathcal{X}_1 but not on \mathcal{X}_3 and \mathcal{X}_2 and cannot be stable equilibrium points.

Finally, we can emphasize that the trajectories may cross each other. That is a consequence of the fact that the dynamics (1) does not depend only on the state, but also on the memory of the trajectory via the nonlinear operator Φ .

VI. CONCLUSION

A piecewise quadratic Lyapunov function approach has been used to cope with the uniform ultimately boundedness property of a system which is a linear plant that is feedback interconnected with a nonlinear operator exhibiting an asymmetric backlash and an asymmetric deadzone. The requirements of a Lyapunov function candidate are imposed thanks to linear matrix inequalities and the exponential uniform ultimately boundedness property is ensured thanks to parameterized linear matrix inequalities. The tractability of our sufficient conditions and the fact that our solution is less conservative than the results available in the literature are discussed on an academic illustration. For future research, we will be focused on the three following points: (i) extend our result in the framework of control design, (ii) alleviate the conservatism of the techniques and (iii) investigate the feasibility of the proposed LMIs.

REFERENCES

- [1] J.-J. E. Slotine and W. Li, *Applied Nonlinear Control*. Prentice Hall, 1991.
- [2] X. Tan and R. V. Iyer, "Modeling and control of hysteresis," *IEEE Control Systems Magazine*, vol. 29, no. 1, pp. 26–28, 2009.
- [3] R. V. Iyer and X. Tan, "Control of hysteretic systems through inverse compensation," *IEEE Control Systems Magazine*, vol. 29, no. 1, pp. 83–99, 2009.

- [4] M. Al Janaideh, M. Al Saaideh, and X. Tan, "The Prandtl–Ishlinskii hysteresis model: Fundamentals of the model and its inverse compensator [Lecture Notes]," *IEEE Control Systems Magazine*, vol. 43, no. 2, pp. 66–84, 2023.
- [5] M. Al Janaideh, R. Naldi, L. Marconi, and P. Krejčí, "A hybrid system for a class of hysteresis nonlinearity: Modeling and compensation," in *2012 IEEE 51st IEEE Conference on Decision and Control (CDC)*, pp. 5380–5385, 2012.
- [6] S. Tarbouriech, I. Queinnec, and C. Prieur, "Stability Analysis and Stabilization of Systems With Input Backlash," *IEEE Transactions on Automatic Control*, vol. 59, pp. 488–494, Feb. 2014.
- [7] A. Pierron, J. Kreiss, M. Jungers, G. Millerioux, and J. Dupont, "Stability analysis for linear systems with asymmetric input backlash and dead-zone through LMI conditions," in *2023 European Control Conference (ECC)*, pp. 1–6, 2023.
- [8] M. Johansson, *Piecewise Linear Control Systems - A Computational Approach*. Lecture Notes in Control and Information Sciences, Heidelberg, Germany: Springer-Verlag, 2002.
- [9] M. Johansson and A. Rantzer, "Computation of piecewise quadratic Lyapunov functions for hybrid systems," *IEEE Transactions on Automatic Control*, vol. 43, pp. 555–559, April 1998.
- [10] A. Hassibi, S. Boyd, and J. How, "A class of Lyapunov functionals for analyzing hybrid dynamical systems," in *Proceedings of the 1999 American Control Conference*, vol. 4, pp. 2455–2460, 1999.
- [11] S. Tarbouriech, G. Garcia, J. M. Gomes da Silva Jr., and I. Queinnec, *Stability and Stabilization of Linear Systems with Saturating Actuators*. Springer, 2011.
- [12] D. Dai, T. Hu, A. R. Teel, and L. Zaccarian, "Piecewise-quadratic Lyapunov functions for systems with deadzones or saturations," *Systems & Control Letters*, vol. 58, no. 5, pp. 365–371, 2009.
- [13] V. Dilda, M. Jungers, and E. B. Castelan, "Uniformly ultimate boundedness analysis and synthesis for linear systems with dead-zone in the actuators," *International Journal of Robust and Nonlinear Control*, vol. 25, no. 14, pp. 2502–2514, 2015.
- [14] M. L. Corradini and G. Orlando, "Robust stabilization of nonlinear uncertain plants with backlash or dead zone in the actuator," *IEEE Transactions on Control System Technology*, vol. 10, no. 1, pp. 158–166, 2002.
- [15] H. K. Khalil, *Nonlinear Systems - Third Edition*. Prentice Hall, 2002.
- [16] A. Hassibi and S. Boyd, "Quadratic stabilization and control of piecewise-linear systems," in *Proceedings of the 1998 American Control Conference (ACC)*, vol. 6, pp. 3659–3664, 1998.