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► **To cite this version:**

Filippo Cacace, Mattia Mattioni, Salvatore Monaco, Dorothée Normand-Cyrot. Consensus and multi-consensus for discrete-time LTI systems. *Automatica*, 2024, 166, pp.111718. 10.1016/j.automatica.2024.111718 . hal-04792564

HAL Id: hal-04792564

<https://hal.science/hal-04792564v1>

Submitted on 20 Nov 2024

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Consensus and multi-consensus for discrete-time LTI systems

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Abstract

Existing consensus control algorithms for networked discrete-time linear systems have slow convergence to consensus trajectories due to limitations on the magnitude of the consensus gain. In this work we propose and analyze a new predictor-based consensus control protocol that recovers the positive features of the continuous-time counterpart in terms of arbitrary rate of convergence to the consensus dynamics. Moreover, it admits a distributed implementation based on multiple consensus steps. Our analysis concerns general weakly connected digraphs and it encompasses both single and multiple consensus.

Key words: Multi-agent systems; multi-consensus; decentralization; digital implementation.

1 Introduction

In this article we study the dynamic consensus of multi-agent systems consisting in a network of linear time-invariant systems. The consensus problem is fundamental to characterize the collective behavior of multi-agent networked systems (Olfati-Saber et al., 2007; Ren and Cao, 2010). The aim in this setting is to study the conditions ensuring that all the agents converge to a consensus that can be either a single point (Ren and Beard, 2008), or a single trajectory (Olfati-Saber et al., 2007; Egerstedt et al., 2012), or a set of trajectories (Chen et al., 2011; Gambuzza and Frasca, 2020; Li et al., 2013; Monaco and Ricciardi Celsi, 2019; Yaghmaie et al., 2017). In this sense, such consensus depends on two main features: the individual dynamics of the agents, homogeneous (Scardovi and Sepulchre, 2008) or heterogeneous (Panteley and Loría, 2017; Mattioni and Monaco, 2022; Xiao and Chen, 2017; Yaghmaie et al., 2017), linear (Dutta et al., 2022; Li et al., 2014) or nonlinear (Li et al., 2012; Wen et al., 2013); the overall graph topology, directed (Dutta et al., 2022; Li et al., 2014; Cai and Ishii, 2012) or undirected (Xiao and

Wang, 2006; Stüdli et al., 2020), fixed or time-varying (Ren and Beard, 2005; Moreau, 2005; Chen et al., 2013). Dynamic consensus is a core problem for networked multi-agent systems, as it is instrumental to analyze and solve many practical problems in several application areas: formation control of mobile agents (Qu, 2009; Ren and Beard, 2008; Ma et al., 2015; Francis and Maggiore, 2016; Roza et al., 2019); synchronization and stability of complex networks (Arenas et al., 2008; Li et al., 2010; Dörfler and Bullo, 2014); collective behaviors (You et al., 2009; Wang and Lu, 2019); opinion dynamics in social networks (Anderson and Ye, 2019; Proskurnikov and Tempo, 2017), just to mention a few. Dynamic consensus is achieved by designing a distributed controller that, based on interactions among neighbouring agents, generates a common behavior that usually corresponds to a suitably weighted average of the dynamics of the interconnected systems (Dutta et al., 2022). The problem is therefore to design the distributed controller as well as to characterize the consensus trajectory.

Quite surprisingly there is a striking difference between the continuous-time and the discrete-time cases. For continuous-time systems, with continuous communications among agents, it is possible to converge to the consensus dynamics with arbitrary speed by means of simple feedback from the state of the neighbors and the agent itself. In particular, by just increasing the scalar weight of the consensus feedback term, an arbitrarily

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fast rate of convergence can in principle be attained (Li et al., 2014; Liu et al., 2021). In contrast to this, when using the same feedback in discrete time (Ren and Cao, 2010; Ma et al., 2015), the scalar weight cannot be too large and the rate of convergence to consensus is limited. The coupling gain is generally inversely proportional to either the largest eigenvalue of the Laplacian matrix of the network or to the number of agents in the network (Mattioni et al., 2022), unless specific graph structures are enforced (Stüdl et al., 2020). In the seminal study Xiao and Boyd (2004) it is shown how to assign weights to the graph connection in order to achieve fast convergence to consensus in the case of averaging of scalar values. However, the fastest convergence rate is determined by the graph structure and it cannot be arbitrarily assigned. The limitations on the consensus gained mentioned above imply that for stability reasons the influence of the network over each agent must be significantly mitigated, which is not desirable in practice because it slows down the convergence to the collective dynamics. Intuitively this difficulty for fast converge to consensus lies in the fact that at discrete-time the exchange information between a pair of agents requires a time equal to the length of the shortest path between them.

In this context we propose a novel consensus controller that has the structure of a predictor-based feedback based on both the state and the input of the neighbours. The approach has already been studied for networks of integrators in (Mattioni et al., 2022; Cacace et al., 2023); in this paper we aim at extending it to the case of general linear systems and graph topologies. This extension, beyond being not trivial from a technical point of view, leads to conditions on the parameters of the algorithm and on the controller gains that are necessary for enforcing convergence to the consensus trajectories. Such conditions depend on both the agent dynamics and the network topology and, hence, were not present in the case of simple integrators (Mattioni et al., 2022; Cacace et al., 2023).

After recalling some preliminary results in Section 2, in Section 3 we prove the crucial property of the new consensus controller, namely the fact that it is capable of achieving an arbitrary rate of convergence to the consensus trajectory for directed graphs with a spanning tree. Similarly to the continuous-time case, the coupling gain can be arbitrarily large and it can have distinct values at each agent, as long as all the values are larger than a lower bound that depends, as in the continuous-time case, on the smallest non zero eigenvalue of the Laplacian matrix (Li et al., 2014). This new feedback is however implicitly defined, *i.e.* it is the solution of an equation that must be solved at each time. In order to make the approach useful in practice, we show that the solution can be approximated arbitrarily well by resorting to multiple interactions among neighboring agents at each time-step. This technique, based on multiple consensus steps, has already employed for distributed filter-

ing (Battistelli et al., 2014; Kamal et al., 2013) and it is here applied for the first time to the consensus control problem. Differently from the case of scalar integrators of (Cacace et al., 2023), we show that there exists a lower bound to the number of consensus steps needed to attain consensus. Section 3.2, that describes the resulting controller operating on two time scales, also provides an estimate of this lower bound that can be computed in a distributed way based on the agents' dynamics.

In Section 4 we extend our approach to more general graph structures. We consider the weakest requirement for consensus on directed graphs, namely that the network is weakly connected. We show that in this situation, analogously to the continuous time case (Cacace et al., 2021), the agents are partitioned into clusters, each converging to its consensus trajectory. We show that there are two type of clusters. The first type converges to a trajectory which is autonomously determined only from the initial conditions of its root set, whereas the second type converges to a convex combination of the consensus trajectories of the clusters of the first type. This result extends the application of consensus control to the problem of formation control and containment (Liu et al., 2012; Li et al., 2013; Cacace et al., 2021) to the discrete-time case.

In Section 5 we validate the theoretical properties of the approach through numerical examples, and conclusions are drawn in Section 6.

Notations. \mathbb{C} , \mathbb{R} and \mathbb{N} denote the set of complex, real and natural numbers including 0 respectively. $\Re(z)$ denotes the real part of $z \in \mathbb{C}$ and z^* is the conjugate of z . I_n and 0_n denote respectively the identity and zero matrices of dimension n . $\mathbb{1}_n$ denotes the vector in \mathbb{R}^n with entries 1. Given a matrix $A \in \mathbb{R}^{n \times n}$, A^* is its transpose conjugate and $\sigma(A) \subset \mathbb{C}$ its spectrum. $\lambda \in \sigma(A)$ is said to be simple if its algebraic and geometric multiplicity is one. A is said to be Schur if its spectrum is included in the open unit circle of the complex plane. Given two matrices A, B of the same size, $A > B$ (resp. $A \geq B$) means that $A - B$ is positive definite (resp. positive semi-definite). $|\cdot| \in \mathbb{R}$ denotes, depending on the argument, either the cardinality of a set \mathcal{S} or the absolute value of a complex number $\lambda \in \mathbb{C}$. Given two matrices $A \in \mathbb{R}^{n_1 \times n_2}$ and $B \in \mathbb{R}^{m_1 \times m_2}$, the Kronecker product is denoted by $A \otimes B \in \mathbb{R}^{n_1 m_1 \times n_2 m_2}$. Throughout the paper the usual product has higher priority, thus $A \otimes BC = A \otimes (BC)$. Given $x : t \mapsto x(t)$ with $t \in \mathbb{N}$ and $x(t) \in \mathbb{R}^n$, we denote for concision $x^+ = x(t+1)$. Finally, $\text{col}_{i=1}^N(x_i) = (x_1, \dots, x_N)^\top$, $\text{row}_{i=1}^N(x_i) = (x_1, \dots, x_N)$, and $\text{diag}_{i=1}^N(x_i)$ denote vertical, horizontal and diagonal composition respectively.

2 Problem statement and recalls

2.1 Preliminaries on graphs

We consider an unweighted directed graph (or digraph for short) $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with $|\mathcal{V}| = N$, $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$. The set of neighbors to a $i \in \mathcal{V}$ is $\mathcal{N}_i = \{j \in \mathcal{V} \text{ s.t. } (j, i) \in \mathcal{E}\}$. For all pairs of distinct nodes $i, j \in \mathcal{V}$, a directed path from i to j is defined as $i \rightsquigarrow j := \{(r, r+1) \in \mathcal{E} \text{ s.t. } \cup_{r=0}^{\ell-1} (r, r+1) \subseteq \mathcal{E} \text{ with } 0 = i, \ell = j \text{ and } \ell > 0\}$.

The Laplacian matrix associated to a digraph \mathcal{G} with $N = |\mathcal{V}|$ nodes is $\mathcal{L} = \mathcal{D} - \mathcal{A}$ with $\mathcal{D} = \text{diag}_k(d_k) \in \mathbb{R}^{N \times N}$ and $\mathcal{A} \in \mathbb{R}^{N \times N}$ being respectively the in-degree and the adjacency matrices. By construction, $\lambda_1 = 0 \in \sigma(\mathcal{L})$ and one right eigenvector of λ_1 is $\mathbf{1}_N$. Classical consensus or multi-consensus is achieved depending on the algebraic multiplicity of the λ_1 eigenvalue (that is the multiplicity with respect to the characteristic polynomial). A digraph is called: *strongly connected* if there is a directed path between each pair of nodes; *weakly connected* if its un-directed version is connected (*i.e.* there is an un-directed path between each pair of nodes); *with one reach* if it is weakly connected and there is a directed path to all nodes from each node of a set of *root* nodes. This last name descends from the fact that the existence of a spanning tree implies that the digraph has just one *reach* (defined below in Definition 4.1). Clearly, weakly connected digraphs include graphs with one reach that in turn include strongly connected digraphs. We recall two simple properties that are used in the sequel.

Proposition 2.1 *The Laplacian matrix of a digraph with one reach has a simple eigenvalue $\lambda_1 = 0$. All the remaining eigenvalues have positive real parts.*

Proposition 2.2 *The set of root nodes of a digraph with one reach forms a strongly connected sub-graph.*

Proposition 2.1 is a consequence of the fact that the existence of just one reach implies that λ_1 is simple (see Theorem 3.2 in (Caughman and Veerman, 2006)). Moreover, an application of the Gerschgorin circles to the rows of \mathcal{L} yields that the eigenvalues are contained in the circles of the complex plane centered in d_k with radius d_k , thus their real parts are always positive (except for $\lambda_1 = 0$ which is simple). Proposition 2.2 descends trivially from the fact that for each node r in the set of root nodes there is a spanning tree rooted in r , thus any other element of the set is reachable and in particular the remaining root nodes. When the set of root nodes of a digraph with one reach contains just one node, the consensus problem is known as *leader-following* problem.

2.2 Problem formulation and overview of the approach

Consider a multi-agent system exchanging information via a communication digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ with Laplacian

\mathcal{L} . Each node $i \in \mathcal{V}$ is a dynamical unit of the form

$$x_i^+ = Ax_i(t) + Bu_i(t) \quad (1)$$

with $x_i \in \mathbb{R}^n$, $u_i \in \mathbb{R}^m$, $x_i^+ := x_i(t+1)$ and $t \in \mathbb{N}$, $i = 1, \dots, N$. We use interchangeably the terms nodes and agents to refer to the nodes in \mathcal{V} and the associated LTI dynamical systems. If we denote

$$\mathbf{x} = \text{col}_{i=1}^N(x_i) \in \mathbb{R}^{nN}, \quad \mathbf{u} = \text{col}_{i=1}^N(u_i) \in \mathbb{R}^{mN} \quad (2)$$

the collective dynamics of the multi-agent systems is represented as

$$\mathbf{x}^+ = (I_N \otimes A)\mathbf{x}(t) + (I_N \otimes B)\mathbf{u}(t). \quad (3)$$

The assumptions we use throughout the work are the following ones:

- A1.** the couple (A, B) is controllable;
- A2.** the digraph \mathcal{G} is weakly connected.

The problem we solve is to determine a control law to make the agents converge with a chosen rate to a set of common trajectories determined by the network topology. Specifically, the following two cases are possible:

- (1) **consensus**, when \mathcal{G} is a graph with one reach, all the trajectories converge asymptotically to the same trajectory x_s , that is, $x_i(t) \rightarrow x_s(t)$ or $\mathbf{x}(t) \rightarrow \mathbf{1}_N \otimes x_s(t)$.
- (2) **multi-consensus**, when \mathcal{G} is an arbitrary weakly connected graph, \mathcal{V} will be partitioned in clusters, each converging to the same trajectory.

The main tool for the general multi-consensus case is the definition of partitions (Cardoso et al., 2007; Monshizadeh et al., 2015; Monaco and Ricciardi Celsi, 2019; Cacace et al., 2021) that will be introduced in Section 4. In Section 3 we study the consensus control problem and in Section 4 the multi-consensus control problem. The main motivation for separating the two cases is that of introducing the new control approach for the simpler case of graphs with one reach without the notational burden of specifying the behavior of the single clusters in the general case. In fact, although the problem has already been solved in the continuous-time case by following the same approach based on graph partitions (Cacace et al., 2021), the control design for the discrete-time case is not a trivial extension of its. In particular, in the discrete-time context the main challenge is to obtain consensus with a pre-defined, arbitrary rate. As argued in (Mattioli et al., 2022; Cacace et al., 2023), an arbitrary speed of convergence to consensus is not possible with a feedback from the classic consensus term $\sum_{j \in \mathcal{N}_i} (x_i - x_j)$ so

that a slight modification

$$u_i = -\kappa \sum_{j \in \mathcal{N}_i} (x_i - x_j), \quad \kappa < \bar{\kappa} = \frac{1}{N} \quad (4)$$

has been proposed and studied in many previous works (Ren and Cao, 2010; Panteley and Loría, 2017; Cai and Ishii, 2012; Liu et al., 2008). In this paper we propose a new control law of the form

$$y_i = Fx_i + Gu_i, \quad (5a)$$

$$\begin{aligned} u_i &= -\kappa_i \sum_{j \in \mathcal{N}_i} (y_i - y_j) \\ &= -\kappa_i \sum_{j \in \mathcal{N}_i} F(x_i - x_j) + G(u_i - u_j) \end{aligned} \quad (5b)$$

where $\kappa_i > 0$, $y_i \in \mathbb{R}^m$, and $F \in \mathbb{R}^{m \times n}$, $G \in \mathbb{R}^{m \times m}$ to be defined. We notice that a seemingly similar consensus control law has been proposed in (Liu and Zhou, 2018) for the case of communication delays, but in that case the present input depends on past state and input. In contrast, the present input of (5b) depends on both the present state and present input of the neighbors. The gains κ_i are distinct across the nodes. We introduce the modified Laplacian

$$\bar{\mathcal{L}} = \text{diag}_{i=1}^N(\kappa_i)\mathcal{L}, \quad (6)$$

that is still row stochastic (i.e. $\bar{\mathcal{L}}\mathbf{1}_N = 0$) and deduced by multiplying all rows of \mathcal{L} by the corresponding κ_i in (5b). Clearly, (5), that reads in compact form

$$\mathbf{u} = -(\bar{\mathcal{L}} \otimes F) \mathbf{x} - (\bar{\mathcal{L}} \otimes G) \mathbf{u}, \quad (7)$$

is an equation that is solved by

$$\mathbf{u} = -(I_{mN} + (\bar{\mathcal{L}} \otimes G))^{-1} (\bar{\mathcal{L}} \otimes F) \mathbf{x}. \quad (8)$$

In order to solve the consensus control problem we need to separately analyze the closed loop dynamics of (3) under (5) and the distributed computation of (8) needed to implement the implicit form (5).

Remark 2.1 *As typical, the consensus problem concerns the stabilization of each agent to a common trajectory. Depending on the agent dynamics (i.e., on A) such trajectory may be bounded or not without affecting the consensus behavior. If one wants that the overall behavior to be not unstable, it is possible to add a local feedback term, $u_i(t) = Kx_i(t) + \bar{u}_i(t)$, so that the matrix $\bar{A} := A + BK$ in (1) possesses the desired stability features. The remaining component \bar{u}_i can be thus designed to enforce consensus using the results to come.*

3 Consensus control for weakly connected digraphs with one reach

3.1 Consensus analysis

Let $\sigma(\mathcal{L}) = \{\lambda_k\}$ and $\sigma(\bar{\mathcal{L}}) = \{\bar{\lambda}_k\}$, for $k = 1, \dots, N$. In this section we analyze the closed-loop stability of (7) for the case of weakly connected digraphs with one reach, that is, when $\lambda_1 = \bar{\lambda}_1 = 0$ is simple (Proposition 2.1). By combining (3) and (8) we obtain the closed-loop dynamics

$$\mathbf{x}^+ = \Theta \mathbf{x}(t) \quad (9)$$

with

$$\Theta = I_N \otimes A - (I_N \otimes B) (I_{mN} + \bar{\mathcal{L}} \otimes G)^{-1} (\bar{\mathcal{L}} \otimes F). \quad (10)$$

Lemma 3.1 *The spectrum of Θ in (10) reads*

$$\sigma(\Theta) = \bigcup_{\bar{\lambda}_k \in \sigma(\bar{\mathcal{L}})} \sigma(\bar{A}_k) \quad (11)$$

$$\bar{A}_k = A - B \left(\frac{1}{\bar{\lambda}_k} I_m + G \right)^{-1} F. \quad (12)$$

Moreover, for all pairs $(\bar{\lambda}_k, z_k)$, $(\mu_{j,k}, w_{j,k})$, $j = 1, \dots, N$, of eigenvalue and eigenvector associated to $\bar{\mathcal{L}}$ and \bar{A}_k respectively, the right eigenvectors of Θ are in the form $z_k \otimes w_{j,k}$; i.e. for $\bar{A}_k w_{j,k} = \mu_{j,k} w_{j,k}$, $\bar{\mathcal{L}} z_k = \bar{\lambda}_k z_k$, one has

$$\Theta (z_k \otimes w_{j,k}) = \mu_{j,k} (z_k \otimes w_{j,k}). \quad (13)$$

Analogously, the left eigenvectors can be represented as

$$(\nu_k^\top \otimes \tilde{w}_{j,k}^\top) \Theta = \mu_{j,k} (\nu_k^\top \otimes \tilde{w}_{j,k}^\top) \quad (14)$$

with $\tilde{w}_{j,k}^\top \bar{A}_k = \mu_{j,k} \tilde{w}_{j,k}^\top$, $\nu_k^\top \bar{\mathcal{L}} = \bar{\lambda}_k \nu_k^\top$.

Proof: To prove (13), we start by showing that, for $\bar{\mathcal{L}} z_k = \bar{\lambda}_k z_k$ and all $w \in \mathbb{R}^m$,

$$(I_{mN} + \bar{\mathcal{L}} \otimes G)^{-1} (z_k \otimes w) = z_k \otimes (I_m + \bar{\lambda}_k G)^{-1} w. \quad (15)$$

In fact, one gets

$$\begin{aligned} & (I_{mN} + \bar{\mathcal{L}} \otimes G) (z_k \otimes (I_m + \bar{\lambda}_k G)^{-1} w) \\ &= z_k \otimes (I_m + \bar{\lambda}_k G)^{-1} w + z_k \otimes \bar{\lambda}_k G (I_m + \bar{\lambda}_k G)^{-1} w \\ &= z_k \otimes w. \end{aligned} \quad (16)$$

By using (15) it easily follows that

$$\begin{aligned}
\Theta(z_k \otimes w_{j,k}) &= z_k \otimes Aw_{j,k} - (I_N \otimes B) (I_{mN} + \bar{\mathcal{L}} \otimes G)^{-1} \\
&\quad \cdot (\bar{\mathcal{L}} z_k \otimes Fw_{j,k}) \\
&= z_k \otimes Aw_{j,k} - (I_N \otimes B) (I_{mN} + \bar{\mathcal{L}} \otimes G)^{-1} (z_k \otimes \bar{\lambda}_k Fw_{j,k}) \\
&= z_k \otimes Aw_{j,k} - (I_N \otimes B) \left(z_k \otimes \left((I_m + \bar{\lambda}_k G)^{-1} \bar{\lambda}_k Fw_{j,k} \right) \right) \\
&= z_k \otimes Aw_{j,k} - z_k \otimes \left(\bar{\lambda}_k B (I_m + \bar{\lambda}_k G)^{-1} Fw_{j,k} \right) \\
&= z_k \otimes (\bar{A}_k w_{j,k}) = \mu_{j,k} (z_k \otimes w_{j,k}). \tag{17}
\end{aligned}$$

The expression above proves (11) and (13). (14) can be proved analogously to (13). \triangleleft

We notice that $\bar{\lambda}_1 = 0 \in \sigma(\bar{\mathcal{L}})$, thus $\sigma(A) = \sigma(\bar{A}_1) \subset \sigma(\Theta)$. Our aim is to design κ_i , F and G so that all the matrices \bar{A}_k are Schur, for $\bar{\lambda}_k \in \sigma(\bar{\mathcal{L}}) \setminus \{0\}$, and to prove that in this situation the multi-agent system will reach consensus on a trajectory specified by A and by a linear combination of the initial conditions of the root nodes of \mathcal{G} . Hence, the following result is instrumental.

Lemma 3.2 *If $M = M^\top \geq 0$ and $\Re(z) \geq 1$, $z \in \mathbb{C}$, then*

$$2(I + M)^{-1} \leq \left(\frac{1}{z} I + M \right)^{-1} + \left(\frac{1}{z^*} I + M \right)^{-1} \tag{18}$$

Proof: Let $\lambda_k \in \sigma(M)$, $\lambda_k \geq 0$. Then, $(\lambda_k + 1)^{-1} \in \sigma((I + M)^{-1})$, $\mu_k = (\lambda + 1/z)^{-1} \in \sigma\left(\left(\frac{1}{z}I + M\right)^{-1}\right)$, and $\mu_k^* \in \sigma\left(\left(\frac{1}{z^*}I + M\right)^{-1}\right)$. It is thus sufficient to prove $\Re(\mu_k) > \frac{1}{\lambda_k + 1}$. Explicit computations yield $\Re(\mu_k) \geq \frac{1}{\lambda_k + 1/\Re(z)} > \frac{1}{\lambda_k + 1}$. \triangleleft

Theorem 3.1 *Let \mathcal{L} be the Laplacian of weakly connected digraph with one reach \mathcal{G} with $\bar{r} = \min_{\lambda \in \sigma(\mathcal{L}) \setminus \{0\}} \Re(\lambda)$. For all $Q = Q^\top > 0$, let $P = P^\top \succ 0 \in \mathbb{R}^{n \times n}$ be the solution of the Discrete Algebraic Riccati Equation (DARE)*

$$A^\top P A - P - F^\top (I_m + G)^{-1} F + Q = 0 \tag{19}$$

with

$$F = B^\top P A \tag{20a}$$

$$G = B^\top P B. \tag{20b}$$

The matrices \bar{A}_k in (12) are Schur if, for all $i = 1, \dots, N$, the following condition holds

$$\kappa_i \cdot \bar{r} > 1. \tag{21}$$

Proof: Since the eigenvalues of \mathcal{L} have nonnegative real parts, it is easy to see that condition (21) implies that $\Re(\bar{\lambda}_k) > 1$ for all $\bar{\lambda}_k \neq 0 \in \sigma(\bar{\mathcal{L}})$. In general, $\bar{\lambda}_k \in \mathbb{C}$. We aim at proving $\bar{A}_k^* P \bar{A}_k - P \leq -Q$, where P is the solution of (19) for a generic $Q = Q^\top > 0$. Replacing (20a) and (20b) in (12),

$$\begin{aligned}
\bar{A}_k^* P \bar{A}_k - P &= A^\top P A - P \\
&\quad - F^\top \left(\left(\frac{1}{\bar{\lambda}_k} I_m + G \right)^{-1} + \left(\frac{1}{\bar{\lambda}_k^*} I_m + G \right)^{-1} \right) F \\
&\quad + F^\top \left(\frac{1}{\bar{\lambda}_k} I_m + G \right)^{-1} G \left(\frac{1}{\bar{\lambda}_k^*} I_m + G \right)^{-1} F. \tag{22}
\end{aligned}$$

Let $S = \left(\frac{1}{\bar{\lambda}_k} I_m + G \right)^{-1}$. By using Lemma 3.2 we get

$$\begin{aligned}
\bar{A}_k^* P \bar{A}_k - P &\leq A^\top P A - P - F^\top (I_m + G)^{-1} F \\
&\quad - F^\top \left(\frac{1}{2} S + \frac{1}{2} S^* - S G S^* \right) F \\
&= -Q - \frac{\Re(\bar{\lambda}_k)}{|\bar{\lambda}_k|^2} F^\top S^* S F \leq -Q. \tag{23}
\end{aligned}$$

\triangleleft

Our next result proves that the trajectories of (3) under (5) converge to consensus. Moreover, the consensus trajectory can be characterized in terms of the dynamic matrix A of the agents and the initial conditions of the root nodes only.

Theorem 3.2 *Under the hypotheses of Theorem 3.1, the feedback (5b) ensures consensus of the network (9); namely, one gets*

$$x_i(t) \rightarrow A^t x_s(0) \tag{24}$$

where $x_s(0) = (\nu_1^\top \otimes I_n) \text{col}_{i=1}^N(x_i(0))$, ν_1^\top is the left eigenvector associated to the zero eigenvalue of \mathcal{L} normalized to $\sum_{k=1}^N \nu_{1,k} = 1$ and $x_s(0)$ depends only on the initial conditions of the root nodes of \mathcal{G} .

Proof: Let, for $k = 1, \dots, N$, $j = 1, \dots, n$, $\tilde{w}_{j,k}^\top$, $w_{j,k}$ be the left and right eigenvectors, respectively, of \bar{A}_k defined in (12), normalized so that $\tilde{w}_{j,k}^\top w_{j,k} = 1$, and let $\mu_{j,k}$ be the relative eigenvalues. Let moreover ν_k^\top , u_k be the left and right eigenvectors, respectively, of $\bar{\mathcal{L}}$, normalized so that $\nu_k^\top z_k = 1$, and let $\bar{\lambda}_k$ be the relative eigenvalues. Since the hypotheses of Theorem 3.1 are satisfied, $|\mu_{j,k}| < 1$ for $k > 1$, thus the asymptotic behavior depends only on the n eigenvalues $\mu_{j,1}$ of A . That is,

denoting for concision $w_{j,1} = w_j$, $\tilde{w}_{j,1}^\top = \tilde{w}_j$, $\mu_{j,1} = \mu_j$

$$\begin{aligned} \mathbf{x}(t) &= \sum_{k=1}^N \sum_{j=1}^n \mu_{j,k}^t (z_k \otimes w_{j,k}) (\nu_k^\top \otimes \tilde{w}_{j,k}^\top) \mathbf{x}(0) \\ &\rightarrow \sum_{j=1}^n \mu_j^t (\mathbf{1}_N \otimes w_j) (\nu_1^\top \otimes \tilde{w}_j^\top) \mathbf{x}(0), \end{aligned} \quad (25)$$

where we have used $u_1 = \mathbf{1}_N$. Recalling that $A = \sum_{j=1}^n \mu_j w_j \tilde{w}_j^\top$,

$$\begin{aligned} \mathbf{x}(t) &\rightarrow \mathbf{1}_N \otimes \sum_{j=1}^n \mu_j^t (\nu_1^\top \otimes (w_j \tilde{w}_j^\top)) \mathbf{x}(0), \\ &= \mathbf{1}_N \otimes \sum_{j=1}^n \mu_j^t (w_j \tilde{w}_j^\top) (\nu_1^\top \otimes I_n) \mathbf{x}(0), \\ &= \mathbf{1}_N \otimes (A^t (\nu_1^\top \otimes I_n) \mathbf{x}(0)). \end{aligned} \quad (26)$$

Finally, since ν_1^\top has non-zero entries only on the root nodes (see Proposition 3 in (Monaco and Ricciardi Celsi, 2019)), the value $\mathbf{x}_s(0) = (\nu_1^\top \otimes I_n) \mathbf{x}(0)$ depends only on the initial conditions of the root nodes. \triangleleft

Remark 3.1 Recalling that $\Re(\lambda_k) > 0$ for $\lambda_k \neq 0$ (Proposition 2.1) it is clear that condition (21) can always be satisfied by suitably fixing the coupling gains κ_i . The advantage of the control law (5) is that it works for any 'sufficiently large' κ_i , a result that makes the discrete-time case entirely analogue to the continuous-time one (see for example (Zhang et al., 2011, Theorem 1)). This significantly simplifies the design since the κ_i are not necessarily identical for all the agents.

Remark 3.2 A further advantage is that the consensus can be reached arbitrarily fast (that is, the disagreement converges to 0 with any desired rate) by suitably choosing $Q \succ 0$ in (23). In fact, assume for simplicity that $\sigma(\bar{\mathcal{L}}) \subset \mathbb{R}$. By Theorem 3.1 and 3.2, one can define the coordinate transformation $\mathbf{x} \mapsto \text{col}(x_s, \varepsilon_1, \dots, \varepsilon_{N-1})$ with

$$x_s = \nu_1^\top \mathbf{x}, \quad \varepsilon_{k-1} = \nu_k^\top \mathbf{x}, \quad k = 2, \dots, N$$

with ν_k^\top and z_k left and right eigenvectors of $\bar{\mathcal{L}}$, normalized so that $\nu_k^\top z_k = 1$. The corresponding dynamics is

$$x_s^+ = A x_s \quad (27a)$$

$$\varepsilon_{k-1}^+ = \bar{A}_k \varepsilon_{k-1} \quad (27b)$$

with \bar{A}_k as in (12) Schur stable for all $k = 2, \dots, N$ by Theorem 3.1. In these coordinates (27a) specifies the consensus dynamics and (27b) is uniquely associated to the disagreement $\mathbf{e} = \mathbf{x} - \mathbf{1}_N \otimes x_s$ that can be represented as $\mathbf{e} = \text{col}_{k=2}^N (z_k) \varepsilon_{k-1}$ so that $\mathbf{e} \rightarrow 0$ if and only if $\varepsilon_{k-1} \rightarrow 0$. With this in mind, by Theorem 3.1

Approximate γ steps implementation of (5) at node i .

- 1: At each time $t \geq 0$, send $x_i(t)$ to the neighbors.
- 2: Receive $x_j(t)$ from the neighbors, $j \in \mathcal{N}_i$, and compute, with $d_i = |\mathcal{N}_i|$,

$$v_i(t, 0) = -\kappa_i (I_m + \kappa_i d_i G)^{-1} \sum_{j \in \mathcal{N}_i} F(x_i - x_j) \quad (28)$$

- 3: For $h = 0, \dots, \gamma - 1$ do:
 - 3.1: Send $v_i(t, h)$ to the neighbors
 - 3.2: Compute

$$\begin{aligned} v_i(t, h+1) &= v_i(t, 0) \\ &\quad + \kappa_i (I_m + \kappa_i d_i G)^{-1} \sum_{j \in \mathcal{N}_i} G v_j(t, h) \end{aligned} \quad (29)$$

- 4: Set $u_i(t) = v_i(t, \gamma)$.

Fig. 1. Distributed approximate implementation of (5)

each component of (27b) is asymptotically stable with Lyapunov function $V(\varepsilon_{k-1}) = \varepsilon_{k-1}^\top P \varepsilon_{k-1}$ that satisfies $\Delta V(\varepsilon_{k-1}) \leq -\varepsilon_{k-1}^\top Q \varepsilon_{k-1}$. This implies that all the ε_{k-1} and the disagreement \mathbf{e} converge to zero with rate that is lower bounded by q , the smallest eigenvalue of $Q \succ 0$. Thus, the lower bound of the convergence rate is regulated by Q and specifically by its smallest eigenvalue: convergence to consensus is as fast as q is large.

3.2 An approximate distributed implementation

The control law in (5) requires to know the input of the neighbors in order to compute the input of each node. The goal of this section is to define a distributed dynamic implementation of the control law (5). As the intuition suggests, such problem is not trivial. Moreover, one must take into account that, contrarily to the centralized implementation (Remark 3.1), arbitrary convergence to consensus cannot be obtained via a distributed implementation due to the intrinsic limits imposed by the discrete-time communication in the information spread.

One possible way to overcome this conceptual limitation is to separate the time-scale of the information exchange and of the system evolution by resorting to a multi-rate controller. The idea of multiple information exchanges per time unit is not new and it has been used extensively in the field of distributed filtering (Battistelli et al., 2014; Kamal et al., 2013). In the present context this intuition can be used to implement the u_i computed by solving the coupling rule defined by (5b). In fact, (5b) with $d_i = |\mathcal{N}_i|$ yields

$$u_i = -\kappa_i (I_m + \kappa_i d_i G)^{-1} \sum_{j \in \mathcal{N}_i} (F(x_i - x_j) - G u_j). \quad (30)$$

which can be approximated making use of the algorithm in Fig. 1 by initially using the available term that depends on $x_i - x_j$ and by iterating the evaluation of u_j for γ steps. The algorithm converges to $\mathbf{u}(t)$ as $\gamma \rightarrow \infty$, as proved by the following result.

Theorem 3.3 *If the graph \mathcal{G} is weakly connected, then at each node i and time t the sequence $v_i(t, \gamma)$ generated by (29) is such that, for all $\kappa_i > 0$, $i = 1, \dots, N$,*

$$\lim_{\gamma \rightarrow \infty} v_i(t, \gamma) = u_i(t), \quad (31)$$

where $u_i(t)$ is given by (5).

Proof: We recall that $\mathbf{u} = \text{col}_{i=1}^N(u_i)$ is given by (8). Let

$$K = \text{diag}_{i=1}^N(\kappa_i) \quad (32)$$

$$\tilde{D} = (I_{mN} + KD \otimes G)^{-1} = \text{diag}_{i=1}^N(I_m + \kappa_i d_i G)^{-1} \quad (33)$$

$$\tilde{G} = \tilde{D}(KA \otimes G). \quad (34)$$

Notice that, since $\bar{\mathcal{L}} = KD - KA$,

$$\begin{aligned} (I_{mN} + \bar{\mathcal{L}} \otimes G) &= I_{mN} + KD \otimes G - KA \otimes G \\ &= \tilde{D}^{-1} \left(I_{mN} - \tilde{D}(KA \otimes G) \right) \\ &= \tilde{D}^{-1} \left(I_{mN} - \tilde{G} \right). \end{aligned} \quad (35)$$

By stacking $\mathbf{v}(t, 0) = \text{col}_{i=1}^N(v_i(t, 0))$ and $\mathbf{v}(t, \gamma) = \text{col}_{i=1}^N(v_i(t, \gamma))$ we obtain

$$\mathbf{v}(t, 0) = -\tilde{D}(\bar{\mathcal{L}} \otimes F)\mathbf{x}(t) \quad (36)$$

$$\begin{aligned} \mathbf{v}(t, \gamma + 1) &= \mathbf{v}_0(t) + \tilde{D}(KA \otimes G)\mathbf{v}_\gamma(t) \\ &= \mathbf{v}(t, 0) + \tilde{G}\mathbf{v}(t, \gamma) = \left(\sum_{\ell=0}^{\gamma+1} \tilde{G}^\ell \right) \mathbf{v}(t, 0). \end{aligned} \quad (37)$$

Since $\rho(\tilde{G}) < 1$ (Lemma 6.1 in Appendix) and by (35)

$$\begin{aligned} \lim_{\gamma \rightarrow \infty} \mathbf{v}(t, \gamma) &= \left(I_{mN} - \tilde{G} \right)^{-1} \mathbf{v}(t, 0) \\ &= - \left(I_{mN} - \tilde{G} \right)^{-1} \tilde{D}(\bar{\mathcal{L}} \otimes F)\mathbf{x}(t) \\ &= \left(I_{mN} + (\bar{\mathcal{L}} \otimes G) \right)^{-1} (\bar{\mathcal{L}} \otimes F)\mathbf{x}(t) \\ &= \mathbf{u}(t). \end{aligned} \quad (38)$$

◁

Corollary 3.1 *The dynamics of (3) under $u_i(t) =$*

$v_i(t, \gamma)$ is given by

$$\mathbf{x}(t + 1) = \Theta_\gamma \mathbf{x}(t) \quad (39)$$

$$\Theta_\gamma = (I_N \otimes A) - (I_N \otimes B) \left(\sum_{h=0}^{\gamma} \tilde{G}^h \right) \tilde{D}(\bar{\mathcal{L}} \otimes F), \quad (40)$$

where \tilde{D} and \tilde{G} are defined in (33) and (34).

In practice however one needs a lower bound for the number of consensus iterations needed to reach consensus. To this end, the following result is instrumental to further represent Θ and Θ_γ in (10) and (40).

Lemma 3.3 *Let Θ and Θ_γ be as in (10) and (40) respectively. Then, with $M = BB^\top P$,*

$$\Theta = \left(I_{nN} + (\bar{\mathcal{L}} \otimes M) \right)^{-1} (I_N \otimes A) \quad (41a)$$

$$\Theta_\gamma = \left(I_{nN} + \tilde{G}_s^{\gamma+1} (\bar{\mathcal{L}} \otimes M) \right) \Theta, \quad (41b)$$

where

$$\tilde{G}_s = \tilde{D}_s(KA \otimes M) \quad (42)$$

$$\tilde{D}_s = (I_{nN} + KD \otimes M)^{-1}. \quad (43)$$

The proof is reported in Appendix 6.1. We can now state a result about the minimum number of consensus iterations that yields consensus with the distributed implementation.

Theorem 3.4 *Assume that the agents (1) exchange information on a weakly connected digraph \mathcal{G} under the hypotheses of Theorem 3.2. Let the control input be computed as in Fig. 1, with F and G in (20a)–(20b). Then the following holds:*

- (i) *the matrix $\bar{A} = (I_n + M)^{-1}A$ is Schur stable with $M = BB^\top P$;*
- (ii) *the network (9) converges to a consensus provided that γ satisfies the bound below*

$$\left(\frac{\max_i \{\kappa_i d_i\} \rho(G)}{1 + \max_i \{\kappa_i d_i\} \rho(G)} \right)^{\gamma+1} < \frac{1 - \rho(\bar{A})}{\|T_n^{-1} A T_n\|_\infty} \quad (44)$$

with $T_n \in \mathbb{R}^{n \times n}$ such that $T_n^{-1} M T_n = \Lambda_M = \text{diag}_{j=1}^n(\lambda_j^M)$ for $\lambda_j^M \in \sigma(M)$.

Remark 3.3 *Since $(I_n + M)^{-1}A$ and G are known to each agent, the only non-local parameter needed to compute the bound (44) is $\max_i \{\kappa_i d_i\}$.*

Proof: From the Riccati equation (19) it follows that when the pair (A, B) is controllable $A - B(I_m + G)^{-1}F$ is Schur. (i) follows from the matrix inversion lemma,

$$A - B(I_m + G)^{-1}F = (I_n - B(I_m + B^\top P B)^{-1}B^\top P)A = (I_n + M)^{-1}A = \bar{A}.$$

As for the point (ii), we re-write Θ_γ by using Lemma 3.3 as

$$\begin{aligned} \Theta_\gamma &= \Theta + \tilde{G}_s^{\gamma+1}(\bar{\mathcal{L}} \otimes M)\Theta \\ &= \Theta + \tilde{G}_s^{\gamma+1}(\bar{\mathcal{L}} \otimes M)(I_{nN} + \bar{\mathcal{L}} \otimes M)^{-1}(I_N \otimes A). \end{aligned}$$

With the change of coordinates $T = I_N \otimes T_n$ we obtain

$$\begin{aligned} T^{-1}\Theta_\gamma T &= T^{-1}\Theta T + \tilde{G}_T^{\gamma+1}(\bar{\mathcal{L}} \otimes \Lambda_M)\tilde{\mathcal{L}}_T(I_N \otimes T_n^{-1}AT_n) \\ \tilde{G}_T &= T^{-1}\tilde{G}_s T = (I_{nN} + KD \otimes \Lambda_M)^{-1}(KA \otimes \Lambda_M), \end{aligned}$$

where $\tilde{\mathcal{L}}_T = (I_{nN} + \bar{\mathcal{L}} \otimes \Lambda_M)^{-1}$. M is positive semi-definite with $\rho(M) = \rho(G)$, thus \tilde{G}_T is non-negative. Moreover, \tilde{G}_T is Schur, and it is immediate to check that

$$\|\tilde{G}_T\|_\infty = \max_i \sum_j |(\tilde{G}_T)_{ij}| \leq \frac{\max_i \{\kappa_i d_i\} \rho(G)}{1 + \max_i \{\kappa_i d_i\} \rho(G)} < 1. \quad (45)$$

It is easy to check that

$$(\bar{\mathcal{L}} \otimes \Lambda_M)\tilde{\mathcal{L}}_T = I_{nN} - \tilde{\mathcal{L}}_T,$$

and therefore

$$T^{-1}\Theta_\gamma T = T^{-1}\Theta T + \tilde{G}_T^{\gamma+1}(I_{nN} - \tilde{\mathcal{L}}_T)(I_N \otimes T_n^{-1}AT_n).$$

Letting $\bar{\mathcal{L}}z_k = \bar{\lambda}_k z_k$ for $\bar{\lambda}_k \neq 0$, then for all $\xi \in \mathbb{R}^n$,

$$\begin{aligned} (I_{nN} - \tilde{\mathcal{L}}_T)(z_k \otimes \xi) &= z_k \otimes \text{diag}_{j=1}^n \left(\frac{\lambda_k \lambda_j^M}{1 + \lambda_k \lambda_j^M} \right) \xi \\ &= z_k \otimes D_{kj} \xi, \end{aligned}$$

where $\lambda_k \lambda_j^M \geq 0$. Clearly, D_{kj} is Schur and $\|D_{kj}\|_\infty < 1$. Let, moreover, $\bar{A}_k w_{jk} = \mu_{jk} w_{jk}$ and $\tilde{w}_{jk} = T_n^{-1}w_{jk}$, where \bar{A}_k is defined in (12). From (13) of Lemma 3.1 follows that

$$\begin{aligned} (T^{-1}\Theta_\gamma T)(z_k \otimes \tilde{w}_{jk}) &= (T^{-1}\Theta T)(z_k \otimes \tilde{w}_{jk}) \\ &\quad + \tilde{G}_T^{\gamma+1}(I_{nN} - \tilde{\mathcal{L}}_T)(I_N \otimes T_n^{-1}AT_n)(z_k \otimes \tilde{w}_{jk}) \\ &= \mu_{jk}(z_k \otimes \tilde{w}_{jk}) + \tilde{G}_T^{\gamma+1}(z_k \otimes D_{kj}T_n^{-1}AT_n \tilde{w}_{jk}), \end{aligned}$$

and taking norms yields

$$\begin{aligned} \|(T^{-1}\Theta_\gamma T)(z_k \otimes \tilde{w}_{jk})\|_\infty &\leq |\mu_{jk}| \|z_k \otimes \tilde{w}_{jk}\|_\infty \\ &\quad + \|\tilde{G}_T\|_\infty^{\gamma+1} \|T_n^{-1}AT_n\|_\infty \|z_k \otimes \tilde{w}_{jk}\|_\infty. \end{aligned} \quad (46)$$

At this point, since $\bar{A} = (I_n + M)^{-1}A$ and $\bar{A}_k = (I_n + \bar{\lambda}_k M)^{-1}A$ and $\Re(\lambda_k) > 1$, one gets $\rho(\bar{A}) > \rho(\bar{A}_k)$ and (44) implies

$$\|\tilde{G}_T\|_\infty^{\gamma+1} = \left(\frac{\max_i \{\kappa_i d_i\} \rho(G)}{1 + \max_i \{\kappa_i d_i\} \rho(G)} \right)^{\gamma+1} < \frac{1 - \rho(\bar{A}_k)}{\|T_n^{-1}AT_n\|_\infty}. \quad (47)$$

Thus, it follows that $(T^{-1}\Theta_\gamma T)$ is a contraction on the subspace orthogonal to $z_1 \otimes \mathbb{R}^n$, that is

$$\|(T^{-1}\Theta_\gamma T)(z_k \otimes \tilde{w}_{jk})\|_\infty \leq \|z_k \otimes \tilde{w}_{jk}\|_\infty. \quad (48)$$

Since for all $w \in \mathbb{R}^n$ one gets

$$(T^{-1}\Theta_\gamma T)(z_1 \otimes w) = z_1 \otimes Aw, \quad (49)$$

thus yielding $x_i(t) \rightarrow A^t x_s(0)$ when (47) holds. \triangleleft

Remark 3.4 We stress that the result above proves the existence of a certain γ^* , computable in a distributed way, such that for all $\gamma \geq \gamma^*$ consensus is achieved over the network under the distributed control law depicted in Figure 1. As it is clear, under stronger demands and depending on the case at hand, an exact or less conservative value of the parameter can be established as we will show later on through a simple example.

Remark 3.5 For the centralized implementation convergence toward the consensus trajectory is faster as κ increases, as one can easily deduce from (21) and the analysis developed in Theorem 3.1. As far as the decentralized implementation is concerned, as the intuition suggests, for a fixed κ satisfying (21) the convergence rate is affected by γ , the number of required inter-consensus steps at each $t \in \mathbb{N}$ for approximating (30). In that case, for a fixed κ , the convergence rate increases as γ does.

4 Multi-consensus control for weakly connected digraphs

4.1 Preliminaries on Graph Partitions

The set of nodes reachable from a node $i \in \mathcal{V}$ is the *reachable set* $R(i) := \{i\} \cup \{j \in \mathcal{V} \text{ s.t. } i \rightsquigarrow j\}$.

Definition 4.1 A set $\mathcal{R} \subseteq \mathcal{V}$ is called a *reach* if it is a maximal reachable set, that is, $\mathcal{R} = R(i)$ for some $i \in \mathcal{V}$ and there is no $j \in \mathcal{V}$ such that $R(i) \subset R(j)$.

Since \mathcal{G} possesses a finite number of vertices, such maximal sets exist and are uniquely determined by the graph itself. We denote by \mathcal{R}_k for $k = 1, \dots, \mu$, the reaches of \mathcal{G} . As proved in (Agaev and Chebotarev, 2005; Caughman and Veerman, 2006), the eigenvalue $\lambda_1 = 0$ of the Laplacian matrix has algebraic and geometric multiplicities equal to μ , the number of reaches of \mathcal{G} . We further

partition the reaches \mathcal{R}_k in their *exclusive part*, defined as $\mathcal{H}_k = \mathcal{R}_k \setminus \cup_{\ell=1, \ell \neq k}^{\mu} \mathcal{R}_\ell$, having cardinality $h_k = |\mathcal{H}_k|$. Finally, the *common part* of \mathcal{G} is given by $\mathcal{C} = \mathcal{V} \setminus \cup_{k=1}^{\mu} \mathcal{H}_k$ and it has cardinality $c = |\mathcal{C}|$. Hence, after a suitable re-labeling of nodes, the Laplacian admits the lower triangular form (Caughman and Veerman, 2006)

$$\mathcal{L} = \begin{pmatrix} \text{diag}_{k=1}^{\mu}(\mathcal{L}_k) & 0 \\ \text{raw}_{k=1}^{\mu}(\mathcal{M}_k) & \mathcal{M} \end{pmatrix} \quad (50)$$

and, equivalently for the modified Laplacian,

$$\bar{\mathcal{L}} = \text{diag}_{i=1}^N(k_i)\mathcal{L} = \begin{pmatrix} \text{diag}_{k=1}^{\mu}(\bar{\mathcal{L}}_k) & 0 \\ \text{raw}_{k=1}^{\mu}(\bar{\mathcal{M}}_k) & \bar{\mathcal{M}} \end{pmatrix} \quad (51)$$

with $\bar{\mathcal{L}}_k = \kappa_k \mathcal{L}_k$, $\bar{\mathcal{M}}_k = \kappa_{\mathcal{C}} \mathcal{M}_k$, $\bar{\mathcal{M}} = \kappa_{\mathcal{C}} \mathcal{M} \kappa_k = \text{diag}_{j=1}^{h_k}(\kappa_{h_1+\dots+h_{k-1}+j})$, $\kappa_{\mathcal{C}} = \text{diag}_{j=1}^c(\kappa_{h_1+\dots+h_{\mu}+j})$ and where:

- each \mathcal{H}_k is a weakly connected sub-graph with one reach;
- $\mathcal{L}_k \in \mathbb{R}^{h_k \times h_k}$ and $\bar{\mathcal{L}}_k \in \mathbb{R}^{h_k \times h_k}$ ($k = 1, \dots, \mu$) are the Laplacians associated to the subgraph \mathcal{H}_k with $0 \in \sigma(\mathcal{L}_k)$ simple;
- $\bar{\mathcal{M}} \in \mathbb{R}^{c \times c}$ satisfies $\sigma(\bar{\mathcal{M}}) \subset \mathbb{C}^+$ and it corresponds to the common component \mathcal{C} ;
- the eigenspace associated to $\lambda_1 = 0$ for $\bar{\mathcal{L}}$ are spanned by the right eigenvectors $\{z_{1,1} \dots z_{\mu,1}\}$ with

$$z_{1,1} = \begin{pmatrix} \mathbb{1}_{h_1} \\ \vdots \\ 0 \\ \gamma^1 \end{pmatrix} \quad \dots \quad z_{1,\mu} = \begin{pmatrix} 0 \\ \vdots \\ \mathbb{1}_{h_\mu} \\ \gamma^\mu \end{pmatrix} \quad (52)$$

$\sum_{k=1}^{\mu} \gamma^k = \mathbb{1}_c$ and $\bar{\mathcal{M}}_k \mathbb{1}_{h_k} + \bar{\mathcal{M}} \gamma^k = 0$ for all $k = 1, \dots, \mu$;

- the left eigenvectors associated to $\lambda_1 = 0$ for $\bar{\mathcal{L}}$ are $\{\tilde{\nu}_1^\top \dots \tilde{\nu}_\mu^\top\}$,

$$\tilde{\nu}_{1,1}^\top = (\nu_{1,1}^\top \dots 0 \ 0) \dots \tilde{\nu}_{1,\mu}^\top = (0 \dots \nu_{1,\mu}^\top \ 0) \quad (53)$$

with $\nu_{1,k}^\top = (\nu_{1,k}^1 \dots \nu_{1,k}^{h_k}) \in \mathbb{R}^{1 \times h_k}$, each entry of $\nu_{1,k}$ positive if the corresponding node is a root node of \mathcal{H}_k and zero otherwise.

The following definitions are then instrumental.

- A partition (or, as an alternative, a cluster) $\pi = \{\rho_1, \dots, \rho_r\}$ of \mathcal{V} is a collection of cells $\rho_k \subseteq \mathcal{V}$ verifying $\rho_k \cap \rho_j = \emptyset$ for all $i \neq j$ and $\cup_{k=1}^r \rho_k = \mathcal{V}$.

- Given two partitions π_1 and π_2 , π_1 is said to be finer than π_2 ($\pi_1 \preceq \pi_2$) if all cells of π_1 are a subset of some cell of π_2 ; equivalently, we say that π_2 is coarser than π_1 ($\pi_2 \succeq \pi_1$).
- We name $\pi = \mathcal{V}$ the trivial partition as composed of a unique cell with all nodes.
- A partition $\pi = \{\rho_1, \dots, \rho_r\}$ of \mathcal{V} is said to be an *almost equitable partition* (AEP, in short) if each node of ρ_k has the same number of neighbors in ρ_ℓ , for all $i, \ell \in \{1, \dots, r\}$ with $i \neq \ell$. The precise definition can be found in (Cardoso et al., 2007; Monshizadeh et al., 2015; Cacace et al., 2021).
- A non trivial partition π^* is the coarsest AEP of \mathcal{G} if for all non trivial π AEP of \mathcal{G} then $\pi^* \succeq \pi$.

Assume now a graph with Laplacian (50). In (Monaco and Ricciardi Celsi, 2019) and for the case of continuous-time scalar integrators, it was proved that nodes split into several multi-consensus according to the partition given by

$$\pi^* = \{\mathcal{H}_1, \dots, \mathcal{H}_\mu, \mathcal{C}_{\mu+1}, \dots, \mathcal{C}_{\mu+p}\} \quad (54)$$

with:

- $\mathcal{C} = \cup_{\ell=1}^p \mathcal{C}_{\mu+\ell}$.
- the nodes in \mathcal{C} belong to the same $\mathcal{C}_{\mu+\ell}$ if and only if the corresponding entries of the vectors $\gamma^k \in \mathbb{R}^c$ of (52) are identical for $k = 1, \dots, \mu$.

Accordingly, with no loss of generality, nodes can be re-labeled and sorted according to the cell of π^* they belong to. By such sorting, the γ^k in (52) get the form, $k = 1, \dots, \mu$,

$$\gamma^k = \begin{pmatrix} \gamma_1^k \mathbb{1}_{c_1} \\ \vdots \\ \gamma_p^k \mathbb{1}_{c_p} \end{pmatrix}, \quad (55)$$

with $c_\ell = |\mathcal{C}_{\mu+\ell}|$, $\gamma_\ell^k \in \mathbb{R}$, $\ell = 1, \dots, p$, and $\sum_{k=1}^{\mu} \gamma_\ell^k = 1$.

Remark 4.1 *It is worth to note that multi-consensus control extends the problem of multi-leader following allowing, within the same cluster, more than one leader. As a matter of fact a graph with μ distinct leader is a graph with μ exclusive reaches, each containing one leader only. Accordingly, the corresponding Laplacian admits the form (50) with $\mathcal{L}_k = 0$, for $k = 1, \dots, \mu$, and $\mathcal{H}_k = \{k\}$ with $k \in \mathcal{V}$ being a leader of the graph.*

Example 4.1 *In the digraph of Fig.2 there are two reaches \mathcal{R}_1 and \mathcal{R}_2 that correspond to the two maximal sets of reachable nodes $\mathcal{R}(1) = \mathcal{R}(2)$ and $\mathcal{R}(4) = \mathcal{R}(5)$ ¹.*

¹ The node reaches $\mathcal{R}(3)$, $\mathcal{R}(6)$, $\mathcal{R}(7)$ and $\mathcal{R}(8)$ are not graph reaches because they are contained in \mathcal{R}_1 and \mathcal{R}_2 .

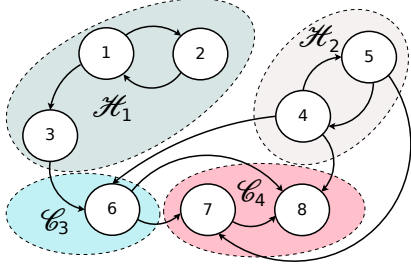


Fig. 2. A weakly connected digraph and its coarsest AEP. $\mathcal{R}_1 = \mathcal{R}(1) = \mathcal{R}(2) = \{1, 2, 3, 6, 7, 8\}$ and $\mathcal{R}_2 = \mathcal{R}(4) = \mathcal{R}(5) = \{4, 5, 6, 7, 8\}$ are the maximal sets of reachable nodes and they thus correspond to the two reaches of the graph. Their exclusive parts are $\mathcal{H}_1 = \{1, 2, 3\}$ and $\mathcal{H}_2 = \{4, 5\}$, while the common part $\mathcal{C} = \{6, 7, 8\}$ contains two cells, $\mathcal{C} = \mathcal{C}_3 \cup \mathcal{C}_4$.

Thus, $\mu = 2$ is the multiplicity of 0 as an eigenvalue of the Laplacian. The exclusive part of \mathcal{R}_1 is $\mathcal{H}_1 = \{1, 2, 3\}$ and its root nodes are $\mathcal{P}_1 = \{1, 2\}$. The exclusive part of \mathcal{R}_2 is $\mathcal{H}_2 = \{4, 5\}$ and its root nodes are $\mathcal{P}_2 = \{4, 5\}$. Notice that the sub-graphs associated to \mathcal{P}_1 and \mathcal{P}_2 are strongly connected. The common part of the reaches is $\mathcal{C} = \overline{\mathcal{R}}_1 = \overline{\mathcal{R}}_2 = \{6, 7, 8\}$ which is composed by the two cells \mathcal{C}_3 and \mathcal{C}_4 of the AEP. Nodes in \mathcal{C}_3 have one neighbor in \mathcal{H}_1 and one neighbor in \mathcal{H}_2 , whereas nodes in \mathcal{C}_4 have one neighbor in \mathcal{C}_3 and one neighbor in \mathcal{H}_2 .

4.2 Multiconsensus analysis and control

With reference to π^* in (54) with $c_\ell = |\mathcal{C}_{\mu+\ell}|$ for $\ell = 1, \dots, p$, $k = 1, \dots, \mu$ and, for simplicity, $h_0 = 0$, we denote the agglomerate vectors

$$\begin{aligned} \mathbf{x}_k &= \text{col}_{j=1}^{h_k} (x_{h_1+\dots+h_{k-1}+j}) \in \mathbb{R}^{h_k n} \\ \mathbf{x}_{\mu+\ell} &= \text{col}_{j=1}^{c_\ell} (x_{N-c_\ell-\dots-c_p+j}) \in \mathbb{R}^{c_\ell n} \\ \mathbf{x}_C &= \text{col}_{k=1}^{\mu+p} (\mathbf{x}_{\mu+\ell}) \in \mathbb{R}^{(N-\mu)n} \\ \mathbf{x} &= \text{col}_{k=1}^{\mu+p} (\mathbf{x}_k) \in \mathbb{R}^{Nn} \\ \mathbf{u}_k &= \text{col}_{j=1}^{h_k} (u_{h_1+\dots+h_{k-1}+j}) \in \mathbb{R}^{h_k m} \\ \mathbf{u}_{\mu+\ell} &= \text{col}_{j=1}^{c_\ell} (u_{N-c_\ell-\dots-c_p+j}) \in \mathbb{R}^{c_\ell m} \\ \mathbf{u}_C &= \text{col}_{k=1}^{\mu+p} (\mathbf{u}_{\mu+\ell}) \in \mathbb{R}^{(N-\mu)m} \\ \mathbf{u} &= \text{col}_{k=1}^{\mu+p} (\mathbf{u}_k) \in \mathbb{R}^{Nm}. \end{aligned}$$

The multi-consensus problem can be then specified requiring to design, if any, a local control law of the form (5) making all agents of the same cell of (54) asymptotically converge to the same trajectory; namely, as $t \rightarrow \infty$

$$\mathbf{x}_k(t) \rightarrow \mathbf{1}_{h_k} \otimes x_{s,i}(t), \quad k = 1, \dots, \mu \quad (56a)$$

$$\mathbf{x}_{\mu+\ell}(t) \rightarrow \mathbf{1}_{c_\ell} \otimes x_{s,\mu+\ell}(t), \quad \ell = 1, \dots, p \quad (56b)$$

for the multiconsensus trajectories $x_{s,i} : \mathbb{N} \rightarrow \mathbb{R}^n$ and $x_{s,\mu+\ell} : \mathbb{N} \rightarrow \mathbb{R}^n$ as specified in the next result, that extends Theorem 3.2 to the general case.

Theorem 4.1 *Let \mathcal{G} be the communication digraph of a network of N agents of the form (1) under the feedback solution to (5). Let \mathcal{L} and (54) be the Laplacian and a partition of \mathcal{G} . Let the control input be computed as in (5), with κ_i , F and G in (20a)–(20b), and P solution of (19) for an arbitrary $Q = Q^\top > 0$. Then, the multiconsensus problem is solved if all κ_i satisfy (21); i.e., for all $k = 1, \dots, \mu$ and $\ell = 1, \dots, p$*

$$\mathbf{x}_k(t) \rightarrow (\mathbf{1}_{h_k} \otimes I_n) A^t x_{s,k}(0) \quad (57a)$$

$$\mathbf{x}_{\mu+\ell}(t) \rightarrow (\mathbf{1}_{c_\ell} \otimes I_n) A^t x_{s,\mu+\ell}(0) \quad (57b)$$

with $x_{s,k}(0) = (\nu_{1,k}^\top \otimes I_n) \mathbf{x}_k(0)$ and $x_{s,\mu+\ell}(0) = \sum_{k=1}^{\mu} \gamma_\ell^k x_{s,k}(0)$.

Proof: First, we rewrite (8) by collecting the terms corresponding to nodes within the same exclusive reach and in the common respectively, that is

$$\begin{aligned} \mathbf{u}_k &= - \left(I_{h_k n} + (\bar{\mathcal{L}}_k \otimes G) \right)^{-1} (\bar{\mathcal{L}}_k \otimes F) \mathbf{x}_k \\ \mathbf{u}_C &= - \left(I_{c_\ell n} + (\bar{\mathcal{M}} \otimes G) \right)^{-1} \left(\sum_{k=1}^{\mu} (\bar{\mathcal{M}}_k \otimes F) \mathbf{x}_k \right. \\ &\quad \left. + (\bar{\mathcal{M}} \otimes F) \mathbf{x}_C \right). \end{aligned}$$

The closed-loop network dynamics (3) gets the cascaded form

$$\mathbf{x}_k^+ = \Theta_k \mathbf{x}_k \quad (58a)$$

$$\mathbf{x}_C^+ = \Theta_C \mathbf{x}_C + \sum_{k=1}^{\mu} \Theta_{k,C} \mathbf{x}_k \quad (58b)$$

with $k = 1, \dots, \mu$ and

$$\begin{aligned} \Theta_k &= I_{h_k} \otimes A - (I_{h_k} \otimes B) (I_{h_k n} + (\bar{\mathcal{L}}_k \otimes G))^{-1} (\bar{\mathcal{L}}_k \otimes F) \\ \Theta_C &= I_C \otimes A - (I_C \otimes B) (I_{c_\ell n} + (\bar{\mathcal{M}} \otimes G))^{-1} (\bar{\mathcal{M}} \otimes F) \\ \Theta_{k,C} &= -(I_C \otimes B) (I_{c_\ell n} + (\bar{\mathcal{M}} \otimes G))^{-1} (\bar{\mathcal{M}}_k \otimes F). \end{aligned}$$

By virtue of the properties of (50) and each \mathcal{L}_k with $k = 1, \dots, \mu$, (57a) follows from Theorem 3.2 so that it remains to prove that (57b) holds. Thus, we focus on (58b) only and introduce the disagreement state over the common as

$$\mathbf{e}_C = \mathbf{x}_C - \sum_{i=1}^{\mu} (\gamma^i \otimes I_n) x_{s,i} \quad (59)$$

with $x_{s,k} = (\nu_{1,k}^\top \otimes I_n) \mathbf{x}_k$. Accordingly, since

$$\begin{aligned} x_{s,k}^+ &= (\nu_{1,k}^\top \otimes I_n) \Theta_k \mathbf{x}_k(t) \\ &= (\nu_{1,k}^\top \otimes A) \mathbf{x}_k(t) = A x_{s,k}(t), \end{aligned}$$

the corresponding dynamics reads

$$\begin{aligned} e_c^+ &= \Theta_C e_c(t) + \sum_{k=1}^{\mu} \Theta_{k,C} x_k(t) \\ &+ \left(\sum_{k=1}^{\mu} \Theta_C (\gamma^k \otimes I_n) - (\gamma^k \otimes A) \right) x_{s,k}(t). \end{aligned}$$

Because $x_k(t) \rightarrow (\mathbb{1}_{h_k} \otimes I_n) x_{s,k}(t)$, one gets

$$\begin{aligned} e_c^+ &\rightarrow \Theta_C e_c(t) + \sum_{k=1}^{\mu} \left(\Theta_{k,C} (\mathbb{1}_{h_k} \otimes I_n) \right. \\ &\quad \left. + (\Theta_C - (I_{h_k} \otimes A)) (\gamma^k \otimes I_n) \right) x_{s,k}(t) \\ &= \Theta_C e_c(t) \end{aligned}$$

where we have exploited that $\bar{\mathcal{M}}_k \mathbb{1}_{h_k} + \bar{\mathcal{M}} \gamma^k = 0$; it follows that

$$\begin{aligned} \Theta_{k,C} (\mathbb{1}_{h_k} \otimes I_n) &= (I_C \otimes B) (I_{Cn} + (\bar{\mathcal{M}} \otimes G))^{-1} (\bar{\mathcal{M}} \gamma^k \otimes F) \\ \Theta_C (\gamma^k \otimes I_n) &= \gamma^k \otimes A - \Theta_{k,C} (\mathbb{1}_{h_k} \otimes I_n). \end{aligned}$$

Thus, it is enough to prove that Θ_C is Schur under the choice of the gains κ_i , F and G of the form (20a)–(20b) for P solution of (19). To this purpose, along the lines of Lemma 3.1 and recalling that $\sigma(\bar{\mathcal{M}}) \subset \mathbb{C}^+$, we get that

$$\Theta_C \sim \text{diag} \left((A - B((\bar{\lambda}_j)^{-1} I_m + G)^{-1} F), \quad \bar{\lambda}_j \in \sigma(\bar{\mathcal{M}}) \right).$$

Because all κ_i satisfy (21) and $\Re(\bar{\lambda}_j) > 0$ then $\bar{\lambda}_j > 1$ and Θ_C using the same arguments as the ones in the proof as the one of Theorem 3.1. We can then conclude that $e_c \rightarrow 0$ and, using (55), (57b). \triangleleft

Remark 4.2 *The result in Theorem 3.3 can be extended to this case along the same lines of the previous one.*

Remark 4.3 *The usual discrete-time feedback for guaranteeing consensus is computed as in (5) with $G = 0$ and $F = (I_m + B^T P B)^{-1} B^T P A$ for $P \succ 0$ the solution to the DARE in (19). However, such a feedback guarantees convergence to consensus only if for all $\lambda_k \in \sigma(\mathcal{L}) \setminus \{0\}$*

$$\kappa < \frac{1}{2\lambda_k - \lambda_k^2} \text{ and } \lambda_k \in (0, 2).$$

Accordingly, even if a solution may exist, small gain is in general necessary. In addition, since the admissible gains are spanned by an interval, a good knowledge of the λ_k 's (or an estimate of theirs) is necessary for computing the controller. Several modifications thus of such a bound have been considered; see for instance (Stüdli et al., 2020), holding for $m = 1$ and undirected connected graphs only, or (Ma et al., 2015) for multi-consensus in a

multi-leader perspective only (i.e., when each reach possesses one root).

As in the simple case, the control law enforcing multi-consensus (5) requires a direct throughput resulting implicitly defined. Again, a distributed computation of the corresponding solution is needed. As the intuition suggests, all results in Section 3.2 hold true over each reach \mathcal{H}_k . In addition, Theorem 3.3, Corollary 3.1 and Lemma 3.3 apply as they are to the case of a general digraph \mathcal{G} . The next result extends Theorem 3.4 to the multiconsensus case.

Theorem 4.2 *Assume that the agents (1) exchange information on a digraph \mathcal{G} under the hypotheses of Theorem 4.1. Let the control input be computed as in Fig. 1, with F and G in (20a)–(20b). Then, the multiconsensus problem is solved and (57) holds provided that γ satisfies the bound in (44).*

Proof: The main part of the proof follows exactly the same lines as the one of Theorem 3.4 by considering, here, $z_1 = \text{col}_{k=1}^{\mu} (z_{1,k})$ so getting that, by the bound (47) and (48), $(T^{-1} \Theta_{\gamma} T)$ is a contraction on the subspace orthogonal to $z_1 \otimes \mathbb{R}^n$. But, for any $w \in \mathbb{R}^n$,

$$(T^{-1} \Theta_{\gamma} T)(z_1 \otimes w) = z_1 \otimes Aw. \quad (60)$$

As a consequence, (57) holds when (47) does. Moreover, since $\bar{A} = (I_n + M)^{-1} A$ and $\bar{A}_k = (I_n + \bar{\lambda}_k M)^{-1} A$, in the hypothesis $\Re(\bar{\lambda}_k) > 1$ it is easy to check that $\rho(\bar{A}) > \rho(\bar{A}_k)$ and (44) implies (47). This concludes the proof. \triangleleft

5 Simulations

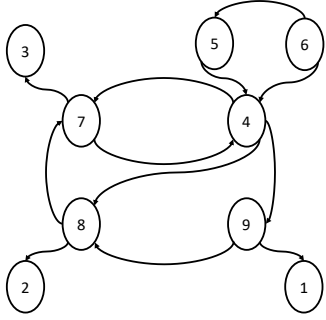
We propose two simulations within the multi-consensus scenario. In the first case, we compare the proposed control law with the one in (Ma et al., 2015) using the same case-study as in (Ma et al., 2015) and within the same working assumptions (i.e., multi-consensus in multi-leader following). Then, we apply the proposed control for solving the formation control problem over a network of continuous-time LTI agents under piecewise constant inputs and sampled-data measurements.

5.1 Multi-leader following

We consider the case of $N = 9$ agents, each of the form (1) with

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad (61)$$

each communicating according to the topology fixed graph depicted in Figure 3 with, in particular, 3 independent leaders and 6 followers yielding for $k = 1, 2, 3$



Norm of the Consensus Error

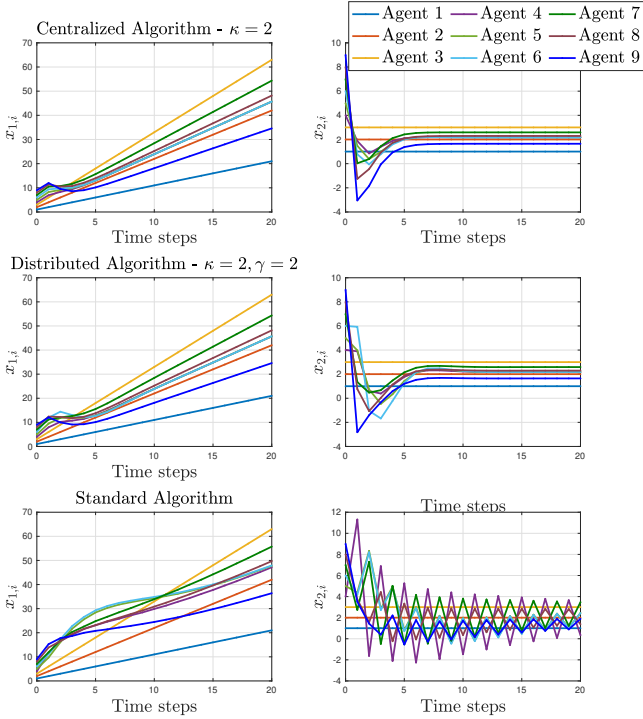
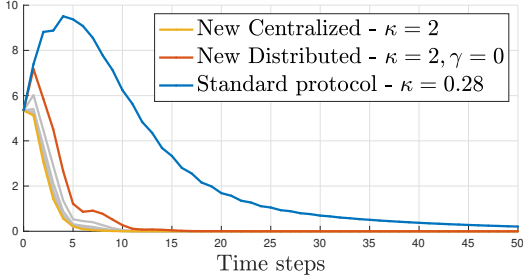


Fig. 3. A network of $N = 9$ identical discrete-time integrators with three leaders. Grey lines indicate the norm of the consensus error for increasing values of γ from γ_{\min} (reddish line) to ∞ (yellow line).

$\mathcal{H}_k = \{k\}$, $\mathcal{C} = \{4, 5, 6, 7, 8, 9\}$ and thus $\mu = 3$; Accordingly, exploiting Theorem 4.1, multi-consensus is dictated by (54) with $p = 4$

$$\mathcal{C}_{\mu+1} = \{4, 5, 6\}, \quad \mathcal{C}_{\mu+2} = \{7\}, \quad \mathcal{C}_{\mu+3} = \{8\} \quad \mathcal{C}_{\mu+4} = \{9\}$$

and is achieved, under the centralized control (5), fixing $Q = I_2$, $G = 4.2973$

$$P = \begin{pmatrix} 2.3671 & 1.1180 \\ 1.1180 & 2.5875 \end{pmatrix}, \quad F = \begin{pmatrix} 2.3016 & 5.4481 \end{pmatrix}$$

as stated by Theorem 4.1 with $\bar{r} = 0.8213$. For the distributed implementation (Theorem 4.2), one gets that the overall network achieves consensus if and only if $\gamma \geq \gamma_{\min} = 0$ steps, as one can analytically prove, in this case. In this particular case, the bound provided by (44) is given by $\gamma^* = 36$. The difference depends on the fact that the bound is obtained through a distributed algorithm, based on local information only.

The results of the simulations are reported in Figure 3 setting the initial condition at $x_i(0) = i\mathbf{1}_2$ and identical gains for all agents ($\kappa = \kappa_i = 2$ for all $i = 1, \dots, 9$).

The centralized implementation of the control law is reported for completeness and it reaches convergence to multi-consensus after around 8 time steps. When fixing $\gamma = \gamma_{\min} = 0$ (reddish line, top-right plot), multi-consensus is achieved in around 16 total time steps. As the number γ of consensus iterations per time step increases, performance improves and tends to the centralized implementation (consensus errors plotted in yellow in the top-right plot). Fig. 3 also reports the results when $\gamma = 2$: the convergence error is close to zero after around 12 time steps corresponding to $12\gamma = 24$ total iterations (at each time step, all agents start an inter-consensus phase of lengths γ -time steps). We highlight that further simulations highlight that performance almost identical to the centralized case are obtained with $\gamma = 12$ and a total convergence time of 144 iterations. In all of those cases, the proposed solutions guarantee better performance than the one proposed in (Ma et al., 2015) that reaches convergence to consensus in around 150 time steps with oscillatory behaviors during the transient (see *Standard Algorithm* in the bottom plots of Fig. 3).

5.2 Formation control under sampling

We consider N continuous-time agents of the form

$$\ddot{\zeta}_i(t) = u_i(t_k), \quad t \in [t_k, t_{k+1}) \subset \mathbb{R} \quad (62)$$

over the digraph \mathcal{G} having one reach only shown in Fig. 4. $\zeta_i = \text{col}(\zeta_{x_i}, \zeta_{y_i}) \in \mathbb{R}^2$ denotes the vector of the Cartesian coordinates of each agent; $u_i = \text{col}(u_{x_i}, u_{y_i}) \in \mathbb{R}^2$ is the vector of the control forces being piece-wise constant signals over the sampling period of length $T > 0$ (i.e., $u(t) = u(t_k)$ for $t \in [t_k, t_{k+1})$ and $t_k = kT$ for $k \in \mathbb{N}$). We solve here the problem of designing a distributed piece-wise control law to force the agents to move with the same velocities and maintain a desired formation specified by the vector $\sigma = \text{col}_{i=1}^N(\sigma_i)$ for $\sigma_i \in \mathbb{R}^2$ and $i = 1, \dots, N$. Specifically, the control law must guarantee

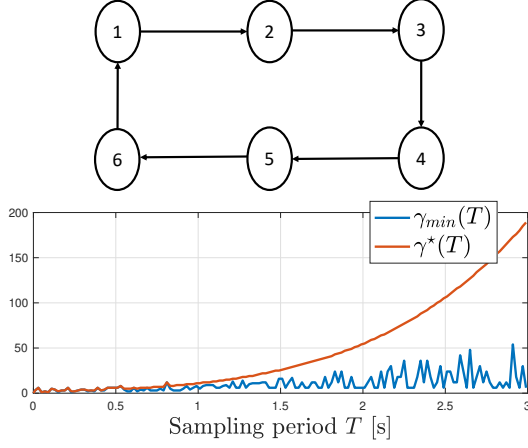


Fig. 4. A network with $\mu = 1$ reach and $N = 6$ continuous-time systems (top). The value of γ_{\min} and its estimate γ^* via Theorem 3.4 as functions of T for $\kappa = 2$ (bottom).

that, for all $i, j = 1, \dots, N$, $\|\zeta_i(t_k) - \sigma_i - \zeta_j(t_k) + \sigma_j\| \rightarrow 0$ as $t_k = kT \rightarrow \infty$ with $k \in \mathbb{N}$. Setting $x_i = \text{col}(\zeta_i - \sigma_i, \dot{\zeta}_i)$ and the state-space representation

$$\dot{x}_i(t) = A_c x_i(t) + B_c u_i(t_k), \quad A_c = \begin{bmatrix} 0 & I_2 \\ 0 & 0 \end{bmatrix}, \quad B_c = \begin{bmatrix} 0 \\ I_2 \end{bmatrix}$$

we introduce the corresponding discrete-time equivalent model of the form (1) with $A = e^{A_c T}$ and $B = \int_0^T e^{A_c s} ds$. Hence we use the result in Theorem 3.4 and the distributed control law in Fig. 1. More in detail, we first partition each sampling interval into γ subintervals, each of length $T_\gamma := \frac{T}{\gamma+1}$; *i.e.*, $[t_k, t_{k+1}) = \cup_{h=0}^{\gamma} [t_k + hT_\gamma, t_k + (h+1)T_\gamma)$. Accordingly, at each sampling instant $t_k = kT$ with $k \in \mathbb{N}$, γ consensus iterations are started to distributedly compute the feedback to apply at $u(t_{k+1})$ as solution to (30); *i.e.*, for $h = 1, \dots, \gamma$, the control is computed as

$$u(t_{k+1}) = v_i(t_k + \gamma T_\gamma), \quad u(t_0) = v_i(t_0) \quad (63)$$

$$v_i(t_k) = -\kappa_i (I_2 + \kappa_i d_i G)^{-1} \sum_{j \in \mathcal{N}_i} F(x_i(t_k) - x_j(t_k))$$

$$v_i(t_k + hT_\gamma) = v_i(t_k) + \kappa_i (I_2 + \kappa_i d_i G)^{-1} \cdot \sum_{j \in \mathcal{N}_i} G v_j(t_k + (h-1)T_\gamma) \quad (64)$$

With this in mind then, because \mathcal{G} contains only one reach, applying the result in Theorem 3.4 each agent asymptotically converges to a single consensus (*i.e.*, $x_i(t_k) - x_j(t_k) \rightarrow 0$ as $t_k \rightarrow \infty$) for all sampling periods $T \geq 0$. As a consequence, the agents converge to the desired formation with identical velocities, *i.e.*, $\|\zeta_i(t_k) - \sigma_i - \zeta_j(t_k) + \sigma_j\| \rightarrow 0$ and $\|\dot{\zeta}_i(t_k) - \dot{\zeta}_j(t_k)\| \rightarrow 0$.

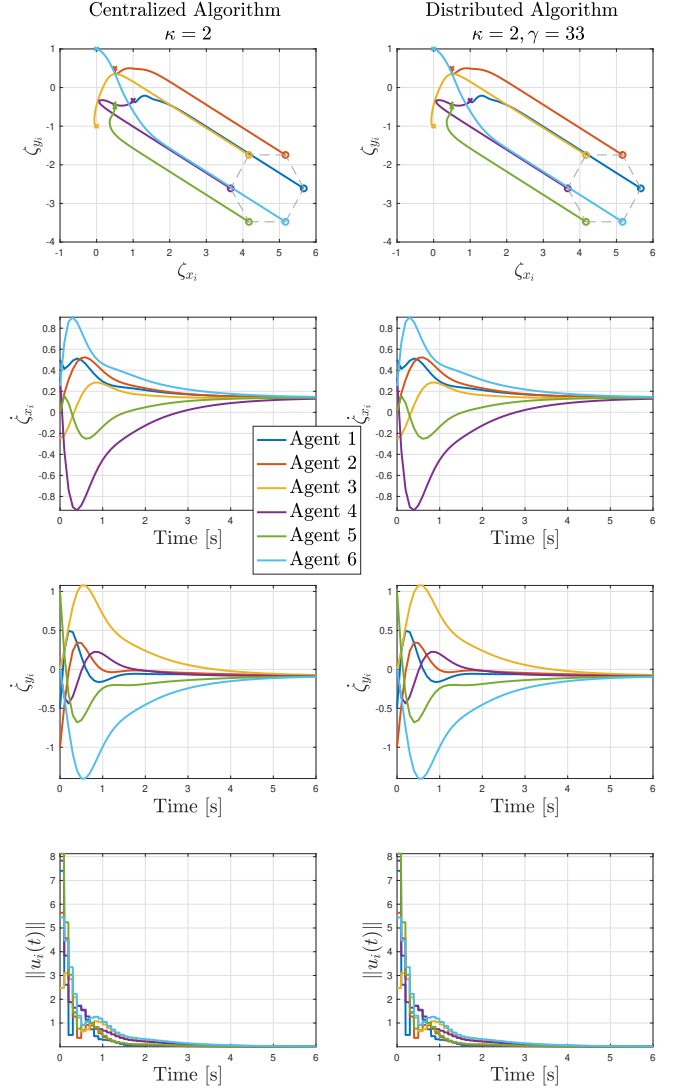


Fig. 5. Formation control for a network of $N = 6$ continuous-time systems with $T = .1$ seconds.

We performed simulations choosing \mathcal{G} as in Fig. 4 and

$$\sigma = \begin{bmatrix} 1 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & -1 & 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}^\top,$$

$$\zeta_1(0) = \begin{bmatrix} 1 & -\frac{1}{3} \end{bmatrix}^\top, \quad \zeta_2(0) = \frac{1}{2} \begin{bmatrix} 1 & 1 \end{bmatrix}^\top, \quad \zeta_3(0) = \begin{bmatrix} 0 & -1 \end{bmatrix}^\top$$

$$\zeta_4(0) = \begin{bmatrix} 1 & -\frac{1}{3} \end{bmatrix}^\top, \quad \zeta_5(0) = \frac{1}{2} \begin{bmatrix} 1 & -1 \end{bmatrix}^\top, \quad \zeta_6(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top$$

$$\dot{\zeta}_1(0) = \frac{1}{2} \begin{bmatrix} 1 & -1 \end{bmatrix}^\top, \quad \dot{\zeta}_2(0) = \begin{bmatrix} 0 & -1 \end{bmatrix}^\top, \quad \dot{\zeta}_3(0) = \begin{bmatrix} -\frac{1}{4} & 0 \end{bmatrix}^\top$$

$$\dot{\zeta}_4(0) = \begin{bmatrix} \frac{1}{3} & 0 \end{bmatrix}^\top, \quad \dot{\zeta}_5(0) = \begin{bmatrix} 0 & 1 \end{bmatrix}^\top, \quad \dot{\zeta}_6(0) = \begin{bmatrix} \frac{1}{4} & 0 \end{bmatrix}^\top.$$

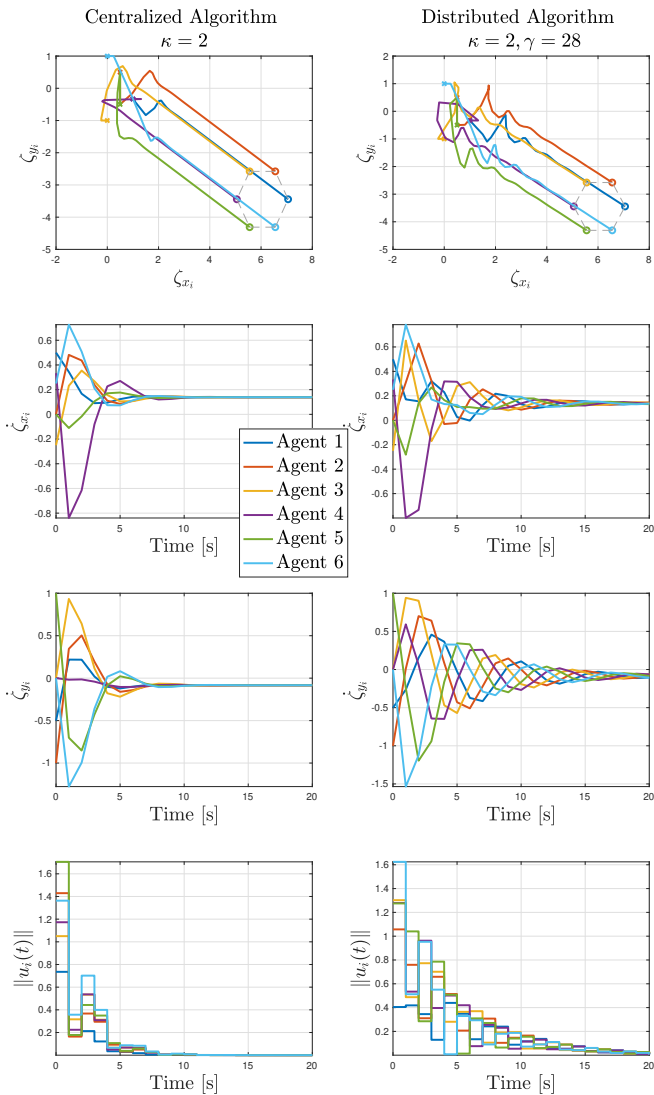


Fig. 6. Formation control for a network of $N = 6$ continuous-time systems with $T = 1$ second.

$\kappa = \kappa_i = 2$, $Q = 5I_4$, $R = I_2$ and computing the corresponding matrices P , F and G , that depend on T . In this case, agents are required to asymptotically evolve over to form an hexagon while moving with uniform velocity. As the intuition suggests, the minimum value γ_{\min} of consensus iterations that guarantees convergence to the desired formation depends on the sampling time via the matrices A and B that depend on T . To show this, Fig. 4 reports the values of γ_{\min} and its estimate γ^* generated via Theorem 3.4 as T increases.

In addition, two simulations were reported in Figs 5 and 6 when applying the piecewise constant control (64) over the continuous-time plant (62) for two values of the

sampling period. In the first case, when $T = 0.1$ and $\gamma = 33$, the performances guaranteed by the distributed implementation of the protocol are very close to the ones of the centralized counterpart in terms of trajectories, convergence rate and control effort. In addition, as T increases (see Fig. 6), convergence to the desired formation with a common velocity is still ensured with good performance and, not surprisingly, a larger transient due to the increased amplitude of the sampling period T .

6 Conclusions and perspectives

We have presented a new fully distributed algorithm for enforcing multi-consensus over a network of discrete homogeneous LTI agents. When specified to one consensus only or multi-leader following (the only cases available in the literature so far), the proposed control law overcomes typical issues raising with standard solutions and recovers the typical features of the continuous-time counterparts. We highlight that the qualitative results we provide hold true in the case of weighted graphs as long as the weights on the edges preserve the properties of the Laplacian we invoke. However, a more general investigation on weighted graph topology represents a challenging and interesting perspective. Perspectives include heterogeneous networks under communication noise and delays.

Appendix

Auxiliary results

Lemma 6.1 For any $\kappa_i > 0$ and $K = \text{diag}_i(\kappa_i)$ the matrix $\tilde{G} = \tilde{D}(KA \otimes G)$ defined in (34) is Schur, i.e. $\rho(\tilde{G}) < 1$.

Proof: We recall that $G = G^\top$ is positive definite. Consider its diagonal version $\Lambda_G = U^{-1}GU$. Since \tilde{D} in (33) is block-diagonal and it can be re-written

$$\tilde{D} = \text{diag}_{i=1}^N \left((I_m + \kappa_i d_i G)^{-1} \right), \quad (65)$$

it follows that G and the blocks on the diagonal of \tilde{D} commute and that the eigenvalues of \tilde{D} are $(1 + \kappa_i d_i \lambda_j^G)^{-1}$, where $i = 1, \dots, N$, $j = 1, \dots, m$ and $\lambda_j^G \in \sigma(G) > 0$. This yields,

$$\Lambda_{\tilde{D}} = (I_N \otimes U^{-1}) \tilde{D} (I_N \otimes U) = \text{diag}_{i=1}^N (\text{diag}_{j=1}^m (1 + \kappa_i d_i \lambda_j^G)^{-1}) \quad (66)$$

$$(I_N \otimes U^{-1}) \tilde{G} (I_N \otimes U) = \Lambda_{\tilde{D}} (KA \otimes \Lambda_G). \quad (67)$$

The matrix above contains $N \times N$ diagonal blocks of size $m \times m$, where the block (i, j) is non zero only when

$\mathcal{A}_{ij} = 1$ and each block has the structure

$$\left[(I_N \otimes U^{-1}) \tilde{G} (I_N \otimes U) \right]_{ij} = \text{diag}_\ell \left(\frac{\kappa_i \lambda_\ell^G}{1 + \kappa_i d_i \lambda_\ell^G} \right), \quad (68)$$

i.e. all the d_i non null blocks on the same block-row are identical and non-negative. Consequently, $(I_N \otimes U^{-1}) \tilde{G} (I_N \otimes U)$ is non-negative and with null main diagonal. Applying the Gerschgorin circles to the rows of this matrix, one gets that its eigenvalues are bounded by the sum of the entries on the rows, that is,

$$|\lambda_{\tilde{G}}| \leq \frac{\kappa_i d_i \lambda_\ell^G}{1 + \kappa_i d_i \lambda_\ell^G} < 1. \quad (69)$$

◁

6.1 Proof of Lemma 3.3

To prove that (10) and (41a) are equivalent, we compute

$$\begin{aligned} \Theta &= I_N \otimes A - (I_N \otimes B) (I_{mN} + \bar{\mathcal{L}} \otimes G)^{-1} (\bar{\mathcal{L}} \otimes F) \\ &= \left(I_{nN} - (I_N \otimes B) (I_{mN} + \bar{\mathcal{L}} \otimes G)^{-1} (\bar{\mathcal{L}} \otimes B^\top P) \right) \\ &\quad \cdot (I_N \otimes A). \end{aligned}$$

By the matrix inversion Lemma, one gets

$$\begin{aligned} I_{nN} - (I_N \otimes B) (I_{mN} + (\bar{\mathcal{L}} \otimes B^\top P) (I_N \otimes B))^{-1} \\ \cdot (\bar{\mathcal{L}} \otimes B^\top P) = (I_{nN} + \bar{\mathcal{L}} \otimes M)^{-1} \end{aligned}$$

so getting (41a). For deducing (41b) from (40), one gets $(I_N \otimes B) \tilde{G}^h = \tilde{G}_s^h (I_N \otimes B)$ and $(I_N \otimes B) \tilde{D} = \tilde{D}_s (I_N \otimes B)$ with $\tilde{D} = I_{mN} - (K D \otimes B^\top P) \tilde{D}_s (I_N \otimes B)$ so yielding

$$\begin{aligned} \Theta_\gamma &= (I_N \otimes A) - (I_N \otimes B) \left(\sum_{h=0}^{\gamma} \tilde{G}^h \right) \tilde{D} (\bar{\mathcal{L}} \otimes F) \\ &= \left(I_{nN} - \left(\sum_{h=0}^{\gamma} \tilde{G}_s^h \right) \tilde{D}_s (\bar{\mathcal{L}} \otimes M) \right) (I_N \otimes A). \end{aligned}$$

At this point, one can easily verify that

$$\begin{aligned} \sum_{h=0}^{\gamma} \tilde{G}_s^h &= (I_{nN} - \tilde{G}_s^{\gamma+1}) (I_{nN} - \tilde{G}_s)^{-1} \\ (I_{nN} - \tilde{G}_s)^{-1} \tilde{D}_s &= (I_{nN} + (\bar{\mathcal{L}} \otimes M))^{-1} \\ \left(\sum_{h=0}^{\gamma} \tilde{G}_s^h \right) \tilde{D}_s &= (I_{nN} - \tilde{G}_s^{\gamma+1}) (I_{nN} - \tilde{G}_s)^{-1} \tilde{D}_s \\ &= (I_{nN} - \tilde{G}_s^{\gamma+1}) (I_{nN} + \bar{\mathcal{L}} \otimes M)^{-1}. \end{aligned}$$

By the last expression above, one obtains

$$\begin{aligned} \Theta_\gamma &= \left(I_{nN} - (I_{nN} - \tilde{G}_s^{\gamma+1}) (I_{nN} + \bar{\mathcal{L}} \otimes M)^{-1} (\bar{\mathcal{L}} \otimes M) \right) \\ &\quad \cdot (I_N \otimes A) \\ &= \left(I_{nN} - (I_{nN} - \tilde{G}_s^{\gamma+1}) (\bar{\mathcal{L}} \otimes M) (I_{nN} + \bar{\mathcal{L}} \otimes M)^{-1} \right) \\ &\quad \cdot (I_N \otimes A) \\ &= \left(I_{nN} + \tilde{G}_s^{\gamma+1} (\bar{\mathcal{L}} \otimes M) \right) \Theta. \end{aligned}$$

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