



HAL
open science

Assessment of approximate models to evaluate transient and cyclical hygrothermoelastic stress in composite plates

Marco Gigliotti, Frédéric Jacquemin, Alain Vautrin

► **To cite this version:**

Marco Gigliotti, Frédéric Jacquemin, Alain Vautrin. Assessment of approximate models to evaluate transient and cyclical hygrothermoelastic stress in composite plates. *International Journal of Solids and Structures*, 2007, 44 (3-4), pp.733-759. 10.1016/j.ijsolstr.2006.05.014 . hal-04791957

HAL Id: hal-04791957

<https://hal.science/hal-04791957v1>

Submitted on 19 Nov 2024

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.



Distributed under a Creative Commons Attribution - NonCommercial 4.0 International License

Assessment of approximate models to evaluate transient and cyclical hygrothermoelastic stress in composite plates

Marco Gigliotti ^{a,*}, Frédéric Jacquemin ^b, Alain Vautrin ^c

^a *Italian Aerospace Research Center (CIRA), Computational Mechanics Lab (LAMC), Via Maiorise, 81043 Capua, Ce, Italy*

^b *Institut de Recherche en Génie Civil et Mécanique (GeM), Université de Nantes, Boulevard de l'Université, BP 406, 44602 Saint Nazaire Cedex, France*

^c *Département Mécanique et Matériaux, Centre SMS, Ecole Nationale Supérieure des Mines de Saint Étienne, 158 Cours Fauriel, 42023 Saint Étienne Cedex 2, France*

The present paper aims at assessing an approximate model to evaluate hygrothermoelastic stress in composite laminated plates. The approximate model is based on straight hypotheses (classical lamination theory of plates) and is able to simulate the effects of complex hygrothermal conditionings, such as those encountered in the life of real structures. For the purpose of the assessment, a 3D analytical solution for hygrothermally stressed plates is employed. The approximate model for plates is also compared to an analytical model for cylinders under transient and cyclical hygrothermal loads [Jacquemin, F., Vautrin, A., 2002a. Modelling of the moisture concentration field due to cyclical hygrothermal conditions in thick laminated pipes. *European Journal of Mechanics A/Solids* 21, 845–855; Jacquemin, F., Vautrin, A., 2002b. A closed form solution for the internal stresses in thick composite cylinders induced by cyclical environmental conditions. *Composite Structures* 58, 1–9]. This is done in order to assess the capabilities and the limits of a simplified model for plates to represent also the behaviour of cylinders.

Keywords: Fick's law; Transient/cyclical hygrothermal stress; Analytical 3D solution; Simplified CLT solution

1. Introduction

Composite structures are very sensitive to hygrothermoelastic stress: this is due to their intrinsic heterogeneity, emerging at the microscopic and at the macroscopic scale and related to the difference in hygrothermoelastic properties of adjacent entities; fibres and matrices, adjacent plies aligned with different fibre orientations with respect to a common reference orientation, etc.

* Corresponding author. Tel.: +39 823 62 3066; fax: +39 823 62 3515.

E mail addresses: m.gigliotti@cira.it (M. Gigliotti), frederic.jacquemin@univ-nantes.fr (F. Jacquemin), vautrin@emse.fr (A. Vautrin).

It is crucial to a good design of composite structures to take into proper account hygrothermoelastic stresses; even within the context of an uncoupled theory, that is, the hygrothermal fields are uncoupled from the stress ones, this task is quite complex.

Analytical solutions are important as benchmark to validate numerical finite element codes; however simplified tools are always needed, since they allow for designing, in very short times, not only structures but also experiments and optimisation inverse procedures.

In the present paper an approximate model for plates under transient and cyclical hygrothermal conditions, based on the hypotheses of the classical lamination theory of plates (CLT), is tested against analytical solutions for plates and cylinders; the aim of the authors is to establish if, and within which limits, the approximate theory rigorously valid only for slender flat plates is capable to represent the behaviour of real structures, spanning from 3D anisotropic plates to cylinders.

In polymeric based composite materials the basic constituents, fibres and matrices, exhibit different swelling strains when exposed to the same amount of moisture (almost all fibres are insensitive to moisture) or temperature differentials. Since they are “bonded” to each other and have to respect displacement/traction continuity, they develop stress. *Hygrothermoelastic stresses are active at the microscopic level.* When adjacent composite plies are arranged with different fibre orientations with respect to a common reference direction, they have a different expansion hygrothermal strain in that direction, thus they again develop internal stress in order to assure continuity at the ply level.

Usually, structural analyses of composite materials subjected to hygrothermal solicitations are performed by considering the actual heterogeneous plies (fibres + matrix) as equivalent homogenized anisotropic materials, thus disregarding the actual microscopic structure.

In most cases the hygrothermal fields are obtained, independently from the stress ones, by employing classical Fourier (see, for instance, the book by Carslaw and Jaeger, 1959) and Fick’s (Crank, 1975) theories for heat/moisture conduction/diffusion, the mathematical structure of the two theories is the same. Then, the obtained fields are used to calculate stress. This approach is *uncoupled* (or *one-way coupled*), since the hygrothermal fields are not influenced by the mechanical ones.

Full coupling (or *two-way coupling*) can be important for large strains, when plasticity/viscoplasticity effects should be taken into account, as shown, for instance, by Allen (1991). For recent development on fully coupled theories the reader also is referred to Chandrasekharaiah (1986) and Weitsman (1987, 1998).

In the present paper, *small strains* are considered so that an uncoupled approach can be employed.

However, the explicit solution of problems of the uncoupled hygrothermoelasticity theory presents still some difficulties for anisotropic bodies.

The case of laminated cylinders is relatively “easy” to handle: such structures exhibit many symmetries. Moreover, a cylinder can be considered with great approximation as a structure “without edges”, thus the usual approach is a *generalized plane strain* one: since the cylinder is approximately infinite along its axis, only a “thin slice” of it is studied.

One of the first papers dedicated to the assessment of internal (thermal) stresses in laminated anisotropic cylinders has been published by Hyer and Rousseau (1987), who used a continuum mechanics approach (generalized plane strain): they considered uniform temperature differentials. Ootao et al. (1991) considered complex thermal conditions in laminated cylinders composed by isotropic plates.

The approach by Hyer and Rousseau was extended by Paul and Vautrin (1995) to the case of transient hygrothermal fields. Jacquemin and Vautrin described analytically the behaviour of laminated cylinders under cyclical hygrothermal conditions, giving the hygrothermal fields (2002a) and the related internal stresses (2002b).

The case of laminated composite plates subjected to hygrothermal stress is different, mainly due to the presence of edges. At the edge of a composite plate, the boundary conditions must be satisfied exactly, thus the stress fields close to the edges can be extremely complicated. These complications increase in the proximity of interfaces between plies, where displacement/traction continuity conditions must be obeyed. According to the Saint-Venant principle, the influence of edges on the state of stress of plates is limited to zones very close to the edges themselves. For anisotropic bodies, the influence of edges could be also more pronounced.

The first studies on hygrothermoelastic stresses in composite plates simply neglected the presence of edges and employed the CLT.

Significant contributions to the assessment of thermoelastic stresses in plates came from Wu and Tauchert, who treated the case of symmetric special orthotropic laminates (1980a) and antisymmetric cross-ply and angle-ply laminates (1980b). They consider plates with simple supported edges and temperature variations which are uniform or linear along the thickness of the structure.

Reissner Mindlin and higher order shear deformation theories were employed, among others, by Reddy and Hsu (1980) and Noor and Burton (1989).

Huang and Tauchert (1988) investigated the buckling and postbuckling response of simply supported laminated plates subjected to gradually increasing uniform temperature fields, while Kao and Pao (1976) studied small amplitude temperature induced vibrations of simply supported symmetric cross-ply laminated plates.

Computational models for high-temperature multi-layered composite plates and shells were extensively reviewed by Tauchert (1991) and Noor and Burton (1992).

These review papers contain more than 200 references each.

All the cited papers contain models which falls into the set of the *single-layer* and *multi-layer* theories. These theories *postulate* a mathematical distribution, usually of polynomial form, for the through-the-thickness strain. Adequate displacement/traction conditions have then to be imposed at the interface of each ply in order to determine the unknown coefficients of the assumed polynomials. Obviously, *multi-layer* theories are more accurate than the *single-layer* ones, but are also more expensive from a computational point of view.

Analytical models follow a different approach. Each ply is considered as a homogenized anisotropic continuum, the displacements along a single ply are unknown and must be determined by solving the strain compatibility equations and the stress balance equations. Interface and boundary conditions give the additional equations to solve the problem. Solving a full 3D problem for plates subjected to hygrothermal loads is a formidable task. The exact form of the 3D hygrothermal fields is cumbersome to determine, the presence of edges make the solution of the elasticity problem extremely difficult to find. A full 3D analytical solution for hygrothermoelastic problems still does not exist.

Tungikar and Rao (1994) were the first to provide an analytical solution for the thermal stresses in rectangular composite plates. However, their solution was limited to special boundary conditions, simply supported edges, cross-ply laminates and steady thermal conditions.

Savoia and Reddy (1995) enlarged the analysis of Tungikar and Rao by taking into account more complex thermal fields, approximated by polynomial distributions.

Vel and Batra (2001) proposed an elegant mathematical procedure to handle thermoelastic problems for anisotropic thick plates. They used a *generalized plane strain* approach and employed the Eshelby Stroh formalism. In this approach, the analytical solution is in terms of an infinite series, the coefficients of the series are determined by imposing the continuity conditions at the interface and boundary conditions at the bounding surfaces. Although, in principle, the method is capable of taking into account different mechanical and thermal boundary conditions, without any restriction, only few examples are worked out.

Precisely, results are given for rigidly clamped, simply supported and traction-free edges, while the temperature distribution along the thickness of the plate is uniform or linear. Predictions of stresses and strains are compared to those of the CLT and the first-order shear deformation theory and boundary layer effects close to the clamped-free edges are also investigated. The agreement between all three solutions is very good only for plates with span-to-thickness ratio higher than 20. For span-to-thickness ratio equal to 5, the solutions can differ by more than 60%. The analysis also reveals the absence of boundary layers for simply supported edges. On the contrary, near to clamped or free edges of $[0/90]_S$ laminates, severe oscillations of the transverse shear stress can be observed at points on the interface between the 0 and the 90 ply: such oscillations may be due to the presence of singularities. The existence of a singularity can be confirmed only by performing an asymptotic analysis.

The analytical approach by Vel and Batra is the most refined, insofar: however, it is limited to generalized plane strain fields and relatively simple thermal distributions.

Recently, Zenkour (2004) proposed a unified shear deformation theory capable of taking into account complex thermal conditions: by this theory he tested several different plate formulations confirming the trend found by Vel and Batra.

The cited investigations were for pure thermal problems.

Analysis of plates under hygrothermal loads were performed by Whitney and Ashton (1971) and Hahn and Kim (1978), within the context of the CLT.

Benkeddad et al. (1996) extended this classical approach in order to take into account transient hygroscopic fields.

Williams (2005) proposed a generalized thermo-diffusional-mechanically coupled approach for laminated plates with delamination, which can be eventually specialised to existing theories. This framework is capable to handle very general conditions, even in the presence of large strains, plasticity and certain forms of damage.

The aim of the present paper is to test the simple approach by Benkeddad et al. (1996) against analytical solutions and to check the capability of such approach to represent more complex structures, such as plates or cylinders. This approach is based on a slight refinement of the CLT to take into account complex hygrothermal fields, each ply is divided into *subplies* and the actual concentration distribution is approximated by a piecewise linear distribution (within each *subply*). In turn the free expansion strains are piecewise linear and the compatibility equations are satisfied within each *subply*.

The method proposed by Benkeddad et al., although relatively simple and computationally very efficient, has been used by Gigliotti et al. (2004) to evaluate internal stresses in composite plates subjected to realistic environments. However, it has never been carefully checked. Especially, it could be argued that the exactness of the method may depend quite dramatically on the number of *subplies* and on the form of the hygrothermal distribution.

Section 2 presents a detailed description of the models: first, the hygrothermal fields due to cyclical conditions are determined through analytical and finite difference techniques. Classical Fourier/Fick models are employed.

The hygrothermal fields give access to the hygrothermal free strains by which internal stresses are then determined. Classical elastic laws are employed, in their three-dimensional form. In this way, the stresses in a 3D plate can be determined. For the cylinder, the assumption of generalized plane strain is made.

It is important to remind that the hygrothermal and the elastic models are uncoupled. On the other hand, the hygroscopic and the thermal field are coupled through an Arrhenius-like dependency of the diffusion coefficient on the temperature, that is, diffusivity of the material is influenced by temperature and this process is modelled by the Arrhenius laws.

2. Model description

2.1. Hygrothermal fields

The ideal Fick's theory of diffusion assumes (Crank, 1975) that the rate of transfer of diffusing substance through unit area of a section of the solvent is proportional to the solute concentration gradient measured normal to the section. The diffusivity coefficient depends only on temperature and the diffusive flux is

$$\mathbf{j} = -\mathbf{D} \cdot \nabla c \quad (1)$$

where c is the solute concentration (in many cases and later in the text the composition $m = c/\rho_s$ is employed in place of c , where ρ_s is the density of the dry solvent) \mathbf{D} is the diffusivity tensor, $\mathbf{D} = \mathbf{D}(T)$. Assuming that diffusion is taking place along three directions x , y and z , to which a Cartesian reference frame is attached, the diffusivity tensor can be expressed in that reference frame as

$$(\mathbf{D}) = \begin{pmatrix} D_{xx} & D_{xy} & D_{xz} \\ D_{yx} & D_{yy} & D_{yz} \\ D_{zx} & D_{zy} & D_{zz} \end{pmatrix} \quad (2)$$

In Eq. (2) (\mathbf{D}) indicates the matrix attached to the tensor \mathbf{D} in the given reference frame. Of course, Eq. (2) refers to anisotropic media, where the diffusivity depends on the direction of propagation of the solute. For isotropic media, obviously

$$\mathbf{D} = D\mathbf{I} \quad (3)$$

By taking into account Eq. (1) and by assuming that the total velocity of the *solute-solvent* mixture system is zero (in fact, it can be neglected), the conservation of mass reads

$$\frac{\partial c}{\partial t} - \mathbf{D} \cdot \nabla \cdot \nabla c = 0 \quad (4)$$

provided the medium is homogeneous and there are no solute sources in it. For heterogeneous media, Eq. (4) should be applied to each component of the medium and adequate interface conditions should be enforced. Eq. (4), solved with the appropriate initial and boundary conditions, gives the solute concentration as a function of time and of the three spatial co-ordinates. The transient temperature fields follow analogous equations.

The concentration at the surfaces, c_∞ , may be related to the external relative humidity through a semi-empirical expression, for instance (Shen and Springer, 1981):

$$c_\infty = C(\text{RH})^b \quad (5)$$

where C and b are material constants and (RH) is the external relative humidity (%). The dependency of the diffusivity coefficient on temperature follows an Arrhenius-like law:

$$D = D_0 e^{\left(\frac{-\Delta H}{RT}\right)} = A e^{\left(\frac{B}{T}\right)} \quad (6)$$

where D_0 is a material constant, ΔH an activation energy and R is the constant of perfect gases. Finally A and B represent the two effective parameters to be identified.

Diffusion processes under real environmental conditions are quite complicate to model. Temperature and moisture variations enter into the field equation (Fick's law) through the diffusion coefficient (which depends on temperature, hence on time) and into the boundary conditions via the external relative humidity, which is time dependent.

In mathematical terms the one-dimensional problem (along the z -direction) for a plate structure is as follows. Given a spatial region plate of thickness e composed by n sub-regions of thickness e_i bounded by two parallel planes $0 < z_i < e_i$ (Fig. 1), the field equation reads

$$\frac{\partial c_i(z_i, t)}{\partial t} = D_i(t) \frac{\partial^2 c_i(z_i, t)}{\partial z_i^2} \quad \forall 0 < z_i < e_i, t > 0 \quad \forall i, i = 1, \dots, n \quad (7)$$

with the following boundary and interface conditions:

$$\begin{aligned} c_i(z_i, 0) &= 0 \\ c_i(e_i, t) &= \alpha_{i,i+1} c_{i+1}(0, t) \quad \forall i, i = 1, \dots, n-1 \\ D_i(t) \frac{\partial c_i(e_i, t)}{\partial z_i} &= D_{i+1}(t) \frac{\partial c_{i+1}(0, t)}{\partial z_{i+1}} \quad \forall i, i = 1, \dots, n-1 \\ c_1(0, t) &= c_a(t) \\ c_n(e_n, t) &= c_b(t) \end{aligned} \quad (8)$$

$D_i(t)$ is the diffusion coefficient of the i th sub-region and depends on temperature according to a relation of the type:

$$D_i(t) = A_i \exp\left(\frac{B_i}{T(t)}\right) \quad (9)$$

with A_i and B_i constants. $\alpha_{i,i+1}$ is a constant which indicates the ratio between the saturation level of the i th sub-region and that of the $(i+1)$ th one. In fact, equilibrium conditions impose at the interfaces between different materials, equality of the solute chemical potential, not of its concentration.

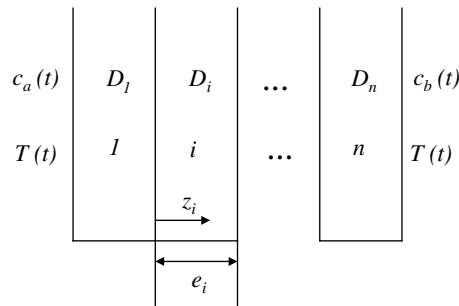


Fig. 1. One dimensional diffusion in a multi layered plate.

Temperature and surface concentrations are periodic functions of time with period τ . Actually, temperature can be assumed to be uniform through the thickness of the body: this hypothesis is plausible since time for thermal diffusion is much faster than time for solute diffusion.

Problem (6) (8) can be solved by numerical methods, finite difference methods, for instance. However an analytical solution can be approached by using average quantities. The methodology was used by Jacquemin and Vautrin (2002a) for laminated cylinders and here is applied to laminated plates. The method consists in reducing the original problem to an *equivalent* problem with constant diffusivity and external concentrations.

By introducing the following change of variables (see also Verchery, 1992):

$$u_i(t) = \frac{\int_0^t D_i(q) dq}{\int_0^\tau D_i(q) dq} = \frac{D_i(t)}{\Delta_i(\tau)} \Rightarrow du_i = \frac{D_i(t)}{\Delta_i(\tau)} dt \quad (10)$$

it follows

$$\frac{\partial c_i}{\partial t} = \frac{\partial c_i}{\partial u_i} \frac{\partial u_i}{\partial t} = \frac{\partial c_i}{\partial u_i} \frac{D_i(t)}{\Delta_i(\tau)} = D_i(t) \frac{\partial^2 c_i}{\partial z_i^2} \quad (11)$$

When the activation energies B_i are the same for each sub-region one has

$$u_i(t) = u_{i+1}(t) \quad \forall i \quad (12)$$

and the problem becomes

$$\frac{\partial c_i(z_i, u)}{\partial u} = \Delta_i(\tau) \frac{\partial^2 c_i(z_i, u)}{\partial z_i^2} \quad \forall 0 < z_i < e_i, u > 0 \quad \forall i, i = 1, \dots, n \quad (13)$$

$$c_i(z_i, 0) = 0$$

$$c_i(e_i, u) = \alpha_{i,i+1} c_{i+1}(0, u) \quad \forall i, i = 1, \dots, n-1$$

$$\Delta_i(\tau) \frac{\partial c_i(e_i, u)}{\partial z_i} = \Delta_{i+1}(\tau) \frac{\partial c_{i+1}(0, u)}{\partial z_{i+1}} \quad \forall i, i = 1, \dots, n-1 \quad (14)$$

$$c_1(0, u) = c_a(u)$$

$$c_n(e_n, u) = c_b(u)$$

where $c_a(u)$ and $c_b(u)$ are cyclical with period equal to 1. Boundary conditions (14) depend on u and have to be converted into equivalent quantities independent of time. To this aim the average mobile concentration is introduced

$$\hat{c}_i(z_i, u) = \int_{u-1}^u c_i(z_i, q) dq \quad (15)$$

from which

$$\hat{c}_1(0, u) = \int_{u-1}^u c_a(q) dq = \int_0^1 c_a(q) dq = \hat{c}_a \quad (16)$$

$$\hat{c}_n(e_n, u) = \int_{u-1}^u c_b(q) dq = \int_0^1 c_b(q) dq = \hat{c}_b$$

The problem has now the following form:

$$\frac{\partial \hat{c}_i(z_i, u)}{\partial u} = \Delta_i(\tau) \frac{\partial^2 \hat{c}_i(z_i, u)}{\partial z_i^2} \quad \forall 0 < z_i < e_i, u > 0 \quad \forall i, i = 1, \dots, n \quad (17)$$

$$\begin{aligned}
\hat{c}_i(z_i, 0) &= 0 \\
\hat{c}_i(e_i, u) &= \alpha_{i,i+1} \hat{c}_{i+1}(0, u) \quad \forall i, i = 1, \dots, n-1 \\
\Delta_i(\tau) \frac{\partial \hat{c}_i(e_i, u)}{\partial z_i} &= \Delta_{i+1}(\tau) \frac{\partial \hat{c}_{i+1}(0, u)}{\partial z_{i+1}} \quad \forall i, i = 1, \dots, n-1 \\
\hat{c}_1(0, u) &= \hat{c}_a(u) = \int_0^1 c_a(q) dq \\
\hat{c}_n(e_n, u) &= \hat{c}_b(u) = \int_0^1 c_b(q) dq
\end{aligned} \tag{18}$$

By introducing adimensional variables

$$\begin{aligned}
\hat{c}_i &= \frac{\hat{c}_i}{c_0}, \quad \hat{c}_a = \frac{\hat{c}_a}{c_0}, \quad \hat{c}_b = \frac{\hat{c}_b}{c_0}, \quad z_i = \frac{z_i}{e}, \quad e_i = \frac{e_i}{e} \\
\Delta_i &= \frac{\Delta_i}{e^2} \\
u &= \frac{D_0 u}{e^2}
\end{aligned} \tag{19}$$

the adimensional problem is as follows:

$$\frac{\partial \hat{c}_i(z_i, u)}{\partial u} = \Delta_i(\tau) \frac{\partial^2 \hat{c}_i(z_i, u)}{\partial z_i^2} \quad \forall 0 < z_i < e_i, u > 0 \quad \forall i, i = 1, \dots, n \tag{20}$$

$$\begin{aligned}
\hat{c}_i(z_i, 0) &= 0 \\
\hat{c}_i(e_i, u) &= \alpha_{i,i+1} \hat{c}_{i+1}(0, u) \quad \forall i, i = 1, \dots, n-1 \\
\Delta_i(\tau) \frac{\partial \hat{c}_i(e_i, u)}{\partial z_i} &= \Delta_{i+1}(\tau) \frac{\partial \hat{c}_{i+1}(0, u)}{\partial z_{i+1}} \quad \forall i, i = 1, \dots, n-1 \\
\hat{c}_1(0, u) &= \hat{c}_a(u) \\
\hat{c}_n(e_n, u) &= \hat{c}_b(u)
\end{aligned} \tag{21}$$

Now boundary conditions are independent of time (t or u): therefore the problem can be solved in a simple way by employing the Laplace transform method. In the Laplace domain equations (20) and (21) become

$$\hat{c}_i^{\text{Lapl}}(z_i, s) = A_i \cos q_i^2 z_i + B_i \sin q_i^2 z_i \tag{22}$$

the Laplace transform of the solution being

$$\begin{aligned}
\hat{c}_i^{\text{Lapl}}(z_i, s) &= \text{Lapl}(\hat{c}_i(z_i, u)) = \int_0^\infty \hat{c}_i(z_i, u) \exp(-su) du \\
q_i^2 &= \frac{-s}{\Delta_i}
\end{aligned} \tag{23}$$

or also

$$q_i^2 = \omega^2 \beta_i^2 \tag{24}$$

with

$$\begin{aligned}
\omega^2 &= -s \\
\beta_i &= \sqrt{\frac{1}{\Delta_i(\tau)}} = \sqrt{\frac{e^2}{\Delta_i(\tau)}}
\end{aligned} \tag{25}$$

Boundary and interface conditions in the Laplace domain read

$$\begin{aligned}
\hat{c}_i^{\text{Lapl}}(e_i) &= \alpha_{i,i+1} \hat{c}_{i+1}^{\text{Lapl}}(0) \quad \forall i, i = 1, \dots, n-1 \\
\Delta_i(\tau) \frac{\partial \hat{c}_{i+1}^{\text{Lapl}}(e_i)}{\partial z_i} &= \Delta_{i+1}(\tau) \frac{\partial \hat{c}_i^{\text{Lapl}}(0)}{\partial z_{i+1}} \quad \forall i, i = 1, \dots, n-1 \\
\hat{c}_1^{\text{Lapl}}(0) - \frac{\hat{c}_a}{s} &= 0 \\
\hat{c}_n^{\text{Lapl}}(e_n) - \frac{\hat{c}_b}{s} &= 0
\end{aligned} \tag{26}$$

By applying the residue theorem for finding the inverse Laplace transform one gets

$$\hat{c}_i(z, u) = \frac{1}{\Delta_s} (A'_i + B'_i z) + \sum_{m=1}^{\infty} \frac{2 \exp(-\omega_m^2 u)}{\omega_m \Delta'_u(\omega_m)} \{A_i^*(\omega_m) \cos(\beta_i \omega_m z) + B_i^*(\omega_m) \sin(\beta_i \omega_m z)\} \tag{27}$$

where Δ_u and Δ_s are the determinants of the $2n \times 2n$ (a) and (b) matrices (see Eqs. (28) and (29)). $A_i^*(\omega_m)$ and $B_i^*(\omega_m)$ are the determinants (calculated at ω_m) of the a_{ij} matrix in which the columns $(2i-1)$ and $2i$ are substituted by the constant vector $\{g\}$ (see Eq. (30)). ω_m is the m th positive root of the equation $\Delta_u(\omega) = 0$. A'_i and B'_i are the determinants of the b_{ij} matrix in which the columns $(2i-1)$ and $2i$ are substituted by the constant vector $\{g\}$. Finally $\Delta'_u(\omega_m)$ is the derivative of Δ_u with respect to ω calculated at ω_m .

The non-zero elements of the two matrices (a) and (b) and of the vector $\{g\}$ are given by

$$\begin{aligned}
a_{11} &= 1 \\
a_{2n2n-1} &= \cos \beta_n \omega e_n, \quad a_{2n2n} = \sin \beta_n \omega e_n \\
a_{2i2i-1} &= \cos \beta_i \omega e_i, \quad a_{2i2i} = \sin \beta_i \omega e_i, \quad a_{2i2i+1} = -\alpha_{i,i+1} \\
a_{2i+12i-1} &= -\Delta_i \beta_i \sin \beta_i \omega e_i, \quad a_{2i+12i} = \Delta_i \beta_i \cos \beta_i \omega e_i, \quad a_{2i+12i+2} = -\Delta_{i+1} \beta_{i+1}
\end{aligned} \tag{28}$$

for $i = 1, \dots, (n-1)$

$$\begin{aligned}
b_{11} &= 1 \\
b_{2n2n-1} &= 1, \quad b_{2n2n} = e_n \\
b_{2i2i-1} &= 1, \quad b_{2i2i} = e_i, \quad b_{2i2i+1} = -\alpha_{i,i+1} \\
b_{2i+12i} &= \Delta_i, \quad b_{2i+12i+2} = -\bar{\Delta}_{i+1}
\end{aligned} \tag{29}$$

for $i = 1, \dots, (n-1)$ and

$$g_1 = \hat{c}_a, \quad g_{2n} = \hat{c}_b \tag{30}$$

If we consider an homogeneous plate the matrices (a) and (b) become:

$$(a) = \begin{pmatrix} 1 & 0 \\ \cos \beta_n \omega e_n & \sin \beta_n \omega e_n \end{pmatrix} \tag{31}$$

$$(b) = \begin{pmatrix} 1 & 0 \\ 1 & e_n \end{pmatrix} \tag{32}$$

The first term of Eq. (27) represents the permanent concentration, which is attained at saturation. The second term is the transient one and is time dependent.

The concentration field can be also written as a function of cycles by employing a recursion technique: after $N - k$ cycles (N is the number of cycles at saturation) is given by

$$c_i(z, N - k) = \frac{1}{\Delta_s} (A'_i + B'_i z) + \sum_{m=1}^{\infty} \frac{2\omega_m}{\Delta'_u(\omega_m)} \left[\sum_{j=0}^{k-1} \exp(-\omega_m^2 (N - j)) \right] \{A_i^*(\omega_m) \cos(\beta_i \omega_m z) + B_i^* \sin(\beta_i \omega_m z)\} \tag{33}$$

It should be noted that Eq. (33) is analogous to that given by Jacquemin and Vautrin (2002a,b) for infinite laminated cylinders, which reads

$$c_i(r, N - k) = \frac{1}{A_s} (A'_i + B'_i \ln r) + \sum_{m=1}^{\infty} \frac{2\omega_m}{A'_u(\omega_m)} \left[\sum_{j=0}^{k-1} \exp(-\omega_m^2(N - j)) \right] \{A_i^*(\omega_m) J_0(\beta_i \omega_m r) + B_i^* Y_0(\beta_i \omega_m r)\} \quad (34)$$

In Eq. (34) the adimensional radial coordinate takes the place of the adimensional z -direction.

The diffusion process is still one-dimensional, along the radial direction. The permanent term in Eq. (34) is logarithmic in r , while that of Eq. (33) is linear in z . Bessel's functions of the first kind, J_0 and Y_0 , which describe the transient behaviour of Eq. (34) are substituted in the transient part of Eq. (33) by sinus and cosinus functions.

Eqs. (34) and (33), as mentioned, employ average boundary conditions and diffusion coefficient, thus they are not able to reproduce the fluctuating concentration regime close to the external surfaces.

Thin composite plates can be considered as homogeneous with respect to the through-the thickness diffusion process, provided their plies are all made with the same material. In this case, defining average quantities, such as the *average* diffusion coefficient and *average* external concentration

$$A(\tau) = \int_0^\tau D(t) dt, \quad \hat{c}_\infty = \frac{1}{A(\tau)} \int_0^\tau D(t) c_\infty(t) dt \quad (35)$$

Eq. (33) can be rewritten in a quite simple form, that is

$$c(z, N - k) = \hat{c}_\infty + \frac{A(\tau)\pi^2}{2(e/2)^3} \sum_{n=0}^{\infty} \sum_{i=0}^{k-1} \left((2n+1)^2 \exp \left(\frac{-(2n+1)^2 \pi^2 A(\tau)(N-i)}{4(e/2)^2} \right) \right) \\ \times \cos \left(\frac{(2n+1)\pi z}{e} \right) \left\{ \frac{2(e/2)(-1)^{n+1} \hat{c}_\infty}{(2n+1)\pi} + \int_0^{e/2} c_i(z') \left(\frac{(2n+1)\pi z'}{2(e/2)} \right) dz' \right\} \\ \forall (-e/2) < z < (e/2) \quad (36)$$

where

$$\beta = \sqrt{\frac{e^2}{A(\tau)}} \quad (37)$$

When the initial concentration (at $\tau = 0$) is $c_i = 0$, Eq. (36) reads

$$c(z, N - k) \\ = \hat{c}_\infty - \frac{4\hat{c}_\infty\pi}{\beta^2} \sum_{n=0}^{\infty} \sum_{i=0}^{k-1} \left((-1)^n (2n+1) \exp \left(-\frac{(2n+1)^2 \pi^2 (N-i)}{\beta^2} \right) \right) \cos \left(\frac{(2n+1)\pi z}{e} \right) \\ \forall (-e/2) < z < (e/2) \quad (38)$$

Eqs. (33), (34), (36) and (38) have been extensively used for calculating moisture fields under fairly general hygrothermal conditions, in most cases representative of realistic exposure to the external environment. Some examples can be found, for instance, in Gigliotti et al. (2004).

2.2. Stress fields: cylinders, 3D plate models

The hygrothermal fields obtained, the hygrothermoelastic stresses can be calculated by applying the usual principles of continuum mechanics, that is, equilibrium and compatibility equations. Indicating with \mathbf{X} the position of a solid particle in the reference configuration, with \mathbf{x} its position in the actual configuration, the displacement \mathbf{u} is defined as

$$\mathbf{u} = \mathbf{x} - \mathbf{X} \quad (39)$$

and, within the context of the small strain theory, the strain \mathbf{E} is given by

$$\mathbf{E} = \frac{1}{2}(\nabla\mathbf{u} + \nabla\mathbf{u}^T) \quad (40)$$

Strains must respect *compatibility* conditions

$$\nabla \cdot \mathbf{E} + \nabla(\nabla \text{tr} \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) + \nabla(\nabla \cdot \mathbf{E})^T \quad (41)$$

while stresses must satisfy *equilibrium* conditions:

$$\nabla \cdot \mathbf{S} + \mathbf{f} = 0 \quad (42)$$

Constitutive laws, interface and boundary conditions have then to be imposed to solve the problem. Concerning constitutive laws, we may attempt the hypothesis that the stress \mathbf{S} is only function of the actual composition m (m is defined as c/ρ_s , where ρ_s is the density of the solid) and the actual strain \mathbf{E} , independently of the path chosen to reach this state. Therefore we may consider the following paths:

- a change of composition from m_0 to m at zero stress, producing a strain \mathbf{E}^{HT} ,
- a change of stress from 0 to \mathbf{S} at constant m .

Assuming the behaviour of the solid to be *elastic*, following the Hooke's law, we have

$$\mathbf{S} = \mathbb{C} : (\mathbf{E} - \mathbf{E}^{\text{HT}}) \quad (43)$$

\mathbb{C} is the usual stiffness tensor, measured at constant composition and dependent on the composition, \mathbf{E} is the measurable strain, \mathbf{E}^{HT} is the hygrothermal free strain given by

$$\mathbf{E}^{\text{HT}} = \alpha \Delta T + \beta \Delta m \quad (44)$$

where α and β are anisotropic tensors of thermal and hygroscopic expansion, while ΔT and Δm are temperature and composition variations with respect to a reference condition. Behind the assumptions made, there is the basic hypothesis of reversible behaviour. Stress development due to hygrothermal fields can be eliminated at a certain value of moisture composition and/or temperature, stress free condition, which is assumed to exist.

For infinite (along their axes) laminated cylinders subjected to transient/cyclical hygrothermal fields, the solution has been given by [Jacquemin and Vautrin \(2002b\)](#). The cylinder being infinite along its axis, the axial deformations can be assumed to be uniform and constant (generalized plane strain condition). Moreover, due to axisymmetry, the mechanical problem, as the hygrothermal one, can be reduced to a one-dimensional partial differential equation in the radial displacement w . Full details are given in [Jacquemin and Vautrin \(2002b\)](#). It is important to remark that, since the analytical hygrothermal fields (33) and (34) employ average boundary conditions, analytical stress calculations, in turn, are not able to reproduce the fluctuating regime close to the external surfaces. Numerical methods (finite differences, for instance) can be then employed for such purpose.

Laminated plates present more serious difficulties, due to the presence of edges. The problem is, theoretically, a full three-dimensional one.

As mentioned in Section 1, [Tungikar and Rao \(1994\)](#), then [Savoia and Reddy \(1995\)](#) proposed an analytical solutions for 3D plates subjected to thermal loads. However, they solved simple cases. In the followings, their model is shortly reviewed and enlarged to fairly general hygrothermal conditions.

We will discuss only the case of orthotropic layers, anisotropic materials can be treated following analogous procedures. For an orthotropic material, the constitutive relations read in matrix notation:

$$\begin{pmatrix} S_{xx} \\ S_{yy} \\ S_{zz} \\ S_{yz} \\ S_{xz} \\ S_{xy} \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ C_{12} & C_{22} & C_{23} & 0 & 0 & 0 \\ C_{13} & C_{23} & C_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66} \end{pmatrix} \cdot \begin{pmatrix} E_{xx} - \alpha_x T - \beta_x m \\ E_{yy} - \alpha_y T - \beta_y m \\ E_{zz} - \alpha_z T - \beta_z m \\ 2E_{yz} \\ 2E_{xz} \\ 2E_{xy} \end{pmatrix} \quad (45)$$

where the C_{ij} s are related to the elastic constants (see, for instance, [Herakovitch, 1997](#)) and $T = T(x, y, z)$ and $m = m(x, y, z)$ are the temperature and composition fields inside the material. Neglecting the body forces and applying equilibrium conditions one gets the following coupled system of partial differential equations:

$$C_{11} \frac{\partial^2 u}{\partial x^2} + C_{66} \frac{\partial^2 u}{\partial y^2} + C_{55} \frac{\partial^2 u}{\partial z^2} + (C_{12} + C_{66}) \frac{\partial^2 v}{\partial x \partial y} + (C_{13} + C_{55}) \frac{\partial^2 w}{\partial x \partial z} = A' \frac{\partial T}{\partial x} + A'' \frac{\partial m}{\partial x} \quad (46)$$

$$C_{66} \frac{\partial^2 v}{\partial x^2} + C_{22} \frac{\partial^2 v}{\partial y^2} + C_{44} \frac{\partial^2 v}{\partial z^2} + (C_{12} + C_{66}) \frac{\partial^2 u}{\partial x \partial y} + (C_{23} + C_{44}) \frac{\partial^2 w}{\partial y \partial z} = B' \frac{\partial T}{\partial y} + B'' \frac{\partial m}{\partial y} \quad (47)$$

$$C_{55} \frac{\partial^2 w}{\partial x^2} + C_{44} \frac{\partial^2 w}{\partial y^2} + C_{33} \frac{\partial^2 w}{\partial z^2} + (C_{13} + C_{55}) \frac{\partial^2 u}{\partial x \partial z} + (C_{23} + C_{44}) \frac{\partial^2 v}{\partial y \partial z} = C' \frac{\partial T}{\partial z} + C'' \frac{\partial m}{\partial z} \quad (48)$$

where A', A'', B', \dots are thermoelastic constants, whose details are given in [Appendix A](#).

The homogeneous system of Eqs. (46)–(48) was employed already by [Pagano \(1969\)](#) to solve analytically the problem of composite plates in bending (one of the first analytical solutions available for composite structures).

The equations above can be modified and largely simplified under generalized plane conditions, for example in the case of plates stretched by a uniform axial force. Generalized plane conditions were used by [Pipes and Pagano \(1970\)](#) to solve (numerically) the state of stress at the free edges of axially strained composite plates. However, a plate stressed by hygrothermal loads cannot be considered, a priori, as being in generalized plane conditions: postulating generalized plate strain conditions implicitly assumes one side of the plate being much bigger than the other (for instance $L_x \gg L_y$): in principle, a full 3-D problem should be solved.

A simple method of solution, already employed by [Pagano \(1969\)](#) for plates in bending, consists in assuming part of the displacements field, satisfying some specific boundary conditions (Navier-like approach). For example:

$$\begin{aligned} u(x, y, z) &= U(z) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \cos \frac{n\pi x}{L_x} \sin \frac{m\pi y}{L_y} \\ v(x, y, z) &= V(z) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sin \frac{n\pi x}{L_x} \cos \frac{m\pi y}{L_y} \\ w(x, y, z) &= W(z) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sin \frac{n\pi x}{L_x} \sin \frac{m\pi y}{L_y} \end{aligned} \quad (49)$$

where now the plate is bounded by the planes $x = 0, L_x, y = 0, L_y, z = -e/2, e/2$. Accordingly, the temperature and composition fields have the form:

$$\begin{aligned} T(x, y, z) &= T(z) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sin \frac{n\pi x}{L_x} \sin \frac{m\pi y}{L_y} \\ m(x, y, z) &= m(z) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sin \frac{n\pi x}{L_x} \sin \frac{m\pi y}{L_y} \end{aligned} \quad (50)$$

The approach above is analogous to that employed by [Vel and Batra \(2001\)](#) for generalized plane strain thermoelastic problems: in that case “infinite” (truncated) series were used to simulate, with a desired level of accuracy, several different boundary conditions.

It is not difficult to see that, on the other hand, Eq. (49) satisfies simply supported boundary conditions at the edges for each term of order n, m , that is

$$S_{xx} = v = w = 0 \quad \text{at } x = 0, L_x \quad (51)$$

$$S_{yy} = u = w = 0 \quad \text{at } x = 0, L_y \quad (52)$$

By substituting expressions (49) and (50) into the equilibrium equations, one gets the following system of partial differential equations in the unknowns $U(z)$, $V(z)$ and $W(z)$. These equations are valid for each layer of the laminated structure, for all x, y and for each term of order n, m :

$$C_{55} \frac{\partial^2 U}{\partial z^2} - (p_n^2 C_{11} + p_m^2 C_{66})U - p_n p_m (C_{12} + C_{66})V + p_n (C_{13} + C_{55}) \frac{\partial W}{\partial z} = p_n A' T(z) + p_n A'' m(z) \quad (53)$$

$$C_{44} \frac{\partial^2 V}{\partial z^2} - (p_n^2 C_{66} + p_m^2 C_{22})V - p_n p_m (C_{12} + C_{66})U + p_m (C_{23} + C_{44}) \frac{\partial W}{\partial z} = p_m B' T(z) + p_m B'' m(z) \quad (54)$$

$$C_{33} \frac{\partial^2 W}{\partial z^2} - (p_n^2 C_{55} + p_m^2 C_{44})W - p_n (C_{13} + C_{55}) \frac{\partial U}{\partial z} - p_m (C_{23} + C_{44}) \frac{\partial V}{\partial z} = A' \frac{\partial T}{\partial z} + C'' \frac{\partial m}{\partial z} \quad (55)$$

where $p_n = n\pi/L_x$ and $p_m = m\pi/L_y$. The system of Eqs. (53)–(55), although quite cumbersome, can be solved by applying the standard methods of differential calculus. The solution is split in a homogeneous solution, dependent on the material properties and the geometrical parameters, and a particular solution, which depends on the hygrothermal fields $T(z)$ and $m(z)$. Of course boundary and interface (in the case of multi-material structures) conditions must be imposed. By indicating with S_E the external surfaces (at $z = e/2$ and $z = -e/2$) and with S_I the interface between adjacent layers these conditions read

$$\begin{aligned} S_{xz} = S_{yz} = S_{zz} &= 0 \quad \text{at } S_E \\ [S_{xz}] = [S_{yz}] = [S_{zz}] &= 0 \quad \text{at } S_I \\ [u] = [v] = [w] &= 0 \quad \text{at } S_I \end{aligned} \quad (56)$$

where the symbol $[\]$ indicated the jump of a given function over the interface. Boundary conditions Eq. (56) translate the fact that the external surfaces are free from transverse stresses and that, at the interfaces, no jumps of displacements and transverse stresses occur. Of course, interfaces are supposed to be perfectly bonded to each other (no cracked or weak interfaces). The system and its boundary conditions have been programmed and solved in MAPLE[®]. Further details about the solutions will be discussed in Appendix A.

2.3. Approximate methods for transient and cyclical hygrothermoelastic problems in composite laminates

Thin plates under load can be considered in a plane stress state, with constitutive equation (including hygrothermoelastic free strains):

$$\mathbf{S} = \mathbb{Q} : (\mathbf{E} - \mathbf{E}^{\text{HT}}) \quad (57)$$

\mathbb{Q} is the stiffness tensor of a ply under plane stress conditions.

The CLT can be completed by the Kirchhoff hypothesis for strains, which postulates (see, for instance, Timoshenko and Woinowsky-Krieger, 1954)

$$E_{xx} = \frac{\partial u}{\partial x} - z \frac{\partial^2 w}{\partial x^2} \quad E_{yy} = \frac{\partial v}{\partial y} - z \frac{\partial^2 w}{\partial y^2} \quad E_{xy} = \frac{1}{2} \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right) - z \frac{\partial^2 w}{\partial x \partial y} \quad (58)$$

where u , v and w are displacements of the mid-surface. By integrating Eq. (57) through the thickness, one gets the laminate constitutive equations:

$$\begin{pmatrix} \mathbf{N} \\ \mathbf{M} \end{pmatrix} = \begin{pmatrix} \mathbb{A} & \mathbb{B} \\ \mathbb{B} & \mathbb{D} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{E}^0 \\ \mathbf{K} \end{pmatrix} - \begin{pmatrix} \mathbf{N}^{\text{HT}} \\ \mathbf{M}^{\text{HT}} \end{pmatrix} \quad (59)$$

where

$$N_{xx} = \int_{-e/2}^{e/2} S_{xx} dz \quad N_{yy} = \int_{-e/2}^{e/2} S_{yy} dz \quad N_{xy} = \int_{-e/2}^{e/2} S_{xy} dz \quad (60)$$

$$M_{xx} = \int_{-e/2}^{e/2} S_{xx} z dz \quad M_{yy} = \int_{-e/2}^{e/2} S_{yy} z dz \quad M_{xy} = \int_{-e/2}^{e/2} S_{xy} z dz$$

$$\mathbb{A} = \int_{-e/2}^{e/2} \mathbb{Q} dz$$

$$\mathbb{B} = \int_{-e/2}^{e/2} \mathbb{Q} z dz \quad (61)$$

$$\mathbb{D} = \int_{-e/2}^{e/2} \mathbb{Q} z^2 dz$$

and

$$\begin{aligned}\mathbf{N}^{\text{HT}} &= \int_{-e/2}^{e/2} (\mathbb{Q} : \alpha T + \mathbb{Q} : \beta m) dz \\ \mathbf{M}^{\text{HT}} &= \int_{-e/2}^{e/2} (\mathbb{Q} : \alpha T + \mathbb{Q} : \beta m) z dz\end{aligned}\quad (62)$$

Solving a hygrothermal problem within the framework of the CLT implies generally the following steps:

- calculation of the appropriate homogenized quantities, $\mathbb{A}, \mathbb{B}, \mathbb{D}$ tensors (Eq. (61)) and hygrothermal force and moment resultants, Eq. (62),
- calculation of the total strains \mathbf{E}^0 and curvatures \mathbf{K} through inversion of Eq. (59),
- calculation of ply residual stresses through application, for each ply, of Eq. (57).

When the hygrothermal fields are given by transient equations, such those presented in Section 2.1, the calculation process is not only complex but also wrong.

In fact, the free strains related to transient hygrothermal fields are not compatible with the kinematics imposed by the Kirchhoff hypothesis, which postulates linear variation of strains over the thickness of the plate.

In principle, displacements should not be assigned *a priori* and should be derived by equilibrium and compatibility conditions. Alternatively, one should modify the Kirchhoff kinematics by means of corrective terms.

Another way to circumvent the problem has been proposed by Benkeddad et al. (1996). The actual hygrothermal fields can be approximated by piecewise linear functions:

$$c(z) = a_k z + b_k \quad (63)$$

where

$$a_k = \frac{c_k - c_{k-1}}{e_k} \quad b_k = c_{k-1} - \frac{c_k - c_{k-1}}{e_k} z_{k-1} \quad (64)$$

The actual hygrothermal fields are discretized over spatial regions called *sub-ply*s, with thickness e_k (see Fig. 2). One sub-ply may be representative of a true physical ply, or of parts of it. In this last case, sub-ply

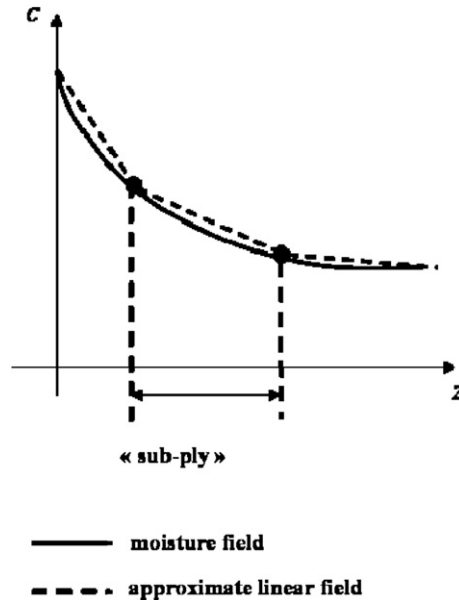


Fig. 2. Piecewise linear discretization of the true hygrothermal field over sub-ply.

do not have an immediate physical sense, but are introduced for discretization purpose only. Of course, according to the discretized hygrothermal fields in Eq. (63), resultant hygrothermal forces and moments (Eq. (62)) reduce to simple summations over M sub-plyes.

The employment of sub-plyes has some clear shortcomings. As the number of sub-plyes is finite, the concentration and the stress fields are approximate: the accuracy increases as the number of sub-plyes increases but the maximum number of sub-plyes is limited due to some physical considerations. In the present paper a macroscopic approach is proposed: thus, physically, a sub-ply must be able to represent the homogenised behaviour of the material (fibres + matrix), this imposes a limitation on its minimum dimensions. In the following examples the thickness of a sub-ply is fixed to 0.1 mm, which, a priori, seems to be a good compromise between accuracy of the method and physical significance of a sub-ply. Obviously, the total number M of sub-plyes is then determined by the total thickness of the laminated plate. The pertinence of the approximate method (modified CLT) will be checked in the following sections.

3. Assessment of the approximate method: comparison with analytical solutions

The approximate method (modified CLT) is tested via the available analytical solutions.

Not only solutions for the 3D plate but also simulations coming from the analytical solution for cylinders given by Jacquemin and Vautrin (2002b) are employed for assessment.

Actually the reasons for discrepancies between an approximate model and a 3D model of plate under hygrothermal stress are different from those which occur between an approximate model of plate and a model for cylinders. Moreover it has to be specified where (at which location) the comparison should be carried out. One could, for instance, investigate the distribution of stress at the centre of the plate and try to establish when stresses at this point start to be well described by models that employ plane stress conditions.

In the first case discrepancies are related to boundary conditions. As long as the boundary conditions are close to the centre of the plate, this point is far from being in plane stress conditions. In fact, 3D effects (transverse shear and out-of-plane normal stresses) are felt close to the external boundaries. This consideration may seem trivial, but it is less trivial to establish when (that is, for which dimensions) this phenomenon occurs.

In cylinders, 3D effects are due to the *slenderness* of the structure but also to the curvature (the radius) of the cylinder: we will try to establish when a cylinder, *also*, may be considered as a flat plate.

In the literature it is often found that plates become *thin* for a value of the ratio $(L_x, L_y)/e$ approximately equal or bigger than 10. Also for cylinders a value of 10 for the ratio R_{int}/e is usually retained as representative of a slender structure, R_{int} being the internal radius of the cylinder.

Material properties used for the simulations are representative of a high modulus IM7/977-2 carbon epoxy material. Its hygroscopic and mechanical properties are summarised in Tables 1 and 2.

For testing the models, three particular conditions which give rise to some extreme anisotropic effects are taken into account as examples.

Looking at laminates built from a given UD ply for instance, it is obvious that UD coupons achieve the maximum anisotropic effects. Furthermore, looking now at angle ply laminates, one key feature is that since the mechanical and thermoelastic ply mismatch is maximum between 0 and 90 plies cross-ply symmetric laminates should be selected to optimize the effects of the internal stresses at the ply level. Finally, since the joint effect of temperature differential and moisture changes should also be investigated, the previous cross-ply

Table 1
Hygroscopic properties of the IM7/977 2 carbon epoxy material

A (mm ² /h)	B (K ⁻¹)	C (%)	b
7.18	2910.2	0.0007	1.6036

Table 2
Mechanical properties of the IM7/977 2 carbon epoxy material

E_1 (GPa)	E_2 (GPa)	G_{12} (GPa)	ν_{12}	α_1 (°C ⁻¹)	α_2 (°C ⁻¹)	β_1	β_2
152	8.4	4.2	0.35	0.09×10^{-6}	28.8×10^{-6}	0	0.6

laminates subjected to transient hygroscopic state at various uniform temperatures are considered. It is clear indeed that effects of concentration gradients would be amplified for that type of laminates due to the ply orientations. Therefore, interesting conditions, based on previous experience of anisotropic effects in laminates and leading to reasonable bounds of the anisotropic behaviour under temperature and moisture changes are the following:

1. transient hygroscopic state in homogeneous materials (hygroscopic stress only),
2. uniform temperature differential in a $[0/90]_S$ multi-layered structure (thermal stress only),

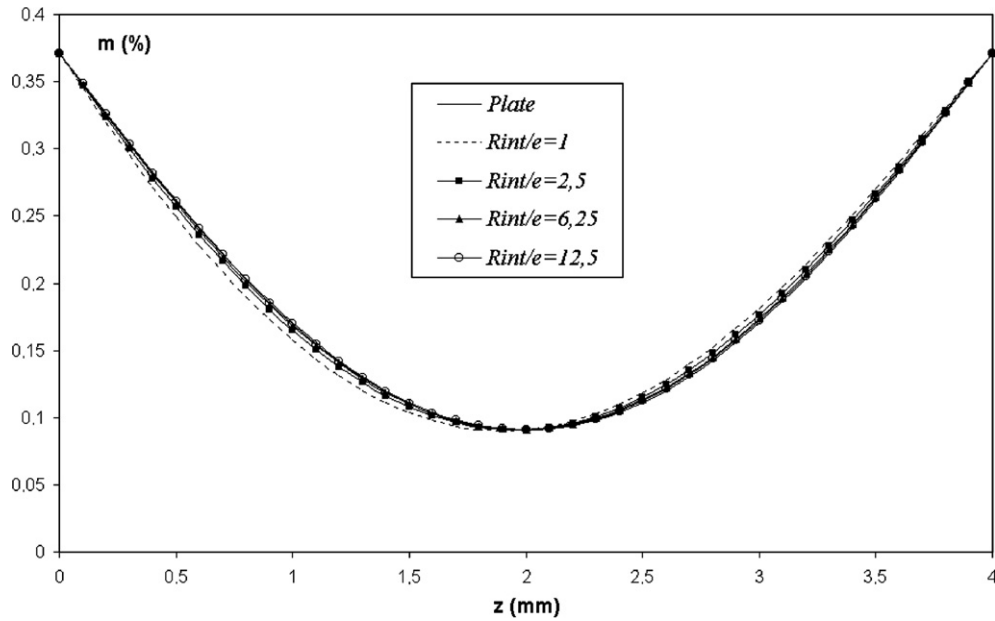


Fig. 3. Plate and cylinder hygroscopic fields after 3 months conditioning at $T = 23\text{ }^\circ\text{C}$ and $\text{RH}\% = 50\%$.

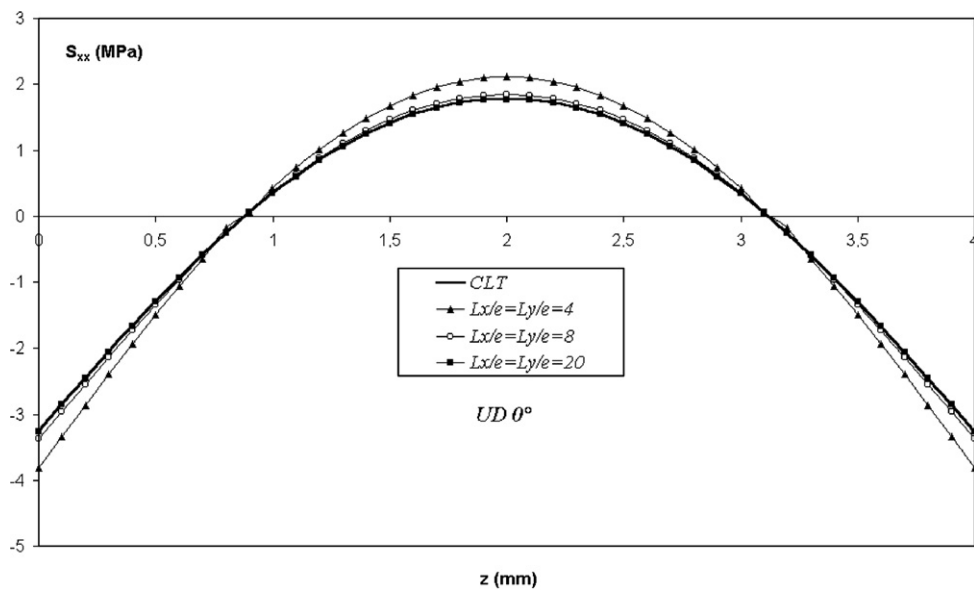


Fig. 4. Hygroscopic stress S_{xx} induced in UD plates by the moisture field (labelled "Plate") in Fig. 3.

3. uniform temperature differential + transient hygroscopic state in a $[0/90]_S$ multi-layered structure (full hygrothermal stress).

Fig. 3 presents through the thickness plate and cylinder hygroscopic fields in transient state after a 3 month conditioning at $T = 23\text{ }^\circ\text{C}$ and $\text{RH}\% = 50\%$ (note: $m(\%) = c/\rho_s(\%)$ where $\rho_s = 1500\text{ kg/m}^3$, which is a typical value for epoxy resins). The thickness of the structure is taken equal to 4 mm, the two external surfaces have the same concentration.

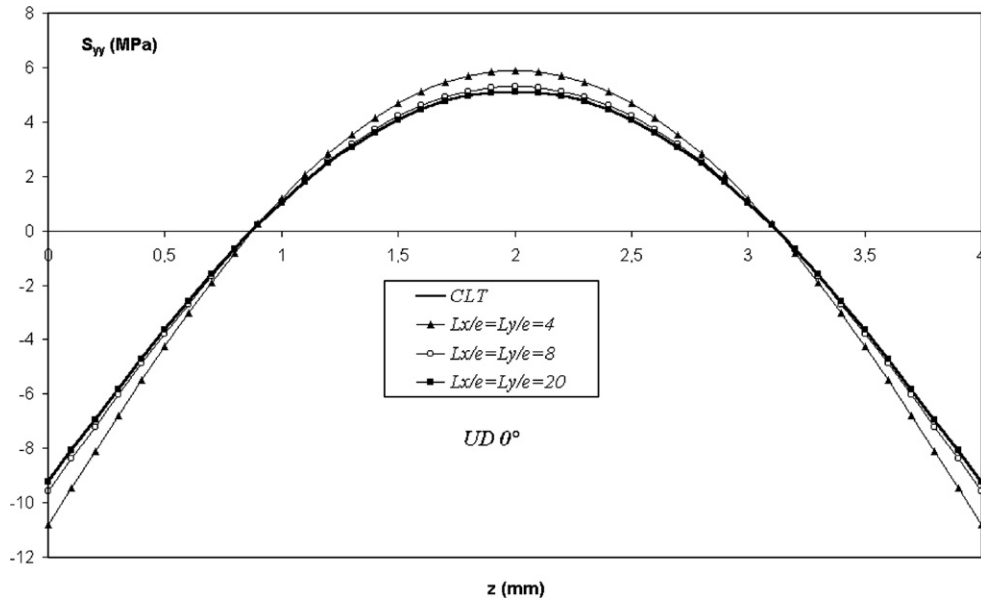


Fig. 5. Hygroscopic stress S_{yy} induced in UD plates by the moisture field (labelled “Plate”) in Fig. 3.

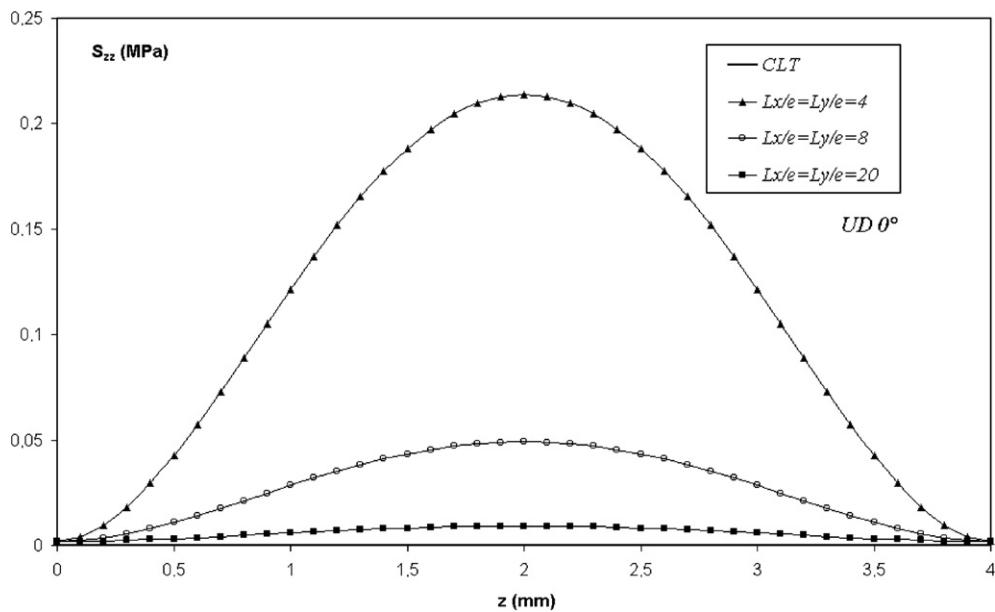


Fig. 6. Hygroscopic stress S_{zz} induced in UD plates by the moisture field (labelled “Plate”) in Fig. 3.

Fig. 3 allows one to estimate the difference between the transient terms of Eqs. (33) and (34). It can be noted that the curves practically overlap even for small values of R_{int}/e . The transient hygroscopic field is almost the same in all the structures under consideration.

Figs. 4-6 show the hygroscopic stresses S_{xx} , S_{yy} and S_{zz} calculated, at the centre of a unidirectional 0° plate (i.e. at $L_x = L_y = 0$), by the analytical solution and by the modified CLT as a function of the thickness for several values of $(L_x, L_y)/e$.

Discrepancies among the different curves are not very marked. From Figs. 4 and 5 it is noted that stresses S_{xx} and S_{yy} are compressive close to external surfaces and tensile for internal zones of the plate: however they

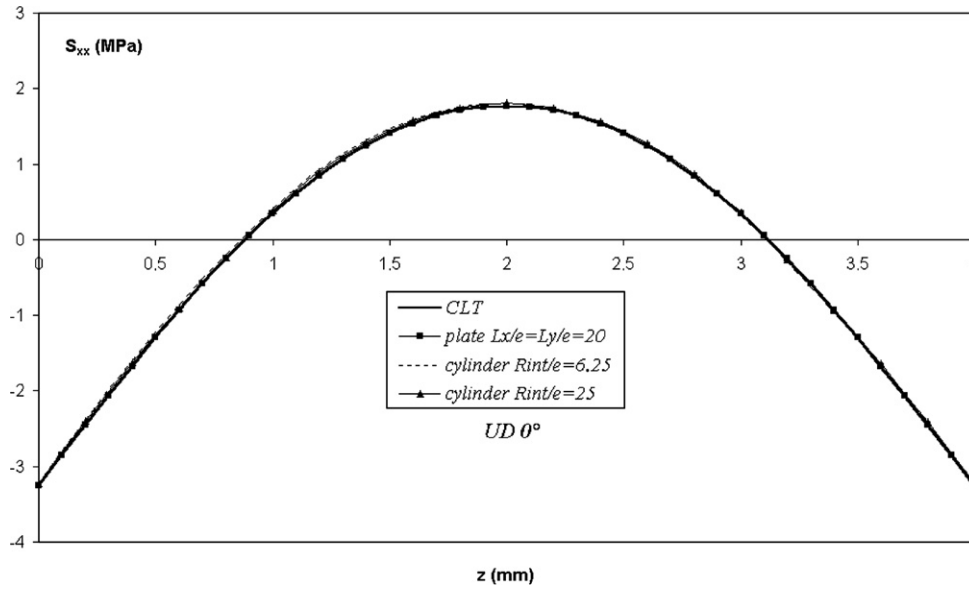


Fig. 7. Hygroscopic stress S_{xx} induced in UD plates and cylinders by the moisture field in Fig. 3.

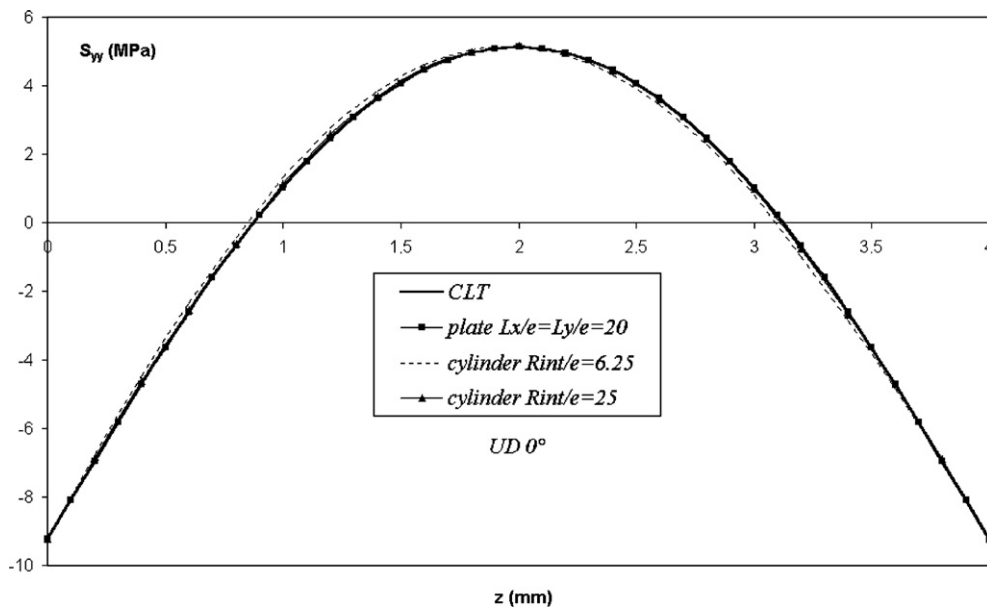


Fig. 8. Hygroscopic stress S_{yy} induced in UD plates and cylinders by the moisture field in Fig. 3.

are self-equilibrated as no external mechanical solicitations are applied in the x, y directions. The 3D analytical model starts to converge to the modified CLT one from $(L_x, L_y)/e$ around 8. However, quite perfect overlap is achieved only for $(L_x, L_y)/e$ equal to 20. In fact, from Fig. 6 it can be seen that the normal stress, S_{zz} , is negligible, approaching plane stress conditions (for the modified CLT $S_{zz} = 0$), for $(L_x, L_y)/e$ equal to about 20, even though its magnitude is quite small also for the other two cases. From the figures above, it can be deduced that assuming an internally stressed plate as being “thin” for $(L_x, L_y)/e > 10$ is a quite good approximation. In turn, the modified approximate CLT, which is based on a discretization procedure, works correctly and is well representative of a thin plate.

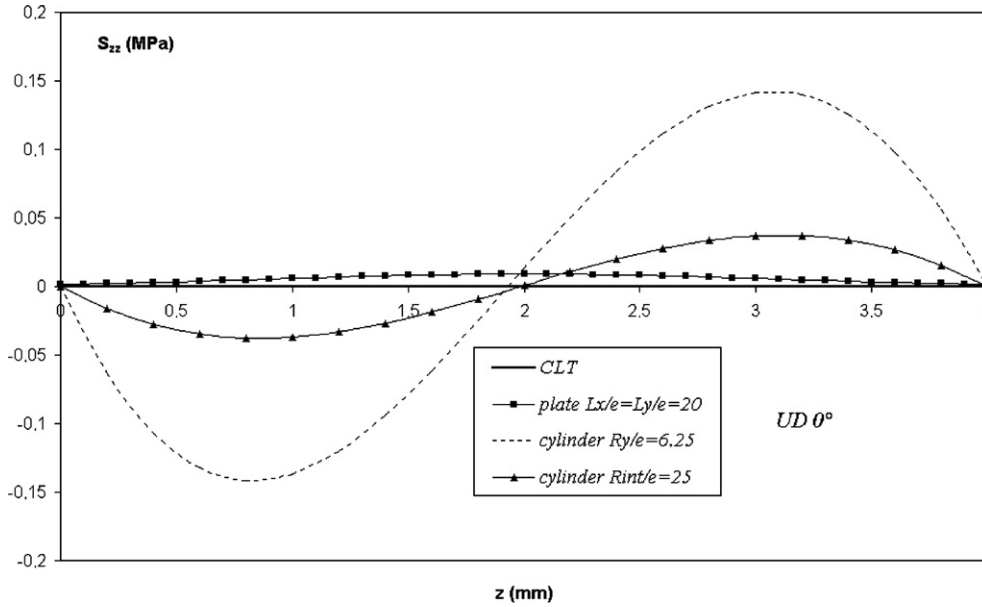


Fig. 9. Hygroscopic stress S_{zz} induced in UD plates and cylinders by the moisture field in Fig. 3.

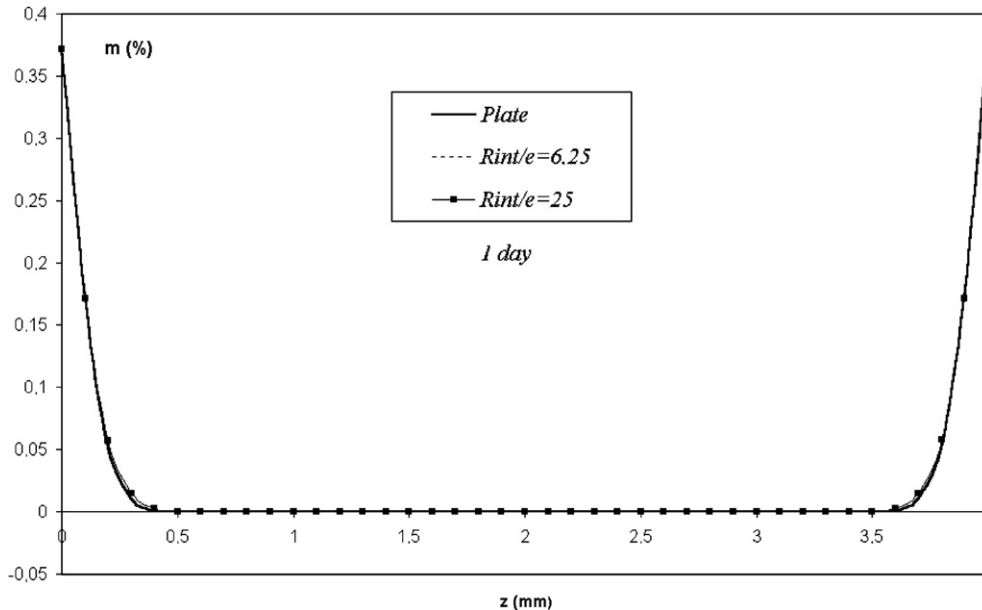


Fig. 10. Plate and cylinder hygroscopic fields after one day conditioning at $T = 23$ °C and HR% = 50%.

Figs. 7-9 present the hygroscopic stresses S_{xx} , S_{yy} and S_{zz} calculated by the modified CLT and compared to those evaluated analytically for a plate with $(L_x, L_y)/e = 20$ and for cylinders with $R_{int}/e = 6.25$ and $R_{int}/e = 25$.

The behaviour of internally stressed unidirectional cylinders is distinctly different from that of 3D plates. The solution converges to that of the modified CLT model, but the convergence is almost not influenced by the ratio R_{int}/e .

This is due to the fact that the *average* value of transverse stress S_{zz} close to zero whatever the value of R_{int}/e (see Fig. 9). S_{zz} stresses are, for this application, almost self-equilibrated. Also, S_{zz} in cylinders reaches magnitudes that are always almost negligible compared to S_{xx} and S_{yy} .

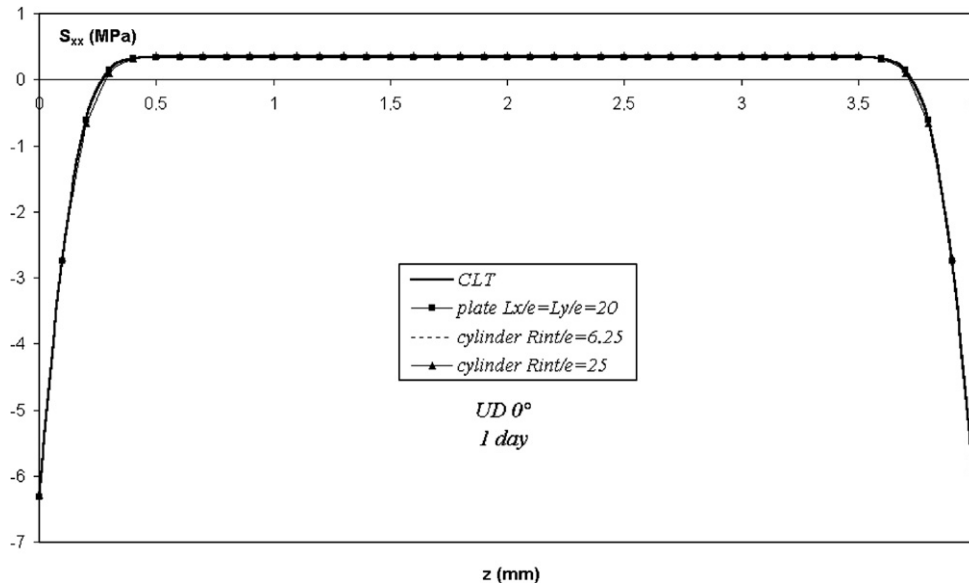


Fig. 11. Hygroscopic stress S_{xx} induced in UD plates and cylinders by the moisture field in Fig. 10.

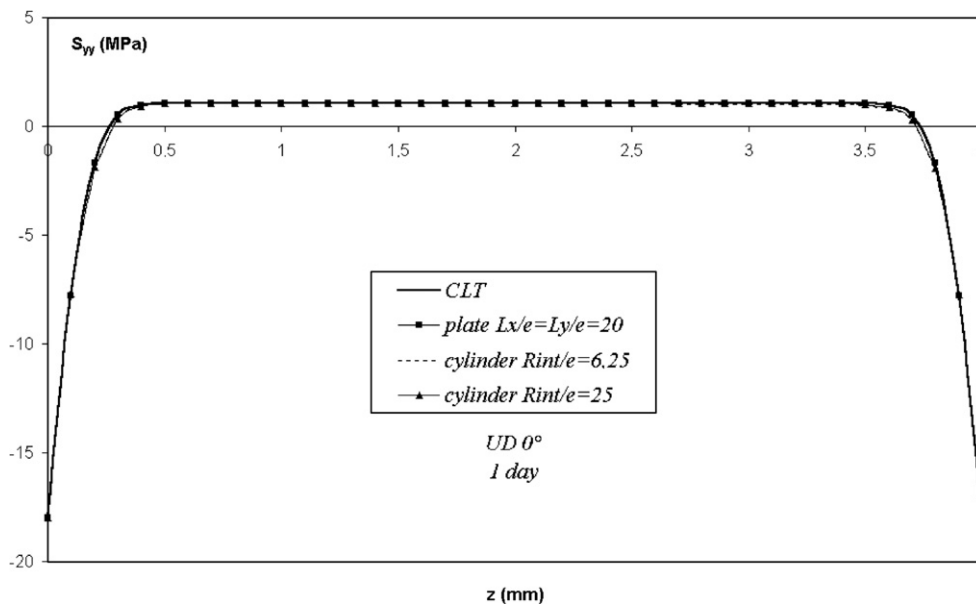


Fig. 12. Hygroscopic stress S_{yy} induced in UD plates and cylinders by the moisture field in Fig. 10.

An important conclusion can be drawn: transient internal stresses in homogeneous or unidirectional composite cylinders, whatever their geometrical arrangement (i.e. the ratio R_{int}/e), can be simulated accurately by the modified CLT model, which is simpler and computationally less expensive than the analytical one.

In order to check the full pertinence of the modified CLT model, “rougher” transient conditions are considered. In Fig. 10 the transient hygroscopic state is simulated after one day conditioning at the same environmental conditions used for the previous transient state.

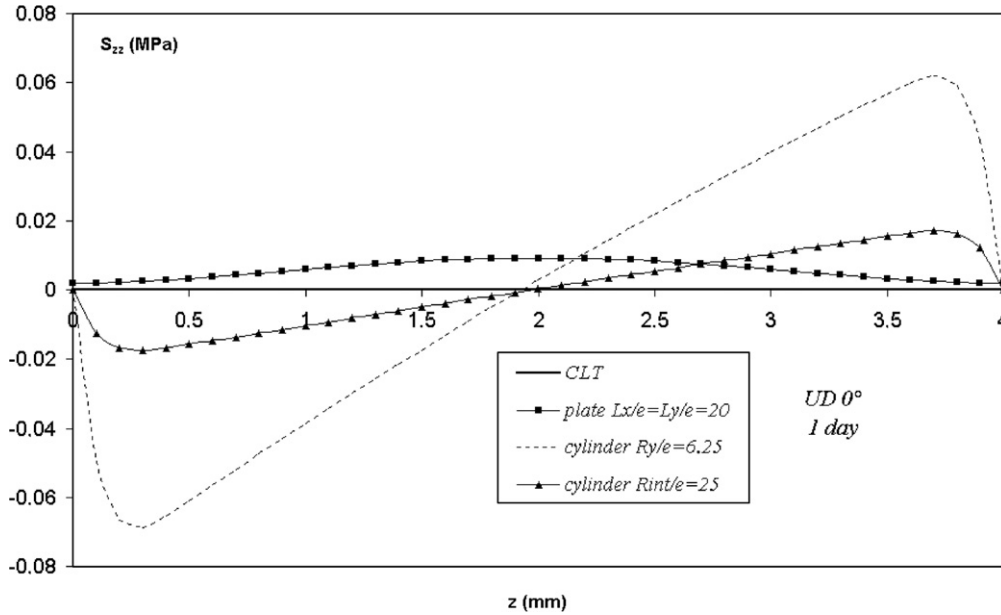


Fig. 13. Hygroscopic stress S_{zz} induced in UD plates and cylinders by the moisture field in Fig. 10.

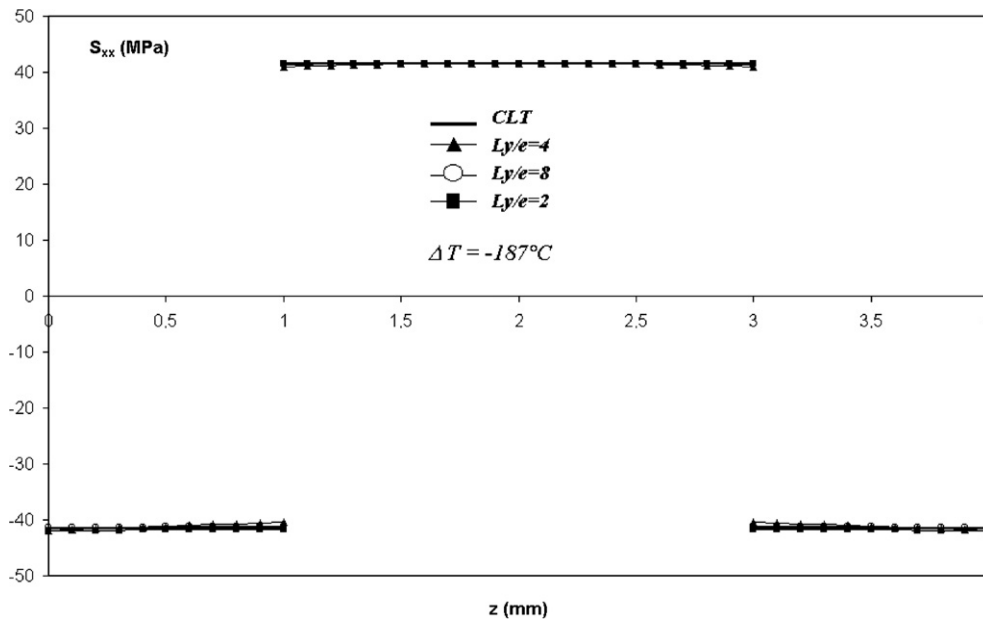


Fig. 14. Thermal stress S_{xx} induced in a $[0/90]_5$ plate by a uniform temperature differential $\Delta T = -187^\circ\text{C}$.

These conditions are “rougher” since they induce a less regular moisture distribution through the thickness of the plate/cylinder.

The moisture gradient close to the external surfaces is pronounced because the concentration drops from relatively high values (at the surface) to zero in a quite small distance. Moreover, analytical moisture profiles and stresses are given by series that converge very slowly for short conditioning times and therefore need to be checked. On the other hand, it is felt that for such “rough” distributions, the modified CLT model could not work well within insufficiently discretized domains.

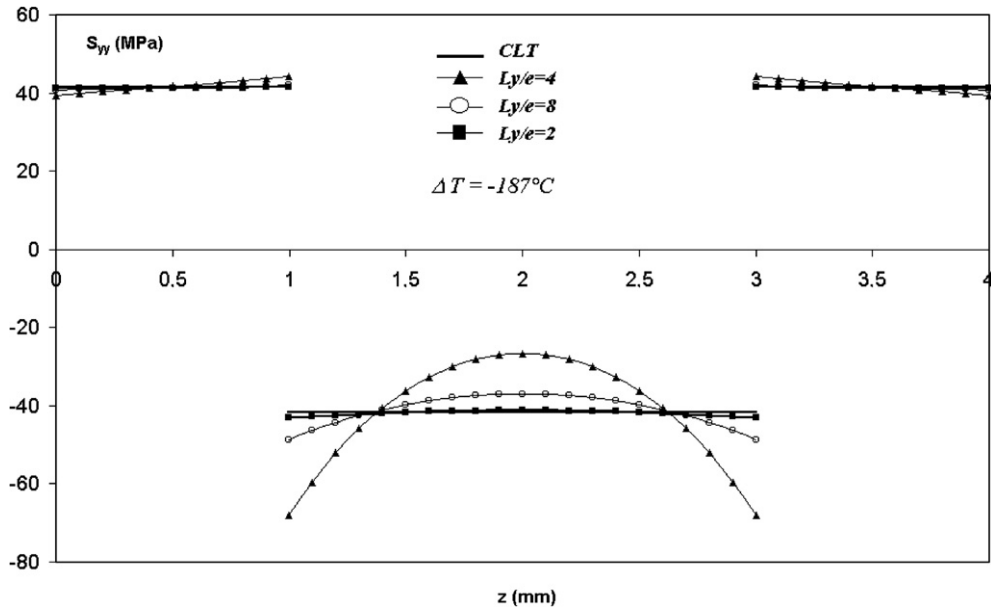


Fig. 15. Thermal stress S_{yy} , induced in a $[0/90]_8$ plate by a uniform temperature differential $\Delta T = -187^\circ\text{C}$.

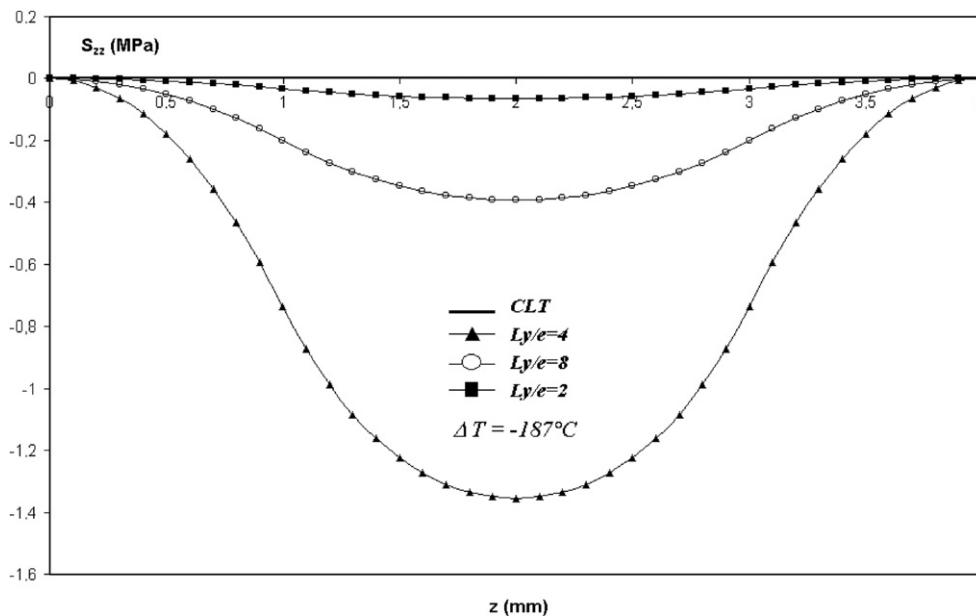


Fig. 16. Thermal stress S_{zz} , induced in a $[0/90]_8$ plate by a uniform temperature differential $\Delta T = -187^\circ\text{C}$.

From Figs. 11–13 it can be noted that again the approximate CLT model (with the chosen discretization) works very well capturing the plate stress fields. Again, the behaviour of orthotropic cylinders is almost independent of the ratio R_{int}/e .

The effect of uniform thermal fields, a uniform temperature differential $\Delta T = -187^\circ\text{C}$, on 4 mm $[0/90]_S$ plates is presented in Figs. 14–16 for several values of $(L_x, L_y)/e$. S_{xx} stresses are compressive in the 0° layers and tensile in the 90° layers. The inverse is true for S_{yy} stresses. Obviously, S_{xx} and S_{yy} are such that their

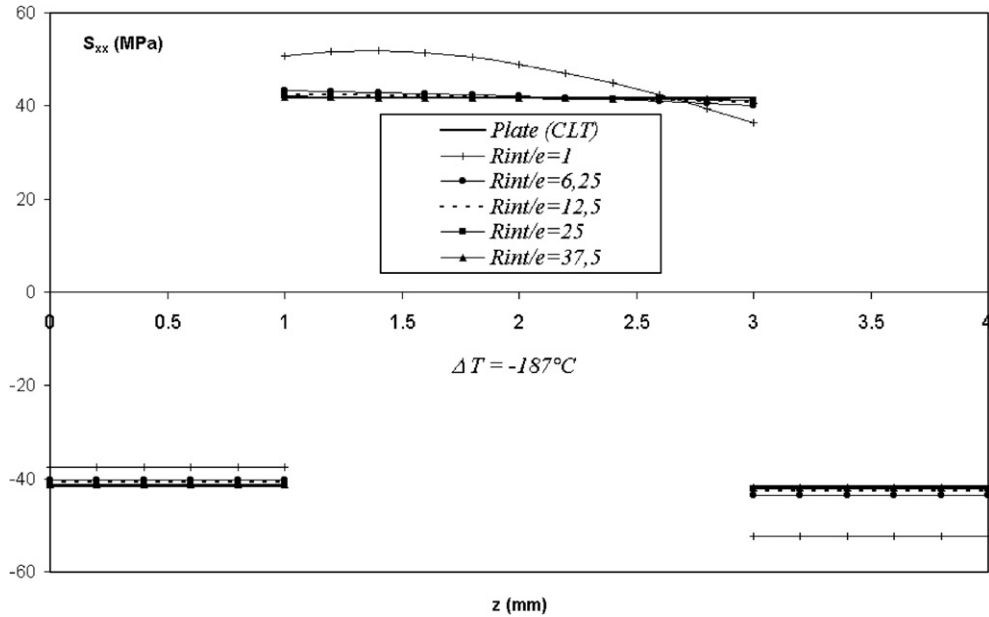


Fig. 17. Thermal stress S_{xx} induced in a $[0/90]_S$ plates and cylinders by a uniform temperature differential $\Delta T = -187^\circ\text{C}$.

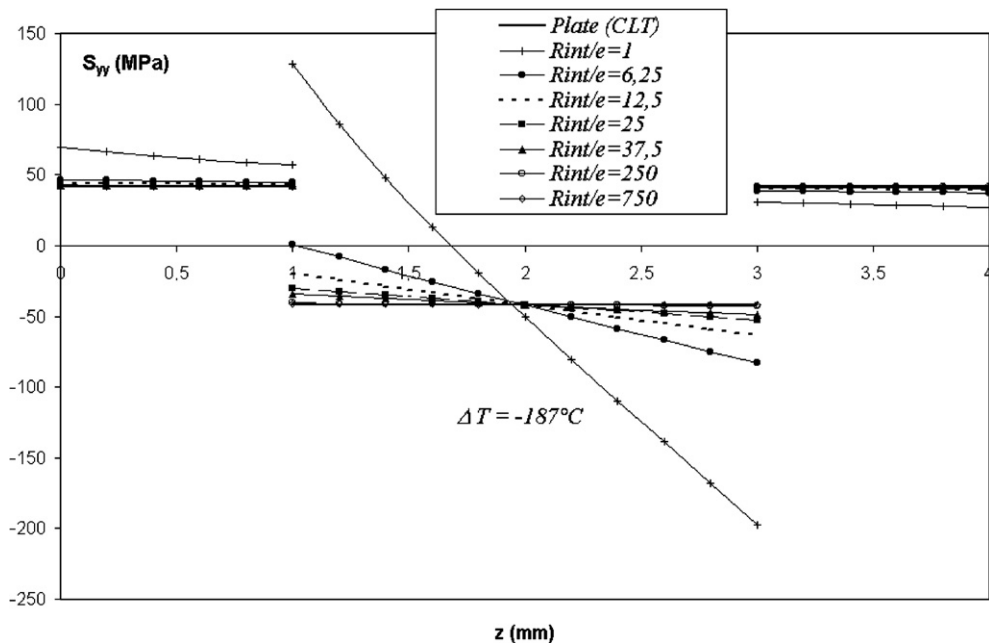


Fig. 18. Thermal stress S_{yy} induced in a $[0/90]_S$ plates and cylinders by a uniform temperature differential $\Delta T = -187^\circ\text{C}$.

resultant and the moments resulting from them are zero as no external loads are applied. Again analytical solutions converge to the modified CLT model for $(L_x, L_y)/e$ around 20. Compared to the purely hygroscopic case bigger discrepancies of the S_{yy} stresses are noted for $(L_x, L_y)/e$ equal to 4 and 8. This is true despite the fact that S_{zz} stresses are very small compared to the corresponding in-plane stresses.

The behaviour of cylinders (with several geometrical arrangements) subjected to the same temperature differential is presented in Figs. 17-19. The trend is very different from that of the purely hygroscopic case. Stresses (in particular S_{yy} , see Fig. 18) converge very slowly to the modified CLT solution and curves tend to

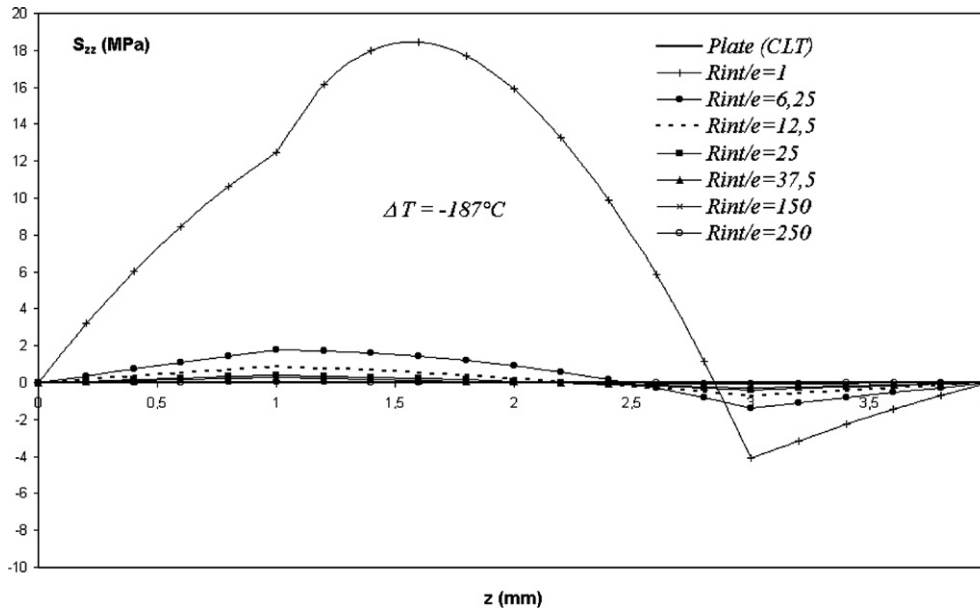


Fig. 19. Thermal stress S_{zz} induced in a $[0/90]_S$ plates and cylinders by a uniform temperature differential $\Delta T = -187^\circ\text{C}$.

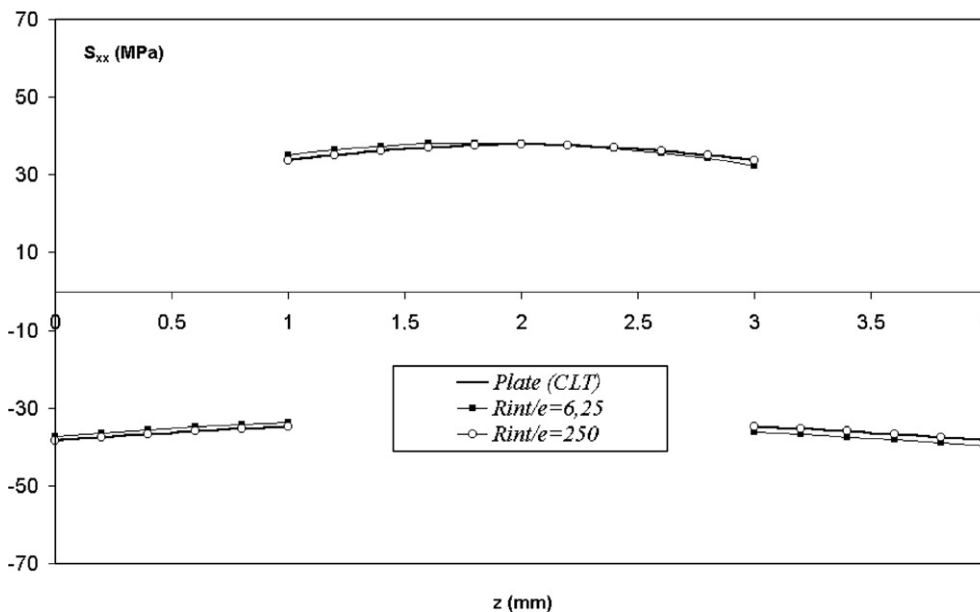


Fig. 20. Hygrothermal stress S_{xx} induced in $[0/90]_S$ plates and cylinders by $\Delta T = -187^\circ\text{C}$ and the moisture fields in Fig. 3.

overlap only for values of R_{int}/e higher than around 40. A steep gradient of the S_{yy} stress can be appreciated in the internal transverse (90°) layers of the cylinder for $R_{\text{int}}/e = 1$ and $R_{\text{int}}/e = 6.25$.

For small values of R_{int}/e the S_{zz} stress is largely far from being self-equilibrated. In Fig. 19, for low R_{int}/e ratios, curves cross the abscissa axis quite far from the mid-thickness.

Analogous considerations hold for a cylinder subjected to mixed hygrothermal conditions, that is, a uniform temperature differential $\Delta T = 187^\circ\text{C}$ and a 3 month moisture conditioning at 23°C and $\text{RH}\% = 50\%$ (case 1 + case 2), see Figs. 20-22.

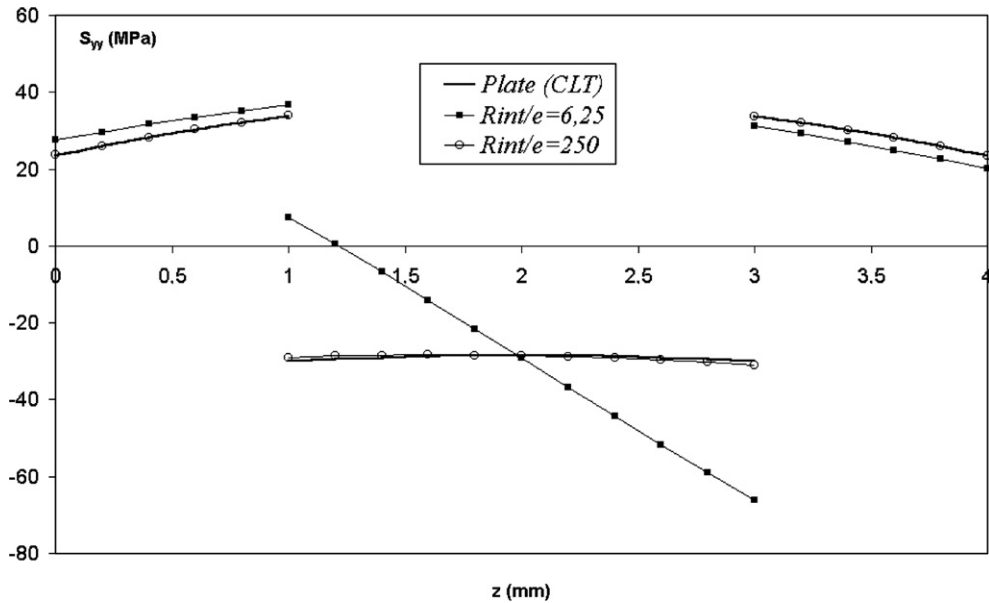


Fig. 21. Hygrothermal stress S_{yy} induced in $[0/90]_S$ plates and cylinders by $\Delta T = 187^\circ\text{C}$ and the moisture fields in Fig. 3.

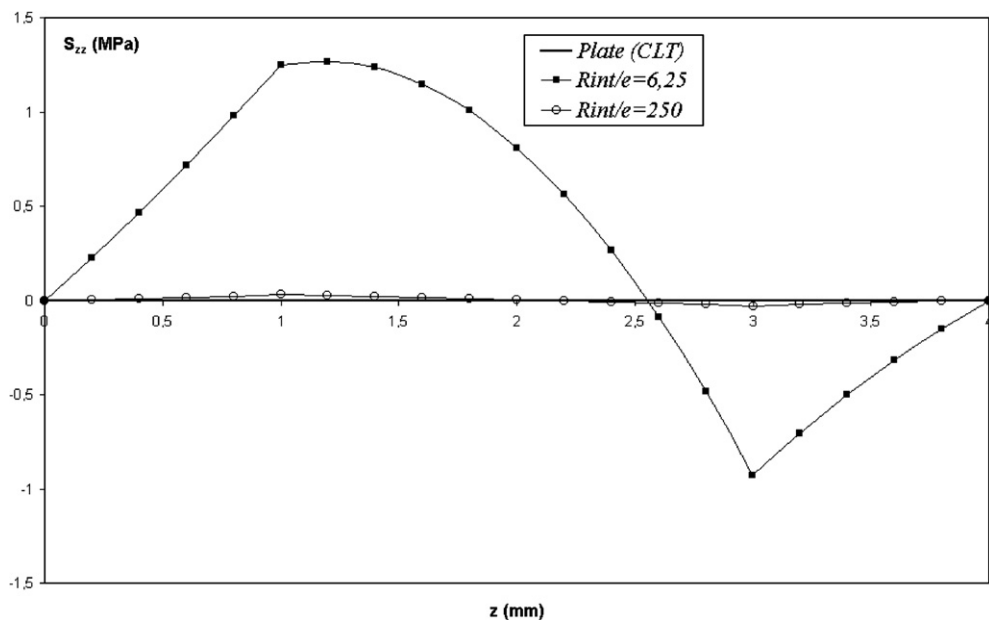


Fig. 22. Hygrothermal stress S_{zz} induced in $[0/90]_S$ plates and cylinders by $\Delta T = 187^\circ\text{C}$ and the moisture fields in Fig. 3.

Moisture conditioning induces stresses that tend to counteract the purely thermal induced ones. Analogous conclusions to those done for plate and cylinder structures subjected to pure thermal loads can be done.

4. Conclusions

The present paper presents a comparison between models of hygrothermoelastic stress in elastic 3D plates, cylinders and an approximate model of plate in plane stress. The approximations reside not only in postulating plane stress conditions, that is, neglecting out-of-plane normal and transverse stresses, by using the Kirchhoff hypothesis but also by approximating the quite complex hygrothermal fields related to transient and cyclical environmental conditions.

Comparisons indicate that the response of a hygrothermally stressed 3D plate tends towards that of slender 2D (plane stress) plates for a ratio $(L_x, L_y)/e > 10$, that is, when the influence of boundary conditions coming from the edges on internal zones of the plate become negligible. This effect *is independent of the type of conditioning and of the level of anisotropy* of the structure under consideration and it is true even under quite rough hygrothermal conditions, where moisture or thermal gradients are important. On the contrary this result depends possibly on the type of boundary conditions at the edges. In the present paper only one kind of edge boundaries is considered (simply supported); further analyses should be conducted by employing, for assessment purposes, analytical solutions for more complex edge boundary conditions. These analytical solutions are still not available, due to great complexity of the equations to be solved. Up to date, the most refined analytical solution seems to be that by Vel and Batra (2001), which is still for generalized plane strain conditions.

Solutions from the approximate plate model are compared to solutions for cylinders. Quite surprisingly, the model works very well whatever the ratio R_{int}/e of the cylinder. However this is true only for certain special conditions, comparisons agree only for homogeneous or unidirectional composite (orthotropic-like) cylinders. As soon as the level of anisotropy of the structure is increased, for example when considering $[0/90]_s$ laminated cylinders, the trend is very different and the two solutions converge only for R_{int}/e approximately equal to 40.

The approximate CLT model has been used to simulate cyclical hygrothermoelastic stresses in composite plates subject to supersonic hygrothermal environments, in which external temperature and relative humidity vary in a complex cyclical manner (Gigliotti et al., 2004), to design experiments and to perform optimization of structures subject to hygrothermoelastic stress.

The easiness of the method makes it an ideal tool for design purposes.

Appendix A

Solving Eqs. (53) (55) in Section 2.2 give the unknowns $U(z)$, $V(z)$ and $W(z)$. These equations are valid for each layer of the laminated structure, for all x , y and for each term of order n , m :

$$C_{55} \frac{\partial^2 U}{\partial z^2} - (p_n^2 C_{11} + p_m^2 C_{66})U - p_n p_m (C_{12} + C_{66})V + p_n (C_{13} + C_{55}) \frac{\partial W}{\partial z} = p_n A' T(z) + p_n A'' m(z) \quad (A.1)$$

$$C_{44} \frac{\partial^2 V}{\partial z^2} - (p_n^2 C_{66} + p_m^2 C_{22})V - p_n p_m (C_{12} + C_{66})U + p_m (C_{23} + C_{44}) \frac{\partial W}{\partial z} = p_m B' T(z) + p_m B'' m(z) \quad (A.2)$$

$$C_{33} \frac{\partial^2 W}{\partial z^2} - (p_n^2 C_{55} + p_m^2 C_{44})W - p_n (C_{13} + C_{55}) \frac{\partial U}{\partial z} - p_m (C_{23} + C_{44}) \frac{\partial V}{\partial z} = A' \frac{\partial T}{\partial z} + C'' \frac{\partial m}{\partial z} \quad (A.3)$$

where

$$\begin{aligned} A' &= (C_{11}\alpha_{xx} + C_{12}\alpha_{yy} + C_{13}\alpha_{zz}); & A'' &= (C_{11}\beta_{xx} + C_{12}\beta_{yy} + C_{13}\beta_{zz}) \\ B' &= (C_{12}\alpha_{xx} + C_{22}\alpha_{yy} + C_{23}\alpha_{zz}); & B'' &= (C_{12}\beta_{xx} + C_{22}\beta_{yy} + C_{23}\beta_{zz}) \\ C' &= (C_{13}\alpha_{xx} + C_{23}\alpha_{yy} + C_{33}\alpha_{zz}); & C'' &= (C_{13}\beta_{xx} + C_{23}\beta_{yy} + C_{33}\beta_{zz}) \end{aligned} \quad (A.4)$$

The homogeneous system can be rewritten as

$$\begin{aligned} a_1 U_{zz} + b_1 W_z - c_1 U - d_1 V &= 0 \\ a_2 V_{zz} + b_2 W_z - c_2 U - d_2 V &= 0 \\ a_3 W_{zz} - b_3 W_z - c_3 U_z - d_3 V_z &= 0 \end{aligned} \quad (\text{A.5})$$

where the subscript ($_z$) indicates derivative of a function with respect to z : U, V, W are unknown functions of z . A solution of Eq. (A.5) is given by

$$\begin{aligned} U &= A \exp^{\lambda z} \\ V &= B \exp^{\lambda z} \\ W &= C \exp^{\lambda z} \end{aligned} \quad (\text{A.6})$$

By substituting Eq. (A.5) in (A.4):

$$\begin{aligned} A(a_1 \lambda^2 - c_1) - B d_1 + C b_1 \lambda &= 0 \\ -A c_2 + B(a_2 \lambda^2 - d_2) + C b_2 \lambda &= 0 \\ -A c_3 \lambda + B d_3 \lambda + C(a_3 \lambda^2 - b_3) &= 0 \end{aligned} \quad (\text{A.7})$$

Eq. (A.7) have solutions if

$$\det \begin{pmatrix} (a_1 \lambda^2 - c_1) & -d_1 & b_1 \lambda \\ -c_2 & (a_2 \lambda^2 - d_2) + b_2 \lambda & b_2 \lambda \\ -c_3 \lambda & -d_3 \lambda & (a_3 \lambda^2 - b_3) \end{pmatrix} = 0 \quad (\text{A.8})$$

Solving Eq. (A.8) gives six values of λ , from which the corresponding (six) values of (A, B, C) can be calculated. The superposition of solutions of the type $U_k = A_k e^{\lambda_k z}$, $k = 1, \dots, 6$ gives the searched *homogeneous* solution of U , the same for V and W .

The particular solutions of U, V and W can be found by employing, for instance, the method of *undetermined coefficients*. This method consists in taking as particular solutions the same functions multiplied by unknown coefficients as those in the left hand side of Eqs. (A.1) (A.3). Substituting these functions in the non-homogeneous system (A.1) (A.3) gives rise to a conceptually simple but quite lengthy system of algebraic equations, which is not detailed here.

References

- Allen, D.H., 1991. Thermo mechanical coupling in inelastic solids. *Applied Mechanics Reviews* 44, 361–373.
- Benkaddad, A., Grédiac, M., Vautrin, A., 1996. Computation of transient hygroscopic stresses in laminated composite plates. *Composites Science and Technology* 56, 869–876.
- Byron Pipes, R., Pagano, N.J., 1970. Interlaminar stresses in composite laminates under uniform axial extension. *Journal of Composite Materials* 4, 538–548.
- Carslaw, H.S., Jaeger, J.C., 1959. *Conduction of Heat in Solids*, second ed. Oxford Science Publication.
- Chandrasekharaiah, D.S., 1986. Thermoelasticity with second sound: a review. *Applied Mechanics Reviews* 39, 355–376.
- Crank, J., 1975. *The Mathematics of Diffusion*. Oxford University Press, London.
- Gigliotti, M., Jacquemin, F., Vautrin, A., 2004. Internal stresses in composite laminates due to cyclical hygrothermal loading. *AIAA Journal* 42, 2600–2605.
- Hahn, H.T., Kim, R.Y., 1978. Swelling of composite laminates. In: Vinson, J.R. (Ed.), *ASTM STP 658*, Philadelphia, PA, pp. 98–120.
- Herakovitch, C., 1997. *Mechanics of Fibrous Composites*. John Wiley and Sons.
- Huang, N.N., Tauchert, T.R., 1988. Postbuckling response of antisymmetric angle ply laminates to uniform temperature loading. *Acta Mechanica* 72, 173–183.
- Hyer, M.W., Rousseau, C.Q., 1987. Thermally induced stresses and deformations in angle ply composite tubes. *Journal of Composite Materials* 21, 454–480.
- Jacquemin, F., Vautrin, A., 2002a. Modelling of the moisture concentration field due to cyclical hygrothermal conditions in thick laminated pipes. *European Journal of Mechanics – A/Solids* 21, 845–855.
- Jacquemin, F., Vautrin, A., 2002b. A closed form solution for the internal stresses in thick composite cylinders induced by cyclical environmental conditions. *Composite Structures* 58, 1–9.

- Kao, W.T., Pao, Y.C., 1976. Thermally induced vibration of simply supported symmetric cross ply plates. *Developments in Theoretical and Applied Mechanics* 8, 331-348.
- Noor, A.K., Burton, W.S., 1989. Assessment of shear deformation theories for multilayered composite plates. *Advanced Mechanics Reviews* 42, 1-13.
- Noor, A.K., Burton, W.S., 1992. Computational models for high temperature multilayered composite plates and shells. *Applied Mechanics Reviews* 45, 419-446.
- Ootao, Y., Tanigawa, Y., Fukuda, T., 1991. Axisymmetric transient thermal stress analysis of a multilayered composite hollow cylinder. *Journal of Thermal Stresses* 14, 201-213.
- Pagano, N.J., 1969. Exact solutions for composite laminates in cylindrical bending. *Journal of Composite Materials* 3, 398-411.
- Paul, D., Vautrin, A., 1995. Transient hygrothermal stresses in laminated cylinders. In: Cardon, A.H., Fukuda, H., Reifsnider, K. (Eds.), *Proceedings of the Second International Conference on Progress in Durability Analysis of Composite Systems, DURACOSYS 95*, pp. 281-288.
- Reddy, J.N., Hsu, Y.S., 1980. Effects of shear deformation and anisotropy on the thermal bending of layered composite plates. *Journal of Thermal Stresses* 3, 475-493.
- Savoia, M., Reddy, J.N., 1995. Three dimensional thermal analysis of laminated composite plates. *International Journal of Solids and Structures* 32, 593-608.
- Shen, C.H., Springer, G.S., 1981. Moisture absorption and desorption of composite materials. In: Springer, G.S. (Ed.), *Environmental Effects on Composite Materials*. Technomic Publisher, pp. 15-33.
- Tauchert, T.R., 1991. Thermally induced flexure, buckling and vibration of plates. *Applied Mechanics Reviews* 44, 347-360.
- Timoshenko, S., Woinowsky Krieger, S., 1954. *Theory of Plates and Shells*. McGraw Hill Book Company, Inc.
- Tungikar, V.B., Rao, K.M., 1994. Three dimensional exact solution of thermal stresses in rectangular composite laminate. *Composite Structures* 27, 419-430.
- Vel, S.S., Batra, R.C., 2001. Generalized plane strain thermoelastic deformation of laminated anisotropic thick plates. *International Journal of Solids and Structures* 38, 1395-1414.
- Verchery, G., 1992. Moisture diffusion in polymeric matrix composites with cyclic environmental conditions. In: Bunsell, A.R., Jamet, J.F., Messiah, A. (Eds.), *Proceedings of the 5th European Conference on Composite Materials (ECCM5)*, Bordeaux, France, pp. 505-510.
- Weitsman, Y.J., 1987. Stress assisted diffusion in elastic and viscoelastic materials. *Journal of the Mechanics and Physics of Solids* 35, 73-93.
- Weitsman, Y.J., 1998. Effects of fluids on polymeric composites. A review. In: Talreja, R. (Ed.), *Polymer Matrix Composite*. Pergamon Press, pp. 369-401.
- Whitney, J.M., Ashton, J.E., 1971. Effect of environment on the elastic response of layered composite plates. *AIAA Journal* 9, 1708-1713.
- Williams, T.O., 2005. A generalised, multilength scale framework for thermo diffusional mechanically coupled, nonlinear, laminated plate theories with delaminations. *International Journal of Solids and Structures* 42, 1465-1490.
- Wu, C.H., Tauchert, T.R., 1980a. Thermoelastic analysis of laminated plates. 1: Symmetric specially orthotropic laminates. *Journal of Thermal Stresses* 3, 247-259.
- Wu, C.H., Tauchert, T.R., 1980b. Thermoelastic analysis of laminated plates. 2: Antisymmetric cross ply and angle ply laminates. *Journal of Thermal Stresses* 3, 365-378.
- Zenkour, A.M., 2004. Analytical solution for bending of cross ply laminated plates under thermo mechanical loading. *Composite Structures* 65, 367-379.