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Hybrid multi-observer for improving estimation performance

E. Petri, R. Postoyan, D. Astolfi, D. Nešić and V. Andrieu

Abstract—Various methods are nowadays available to design observers for broad classes of systems, where the primary focus is on establishing the convergence of the estimated states. Nevertheless, the question of the tuning of the observer to achieve satisfactory estimation performance remains largely open. In this context, we present a general design framework for the online tuning of the observer gains. Our starting point is a robust nominal observer designed for a general nonlinear system, for which an input-to-state stability property can be established. Our goal is then to improve the performance of this nominal observer. We present for this purpose a new hybrid multi-observer scheme, whose flexibility can be exploited to enforce various desirable properties, e.g., fast convergence and good sensitivity to measurement noise. We prove that an input-to-state stability property also holds for the proposed scheme and, importantly, we ensure that the estimation performance in terms of a quadratic cost is (strictly) improved. We illustrate the efficiency of the approach in improving the performance of given nominal observers in two numerical examples (Van der Pol oscillator and Lithium-Ion (Li-Ion) battery model).

I. INTRODUCTION

State estimation of dynamical systems is a central theme in control theory, whereby an observer is designed to estimate the unmeasured system states exploiting the knowledge of the system model and input and output measurements. Many techniques are available in the literature for the observer design of linear and nonlinear systems, see [1] and the references therein. The vast majority of these works focuses on designing the observer so that the origin of the associated estimation error system is (robustly) asymptotically stable. A critical and largely open question is how to tune the observer to obtain desirable performance (e.g. convergence speed and overshoot in the transient response) in the presence of model uncertainties, measurement noise and disturbances. This question obviously also arises in the context of control.

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One of the challenges in observer tuning is that there exist different trade-offs between the desired properties. Indeed, a standard approach to observer design consists in using a copy of the plant (in some coordinates that may be different from the original ones) and then adding a correction term, often denoted as the output injection term. This term is designed by multiplying the difference between the measured output and the estimated one by a (possibly nonlinear) gain, whose tuning produces different estimation performance. Typically, the output injection term with small gains produces an observer robust to measurement noise, but its convergence is very slow. On the contrary, an observer with a large gain usually has a fast convergence, but is more sensitive to noise. An answer to the question on how to tune the observer gain in the special context of linear systems affected by additive Gaussian noise impacting the dynamics and the output is the celebrated Kalman filter [2]. For general nonlinear systems and noise/disturbances, optimal observer design with global stability guarantees is notoriously hard. For instance, in the context of nonlinear systems, optimal state estimation requires solving challenging partial differential equations [3]. An alternative consists in aiming at *improving* the estimation performance of a given observer. To the best of the authors' knowledge, existing works in this direction either concentrate on specific classes of systems (see, e.g., [4]–[7] for linear systems or, e.g., [8]–[12] in the context of high-gain observers), specific observers, e.g., [13], [14] or specific properties like robustness to measurement noise (see, e.g., [15], [16]) or the reduction of the undesired effect of the peaking phenomenon (see, e.g., [17] for a general approach and [9], [11] for specific solutions in the context of high-gain observer). An exception is [18], where two observers designed for a general nonlinear system are “united” to exploit the good properties of each. However, the design in [18] is not always easy to implement as it requires knowledge of various properties of the observers (basin of attraction, ultimate bound), which may be difficult to obtain. Alternative methods proposed in the literature are particle filters and unscented Kalman filters, see, e.g., [19], which combine observers to obtain good estimation performance, but are not endowed with robust stability guarantees in general. Recently, in [20], [21] suboptimal moving horizon estimation schemes have been proposed for general discrete-time nonlinear systems, where the performance of a given auxiliary observer is improved. There is thus a need for performant estimation schemes with convergence guarantees that are both easy to design and applicable to general nonlinear systems.

In this work we present a new, flexible and general observer

design methodology based on supervisory multi-observer ideas that can be used to address various trade-offs between robustness to modeling errors and measurement noise, and convergence speed. A multi-observer consists of a bank of observers that run in parallel. It has been used in the literature in a range of different contexts, including improvement of the sensitivity to measurement noise and/or reduction of the undesired overshoot during the transient (e.g., [10], [11] for high-gain observer and [22] for KKL observer), joint state-parameter estimation (e.g., [23]–[26]) or distributed observers (e.g., [27], [28]). In this paper, we propose a new problem formulation that we believe has not yet been addressed in the literature. Our starting point is the knowledge of a nominal observer, which ensures that the associated state estimation error system satisfies an input-to-state stability property with respect to measurement noise and disturbances. Various methods from the literature can be used to design the nominal observer, see [15], [29] and the references therein. Then, we construct a multi-observer, composed of the nominal observer and additional dynamical systems, all together called *modes*, that have the same structure as the nominal observer, but different gains. It is important to emphasize that the number of modes and the associated gains can be freely assigned (no specific stability/convergence property is required). Because the gains are different, each mode exhibits different properties in terms of speed of convergence and robustness to measurement noise. The latter point is relevant as we can heuristically design the extra modes for the application at hand without having to worry about the proof of their convergence. For instance, we can select one of these modes to have a zero gain, which is advantageous when measurement noise is an issue, as we illustrate on simulations. We will provide more general guidelines for the design of the extra modes. We run all modes in parallel and we evaluate their estimation performance in terms of a quadratic cost using monitoring variables, inspired by supervisory control and estimation techniques, see, e.g., [23], [26], [30]–[33]. Based on these running costs (i.e., monitoring variables), we design a switching rule that selects, at any time instant, the mode which is providing the best performance. When a new mode is selected, the other ones may reset or not their current state estimate (and their monitoring variable) to it.

We model the overall system as a hybrid system using the formalism of [34]. We prove that the proposed hybrid estimation scheme satisfies an input-to-state stability property with respect to disturbance and measurement noise. Note that such a property is not obvious as we do not require any convergence guarantee on the additional modes, but only for the nominal one. We also guarantee the existence of a (semiglobal uniform) average dwell-time thereby ruling out Zeno phenomenon. The performance of the proposed hybrid multi-observer in terms of the cost associated with the designed monitoring variables is guaranteed to be, at least, as good as the performance of the nominal observer by design. Moreover, we provide extra conditions under which the proposed hybrid multi-observer produces a strict performance improvement compared to the nominal one in terms of an integral cost. The efficiency of the proposed technique is illustrated on two numerical examples.

In the first one, the proposed approach is used to improve the estimation performance of a high-gain observer used to estimate the state of a Van der Pol oscillator. In the second example, we consider an equivalent circuit model for an Li-Ion battery, whose state is estimated with an observer designed using a polytopic approach and we implement the hybrid estimation scheme to improve its performance. Another example is given in [35], where a previous version of the hybrid estimation scheme [36] is applied for the state estimation of a more advanced Li-Ion battery model, namely a single-particle electrochemical model.

To summarize, the main contributions are: (i) a general and flexible design methodology to improve the performance of a nominal observer; (ii) the guarantee that the obtained state estimate ensures an input-to-state stability property; (iii) the proof of the existence of an average dwell-time, which excludes the Zeno phenomenon; and (iv) the performance analysis of the proposed hybrid scheme and the conditions under which strict performance improvement occurs. Items (iii)-(iv) are new compared to the preliminary version in [36]. **Notation and Preliminaries.** \mathbb{R} stands for the set of real numbers, $\mathbb{R}_{\geq 0} := [0, +\infty)$, \mathbb{Z} is the set of integers, $\mathbb{Z}_{\geq 0} := \{0, 1, 2, \dots\}$ and $\mathbb{Z}_{>0} := \{1, 2, \dots\}$. For a vector $x \in \mathbb{R}^n$, $\|x\|$ denotes its Euclidean norm. For a matrix $A \in \mathbb{R}^{n \times m}$, $\|A\|$ stands for its 2-induced norm. For a signal $v : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_v}$ with $n_v \in \mathbb{Z}_{>0}$, and $t_1, t_2 \in \mathbb{R}_{\geq 0} \cup \{\infty\}$ with $t_1 \leq t_2$, $\|v\|_{[t_1, t_2]} := \text{ess. sup}_{t \in [t_1, t_2]} |v(t)|$. Given a real, symmetric matrix P , its maximum and minimum eigenvalues are denoted by $\lambda_{\max}(P)$ and $\lambda_{\min}(P)$ respectively. The notation I_N stands for the identity matrix of dimension $N \in \mathbb{Z}_{>0}$, while $0_{N \times M}$ stands for the null matrix of dimension $N \times M$, with $N, M \in \mathbb{Z}_{>0}$. We use $\mathbb{S}_{>0}^N$ ($\mathbb{S}_{\geq 0}^N$) to denote the set of real symmetric positive definite (semidefinite) matrices of dimension N . The notation $\delta_{i,j}$, with $i, j \in \mathbb{Z}_{>0}$ denotes the Kronecker delta defined as $\delta_{i,j} = 0$ if $i \neq j$ and $\delta_{i,j} = 1$ if $i = j$. A continuous function $\alpha : [0, \infty) \rightarrow [0, \infty)$ is of class \mathcal{K} if $\alpha(0) = 0$ and it is strictly increasing. Moreover, α is of class \mathcal{K}_{∞} if $\alpha \in \mathcal{K}$ and $\lim_{r \rightarrow \infty} \alpha(r) = \infty$. A continuous function $\beta : [0, \infty) \times [0, \infty) \rightarrow [0, \infty)$ is of class $\mathcal{K}\mathcal{L}$ if, for any fixed $s \in \mathbb{R}_{\geq 0}$, $\beta(\cdot, s) \in \mathcal{K}$ and, for each fixed $r \in \mathbb{R}_{\geq 0}$, $\beta(r, \cdot)$ is non-increasing and satisfies $\lim_{s \rightarrow \infty} \beta(r, s) = 0$. Given a function $f : \mathcal{S}_1 \rightarrow \mathcal{S}_2$ with sets $\mathcal{S}_1, \mathcal{S}_2$, $\text{dom } f := \{z \in \mathcal{S}_1 : f(z) \neq \emptyset\}$. Based on the formalism of [34], we will model the proposed estimation scheme together with the plant as a hybrid system with inputs of the form

$$\mathcal{H} : \begin{cases} \dot{x} &= F(x, u), & x \in \mathcal{C}, \\ x^+ &\in G(x, u), & x \in \mathcal{D}, \end{cases} \quad (1)$$

where $\mathcal{C} \subseteq \mathbb{R}^{n_x}$ is the flow set, $\mathcal{D} \subseteq \mathbb{R}^{n_x}$ is the jump set, F is the flow map and G is the jump map. We consider hybrid time domains as defined in [34]. The notation $(t, j) \geq (t^*, j^*)$ means that $t \geq t^*$ and $j \geq j^*$, where $(t, j), (t^*, j^*) \in \mathbb{R}_{\geq 0} \times \mathbb{Z}_{\geq 0}$. We use the notion of solution for system (1) as given in [37, Definition 4]. Given a set $\mathcal{U} \subseteq \mathbb{R}^{n_u}$, $\mathcal{L}_{\mathcal{U}}$ is the set of all functions from $\mathbb{R}_{\geq 0}$ to \mathcal{U} that are Lebesgue measurable and locally essentially bounded. Given a set $\mathcal{C} \subseteq \mathbb{R}^{n_x}$, the tangent cone to the set \mathcal{C} at a point $x \in \mathbb{R}^{n_x}$, denoted $T_{\mathcal{C}}(x)$, is the set of all vectors $v \in \mathbb{R}^{n_x}$ for which there exist $x_i \in \mathcal{C}$,

$\tau_i > 0$, with $x_i \rightarrow x$, $\tau_i \rightarrow 0$, and $v = \lim_{i \rightarrow \infty} \frac{x_i - x}{\tau_i}$. Given a solution q for system (1) and its hybrid time domain $\text{dom } q$, $I^j := \{t : (t, j) \in \text{dom } q\}$ is the flow interval for each $j \in \mathbb{Z}_{\geq 0}$ and $\text{int } I^j$ denotes its interior. Finally, we use $U^\circ(x; v) := \limsup_{h \rightarrow 0^+, y \rightarrow x} \frac{U(y+hv) - U(y)}{h}$ to denote the Clarke generalized directional derivative of a Lipschitz function U at x in the direction v [38], which reduces to the standard directional derivative $\langle \nabla U(x), v \rangle$ when U is continuously differentiable.

II. PROBLEM STATEMENT

The aim of this work is to improve the estimation performance of a given nonlinear nominal observer by exploiting a novel hybrid estimation scheme that is presented in the next section. We consider the plant model

$$\begin{aligned} \dot{x} &= f_p(x, u, v) \\ y &= h(x, w), \end{aligned} \quad (2)$$

where $x \in \mathbb{R}^{n_x}$ is the state to be estimated, $u \in \mathbb{R}^{n_u}$ is the measured input, $y \in \mathbb{R}^{n_y}$ is the measured output, $v \in \mathbb{R}^{n_v}$ is an unknown disturbance input and $w \in \mathbb{R}^{n_w}$ is an unknown measurement noise, with $n_x, n_y \in \mathbb{Z}_{>0}$ and $n_u, n_v, n_w \in \mathbb{Z}_{\geq 0}$. The input signal $u : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_u}$, the unknown disturbance input $v : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_v}$ and the measurement noise $w : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}^{n_w}$ are such that $u \in \mathcal{L}_U$, $v \in \mathcal{L}_V$ and $w \in \mathcal{L}_{WV}$ for closed sets $\mathcal{U} \subseteq \mathbb{R}^{n_u}$, $\mathcal{V} \subseteq \mathbb{R}^{n_v}$ and $\mathcal{W} \subseteq \mathbb{R}^{n_w}$.

We consider a so-called nominal observer for system (2) of the form

$$\begin{aligned} \dot{\hat{x}}_1 &= f_o(\hat{x}_1, u, L_1(y - \hat{y}_1)) \\ \hat{y}_1 &= h(\hat{x}_1, 0), \end{aligned} \quad (3)$$

where $\hat{x}_1 \in \mathbb{R}^{n_x}$ is the state estimate, $\hat{y}_1 \in \mathbb{R}^{n_y}$ is the output estimate and $L_1 \in \mathbb{R}^{n_{L_1} \times n_y}$ is the observer output injection gain with $n_{L_1} \in \mathbb{Z}_{>0}$. We define the estimation error as $e_1 := x - \hat{x}_1 \in \mathbb{R}^{n_x}$ and introduce a *perturbed* version of the error dynamics, following from (2) and (3), as

$$\begin{aligned} \dot{e}_1 &= f_p(x, u, v) - f_o(\hat{x}_1, u, L_1(y - \hat{y}_1) + d) \\ &=: \tilde{f}(e_1, x, u, v, w, d) \end{aligned} \quad (4)$$

where $d \in \mathbb{R}^{n_{L_1}}$ represents an additive perturbation on the output injection term $L_1(y - \hat{y}_1)$. We assume that observer (3) is designed such that system (4) is input-to-state stable with respect to v , w and d , uniformly in u and x , as formalized next.

Assumption 1. *There exist $\underline{\alpha}, \bar{\alpha}, \psi_1, \psi_2 \in \mathcal{K}_{\mathcal{L}}$, $\alpha \in \mathbb{R}_{>0}$, $\gamma \in \mathbb{R}_{\geq 0}$ and $V : \mathbb{R}^{n_x} \rightarrow \mathbb{R}_{\geq 0}$ continuously differentiable, such that for all $x \in \mathbb{R}^{n_x}$, $e_1 \in \mathbb{R}^{n_x}$, $d \in \mathbb{R}^{n_{L_1}}$, $u \in \mathcal{U}$, $v \in \mathcal{V}$, $w \in \mathcal{W}$,*

$$\underline{\alpha}(|e_1|) \leq V(e_1) \leq \bar{\alpha}(|e_1|) \quad (5)$$

$$\begin{aligned} \left\langle \nabla V(e_1), \tilde{f}(e_1, x, u, v, w, d) \right\rangle &\leq -\alpha V(e_1) + \psi_1(|v|) \\ &\quad + \psi_2(|w|) + \gamma |d|^2. \end{aligned} \quad (6)$$

A large number of observers in the literature have the form of (3) and satisfy Assumption 1, possibly after a change of coordinates, see [15], [17], [29] and the references therein for more details. Examples are also provided in Section VIII.

Assumption 1 implies that there exist $\beta \in \mathcal{K}_{\mathcal{L}}$ and $\rho \in \mathcal{K}_{\mathcal{L}}$ such that, for any $u \in \mathcal{L}_U$, $v \in \mathcal{L}_V$, $w \in \mathcal{L}_W$ and $d \in \mathcal{L}_{\mathbb{R}^{n_{L_1}}}$, any corresponding solution (x, e_1) to systems (2) and (4) verifies, for all $t \in \text{dom}(x, e_1)$,

$$|e_1(t)| \leq \beta(|e_1(0)|, t) + \rho(\|v\|_{[0,t]} + \|w\|_{[0,t]} + \|d\|_{[0,t]}). \quad (7)$$

Inequality (7) provides a desirable robust stability property of the estimation error associated with observer (3). However, this property may not be satisfactory in terms of performance, like convergence speed and noise/disturbance rejection. To tune L_1 to obtain desirable performance properties is highly challenging in general and even impossible in some cases when the desired properties are conflicting like high convergence speed and efficient noise rejection, see, e.g., [39]. To address this challenge, we propose a hybrid redesign of observer (3), which aims at improving its performance, in a sense made precise in the following, while preserving an input-to-state stability property for the obtained estimation error system.

Remark 1. *The results of the paper apply mutatis mutandis to the case where Assumption 1 holds semiglobally or when the Lyapunov function V depends on both x and e_1 , which allow to cover an even broader class of observers [1, Section V]. We do not consider these in this paper to not over-complicate the exposition and to not blur the main message of the work.*

In the following we also need the next technical assumption on the output map h in (2).

Assumption 2. *There exist $\delta_1, \delta_2 \in \mathbb{R}_{>0}$ such that for all $x, x' \in \mathbb{R}^{n_x}$, $w, w' \in \mathcal{W}$,*

$$|h(x, w) - h(x', w')|^2 \leq \delta_1 V(x - x') + \delta_2 |w - w'|^2, \quad (8)$$

where V comes from Assumption 1.

Assumption 2 holds in the common case where V in Assumption 1 is quadratic and h is globally Lipschitz. Indeed, in this case, $V(x - x') := (x - x')^\top P(x - x')$, with $P \in \mathbb{R}^{n_x \times n_x}$ symmetric, positive definite, and $|h(x, w) - h(x', w')| \leq K|(x - x', w - w')|$ for any $x, x' \in \mathbb{R}^{n_x}$, $w, w' \in \mathcal{W}$ and some $K \in \mathbb{R}_{\geq 0}$, then (8) holds with $\delta_1 = \frac{K^2}{\lambda_{\min}(P)}$ and $\delta_2 = K^2$. Note that h globally Lipschitz covers the common case where $h(x, w) = Cx + Dw$ with $C \in \mathbb{R}^{n_y \times n_x}$ and $D \in \mathbb{R}^{n_y \times n_w}$.

III. HYBRID ESTIMATION SCHEME

The hybrid estimation scheme we propose consists of the following elements, see Fig. 1:

- *nominal observer* given in (3);
- N additional dynamical systems of the form (3) but with a different output injection gain, where $N \in \mathbb{Z}_{>0}$. Each of these systems, as well as the nominal observer, is called *mode* for the sake of convenience;
- $N+1$ *monitoring variables* used to evaluate the performance of each mode of the multi-observer;
- *a selection criterion*, that switches between the state estimates produced by the different modes exploiting the performance knowledge given by the monitoring variables;
- *a reset rule*, that defines how the estimation scheme may be updated when the selected mode switches.

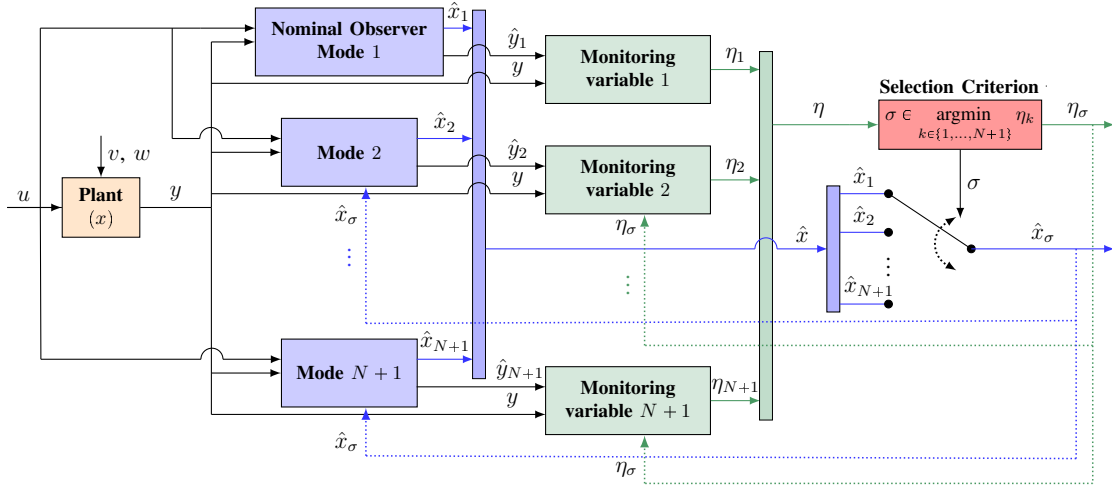


Fig. 1. Block diagram representing the system architecture with $\eta := (\eta_1, \dots, \eta_{N+1})$, $\hat{x} := (\hat{x}_1, \dots, \hat{x}_{N+1})$.

All these elements together form the hybrid multi-observer. We describe each component in the sequel.

A. Additional modes

We consider N additional dynamical systems, where the integer $N \in \mathbb{Z}_{>0}$ is arbitrarily selected by the user. These N extra systems are of the form of (3) but with a different output injection gain, i.e., for any $k \in \{2, \dots, N+1\}$, the k^{th} mode of the multi-observer is given by

$$\begin{aligned} \dot{\hat{x}}_k &= f_o(\hat{x}_k, u, L_k(y - \hat{y}_k)) \\ \hat{y}_k &= h(\hat{x}_k, 0), \end{aligned} \quad (9)$$

where $\hat{x}_k \in \mathbb{R}^{n_x}$ is the k^{th} mode state estimate, $\hat{y}_k \in \mathbb{R}^{n_y}$ is the k^{th} mode output and $L_k \in \mathbb{R}^{n_{L_1} \times n_y}$ is its gain. It is important to emphasize that we make no assumptions on the convergence properties of the solution to system (9) contrary to observer (3). There is thus full freedom for selecting the gains $L_k \in \mathbb{R}^{n_{L_1} \times n_y}$, with $k \in \{2, \dots, N+1\}$. We will elaborate more on the choice of the L_k 's in Section IV.

Remark 2. *There is also full freedom in the choice of the initial conditions of all modes in (3) and (9). We can therefore select all of them equal, but this is not necessary for the stability results in Section V to hold. See Remark 4 in the sequel for more details.*

Remark 3. *The nominal observer (3) and the additional modes (9) have constant gains L_k , but it is also possible to consider time-varying gains $L_k(t)$. In this case, if all the gains are uniformly bounded, i.e., there exists $\mathcal{M} > 0$ such that $|L_k(t)| \leq \mathcal{M}$ for all $t \geq 0$ and all $k \in \{1, \dots, N+1\}$, then the results in this paper hold mutatis mutandis. In the numerical example in Section VIII-B one of the modes is an extended Kalman filter, which thus has a time-varying gain. We recall that, in general, the extended Kalman filter does not ensure a global convergence property of the estimation error. However, this is not an issue since no stability guarantee is required for the additional modes.*

B. Monitoring variables

Given the $N+1$ modes, our goal is now to find a way to select the “best” between them, namely the one providing a better estimate, possibly improving the estimation given by the nominal observer (3). Ideally, the criterion used to evaluate the performance of each mode would depend on the estimation errors $e_k = x - \hat{x}_k$, with $k \in \{1, \dots, N+1\}$. However, since the state x is unknown, e_k is unknown and any performance criterion involving e_k would not be implementable. As a consequence, as done in other contexts, see, e.g., [40], [41], we rely on the knowledge of the output y and the estimated outputs \hat{y}_k for $k \in \{1, \dots, N+1\}$. In particular, inspired by [40], in order to evaluate the performance of each mode, we introduce the $N+1$ monitoring variables $\eta_k \in \mathbb{R}_{\geq 0}$ for any $k \in \{1, \dots, N+1\}$, with dynamics given by

$$\begin{aligned} \dot{\eta}_k &= -\nu\eta_k + (y - \hat{y}_k)^\top (\Lambda_1 + L_k^\top \Lambda_2 L_k) (y - \hat{y}_k) \\ &=: g(\eta_k, L_k, y, \hat{y}_k), \end{aligned} \quad (10)$$

with $\Lambda_1 \in \mathbb{S}_{\geq 0}^{n_y}$ and $\Lambda_2 \in \mathbb{S}_{\geq 0}^{n_x}$ with at least one of them positive definite and $\nu \in (0, \alpha]$ a design parameter, where α comes from Assumption 1. The term $(y - \hat{y}_k)^\top \Lambda_1 (y - \hat{y}_k)$ in (10) is related to the output estimation error, while $(y - \hat{y}_k)^\top L_k^\top \Lambda_2 L_k (y - \hat{y}_k)$ takes into account the correction effort of the observer, also called latency in [40]. Note that the monitoring variable η_k in (10) for all $k \in \{1, \dots, N+1\}$ is implementable since we have access to the output y and all the estimated outputs \hat{y}_k at all time instants. The monitoring variables η_k , with $k \in \{1, \dots, N+1\}$, provide evaluations of the performance of all the modes of the multi-observer. Indeed, by integrating (10) between time 0 and $t \in \mathbb{R}_{\geq 0}$, we obtain that for any $k \in \{1, \dots, N+1\}$, for any initial condition $\eta_k(0) \in \mathbb{R}_{\geq 0}$, for any $y, \hat{y}_k \in \mathcal{L}_{\mathbb{R}^{n_y}}$, and any $t \geq 0$,

$$\begin{aligned} \eta_k(t) &= e^{-\nu t} \eta_k(0) + \int_0^t e^{-\nu(t-\tau)} ((y(\tau) - \hat{y}_k(\tau))^\top (\Lambda_1 \\ &\quad + L_k^\top \Lambda_2 L_k) (y(\tau) - \hat{y}_k(\tau))) d\tau. \end{aligned} \quad (11)$$

Equation (11) is a finite-horizon discounted cost, which depends on the output estimation error.

The results we will show in this paper apply for any $\Lambda_1 \in \mathbb{S}_{\geq 0}^{n_y}$ and $\Lambda_2 \in \mathbb{S}_{\geq 0}^{n_x}$ with at least one of them being positive definite. However, their tuning impacts when a switch of the selected mode occurs and which mode is chosen. Indeed, Λ_1 and Λ_2 are the weights of the two terms in (10) and thus their values reflect how much we take into account the output estimation error and the correction effort of the observer in the monitoring variables. In particular, selecting Λ_1 with a norm bigger than Λ_2 implies that we weight more the term related to the output estimation error compared to the term related to the correction effort of the observer in the design of the monitoring variables and vice-versa. In addition, note that, Λ_2 multiplies the mode gains L_k and thus it implicitly considers also the effect of the measurement noise in the estimation error.

C. Selection criterion

Based on the monitoring variables η_k , with $k \in \{1, \dots, N+1\}$, we define a criterion to select the state estimate to look at. We use a variable $\sigma : \mathbb{R}_{\geq 0} \rightarrow \{1, \dots, N+1\}$ for this purpose, and we denote the selected state estimate mode \hat{x}_σ and the associated monitoring variable η_σ . The criterion consists in selecting the mode with the minimal monitoring variable, which implies minimizing the cost (11) over the modes $k \in \{1, \dots, N+1\}$. When several modes produce the same minimum monitoring variable at a given time, we select the mode, between the ones with the minimum monitoring variables, with the smaller derivative of η_k (which is given by $g(\eta_k, L_k, y, \hat{y}_k)$ from (10)). Moreover, if two or more modes have the same minimum monitoring variable and the same minimum derivative of the monitoring variable, then the proposed technique selects randomly one of them and this is not an issue to obtain the results in Sections V, VI and VII. Thus, we switch the selected mode only when there exists $k \in \{1, \dots, N+1\} \setminus \{\sigma\}$ such that¹ $\eta_k \leq \eta_\sigma$. In that way, at the initial time $t_0 = 0$, we take $\sigma(0) \in \underset{k \in \Pi}{\operatorname{argmin}} (g(\eta_k(0), L_k, y(0), \hat{y}_k(0)))$, where $\eta := \{\eta_1, \dots, \eta_{N+1}\}$ and $\Pi(\eta) := \underset{k \in \{1, \dots, N+1\} \setminus \{\sigma\}}{\operatorname{argmin}} \eta_k$, for all $\eta \in \mathbb{R}_{\geq 0}^{N+1}$. Then, σ is kept constant, i.e., $\dot{\sigma}(t) = 0$ for all $t \in (0, t_1)$, with $t_1 := \inf\{t \geq 0 : \exists k \in \{1, \dots, N+1\} \setminus \{\sigma(t)\} \text{ such that } \eta_k(t) \leq \eta_{\sigma(t)}(t)\}$. At time t_1 , we switch the selected mode according to $\sigma(t_1^+) \in \underset{k \in \Pi}{\operatorname{argmin}} (g(\eta_k(t_1), L_k, y(t_1), \hat{y}_k(t_1)))$. We repeat these steps iteratively and we denote with $t_i \in \mathbb{R}_{\geq 0}$, $i \in \mathbb{Z}_{>0}$ the i^{th} time when the selected mode changes (if it exists), i.e., $t_i := \inf\{t \geq t_{i-1} : \exists k \in \{1, \dots, N+1\} \setminus \{\sigma(t)\} \text{ such that } \eta_k(t) \leq \eta_{\sigma(t)}(t)\}$. Consequently, for all $i \in \mathbb{Z}_{>0}$, $\dot{\sigma}(t) = 0$ for all $t \in (t_{i-1}, t_i)$ and

$$\sigma(t_i^+) \in \underset{k \in \Pi}{\operatorname{argmin}} (g(\eta_k(t_i), L_k, y(t_i), \hat{y}_k(t_i))), \quad (12)$$

¹In [36] a switch of the selected mode occurs when there exists $k \in \{1, \dots, N+1\} \setminus \{\sigma\}$ such that $\eta_k \leq \epsilon \eta_\sigma$, where the parameter $\epsilon \in (0, 1]$ was introduced to mitigate the occurrence of infinitely fast switching. However, this does not allow to rule out the Zeno phenomenon. To solve this issue, we propose a different jump map in this work, which does not require introducing parameter ϵ in the switching criterion as in [36].

where we recall that $\eta = \{\eta_1, \dots, \eta_{N+1}\}$ and $\Pi(\eta) = \underset{k \in \{1, \dots, N+1\} \setminus \{\sigma\}}{\operatorname{argmin}} \eta_k$, for all $\eta \in \mathbb{R}_{\geq 0}^{N+1}$. We also argue that, implementing (12) online, which requires the knowledge of the derivative of the monitoring variables, is not an issue.

Remark 4. *The scheme proposed in this paper works for any initial condition $\eta_k(0) \in \mathbb{R}_{\geq 0}$, for all $k \in \{1, \dots, N+1\}$, which corresponds to the initial cost of each mode of the multi-observer. Consequently, the choice of $\eta_k(0)$ is an extra degree of freedom that can be used to initially penalize the modes when there is a prior knowledge of which mode should be initially selected, as done in [35]. Conversely, in the case where there is no prior knowledge on which mode should be chosen at the beginning, all η_k , with $k \in \{1, \dots, N+1\}$, can be initialized at the same value such that the term $e^{-\nu t} \eta_k(0)$ in (11) is irrelevant for the minimization.*

Remark 5. *The results in Sections V and VI also apply with $\sigma(t_i^+) \in \underset{k \in \{1, \dots, N+1\} \setminus \{\sigma\}}{\operatorname{argmin}} \eta_k$ instead of (12), as in [36]. To select the mode with the minimum derivative of the monitoring variable, among those with the minimum η_k , allows us to prove a strict performance improvement in Section VII.*

D. Reset rule

When a switching occurs, i.e., when a different mode is selected, we propose two different options to update the hybrid estimation scheme. The first one, called *without resets*, consists in only updating σ , and consequently, we only update the state estimate based on the new value of σ . Conversely, the second option, called *with resets*, consists in not only switching the mode that is considered, but also resetting the state estimates and the monitoring variables of all the modes $k \in \{2, \dots, N+1\}$ to the updated \hat{x}_σ and η_σ , respectively. The state estimate and the monitoring variable of the nominal observer (3), corresponding to mode 1, are never reset. Since the reset option re-initializes the modes states at each switching time, it can be useful when one or more modes are local observers, and thus they guarantee a convergence property of the estimation error only if they are initialized sufficiently close to the origin, or when the null gain is selected for one of the additional modes. However, the reset case requires communication between the modes, which may not be always implementable in practice.

To avoid infinitely fast switching, we introduce a regularization parameter $\varepsilon \in \mathbb{R}_{>0}$. In particular, when a switch of the selected mode occurs, the value of monitoring variables η_k , with $k \in \{2, \dots, N+1\} \setminus \{\sigma\}$, is increased by ε , both in the case without and with resets. The idea is to penalize the unselected modes and to allow the selected one to run for some amount of time before a new switch occurs. The analysis of the properties of the hybrid time domains of the overall solutions and the existence of a uniform semiglobal average dwell-time are presented Section VI.

We use the parameter $r \in \{0, 1\}$ to determine which option is selected, where $r = 0$ corresponds to the case without resets, while $r = 1$ corresponds to the case where the resets are implemented. Note that, the parameter r needs to be chosen off-line and it is fixed for each considered observation

process. When a switch of the considered mode occurs, the state estimate \hat{x}_k of the k^{th} mode is defined as, at a switching time $t_i \in \mathbb{R}_{\geq 0}$,

$$\hat{x}_1(t_i^+) := \hat{x}_1(t_i) \quad (13)$$

and, for all $k \in \{2, \dots, N+1\}$,

$$\begin{aligned} \hat{x}_k(t_i^+) &\in \{(1-r)\hat{x}_k(t_i) + r\hat{x}_{k^*}(t_i) : k^* \in \underset{j \in \Pi}{\operatorname{argmin}} (g(\eta_j(t_i), \\ &L_j, y(t_i), \hat{y}_j(t_i)))\} \\ &=: \hat{\ell}_k(\hat{x}(t_i), \eta(t_i), L, y(t_i), \hat{y}(t_i)), \end{aligned} \quad (14)$$

where $\hat{x} := (\hat{x}_1, \dots, \hat{x}_{N+1})$, $\eta = (\eta_1, \dots, \eta_{N+1})$, $L := (L_1, \dots, L_{N+1})$ and $\hat{y} := (\hat{y}_1, \dots, \hat{y}_{N+1})$. Similarly, at a switching time $t_i \in \mathbb{R}_{\geq 0}$, the monitoring variables are defined as,

$$\eta_1(t_i^+) := \eta_1(t_i), \quad (15)$$

$$\eta_\sigma(t_i^+) := \eta_\sigma(t_i) \quad (16)$$

and, for all $k \in \{2, \dots, N+1\} \setminus \{\sigma\}$,

$$\eta_k(t_i^+) = (1-r)\eta_k + r\eta_{k^*} + \varepsilon \quad (17)$$

where $\varepsilon \in \mathbb{R}_{>0}$ and $\eta_{k^*} = \min_{j \in \{1, \dots, N+1\} \setminus \{\sigma\}} \eta_j$. Note that, if the monitoring variables of more than one mode have the same value and it is the minimum between all the η_k , with $k \in \{1, \dots, N+1\}$, then, from (14), the modes may be reset with different state estimates. Merging (16) and (17) and using the Kronecker delta definition, we obtain, at a switching time $t_i \in \mathbb{R}_{\geq 0}$, for all $k \in \{2, \dots, N+1\}$,

$$\begin{aligned} \eta_k(t_i^+) &= \delta_{k,\sigma}\eta_k + (1-\delta_{k,\sigma})((1-r)\eta_k + r\eta_{k^*} + \varepsilon) \\ &=: p_k(\eta(t_i)), \end{aligned} \quad (18)$$

where $\varepsilon \in \mathbb{R}_{>0}$ and $\eta_{k^*} = \min_{j \in \{1, \dots, N+1\} \setminus \{\sigma\}} \eta_j$.

We can already note that, with the proposed technique, $\eta_{\sigma(t)}(t) \leq \eta_1(t)$ for all $t \geq 0$, both in the case without and with resets. Therefore, the estimation performance of the proposed hybrid multi-observer is always not worse than the performance of the nominal one according to the monitoring variables that we consider. We will study the performance of the estimation scheme in more depth in Section VII.

E. Hybrid model

To proceed with the analysis of the hybrid estimation scheme presented so far, we model the overall system as a hybrid system of the form of [34], where a jump corresponds to a switch of the selected mode and a possible reset as explained in Section III-D. We define the overall state as $q := (x, \hat{x}_1, \dots, \hat{x}_{N+1}, \eta_1, \dots, \eta_{N+1}, \sigma) \in \mathcal{Q} := \mathbb{R}^{n_x} \times \mathbb{R}^{(N+1)n_x} \times \mathbb{R}_{\geq 0}^{N+1} \times \{1, \dots, N+1\}$, and we obtain the hybrid system

$$\begin{cases} \dot{q} = F(q, u, v, w), & q \in \mathcal{C} \\ q^+ \in G(q), & q \in \mathcal{D}, \end{cases} \quad (19)$$

where flow map is defined as, for any $q \in \mathcal{C}$, $u \in \mathcal{U}$, $v \in \mathcal{V}$ and $w \in \mathcal{W}$, from (2), (3), (9), (10),

$$\begin{aligned} F &:= (f_p, f_{o,1}, \dots, f_{o,N+1}, g_1, \dots, g_{N+1}, 0) \\ G &:= (x, \hat{x}_1, \hat{\ell}_2, \dots, \hat{\ell}_{N+1}, \eta_1, p_2, \dots, p_{N+1}, \underset{k \in \Pi}{\operatorname{argmin}} g_k) \end{aligned}$$

with the short notation $f_{o,k} = f_o(\hat{x}_k, u, L_k(y - y_k))$, $g_k = g(\eta_k, L_k, y, y_k)$, $\hat{\ell}_k = \hat{\ell}_k(\hat{x}, \eta, L_k, y, \hat{y}_k)$ and $p_k = p_k(\eta)$, for all $k \in \{1, \dots, N+1\}$, where $\Pi(q) = \underset{k \in \{1, \dots, N+1\} \setminus \{\sigma\}}{\operatorname{argmin}} \eta_k$ for all $q \in \mathcal{D}$. In view of Section III-C, the flow and jump sets \mathcal{C} and \mathcal{D} in (19) are defined as

$$\mathcal{C} := \{q \in \mathcal{Q} : \forall k \in \{1, \dots, N+1\} \ \eta_k \geq \eta_\sigma\}, \quad (20)$$

$$\mathcal{D} := \{q \in \mathcal{Q} : \exists k \in \{1, \dots, N+1\} \setminus \{\sigma\} \ \eta_k \leq \eta_\sigma\}. \quad (21)$$

Remark 6. The state estimate \hat{x}_σ of the hybrid multi-observer is subject to jumps, which may not be suitable in some applications. For this reason, as done in [35] for the state estimation of Li-Ion batteries, it is possible to add a filtered version of \hat{x}_σ , denoted \hat{x}_f , whose dynamics between two successive switching instants is $\dot{\hat{x}}_f = -\zeta\hat{x}_f + \zeta\hat{x}_\sigma$, where $\zeta > 0$ is an additional design parameter and, \hat{x}_f does not change at switching times $t_i \in \mathbb{R}_{\geq 0}$, i.e., $\hat{x}_f(t_i^+) = \hat{x}_f(t_i)$. Note that, when the filtered version of the hybrid multi-observer state estimate is also considered, similar stability results can be proved, see [42, Section 5.3.6].

IV. DESIGN GUIDELINES

We summarize the procedure to follow to design the hybrid estimation scheme.

- 1) Design the nominal observer (3) such that Assumption 1 holds.
- 2) Select N gains L_2, \dots, L_{N+1} for the N additional modes in (9).
- 3) Implement in parallel the $N+1$ modes of the multi-observer.
- 4) Generate the monitoring variables η_k , with $k \in \{1, \dots, N+1\}$.
- 5) Evaluate the variable σ .
- 6) Select $\varepsilon \in \mathbb{R}_{>0}$ and run the hybrid scheme without or with resets.
- 7) \hat{x}_σ is the state estimate to be considered for estimation purpose.

There is a lot of flexibility in the number of additional modes N and the selection of the gains L_k , with $k \in \{2, \dots, N+1\}$. This allows to address the different trade-offs of the state estimation of nonlinear systems. To present a systematic procedure to design the additional gains is very challenging in view of the generality of the considered classes of systems and observers. However, we can provide general principles to be followed.

A. General guidelines for gains selection

1) *Null gain:* We can always select one of the additional mode gains as the null gain, namely $L_k = 0_{n_x \times n_y}$, for some $k \in \{2, \dots, N+1\}$. This choice will produce an open-loop observer, which therefore typically does not have any stability guarantee. However, it is the best gain choice to annihilate the effect of the measurement noise on the estimation error and thus this gain can be very useful to improve the estimation performance, especially when the resets are implemented, as we will show on numerical examples in Section VIII.

2) *Optimization-based design*: The additional modes gains can be designed by solving optimization problems off-line for different possible sets of initial conditions and classes of inputs and disturbances. Typically, the gains obtained in this way do not guarantee any stability property of the estimation error. However, this is not required for the proposed hybrid estimation scheme, as testified by Theorem 1. Among the different cost functions that can be considered to solve the optimization problem, one possible option is described in the following. Given $x_0, \hat{x}_0 \in \mathbb{R}^{n_x}$, $u \in \mathcal{U}$, $v \in \mathcal{V}$ and $w \in \mathcal{W}$ we define, for a given $t \in \mathbb{R}_{\geq 0}$,

$$\mathcal{J}(x_0, \hat{x}_0, u, v, w, L_k, t) := \int_0^t e^{-\vartheta s} (x(s) - \hat{x}_k(s))^\top Q (x(s) - \hat{x}_k(s)) ds, \quad (22)$$

where $\vartheta \in \mathbb{R}_{\geq 0}$, $Q \in \mathbb{S}_{\geq 0}^{n_x}$ is a weight matrix, $k \in \{2, \dots, N+1\}$, and $x(t)$, $\hat{x}_k(t)$ represent the solutions to systems (2) and (3), with $x(0) = x_0$ and $\hat{x}_k(0) = \hat{x}_0$, for all $t \in \mathbb{R}_{\geq 0}$, respectively. The optimization problem to be solved to obtain the gain $L_k \in \mathbb{R}^{n_x \times n_y}$ is given by

$$\min_{L_k \in \mathbb{R}^{n_x \times n_y}} \max_{v \in \mathcal{V}, w \in \mathcal{W}} \mathcal{J}(x_0, \hat{x}_0, u, v, w, L_k, t). \quad (23)$$

Cost (22) is a quadratic cost of the state estimation error, which is available off-line in simulations. By solving the optimization problem (23), we obtain the gain L_k , $k \in \{2, \dots, N+1\}$, that minimizes cost (22) for the worst case scenario for the disturbance $v \in \mathcal{V}$ and measurement noise $w \in \mathcal{W}$, for the considered input $u \in \mathcal{U}$ and initial conditions x_0 and $\hat{x}_0 \in \mathbb{R}^{n_x}$ for systems (2) and (3), respectively. By solving (23) for various initial conditions, inputs u and classes of disturbances and noises and final time t , we obtain a bank of observer gains, which can be used in each mode of the multi-observer.

3) *Adjusting L_1* : We can also select the additional gains in a neighborhood of the nominal one, or to scale the nominal gain by some factors. This gain selection will produce systems with different behaviors and switching between them should allow an improvement of the estimation performance.

B. Exploiting system and nominal observer structure and/or behaviour

The gain selection can be also done by exploiting the knowledge of the system and observer structures. For example, when the nominal observer (3) is a high-gain observer, see, e.g., [8], [43] or, more generally, an infinite gain margin observer [1, Section 3.4], we typically need to select a very large gain based on a conservative bound to ensure Assumption 1, which would result in fast convergence of the estimation error, but, unfortunately, it will be very sensitive to measurement noise. In this case, to overcome the conservatism of the theory, an option is to select the L_k gains (much) smaller than the nominal one, even though there is no convergence proof for these choices, in order to obtain a state estimate which is more robust to measurement noise. This is the approach followed in Section VIII-A on an example.

Another possible approach is to select the additional gains L_k 's considering the behavior of the nominal observer in simulation and choose them based on the properties we want to

improve. For instance, similarly to the case where the nominal observer is an high-gain observer, when the convergence speed of the nominal observer is satisfactory, but its estimation error is very sensitive to noises, the gains L_k 's should be selected smaller than the nominal one L_1 . On the other hand, if the convergence speed of the estimation error of the nominal observer is too slow, the additional gains may be chosen bigger than L_1 . This approach to select the additional gains was used in [35], where the hybrid multi-observer presented in [36] was implemented to improve the estimation performance of an electrochemical Li-Ion battery model.

V. STABILITY GUARANTEES

The goal of this section is to prove that the proposed hybrid estimation scheme satisfies an input-to-state stability property. Even though the nominal observer satisfies an input-to-state stability property by Assumption 1, it is not obvious that so does system (19)-(21), as the extra modes are designed with no convergence guarantees.

A. Input-to-state stability

In the next theorem we prove that system (19)-(21) satisfies an input-to-state stability property.

Theorem 1. *Consider system (19)-(21) and suppose Assumptions 1-2 hold. Then there exist $\beta_U \in \mathcal{KL}$ and $\gamma_U \in \mathcal{K}_\infty$ such that for any input $u \in \mathcal{L}_U$, disturbance input $v \in \mathcal{L}_V$ and measurement noise $w \in \mathcal{L}_W$, any solution q satisfies*

$$\begin{aligned} & |(e_1(t, j), \eta_1(t, j), e_\sigma(t, j), \eta_\sigma(t, j))| \\ & \leq \beta_U(|(e(0, 0), \eta(0, 0))|, t) + \gamma_U(\|v\|_{[0, t]} + \|w\|_{[0, t]}) \end{aligned} \quad (24)$$

for all $(t, j) \in \text{dom } q$, with $e := (e_1, \dots, e_{N+1})$ and $\eta := (\eta_1, \dots, \eta_{N+1})$.

Sketch of proof: The proof consists in exploiting the Lyapunov properties in Proposition 1 and follows very similar steps as [44, proof of Theorem 1]. The full proof of Theorem 1 is available in [42, proof of Theorem 5.1]. ■

Theorem 1 guarantees a two-measure input-to-state stability property [45]. In particular, (24) ensures that e_1 , η_1 , e_σ and η_σ converge to a neighborhood of the origin, whose ‘‘size’’ depends on the \mathcal{L}_∞ norm of v and w . Note that we do not guarantee any stability property for the modes $k \neq \sigma$, but this is not needed for the convergence of the hybrid observer estimation error e_σ . Hence, the convergence of the estimated state vector of the selected mode is guaranteed by Theorem 1. We would like to emphasize that (24) does not inform us about the performance improvement of the hybrid scheme, this will be addressed in Section VII.

B. Lyapunov properties

In this section we state the Lyapunov properties, which are employed to prove Theorem 1. Based on Assumption 1, we first prove an input/output-to-state stability property [46] for the generic estimation error system $e := x - \hat{x} \in \mathbb{R}^{n_x}$ associated with (2) and (9), whose dynamics is defined as $\dot{e} = f_p(x, u, v) - f_o(\hat{x}, u, L(y - \hat{y}))$ with $L \in \mathbb{R}^{n_{L_1} \times n_y}$.

Lemma 1. *Suppose Assumption 1 holds. Then, for any $x \in \mathbb{R}^{n_x}$, $u \in \mathcal{U}$, $v \in \mathcal{V}$, $w \in \mathcal{W}$, $\hat{x} \in \mathbb{R}^{n_x}$ and any $L \in \mathbb{R}^{n_{L_1} \times n_y}$,*

$$\begin{aligned} & \langle \nabla V(e), f_p(x, u, v) - f_o(\hat{x}, u, L(y - \hat{y})) \rangle \\ & \leq -\alpha V(e) + \psi_1(|v|) + \psi_2(|w|) + \gamma \|L - L_1\|^2 |y - \hat{y}|^2, \end{aligned} \quad (25)$$

with $\hat{y} = h(\hat{x}, 0) \in \mathbb{R}^{n_y}$ and $\alpha, \psi_1, \psi_2, \gamma, V$ come from Assumption 1.

Proof: Let $x \in \mathbb{R}^{n_x}$, $u \in \mathcal{U}$, $v \in \mathcal{V}$, $w \in \mathcal{W}$, $\hat{x} \in \mathbb{R}^{n_x}$ and $L \in \mathbb{R}^{n_{L_1} \times n_y}$, we have that $\langle \nabla V(e), f_p(x, u, v) - f_o(\hat{x}, u, L(y - \hat{y})) \rangle = \langle \nabla V(e), f_p(x, u, v) - f_o(\hat{x}, u, L_1(y - \hat{y}) - L_1(y - \hat{y}) + L(y - \hat{y})) \rangle = \langle \nabla V(e), f_p(x, u, v) - f_o(\hat{x}, u, L_1(y - \hat{y})) + (L - L_1)(y - \hat{y}) \rangle$. Applying Assumption 1 with $d = (L - L_1)(y - \hat{y})$ we obtain $\langle \nabla V(e), f_p(x, u, v) - f_o(\hat{x}, u, L(y - \hat{y})) \rangle \leq -\alpha V(e) + \psi_1(|v|) + \psi_2(|w|) + \gamma \|(L - L_1)(y - \hat{y})\|^2$, which implies (25). We have obtained the desired result. ■

Lemma 1 implies that, for $e_k := x - \hat{x}_k$ for any $k \in \{2, \dots, N + 1\}$, the e_k -system, which follows from (2) and (9), satisfies an input/output-to-state property [46] with the same Lyapunov function as in Assumption 1 for any choice for the observer gain $L_k \in \mathbb{R}^{n_{L_1} \times n_y}$. The major difference between (6) and (25) is the term $\gamma \|(L - L_1)\|^2 |y - \hat{y}|^2$ in (25), which may have a destabilizing effect and may thus prevent the e_k -system to exhibit input-to-state stability properties similar to (7).

In the next proposition, we state Lyapunov properties for system (19)-(21), whose proof is postponed to Appendix A.

Proposition 1. *Suppose Assumptions 1-2 hold. Given any sets of gains $L_k \in \mathbb{R}^{n_{L_1} \times n_y}$, with $k \in \{2, \dots, N + 1\}$, any $\nu \in (0, \alpha]$, any $\varepsilon > 0$ and any $\Lambda_1 \in \mathbb{S}_{\geq 0}^{n_y}$, $\Lambda_2 \in \mathbb{S}_{\geq 0}^{n_x}$ with at least one of them positive definite, there exist $U : \mathcal{Q} \rightarrow \mathbb{R}_{\geq 0}$ locally Lipschitz, and $\underline{\alpha}_U, \bar{\alpha}_U \in \mathcal{K}_{\mathcal{X}}$, $\alpha_0 \in \mathbb{R}_{> 0}$, $\phi_1, \phi_2 \in \mathcal{K}_{\mathcal{X}}$, such that the following properties hold.*

- (i) $\underline{\alpha}_U(|(e_1, \eta_1, e_\sigma, \eta_\sigma)|) \leq U(q) \leq \bar{\alpha}_U(|(e, \eta)|)$ for any $q \in \mathcal{Q}$, with $e = (e_1, \dots, e_{N+1})$ and $\eta = (\eta_1, \dots, \eta_{N+1})$.
- (ii) $U^\circ(q; F(q, u, v, w)) \leq -\alpha_0 U(q) + \phi_1(|v|) + \phi_2(|w|)$ for any $q \in \mathcal{C}$, $u \in \mathcal{U}$, $v \in \mathcal{V}$ and $w \in \mathcal{W}$, such that $F(q, u, v, w) \in T_{\mathcal{C}}(q)$, where we recall that U° denotes the Clarke generalized directional derivative of U , as defined in Section I.
- (iii) $U(\mathbf{g}) \leq U(q)$ for any $q \in \mathcal{D}$ and any $\mathbf{g} \in G(q)$.

Proposition 1 shows the existence of a Lyapunov function U for system (19)-(21), which is used to prove the input-to-state stability property in Theorem 1.

VI. PROPERTIES OF THE SOLUTION DOMAINS

An input-to-state stability property is established in Theorem 1 but nothing is said about the completeness and more generally about the properties of the solutions time domains. In this section, we address these points. In Section VI-A, we show that maximal solutions are complete, while in Section VI-B we ensure the existence of a uniform semiglobal average dwell-time thereby ruling out Zeno phenomenon.

A. Completeness of maximal solutions

The goal of this section is to show that maximal solutions to system (19)-(21) are complete, which means that their domains are unbounded. For this purpose, we need that the system plant (2) is complete, as stated in the next assumption.

Assumption 3. *Any maximal solution to (2) with u in $\mathcal{L}_{\mathcal{U}}$, v in $\mathcal{L}_{\mathcal{V}}$ and w in $\mathcal{L}_{\mathcal{W}}$ is complete.*

Before proving the main result of this section, we show in the next lemma that maximal solutions to the additional modes (9) are complete.

Lemma 2. *Consider systems (2) and (9). Suppose Assumptions 1, 2 and 3 hold. Then, for any inputs $u \in \mathcal{L}_{\mathcal{U}}$, $v \in \mathcal{L}_{\mathcal{V}}$, $w \in \mathcal{L}_{\mathcal{W}}$ and $y \in \mathcal{L}_{\mathbb{R}^{n_y}}$, any corresponding maximal solution to (9) is complete.*

Proof: Let $k \in \{2, \dots, N + 1\}$ and let $x \in \mathbb{R}^{n_x}$, $u \in \mathcal{U}$, $v \in \mathcal{V}$, $w \in \mathcal{W}$, $\hat{x}_k \in \mathbb{R}^{n_x}$ and any $L_k \in \mathbb{R}^{n_{L_1} \times n_y}$. From Lemma 1, we have, for all $k \in \{2, \dots, N + 1\}$,

$$\begin{aligned} & \langle \nabla V(e_k), f_p(x, u, v) - f_o(\hat{x}_k, u, L_k(y - \hat{y}_k)) \rangle \\ & \leq -\alpha V(e_k) + \psi_1(|v|) + \psi_2(|w|) + \gamma \|L_k - L_1\|^2 |y - \hat{y}_k|^2 \\ & \leq -\alpha V(e_k) + \psi_1(|v|) + \psi_2(|w|) + \theta |y - \hat{y}_k|^2, \end{aligned} \quad (26)$$

with $\theta := \gamma \max_{k \in \{1, \dots, N+1\}} \|L_k - L_1\|^2 \in \mathbb{R}_{\geq 0}$. Using Assumption 2 we have $|y - \hat{y}_k|^2 = |h(x, w) - h(\hat{x}_k, 0)|^2 \leq \delta_1 V(e_k) + \delta_2 |w|^2$, for all $k \in \{2, \dots, N + 1\}$. Thus, from (26) we obtain,

$$\begin{aligned} & \langle \nabla V(e_k), f_p(x, u, v) - f_o(\hat{x}_k, u, L_k(y - \hat{y}_k)) \rangle \\ & \leq -\alpha V(e_k) + \psi_1(|v|) + \psi_2(|w|) + \theta \delta_1 V(e_k) + \theta \delta_2 |w|^2 \\ & = \mathbf{a} V(e_k) + \psi_1(|v|) + \psi_2^*(|w|), \end{aligned} \quad (27)$$

with $\mathbf{a} := \theta \delta_1 - \alpha \in \mathbb{R}$ and $\psi_2^* : s \mapsto \psi_2(|s|) + \theta \delta_2 |s|^2 \in \mathcal{K}_{\mathcal{X}}$.

Let $u \in \mathcal{L}_{\mathcal{U}}$, $v \in \mathcal{L}_{\mathcal{V}}$, $w \in \mathcal{L}_{\mathcal{W}}$ and x and \hat{x}_k be solutions to systems (2) and (9) respectively, for $k \in \{2, \dots, N + 1\}$. We have, by definition, $e_k(t) = x(t) - \hat{x}_k(t)$, for all $k \in \{2, \dots, N + 1\}$ and all $t \in \text{dom}(x, \hat{x}_k)$. Pick any $k \in \{2, \dots, N + 1\}$, for all $t \in [0, \infty)$, we have from (27),

$$\frac{d}{dt} V(e_k(t)) \leq \mathbf{a} V(e_k(t)) + \psi_1(|v(t)|) + \psi_2^*(|w(t)|). \quad (28)$$

Applying the comparison principle [47, Lemma 3.4], we obtain, for all $t \in [0, \infty)$, $V(e_k(t)) \leq e^{\mathbf{a}t} V(e_k(0)) + \int_0^t e^{\mathbf{a}(t-s)} (\psi_1(|v(s)|) + \psi_2^*(|w(s)|)) ds$. From (5) and the last inequality, e_k cannot blow up in finite time as V is positive definite and the right-hand side is finite for any $t \geq 0$. Moreover, from Assumption 3, x cannot blow up in finite time. Consequently, since $\hat{x}_k = x - e_k$ and both x and e_k cannot explode in finite time, \hat{x}_k cannot as well. Thus, for any $k \in \{2, \dots, N + 1\}$, any maximal solution to system (9) is complete. ■

We are now ready to prove the completeness of maximal solution of system (19)-(21).

Proposition 2. *Under Assumptions 1, 2 and 3, for any inputs $u \in \mathcal{L}_{\mathcal{U}}$, $v \in \mathcal{L}_{\mathcal{V}}$, $w \in \mathcal{L}_{\mathcal{W}}$, any maximal solution to system (19)-(21) is complete.*

Proof: We use [37, Proposition 6] to prove Proposition 2. Let $u \in \mathcal{L}_U$, $v \in \mathcal{L}_V$, $w \in \mathcal{L}_W$ and q be a maximal solution to (19)-(21). In view of the definition of the flow and jump sets, \mathcal{C} and \mathcal{D} , in (20)-(21), we have that $q(0,0) \in \mathcal{C} \cup \mathcal{D}$. Suppose $q(0,0) \in \mathcal{C} \setminus \mathcal{D}$, we want to prove that q is not trivial, i.e., its domain contains at least two points. For this purpose we need to show that the viability condition in [37, Proposition 6] is satisfied. Since the flow map F is continuous and $u \in \mathcal{L}_U$, $v \in \mathcal{L}_V$ and $w \in \mathcal{L}_W$, from [48, Proposition S1] there exists $\epsilon > 0$ and an absolutely continuous function $z : [0, \epsilon] \rightarrow \mathcal{Q}$ such that $z(0) = q(0,0)$ and $\dot{z}(t) = F(z(t), u(t), v(t), w(t))$ for almost all $t \in [0, \epsilon]$. We write $z = (z_x, z_{\hat{x}_1}, \dots, z_{\hat{x}_{N+1}}, z_{\eta_1}, \dots, z_{\eta_{N+1}}, z_\sigma)$. Since $q(0,0) \in \mathcal{C} \setminus \mathcal{D}$, with $\mathcal{C} \setminus \mathcal{D}$ open, and z is absolutely continuous, there exists $\epsilon' \in (0, \epsilon]$ such that, for all $k \in \{1, \dots, N+1\}$, $z_{\eta_k}(t) \geq z_{\eta_\sigma}(t)$ for almost all $t \in [0, \epsilon']$. Thus, $z(t) \in \mathcal{C}$ for almost all $t \in [0, \epsilon']$ and the viability condition in [37, Proposition 6] holds, which implies that q is a non-trivial solution.

To prove that q is complete we need to exclude items (b) and (c) in [37, Proposition 6]. Item (b) in [37, Proposition 6] occurs when at least one component of q blows up in finite time, and consequently q blows up in finite time. Hence, to exclude (b) in [37, Proposition 6] we need to show that each component of q must not explode in finite time. Let $q = (x, \hat{x}_1, \dots, \hat{x}_{N+1}, \eta_1, \dots, \eta_{N+1}, \sigma)$. From Assumption 3, x cannot blow up in finite time. Moreover, \hat{x}_1 cannot do so as well in view of Theorem 1 and since x cannot. In addition, \hat{x}_k , for all $k \in \{2, \dots, N+1\}$ cannot blow up in finite time in view of Lemma 2 and η_k , with $k \in \{1, \dots, N+1\}$ cannot as well in view of its dynamics (10) and because $y - \hat{y}_k$ does not since both x and \hat{x}_k do not, for all $k \in \{1, \dots, N+1\}$. Finally, σ is constant in \mathcal{C} , consequently, it does not blow up in finite time. Thus, item (b) in [37, Proposition 6] cannot occur. On the other hand, since $G(\mathcal{D}) \subseteq \mathcal{C} \cup \mathcal{D}$ and the jump set does not impose conditions on u , v and w , item (c) in [37, Proposition 6] cannot occur. Consequently, any maximal solution to system (19)-(21) is complete. This concludes the proof. ■

B. Average dwell-time

Proposition 2 ensures the completeness of maximal solutions under Assumptions 1-3, still, Zeno phenomenon has not been ruled out yet. In the next proposition, we prove the existence of a uniform semiglobal average dwell-time for the solution to system (19)-(21), which thus excludes the Zeno phenomenon. Its proof is given in Appendix B.

Proposition 3. *Suppose Assumptions 1, 2 hold and the sets \mathcal{V} and \mathcal{W} are compact. Then, system (19)-(21) has a uniform semiglobal average dwell-time, i.e., for any $M \in \mathbb{R}_{>0}$ there exists $c > 0$ such that any corresponding solution q with $|q(0,0)| \leq M$ and $u \in \mathcal{L}_U$, $v \in \mathcal{L}_V$ and $w \in \mathcal{L}_W$, is such that for any (t, j) , $(t', j') \in \text{dom } q$ with $(t, j) \leq (t', j')$, $j' - j \leq \frac{1}{\tau}(t' - t) + 2$ with $\tau := -\frac{1}{2\nu} \ln\left(\frac{\epsilon}{\epsilon + \nu}\right)$, where ν comes from (10) and ϵ is the design parameter in (17).*

We see the importance of the parameter $\epsilon \in \mathbb{R}_{>0}$, used in the

jump map for the monitoring variables (17), in the expression of τ . Indeed, if we would allow ϵ to be equal to 0 (which we do not), τ would have been equal to 0. In addition, Proposition 3 shows that any solution q to (19)-(21) can exhibit at most two instantaneous jumps. Note that to obtain the results of Proposition 3 we do not need Assumption 3. However, in view of Propositions 2 and 3 we have that under Assumptions 1-3, for any inputs $u \in \mathcal{L}_U$, $v \in \mathcal{L}_V$, $w \in \mathcal{L}_W$, any maximal solution q to system (19)-(21) is t -complete, namely $\sup_t \text{dom } q = +\infty$.

Now that we have established robust stability properties and the properties of the hybrid time domain of the solutions for the hybrid estimation scheme, we focus on its performance in the next section.

VII. PERFORMANCE IMPROVEMENT

The goal of this section is to establish the estimation performance improvement given by the proposed hybrid multi-observer. We recall that with the proposed technique we have $\eta_{\sigma(t,j)}(t, j) \leq \eta_1(t, j)$ for all $(t, j) \in \text{dom } q$, for any solution q to (19)-(21) with inputs $u \in \mathcal{L}_U$, $v \in \mathcal{L}_V$ and $w \in \mathcal{L}_W$, both in the case without and with resets. Therefore, the estimation performance of the proposed hybrid multi-observer is always not worse than the performance of the nominal one according to the monitoring variables we consider.

Variable η_σ defined in Section III-B is a performance variable that considers the “best” mode among the $N+1$ at any time instant: this is an instantaneous performance, which ignores the past behavior in terms of the monitoring variable. For this reason, to evaluate the performance of the proposed hybrid multi-observer, we also propose the following cost, for any solution q to (19)-(21) with inputs $u \in \mathcal{L}_U$, $v \in \mathcal{L}_V$ and $w \in \mathcal{L}_W$, for all $(t, j) \in \text{dom } q$,

$$J_{\sigma(t,j)}(t, j) := \sum_{i=0}^j \left(\int_{t_i}^{t_{i+1}} \eta_{\sigma(s,i)}(s, i) ds \right), \quad (29)$$

with $0 = t_0 \leq t_1 \leq \dots \leq t_{j+1} = t$ satisfying $\text{dom } q \cap ([0, t] \times \{0, 1, \dots, j\}) = \bigcup_{i=0}^j [t_i, t_{i+1}] \times \{i\}$.

Similarly, we define the performance cost of the nominal observer, for all $(t, j) \in \text{dom } q$, as

$$J_1(t, j) := \sum_{i=0}^j \left(\int_{t_i}^{t_{i+1}} \eta_1(s, i) ds \right), \quad (30)$$

with $0 = t_0 \leq t_1 \leq \dots \leq t_{j+1} = t$ satisfying $\text{dom } q \cap ([0, t] \times \{0, 1, \dots, j\}) = \bigcup_{i=0}^j [t_i, t_{i+1}] \times \{i\}$.

In the next theorem we prove, that the hybrid scheme in Section III strictly improves the performance J_1 in (30), under some conditions on the gain selection and on the initial conditions of the modes of the multi-observer and monitoring variables.

Theorem 2. *Consider system (19)-(21) under Assumptions 1-3. Let q be a maximal solution with inputs $u \in \mathcal{L}_U$, $v \in \mathcal{L}_V$ and $w \in \mathcal{L}_W$ and for which the initial conditions of the monitoring variables are all the same, namely $\eta_k(0,0) = \eta_0$ for all $k \in \{1, \dots, N+1\}$ for some $\eta_0 \in \mathbb{R}$. Then, for any $(t, j) \in \text{dom } q$,*

$$J_{\sigma(t,j)}(t, j) \leq J_1(t, j), \quad (31)$$

with J_σ and J_1 defined in (29) and (30), respectively. Moreover, if there exists $(t^*, j^*) \in \text{dom } q$ such that

$$\eta_{\sigma(t^*, j^*)}(t^*, j^*) < \eta_1(t^*, j^*), \quad (32)$$

then there exists $j^{*'} \geq j^*$ such that

$$J_{\sigma(t, j)}(t, j) < J_1(t, j) \quad (33)$$

for all $(t, j) \geq (t^*, j^{*'})$, with $(t, j) \in \text{dom } q$.

Proof: Consider system (19)-(21) and let q be a maximal solution to system (19)-(21) with inputs $u \in \mathcal{L}_U$, $v \in \mathcal{L}_V$ and $w \in \mathcal{L}_W$. From (10), (15), (16), (20), (21) and $\eta_k(0, 0) = \eta_0 \in \mathbb{R}$ for all $k \in \{1, \dots, N+1\}$, we have, for all $(t, j) \in \text{dom } q$,

$$\eta_{\sigma(t, j)}(t, j) \leq \eta_1(t, j). \quad (34)$$

We then derive from (29) and (30) that $J_{\sigma(t, j)}(t, j) \leq J_1(t, j)$, for all $(t, j) \in \text{dom } q$, which concludes the first part of the proof.

In the second part of the theorem, we have that there exists $(t^*, j^*) \in \text{dom } q$ such that

$$\eta_{\sigma(t^*, j^*)}(t^*, j^*) < \eta_1(t^*, j^*). \quad (35)$$

We now consider two cases. If $t^* \in \text{int } I^{j^*}$, using (29), (30), (34), (35), since no jump occurs at (t^*, j^*) and η_1 and η_σ are not affected by jumps, by continuity of η_σ and η_1 on I^{j^*} , (33) is obtained by integration of (34) for all $(t, j) \geq (t^*, j^*)$, with $(t, j) \in \text{dom } q$. On the other hand, if I^{j^*} is empty, since q is maximal, it is t -complete by Propositions 2 and 3 as explained in Section VI-B and thus we have that there exists $j^{*'} > j^*$ such that $(t^*, j^{*'}) \in \text{dom } q$ and $\eta_{\sigma(t^*, j^{*'})} \leq \eta_{\sigma(t^*, j^*)} < \eta_1(t^*, j^{*'})$ with $I^{j^{*'}}$ non empty. Following similar step as before we have that (33) holds for all $(t, j) \geq (t^*, j^{*'})$, with $(t, j) \in \text{dom } q$. This concludes the proof. ■

Theorem 2 shows that, if the condition in (32) holds, then the cost of the proposed hybrid multi-observer J_σ is strictly smaller than the one of the nominal observer J_1 and thus, the estimation performance in terms of costs J_σ and J_1 is strictly improved.

In the next proposition, we give the conditions to guarantee that (32) is satisfied and consequently, from Theorem 2, that the estimation performance is strictly improved with the hybrid multi-observer (19)-(21).

Proposition 4. Consider system (19)-(21) with $\Lambda_2 \in \mathbb{S}_{>0}^{n_x}$ and suppose Assumptions 1-3 hold. Select the gains L_k , with $k \in \{2, \dots, N+1\}$, in (9) such that there exists $k^* \in \{2, \dots, N+1\}$ satisfying $L_{k^*}^\top \Lambda_2 L_{k^*} < L_1^\top \Lambda_2 L_1$. Let q be a maximal solution with inputs $u \in \mathcal{L}_U$, $v \in \mathcal{L}_V$ and $w \in \mathcal{L}_W$ and initial condition $q(0, 0)$ satisfying the following properties.

- (i) $\hat{x}_k(0, 0) = \hat{x}_0$ for all $k \in \{1, \dots, N+1\}$ for some $\hat{x}_0 \in \mathbb{R}^{n_x}$.
- (ii) $\eta_k(0, 0) = \eta_0$ for all $k \in \{1, \dots, N+1\}$ for some $\eta_0 \in \mathbb{R}$.
- (iii) $\hat{y}_k(0, 0) \neq y(0, 0)$ for all $k \in \{1, \dots, N+1\}$.

Then, there exists $(t^*, j^*) \in \text{dom } q$ such that

$$\eta_{\sigma(t^*, j^*)}(t^*, j^*) < \eta_1(t^*, j^*). \quad (36)$$

Proof: Let q be a maximal solution to system (19)-(21) with inputs $u \in \mathcal{L}_U$, $v \in \mathcal{L}_V$ and $w \in \mathcal{L}_W$ satisfying items (i)-(iii). We define $\Delta_k := y - \hat{y}_k \in \mathbb{R}^{n_y}$ for all $k \in \{1, \dots, N+1\}$ for the sake of convenience. Note that, thanks to item (i), $\Delta_1(0, 0) = y(0, 0) - \hat{y}_1(0, 0) = y(0, 0) - h(\hat{x}_1(0, 0), 0) = y(0, 0) - h(\hat{x}_k(0, 0), 0) = y(0, 0) - \hat{y}_k(0, 0) = \Delta_k(0, 0)$, for any $k \in \{1, \dots, N+1\}$. On the other hand, from (10) we have, for all $k \in \{1, \dots, N+1\}$,

$$\begin{aligned} \dot{\eta}_k &= -\nu\eta_k + (y - \hat{y}_k)^\top (\Lambda_1 + L_k^\top \Lambda_2 L_k) (y - \hat{y}_k) \\ &= -\nu\eta_k + \Delta_k^\top (\Lambda_1 + L_k^\top \Lambda_2 L_k) \Delta_k. \end{aligned} \quad (37)$$

We evaluate (37) for $k = 1$ at $(t, j) = (0, 0)$. As $\Delta_1(0, 0) = \Delta_k(0, 0)$, from item (ii) of Proposition 4 and since Λ_2 is positive definite, we obtain

$$\begin{aligned} \dot{\eta}_1(0, 0) &= -\nu\eta_1(0, 0) + \Delta_1(0, 0)^\top (\Lambda_1 + L_1^\top \Lambda_2 L_1) \Delta_1(0, 0) \\ &= -\nu\eta_{k^*}(0, 0) + \Delta_{k^*}(0, 0)^\top (\Lambda_1 + L_1^\top \Lambda_2 L_1) \Delta_{k^*}(0, 0) \\ &> -\nu\eta_{k^*}(0, 0) + \Delta_{k^*}(0, 0)^\top (\Lambda_1 + L_{k^*}^\top \Lambda_2 L_{k^*}) \Delta_{k^*}(0, 0) \\ &= \dot{\eta}_{k^*}(0, 0) \end{aligned} \quad (38)$$

for any $k^* \in \{2, \dots, N+1\}$ such that $L_{k^*}^\top \Lambda_2 L_{k^*} < L_1^\top \Lambda_2 L_1$. The strict inequality in (38) comes from the condition $L_{k^*}^\top \Lambda_2 L_{k^*} < L_1^\top \Lambda_2 L_1$ on the observer gain selection, with $\Lambda_2 \in \mathbb{S}_{>0}^{n_x}$, and $\Delta_{k^*}(0, 0) \neq 0$ by item (iii) of Proposition 4. Since q is maximal it is t -complete by Propositions 2 and 3. Moreover, q can exhibit at most two instantaneous jumps as explained in Section VI-B and thus there exists $j^* \in \{0, 1\}$ such that $\sigma(0, j^*) = \bar{k}$ with $\bar{k} \in \text{argmin}_{k \in \Pi} \dot{\eta}_k(0, j^*)$, with $\Pi = \text{argmin}_{k \in \{1, \dots, N+1\}} \eta_k(0, j^*)$ and $(0, j^* + 1) \notin \text{dom } q$. Note that $L_{\bar{k}}^\top \Lambda_2 L_{\bar{k}} < L_1^\top \Lambda_2 L_1$ and $\dot{q}(0, j^*) \in T_C(q) := \{q \in \mathcal{Q} : \dot{\eta}_k \geq \eta_\sigma, \forall k \in \{1, \dots, N+1\}\}$, where $\dot{\eta}_k = -\nu\eta_k + (y - \hat{y}_k)^\top (\Lambda_1 + L_k^\top \Lambda_2 L_k) (y - \hat{y}_k)$ and $\dot{\eta}_\sigma = -\nu\eta_\sigma + (y - \hat{y}_\sigma)^\top (\Lambda_1 + L_\sigma^\top \Lambda_2 L_\sigma) (y - \hat{y}_\sigma)$, for all $(t, j) \in \text{dom } q$ with some abuse of notation, in view of [42, Lemma 2.1]. From (17) and (38), we obtain

$$\dot{\eta}_{\sigma(0, j^*)}(0, j^*) = \dot{\eta}_{\bar{k}}(0, j^*) < \dot{\eta}_1(0, j^*). \quad (39)$$

Moreover, since q is t -complete, we have that there exists $\epsilon > 0$ such that $q(t, j^*) \in \mathcal{C}$ for all $t \in [0, \epsilon]$. Consequently, $(t^*, j^*) \in \text{dom } q^*$ for all $t^* \in [0, \epsilon]$. In addition, we have $\eta_{\sigma(0, j^*)}(0, j^*) = \eta_1(0, j^*) = \eta_0$ both when $j^* = 0$ and $j^* = 1$ from (15) and (16). Using the last equation, from (39) we obtain $\eta_{\sigma(t^*, j^*)}(t^*, j^*) < \eta_1(t^*, j^*)$. This concludes the proof. ■

Note that, the conditions in items (i) and (ii) of Proposition 4 can always be ensured by designing the same initial condition for the state estimate and monitoring variables for all the modes. Moreover, condition in item (iii) is verified almost everywhere (it is a set of null measure). We also acknowledge that we state the performance improvement with respect to costs J_1 and J_σ , and that it would be interesting to state properties for a cost, which involves the state estimation errors e_1 and e_σ . This is a challenging question, which goes beyond the scope of this work.

VIII. NUMERICAL CASE STUDIES

A. Van der Pol oscillator

We consider

$$\dot{x} = Ax + B\varphi(x), \quad y = Cx + w \quad (40)$$

where $x = (x_1, x_2) \in \mathbb{R}^2$ is the system state to be estimated, $y \in \mathbb{R}$ is the measured output and $w \in \mathbb{R}$ is the measurement noise. The system matrices are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C = [1 \quad 0] \quad (41)$$

and $\varphi(x) = \text{sat}(-x_1 + 0.5(1 - x_1^2)x_2)$ for any $x \in \mathbb{R}^2$, where the saturation level is symmetric and equal to 10. We consider the measurement noise w generated from a random sequence of points, occurring at intervals of 0.01 s, with amplitude ranging between -0.1 and 0.1 and with linear interpolation applied between any two consecutive points.

We design a nominal high-gain observer for system (40)

$$\dot{\hat{x}}_1 = A\hat{x}_1 + B\varphi(\hat{x}_1) + L_1(y - \hat{y}_1), \quad \hat{y}_1 = C\hat{x}_1 \quad (42)$$

where \hat{x}_1 is the state estimate, \hat{y}_1 is the estimated output and $L_1 \in \mathbb{R}^{2 \times 1}$ is the output injection gain, which is defined as $L_1 := H_1 D$, where $D \in \mathbb{R}^{2 \times 1}$, $H_1 = \text{diag}(h_1, h_1^2) \in \mathbb{R}^{2 \times 2}$, with $h_1 \in \mathbb{R}_{>0}$ the high-gain design parameter. To satisfy Assumption 1, $D \in \mathbb{R}^{2 \times 1}$ is selected such that the matrix $A - DC$ is Hurwitz and the parameter h_1 is taken sufficiently large, i.e., $h_1 \geq h_1^*$, where h_1^* is equal to $2\lambda_{\max}(P)K$, where $P \in \mathbb{R}^{2 \times 2}$ is the solution of the Lyapunov equation $P(A - DC) + (A - DC)^\top P = -I_2$ and $K = 58.25$ is the Lipschitz constant of the function φ . We select D such that the eigenvalues of $A - DC$ are equal to -1 and -2 and we obtain $D = [3, 2]$, while the parameter h_1 is selected equal to $200 > h_1^* = 152.50$. With this choice of h_1 , Assumption 1 is satisfied with a quadratic Lyapunov function and $\alpha = 53.28$. Furthermore, since the output is linear, also Assumption 2 is satisfied.

We consider $N = 4$ additional modes, with the same structure as the nominal one in (42). The only difference is the output injection gain $L_k \in \mathbb{R}^{2 \times 1}$, which is defined as $L_k := H_k D$, with $H_k = \text{diag}(h_k, h_k^2) \in \mathbb{R}^{2 \times 2}$, with $k \in \{2, \dots, 5\}$. We select $h_2 = 20$, $h_3 = 1$, $h_4 = 0$ and $h_5 = -1$. Note that $h_k \leq h_1^*$, for all $k \in \{2, \dots, 5\}$. Therefore, we have no guarantees that these modes satisfy Assumption 1, and consequently, that they converge. Simulations suggest that the modes with L_2 and L_3 converge, while the ones with L_4 and L_5 do not. Note that, the gain $L_4 = 0_{2 \times 1}$ is the best choice to annihilate the effects of the measurement noise.

We simulate the proposed estimation technique considering the initial conditions $x(0, 0) = (1, 1)$, $\hat{x}_k(0, 0) = (0, 0)$, $\eta_k(0, 0) = 10$ for all $k \in \{1, \dots, 5\}$ and $\sigma(0, 0) = 1$. Both cases, without and with resets, are simulated with $\nu = 5$, $\Lambda_1 = 1$, $\lambda_2 = 0.1 \cdot I_2$ and $\varepsilon = 10^{-4}$. Note that the condition $\nu \in (0, \alpha]$ is satisfied.

The evolution of $|e_1|$ and $|e_\sigma|$, obtained with or without resets, is shown in Fig. 2, together with the nominal performance cost J_1 and the costs J_σ obtained both in the case without and with resets. Fig. 2 shows that both solutions (without resets and with resets) improve the estimation performance compared to

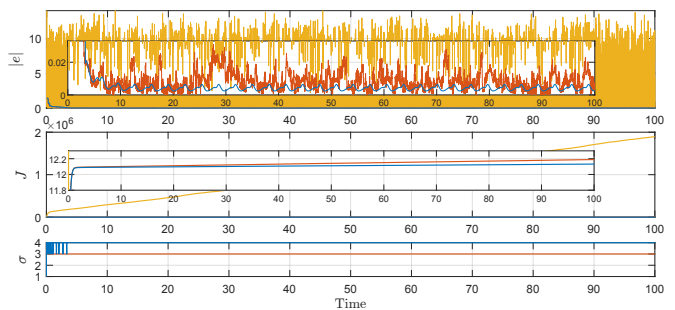


Fig. 2. Van der Pol oscillator. Norm of the estimation error $|e|$ (top figure), performance cost J (middle figure) and σ (bottom figure). Nominal (yellow), without resets (red), with resets (blue).

TABLE I

VAN DER POL OSCILLATOR. AVERAGE MAE AND RMSE.

	no reset			reset		
	e_1	e_σ	% improv.	e_1	e_σ	% improv.
MAE	3.505	0.031	99.11	3.507	0.035	98.99
RMSE	3.519	0.033	99.05	3.516	0.036	98.97

the nominal one. Zooms of the estimation error and cost plots are also shown in Fig. 2 to highlight the difference, and thus the performance improvement, between the nominal observer and the hybrid multi-observer. The last plot in Fig. 2 represents σ and indicates which mode is selected at every time instant both in the case without and with resets. Interestingly, when the resets are considered, the fourth mode (with $L_4 = [0, 0]^\top$), that is not converging, is selected.

To further evaluate the performance improvement given by the hybrid multi-observer, we run 100 simulations with different initial conditions for the state estimate of all the modes of the multi-observer, both in the case without and with resets. In particular, both components of $\hat{x}_k(0, 0) \in \mathbb{R}^2$, for all $k \in \{1, \dots, 5\}$, were selected randomly in the interval $[-2, 2]$ and, in each simulation, all modes of the multi-observer were initialized with the same state estimate. The system state, the monitoring variables and the variable σ were always initialized at $x(0, 0) = (1, 1)$, $\eta_k(0, 0) = 10$, for all $k \in \{1, \dots, 5\}$, and $\sigma(0, 0) = 1$. We considered the same choice of design parameters as before. To quantify the performance improvement, we evaluate the mean absolute error (MAE) and the root mean square error (RMSE), averaged over all the simulations, of the state estimation error obtained with the nominal observer and the hybrid multi-observer both in the case without and with resets. The obtained data are given in Table I. Note that the data for e_1 without and with resets are slightly different because the 100 initial conditions were randomly selected and thus they may be different in the simulations without and with resets. Table I shows that the proposed technique, both without and with resets, highly improves the estimation performance compared to the nominal one. Indeed, both the MAE and the RMSE are improved by more than 99% both in the case without and with resets. Moreover, in this example, the performance of the hybrid multi-observer without and with resets are very similar, with the case without resets that slightly outperforms the case where the resets are implemented, both in term of MAE and RMSE.

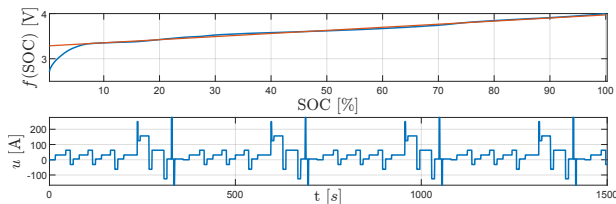


Fig. 3. $f(\text{SOC})$ (blue) with its linearization (red) and PHEV current input.

B. Electric circuit model of a lithium-ion battery

We consider an electric circuit model of 1-cell lithium-ion battery shown in [49, Fig. 2], with a nonlinear output map. From the circuit the following system model is derived

$$\dot{x} = Ax + Bu, \quad y = Cx + f(Hx) + Du + w. \quad (43)$$

The state is $x := (U_{\text{RC}}, \text{SOC}) \in \mathbb{R}^2$, where U_{RC} is the voltage of the RC circuit and SOC is the state of charge of the battery. The output y is the output voltage, the input u is the current and w is the measurement noise. The system matrices are

$$A = \begin{bmatrix} -\frac{1}{\tau} & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} -\frac{1}{c} \\ \frac{1}{Q} \end{bmatrix}, \quad C = [-1 \quad 0], \quad (44)$$

$$H = [0 \quad 1], \quad D = [-R_{\text{int}}].$$

Considering the temperature to be constant and equal to 25°C, the parameters values are $\tau = 7$ s, $R = 0.5 \cdot 10^{-3}$ Ω , $c = \frac{\tau}{R}$ F, $Q = 25$ Ah and $R_{\text{int}} = 1$ m Ω . The function f and its linearization are shown in Fig. 3 on the interval $[0, 100]$ % and we consider a first order approximation outside the interval $[0, 100]$ %. The function f satisfies Assumption 2 since it has bounded derivatives. The input u is given by a plug-in hybrid electric vehicle (PHEV) current profile, see Fig. 3, and the measurement noise is given by $w(t) = 0.01 \sin(10t)$, for all $t \geq 0$. We design the nominal observer

$$\dot{\hat{x}}_1 = A\hat{x}_1 + Bu + L_1(y - \hat{y}_1), \quad \hat{y}_1 = C\hat{x}_1 + f(H\hat{x}_1) + Du, \quad (45)$$

where \hat{x}_1 is the state estimate, \hat{y}_1 is the output estimate and $L_1 = [-2.07, 2.48]^\top \in \mathbb{R}^{2 \times 1}$ is the observer gain that is designed following a polytopic approach like in [50]. Observer (45) satisfies Assumption 1 with $\alpha = 0.1$.

To improve the estimation performance, we design the hybrid multi-observer considering $N = 3$ additional modes. To select L_2 , we linearize the output map and we design a Luenberger observer with eigenvalues in $[-0.2, -0.3]$ and we obtain $L_2 = [0.06, 61.25]^\top$. Note that, since this observer is designed for the linearized system, we have no guarantees that it satisfies an input-to-state stability property for the nonlinear system. Moreover, we chose $L_3 = [0, 0]^\top$ and we designed an extended Kalman filter [13], with $R_{\text{EKF}} = 1$, $Q_{\text{EKF}} = 0.1 \cdot I_2$ and $\alpha_{\text{EKF}} = 0.01$, to obtain L_4 , which is thus a time-varying gain. Note that, in view of Remark 3, the results presented in this paper hold also in this case.

We simulate the proposed hybrid multi-observer, both without and with resets, considering the initial conditions $x(0, 0) = (1, 100)$, $\hat{x}_k(0, 0) = (0.5, 50)$, $\eta_k(0, 0) = 0$ for all $k \in \{1, \dots, 4\}$ and $\sigma(0, 0) = 1$. The design parameters are selected

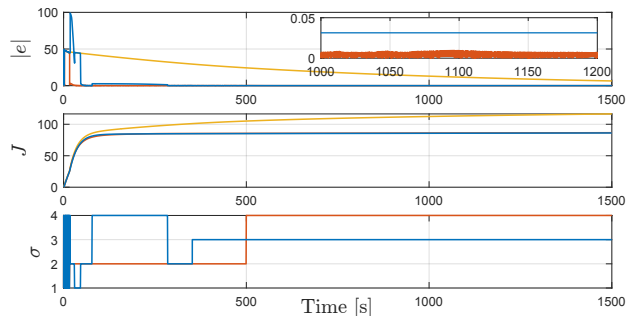


Fig. 4. Battery example. Norm of the estimation error $|e|$ (top figure), performance cost J (middle figure) and σ (bottom figure). Nominal (yellow), without resets (red), with resets (blue).

TABLE II
BATTERY EXAMPLE. AVERAGE MAE AND RMSE.

	no reset			reset		
	e_1	e_σ	% improv.	e_1	e_σ	% improv.
MAE	28.10	3.37	87.99	27.30	1.65	93.94
RMSE	30.87	8.66	71.95	29.90	5.99	79.98

$\nu = 0.05$, $\Lambda_1 = 1$, $\Lambda_2 = \begin{bmatrix} 1 & 0 \\ 0 & 10^{-4} \end{bmatrix}$ and $\varepsilon = 10^{-2}$. Note that the condition $\nu \in (0, \alpha]$ is satisfied.

Fig. 4 shows the norm of the nominal estimation error, namely $|e_1|$, as well as $|e_\sigma|$, obtained with or without resets. Moreover, the nominal performance cost J_1 and the costs J_σ obtained both in the case without and with resets are shown in Fig. 4, together with the variable σ , which indicates the selected mode at every time instant. Fig. 4 shows that both solutions (without resets and with resets) significantly improve the estimation performance compared to the nominal one.

As in the example in Section VIII-A, we run 100 simulations with different initial conditions for the state estimate of all modes of the multi-observer. In particular the first component of $\hat{x}_k(0, 0)$ was selected randomly in the interval $[0, 3]$ [V], while the second component of $\hat{x}_k(0, 0)$ was selected randomly in the interval $[1, 100]$ [%], for all $k \in \{1, \dots, 4\}$. All the other initial conditions and the design parameters were selected as before. We evaluate the MAE and the RMSE as in the example in Section VIII-A and the obtained results, given in Table II, show the estimation performance improvement. In this example, the case with resets outperforms the case without resets, both in term of MAE and RMSE, according to the average results of 100 simulations with different initial conditions showed in Table II. However, for the specific choice of initial conditions considered to obtain the plots in Fig. 4 the case without resets outperforms the one with resets.

IX. CONCLUSION

We have presented a novel hybrid multi-observer that can improve the state estimation performance of a given nominal nonlinear observer. Each additional mode of the multi-observer differs from the nominal one only in its output injection gain, that can be freely selected as no convergence property is required for these modes. Inspired by supervisory control/observer approaches, we have designed a switching criterion, based on monitoring variables, that selects one mode

at any time instant by evaluating their performance. We have proved an input-to-state stability property of the estimation error and the estimation performance improvement. Finally, numerical examples confirm the efficiency of the proposed approach.

We believe that the flexibility of the presented framework leads to a range of fascinating research questions among which the off-line tuning of the additional modes gains, for which various techniques (learning based, dynamic programming) may be envisioned. Some first ideas are drawn in Section IV-A.2 but more could be done in this direction. The question of the link between the considered cost function, which involves the output estimation error, and a cost based on the state estimation error is also a relevant challenge to unravel. It would also be interesting to extend the results to more general classes of systems and observers. This might involve investigating observability/detectability singularities or examining observers derived through immersion techniques, see, e.g., [51], or observers given by differential inclusions like sliding modes observers, Levant's observers, and super-twisting observers, see, e.g., [1, Sections 6.3 and 6.4] and references therein. Finally, merging the approach presented in this work with the one in, e.g., [23], [26], addressing the case of systems with unknown parameters is also a challenging research question.

APPENDIX

A. Proof of Proposition 1

The Lyapunov function of Proposition 1 is defined as $U(q) := c_1(aV(e_1) + \eta_1) + c_2 \max_{k \in \{1, \dots, N+1\}} \{bV(e_k) - \eta_k, 0\} + c_3 \max\{\eta_\sigma - \eta_1, 0\}$, for any $q \in \mathcal{Q}$, where $c_1, c_2, c_3, a, b \in \mathbb{R}_{>0}$ are selected such that $c_2 < c_3 < c_1$, $a > \bar{a}$ where $\bar{a} := \frac{\delta_1(\lambda_{\max}(\Lambda_1) + \lambda_{\max}(L_1^\top \Lambda_2 L_1))}{\alpha} \geq 0$ and $b \in (0, \bar{b})$ with $\bar{b} := \frac{\lambda_{\min}(\Lambda_1)}{\theta}$, with $\theta := \gamma \max_{k \in \{1, \dots, N+1\}} \|(L_k - L_1)\|^2 \in \mathbb{R}_{\geq 0}$, where γ comes from Assumption 1. Note that U is locally Lipschitz as V is continuously differentiable.

We prove the three items of Proposition 1 separately.

Proof of item (i). We first show the upper-bound. Let $q \in \mathcal{Q}$, using (5) we have $U(q) \leq c_1(a\bar{\alpha}(|e_1|) + \eta_1) + c_2 \max_{k \in \{1, \dots, N+1\}} \{b\bar{\alpha}(|e_k|) - \eta_k, 0\} + c_3 \max\{\eta_\sigma - \eta_1, 0\} \leq c_1(a\bar{\alpha}(|e_1|) + \eta_1) + c_2 \sum_{k=1}^{N+1} (b\bar{\alpha}(|e_k|) + \eta_k) + c_3(\eta_\sigma + \eta_1) := \bar{\alpha}_U(|(e, \eta)|)$, for some $\bar{\alpha}_U \in \mathcal{K}_\mathcal{I}$.

We now prove the lower-bound of item (i) of Proposition 1. We have that $\max_{k \in \{1, \dots, N+1\}} \{bV(e_k) - \eta_k, 0\} \geq bV(e_\sigma) - \eta_\sigma$ as $\sigma \in \{1, \dots, N+1\}$. Hence, since $\max(\eta_\sigma - \eta_1, 0) \geq \eta_\sigma - \eta_1$, in view of (5), $U(q) \geq c_1(a\underline{\alpha}(|e_1|) + \eta_1) + c_2(b\underline{\alpha}(|e_\sigma|) - \eta_\sigma) + c_3(\eta_\sigma - \eta_1) = c_1a\underline{\alpha}(|e_1|) + (c_1 - c_3)\eta_1 + c_2b\underline{\alpha}(|e_\sigma|) + (c_3 - c_2)\eta_\sigma$. Since $c_1 - c_3 > 0$ and $c_3 - c_2 > 0$, there exists $\underline{\alpha}_U \in \mathcal{K}_\mathcal{I}$ such that $U(q) \geq \underline{\alpha}_U(|(e_1, \eta_1, e_\sigma, \eta_\sigma)|)$.

Proof of item (ii). For the sake of convenience we write $U(q) = U_1(q) + U_2(q) + U_3(q)$, for any $q \in \mathcal{C}$, where $U_1(q) = c_1(aV(e_1) + \eta_1)$, $U_2(q) = c_2 \max_{k \in \{1, \dots, N+1\}} \{bV(e_k) - \eta_k, 0\}$ and $U_3(q) = c_3 \max\{\eta_\sigma - \eta_1, 0\}$. We introduce here the compact notation $F \triangleq F(q, u, v, w)$ for the sake of

convenience. Let $q \in \mathcal{C}$, $u \in \mathcal{U}$, $v \in \mathcal{V}$ and $w \in \mathcal{W}$, in view of (6) and (10), $U_1^\circ(q; F) \leq -c_1a\alpha V(e_1) + c_1a\psi_1(|v|) + c_1a\psi_2(|w|) - c_1\nu\eta_1 + c_1(\lambda_{\max}(\Lambda_1) + \lambda_{\max}(L_1^\top \Lambda_2 L_1))|y - \hat{y}_1|^2$. Then, using Assumption 2 we have $|y - \hat{y}_1|^2 = |h(x, w) - h(\hat{x}_1, 0)|^2 \leq \delta_1 V(e_1) + \delta_2 |w|^2$. Thus, $U_1^\circ(q; F) \leq -c_1(a\alpha - \lambda_{\max}(\Lambda_1)\delta_1 - \lambda_{\max}(L_1^\top \Lambda_2 L_1)\delta_1)V(e_1) - c_1\nu\eta_1 + c_1a\psi_1(|v|) + c_1a\psi_2(|w|) + c_1(\lambda_{\max}(\Lambda_1) + \lambda_{\max}(L_1^\top \Lambda_2 L_1))\delta_2|w|^2$. Since $a > \bar{a} = \frac{\delta_1(\lambda_{\max}(\Lambda_1) + \lambda_{\max}(L_1^\top \Lambda_2 L_1))}{\alpha} \geq 0$ and defining $a_1 := \min \left\{ \frac{a\alpha - \delta_1\lambda_{\max}(\Lambda_1) - \delta_1\lambda_{\max}(L_1^\top \Lambda_2 L_1)}{a}, \nu \right\} > 0$ we obtain

$$U_1^\circ(q; F) \leq -a_1 U_1(q) + c_1 a \psi_1(|v|) + c_1 a \psi_2(|w|) + c_1 (\lambda_{\max}(\Lambda_1) + \lambda_{\max}(L_1^\top \Lambda_2 L_1)) \delta_2 |w|^2. \quad (46)$$

We now consider U_2 . We need to distinguish four cases.

Case a). Suppose there exists a unique $j \in \{1, \dots, N+1\}$ such that $\max_{k \in \{1, \dots, N+1\}} \{bV(e_k) - \eta_k, 0\} = bV(e_j) - \eta_j$ and $bV(e_j) - \eta_j > 0$. Then, by applying Lemma 1 to the j -th dynamics, and by recalling the definition of θ given at the beginning of the proof, we obtain $U_2^\circ(q; F) = c_2 \langle b \langle \nabla V(e_j), f_p(x, u, v) - f_o(\hat{x}_j, u, L_j(y - \hat{y}_j)) \rangle + \nu \eta_j - (y - \hat{y}_j)^\top (\Lambda_1 + L_j^\top \Lambda_2 L_j) (y - \hat{y}_j) \rangle \leq -c_2 b \alpha V(e_j) + c_2 b \psi_1(|v|) + c_2 b \psi_2(|w|) + c_2 b \theta |y - \hat{y}_j|^2 + c_2 \nu \eta_j - c_2 (y - \hat{y}_j)^\top \Lambda_1 (y - \hat{y}_j) - c_2 (y - \hat{y}_j)^\top L_j^\top \Lambda_2 L_j (y - \hat{y}_j) \leq -c_2 (b \alpha V(e_j) - \nu \eta_j) + c_2 b \psi_1(|v|) + c_2 b \psi_2(|w|) - c_2 (\lambda_{\min}(\Lambda_1) - b \theta) |y - \hat{y}_j|^2$. Since $\nu \in (0, \alpha]$ and $b \in (0, \bar{b})$ with $\bar{b} = \frac{\lambda_{\min}(\Lambda_1)}{\theta}$, $\lambda_{\min}(\Lambda_1) - b \theta > 0$ and thus, defining $a_2 := c_2 \alpha$, we have $U_2^\circ(q; F) \leq -a_2 U_2(q) + c_2 b \psi_1(|v|) + c_2 b \psi_2(|w|)$.

Case b). If for all $k \in \{1, \dots, N+1\}$, $bV(e_k) - \eta_k < 0$, then $U_2(q) = 0$ and $U_2^\circ(q; F) = 0 = -a_2 U_2(q)$.

Case c). If there exists a subset $\mathcal{S} \subseteq \{1, \dots, N+1\}$ such that, for all $i \in \mathcal{S}$, $bV(e_i) - \eta_i = 0$ and for all $j \in \{1, \dots, N+1\} \setminus \mathcal{S}$, $bV(e_j) - \eta_j < 0$, then $U_2(q) = 0$. Following similar steps as in *case a)*, we obtain $U_2^\circ(q; F) = \max_{i \in \mathcal{S}} \left\{ -a_2 (bV(e_i) - \eta_i) + c_2 b \psi_1(|v|) + c_2 b \psi_2(|w|), 0 \right\} \leq c_2 b \psi_1(|v|) + c_2 b \psi_2(|w|) = -a_2 U_2(q) + c_2 b \psi_1(|v|) + c_2 b \psi_2(|w|)$.

Case d). If there exists a subset $\tilde{\mathcal{S}} \subseteq \{1, \dots, N+1\}$ such that, for all $i, j \in \tilde{\mathcal{S}}$, $\max_{k \in \{1, \dots, N+1\}} \{bV(e_k) - \eta_k, 0\} = bV(e_i) - \eta_i = bV(e_j) - \eta_j > 0$. Following similar steps as in *case a)*, we obtain $U_2^\circ(q; F) \leq \max_{i \in \tilde{\mathcal{S}}} \left\{ -a_2 (bV(e_i) - \eta_i) + c_2 b \psi_1(|v|) + c_2 b \psi_2(|w|) \right\}$. Then, for any $i \in \tilde{\mathcal{S}}$, by the definition of $\tilde{\mathcal{S}}$, we have $U_2^\circ(q; F) \leq -a_2 (bV(e_i) - \eta_i) + c_2 b \psi_1(|v|) + c_2 b \psi_2(|w|) = -a_2 U_2(q) + c_2 b \psi_1(|v|) + c_2 b \psi_2(|w|)$. Merging cases a), b), c) and d), we have that, for any $q \in \mathcal{C}$, $u \in \mathcal{U}$, $v \in \mathcal{V}$ and $w \in \mathcal{W}$,

$$U_2^\circ(q; F) \leq -a_2 U_2(q) + c_2 b \psi_1(|v|) + c_2 b \psi_2(|w|). \quad (47)$$

We now consider U_3 . Since $q \in \mathcal{C}$, from (20) we have that $\eta_\sigma \leq \eta_k$ for all $k \in \{1, \dots, N+1\}$. Therefore, $\eta_\sigma \leq \eta_1$. When $\eta_\sigma < \eta_1$ we have that $U_3(q) = 0$ and

$$U_3^\circ(q; F) = 0 = -a_3 U_3(q), \quad (48)$$

for any $a_3 \in \mathbb{R}_{>0}$. When $\eta_\sigma = \eta_1$, since $F(q, u, v, w) \in T_{\mathcal{C}}(q)$ and $T_{\mathcal{C}}(q) := \{q \in \mathcal{Q} : \dot{\eta}_1 \geq \dot{\eta}_\sigma\}$, where we use $\dot{\eta}_1 = -\nu \eta_1 +$

$(y - \hat{y}_1)^\top (\Lambda_1 + L_1^\top \Lambda_2 L_1) (y - \hat{y}_1)$ and $\dot{\eta}_\sigma = -\nu \eta_\sigma + (y - \hat{y}_\sigma)^\top (\Lambda_1 + L_\sigma^\top \Lambda_2 L_\sigma) (y - \hat{y}_\sigma)$ for the sake of convenience, we have

$$U_3^\circ(q; F) \leq \max\{\dot{\eta}_\sigma - \dot{\eta}_1, 0\} = 0 = -a_3 U_3(q), \quad (49)$$

for any $a_3 \in \mathbb{R}_{>0}$. Consequently, from (48) and (49), we obtain that,

$$U_3^\circ(q; F) \leq 0 = -a_3 U_3(q), \quad (50)$$

for all $a_3 \in \mathbb{R}_{>0}$.

From (46), (47) and (50) we have that, for any $q \in \mathcal{C}$, $u \in \mathcal{U}$, $v \in \mathcal{V}$ and $w \in \mathcal{W}$, such that $F(q, u, v, w) \in T_{\mathcal{C}}(q)$, $U^\circ(q; F) = U_1^\circ(q; F) + U_2^\circ(q; F) + U_3^\circ(q; F) \leq -a_1 U_1(q) + c_1 a \psi_1(|v|) + c_1 a \psi_2(|w|) + c_1 (\lambda_{\max}(\Lambda_1) + \lambda_{\max}(L_1^\top \Lambda_2 L_1)) \delta_2 |w|^2 - a_2 U_2(q) + c_2 b \psi_1(|v|) + c_2 b \psi_2(|w|) - a_3 U_3(q)$. Defining $\alpha_0 := \min\{a_1, a_2, a_3\} \in \mathbb{R}_{>0}$, we obtain $U^\circ(q; F) \leq -\alpha_0 U(q) + \phi_1(|v|) + \phi_2(|w|)$, where $\phi_1(s) := (c_1 a + c_2 b) \psi_1(s)$ and $\phi_2(s) := (c_1 a + c_2 b) \psi_2(s) + (c_1 (\lambda_{\max}(\Lambda_1) + \lambda_{\max}(L_1^\top \Lambda_2 L_1)) \delta_2) s^2$, for any $s \geq 0$.

Proof of item (iii). As in the proof of item (ii), for the sake of convenience we write $U(q) = U_1(q) + U_2(q) + U_3(q)$, for any $q \in \mathcal{D}$, where $U_1(q) = c_1 (aV(e_1) + \eta_1)$, $U_2(q) = c_2 \max_{k \in \{1, \dots, N+1\}} \{bV(e_k) - \eta_k, 0\}$ and $U_3(q) = c_3 \max\{\eta_\sigma - \eta_1, 0\}$. In addition, we use σ^+ to denote the selected mode after a jump, in view of the hybrid system notation described in the preliminaries.

Let $q \in \mathcal{D}$ and $\mathbf{g} \in G(q)$. Then, from (13) and (15) we have

$$U_1(\mathbf{g}) = c_1 (aV(e_1) + \eta_1) = U_1(q). \quad (51)$$

We now consider U_2 . We need to distinguish the case without resets and the case with resets. We first consider the case without resets. From (13)-(18), we obtain

$$\begin{aligned} U_2(\mathbf{g}) &= c_2 \max_{k \in \{2, \dots, N+1\} \setminus \{\sigma^+\}} \{bV(e_1) - \eta_1, bV(e_{\sigma^+}) - \eta_{\sigma^+}, \\ &\quad bV(e_k) - \eta_k - \varepsilon, 0\} \\ &\leq c_2 \max_{k \in \{2, \dots, N+1\} \setminus \{\sigma^+\}} \{bV(e_1) - \eta_1, bV(e_{\sigma^+}) - \eta_{\sigma^+}, \\ &\quad bV(e_k) - \eta_k, 0\} \\ &= c_2 \max_{k \in \{1, \dots, N+1\}} \{bV(e_k) - \eta_k, 0\} = U_2(q). \end{aligned} \quad (52)$$

On the other hand, when the resets are implemented, we need to distinguish the case where $\sigma^+ = 1$ and the case when $\sigma^+ \in \{2, \dots, N+1\}$. Suppose first that $\sigma^+ = 1$. Then, from (13)-(18), we have

$$\begin{aligned} U_2(\mathbf{g}) &= c_2 \max\{bV(e_1) - \eta_1, bV(e_{k^*}) - \eta_{k^*} - \varepsilon, 0\} \\ &\leq c_2 \max\{bV(e_1) - \eta_1, bV(e_{k^*}) - \eta_{k^*}, 0\} \end{aligned} \quad (53)$$

with $k^* \in \operatorname{argmin}_{k \in \Pi} (-\nu \eta_k + (y - \hat{y}_k)^\top (\Lambda_1 + L_k^\top \Lambda_2 L_k) (y - \hat{y}_k))$,

where $\Pi(q) = \operatorname{argmin}_{k \in \{1, \dots, N+1\} \setminus \{\sigma\}} \eta_k$, for all $q \in \mathcal{D}$ and $\eta_{k^*} = \min_{k \in \{1, \dots, N+1\} \setminus \{\sigma\}} \eta_k$. Note that, if different observers generate the same minimum η_k , with the same minimum derivative, then k^* may be different from σ^+ . However, $\eta_{\sigma^+} = \eta_{k^*} = \tilde{\eta}_{k^*}$. Consequently, from (53) and since $k^* \in \{1, \dots, N+1\}$,

$$\begin{aligned} U_2(\mathbf{g}) &\leq c_2 \max\{bV(e_1) - \eta_1, bV(e_{k^*}) - \eta_{k^*}, 0\} \\ &\leq c_2 \max_{k \in \{1, \dots, N+1\}} \{bV(e_k) - \eta_k, 0\} = U_2(q). \end{aligned} \quad (54)$$

On the contrary, when $\sigma^+ \in \{2, \dots, N+1\}$, from (13)-(18), we have

$$\begin{aligned} U_2(\mathbf{g}) &= c_2 \max\{bV(e_1) - \eta_1, bV(e_{k^*}) - \eta_{\sigma^+}, bV(e_{k^*}) - \eta_{k^*} \\ &\quad - \varepsilon, 0\} \\ &\leq c_2 \max\{bV(e_1) - \eta_1, bV(e_{k^*}) - \eta_{\sigma^+}, bV(e_{k^*}) - \eta_{k^*}, \\ &\quad 0\}, \end{aligned} \quad (55)$$

with $k^* \in \operatorname{argmin}_{k \in \Pi} (-\nu \eta_k + (y - \hat{y}_k)^\top (\Lambda_1 + L_k^\top \Lambda_2 L_k) (y - \hat{y}_k))$, where $\Pi(q) = \operatorname{argmin}_{k \in \{1, \dots, N+1\} \setminus \{\sigma\}} \eta_k$, for all $q \in \mathcal{D}$ and $\eta_{k^*} = \min_{k \in \{1, \dots, N+1\} \setminus \{\sigma\}} \eta_k$. Note that, if different observers generate the same minimum η_k , with the same minimum derivative, then k^* may be different from σ^+ . However, $\eta_{\sigma^+} = \eta_{k^*} = \tilde{\eta}_{k^*}$. Consequently, from (55) and since $k^* \in \{1, \dots, N+1\}$, $U_2(\mathbf{g}) \leq c_2 \max\{bV(e_1) - \eta_1, bV(e_{k^*}) - \eta_{k^*}, 0\} \leq c_2 \max_{k \in \{1, \dots, N+1\}} \{bV(e_k) - \eta_k, 0\} = U_2(q)$. As a result, from (52), (54) and the last inequality we have, for any $q \in \mathcal{D}$ and any $\mathbf{g} \in G(q)$,

$$U_2(\mathbf{g}) \leq U_2(q). \quad (56)$$

We now consider U_3 . From (15) and (16) we have $U_3(\mathbf{g}) = c_3 \max\{\eta_{\sigma^+} - \eta_1, 0\} = c_3 \max\{\eta_\sigma - \eta_1, 0\} = U_3(q)$.

Merging (51), (56) and the last equation we obtain, for any $q \in \mathcal{D}$ and any $\mathbf{g} \in G(q)$, $U(\mathbf{g}) \leq U(q)$, which concludes the proof of item (iii) of Proposition 1. This complete the proof.

B. Proof of Proposition 3

Let $u \in \mathcal{L}_{\mathcal{U}}$ and $v \in \mathcal{L}_{\mathcal{V}}$, $w \in \mathcal{L}_{\mathcal{W}}$ with \mathcal{V} and \mathcal{W} compact set. Let $\bar{M} \geq M$ such that $\|v\|_{\mathcal{X}} \leq \bar{M}$ and $\|w\|_{\mathcal{X}} \leq \bar{M}$. Let q be a solution to system (19) with $|q(0, 0)| \leq M \leq \bar{M}$. Pick any $(t, j) \in \operatorname{dom} q$ and let $0 = t_0 \leq t_1 \leq \dots \leq t_{j+1} = t$ satisfy $\operatorname{dom} q \cap ([0, t] \times \{0, 1, \dots, j\}) = \bigcup_{i=0}^j [t_i, t_{i+1}] \times \{i\}$. For each $i \in \{0, \dots, j\}$ and almost all $s \in [t_i, t_{i+1}]$, $q(s, i) \in \mathcal{C}$. Then, from (10), for all $k \in \{1, \dots, N+1\}$, for all $s \in (t_i, t_{i+1})$, (we omit the dependency on (s, i) below), $\dot{\eta}_k - \dot{\eta}_\sigma = -\nu(\eta_k - \eta_\sigma) + (y - \hat{y}_k)^\top (\Lambda_1 + L_k^\top \Lambda_2 L_k) (y - \hat{y}_k) - (y - \hat{y}_\sigma)^\top (\Lambda_1 + L_\sigma^\top \Lambda_2 L_\sigma) (y - \hat{y}_\sigma) \geq -\nu(\eta_k - \eta_\sigma) - (y - \hat{y}_\sigma)^\top (\Lambda_1 + L_\sigma^\top \Lambda_2 L_\sigma) (y - \hat{y}_\sigma) \geq -\nu(\eta_k - \eta_\sigma) - (\lambda_{\max}(\Lambda_1) + \lambda_{\max}(L_\sigma^\top \Lambda_2 L_\sigma)) |y - \hat{y}_\sigma|^2$. Then, using Assumption 2 we have $|y - \hat{y}_\sigma|^2 = |h(x, w) - h(\hat{x}_\sigma, 0)|^2 \leq \delta_1 V(e_\sigma) + \delta_2 |w|^2$. Thus, we obtain, for all $s \in (t_i, t_{i+1})$,

$$\begin{aligned} \dot{\eta}_k - \dot{\eta}_\sigma &\geq -\nu(\eta_k - \eta_\sigma) - (\lambda_{\max}(\Lambda_1) + \lambda_{\max}(L_\sigma^\top \Lambda_2 L_\sigma)) \\ &\quad (\delta_1 V(e_\sigma) + \delta_2 |w|^2). \end{aligned} \quad (57)$$

Using (24), from Theorem 1, we obtain, for all $(t, j) \in \operatorname{dom} q$, $|e_\sigma(t, j)| \leq \beta_U(|q(0, 0)|, t) + \gamma_U(\|v\|_{\mathcal{X}} + \|w\|_{\mathcal{X}})$, with $\beta_U \in \mathcal{KL}$ and $\gamma_U \in \mathcal{K}_{\mathcal{X}}$. Then, using $|q(0, 0)| \leq \bar{M}$, $\|v\|_{\mathcal{X}} \leq \bar{M}$ and $\|w\|_{\mathcal{X}} \leq \bar{M}$ we obtain, for all $(t, j) \in \operatorname{dom} q$, $|e_\sigma(t, j)| \leq \beta_U(\bar{M}, t) + \gamma_U(2\bar{M}) \leq \beta_U(\bar{M}, 0) + \gamma_U(2\bar{M})$. From Assumption 1, for all $e_\sigma \in \mathbb{R}^{n_x}$, $V(e_\sigma) \leq \bar{\alpha}(|e_\sigma|)$, where $\bar{\alpha} \in \mathcal{K}_{\mathcal{X}}$ comes from Assumption 1. From the last two inequalities we have, for all $(t, j) \in \operatorname{dom} q$,

$$\begin{aligned} V(e_\sigma(t, j)) &\leq \bar{\alpha}(\beta_U(\bar{M}, t) + \gamma_U(2\bar{M})) \\ &\leq \check{\beta}_U(\bar{M}, 0) + \check{\gamma}_U(2\bar{M}), \end{aligned} \quad (58)$$

where $\check{\beta}_U := \bar{\alpha} \circ \beta_U \in \mathcal{K}\mathcal{L}$ and $\check{\gamma}_U := \bar{\alpha} \circ \gamma_U \in \mathcal{K}_{\mathcal{L}}$. Combining (57) with (58) we obtain, for all $k \in \{1, \dots, N+1\}$, for all $s \in [t_i, t_{i+1}]$,

$$\begin{aligned} \dot{\eta}_k - \dot{\eta}_\sigma &\geq -\nu(\eta_k - \eta_\sigma) - (\lambda_{\max}(\Lambda_1) + \lambda_{\max}(L_\sigma^\top \Lambda_2 L_\sigma)) \\ &\quad (\delta_1(\check{\beta}_U(\bar{M}, 0) + \check{\gamma}_U(2\bar{M})) + \delta_2\bar{M}^2) \\ &\geq -\nu(\eta_k - \eta_\sigma) - c, \end{aligned} \quad (59)$$

with $c := (\lambda_{\max}(\Lambda_1) + \max_{k \in \{1, \dots, N+1\}} \lambda_{\max}(L_k^\top \Lambda_2 L_k))(\delta_1(\check{\beta}_U(\bar{M}, 0) + \check{\gamma}_U(2\bar{M})) + \delta_2\bar{M}^2) \in \mathbb{R}_{>0}$. Integrating (59) and applying the comparison principle [47, Lemma 3.4] we obtain, for all $s \in [t_i, t_{i+1}]$, for all $k \in \{1, \dots, N+1\}$,

$$\begin{aligned} \eta_k(s, i) - \eta_\sigma(s, i) &\geq e^{-\nu(s-t_i)}(\eta_k(t_i, i) - \eta_\sigma(t_i, i)) \\ &\quad - \frac{c}{\nu}(1 - e^{-\nu(s-t_i)}). \end{aligned} \quad (60)$$

On the other hand, from (21), we have

$$t_{i+1} := \inf\{t \geq t_i : \min_{k \in \{1, \dots, N+1\} \setminus \{\sigma\}} \eta_k(t, i) = \eta_\sigma(t, i)\}. \quad (61)$$

We define $k^* := \sigma(t_{i+1}, i+1) \in \operatorname{argmin}_{k \in \Pi} (-\nu\eta_k(t_{i+1}, i+1) + (y(t_{i+1}, i+1) - \hat{y}_k(t_{i+1}, i+1))^\top (\Lambda_1 + L_k^\top \Lambda_2 L_k)(y(t_{i+1}, i+1) - \hat{y}_k(t_{i+1}, i+1))))$, where $\Pi(q) = \operatorname{argmin}_{k \in \{1, \dots, N+1\} \setminus \{\sigma\}} \eta_k$.

Evaluating (60) for $s = t_{i+1}$ and $k = k^*$, from (61), we have

$$\begin{aligned} 0 &= \eta_{k^*}(t_{i+1}, i) - \eta_\sigma(t_{i+1}, i) \\ &\geq e^{-\nu(t_{i+1}-t_i)}(\eta_{k^*}(t_i, i) - \eta_\sigma(t_i, i)) - \frac{c}{\nu}(1 - e^{-\nu(t_{i+1}-t_i)}). \end{aligned} \quad (62)$$

We first consider the case where $k^* \neq 1$. Note that $\sigma(s, i) \neq k^*$ by the definition of k^* , for all $s \in [t_i, t_{i+1}]$. We now consider the cases without and with resets separately. From (16), (17) and (21) we have in the case without resets, for all $i \in \mathbb{Z}_{>0}$, $\eta_{k^*}(t_i, i) = \eta_{k^*}(t_i, i-1) + \varepsilon \geq \eta_\sigma(t_i, i) + \varepsilon$, while in the case with resets, $\eta_{k^*}(t_i, i) = \eta_\sigma(t_i, i) + \varepsilon$. As a result, $\eta_{k^*}(t_i, i) \geq \eta_\sigma(t_i, i) + \varepsilon$, both in the case without and with resets. Thus, $\eta_{k^*}(t_i, i) - \eta_\sigma(t_i, i) \geq \varepsilon$ and from (62), $0 \geq e^{-\nu(t_{i+1}-t_i)}\varepsilon - \frac{c}{\nu}(1 - e^{-\nu(t_{i+1}-t_i)})$, which can be rewritten as $e^{-\nu(t_{i+1}-t_i)}(\varepsilon + \frac{c}{\nu}) \leq \frac{c}{\nu}$, that implies

$$t_{i+1} - t_i \geq -\frac{1}{\nu} \ln\left(\frac{\frac{c}{\nu}}{\varepsilon + \frac{c}{\nu}}\right) \in \mathbb{R}_{>0}. \quad (63)$$

On the other hand, when $k^* = 1 = \sigma(s, i+1)$, for all $s \in [t_{i+1}, t_{i+2}]$, we have that $\sigma(t_{i+2}, i+2) = \operatorname{argmin}_{k \in \{2, \dots, N+1\}} \eta_k(t_{i+1}, i) \neq 1$. Therefore, from (63), we obtain

$t_{i+2} - t_{i+1} \geq -\frac{1}{\nu} \ln\left(\frac{\frac{c}{\nu}}{\varepsilon + \frac{c}{\nu}}\right) \in \mathbb{R}_{>0}$. Consequently, for all switching times $(t_i, i) \in \operatorname{dom} q$, we have $t_{i+2} - t_i \geq -\frac{1}{\nu} \ln\left(\frac{\frac{c}{\nu}}{\varepsilon + \frac{c}{\nu}}\right) \in \mathbb{R}_{>0}$. Pick any $(t, j), (t', j') \in \operatorname{dom} q$ such that $(t, j) \leq (t', j')$, from the last inequality and using $\tau = -\frac{1}{2\nu} \ln\left(\frac{\frac{c}{\nu}}{\varepsilon + \frac{c}{\nu}}\right)$ we obtain $j' - j \leq \frac{1}{\tau}(t' - t) + 2$, which concludes the proof.

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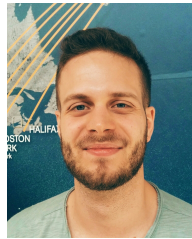
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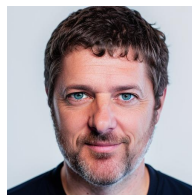
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