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Philipp Weißgraeber, Dominique Leguillon, Wilfried Becker. A review of Finite Fracture Mechanics: crack initiation at singular and non-singular stress raisers. *Archive of Applied Mechanics*, 2015, 86 (1-2), pp.375-401. 10.1007/s00419-015-1091-7. hal-04794172

HAL Id: hal-04794172

<https://hal.science/hal-04794172v1>

Submitted on 29 Dec 2024

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A review of Finite Fracture Mechanics: crack initiation at singular and non-singular stress raisers

Philipp Weißgraeber, Dominique Leguillon, Wilfried Becker

Abstract Crack initiation in brittle materials is not covered by classical fracture mechanics that deals only with the growth of pre-existing cracks. In order to overcome this deficiency, the Finite Fracture Mechanics concept assumes the instantaneous formation of cracks of finite size at initiation. Within this framework, a coupled criterion was proposed at the beginning of the 2000's requiring two necessary conditions to be fulfilled simultaneously. The first one compares the tensile stress to the tensile strength, while the other uses an energy balance and the material toughness. The present analysis is restricted to the 2D case, and, through a wide list of references, it is shown that this criterion gives predictions in agreement with experiments in various cases of stress concentration, which can be classified in two categories: the singularities, i.e. indefinitely growing stresses at a point, and the non-singular stress raisers. It is applied to different materials and structures: notched specimens, laminates, adhesive joints or embedded inclusions. Of course, a lot of work remains to do in these domains but also in domains that are almost not explored such as fatigue loadings and dynamic loadings as well as a sound 3D extension. Some ideas in these directions are issued before concluding that FFM and the coupled criterion have filled a gap in fracture mechanics.

Résumé L'approche classique de la mécanique de la rupture des matériaux fragiles n'aborde pas les problèmes d'initiation de nouvelles fissures, elle ne traite que la croissance de fissures préexistantes. Afin de surmonter cette déficience, l'approche connue sous la désignation anglo-saxonne de Finite Fracture Mechanics suppose la formation instantanée de fissures de taille finie à l'initiation. Développé dans ce cadre, le critère couplé requiert la vérification simultanée de deux conditions nécessaires : la première compare la contrainte de traction à la résistance en traction du matériau tandis que l'autre utilise une équation de conservation de l'énergie et fait appel à la ténacité. La présente analyse se limite au cas 2D et, à travers une longue liste de références, il est montré que le critère couplé donne, aux points de concentration de contraintes, des prédictions d'amorçage de fissures qui sont en accord avec les expériences. On peut distinguer deux catégories : les singularités, lorsque les contraintes croissent indéfiniment en s'approchant d'un point, et les simples concentrations de contraintes, lorsque celles-ci tout en étant élevées restent bornées. Le critère est appliqué à différents matériaux et structures: éprouvettes entaillées, composites stratifiés, joints adhésifs, inclusions. Bien sûr, beaucoup de travail reste à faire dans ces domaines, mais il existe aussi des sujets qui ne sont pratiquement pas explorés comme la fatigue,

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les chargements dynamiques ainsi qu'une généralisation aux situations tridimensionnelles. Quelques idées sont émises dans ces directions avant de conclure que la FFM et le critère couplé ont comblé une lacune en mécanique de la rupture.

Zusammenfassung Die Initiierung von Rissen wird in der klassischen Bruchmechanik nicht umfasst, da sich diese auf die Beschreibung des Verhaltens vorhandener Risse beschränkt. In der Bruchmechanik finiter Risse wird diese Einschränkung durch die Betrachtung der instantanen Entstehung von Rissen endlicher Länge aufgehoben. Im Rahmen dieses Konzeptes wurde ein gekoppeltes Kriterium vorgeschlagen, das eine hinreichende Versagensbedingung in Form zweier gleichzeitig zu erfüllender notwendiger Bedingungen darstellt: eine Bedingung der Festigkeitsmechanik und eine Energiebedingung für den Bruchprozess. Die gegenwärtige Formulierung ist auf zweidimensionale Modelle beschränkt und ist, wie anhand zahlreicher Referenzen dargestellt, geeignet experimentelle Ergebnisse über Versagen an unterschiedlichsten Spannungskonzentratoren korrekt abzubilden. Diese lassen sich klassifizieren in singuläre Spannungskonzentratoren mit lokal unendlich hohen Spannungen und in nicht-singuläre Spannungskonzentratoren mit lokal stark erhöhten aber endlichen Spannungen. Das Kriterium wurde auf verschiedenste Struktursituationen und Materialien angewendet: gekerbte Bauteile, Laminat, Klebverbindungen oder auch Materialeinschlüsse. Jedoch verbleiben weitere unerschlossene Felder für weitere Untersuchungen wie etwa die Betrachtung von zyklischen und dynamischen Lasten oder eine gründliche Erweiterung auf dreidimensionale Risse. Mögliche Ansätze für solche Erweiterungen werden vorgestellt bevor der Schluss gezogen wird, dass die Bruchmechanik finiter Risse mit dem gekoppelten Kriterium eine Lücke in der Bruchmechanik geschlossen hat.

Keywords Finite Fracture Mechanics · Crack initiation · Stress singularity · Weak singularity · Size effects

1 Introduction

Concepts of non-local evaluation of stresses or fracture mechanics quantities have now been used for more than a hundred years. Since the pioneering works of Wieghardt [212], Neuber [156, 157] or Peterson [164], many analysis approaches using a non-local evaluation of stresses near stress concentrations have been proposed and applied to many different structural situations [192]. With increasing development in the field of fracture mechanics, non-local concepts were proposed to tackle the problem of crack initiation that is not covered by classical fracture mechanics approaches. One of the first works following such concepts was given by Novozhilov [159] and Waddoups et al. [201] that aimed at developing a framework for the modelling of crack initiation. In these models, it was found that a “discreteness” or microflaw associated with a certain dimension must be assumed. These ideas were developed further allowing for the analysis of cracking of inner laminate plies (e.g. [163]) or at sharp notches (e.g. Seweryn [184]). Hashin [73] has then developed the framework of finite fracture mechanics by considering “fracture events” associated with cracks of finite size: “New cracks appear in a very short time and it is not possible or of interest to follow the history of their development”. He analysed the formation of cracks in laminates and showed that the energy release rate of finite cracks can be used for a Griffith-like fracture condition. He showed that the value of the energy release rate is bounded by a strength criterion in the sub-ply. This work led to several concepts for the analysis of laminate microcracking [132, 154]. The concept of Finite Fracture Mechanics was used by Leguillon [110] to propose a coupled stress and energy criterion for brittle fracture.¹ The coupled criterion allows for the general analysis of arbitrary stress concentrations in sufficiently brittle materials and has been used by many researchers to establish general failure criteria for a wide range of engineering problems in the last 14 years. It is the aim of the present paper to give a comprehensive overview on the concept of the coupled criterion, its applications and its ongoing development.

2 Theory

2.1 Theoretical framework of Finite Fracture Mechanics

The framework of Finite Fracture Mechanics [73] assumes the instantaneous formation of cracks of finite size. Within this framework, Leguillon [110] has proposed a coupled stress and energy criterion to identify the critical

¹ It should be noted that Li, Hong and Bazant have proposed a related criterion [19, 130] some years earlier. They developed a crack initiation criterion that also uses the concept of finite cracks to analyse the formation of crack patterns on smooth surfaces. A detailed discussion of the relationship and the fundamental differences to Leguillon's coupled criterion is given in Sect. 5.1.4.

loading and the corresponding finite crack size. Two necessary conditions must be fulfilled simultaneously, constituting a sufficient criterion for crack initiation. The stress condition requires that the whole surface of the crack is overloaded prior to crack initiation. Using an appropriately chosen function of the stresses, e.g. maximum principal stress, the stress condition for the formation of a crack of finite size ΔA with its surface $\Omega_c(\Delta A)$ can be written as

$$f(\sigma_{ij}(x)) \geq \sigma_c \quad \forall x \in \Omega_c, \quad (1)$$

with σ_c being the strength of the material.

The second necessary condition, the energy condition, is obtained by consideration of the change of the total potential energy during crack formation. Following a Griffith-type approach with the fracture energy being proportional to the crack surface and to a material constant—the fracture toughness \mathcal{G}_c —from the law of conservation of energy for a purely mechanically loaded system, we obtain

$$\Delta \Pi + \Delta K + \mathcal{G}_c \Delta A = 0. \quad (2)$$

Here, $\Delta \Pi$ is the potential energy change between the uncracked and the cracked states, ΔK is the kinetic energy change and $\mathcal{G}_c \Delta A$ the dissipated energy to create a new crack with surface ΔA . The scaling coefficient \mathcal{G}_c is called the fracture toughness. The uncracked body being initially in rest, the kinetic energy can only grow: $\Delta K \geq 0$, leading to the energy condition for the formation of a finite crack

$$-\frac{\Delta \Pi(\Delta A)}{\Delta A} \geq \mathcal{G}_c. \quad (3)$$

In the following, we will restrict ourselves to two-dimensional modelling. Hence, $\Delta A = t \Delta a$, with t being the thickness of the structure and Δa being the length of the crack. The left-hand side of the former equation is typically denoted as the incremental energy release rate $\bar{\mathcal{G}}$, which is the differential energy release rate averaged over the finite crack length

$$\bar{\mathcal{G}} = \frac{1}{\Delta a} \int_0^{\Delta a} \mathcal{G}(\tilde{a}) d\tilde{a}. \quad (4)$$

The energy condition can also be evaluated using stress intensity factors and relating them to a critical stress intensity factor, the fracture toughness K_c . For instance, in case of pure mode I, this reads:

$$\frac{1}{\Delta a} \int_0^{\Delta a} K_I(\tilde{a})^2 d\tilde{a} \geq K_{Ic}^2. \quad (5)$$

Both necessary conditions (1) and (3) must be fulfilled simultaneously, leading to the sufficient condition for finite crack initiation, the so-called coupled (stress and energy) criterion:

$$f(\sigma_{ij}(\mathbf{x})) \geq \sigma_c \quad \forall \mathbf{x} \in \Omega_c(\Delta a) \quad \wedge \quad \bar{\mathcal{G}}(\Delta a) \geq \mathcal{G}_c. \quad (6)$$

This criterion is sometimes also referred to as the “hybrid criterion” [71, 77, 152].

Instead of using the above stress condition requiring the overloading on the whole crack surface, Cornetti et al. [38] have proposed a coupled criterion that considers the average stress on the crack surface prior to crack formation:

$$\frac{1}{\Delta a} \int_0^{\Delta a} f(\sigma_{ij}(\tilde{x})) d\tilde{x} \geq \sigma_c \quad \wedge \quad \bar{\mathcal{G}}(\Delta a) \geq \mathcal{G}_c. \quad (7)$$

Following the terminology by Taylor [192], this variant of the coupled criterion will be referred to as Line Method (FFM-LM), whereas (6) is called Point Method (FFM-PM).

To take into account the fracture toughness variation along the crack length, Catalanotti and Camanho [34] have proposed to extend the energy condition by the consideration of a crack growth resistance curve (R-curve) [9], that gives a dependence of the fracture toughness on the crack advance. The coupled criterion then reads:

$$\bar{\mathcal{G}}(\Delta a) \geq \frac{1}{\Delta a} \int_0^{\Delta a} R(\tilde{a}) d\tilde{a}. \quad (8)$$

In most crack initiation problems, the stress function and the energy release rate functions show a monotonic decrease and increase, respectively, in the relevant domain close to the considered stress concentrator. These structural problems are sometimes referred to as positive geometries [16,18,92]. Then, the coupled criterion simplifies: the stress condition reverts to the condition that the stress at the most distant point must equal to the strength and also the energy condition reverts to an equality. Hence, the stress condition yields an upper bound of admissible crack lengths and the energy condition yields a lower bound. The lowest and therefore relevant crack initiation load is found when both bounds coincide. If the mechanical system has a linear response to loading, the division of the energy condition by the square of the stress condition yields an equation that only depends on the (dimensionless) crack length. In most cases, the solution to this equation can only be found numerically.

In the most general case, an optimization problem has to be solved to identify the lowest load satisfying both conditions for all considered crack configurations. Then, typically the set of considered crack configurations has to be defined during the analysis. Typical assumptions are straight cracks originating at the origin of the stress concentrator. The considered crack configurations are not necessarily single cracks; in some cases, symmetric or asymmetric cracks or even crack patterns have to be considered [112,177]. When the loading of the structure can be described by a single scalar loading parameter P , the optimization problem can be written as:

$$P_f = \min_{P, \Delta a} \{ P \mid f(\sigma_{ij}(\mathbf{x})) \geq \sigma_c \forall \mathbf{x} \in \Omega_c(\Delta a) \wedge \bar{\mathcal{G}}(\Delta a) \geq \mathcal{G}_c \}. \quad (9)$$

In this optimization problem derived from the coupled criterion, there are two free parameters, the load and the finite crack configuration. The solution of the optimization problem yields the critical failure load and the associated finite crack length.

2.2 Related models for fracture analysis

It may be noted that other concepts are to avail for the analysis of brittle failure at arbitrary stress raisers. Two methods appear most relevant to this discussion of Finite Fracture Mechanics: the cohesive zone models (CZM) and phase-field models of fracture. Both also allow for analysis of crack initiation.

CZM are based on damage mechanics and describe a traction/separation law prior to the final opening of a crack; this law is supposed to model a process zone. The pioneering works are due to Barenblatt [15], Dugdale [49] and Leonov and Panasyuk [126]. The degradation law describing the damaging of the process zone leads to a nonlinear behaviour of the structure and requires appropriate solution techniques. The Finite Element Method is commonly used [54,84,101]. CZM have been extensively used, especially to describe the failure of interfaces [5,6]. Analogously to the coupled criterion, the CZM require two parameters to be implemented. Often they are: (i) a peak stress which, when reached, causes a softening of the process zone due to damage, and (ii) a critical opening beyond which the crack permanently opens. But, clearly, another pair of parameters is equivalent to these: (i) the peak stress and (ii) the amount of energy dissipated before reaching the critical opening. This latter situation is very similar to that of the coupled criterion as shown in [153]. But, despite these similarities, we must emphasize on the fact that there is a very significant difference between the coupled criterion and CZM. The coupled criterion only requires solving one (or sometimes several) linear problem, while CZM lead to nonlinear problems with all the difficulties that have to be overcome to ensure convergence.

Phase-field models for fracture are based upon the variational approaches to fracture that study failure of a brittle body by considering a global energy functional [23,58]. In phase field models, the cracks are represented as continuous field variables. Crack initiation and growth (including branching and coalescence) can be modelled by evaluating the energy functional or equivalent phase-field evolution equations. The solution of phase field models is typically performed within the Finite Element Method, providing solution schemes for this problem with spatial and time discretization. To allow for a numerical solution, a regularization parameter

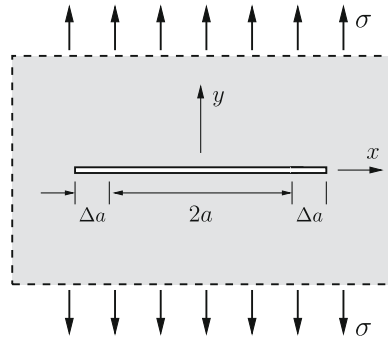


Fig. 1 Canonical problem of a through crack in a plate. Consideration of formation of two symmetric finite cracks

is required that controls the transition from unfractured to fully fractured material. The identification of this parameter in terms of physical quantities [100] and the extension to dynamic [86, 183] and ductile fracture problems [199] are current fields of research. Since the first works by Aranson et al. [11] and Eastgate et al. [52], phase-field models of fracture have been applied to a wide range of fracture analysis [1, 8, 98, 99, 143, 215].

Besides these and other approaches for modelling failure in solids, Finite Fracture Mechanics can be understood as an independent approach in the class of two parameter models. Using linear models, it allows for a straightforward analysis and, as shown in this review, it can be readily applied to a wide range of engineering problems.

3 Canonical problem of a through crack in a plate

We consider the canonical problem of fracture mechanics: a through crack of length $2a$ in an infinite plate under tensile loading σ (Fig. 1). For the case of isotropy, the Westergaard solution [210] yields the crack opening displacement v and the stresses on the ligament ahead of the crack tips σ_y in closed form:

$$v = \frac{2\sigma}{E} \sqrt{a^2 - x^2}, \quad |x| < a \quad (10)$$

$$\sigma_y = \sigma \frac{|x|}{\sqrt{x^2 - a^2}}, \quad |x| > a \quad (11)$$

Correspondingly, the stress intensity factor is $K_I = \sigma \sqrt{\pi a}$. According to Linear Elastic Fracture Mechanics, the critical stress is

$$\sigma_f^{\text{LEFM}} = \frac{K_{Ic}}{\sqrt{\pi a}}. \quad (12)$$

We consider the symmetric formation of finite cracks Δa in mode I, such that the crack extends to a length of $2(a + \Delta a)$. Both the stresses on the ligament and the energy release rate are monotonic functions. Hence, using the relations (10) and (11), the coupled criterion with PM (6) can be written as

$$\sigma \frac{a + \Delta a}{\sqrt{\Delta a (2a + \Delta a)}} = \sigma_c \quad \wedge \quad \pi \sigma^2 \left(a + \frac{\Delta a}{2} \right) = K_{Ic}^2 \quad (13)$$

and in the case of the coupled criterion using the LM (7)

$$\sigma \sqrt{1 + 2 \frac{a}{\Delta a}} = \sigma_c \quad \wedge \quad \pi \sigma^2 \left(a + \frac{\Delta a}{2} \right) = K_{Ic}^2. \quad (14)$$

Here, σ_c is the tensile strength and K_{Ic} is the fracture toughness of the material.

When these sets of equations are solved for the critical stress and the finite crack size and by using the plastic zone size for plane stress for the upcoming constant

$$r_p = \frac{1}{2\pi} \left(\frac{K_{Ic}}{\sigma_c} \right)^2, \quad (15)$$

the solution for the coupled criterion with PM is

$$\Delta a^{\text{PM}} = \frac{a}{3} \left(4 \left(\frac{r_p}{a} - 1 \right) + \frac{2 \cdot 2^{\frac{3}{2}} \left(1 - 2 \frac{r_p}{a} + 4 \left(\frac{r_p}{a} \right)^2 \right)}{\sqrt[3]{2\omega}} + \sqrt[3]{2\omega} \right), \quad (16)$$

$$\begin{aligned} \sigma_f^{\text{PM}} = & \frac{K_{Ic}}{\sqrt{\pi a}} \frac{1}{\sqrt[4]{36 \frac{r_p}{a} \omega^2}} \left(256 \left(\frac{r_p}{a} \right)^{\frac{7}{2}} - 32 \left(\frac{r_p}{a} \right)^{\frac{5}{2}} \left(6 + \sqrt[3]{2\omega} \right) \right. \\ & + \sqrt{\rho} \left(\sqrt[3]{2\omega} - 2 \right) \left(\sqrt[3]{2\omega} \left(\sqrt{3} + 4 \frac{r_p}{\rho} \right) - 16 \frac{r_p}{\rho a} \right) \\ & \left. - 4 \frac{r_p}{a} \sqrt{\rho} \left(\sqrt[3]{2\omega} - 6 \right) \left(\sqrt{3} + 5 \sqrt{\frac{r_p}{\rho a}} + 2 \sqrt[3]{2\omega} \sqrt{\frac{r_p}{\rho a}} \right) \right)^{\frac{1}{2}} \end{aligned} \quad (17)$$

with

$$\rho = 8 - 13 \frac{r_p}{a} + 16 \left(\frac{r_p}{a} \right)^2, \quad (18)$$

$$\omega = 4 + 15 \frac{r_p}{a} - 24 \left(\frac{r_p}{a} \right)^2 + 32 \left(\frac{r_p}{a} \right)^3 - 3\sqrt{3} \sqrt{\frac{r_p}{a} \left(8 - 13 \frac{r_p}{a} + 16 \left(\frac{r_p}{a} \right)^2 \right)} \quad (19)$$

and with LM

$$\Delta a^{\text{LM}} = 4r_p, \quad (20)$$

$$\sigma_f^{\text{LM}} = \frac{K_{Ic}}{\sqrt{\pi a}} \frac{1}{\sqrt{1 + 2 \frac{r_p}{a}}} \quad (21)$$

can be given.

The latter solution is not only a lot more compact, but also clearly shows the relationship to the well-known concept of Irwin's crack tip plasticity. In this framework, the critical loading is calculated under the assumption of a small plastic zone at the notch tip, in which the stresses cannot exceed the plastic limit (strength). The critical stress of the present problem of a through crack then reads

$$\sigma_f^{\text{Irwin}} = \frac{K_{Ic}}{\sqrt{\pi(a + r_p)}} = \frac{K_{Ic}}{\sqrt{\pi a}} \frac{1}{\sqrt{1 + \frac{r_p}{a}}}. \quad (22)$$

In the case of a perfectly brittle material, the size of the plastic zone must be very small compared to the only structural dimension, which is the half crack length a , such that $\frac{r_p}{a} \rightarrow 0$ holds. Then, both solutions of the coupled criterion and Irwin's concept of crack tip plasticity yield the critical stress obtained from Linear Elastic Fracture Mechanics (12). The corresponding finite crack lengths are:

$$\lim_{\frac{r_p}{a} \rightarrow 0} \Delta a^{\text{PM}} = r_p, \quad (23)$$

$$\lim_{\frac{r_p}{a} \rightarrow 0} \Delta a^{\text{LM}} = 4r_p. \quad (24)$$

With the condition of perfectly brittle material behaviour ($\frac{r_p}{a} \rightarrow 0$), these finite crack advancements vanish in comparison with the initial crack length and continuous crack growth is predicted. However, whenever the material behaviour is not perfectly brittle, crack growth in finite increments is predicted by FFM. In case of negative geometries, which show a decrease in the energy release rate with increasing crack lengths, a different behaviour is predicted by FFM approaches [208].

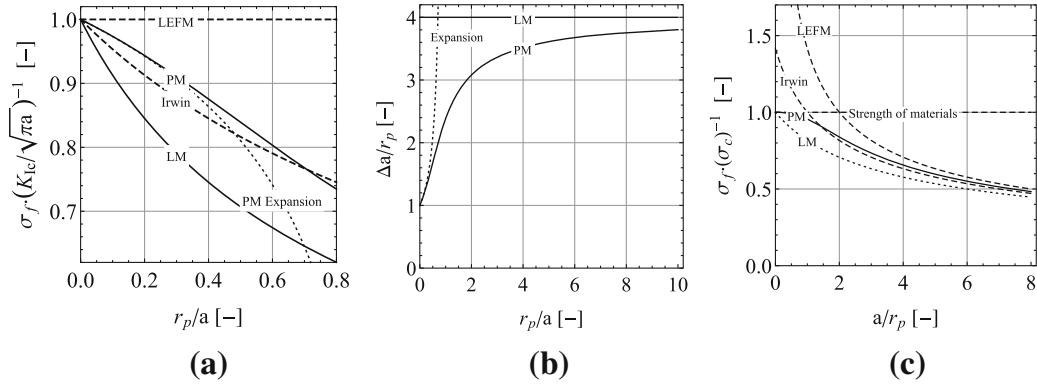


Fig. 2 FFM solution for the case of a central crack under tensile loading. Effect of the size of the plastic zone on the failure load and the corresponding finite crack lengths. Comparison to Linear Elastic Fracture Mechanics (LEFM) and Irwin's concept of crack tip plasticity

In order to obtain a more compact solution in the case of the coupled criterion with PM, one can use asymptotic expansions of the true solution of the stresses on the ligament (11). Using the first two terms of the expansion

$$\sigma_y = \frac{\sigma}{\sqrt{2}} \left(\sqrt{\frac{a}{x}} + \frac{3}{4} \sqrt{\frac{x}{a}} - \frac{5}{32} \left(\frac{x}{a} \right)^{\frac{3}{2}} + \mathcal{O} \left(\left(\frac{x}{a} \right)^{\frac{5}{2}} \right) \right). \quad (25)$$

the following approximate solution is obtained:

$$\Delta a^{\text{PM, T2}} = r_p \frac{4 \left(2\sqrt{1 - \frac{r_p}{a}} + 3\frac{r_p}{a} - 2 \right)}{\frac{r_p}{a} (8 - 9\frac{r_p}{a})}, \quad (26)$$

$$\sigma_f^{\text{PM, T2}} = \frac{K_{Ic}}{\sqrt{\pi a}} \sqrt{\sqrt{\left(\frac{a}{r_p} \right)^2 - \frac{a}{r_p}} - \frac{a}{r_p} + \frac{3}{4}}. \quad (27)$$

Of course, this solution is only valid for small ratios of the plastic zone size to the crack length (Here, less than 0.1% error for $\frac{r_p}{a} < 0.2$).

The transition to LEFM is shown in Fig. 2a, b. For small ratios of the plastic zone size to the crack length, both FFM models reduce to the LEFM approach. For increasing values of this ratio, the bearable load decreases. The decrease shows good agreement with the classical concept of crack tip plasticity by Irwin [24,90]. Results for very small crack lengths are shown in Fig. 2c. Whereas the LEFM predicts infinite values of the critical load, the FFM solutions approach the limit value given by strength of materials considerations. In this case, Irwin's concept overestimates the failure loads by 41%. The analysis of this canonical problem has also been addressed in the works of Cornetti et al. [38] and Mantič [134].

4 Crack initiation at weak stress singularities

4.1 Theoretical background

Stress singularities can arise in many structural situations; in most cases, they are weak, i.e. less singular than at a crack tip. They arise, for instance, at sharp notches [213], at multi-material joints [22] or at the free edge of laminates [145, 146, 202]. At weak stress singularities, the asymptotic stress field can be given as a Williams series expansion [213]

$$\sigma \propto K r^{\lambda-1} \mathbf{g}(\varphi) + \dots \quad (28)$$

with the singularity order λ being larger than 0.5, the generalized stress intensity factor K and the angular function $\mathbf{g}(\varphi)$. The singularity order must be identified for the specific structural situation. Closed-form analytical approaches [79, 194, 213] or numerical solution schemes [48, 67, 80, 140] are to avail for many structural

situations. But even in former case, a numerical solution of implicit equations is typically required. However, Sator and Becker [182] provide a full closed-form expression to calculate the singularity order at bi-material notches by providing an explicit expression on the dependence of the ratio of Young's moduli on the singularity order as an inverse function.

Due to the nature of the singular stresses, the evaluation of the maximum value of the stress is not possible as this would lead to a vanishing failure load. On the other hand, Fracture Mechanics approaches are not directly applicable to such structural situations as well. To calculate the energy release rate of a crack, the virtual crack closure technique (VCCT) can be used readily if the stress field and the crack opening are known. We consider a small crack of length Δa emanating from a sharp notch under mode I loading. In that case, the asymptotic stress field prior to crack initiation behaves like $\sigma \propto r^{\lambda-1}$ while the crack opening displacement after crack initiation is $\mathbf{u} \propto r^\lambda$. This latter point is not straightforward; the intensity factor K_I at the tip of the new crack behaves as $K_I \propto r^{\lambda-\frac{1}{2}}$ (see, e.g. [119]); as a consequence, the opening behaves as $r^{\lambda-\frac{1}{2}} r^{\frac{1}{2}} = r^\lambda$. Then, the differential energy release rate can be calculated using VCCT

$$\mathcal{G} = \lim_{\Delta a \rightarrow 0} \frac{1}{\Delta a} \int_{\Delta a} \frac{1}{2} \sigma u \, da. \quad (29)$$

Using the asymptotics of a notch and a crack and omitting upcoming constants, we obtain

$$\begin{aligned} \mathcal{G} &\propto \lim_{\Delta a \rightarrow 0} \frac{1}{\Delta a} \int_0^{\Delta a} r^{\lambda-1} r^\lambda \, dr \\ &\propto \lim_{\Delta a \rightarrow 0} \frac{1}{\Delta a} [r^{2\lambda}]_0^{\Delta a} \end{aligned} \quad (30)$$

$$\propto \lim_{\Delta a \rightarrow 0} \Delta a^{2\lambda-1}. \quad (31)$$

The differential energy release rate only attains finite values in the case of a crack with $\lambda = 1/2$. Weak singularities ($\lambda > 1/2$) lead to vanishing energy release rates, whereas strong singularities ($\lambda < 1/2$) result in infinite energy release rates. A more rigorous consideration using an asymptotic expansion of the energy during crack formation at a notch is provided by [109, 141, 200]. It is used by Francfort and Marigo [58] in their variational framework to fracture, to prove that crack growth must be finite (they call it “brutal”) at weak stress singularities.

Since the differential energy release rate vanishes at points of weak singularities, LFM approaches cannot be used. Coupled criteria in the framework of FFM overcome these problems as they evaluate the energy release rate of finite cracks and consider the stresses at a certain, but not empiric, distance from the point of singularity.

4.2 Application of FFM

4.2.1 V-notches

Corner singularities as they arise in sharp notches have been studied by many researchers. The failure behaviour of such notches with a sharp notch root, so-called V-notches, shows a distinct dependence on the notch opening angle, the mode mixity and the size of the structure. It has been studied experimentally by many researchers [50, 57, 174, 184], and several failure criteria and modelling approaches have been proposed (e.g. [50, 51, 70, 85, 184]). But many of these criteria typically lack a physical explanation and experimental parameter determination can be necessary. For instance, the critical generalized stress intensity factor (GSIF)² depends on the notch opening and its loading and must be obtained experimentally.

In the past 14 years, V-notches have been considered in many studies by means of FFM approaches that make use of coupled criteria. Brittle failure of V-notches has been studied by Leguillon [110], along with the proposition of the coupled criterion. The specific case of a V-notched beam under three-point bending was studied showing good agreement with experimental results on PMMA specimens [50]. Further comparison to experimental results was performed in the subsequent study [119, 165], and a discussion of the effect of a

² Sometimes also denoted as Notch Stress Intensity Factor.

notch tip radius was provided, again by means of asymptotical analysis. Sharp V-notches were also studied by Cornetti et al. [38,193]. As outlined previously, they proposed an evaluation of the averaged stresses prior to crack initiation (LM) in the coupled criterion. The case of V-notches under mixed-mode loading was analysed by Yosibash et al. [217] by means of an asymptotic analysis that uses generalized stress intensity factors obtained numerically. The predicted crack initiation loads and crack initiation angles show a good agreement to experimental results, at least for sufficiently low mode mixity. For this structural situation, a comparison of FFM models to a Leonov–Panasyuk–Dugdale strip yield model based on matched asymptotics was given by Henninger et al. [83]. Their results indicate an excellent agreement of the crack onset predictions of both failure models. A V-notch with unsymmetrical loading was studied by Hebel and Becker [77] showing very good agreement to mixed-mode experiments [185]. Carpinteri et al. [30] studied V-notched plates under mode I loading and discussed the problem of a critical angle leading to the lowest failure load. In the study by Garcia and Leguillon [59], the case of a V-notch in the vicinity of the elastic contrast of an adhesive bond was discussed for the general case of mixed-mode loading. The case of a crack emanating from a V-notch along a weakly bonded interface has been studied by Tran et al. [196] experimentally and by means of an FFM approach that accounts for the mixed-mode loading. They addressed the inverse identification of the fracture toughness of the adhesive bonding and discussed the appearing effect of the mode mixity. Crack initiation at V-notches under mixed-mode in-plane loading of a V-notch has been analysed by Hebel and Becker [77] using an asymptotic solution and a direct numerical approach using Finite Element Analysis. Both approaches showed good agreement to experimental results [185]; only the predicted finite crack lengths differed. The crack initiation angle was predicted in very good accordance with the experimental results by both approaches. Similar results were obtained by Sun et al. [188], who developed a direct numerical approach that uses the Scaled Boundary Finite Element Method [186,214] to analyse the energy release rates of the considered crack configurations.

The effect of mode mixity was also studied by Cornetti et al. in their studies [40,181] that comprise an analysis and discussion of size effects. They were using the Williams solution along with discrete values of generalized stress intensity factors from the literature [20,217] to establish their FFM solution. Besides the crack initiation load, the angle of crack initiation is predicted by their FFM failure model. Results of the comparison of their predictions to the experiments by Seweryn et al. [185] are shown in Fig. 3. In particular, the crack initiation angle showed good agreement to experimental results. The size effect of the notch depth is studied in detail.

A comprehensive discussion of the present models for fracture of V-notches has been given by Lazzarin et al. [107]. In their work, they compare the FFM solutions of Leguillon [110] with a point-wise stress criterion of Carpinteri et al. [29] and with a stress criterion using averaged stresses. Besides these FFM solutions,

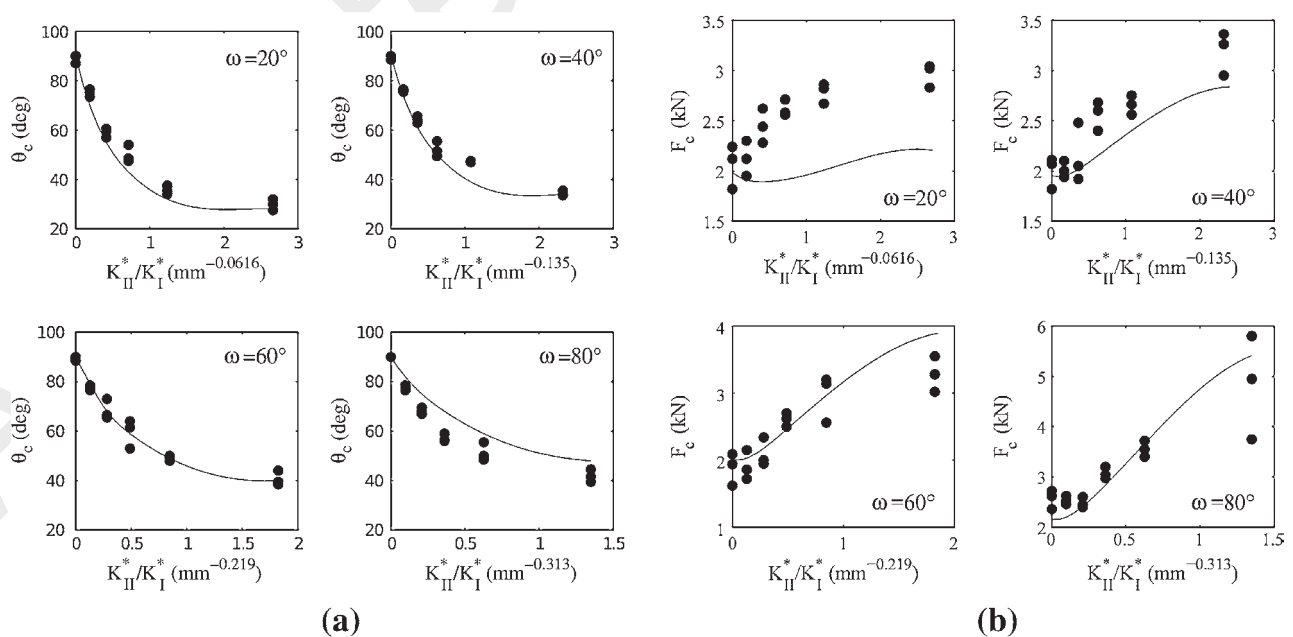


Fig. 3 Comparison of the FFM model for V-notches by Cornetti et al. [40,181] to experimental results on PMMA specimens [185]. The opening angle of the notch is denoted ω . (Reprinted from [181] with permission from Elsevier Ltd.). **a** Angle of crack initiation. **b** Crack initiation load

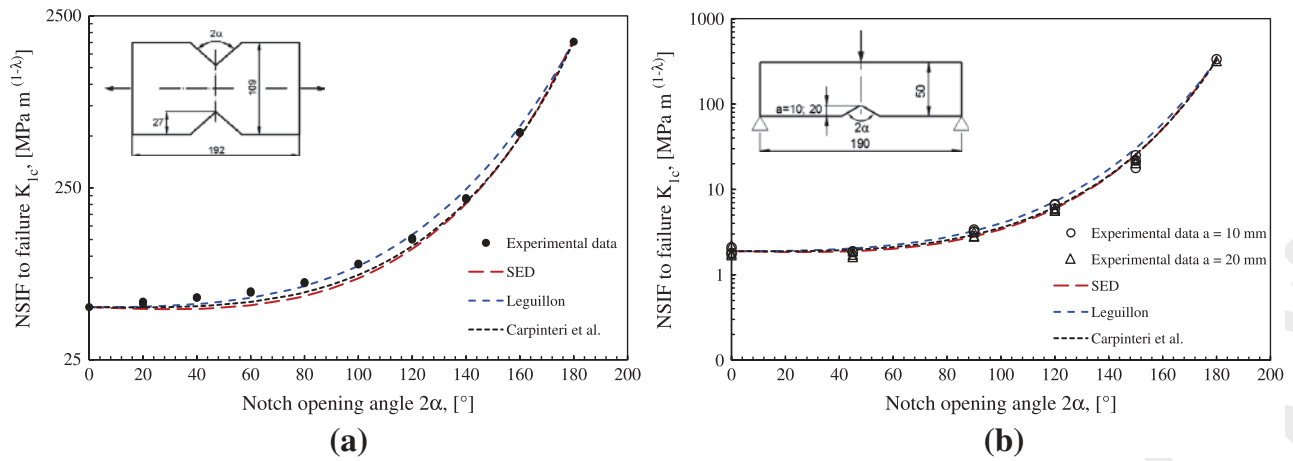


Fig. 4 Lazzarin et al. [107]: Comparison of FFM failure models [29, 110] to experimental results. Dependence of the critical notch stress intensity factor on the notch opening angle 2α . (Reprinted from [107] with permission from Elsevier Ltd.). **a** Tensile loading experiments by Seweryn et al. [185]. **b** Three-point bending experiments by Carpinteri [28]

they compare the predictions of an averaged strain energy density (SED) criterion [105] that makes use of an experimentally obtained finite volume for averaging. They considered experimental results on notched specimens under tensile and bending loading for a comparison of the failure models. The critical notch stress intensity factors predicted by the FFM approaches were compared to experimental data. The results shown in Fig. 4 indicate that both approaches can cover the effect of the notch opening angle on the critical crack initiation load in the case of mode I loading. A good agreement to experimental results is obtained for both modelling approaches with the averaged stress FFM (and the closely related SED criterion by Carpinteri et al. [106]). The PM model yields more conservative failure load predictions.

4.2.2 Adhesive joints

In bi-material corners of adhesive joints, weak stress singularities occur, too, that can lead to crack initiation in the case of overloading of the adhesive joint. The precise determination of failure loads is of high importance for the widespread application of this advantageous joining method [2]. The effect of material and geometry parameters of the joint configuration on the failure behaviour has been in the focus of many studies (for overview cf. [42, 44–46]). In particular, the effect of the adhesive layer thickness that leads to a size effect such that the failure load decreases with increasing adhesive layer thickness is of high interest and many different explanations are provided in the literature, cf. [65]. A FFM model for the analysis has been developed by Leguillon et al. [121] in the framework of a matched asymptotic analysis. The approach uses two different analyses at the “microscopic” scale and the “macroscopic” scale, to allow for a detailed but yet efficient analysis. The comparison to experimental results from three-point bending specimens with adhesively bonded patches shows that the model underestimates the bearable loads of the joint configuration. This analysis has been extended by Henninger et al. [83] and Moradi et al. [151] to incorporate the effect of thermal loading (see below in Sect. 4.2.5) and the effect of the adhesive layer thickness in detail.

Besides these approaches, several direct numerical implementations of FFM have been proposed for the analysis of crack initiation in bonded joints. Müller et al. [152] have analysed the crack initiation at the bi-material points by means of a detailed FEA for metal ceramic joint configurations that are found in high-temperature fuel cell technologies. Their main result is the identification of possible finite cracks due to overloading caused by thermal, mechanical or shock loading that can be the starting point of subsequent failure processes. Another FFM model for the analysis of fuel cell joints was provided by Hebel and Becker [77]. In their work, they also used a very finely meshed FEA to analyse the stress concentration in the vicinity of the bi-material points in the metal ceramic joint and to analyse the release of strain energy due to the formation of cracks within the bondline. They considered the loading case of thermal loading during manufacturing and conducted a comprehensive study of possible parameter effects. Structural adhesive joints have been studied in detail with a numerical implementation of FFM by Mendoza-Navarro et al. [142], Moradi et al. [150] and Hell et al. [81]. In these studies, different adhesive joint configurations were analysed by means of a FFM model that utilizes comprehensive FEA. To correctly cover the local stress concentration and the energy release rate of the considered crack configurations in the adhesive layer, very fine meshes were employed. Mendoza-Navarro

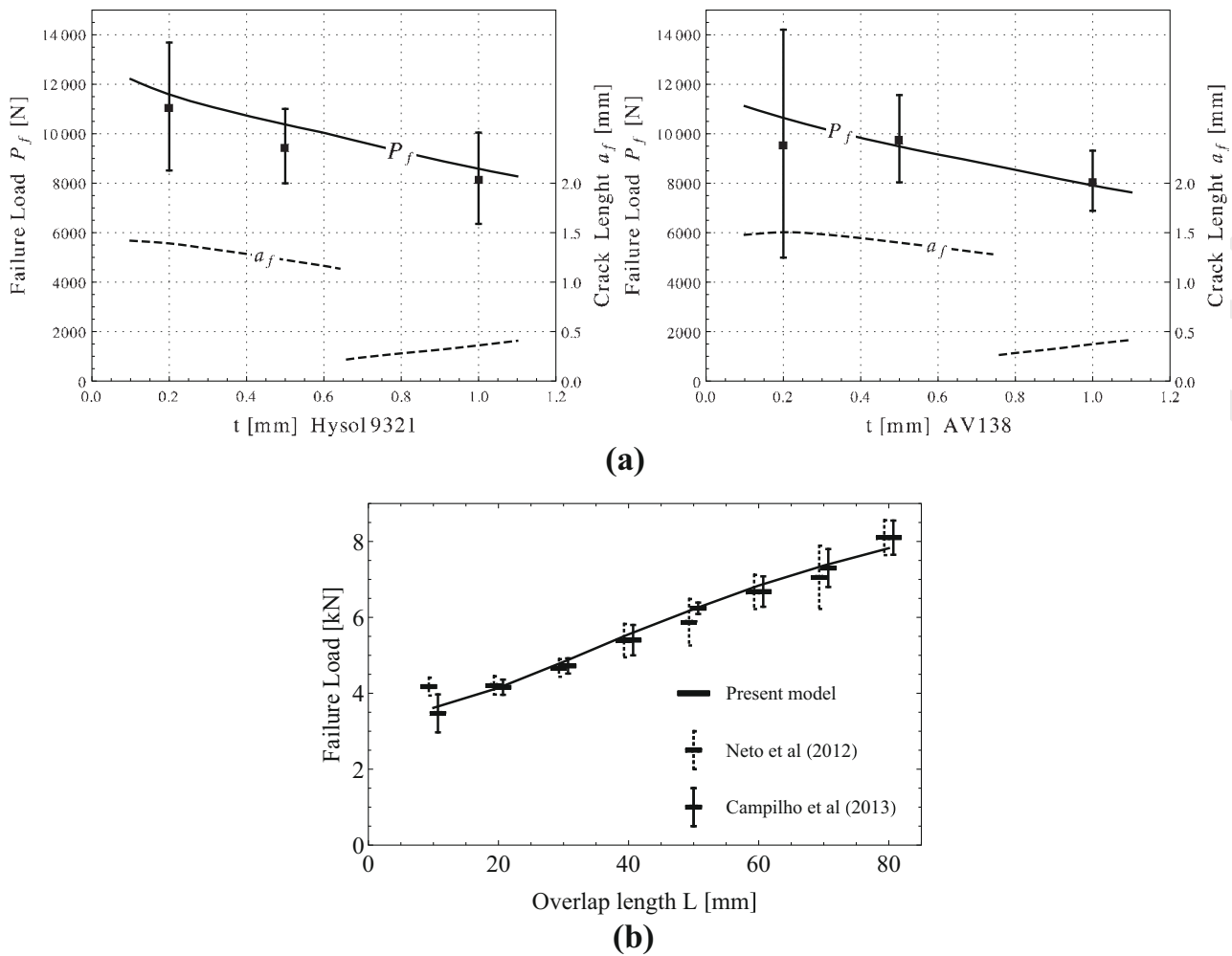


Fig. 5 Results of the crack initiation load prediction by direct numerical implementation of FFM and comparison to experimental results on single-lap joints under tensile loading. (Reprinted from [81] with permission from Elsevier Ltd. and from [207] with permission from Springer Science + Business Media.). **a** Effect of adhesive layer thickness (Experiments: [43]). **b** Effect of bondline length (Experiments: [27,155])

et al. [142] and Moradi et al. [150] only considered cracks in the interface of the substrate, and the adhesive and their analyses did not take into account large bending deformations that appear in the unbonded part of the substrates. These bending deformations lead to altered loading conditions of the bonding region during load increase. Hell et al. [81] have considered straight cracks in the adhesive without a priori defining the angle to the interface. But the angles of crack initiation obtained from the solution of the coupled criterion indicate that only cracks close to the interface appear and that only a small error ($<4\%$) is induced when cracks on the interface are considered only. However, the results of Hell et al. [81] and Weißgraeber et al. [207] clearly show that the large bending deformations of the unbonded regions have a strong effect on the predicted crack initiation loads. A comparison study (cf. Fig. 5) shows a very good agreement to experimental results obtained with typical single-lap joint configurations with metallic adherends and brittle structural adhesives. The effect of the parameter dependencies is covered very well, and no fitting procedure was performed by the authors to obtain the required material parameters strength and toughness. As already reported by Müller et al. [152], non-monotonic energy release rates may occur for finite cracks in the vicinity of the bondline for certain joint configurations and loading cases. In the case of single-lap joints under tensile loading, such non-monotonic energy release rates occur as well [81,142,151] and can lead to certain properties of the solution of the coupled criterion. Certain finite crack lengths are not admissible and a range of finite crack lengths—controlled by the position of the local maximum—is then excluded from the solutions attributed to the lowest crack initiation load. The position of the local maximum is mainly dependent on the adhesive layer thickness. Due to this, the predicted finite crack lengths a_f in the analysis of the effect of the adhesive layer thickness in single-lap joints (Fig. 5a) are non-continuous, whereas the corresponding crack initiation loads are. Carrere et al. [33] have shown that a numerical implementation of FFM with linear kinematics of the same joint configuration shows good agreement to a continuum damage model if the effect of geometrical nonlinearities is small.

Besides comprehensive numerical models, simplified models on the basis of closed-form analytical solutions have been proposed. In all of these models, the adhesive layer is not modelled as a full continuum but rather considered as weak interfaces [125, 173]. This is justified by the elastic contrast of substrate and adhesive that is typically very large and the dimensions of the adherend and the adhesive layer. Simple shear lag models were used to analyse single-lap joints [203, 204] and the pull–push shear test of adhesive joints [39]. In these models, only the shear stresses within the adhesive layer (constant over the thickness) are considered to analyse debonding of the bondline by means of the coupled criterion. (A similar modelling approach has been used to study the triggering of snow-slab avalanches by means of a FFM approach [35]. The same assumptions of a weak interface were used. In this model, a simple shear lag model was used to analyse the shear stresses in the weak interface of a snow pack.)

To account for the important effect of mixed-mode loading of the adhesive layer, Weißgraeber and Becker [205] proposed a FFM failure model for single-lap joints that uses an improved closed-form weak interface model [161] for the stress analysis of the joint. This model yields a peel and shear stress distribution in the interface. The effect of nonlinear bending deformations is taken into account by a nonlinear moment factor for single-lap joints [66, 197]. A comparison to several experimental studies shows good agreement of the predicted crack initiation loads to the measured failure loads. The previously addressed effects of the geometrical parameters on the crack initiation load are fully covered by the simplified analysis. Due to the underlying closed-form analytical solution, a very efficient solution procedure is obtained. A generalization of this approach to all lap joint configurations with planar substrates has been proposed by Stein et al. [187] that makes use of the general sandwich-type solution by Weißgraeber et al. [206] and an improved formulation of nonlinear moment factors [104]. It allows for the efficient analysis of several joint configurations, such as single-lap joints, T-peel joints, L-joints or double-lap joints under arbitrary in-plane loading. A comparison to experimental results and results from FEA with cohesive zone modelling shows a very good agreement for a wide range of parameters.

4.2.3 Failure behaviour of laminates

Failure onset of laminates has also been studied by means of FFM approaches that make use of the coupled criterion. At the edges (or corners) of laminates, strong stress concentrations can occur. This so-called laminate edge effect occurs due to the elastic mismatch of adjacent plies [94, 145, 148] and leads to weak stress singularities (cf. [79]). This structural situation has been analysed by Hebel et al. [78] and Martin et al. [138] using the coupled criterion and the framework of FFM. Hebel et al. [78] have used a direct numerical formulation on the basis of a comprehensive Finite Element routine. To take into account possible mode III contributions, they modelled the laminate edge in such a manner that a generalized plane strain state is employed. A uniaxial strain loading and a manufacturing-induced temperature load are considered. Their results show good agreement of the laminate failure loads to experimental results for several cross-ply laminates [72]. The model predicts a decrease in the bearable loads with increasing layer thickness, thus correctly rendering the known size effects in laminates [103, 163].

In the FFM model by Martin et al. [138], an FEM solution of the variational formulation [167] for determination of the stress distribution near the free edge is used. Again a generalized plane strain state is assumed in this model. A comparison to experimental results on angle-ply laminates shows that the FFM failure model predicts the onset of mode III delamination in good agreement with experimental results. The effect of the ply thickness on the critical longitudinal stress is covered very well as it is shown in Fig. 6. However, it was not the same couple of strength and toughness values, leading to the lowest deviation to experimentally obtained delamination onset loads for the different angle-ply laminates. The authors conclude that this finding corresponds to the observation of Liao and Sun [131] that the parameters of interfacial fracture depend on the ply angle. The work by Saeedi et al. [180] follows the same FFM methodology using a layerwise stress solution [179] allowing for a more efficient analysis.

The onset of cracks in inner plies of laminates has been studied by García et al. [60] by means of a coupled criterion. Classical laminated plate theory is used to study the uncracked state prior to crack nucleation, and the energy release during the fracture events is obtained by a numerical analysis using the Boundary Element Method and a dimensional analysis. The failure model shows good agreement with experimental results and allows for a physical explanation of the effect of the thickness of the inner ply on the critical strain observed in experiments [14, 64, 163, 201].

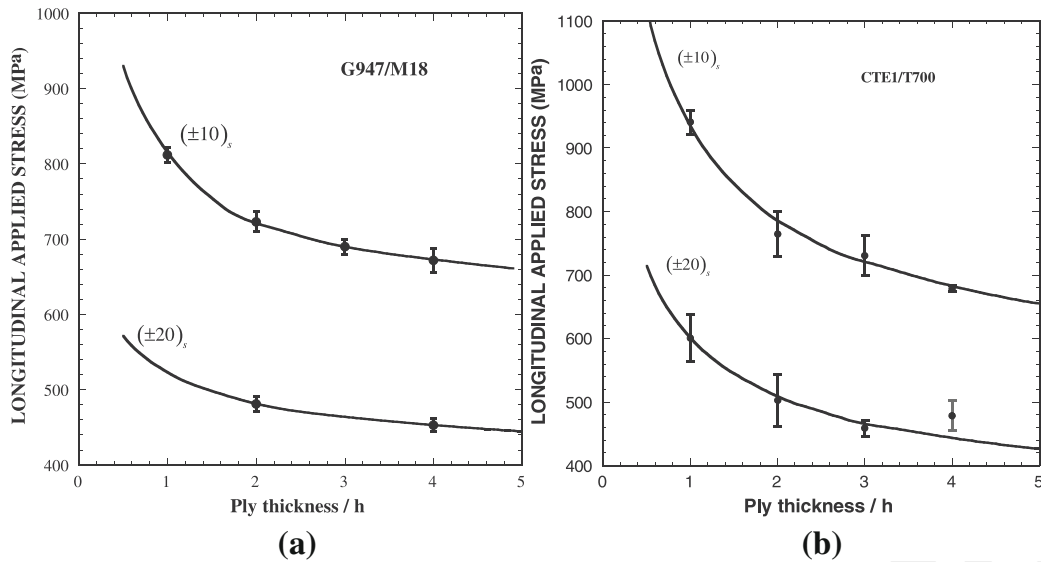


Fig. 6 Predicted delamination onset loads in comparison with experimental results. (Reprinted from [138] with permission from Elsevier Ltd.). **a** G947/M18 laminate (Experiments by Lagunegrand et al. [103]). **b** CTE1/T700 laminate (Experiments by Diaz and Caron [47])

4.2.4 Cracks at interfaces

When cracks in a solid approach an interface with a change of the materials stiffness, the associated crack tip singularity changes. Instead of the classical square root singularity, a singularity arises with a different singularity order when the crack reaches the interface. Depending on the stiffness ratio, weak or strong singularities can occur. Hence, other crack extension mechanisms than classical crack growth must occur. It is well known from experiments that interfaces crossing the crack path increase the effective toughness of a body. Cracks approaching interfaces can lead to debonding on the interface before the crack reaches the interface [195,216]. The analysis of cracks at interfaces and their possible behaviour as deflection at the interface or penetration of the interface has been analysed by many researchers, e.g. [3,37,74,75,108]. A comprehensive overview of literature on experimental and theoretical studies is given by Martin and Leguillon [136]. In many analyses, the concept of non-local evaluation of the stresses and the stored strain energy is used to model the behaviour of the crack. One of the main questions is whether the crack will impinge the interface or will be deflected onto the interface. In the latter case, the corresponding question arises: under what circumstances will the crack be deflected out of the interface? Such questions are of special importance for the development and understanding of improved composite materials. Crack deflection is the desired mechanism to obtain tough and damage tolerant composite materials [56]. In the following, a brief overview shall be given on FFM models that make use of the coupled criterion to study the debonding of interfaces.

In the study by Martin and Leguillon [136], the formation of an interfacial crack on an interface ahead of an approaching crack tip is studied. In particular, the case of a matrix crack approaching a fibre in a composite material is considered. The finite length of the interfacial crack is obtained by evaluation of the coupled criterion. Depending on the ratio of the elastic stiffnesses, different crack mechanisms are predicted. The question whether the crack is deflected at the (fibre/matrix) interface or whether the crack will penetrate the interface has been studied in detail in a subsequent analysis by Martin et al. [137]. They discussed the competition between crack penetration and crack deflection at a bi-material interface at hand of a composite cell of matrix embedding a single fibre. The coupled criterion allows for a detailed analysis and allows to define threshold ratios of the strength and the toughness of both materials. The case of cracks “stepping over” interfaces due to inhomogeneities in rock fracture has been analysed by Quesada et al. [172] in the context of low-porosity reservoirs. The effect of the thickness of the inhomogeneities on the crack path (referred to as interbed thickness) has been discussed in detail revealing a pronounced size effect. Different size effect mechanisms under tensile and compressive loading are predicted allowing for a more detailed understanding of experimental findings.

The effect of T-stresses³ on the phenomena of crack kinking [41] has been studied by Leguillon and Murer [117]. In their study, the coupled criterion is used to study this effect in case of a crack under pure mode-I

³ First non-singular term of the asymptotic expansion corresponding to tensile normal stresses parallel to the crack.

loading. A threshold value of T-stresses is identified that must be exceeded to allow for crack branching when microflaws are neglected.

The general case of a crack approaching an interface with an elastic contrast or a thin stiff layer has been studied by Leguillon and Martin [114,115]. The use of the coupled criterion allows them to identify two different possible mechanisms of a crack crossing an interface: the so-called jump-through and step-over. A three-point bending test is used for their study of toughening effects due to the elastic contrast and the corresponding effect of vanishing energy release rates. A comparison to a homogeneous body is used to discuss and quantify the possible toughening by single or multiple interfaces.

4.2.5 Thermal residual stresses

The examples above are mainly related to mechanical loadings. However, this is not a limitation of the approach. Residual stresses due to thermal loading (or from another origin) may also be considered. They mainly arise in composite materials where the elastic contrast between components together with the mismatch between the coefficients of thermal expansion generates residual stresses after fabrication or in service. These residual stresses play a crucial role in fracture mechanics, when present. They are often detrimental but can also sometimes enhance the resistance of a structure [4,7,21,88,147,170].

There are few changes from the theoretical point of view. The series expansion (28) has to be enriched by a non-singular term

$$\sigma \propto Kr^{\lambda-1}\mathbf{g}(\varphi) + \Theta\mathbf{h}(\varphi) + \dots \quad (32)$$

where Θ is the temperature change between the stress-free state and the current one and $\mathbf{h}(\varphi)$ is an angular function [82,120]. Now the generalized stress intensity factor K depends on both the mechanical and the thermal loadings

$$K = K_m + \Theta K_t$$

where the indices m and t hold for mechanical and thermal. After these changes, the coupled criterion evidences a threshold that depends on the level of residual stresses. Henninger and Leguillon [82] implemented the coupled criterion in the case of two aluminium plates bonded together by an adhesive layer according to the experiments of Qian and Akisanya [170]. The assembly was made at high temperature (120 °C) and then cooled to room temperature (20 °C) and exposed to a tension. The effects of the residual stresses are shown in Fig. 7. In the first case, the adhesive joint is perpendicular to the edge (butt joint, $\alpha = 90^\circ$) and in the second one it is inclined (scarf joint, $\alpha = 105^\circ$).

A similar analysis was carried out in V-notched laminated ceramic specimens under four-point bending loading. These specimens are made of alternate layers: thick layers in tension and thin layers in compression [21,124] after cooling from the sintering temperature to room temperature. The compression layers are expected to trap the cracks growing in the layers in tension. A more complex situation encountered in natural reservoirs for CO₂ sequestration can also be mentioned [122]. The question is whether the cracks in the reservoir can be activated or not by the fluid injection and then cross the interface between the reservoir and the caprock

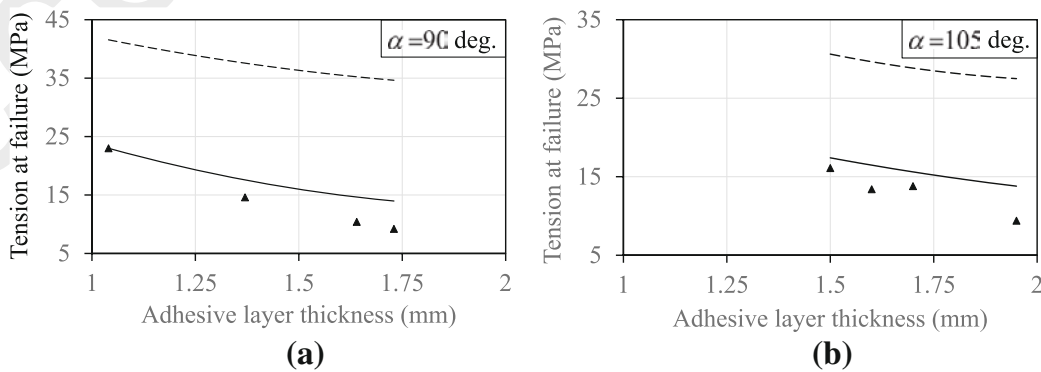


Fig. 7 Predicted tension at failure of adhesive joints as a function of the adhesive layer thickness for a cooling $\Theta = -100^\circ\text{C}$. FFM model by Henninger and Leguillon [82] compared to experiments by Qian and Akisanya [170]: taking into account the thermal residual stresses (*solid line*) and without thermal residual stresses (*dashed line*). (Reprinted from Henninger and Leguillon [82] with permission from the Taylor & Francis Group.). **a** Butt joint. **b** Scarf joint

(see Sect. 4.2.4) triggering a leakage path. In this latter case, many different terms must be added to the series expansion (32) to take into account various effects: the swelling of the reservoir, the fluid pressure on the crack faces and the overburden load.

5 Crack initiation at non-singular stress raisers

Even in the case of stress raisers with non-singular stress fields, as rounded notches or open holes, strength of material approaches can be troublesome to apply. When strength of materials approaches are applied, the following issues arise: in contrast to the previously discussed case of singularities, the evaluation of strength conditions is possible and will not lead to vanishing failure loads. But with increasing stress concentration factors, the quality of the failure load prediction reduces. A prominent example is the analysis of plates with open holes. In this case, a factor of reduction in the bearable load is expected that just matches the stress concentration factor. But experiments show that the failure loads exceed this value. The critical load not only depends on the theoretical dimensionless stress concentration factor k_t but also depends on a characteristic size of the structure, e.g. the hole diameter. To take such effects into account, the concept of notch sensitivity factors has been introduced. The theoretical stress concentration factors k_t obtained from linear elastic analysis is reduced by a heuristic notch sensitivity factor q to obtain an effective stress concentration factor [158, 166]:

$$k_e = q (k_t - 1) + 1. \quad (33)$$

But the proper notch sensitivity factor has to be identified for each material and structural situation. The notch sensitivity factor takes values between unity (full notch sensitivity as theoretically calculated) to zero (no decrease in the bearable loads due to the notch). In the analysis of size effects, it is found that the notch sensitivity approaches unity with increasing structural size. In the literature, notch sensitivity charts and approximate formulae can be found for several structural situations and different materials.

Size effects can also be captured by using modified strength of material approaches that do not evaluate the point of stress concentration but use the stress values at a certain distance, or stress values averaged over a certain length or area instead. Such approaches date back to the 1930s when Neuber and Peterson developed analysis approaches for fatigue failure. A comprehensive overview of such approaches can be found in the textbook by Taylor [192] that summarizes such approaches under the term *Theory of Critical Distances*. Size effects of failure processes at stress concentrations can be analysed by these approaches by means of an empiric length parameter. However, the employed empiric length parameter must be obtained from additional experiments and depends on the structural situation and loading. Although several approximate formulae of the length parameter as functions of structural parameters and material parameters have been proposed, no general dependence can be given.

Using the coupled criterion in the framework of FFM, such structural situations can be analysed very advantageously. As outlined previously, no empiric length scale or notch sensitivity factors are necessary. Instead, the knowledge on the strength and toughness parameters of the material is sufficient. Whereas strength of materials essentially predicts a size effect law of L^0 , LEFM will always predict size effects of $L^{1/2}$ [17]. FFM approaches are able to cover intermediate size effects that are dependent on the type of the stress concentration and the structural situation.

5.1 Application of FFM

5.1.1 U-notches

Rounded V-notches and U-notches are widespread in engineering structures. In design, sharp notches will always be avoided and during manufacturing; in most cases, a certain rounding radius will always be present. These structures closely related to the V-notch that exhibits singular stresses have been addressed by several researchers by means of FFM and the coupled criterion. Taylor et al. [193] have used closed-form approximate solutions of stress intensity factors of cracks emerging from the notch and stress concentration factors of the notch to set up a failure model for U-notches. They obtain a solution that shows a strong dependence on the stress concentration factor of the notch that is essentially controlled by the notch radius. Comparison to results of experiments with U-notched plates with different materials shows a good agreement of the measured and predicted toughness. They give a detailed discussion of the similarity of the FFM approach to classical

approaches like Neuber's concept of microstructural support. An asymptotic approach for the analysis of U-notches using asymptotic solutions was proposed by Carpinteri et al. [31]. The analysis makes use of numerically obtained generalized stress intensity factors of cracks emerging from the notch root. The analyses comprise the effect of the notch radius and the notch opening angle. Again a good agreement to experimental results is obtained. A comparison of FFM solutions and two approaches in the framework of the theory of critical distances was conducted by Cicero et al. [36]. Besides a discussion of the differences and similarities of the failure model, they provide a comparison with experimental results on U-notched three-point bending specimens. The comparison shows good agreement with the experimental results for all considered failure models, whereas only the FFM model does not require the definition of an empiric length scale.

Crack initiation at U-notches in a unidirectional GFRP laminate was studied by Andersons et al. [10]. Tensile loading in an angle to the fibre direction was considered in their analysis. A direct numerical formulation using the FEM was chosen to evaluate the coupled criterion. As in the work by Hebel and Becker [77], the energy release rate of the finite cracks was computed using a discretized form of the Virtual Crack Closure Technique for finite cracks [178]. The failure model yields a good agreement with experimental results giving conservative failure load estimates.

5.1.2 *Debonding of inclusions*

The debonding of a single fibre due to transversal loading has been studied in detail by Mantič [133]. The plane strain model is analysed by means of classical analytical solutions allowing for a straightforward evaluation of the coupled criterion. The formation of a finite interface debond at the interface of the inclusion and the matrix is predicted. The analysis comprises the effect of the elastic mismatch, the interface fracture parameters and the size of the inclusion. The analysis shows that for large inclusions the size of the debond and the critical load at onset are independent of the inclusion size. Good agreement of this failure model with a Cohesive Zone Model for inclusion debonding [198] is obtained for sufficiently large sizes of the inclusion [61]. For small diameters of the inclusion, different size effect laws are predicted. In the work by Quesada et al. [171], the fracture of stiff inclusions under compressive loading is considered. By means of an asymptotic matching analysis, the coupled criterion is evaluated and allows for the discussion of failure modes. Failure of the inclusion with possibly multiple cracks parallel to the compressive loading, failure of the surrounding matrix and interface debonding have been discussed. The competition of these different failure modes is discussed at hand of examples, and it is shown that the failure modes are governed by the size of the inclusion. Biaxial loading of inclusions and possible interface debonding has been studied by means of the coupled criterion in the work by Mantič and García [135]. The effect of the biaxial loading on the occurring size effect has been studied, and failure envelopes of critical loads are given for a wide range of a dimensionless interface failure quantity. The same case of mixed-mode loading of an inclusion has been studied by Greco et al. [68] by means of a numerical analysis to allow for the study of the nonlinear constitutive response of composite materials. They propose a numerical micromechanics model that allows for the prediction of microcrack evaluation and a homogenization analysis. Fibre matrix debonding under a combination of in-plane tension and out-of-plane shear loading has been studied by Carraro and Quaresimin [32]. The aim of their study was to improve the understanding of the debonding of fibres in unidirectional laminates. A comparison to experimental results on GFRP laminates under mixed-mode loading [160] shows that their model on the basis of the coupled criterion can give good predictions of the critical loads. García et al. [62,63] have studied debonding of spherical inclusions by means of FFM and the coupled criterion. Motivated by the problem of failure of particle-reinforced composites, they developed a failure model using the Boundary Element Method to calculate the critical uniaxial tension. The effect of the size of the particles is discussed as well as the question whether symmetric or asymmetric debonding of the inclusion occurs. The results indicate that an asymmetric debonding is preferential due to the increased energy release in comparison with a symmetric debonding.

5.1.3 *Plates with open holes*

A common engineering problem is plates necessarily notched with open holes. Although no infinite stresses are predicted, strength of materials approaches do not suffice for a comprehensive analysis of failure. The failure load exhibits a distinct dependence on the size of the hole, even when the outer dimensions are significantly larger than the hole diameter [69,95,211]. Several researchers have addressed this problem by means of the coupled criterion obtaining good agreement to experimental results in general.

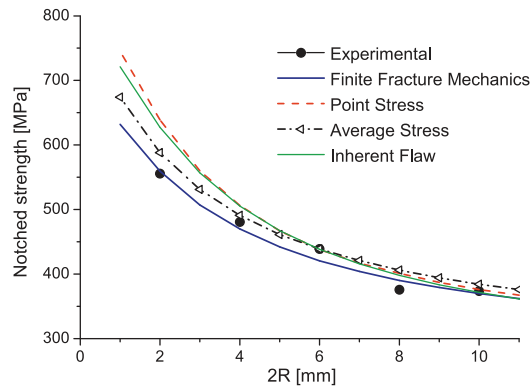


Fig. 8 Size effect analysis of a quasi-isotropic $[0/90/\pm 45]_s$ -laminate notched with an open hole of size R . Comparison to experimental results [26]) (Reprinted from [25] with permission from Elsevier Ltd.)

Li and Zhang [127] have addressed the problem of an open hole in an isotropic, infinite plate by using Kirsch's classical stress solution [96] and an approximate tabulated stress intensity factor of cracks emanating from the hole [191]. To obtain good agreement to experimental results, they propose an extension of the coupled criterion, introducing a specific, size-dependent fracture energy. Their proposed 3-parameter model shows good agreement with experimental results on PMMA specimens. The same experiments were used by Hebel and Becker [77] in their failure analysis that uses a direct numerical formulation of the coupled criterion. They considered the effect of finite width corrections and obtained good agreement to the experimental data with Leguillon's coupled criterion using two parameters by adjusting the fracture toughness to higher values. They assume that nonlinear effects that are not covered by the linear elastic analysis make this necessary. Camanho et al. [25] have developed a closed-form analytical FFM failure model for plates with finite width. In their work, they apply this failure model to the failure analysis of laminated plates with open holes. Quasi-isotropic CFRP laminates under uniaxial tension are studied. Good agreement with experimental results [26,69] is obtained, cf. Fig. 8. A comparison to existing approaches as concepts in the Theory of Critical Distances and Inherent Flaw models that use an empiric length scale is also given. In their work, they discuss the effect of the fracture toughness and the hole diameter on the notch sensitivity of the notched plate. In their failure model, the laminate strength and the laminate toughness obtained from tensile tests of notched and unnotched specimens are used. To obtain the fracture toughness from the experiments, Camanho et al. [25] use an inverse identification by means of the coupled criterion. Arteiro et al. [12] propose a similar inverse identification for composite laminates with non-crimp fabrics but propose to model the crack in the specimens to be of elliptical shape. Hence, they use the corresponding stress solution [89] in their coupled criterion for inverse evaluation.

The model by Camanho et al. [25] was applied to another set of experiments and also to the case of compressive loading in [55]. Other failure models are studied for comparison, and it is shown that the use of the coupled criterion leads to the best failure load predictions, and the case of compressive loading is also covered by this approach. Damage initiation under compressive loading is also studied by Romani et al. [176]. Using Digital Image Correlation, they obtain experimental results on the initiation of damage. The predicted crack initiation load and the size effect show good agreement with the experimentally obtained failure loads.

Martin et al. [139] provide an analysis of an anisotropic plate of finite width notched with a central open hole. Crack initiation under tensile loading is analysed using the coupled criterion. Two modelling approaches using closed-form analytical solutions and a direct numerical implementation by means of FEM are used. Good agreement of both approaches with experimental results on notched laminates [95] is found, showing a distinct size effect. Laminates loaded in an angle to the fibre direction were studied by Modniks et al. [149]. The evaluation of the coupled criterion was also done by using a direct numerical implementation using the FEM. Good agreement to experimental results was obtained for a wide range of mixed-mode loading conditions.

The case of a bolt-loaded hole and the corresponding failure analysis has been studied by Catalanotti and Camanho [34]. By means of a comprehensive FEM, they obtain approximate functions of the stress distribution and the stress intensity factor for the case of an isotropic plate with finite width with a bolt-loaded central hole. Their implementation of the coupled criterion, which incorporates the effect of a crack length-dependent fracture toughness (R-curve), shows good agreement to experimental results on quasi-isotropic laminates.

Weißgraeber et al. [209] have proposed an FFM analysis of an elliptical hole in an infinite body. Using the closed-form analytical stress solution [89] and a proposed new stress intensity factor of mode I cracks emanating from the hole, they provide an implementation of the coupled criterion. The model allows for an

analysis of the effect of the aspect ratio of the ellipse. A transition from very low stress concentrations (slender ellipse aligned with loading direction) to very high stress concentrations (slender ellipse normal to loading direction) is studied. The results show a smooth transition of the FFM solution from strength of materials to LEFM. A comparison to experimental results [26] shows good agreement, and the effect of increasing size effects depending on the stress concentration factor is discussed.

5.1.4 Formation of crack patterns

In several structural situations, failure is associated with the formation of crack patterns. A commonly known example are crack patterns due to shrinkage (e.g. thermal/drying processes), as, e.g. during manufacturing of functional substrates, drying of soil in lake beds or in thermally loaded road pavements. Another case is fracturing of floating ice plates under out-of-plane loading.

In the analysis of crack patterns of floating ice plates under penetration, loading was studied by Bažant and Li [19, 129]. They found that Fracture Mechanics (Griffith's criterion) cannot be applied and strength of materials considerations can describe a critical loading but the number of radial cracks cannot be provided. To allow for an analysis of the crack patterns and their onset load, they proposed a three-part criterion. 1) To allow for onset of failure the strength must be exceeded at least at one point in the crack pattern. 2) After initiation of finite cracks (denoted as "dynamic jumps") crack growth occurs until the equality of Griffith's condition is just met. 3) The energy release during formation of the finite cracks must equal the energy needed to form the finite cracked area.

In subsequent works, they applied this three-part criterion to the formation of crack patterns in elastic halfspaces [130], especially the case of crack patterns in road pavements [87]. The analysis allowed for determination of the finite crack depth and the crack spacing as a function of the material parameters and the penetration depth of the shrinkage loading. They took into account the stiffness of the foundation in their model and obtained realistic agreement to typical crack pattern examples.

The concept of their threefold criterion is related to the coupled criterion independently proposed by Leguillon [110] seven years later. The major difference is that the condition of local overloading is just required at one point of the considered finite crack in the threefold criterion by Bažant and Li, whereas the FFM-PM criterion (6) in FFM requires the whole area of finite crack to be overloaded prior to fracture. So in the threefold criterion parts of the structure are predicted to fracture that are not stressed beyond the strength. The evaluation at a single point can be advantageous in contrast to the analysis of a stress distribution, but typically the determination of the energy release rate, which is required in both criteria, is significantly more complex and rules the efficiency. As the one-point strength condition of Bažant and Li does not bound the finite crack lengths, the criterion can only be applied when the considered geometries are negative. In negative geometries, the energy release rate is a globally decreasing function of the crack length. As they considered structural situations with an initial strain that is assumed to decline with the penetration depth, the considered case of formation of surface cracks is negative. If the energy release rate does not decrease, the second condition can never be fulfilled and the predicted finite cracks are infinitely large.

In the work by Leguillon [112] and Leguillon et al. [123] the coupled criterion in the framework of FFM is applied to the analysis of crack pattern formation during thermal loading. He considered patterns of parallel cracks in a semi-infinite plane. Using plane strain assumptions, the model rests on approximate closed-form solutions for stress intensity factors of cracks in semi-infinite planes. There is a strong dependence of the resulting crack patterns on the strength of the material. The subdivision of crack lattices during slow cooling processes is discussed as well as size effects known from temperature loaded structural situations.

It remains a lot to do in this topic, with applications to the surface cracking of materials during oxidation and ageing. Of course, a main issue is to obtain experimental validations.

6 Outlook on possible extensions

It is always difficult to suggest prospects within a scientific theory, opportunities often arise unexpectedly. However, there are some few topics within the FFM framework that would deserve deeper investigations right now. They are proposed below in ascending order of lack of knowledge.

6.1 Crack kinking out of an interface

Although the crack propagation of an interface crack, straight along this interface is quite well modelled [175], the conditions that govern the crack kinking out of the interface are not so properly established. When using the Williams series expansion (28), a difficulty arises because complex exponents are involved. Calculating the energy release rate, the oscillating terms vanish for the propagation along the interface but not if the crack kinks out of the interface. In the pioneering work of He and Hutchinson [75], the analysis is conducted out of the oscillation zone, i.e. the distance at which the calculations are carried out is more or less fixed, which amounts in some way to neglect the oscillations. Then, the few authors who have addressed the problem [13, 76, 93, 128] all emphasized the role played by the T-stress in the deflection mechanism, focusing on an energy criterion. But they still have the problem of arbitrarily defining a distance.

Clearly, considering the T-stress in the series expansion (28) for an interface crack leads to (the upper bar denotes the complex conjugate)

$$\sigma \propto Kr^{-1/2+i\varepsilon} \mathbf{g}(\varphi) + \overline{K}r^{-1/2-i\varepsilon} \overline{\mathbf{g}(\varphi)} + T\mathbf{k}(\varphi) + \dots \quad (34)$$

where the stress intensity factor K is complex as well as the angular function $\mathbf{g}(\varphi)$ and where the parameter ε depends on the contrast between the two adjacent materials [175]. The angular function $\mathbf{k}(\varphi)$ is associated with the T-stress whose generalized intensity factor is denoted by T . Then, despite a leading exponent with a real part equal to $-1/2$, using the coupled criterion amounts to a reasoning similar to that of a weak singularity [117]. Thus, the distance is no longer arbitrary; it is a consequence of the FFM theory. However, note that it does not completely rule out the problem of oscillations. An attempt in this direction was proposed by Leguillon and Murer [116] but is still awaiting for an experimental confirmation.

Anyway, the above results assume a long interface crack, it remains to describe the step-like pattern of some cracks in laminates: the crack impinges on the interface, then deflects along the interface on a short distance and then kinks out in the next layer.

6.2 Fatigue loadings

Accidental overloads are not the primary cause of failure in structures. Often fatigue, i.e. repeated loads at a low level with cyclic variations of the amplitude, is responsible for the appearance of cracks detrimental to the life time of these structures. Like the Griffith criterion in fracture mechanics, the Paris law addresses pre-existing long cracks and expresses their growth rate. But there is no macroscopic law allowing to describe the initiation of a crack at a stress concentration point under fatigue loading. The only existing descriptions make use of micromechanisms such as, in metal, the indentations created by emerging dislocations at a free surface. Moreover, the concept of short cracks remains very ambiguous in fatigue [189].

Clearly, the coupled criterion addresses brittle materials while fatigue assumes damage accumulation. Then, the question is: can the coupled criterion be extended to the initiation of cracks under fatigue loading? Starting from an idea of Jaubert and Marigo [91] and an analogy between the results of the Dugdale cohesive zone model [49] and the coupled criterion [83] a first model for crack initiation at a V-notch under fatigue loading was proposed in [153]. The Paris-like law for a crack initiation at a V-notch in a homogeneous material under symmetric loading takes the form [153]

$$\frac{da}{dN} = D \left(\frac{K_m}{K_c} \right)^{\frac{p}{2(1-\lambda)}}. \quad (35)$$

where $\frac{da}{dN}$ denotes the growth rate, i.e. the crack advance per cycle. The parameter D is a scaling coefficient depending on the tensile strength, K_m is the maximum value of the generalized stress intensity factor of the V-notch singularity (see (28)) reached during one cycle, K_c its critical value derived from the coupled criterion for a monotonic loading, λ the singularity exponent and p the Paris law exponent. This law was established so that it coincides with the Paris law for long cracks ($\lambda = 1/2$).

This result was rewritten in the formalism of the coupled criterion and extended to more complex loadings and mixed-mode growth in [118] with a slightly different exponent: $p - 4 + 2/(1 - \lambda)$. But again these results are still awaiting, first of all, for an experimental confirmation and for an extension to the crack initiation along an interface between dissimilar materials.

In the analyses by Pugno et al. [168,169], the concept of Finite Fracture Mechanics has been utilized to model the transition of microscopic short cracks to macroscopic large cracks. They obtained a unified ‘law’ for this transition that covers the Wöhler domain and the Paris-Edorgan domain of crack lengths [162,190]. The analysis allows for a first discussion of the concept of fatigue limit and the well-established models of Kitagawa and Takahashi [97] and El Haddad et al. [53].

6.3 Extension to the 3D case

Extending the coupled criterion to the 3D case is a big issue. In [111,113], an extension of the 2D coupled criterion based on the joint use of the 3D singularity theory and matched asymptotic expansions has been proposed. A Williams series expansion similar to (28) holds in 3D, the angular functions depending now on two angles θ and φ (spherical coordinates). Formally, the formulation of the 3D criterion is similar to that of the 2D case, although some differences appear to settle it [144]. The crack length Δa is replaced with the diameter (in the broad sense), still called Δa , of the surface of the new crack and $\Delta \Pi$ varies as $\Delta a^{2\lambda+1}$, whereas it was $\Delta a^{2\lambda}$ in 2D.⁴

As in 2D, satisfying the coupled criterion requires two conditions in stress and energy. But additional hypotheses on the geometry of the crack extensions are needed even if only plane crack extensions are considered. Indeed, in 2D, a straight crack extension can be defined by its length and its direction, i.e. two parameters, whereas in 3D the exact shape of the crack extension is not only defined by its surface and the direction of the fracture plane but by an infinite number of variables describing the crack front. To overcome this difficulty, additional assumptions are made: (i) the fracture plane is defined by the maximum normal stress σ [111]; (ii) the shape of the new crack is defined by the isovalue $\sigma = \sigma_c$ (the isovalues are homothetic curves when retaining only the leading term in the Williams series expansion); (iii) the surface of the new crack is defined by the energy condition $\Delta A = -\Delta \Pi / \mathcal{G}_c$ [113].

There are few sufficiently complete data in the literature to enable validation. In [113], results are compared to a single family of fracture tests carried out on bi-material samples [102] leading to an indirect validation: it provides consistent fracture properties of the interface. As in the two above sections, a definitive experimental confirmation is still awaited. Moreover, it seems necessary to use strong 3D numerical and visualization tools to go further in this direction.

6.4 Dynamic loadings

As claimed at the beginning of this section, this is one of the least explored issues in FFM. In this regard, according to the fundamental equation (2), a production of kinetic energy, i.e. $\Delta K > 0$, does not fall strictly within the framework of dynamic loadings but creates an imbalance in (2). This can occur if the toughness is low while the tensile strength is high, then the energy condition can be fulfilled, whereas the stress one is not. As a consequence, the load can be increased before the tensile stress reaches the tensile strength, increasing the stored energy. Thus, at failure, it may be a large amount of excess energy: $\Delta \Pi - \mathcal{G}_c \Delta A$. As discussed in [171], if no pieces can be ejected, then the largest part of the available energy can be used to increase ΔA , i.e. to create multiple cracks instead of one. In [124], this imbalance intervenes on the crack arrest length which is no longer determined by the relation $K_I \leq K_{Ic}$ but by a complete closure condition since there is some available energy for the crack to grow beyond the theoretical arrest point. This is a finding also established in classical dynamic fracture mechanics and leads to the definition of a crack arrest toughness [24,71]. This topic clearly deserves more thorough investigations, knowing that consuming whole or a part of this excess energy depends strongly on the situation.

The problem of dynamic loadings, *stricto sensu*, is completely open within the framework of FFM. It calls into question the strong assumption that the initial state before the crack nucleation is quasi-static (see Sect. 2) and all remains to be done. As in the above cases, dynamic loadings can bring an excess energy and be responsible for multi-cracking as often observed.

7 Conclusion

The concept of Finite Fracture Mechanics (FFM) has filled a gap in fracture mechanics, since it is able to offer a framework and criteria for predicting the crack initiation in brittle materials. Classical fracture mechanics is

⁴ See Eq. (31) for $\mathcal{G} = \lim(\Delta a \rightarrow 0) \Delta \Pi / \Delta a$ in the 2D case.

restricted to the analysis of growth of existing cracks. Instead of the consideration of infinitesimal cracks, as in fracture mechanics, cracks of finite length are considered within FFM. The formation of such finite cracks is studied by special crack initiation criteria. These criteria are twofold and can all be identified under the single term “coupled criterion”, since they only marginally differ. This so-called coupled criterion has a pronounced macroscopic nature; it involves few macroscopic parameters. In a homogeneous isotropic material, for instance, they are: the elastic constants (Young’s modulus and Poisson’s ratio) and the material toughness, like Griffith’s criterion for crack propagation, and the tensile strength in addition. In case of complex loadings, composite materials, etc., the number of parameters should be slightly extended while remaining very limited: elastic moduli, mode I and mode II toughness, shear strength for the various materials in presence but also for interfaces in some cases of cracks interacting with interfaces. Moreover, this criterion does not involve any adjustable parameter, which is an indisputable advantage. Another important feature is that the analyses do not require the consideration of nonlinear degradations as pure elastic-brittle failure is considered. Hence, no convergence issues emerge.

The coupled criterion does not refer to microscopic mechanisms to predict the crack nucleation although it still has physical bases. The start point is, first, an indisputable energy balance equation and, second, a more intuitive concept that is well founded in experiments, the maximum bearable stress before failure. In particular, emphasis must be put on the fact that the coupled criterion recovers the Griffith criterion when applied to the crack growth, i.e. only an energy condition, and the only maximum stress rule in a pure tensile test without singularity or stress-raiser. Cracks are likely to appear at stress concentration points and primarily at singular points. In these latter cases, an asymptotic approach allows deriving quasi-analytical relationships when the asymptotic assumption, the crack initiation length is small, can be verified a posteriori. If not, or in the absence of a singularity, full FE calculations can still be conducted.

The coupled criterion, in its various forms, has been successfully applied to the prediction of crack initiation in many different cases from homogeneous to composite materials and structures. The failure of a wide range of brittle materials has been compared to experiments: polymers, glass, graphite, carbon, ceramics, plaster, concrete, rocks.... Some original mechanisms, such as those related to strong singularities or the formation of crack patterns, have been highlighted. But of course, like any scientific result, the concept of FFM could be refuted in the future. Until that date which is impossible to anticipate, according to the wide agreement met with many experiments, the coupled criterion can be used with some confidence to predict the crack initiation in brittle materials in new situations never faced before.

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