



**HAL**  
open science

# NONLOCAL HAMILTON-JACOBI EQUATIONS ON A NETWORK WITH KIRCHHOFF TYPE CONDITIONS

Guy Barles, Olivier Ley, Erwin Topp

► **To cite this version:**

Guy Barles, Olivier Ley, Erwin Topp. NONLOCAL HAMILTON-JACOBI EQUATIONS ON A NETWORK WITH KIRCHHOFF TYPE CONDITIONS. 2024. hal-04791608

**HAL Id: hal-04791608**

**<https://hal.science/hal-04791608v1>**

Preprint submitted on 19 Nov 2024

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# NONLOCAL HAMILTON-JACOBI EQUATIONS ON A NETWORK WITH KIRCHHOFF TYPE CONDITIONS

GUY BARLES, OLIVIER LEY, AND ERWIN TOPP

ABSTRACT. In this article, we consider nonlocal Hamilton-Jacobi Equations on networks with Kirchhoff type conditions for the interior vertices and Dirichlet boundary conditions for the boundary ones: our aim is to provide general existence and comparison results in the case when the integro-differential operators are of order strictly less than 1. The main originality of these results is to allow these nonlocal terms to have contributions on several different edges of the network. The existence of Lipschitz continuous solutions is proved in two ways: either by using the vanishing viscosity method or by the usual Perron's method. The comparison proof relies on arguments introduced by Lions and Souganidis. We also introduce a notion of flux-limited solution, nonlocal analog to the one introduced by Imbert and Monneau, and prove that the solutions of the Kirchhoff problem are flux-limited solutions for a suitable flux-limiter. After treating in details the case when we only have one interior vertex, we extend our approach to treat general networks.

## 1. INTRODUCTION.

**1.1. General description of the problem.** In this article, we are interested in nonlocal Hamilton-Jacobi Equations (NLHJE) posed on a general network  $\Gamma$  with Kirchhoff type conditions for the interior vertices and Dirichlet (or other) boundary conditions for the boundary ones. Our aim is to provide a complete approach in the case when the nonlocal term is of order strictly less than 1 but may involve integral terms on several edges, and not only the current one. This means existence and uniqueness results but also a connection with the notion of flux-limited solutions of Imbert and Monneau [27].

A general network  $\Gamma$  in  $\mathbb{R}^d$  is made of a finite number of vertices  $\bar{\mathbf{v}} \in \mathbf{V}$  connected with a finite number of edges  $E \in \mathbf{E}$ . Each edge has the differential structure of a curve and, by using a suitable parametrization, we can define an Hamilton-Jacobi Equation on it. In general, these equations are edge-by-edge unrelated, but after adding Kirchhoff-type conditions on some *interior vertices*  $\bar{\mathbf{v}} \in \mathbf{V}_i$  (those which are connected to several edges),

---

*Date:* November 19, 2024.

*2010 Mathematics Subject Classification.* 35K55, 35R09, 47G20, 35B40, 33B20.

*Key words and phrases.* Nonlocal Equations, Hamilton-Jacobi Equations, Networks, Kirchhoff conditions, Existence, Comparison, Regularity, Junction viscosity solutions, Flux-limited solutions.

and suitable boundary conditions on the remaining *boundary vertices*  $\bar{\mathbf{v}} \in \mathbf{V}_b$ , with  $\mathbf{V} = \mathbf{V}_i \cup \mathbf{V}_b$ ,  $\mathbf{V}_i \cap \mathbf{V}_b = \emptyset$ , we are lead to a system of equations on the whole network which may be well-posed.

We first describe the type of problem we have in mind at a formal level. We consider the stationary Kirchhoff-Dirichlet problem

$$(1.1) \quad \begin{cases} \lambda u - \mathcal{I}u(x) + H(x, u_x) = 0, & x \in \Gamma \setminus \mathbf{V} \quad (\text{NLHJE}), \\ \sum_{E \in \text{Inc}(\bar{\mathbf{v}})} -\partial_E u(\bar{\mathbf{v}}) = B_{\bar{\mathbf{v}}}, & \bar{\mathbf{v}} \in \mathbf{V}_i \quad (\text{Kirchhoff condition}), \\ u(\bar{\mathbf{v}}) = h_{\bar{\mathbf{v}}}, & \bar{\mathbf{v}} \in \mathbf{V}_b \quad (\text{Dirichlet condition}). \end{cases}$$

Here  $\lambda > 0$  is a constant, the nonlocal operator  $\mathcal{I}$  and the Hamiltonian  $H$  are, in fact, some collections  $\{\mathcal{I}_E\}_{E \in \mathbf{E}}$ ,  $\{H_E\}_{E \in \mathbf{E}}$  such that, on the one hand,

$$(1.2) \quad \mathcal{I}_E u(x) = \int_{\Gamma} [u(z) - u(x)] \nu_E(x, z) dz, \quad x \in E,$$

is an integro-differential operator whose kernel  $\nu_E$  satisfies a Lévy-type integrability condition, namely

$$(1.3) \quad \int_{\Gamma} \min\{\text{dist}(x, z)^{\sigma_E}, 1\} \nu_E(x, z) dz < +\infty \quad \text{for some } 0 < \sigma_E < 1,$$

where we have identified  $dz$  with  $d\mathcal{H}^1(z)$ , the 1-dimensional Hausdorff measure in  $\mathbb{R}^d$ .

For a general network, we have no intrinsic definition of  $u_x$  and therefore no intrinsic definition of  $H_E$ . This requires a parametrization of each edge, a definition of  $u_x$  and then of  $H_E$  for any  $E$ . We do not want to enter into details here and we refer the reader to Section 7 for a complete description of this general case.

In this introduction, and in most of our article, we are going to consider the case of a simple junction, i.e., the case when there is only one interior point and the different edges are segments. More specifically,  $\mathbf{V}_i = \{O\}$ , where  $O$  is the origin in  $\mathbb{R}^d$ , connected to a finite number of finite length edges  $\{E_i\}_{1 \leq i \leq N}$  that link it to a finite set of exterior vertices  $\mathbf{V}_b = \{\bar{\mathbf{v}}_i\}_{1 \leq i \leq N}$ . In this setting, we write

$$E_i = \{t\bar{\mathbf{v}}_i, t \in (0, 1)\},$$

and, for  $x \in E_i$ , we set  $x_i := |x|$ , which is the parametrization by arc length of the (open) segment  $E_i$ . Finally we define  $u_i : [0, a_i] \rightarrow \mathbb{R}$ , where  $a_i = |\bar{\mathbf{v}}_i|$ , by

$$u_i(x_i) := u(x) \quad \text{if } x \in E_i,$$

and we use the notation  $u_{x_i}(x) = u'_i(x_i)$ , again if  $x \in E_i$ . With these notations, we can define in a proper way the Hamiltonians  $H_i : E_i \times \mathbb{R} \rightarrow \mathbb{R}$

and our problem can be written as

$$(1.4) \quad \begin{cases} \lambda u(x) - \mathcal{I}_i u(x) + H_i(x, u_{x_i}) = 0, & x \in E_i, \ 1 \leq i \leq N, \\ \sum_{1 \leq i \leq N} -u_{x_i}(O) = B, \\ u(a_i) = h_i, & 1 \leq i \leq N, \end{cases}$$

where, for the sake of notations, we replace the dependence with respect to the edge  $E_i$  by the simpler subscript  $i$ .

We refer the reader to Sections 2, 3 and 7 for precise definitions and, in particular for the notion of viscosity solutions: this notion of solutions is the one which is used by Lions and Souganidis [31, 32] and which is called “junction viscosity solutions” in the book of Barles and Chasseigne [11].

We stress on the fact that even in this simplest version of a single interior vertex, the main difficulties of the problem are still there and for this reason we make an exhaustive study of this case. We also mention that it is possible to implement other boundary conditions on the vertices  $\mathbf{V}_b$  such as Neumann, or state-constraint conditions, as well as exterior conditions which are natural in nonlocal problems, see Section 6 for more details.

**1.2. Previous results.** We prove the well-posedness of this simplified problem (1.4) by using this notion of (junction) viscosity solutions. Though there is an intense research activity on viscosity solutions for local Hamilton-Jacobi Equations (HJ-Equations for short) on networks and, more generally, on stratified structures, few is known about nonlocal equations and this paper intends to be one of the first studying nonlocal equations in this framework.

Thus, before describing more precisely our results, we briefly review the literature for local equations.

Despite we use a pure PDE approach, (1.1) is intimately linked with optimal control problems when the Hamiltonians  $H_E$  are convex (or concave). In the case of networks, the analysis of deterministic optimal control problems posed on a simple junction and its relation with time-dependent HJ-Equations can be found in the seminal works Achdou et al. [1] (see also [4]) and Imbert et al. [28]. Then, Imbert and Monneau in [27] introduced the notion of *flux-limited solutions*, a particular notion at the junction point whose properties makes it compatible with the optimal control perspective, and allows to weaken some of the standing assumptions on the problem. In parallel, Lions and Souganidis [31, 32] (see also Morfe [33]) provide well-posedness for HJ-Equations with Kirchhoff-type condition on the junction (as in (1.1)), providing well-posedness for the problem and investigated its relation with flux-limited solutions (See also [7] for more on the relation among these two types of solutions). In the case of more general networks, we can mention the papers of Camilli and Schieborn [35] and Siconolfi [36] where some of their ideas are used in the treatment of general networks in Section 7.

As far as elliptic or parabolic *nondegenerate* equations on networks are concerned, existence and uniqueness of classical solutions can be found in the work of Von Below [39] for linear parabolic equations, while weak and classical solutions for nonlinear equations are studied in [20, 19, 2, 3, 34]. The case of degenerate equations is more delicate and is considered in [29, 32] using the notion of viscosity solutions.

For the readers who may be interested in stratified media, a pioneering work in this direction is Bressan and Hong [17]: they show that the value function of a certain optimal control problem is the viscosity solution of an associated HJ-Equation posed on the Euclidean space, but whose Hamiltonian encodes the main features of the stratification. The book of Barles and Chasseigne [11] provides a more general version of their results and, even if they do not really consider problems set on networks, their study of HJ-Equations with a co-dimension one discontinuity contains basic ideas to compare “junction viscosity solutions” and “flux-limited solutions”; we borrow several of their technical arguments in this article.

Let us mention that the analysis of equations on junctions has gone beyond existence and uniqueness, and it have also been addressed in more involved settings such as Mean Field Systems on networks [19, 2, 3], and homogenization on networks [23].

**1.3. Main results.** In Section 5, we prove a (strong) comparison result for problem (1.4), from which classical Perron’s method leads us to the well-posedness of a viscosity solution  $u \in C(\Gamma)$  to the problem.

However, we have decided to present a constructive approach that gives us a better understanding of the problem. This is one of the aim of Section 4. The idea is to exploit the well-known vanishing viscosity method. We replace the PDE in the edges  $E_i$  in (1.4) with the approximating equation

$$(1.5) \quad \lambda u^\epsilon - \epsilon u_{x_i x_i}^\epsilon - \mathcal{I}_i u^\epsilon + H_i(x, u_{x_i}^\epsilon) = 0 \quad \text{in } E_i,$$

for  $\epsilon \in (0, 1)$ , where  $u_{x_i x_i}^\epsilon := (u_i^\epsilon)''(x_i)$  for  $1 \leq i \leq N$ .

As a first step, we construct a viscosity solution  $u^\epsilon \in C^{1,1}(\bar{E}_i) \cap C^2(E_i)$  following an idea of Ohavi [34]. It consists in solving the system with a Dirichlet boundary condition at the junction  $O$  instead of the Kirchhoff condition. By a continuous dependence result and the Intermediate Value Theorem, we prove we can choose the value of the Dirichlet condition at the junction in order to recover the Kirchhoff condition.

We remark that our approximate solution  $u^\epsilon$  satisfies the Kirchhoff condition in the classical sense (the derivative at the end of each edge exists). The Kirchhoff junction condition together with the leading effect of the Hamiltonian allows us to prove uniform Lipschitz estimates on a neighborhood of  $O$  for the family  $\{u^\epsilon\}_\epsilon$ . By stability, a junction viscosity solution to (1.4) is obtained in the passage to the limit  $\epsilon \rightarrow 0$ , and this solution is Lipschitz continuous at the junction point. That is the content of our existence result Theorem 4.4.

Another advantage to use this approach, instead of the direct application of Perron's method, is that we obtain, as a by-product, a general well-posedness result for nonlinear viscous nonlocal Hamilton-Jacobi Equations with Kirchhoff conditions on general networks (see Theorem 7.5). Here, the Intermediate Value Theorem is replaced by its higher dimensional version encoded by Poincaré-Miranda Theorem, see Section 7. This is a new result, extending in the framework of nonlocal equations some previous works on local equations [20, 19, 32, 29, 2].

The next step is the uniqueness of the solution, which is presented in Theorem 5.1. The proof of this result is a direct adaptation of the arguments presented in Lions and Souganidis [32]. As it is usual in problems on networks, all the difficulties are related to the vertices. Here, the Lipschitz continuity of the (sub)solution allows us to avoid the usual doubling variables procedure in the comparison proof, which is the main difficulty here since the problem is naturally discontinuous at  $O$ . In general,  $H_i(O, p) \neq H_j(O, p)$  and  $\mathcal{I}_i u(O) \neq \mathcal{I}_j u(O)$  for  $i \neq j$ .

Let us point out that the evaluation of the nonlocal operator at the junction is a key point in our work. It requires the assumption (1.3) for  $\sigma_E < 1$ . The case  $\sigma_E \geq 1$  is more delicate, starting with the evaluation of higher-order nonlocal operators at the junction point, and its viscosity formulation. One particular case that is considered here is when the nonlocality is *censored* to the edge, meaning that the integration in (1.2) occurs only on  $E_i$ . Even in this simpler scenario, the evaluation of  $\mathcal{I}_i u(O)$  necessarily requires that  $u_{x_i}(O) = 0$ , see Guan and Ma [25, Theorem 5.3], which is not always compatible with prescribed Kirchhoff conditions at  $O$ . The nonlocal operators studied in [25] have a probabilistic interpretation and may arise in possible applications of stochastic optimal control problems on networks. This kind of difficulty also appear in the context of (local) second-order problems. In most of the cases the authors require the ellipticity degenerates on the junction, see [29, 32].

We address the connections with the notion of flux limited solutions in Section 8, proving its equivalence with Kirchhoff-type solutions, once an appropriate notion of flux limiter is defined. In this task, Lipschitz regularity of subsolutions and a relaxed evaluation of the nonlocal operator in the extended real line  $\mathbb{R} \cup \{-\infty, +\infty\}$  plays a key role to isolate the role of the critical slopes in the flux limiter, making the problem closer to the pure first-order case. That is the reason the ideas presented here hardly can be used on equations with higher-order nonlocal operators.

The paper is organized as follows. In Section 2 we provide the details of the definition of the junction and the PDE we consider, as well as the standing assumptions. In Section 3 we introduce the notion of solution and basic properties. In Section 4 we prove the existence of a viscosity solution to the Kirchhoff problem, and provide some regularity estimates. In Section 5 we prove a strong comparison result. We list some possible

extensions concerning the boundary conditions in Section 6. Section 7 is devoted to the case of general networks where we extend all the above results to this more complicated setting; the aim task is to introduce a suitable parametrization of each curve to be able to properly define  $u_x$  and then  $H_E$ . In Section 8 we introduce the notion of flux limited solution and prove its equivalence with the Kirchhoff-type solutions in the context of junctions. In Appendix A we present the proofs of some auxiliary results that we use in the body of the paper.

**Acknowledgement.** During the preparation of this work, several research visits have been realized, respectively to USACH and IRMAR. In each case the concerned author wishes to acknowledge their hosts for their hospitality and supports. O.L. is partially supported by the ANR (Agence Nationale de la Recherche) through the COSS project ANR-22-CE40-0010 and the Centre Henri Lebesgue ANR-11-LABX-0020-01. E.T. was supported by CNPq Grant 306022/2023-0, CNPq Grant 408169/2023-0, and FAPERJ APQ1 210.573/2024.

## 2. EQUATION ON A JUNCTION WITH KIRCHHOFF CONDITION AND DIRICHLET BOUNDARY CONDITION.

**2.1. Junction.** We quickly recall what we already explain in the introduction: we consider the case of a star-shaped network  $\Gamma$  embedded in  $\mathbb{R}^d$ ,  $d \geq 2$ , where  $O = (0, \dots, 0) \in \mathbb{R}^d$  is the (unique) junction point—in other words,  $\mathbf{V}_i = \{O\}$ —and we have a family of  $N$  “boundary vertices”  $\mathbf{V}_b = \{\bar{\mathbf{v}}_1, \dots, \bar{\mathbf{v}}_N\}$ , hence of  $N$  (open) edges  $\{E_i\}_{1 \leq i \leq N}$  given by

$$E_i = \{t\bar{\mathbf{v}}_i, t \in (0, 1)\}.$$

Of course, we assume that, for any  $i, j$ ,  $\bar{\mathbf{v}}_i \neq O$  and  $\bar{\mathbf{v}}_i, \bar{\mathbf{v}}_j$  are not collinear; as a consequence the  $E_i$  are non-empty and, if  $i \neq j$ ,  $E_i \cap E_j = \emptyset$ . With these notations, we have

$$(2.1) \quad \Gamma := \bigcup_{i=1}^N \bar{E}_i,$$

where  $\bar{E}_i$  denotes the closure of the set  $E_i$  in  $\mathbb{R}^d$ .

We now fix a natural parametrization of  $\Gamma$  by arc length, namely we set  $x_i = |x|$  on  $E_i$ ,  $a_i = |\bar{\mathbf{v}}_i|$  and  $J_i = (0, a_i)$ , where  $|\cdot|$  denotes the usual Euclidean norm in  $\mathbb{R}^d$ . We denote by  $\gamma_i : \bar{J}_i = [0, a_i] \rightarrow \bar{E}_i$  the canonical bijection  $\gamma_i(x_i) = x$  if  $x \in \bar{E}_i$  and  $|x| = x_i$ . In particular, for each  $x \in \Gamma \setminus \{O\}$ , there exists a unique  $i$  such that  $x \in E_i$ , and in this case we write  $x_i = \gamma_i^{-1}(x) \in J_i$ . Throughout the article we will often make the abuse of notation by identifying  $x \in E_i$  and  $x_i \in J_i$ . We also mention that the parametrization we choose is coherent with the one we will choose in the case of general networks (see Section 7 for details).

We are going to consider the geodesic distance  $\rho$  on  $\Gamma$ . Given  $x, y \in \Gamma$ , we define

$$\rho(x, y) = \begin{cases} |x - y| & \text{if } x, y \in \bar{E}_i, \\ |x| + |y| & \text{if } x \in \bar{E}_i, y \in \bar{E}_j \text{ with } i \neq j. \end{cases}$$

Taking into account the way we parametrize the edges, we have  $\rho(x, y) = |x_i - y_j|$  if  $i = j$ , and  $\rho(x, y) = x_i + y_j$  if  $i \neq j$ . Thus, by abuse of notation, by writing  $\rho(x_i, y_j)$  we mean  $\rho(\gamma_i(x_i), \gamma_j(y_j))$  for  $x_i \in \bar{J}_i$  and  $y_j \in \bar{J}_j$ . Notice that the geodesic distance  $\rho$  and the distance induced on  $\Gamma$  by the Euclidean norm are equivalent.

**2.2. Function spaces.** For a function  $u : \Gamma \rightarrow \mathbb{R}$ , we define  $u_i = u \circ \gamma_i : \bar{J}_i \rightarrow \mathbb{R}$ , from which we have  $u(x) = u_i(x_i)$  for  $x \in \bar{E}_i$ . The function  $u_i$  depends on the parametrization we chose but not its regularity.

We denote by  $USC(\Gamma)$  (respectively  $LSC(\Gamma)$ ) the subset of functions  $u : \Gamma \rightarrow \mathbb{R}$  which are upper-semicontinuous (respectively lower-semicontinuous) on  $\Gamma$ , that is, for all  $i$ ,  $u_i \in USC([0, a_i])$  (respectively  $u_i \in LSC([0, a_i])$ ). The subset  $C(\Gamma) = USC(\Gamma) \cap LSC(\Gamma)$  is the set of continuous functions on  $\Gamma$ . It coincides with the usual notion of continuity on  $\Gamma$  induced by the geodesic distance. For further purpose, we also introduce the subset  $SC(\Gamma) = USC(\Gamma) \cup LSC(\Gamma)$ .

For  $u : \Gamma \rightarrow \mathbb{R}$ , we say that  $u$  is differentiable at  $x \in E_i$  if  $u_i$  is differentiable at  $x_i$ , and in that case we denote

$$u_x(x) = u_{x_i}(x) = u'_i(x_i).$$

For  $m \in \mathbb{N}$ , the space of  $m$ -times continuously differentiable functions on  $\Gamma$  is defined by

$$C^m(\Gamma) := \{u \in C(\Gamma) : u_i \in C^m([0, a_i]) \text{ for all } i\}.$$

Notice that  $u \in C^m(\Gamma)$  is assumed to be continuous on  $\Gamma$ , all the  $u_i$  are  $C^m$ -continuously differentiable inside the edges and all their derivatives of order less than  $m$  can be extended by continuity to  $[0, a_i]$ . More precisely, when  $u \in C^1(\Gamma)$ , we define

$$(2.2) \quad u_{x_i}(O) := \lim_{x \rightarrow O, x \in E_i} u_x(x) = \lim_{x \rightarrow O, x \in E_i} \frac{u(x) - u(O)}{\rho(x, O)},$$

$$(2.3) \quad u_{x_i}(\bar{\mathbf{v}}_i) := \lim_{x \rightarrow \bar{\mathbf{v}}_i, x \in E_i} u_x(x) = \lim_{x \rightarrow \bar{\mathbf{v}}_i, x \in E_i} -\frac{u(x) - u(\bar{\mathbf{v}}_i)}{\rho(x, \bar{\mathbf{v}}_i)}.$$

The above derivatives depend on the parametrization through the orientation we chose for the edges (notice the minus sign in the definition of the second one).

To define more intrinsically the Kirchhoff condition, we may also use the notion of *inward derivative* of  $u$  with respect to  $E_i$  at  $O$ , denoted by  $\partial_i u(O)$ ; this derivative has the advantage to be independent of the chosen orientation of the edge but, of course, it is not independent of the parametrization. The inward derivative is the right one to consider when dealing with Kirchhoff



condition: in our simple framework,  $\partial_i u(O) = u_{x_i}(O)$ , but we underline that it can be different to  $u_{x_i}(O)$  for general networks because of the different possible orientation.

In the sequel, for the sake of notations, we use the notation  $u_{x_i}(x)$  for the derivatives of  $u$  in  $\bar{E}_i$ . We point out again that this simplification of notation will not be anymore possible for general networks in Section 7.

We finally recall that  $u_i \in C^{0,\alpha_i}([0, a_i])$ ,  $0 < \alpha_i \leq 1$ , if

$$[u_i]_{C^{0,\alpha_i}([0,a_i])} := \sup_{x_i, y_i \in [0, a_i], x_i \neq y_i} \frac{|u_i(x_i) - u_i(y_i)|}{|x_i - y_i|^{\alpha_i}} < +\infty.$$

It allows to define the set of Hölder continuous functions on  $\Gamma$  by

$$C^{0,\alpha}(\Gamma) := \{u \in C(\Gamma) : u_i \in C^{0,\alpha_i}([0, a_i]) \text{ for } 0 < \alpha \leq \alpha_i \leq 1\}.$$

Notice that, thanks to the inequality

$$(2.4) \quad a^\alpha + b^\alpha \leq 2^{1-\alpha}(a+b)^\alpha \text{ for all } a, b \geq 0, 0 < \alpha \leq 1,$$

if  $u \in C^{0,\alpha}(\Gamma)$  and  $K := \max_i [u_i]_{C^{0,\alpha_i}([0,a_i])}$ , then

$$(2.5) \quad |u(x) - u(y)| \leq 2^{1-\alpha} K \rho(x, y)^\alpha, \text{ for all } x, y \in \Gamma.$$

**2.3. Hamiltonian.** Now we explain the operators involved in our equation (1.1), starting with the Hamiltonian  $H$ . We assume the existence of a collection  $\{H_i\}_{1 \leq i \leq N}$  such that  $H_i \in C(\bar{E}_i \times \mathbb{R})$  for all  $i$ , and that satisfy

$$(2.6) \quad \begin{aligned} (i) \quad & |H_i(x, p) - H_i(y, p)| \leq C_H(1 + |p|)|x - y|, \quad x, y \in \bar{E}_i, p \in \mathbb{R}, \\ (ii) \quad & |H_i(x, p) - H_i(x, q)| \leq C_H|p - q|, \quad x \in \bar{E}_i, p, q \in \mathbb{R}, \\ (iii) \quad & C_H^{-1}|p| - C_H \leq H_i(x, p) \leq C_H(1 + |p|), \quad x \in \bar{E}_i, p \in \mathbb{R}, \end{aligned}$$

for some  $C_H > 1$ .

For simplicity, we choose here to deal with coercive Hamiltonians satisfying the classical assumptions coming from Optimal Control, see [8] for instance. Nevertheless, several results presented here can be readily applied to Hamiltonians with superlinear growth in the gradient, and/or less regularity on the state variable.

**Remark 2.1.** With these notations, given  $x \in \bar{E}_i$  and  $p \in \mathbb{R}$ , we naturally identify  $H_i(x, p)$  and  $H_i(\gamma_i^{-1}(x), p)$ , meaning that the HJ Equation depends on the parametrization of the network. For instance, if we change our parametrization in Section 2.1 with the opposite orientation,  $u_{x_i}$  changes sign and  $H_i(x, p)$  is changed into  $H_i(x, -p)$ . When thinking to applications, in optimal control problems for example, we often first parametrize the network to be able to write the dynamic and the cost and then we derive the family of Hamiltonians  $H_i$ , which allows to define the ‘‘abstract’’ Hamiltonian  $H$  on  $\Gamma$ .

**2.4. Nonlocal operator.** Next, we define the nonlocal operator  $\mathcal{I}u(x)$  for  $x \in \Gamma$  and  $u : \Gamma \rightarrow \mathbb{R}$ .

We consider a family of two-parametric measurable functions  $\{\nu_{ij}\}_{1 \leq i, j \leq N}$  with the form  $\nu_{ij} : \bar{E}_i \times (0, \infty) \rightarrow [0, +\infty)$ ,  $1 \leq i, j \leq N$ , satisfying the following conditions:

There exist  $\Lambda > 0$  and  $0 < \sigma < 1$  such that for all  $x, y \in \bar{E}_i$ ,  $r > 0$

$$(2.7) \quad 0 \leq \nu_{ij}(x, r) \leq \frac{\Lambda}{r^{1+\sigma}}, \quad |\nu_{ij}(x, r) - \nu_{ij}(y, r)| \leq \frac{\Lambda}{r^{1+\sigma}}|x - y|.$$

In particular, this means that the following Lévy integrability condition (see (1.3)) takes place

$$(2.8) \quad \sup_{1 \leq i, j \leq N} \sup_{x \in \bar{E}_i} \int_0^\infty \min\{r^\gamma, 1\} \nu_{ij}(x, r) dr < +\infty,$$

for all  $\gamma > \sigma$ .

Then, setting, for all  $x \in \bar{E}_i$ ,

$$\nu_i(x, z) = \begin{cases} \nu_{ii}(x, \rho(x, z)) = \nu_{ii}(x, |x_i - z_i|), & z \in \bar{E}_i, \\ \nu_{ij}(x, \rho(x, z)) = \nu_{ij}(x, x_i + z_j), & z \in \bar{E}_j, j \neq i, \end{cases}$$

we define

$$(2.9) \quad \begin{aligned} \mathcal{I}_{ii}u(x) &:= \int_{E_i} [u(z) - u(x)] \nu_{ii}(x, \rho(x, z)) dz, \\ &= \int_{J_i} [u_i(z_i) - u_i(x_i)] \nu_{ii}(x, |x_i - z_i|) dz_i. \end{aligned}$$

It is convenient to introduce the further notation

$$(2.10) \quad \begin{aligned} \mathcal{I}_{ij}u(x) &:= \int_{E_j} [u(z) - u(x)] \nu_{ij}(x, \rho(x, z)) dz \\ &= \int_{J_j} [u_j(z_j) - u_i(x_i)] \nu_{ij}(x, x_i + z_j) dz_j. \end{aligned}$$

in order that we can write

$$(2.11) \quad \mathcal{I}_i u(x) = \mathcal{I}_{ii}u(x) + \sum_{j \neq i} \mathcal{I}_{ij}u(x) \quad \text{for } x \in \bar{E}_i.$$

Notice that the nonlocal operator  $\mathcal{I}_i u$  requires the values of  $u$  on all  $\Gamma$  in order to be evaluated. Writing  $\mathcal{I}_i u$  as in (2.11), we call  $\mathcal{I}_{ii}u$  the *censored part* in  $\bar{E}_i$  (which “does not see” the other edges) and  $\sum_{j \neq i} \mathcal{I}_{ij}u$  is the *exterior part* (which “interacts” with other edges). We finally say that  $\mathcal{I}_i u$  is *censored to  $\bar{E}_i$*  if  $\mathcal{I}_{ij}u = 0$  for  $j \neq i$  (all the  $\nu_{ij}$ ,  $i \neq j$ , are zero).

Thanks to (2.7)-(2.8), the nonlocal operator is well-defined as soon as  $u$  is Hölder continuous on  $\Gamma$  with an exponent larger than  $\sigma$ . More precisely, we have the following estimate.

**Lemma 2.2.** *Consider the nonlocal operator  $\mathcal{I}$  given by (2.9) under Assumption (2.7). If  $u \in C^{0,\gamma}(\Gamma)$  for some  $\gamma > \sigma$ , then the map  $x \mapsto \mathcal{I}_i u(x)$  is in  $C^{0,\gamma-\sigma}(\bar{E}_i)$  for each  $i$ .*

We present a proof of this lemma in Appendix A.

**2.5. PDE on the junction.** We end this section by presenting the system (1.4) we want to address. We are interested in the existence and uniqueness of a function  $u \in C(\Gamma)$ , solving in each edge  $E_i$ ,  $1 \leq i \leq N$ , the nonlocal Hamilton-Jacobi equation

$$(2.12) \quad \lambda u(x) - \mathcal{I}_i u(x) + H_i(x, u_{x_i}(x)) = 0, \quad \text{for } x \in E_i.$$

We complement the equations with a Kirchhoff-type condition on the junction  $O$ ,

$$(2.13) \quad \sum_{i=1}^N -u_{x_i}(O) = B,$$

for some  $B \in \mathbb{R}$ , and Dirichlet boundary conditions at the end of each edge, that is

$$(2.14) \quad u_i(a_i) = h_i, \quad 1 \leq i \leq N.$$

The following **steady assumption** is in force in all the paper:

$$(2.15) \quad \lambda > 0, H \text{ satisfies (2.6), } \nu_{ij} \text{ satisfy (2.7), } B, h_i \in \mathbb{R} \text{ are given.}$$

### 3. DEFINITION OF VISCOSITY SOLUTIONS ON A JUNCTION

Using the definitions introduced in the previous section, we are now in position to state the notion of solution of (1.1). We give the definition for the junction problem (1.4), leaving to the reader the extension to general networks.

Given  $\varphi \in SC(\Gamma)$ , a measurable subset  $A \subseteq \Gamma$ , and  $x \in \bar{E}_i$ ,  $1 \leq i \leq N$ , we write

$$\mathcal{I}_i[A]\varphi(x) = \sum_{j=1}^N \int_{\bar{E}_j \cap A} [\varphi(z) - \varphi(x)] \nu_{ij}(x, \rho(x, z)) dz,$$

whenever the integrals make sense. Notice that

$$\mathcal{I}_i \varphi(x) = \mathcal{I}_i[A]\varphi(x) + \mathcal{I}_i[\Gamma \setminus A]\varphi(x).$$

In what follows,  $B_\delta(x) = \{z \in \Gamma : \rho(x, z) < \delta\}$  is the open ball of radius  $\delta$  centered at  $x \in \Gamma$  induced by the geodesic distance  $\rho$ , and  $B_\delta^c(x) = \Gamma \setminus B_\delta(x)$  is its complementary in  $\Gamma$ .

For  $u \in SC(\Gamma)$ ,  $\varphi \in C^1(\Gamma)$  and every  $\delta \in (0, 1)$ , we write

$$(3.1) \quad G_i^\delta(u, \varphi, p, x) := \lambda u(x) - \mathcal{I}_i[B_\delta^c(x)]u(x) - \mathcal{I}_i[B_\delta(x)]\varphi(x) + H_i(x, p)$$

is well-defined for every  $p \in \mathbb{R}$ ,  $x \in \bar{E}_i$ ,  $1 \leq i \leq N$ .

Recalling (2.2), we introduce the following

**Definition 3.1.** We say that  $u \in USC(\Gamma)$  is a viscosity subsolution to problem (2.12)-(2.13)-(2.14) if, for each  $x \in \Gamma$ , each  $\delta > 0$  and each  $\varphi \in C^1(\Gamma)$  such that  $x$  is a maximum point of  $u - \varphi$  in  $B_\delta(x)$ , we have

$$\begin{aligned} & G_i^\delta(u, \varphi, \varphi_{x_i}(x), x) \leq 0 \quad \text{if } x \in E_i, \\ \min \left\{ \min_{1 \leq i \leq N} G_i^\delta(u, \varphi, \varphi_{x_i}(O), O), \sum_{1 \leq i \leq N} -\varphi_{x_i}(O) - B \right\} & \leq 0 \quad \text{if } x = O, \\ \min \left\{ G_i^\delta(u, \varphi, \varphi_{x_i}(x), x), u(x) - h_i \right\} & \leq 0 \quad \text{if } x = \bar{v}_i. \end{aligned}$$

We say that  $u \in LSC(\Gamma)$  is a viscosity supersolution to problem (2.12)-(2.13)-(2.14) if, for each  $x \in \Gamma$ , each  $\delta > 0$  and each  $\varphi \in C^1(\Gamma)$  such that  $x$  is a minimum point of  $u - \varphi$  in  $B_\delta(x)$ , we have

$$\begin{aligned} & G_i^\delta(u, \varphi, \varphi_{x_i}(x), x) \geq 0 \quad \text{if } x \in E_i, \\ \max \left\{ \max_{1 \leq i \leq N} G_i^\delta(u, \varphi, \varphi_{x_i}(O), O), \sum_{1 \leq i \leq N} -\varphi_{x_i}(O) - B \right\} & \geq 0 \quad \text{if } x = O, \\ \max \left\{ G_i^\delta(u, \varphi, \varphi_{x_i}(x), x), u(x) - h_i \right\} & \geq 0 \quad \text{if } x = \bar{v}_i. \end{aligned}$$

A viscosity solution is a continuous function on  $\Gamma$  which is simultaneously a viscosity sub and supersolution in the sense above.

As it is usual in the theory of viscosity solutions, we can always assume that  $u(x) = \varphi(x)$  (that is,  $\varphi$  ‘‘touches’’  $u$  at  $x$ , from above or below depending the case) and that  $x$  is either a strict global maxima or minima.

Next, we would like to provide an equivalent notion of solution for which we can get rid of the viscosity evaluation on the nonlocal operator and work directly with the function  $u$ . For this purpose, we introduce the subset of functions  $u$  of  $SC(\Gamma)$  such that the nonlocal operator  $\mathcal{I}u$  may be evaluated at  $x \in \Gamma$  (with possibly infinite value),

$$(3.2) \quad \mathcal{F}_x = \left\{ u \in SC(\Gamma) : \lim_{\delta \downarrow 0} \mathcal{I}_i[B_\delta^c(x)]u(x) \in [-\infty, +\infty], \right. \\ \left. \text{for all } i \text{ such that } x \in \bar{E}_i \right\}.$$

When the limit above exists in  $[-\infty, +\infty]$ , we denote it by  $\mathcal{I}_i u(x)$ . Notice that it does not depend on the function  $\varphi$  chosen for computing the limit. In the case of interest  $x = O$ , all the limits  $\mathcal{I}_i u(O)$  for  $1 \leq i \leq N$ , must exist in order that  $u \in \mathcal{F}_O$ .

**Lemma 3.2.** Let  $u \in USC(\Gamma)$  (resp.  $LSC(\Gamma)$ ) and  $\varphi \in C^1(\Gamma)$  such that  $u - \varphi$  has a local maximum (resp. minimum) at  $x \in \bar{E}_i$  for some  $i$ . Then  $u \in \mathcal{F}_x$  and  $\mathcal{I}_i u(x) \in [-\infty, +\infty)$  (resp.  $(-\infty, +\infty]$ ).

*Proof.* We follow the lines of [18, Lemma 3.3]. We only prove the first statement when  $u \in USC(\Gamma)$ , the proof when  $u \in LSC(\Gamma)$  being similar.

Assume that  $u - \varphi$  has a local maximum at  $x \in \bar{E}_i$  in  $B_{\delta_0}(x)$  for some  $1 \leq i \leq N$ ,  $\delta_0 > 0$  and  $\varphi \in C^1(\Gamma)$  with  $u(x) = \varphi(x)$ . We have

$$u(z) - \varphi(z) \leq u(x) - \varphi(x) = 0, \quad z \in B_{\delta_0}(x).$$

Denote  $w_\delta = \varphi\chi_{B_\delta(x)} + u\chi_{B_\delta^c(x)}$ , where  $\chi_A$  is the indicator function of  $A$ . Notice that for each  $j$ ,

$$\mathcal{I}_{ij}w_\delta(x) = \mathcal{I}_{ij}[B_\delta(x)]\varphi(x) + \mathcal{I}_{ij}[B_\delta^c(x)]u(x),$$

is well-defined and finite for each  $\delta \in (0, \delta_0)$ , and it is nonincreasing as  $\delta \rightarrow 0$ .

Since  $w_{\delta_0} - w_\delta \geq 0$  in  $\Gamma$  and it is nondecreasing with  $\delta$ , using Monotone Convergence Theorem we get

$$\lim_{\delta \downarrow 0} \mathcal{I}_{ij}(w_{\delta_0} - w_\delta)(x) \in (0, +\infty],$$

meanwhile, by Dominated Convergence Theorem, we have

$$\lim_{\delta \downarrow 0} \mathcal{I}_{ij}[B_\delta(x)]\varphi(x) = 0.$$

Thus, writing

$$\mathcal{I}_{ij}[B_\delta^c(x)]u(x) = -\mathcal{I}_{ij}(w_{\delta_0} - w_\delta)(x) + \mathcal{I}_{ij}w_{\delta_0}(x) - \mathcal{I}_{ij}[B_\delta(x)]\varphi(x),$$

we get

$$\lim_{\delta \downarrow 0} \mathcal{I}_{ij}[B_\delta^c(x)]u(x) \in [-\infty, +\infty)$$

for all  $j$ , from which  $u \in \mathcal{F}_x$ .  $\square$

In order to state the equivalent definition of viscosity solutions we need to introduce sub/superdifferentials and the term (3.1) with  $\delta = 0$ .

Given an interval  $I \subset \mathbb{R}$  and a function  $u : I \rightarrow \mathbb{R}$ , we recall that  $p \in \mathbb{R}$  is in the superdifferential  $D_I^+u(x_0)$  of  $u$  at  $x_0 \in I$  if

$$u(x) \leq u(x_0) + p(x - x_0) + o(|x - x_0|) \quad \text{for all } x \text{ in an } I\text{-neighborhood of } x_0.$$

In the case of junctions, we say that  $\mathbf{p} = (p_1, \dots, p_N)$  is in the superdifferential  $D_\Gamma^+u(O)$  of  $u$  at  $O$  if, for each  $1 \leq i \leq N$ , then we have

$$u_i(x_i) \leq u(O) + p_i x_i + o(x_i) \quad \text{for all } x_i \text{ in an } \bar{J}_i\text{-neighborhood of } 0.$$

Similarly to the classical case, if  $\mathbf{p} \in D_\Gamma^+u(O)$ , then there exists a function  $\varphi \in C^1(\Gamma)$  such that  $\varphi_{x_i}(O) = p_i$  and  $u - \varphi$  has a local maximum at  $O$ . The above definitions can be stated for the subdifferential  $D_\Gamma^-u(O)$ .

We denote  $G_i(u, p, x) = \lim_{\delta \downarrow 0} G_i^\delta(u, p, x)$  whenever the limit exists in  $\mathbb{R} \cup \{-\infty, +\infty\}$  and meets the value

$$(3.3) \quad G_i(u, p, x) := \lambda u(x) - \mathcal{I}_i u(x) + H_i(x, p).$$

Notice that the nonlocal term decides whether  $G_i(u, p, x) \in \mathbb{R}$  or not. It is the case when  $u$  is smooth enough at  $x$  (Lemma 2.2) but, in general, infinite values are possible.

**Lemma 3.3.** *A function  $u \in USC(\Gamma)$  is a viscosity subsolution to problem (2.12)-(2.13)-(2.14) at the point  $x \in \Gamma$  if and only if*

$$\begin{aligned} G_i(u, p_i, x) &\leq 0 \quad \text{if } x \in E_i, p_i \in D_{J_i}^+ u_i(x_i), \\ \min \left\{ \min_{1 \leq i \leq N} G_i(u, p_i, O), \sum_{1 \leq i \leq N} -p_i - B \right\} &\leq 0 \quad \text{if } x = O, \mathbf{p} \in D_\Gamma^+ u(O), \\ \min \left\{ G_i(u, p_i, x), u(x) - h_i \right\} &\leq 0 \quad \text{if } x = \bar{\mathbf{v}}_i, p_i \in D_{J_i}^+ u_i(a_i). \end{aligned}$$

*The same equivalence holds for a LSC supersolution with usual adaptations (opposite inequalities, “min” replaced with “max” and the superdifferential  $D^+$  is replaced with the subdifferential  $D^-$ ).*

*Proof.* We only provide the proof for subsolutions at  $x = O$ , the other cases being similar or simpler.

Assume that  $u$  is a subsolution at  $O$  in the sense of Definition 3.1. Let  $\mathbf{p} \in D_\Gamma^+ u(O)$ . There exists  $\varphi \in C^1(\Gamma)$  such that  $u - \varphi$  has a local maximum at  $O$  with  $\varphi_{x_i}(O) = p_i$  for all  $i$ . From Lemma 3.2,  $u \in \mathcal{F}_O$  and  $-\mathcal{I}_i u(O) \in (-\infty, +\infty]$  for all  $i$ . Using the subsolution characterization of Definition 3.1 at  $x = O$ , we have that, either  $\sum_i -\varphi_{x_i}(O) - B \leq 0$ , or, for every  $\delta > 0$ , there exists  $1 \leq i \leq N$ ,  $i = i(\delta)$ , such that  $G_i^\delta(u, \varphi, \varphi_{x_i}(O), O) \leq 0$ . In the first case, we conclude that  $\sum_i -p_i - B \leq 0$  since  $\varphi_{x_i}(O) = p_i$ . We now suppose that the second case holds. Since there is a finite number of edges, there exists  $1 \leq i_0 \leq N$  and a subsequence  $\delta_k \downarrow 0$  such that  $G_{i_0}^{\delta_k}(u, \varphi, \varphi_{x_{i_0}}(O), O) \leq 0$ . Since  $u \in \mathcal{F}_O$ , we can send  $\delta_k$  to 0 in the previous inequality to obtain  $G_{i_0}(u, \varphi_{x_{i_0}}(O), O) \leq 0$ , which is the expected inequality in Lemma 3.3. Note that, in this case, we obtain that  $-\mathcal{I}_{i_0} u(O)$  is finite.

Conversely, assume that  $u$  satisfies the inequality of Lemma 3.3 at  $x = O$ . Let  $\varphi \in C^1(\Gamma)$  such that  $u - \varphi$  has a global maximum at  $O$ . Then  $\mathbf{p} := (\varphi_{x_1}(O), \dots, \varphi_{x_N}(O)) \in D_\Gamma^+ u(O)$ . If  $\sum_i -\varphi_{x_i}(O) - B \leq 0$ , then the viscosity inequality for the subsolution at  $x = O$  in Definition 3.1 holds and we are done. Otherwise, there exists  $i$  such that  $G_i(u, \varphi_{x_{i_0}}(O), O) \leq 0$ . In particular, from Lemma 3.2,  $-\mathcal{I}_i u(O)$  is finite. Then, for all  $\delta > 0$ , we have

$$\begin{aligned} G_i^\delta(u, \varphi, \varphi_{x_i}(O), O) &= G_i(u, \varphi_{x_{i_0}}(O), O) \\ &\quad \mathcal{I}_i u(O) - \mathcal{I}_i[B_\delta^\varepsilon(O)]u(O) - \mathcal{I}_i[B_\delta(O)]\varphi(O) \\ &\leq \mathcal{I}_i[B_\delta(O)]u(O) - \mathcal{I}_i[B_\delta(O)]\varphi(O) \leq 0, \end{aligned}$$

the last inequality coming from the fact that,  $O$  being a maximum of  $u - \varphi$ , we have  $u - u(O) \leq \varphi - \varphi(O)$  in  $B_\delta(O)$ , hence  $\mathcal{I}_i[B_\delta(O)]u(O) \leq \mathcal{I}_i[B_\delta(O)]\varphi(O)$ . We conclude that  $u$  is a subsolution in the sense of Definition 3.1.  $\square$

Finally, the above definitions can be extended to second-order equations like (1.5), with test functions  $\varphi \in C^2(\Gamma)$ . Moreover, every classical solution

to the problem, that is, a function  $C^2(\Gamma)$  such that (2.12) or (1.5) complemented by (2.13)-(2.14) holds pointwisely, is a viscosity solution in both senses above.

#### 4. EXISTENCE FOR THE KIRCHHOFF-TYPE PROBLEM ON A JUNCTION.

Let  $\theta \in \mathbb{R}$  and  $\epsilon \in (0, 1)$ . The purpose of this section is to construct a function  $u^\epsilon \in C(\Gamma)$  solving the viscous Dirichlet problem

$$(4.1) \quad \begin{cases} \lambda u - \epsilon u_{x_i x_i} - \mathcal{I}_i u + H_i(x, u_{x_i}) = 0 & \text{on } E_i, 1 \leq i \leq N, \\ u(O) = \theta, \\ u_i(a_i) = h_i, & 1 \leq i \leq N. \end{cases}$$

The boundary condition at  $a_i$  is considered fixed since it coincides with the boundary condition of the original problem. The more careful analysis is performed on the dependence of the solution in terms of  $\theta$ , since we will require to fix it appropriately to find a solution satisfying the Kirchhoff condition at  $O$ .

We start by constructing viscosity sub- and supersolutions to (4.1), which achieve the boundary conditions pointwisely.

We fix  $\alpha \in (0, 1)$ , and, for all  $L, \delta > 0$ , we define

$$(4.2) \quad \psi_{L,\delta}(x) = \begin{cases} L(x - x^{1+\alpha}), & \text{for } 0 < x \leq \delta, \\ \psi(\delta) & \text{for } x > \delta, \end{cases}$$

and we introduce

$$(4.3) \quad C_0 := \frac{1}{\min\{\lambda, 1\}} \left( \max_{1 \leq i \leq N, x \in \bar{J}_i} |H_i(x, 0)| + |\theta| + \max_{1 \leq i \leq N} |h_i| \right).$$

**Lemma 4.1.** *Under the assumption (2.15), there exist  $L, \delta$  depending on the data, and  $L_\epsilon, \delta_\epsilon$  depending on the data and the ellipticity constant  $\epsilon$ , such that the functions  $\psi^+, \psi^- : \Gamma \rightarrow \mathbb{R}$  defined by*

$$\begin{aligned} \psi_i^+(x) &= \min\{C_0, \theta + \psi_{L,\delta}(x), h_i + \psi_{L,\delta}(a_i - x)\}, & x \in \bar{J}_i, \\ \psi_i^-(x) &= \max\{-C_0, \theta - \psi_{L_\epsilon, \delta_\epsilon}(x), h_i - \psi_{L_\epsilon, \delta_\epsilon}(a_i - x)\}, & x \in \bar{J}_i, \end{aligned}$$

are continuous viscosity supersolution and subsolution, respectively, of (4.1), satisfying  $\psi^\pm(O) = \theta$  and  $\psi_i^\pm(a_i) = h_i$ .

*Proof.* Throughout the proof, we systematically identify  $\bar{E}_i$  with  $\bar{J}_i$ ,  $x$  with  $x_i$ , and  $\psi$  with  $\psi_i$ .

At first, notice that the constant functions  $x \in \Gamma \mapsto \pm C_0$ , where  $C_0$  is defined by (4.3), are obviously supersolution and subsolution of (4.1), respectively. But they do not satisfy the boundary condition pointwisely.

We divide the proof in several parts.

*Supersolution property for  $\psi^+$ .* Since the minimum of supersolutions is still a supersolution, it is enough to prove that, for  $1 \leq i \leq N$ , both  $\theta + \psi_{L,\delta}(x)$  and  $h_i + \psi_{L,\delta}(a_i - x)$  are supersolutions in  $\bar{J}_i$  for  $L$  enough large and  $\delta$  enough small. The two proofs being similar, we concentrate on  $\theta + \psi_{L,\delta}(x)$ .

For simplicity, we write  $\psi$  for  $\psi_{L,\delta}$ . Straightforward computations drive us to

$$\psi_x(x) = L(1 - (1 + \alpha)x^\alpha), \quad \psi_{xx}(x) = -L(1 + \alpha)\alpha x^{\alpha-1}, \quad x \in [0, \delta].$$

Set  $\delta < (2(1 + \alpha))^{-1/\alpha}$  and  $\delta < \min_i a_i$ . Then, we have  $L/2 \leq \psi_x(x) \leq L$  for  $x \in [0, \delta]$  and  $L\delta/2 \leq \psi(x) = \psi(\delta) \leq L\delta$  for  $x \in [\delta, a_i]$ . In particular,  $\psi$  is nondecreasing and Lipschitz continuous with constant  $L$  in each  $\bar{J}_i$ .

Now we estimate the nonlocal operators. Let  $x \in \bar{E}_i$  with  $x_i \in \bar{J}_i \cap [0, \delta]$ . We have that

$$\begin{aligned} |\mathcal{I}_{ii}\psi(x)| &\leq \int_{J_i \cap \{\rho(x, \gamma_i(z)) \leq \delta\}} |\psi_i(z) - \psi(x)| \nu_{ii}(x, \rho(x, \gamma_i(z))) dz \\ &\quad + \int_{J_i \cap \{\rho(x, \gamma_i(z)) > \delta\}} |\psi_i(z) - \psi(x)| \nu_{ii}(x, \rho(x, \gamma_i(z))) dz \\ &\leq \Lambda L \int_0^{x_i + \delta} |x_i - z| |x_i - z|^{-1-\sigma} dz \\ &\quad + \Lambda \psi(\delta)^{1-\sigma} \int_{x_i + \delta}^{a_i} (L|x_i - z|)^\sigma |x_i - z|^{-1-\sigma} dz \\ &\leq \Lambda L \int_0^{x_i + \delta} |x_i - z|^{-\sigma} dz + \Lambda L \delta^{1-\sigma} \int_\delta^{a_i} |x_i - z|^{-1} dz \\ &\leq \Lambda L \left( \frac{2^{1-\sigma}}{1-\sigma} + \log a_i - \log \delta \right) \delta^{1-\sigma}. \end{aligned}$$

For  $1 \leq j \leq N$ ,  $j \neq i$ , similarly we have

$$\begin{aligned} |\mathcal{I}_{ij}\psi(x)| &\leq \int_{J_j \cap \{\rho(x, \gamma_j(z)) \leq 2\delta\}} |\psi_j(z) - \psi_i(x_i)| \nu_{ij}(x, \rho(x, \gamma_j(z))) dz \\ &\quad + \int_{J_j \cap \{\rho(x, \gamma_j(z)) > 2\delta\}} |\psi_j(z) - \psi_i(x_i)| \nu_{ij}(x, \rho(x, \gamma_j(z))) dz \\ &\leq \Lambda L \int_0^{2\delta - x_i} |x_i + z|^{-\sigma} dz + \Lambda L \delta^{1-\sigma} \int_\delta^{a_j} |x_i + z|^{-1} dz \\ &\leq \Lambda L \left( \frac{2^{1-\sigma}}{1-\sigma} + \log a_i - \log \delta \right) \delta^{1-\sigma}. \end{aligned}$$

We summarize the nonlocal estimates as

$$(4.4) \quad |\mathcal{I}_i\psi(x)| \leq C_\sigma (1 + |\log \delta|) L \delta^{1-\sigma}, \quad 1 \leq i \leq N,$$

where  $C_\sigma > 0$  depending on the data is of order  $(1 - \sigma)^{-1}$  as  $\sigma \rightarrow 1^-$  but remains stable as  $\sigma \rightarrow 0^+$ .

We use the notation  $G_i^\delta, G_i$  of Section 3, adding the vanishing viscosity term accordingly.



From the above computations together with the coercivity assumption in (2.6), for all  $i$  and all  $x \in J_i \cap (0, \delta)$ , we have

$$\begin{aligned} & G_i(\theta + \psi, \psi_{x_i}(x), x) \\ &= \lambda(\theta + \psi) - \epsilon\psi_{x_i x_i} - \mathcal{I}_i\psi(x) + H_i(x, \psi_{x_i}(x)) \\ &\geq \lambda\theta + \epsilon L(1 + \alpha)\alpha x^{\alpha-1} - C_\sigma(1 + |\log \delta|)L\delta^{1-\sigma} + C_H^{-1}\frac{L}{2} - C_H. \end{aligned}$$

Since  $\sigma < 1$ , it is possible to choose  $\delta$  possibly smaller in order to

$$C_\sigma(1 + |\log \delta|)\delta^{1-\sigma} \leq \frac{1}{4C_H}.$$

Under this choice, and taking  $L \geq 8C_H^2$ , we obtain

$$G_i(\theta + \psi, \psi_{x_i}(x), x) \geq \lambda\theta + \frac{1}{8C_H}L \quad \text{on } J_i \cap (0, \delta).$$

Choosing  $L > 8C_H\lambda|\theta|$ , the left-hand side of the above inequality is nonnegative, from which  $\theta + \psi$  is a viscosity supersolution in  $J_i \cap (0, \delta)$ .

Similar computations yield that  $h_i + \psi(a_i - x)$  is a supersolution on  $(a_i - \delta, a_i)$ , choosing  $L$  large and  $\delta$  small enough. Enlarging  $L$  if necessary, we can assume that

$$\theta + \psi(\delta), h_i + \psi_L(\delta) \geq C_0.$$

Then, the value  $C_0$  is active in the min defining  $\psi^+$  on  $(\delta, a_i - \delta)$ , from which we infer the viscosity inequality on the full interval  $J_i$ . Thus,  $\psi^+$  is a viscosity supersolution for the problem with  $\psi^+(O) = \theta$  and  $\psi_i^+(a_i) = h_i$ .

*Subsolution property for  $\psi^-$ .* As, for the construction of the supersolution, the property that a maximum of subsolutions is a subsolution allows to check the subsolution property for  $\theta - \psi(x)$  and  $h_i - \psi(a_i - x)$  in  $\bar{J}_i$  separately. Here also, the computations being similar, we make them in the case of  $\theta - \psi(x)$ .

Recalling (4.4), for all  $x \in J_i \cap (0, \delta)$  we have

$$\begin{aligned} & G_i(\theta - \psi, -\psi_{x_i}(x), x) \\ &= \lambda(\theta - \psi) - \epsilon\psi_{x_i x_i} - \mathcal{I}_i\psi(x) + H_i(x, -\psi_{x_i}) \\ &\leq \lambda\theta - \epsilon L(\alpha + 1)\alpha x^{\alpha-1} + C_\sigma\delta^{1-\sigma} + C_H(1 + L) \\ &\leq \lambda\theta + C_H + L(-\epsilon(\alpha + 1)\alpha x^{\alpha-1} + C_\sigma\delta^{1-\sigma} + C_H). \end{aligned}$$

Here, we need to use the ellipticity of the PDE (4.6), which will provide a subsolution depending on  $\epsilon$ . We first choose

$$\delta = \delta_\epsilon \leq \left( \frac{\epsilon(1 + \alpha)\alpha}{C_\sigma + C_H + 1} \right)^{\frac{1}{1-\alpha}},$$

yielding

$$G_i(\theta - \psi, -\psi_{x_i}(x), x) \leq \lambda\theta + C_H - L, \quad x \in J_i \cap (0, \delta_\epsilon).$$

Choosing  $L \geq \lambda|\theta| + C_H$ , the left-hand side of the above inequality becomes nonpositive. Therefore  $\theta - \psi$  is subsolution in  $(0, \delta_\epsilon)$ . Similar computations can be made with  $h_i - \psi$  on  $(a_i - \delta_\epsilon, a_i)$ .

Enlarging  $L = L_\epsilon \geq 2\delta_\epsilon^{-1}(\theta + C_0)$ , we get

$$\theta - \psi(\delta), h_i - \psi(\delta) \leq -C_0,$$

from which, as before, we get  $\psi_i^-$  is a viscosity subsolution for the problem on  $\Gamma$ , with  $\psi^-(O) = \theta$  and  $\psi_i^-(a_i) = h_i$ .  $\square$

Thanks to the above lemma, the existence of a unique viscosity solution to (4.1) is an immediate consequence of Perron's method and comparison principles for this type of Dirichlet problem on  $\Gamma$ , see Theorem B.3 in the Appendix.

However, we give an alternative proof, which consists, roughly speaking, to construct a solution branch by branch. The proof uses only Theorem B.3 in the very particular case of the nonlocal Dirichlet problem censored to a unique interval,

$$(4.5) \quad \begin{cases} \lambda u - \epsilon u_{xx} - \mathcal{I}u + H(x, u_x) = 0 & \text{on } (0, 1), \\ u(0) = a, \quad u(1) = b, \end{cases}$$

where  $\mathcal{I}u = \mathcal{I}_{ii}u$  with  $\Gamma = J_i = (0, 1)$  (in the sense of Section 2.4). We think that our proof below has its own interest since it is constructive once we can solve (4.5) efficiently.

Let us turn to our construction. For  $\eta \in (0, 1)$ , for each  $1 \leq i, j \leq N$  and  $x \in \bar{E}_i$ ,  $r > 0$ , we consider the modified kernel

$$\nu_{ij}^\eta(x, r) = \min\{\eta^{-1}, \nu_{ij}(x, r)\},$$

and the nonlocal operator  $\mathcal{I}_i^\eta$  defined as in (2.9) but taking into account this kernel. It is easy to see that  $\nu_{ij}^\eta(x, \rho(x, z))dz_j$  is a bounded measure on  $J_j$ , which satisfies (2.7) with the same constants  $\Lambda, \sigma$ , independently of  $\eta$ . Moreover, if  $u \in C^{0,\gamma}(\Gamma)$ ,  $\gamma > \sigma$ , then we have  $\mathcal{I}^\eta u \rightarrow \mathcal{I}u$  uniformly in  $\Gamma$  as  $\eta \rightarrow 0$  (the proof is similar to the one of Lemma 2.2).

We consider the following approximate problem

$$(4.6) \quad \begin{cases} \lambda u^{\epsilon,\eta} - \epsilon u_{x_i x_i}^{\epsilon,\eta} - \mathcal{I}_i^\eta u^{\epsilon,\eta} + H_i(x, u_{x_i}^{\epsilon,\eta}) = 0 & \text{on } E_i, 1 \leq i \leq N, \\ u^{\epsilon,\eta}(O) = \theta, \\ u_i^{\epsilon,\eta}(a_i) = h_i, \quad 1 \leq i \leq N. \end{cases}$$

**Lemma 4.2.** *Under the assumption (2.15), there exists a viscosity solution  $u^{\epsilon,\eta} \in C(\Gamma)$  to problem (4.6), and we have the estimate*

$$(4.7) \quad \|u^{\epsilon,\eta}\|_{L^\infty(\Gamma)} \leq C_0,$$

where  $C_0$  is defined in (4.3).

*Proof.* Since  $\nu_{ij}^\eta$  are bounded measures, we can set

$$\lambda_{ij}^\eta(x) := \int_{J_j} \nu_{ij}^\eta(x, \rho(x, z))dz_j \text{ for } x \in \bar{E}_i,$$

from which we can write

$$(4.8) \quad \mathcal{I}_{ij}^\eta u(x) = \int_{J_j} u_j(z_j) \nu_{ij}^\eta(x, \rho(x, z)) dz_j - \lambda_{ij}^\eta(x) u_i(x).$$

We construct the solution of the problem (4.6) as the uniform limit of a sequence  $\{u^k\}_k$  that we define inductively as follows. After relabeling the set of indices  $1 \leq i \leq N$  if necessary, we split it as  $\{1, \dots, N', N' + 1, \dots, N\}$  such that, for all  $N' + 1 \leq i \leq N$ ,  $\mathcal{I}_i^\eta = \mathcal{I}_{ii}^\eta$  is censored to  $\bar{E}_i$ , in other words,

$$\lambda_{ij}^\eta(x) = 0, \quad \text{for all } x \in \bar{E}_i.$$

For each  $N' + 1 \leq i \leq N$ , we solve the censored Dirichlet problem

$$(4.9) \quad \begin{cases} \lambda u - \epsilon u_{x_i x_i} - \mathcal{I}_{ii}^\eta u(x) + H_i(x, u_{x_i}) = 0 & \text{in } J_i, \\ u(0) = \theta, u(a_i) = h_i, \end{cases}$$

which has a unique viscosity solution  $u_i \in C([0, a_i])$  in view of Theorem B.3. For all  $k \in \mathbb{N}$  and  $N' + 1 \leq i \leq N$ , we set  $u_i^k := u_i$ .

Now we deal with the edges  $1 \leq i \leq N'$ . We initialize the sequence  $\{u_i^k\}_{k, 1 \leq i \leq N'}$  with affine functions  $u_i^0 \in C(\bar{J}_i)$  taking values  $u_i^0(0) = \theta$ ,  $u_i^0(a_i) = h_i$  for each  $1 \leq i \leq N'$ . This concludes the definition of  $u^0 \in C(\Gamma)$ .

For  $k \geq 0$  and  $u^k$  already defined, using again Theorem B.3, we construct  $u_i^{k+1} \in C(\bar{J}_i)$ ,  $1 \leq i \leq N'$  as the unique solution to the censored Dirichlet problem

$$(4.10) \quad \begin{cases} \Lambda_i^\eta(x) u - \epsilon u_{x_i x_i} - \mathcal{I}_{ii}^\eta u(x) - \sum_{j \neq i} f_{ij}^k(x) + H_i(x, u_{x_i}) = 0 & \text{in } J_i, \\ u(0) = \theta, u(a_i) = h_i, \end{cases}$$

where we have denoted

$$\Lambda_i^\eta(x) = \lambda + \lambda_i^\eta(x) := \lambda + \sum_{j \neq i} \lambda_{ij}^\eta(x),$$

$$f_{ij}^k(x) = \int_{J_j} u_j^k(z_j) \nu_{ij}^\eta(x, \rho(x, z)) dz_j, \quad j \neq i.$$

In the construction of each  $u_i^{k+1}$ , we only require the information of the continuous functions  $u^k$  the functions  $f_{ij}^k$ . Notice that  $f_{ij}^k$  is continuous by Lebesgue Theorem.

The key step on the proof is the following

**Claim.** *There exists  $\lambda^* \in (0, 1)$  such that, for all  $k$  large enough we have*

$$(4.11) \quad \max_{1 \leq i \leq N} \|u_i^k\|_{L^\infty(\bar{J}_i)} \leq C_0,$$

$$(4.12) \quad \|u_i^{k+1} - u_i^k\|_{L^\infty(\bar{J}_i)} \leq \lambda^* \|u^k - u^{k-1}\|_{L^\infty(\Gamma)}, \quad \text{for all } 1 \leq i \leq N,$$

where  $C_0$  is the constant in (4.3).

We start with (4.11). For  $k = 0$  this is evident, as well as for each  $u_i^k$  with  $N' + 1 \leq i \leq N$  by the comparison principle Lemma B.1 for (4.9). If

the bound is true for  $u^k$ , then we have that for each  $1 \leq i \leq N'$ ,  $u_i^{k+1}$  is a viscosity subsolution to

$$\Lambda_i^\eta u - \epsilon u_{x_i x_i} - \mathcal{I}_{ii}^\eta u + H_i(x, u_{x_i}) \leq C_0 \lambda_i^\eta(x) \quad \text{in } J_i,$$

with its respective boundary conditions. From here, recalling that  $\Lambda_i^\eta = \lambda + \lambda_i^\eta$ , it is easy to see that the constant function equal to  $C_0$  in  $J_i$  is a supersolution for the problem solved by  $u_i^{k+1}$ , from which  $u_i^{k+1} \leq C_0$ . The lower bound can be found in the same way, and (4.11) follows. Notice that the estimate in (4.11) does not depend neither on  $k, \epsilon$  nor on  $\eta$ .

Next, we prove (4.12). Notice that this inequality trivially holds for  $N' + 1 \leq i \leq N$  since in that case we have  $u_i^{k+1} = u_i^k$  for all  $k \geq 0$ . Thus, we concentrate on the case  $1 \leq i \leq N'$ .

For each  $k$ , denote  $w_i^k = u_i^{k+1} - u_i^k \in C(\bar{J}_i)$ . Under the Lipschitz assumption on  $H$  in (2.6),  $w_i^k$  solves the problem

$$\begin{cases} \Lambda_i^\eta w^k - \epsilon w_{x_i x_i}^k - \mathcal{I}_{ii}^\eta w^k - C_H |w_{x_i}^k| \leq \lambda_i^\eta \|w^{k-1}\|_{L^\infty(\Gamma)}, & \text{in } J_i, \\ w^k(0) = w^k(a_i) = 0, \end{cases}$$

see [8, Lemma 5.3] or [9, Lemma 3.7]. Denote

$$\lambda^* = \max_{1 \leq i \leq N'} \max_{x \in \bar{J}_i} \frac{\lambda_i^\eta(x)}{\Lambda_i^\eta(x)} \in (0, 1).$$

It is easy to see that the constant function  $x \mapsto \lambda^* \|w^{k-1}\|_{L^\infty(\Gamma)}$  is a supersolution for the problem solved by  $w_i^k$ , from which by comparison we get

$$w_i^k \leq \lambda^* \|w^{k-1}\|_{L^\infty(\Gamma)} \quad \text{in } \bar{J}_i,$$

and since a lower bound can be established in the same way, we arrive at

$$\|w_i^k\|_{L^\infty(\bar{J}_i)} \leq \lambda^* \|w^{k-1}\|_{L^\infty(\Gamma)},$$

which concludes (4.12).

Thus, we can pass to the limit as  $k \rightarrow +\infty$  in (4.10), and by stability properties of viscosity solutions we arrive at a solution to (4.6).  $\square$

We now deduce the well-posedness of (4.1) from the above lemma and give properties of the solution.

**Proposition 4.3.** *Assume (2.15). For each  $\theta \in \mathbb{R}$ , there exists a unique viscosity solution  $u^\epsilon \in C(\Gamma)$  to problem (4.1). For each  $1 \leq i \leq N$ , we have*

(i) *There exist  $L, \delta > 0$ , not depending on  $\epsilon$ , such that*

$$(4.13) \quad u_i^\epsilon(x) \leq \theta + Lx, \quad x \in [0, \delta],$$

$$(4.14) \quad u_i^\epsilon(x) \leq h_i + L(a_i - x), \quad x \in [a_i - \delta, a_i].$$

(ii) *There exist  $L_\epsilon, \delta_\epsilon > 0$  such that*

$$(4.15) \quad \theta - L_\epsilon x \leq u_i^\epsilon(x), \quad x \in [0, \delta_\epsilon],$$

$$(4.16) \quad h_i - L_\epsilon(a_i - x) \leq u_i^\epsilon(x), \quad x \in [a_i - \delta_\epsilon, a_i].$$

(iii) For every  $r > 0$ , there exists  $C_r > 0$ , independent of  $\epsilon$ , such that

$$(4.17) \quad |u_i^\epsilon(x) - u_i^\epsilon(y)| \leq C_r |x - y|, \quad x, y \in [r, a_i - r].$$

If  $C_\epsilon = \max\{L, L_\epsilon, C_{\delta \wedge \delta_\epsilon}\}$ , then

$$(4.18) \quad |u_i^\epsilon(x) - u_i^\epsilon(y)| \leq C_\epsilon |x - y|, \quad x, y \in \bar{J}_i.$$

(iv)  $u^\epsilon \in C^{2,1-\sigma}(\Gamma)$ .

(v) The map  $\theta \in \mathbb{R} \mapsto u^\epsilon \in C^{2,\alpha}(\Gamma)$  is continuous for all  $\alpha < 1 - \sigma$ .

*Proof.* Let  $u^{\epsilon,\eta}$  be the solution found in Lemma 4.2. In what follows, we are going to prove that this solution satisfies the estimates given in the statement of this proposition, and that these estimates are independent of  $\eta$ . Thus, the results for the problem (4.1) follow by Ascoli Theorem and stability, sending  $\eta \rightarrow 0$ . The uniqueness of  $u^\epsilon$  comes from Lemma B.1.

(i) and (ii). The proofs of the barriers are a direct application of the comparison principle Lemma B.1, which yields  $\psi^- \leq u^{\epsilon,\eta} \leq \psi^+$  in  $\Gamma$ , where  $\psi^\pm$  are the sub and supersolution constructed in Lemma 4.1. Notice that (4.15) and (4.16) depend on the ellipticity constant  $\epsilon$  since  $\psi^-$  depend on  $\epsilon$  in Lemma 4.1.

(iii) *Lipschitz estimates.*

*Step 1. Claim: There exists  $K_0 = K_0(N, \lambda, \Lambda, \sigma, C_H, C_0, \text{osc}_\Gamma\{u^{\epsilon,\eta}\})$  such that, for any  $x_0 \in \Gamma$  and  $\varphi(x) = K\rho(x, x_0)$  with  $K > K_0$ ,*

$$(4.19) \quad \sup_{x \in \Gamma} \{u^{\epsilon,\eta}(x) - u^{\epsilon,\eta}(x_0) - \varphi(x)\}$$

*is achieved either at  $\bar{x} = x_0$ , or at a vertex  $\bar{x} \in \mathbf{V}$ .*

Let us underline that this claim relies on the coercivity of the Hamiltonian and does not depend neither on the ellipticity constant  $\epsilon$  nor on  $\eta$  since  $u^{\epsilon,\eta}$  and  $\text{osc}_\Gamma\{u^{\epsilon,\eta}\}$  are bounded independently of  $\epsilon, \eta$ .

We prove the claim, assuming without loss of generality, that  $\bar{x} \in E_i$  and  $\bar{x} \neq x_0$ . It follows that we can use  $\varphi$  as a test-function for the subsolution  $u^\epsilon$  of (4.1) at  $\bar{x}$ . For any  $\rho > 0$  we have

$$(4.20) \quad G_i^\rho(u^{\epsilon,\eta}, \varphi, \varphi_{x_i}(\bar{x}), \bar{x}) \leq 0.$$

We estimate the different terms: we have

$$u^{\epsilon,\eta}(\bar{x}) \geq -C_0,$$

$$H(\bar{x}, \varphi_{x_i}(\bar{x})) \geq C_H^{-1} |\varphi_{x_i}(\bar{x})| - C_H = C_H^{-1} K - C_H, \text{ and}$$

$$\varphi_{x_i x_i}(\bar{x}) = 0.$$

On the other hand, we have

$$\begin{aligned}
|\mathcal{I}_i[B_\varrho^c(\bar{x})]u^\epsilon(\bar{x})| &\leq \operatorname{osc}_\Gamma\{u^{\epsilon,\eta}\} \sum_{j=1}^N \int_{J_j \cap \{\rho(\gamma_j(z), \bar{x}) \geq \varrho\}} \nu_{ij}^\eta(\bar{x}, \rho(\gamma_j(z), \bar{x})) dz \\
&\leq \Lambda \operatorname{osc}_\Gamma\{u^{\epsilon,\eta}\} \sum_{j=1}^N \int_{J_j \cap \{\rho(\gamma_j(z), \bar{x}) \geq \varrho\}} \rho(\gamma_j(z), \bar{x})^{-1-\sigma} dz \\
&\leq 2 \operatorname{osc}_\Gamma\{u^{\epsilon,\eta}\} N \Lambda \sigma^{-1} \varrho^{-\sigma},
\end{aligned}$$

and using that  $\varphi$  is  $K$ -Lipschitz, we have

$$\begin{aligned}
|\mathcal{I}_i[B_\varrho(\bar{x})]\varphi(\bar{x})| &\leq \sum_{j=1}^N \int_{J_j \cap \{\rho(\gamma_j(z), \bar{x}) < \varrho\}} |\varphi_j(z) - \varphi(\bar{x})| \nu_{ij}^\eta(\bar{x}, \rho(\gamma_j(z), \bar{x})) dz \\
&\leq K \Lambda \sum_{j=1}^N \int_{J_j \cap \{\rho(\gamma_j(z), \bar{x}) < \varrho\}} \rho(\gamma_j(z), \bar{x})^{-\sigma} dz \\
&\leq \frac{N K \Lambda \varrho^{1-\sigma}}{1-\sigma}.
\end{aligned}$$

Plugging these estimates into the viscosity inequality (4.20), we arrive at

$$C_H^{-1} K \leq 2 \operatorname{osc}_\Gamma\{u^{\epsilon,\eta}\} N \Lambda \sigma^{-1} \varrho^{-\sigma} + \frac{N K \Lambda \varrho^{1-\sigma}}{1-\sigma} + C_H + \lambda C_0.$$

At this point, we fix  $\varrho$  small enough in order that  $\frac{N \Lambda \varrho^{1-\sigma}}{1-\sigma} \leq \frac{C_H^{-1}}{2}$ , from which we get

$$\frac{C_H^{-1} K}{2} \leq 2 \operatorname{osc}_\Gamma\{u^{\epsilon,\eta}\} N \Lambda \sigma^{-1} \varrho^{-\sigma} + C_H + \lambda C_0.$$

Therefore, choosing  $K \geq K_0$ , with

$$K_0 := 2 C_H (2 \operatorname{osc}_\Gamma\{u^{\epsilon,\eta}\} N \Lambda \sigma^{-1} \varrho^{-\sigma} + C_H + \lambda C_0) + 1,$$

we reach a contradiction with (4.20).

*Step 2. Interior Lipschitz estimates.* Let  $1 \leq i \leq N$ ,  $r > 0$  and  $x_0 \in E_i$  such that  $(x_0)_i \in [r, a_i - r]$ . We consider

$$(4.21) \quad K > \max\left\{\frac{4C_0}{r}, K_0\right\},$$

with  $K_0$  as in Step 1.

The supremum in (4.19) is nonnegative and it is achieved at some  $\bar{x} \in \Gamma$  such that

$$(4.22) \quad K \rho(\bar{x}, x_0) \leq u^{\epsilon,\eta}(\bar{x}) - u^{\epsilon,\eta}(x_0) \leq 2C_0,$$

from which, by the choice of  $K$ , we conclude that  $\bar{x}_i \in J_i \cap [r/2, a_i - r/2]$ . Thanks to the claim, we get  $\bar{x} = x_0$  and therefore the supremum is 0. Since  $x_0$  is arbitrary in the set of points at distance at least  $r$  to the vertices, we conclude the interior Lipschitz estimates (4.17).

*Step 3. Lipschitz estimates near the vertices and global Lipschitz estimates.* These estimates rely on the barriers and will concentrate on  $O$ , the estimates near other vertices being similar.

Consider the constants  $L, \delta, L_\epsilon, \delta_\epsilon$  appearing in (4.13)-(4.14)-(4.15)-(4.16) and now choose

$$K > \max\{L, L_\epsilon, \frac{4C_0}{\delta_\epsilon}, K_0\}$$

and  $x_0 \in \Gamma \cap B_{\delta_\epsilon/2}(O)$  in (4.19).

From (4.22) and the choice of  $K$ , we obtain that the maximum in (4.19) is achieved at  $\bar{x} \in B_{\delta_\epsilon}(O)$ . From the claim in Step 1, we have that either  $\bar{x} = x_0$  and in this case we are done since the maximum is 0, or  $\bar{x} = O$ . In this latter case, using the barriers (4.13) and (4.15), it follows that

$$u^{\epsilon, \eta}(O) - u^{\epsilon, \eta}(x_0) - K\rho(O, x_0) \leq (\max\{L, L_\epsilon\} - K)\rho(O, x_0) \leq 0,$$

and the maximum is also 0. This proves the Lipschitz estimates near the vertices.

Putting the previous results together, we conclude for the global Lipschitz bound (4.18) (depending on  $\epsilon$  since the barriers (ii) depend on  $\epsilon$ , but not depending on  $\eta$  since  $\nu_{ij}^\eta$  satisfies (2.7) uniformly wrt  $\eta$ ).

Let us mention that the proofs of the Lipschitz estimates work readily when  $u$  is a *USC* viscosity subsolution of (4.1). In fact, Steps 1 and 2 do not rely on the ellipticity constant  $\epsilon$  in (4.1), so we can take  $\epsilon = 0$  in (4.1), or consider (2.12)-(2.13)-(2.14), see Lemma 5.2 below. Step 3 needs the whole set of barriers (4.13)-(4.14)-(4.15)-(4.16), and the last two may be lost when the ellipticity vanishes.

(iv)  *$C^{2,1-\sigma}$  regularity.* The function  $u_i^{\epsilon, \eta}$  is a continuous viscosity solution of

$$(4.23) \quad -\epsilon u_{x_i x_i}^{\epsilon, \eta} = f := -\lambda u_i^{\epsilon, \eta} + \mathcal{I}_i^\eta u_i^{\epsilon, \eta} - H_i(x, u_{x_i}^{\epsilon, \eta}) \text{ in } (0, a_i).$$

Since  $u_i^{\epsilon, \eta}$  is  $C_\epsilon$ -Lipschitz continuous on  $[0, a_i]$ , thanks to Lemma 2.2, we obtain that  $f \in L^\infty([0, a_i])$  and the  $L^\infty$  bound depends on  $C_0$  and the Lipschitz constant  $C_\epsilon$  of  $u_i^{\epsilon, \eta}$ . Therefore, by Han-Lin [26, Theorem 5.22], we obtain that  $u_i^{\epsilon, \eta} \in W_{\text{loc}}^{2,p}([0, a_i])$  for all  $1 < p < \infty$ . By Sobolev inequalities (see e.g., Evans [22, Theorem 6, p.270]), it follows that  $u_i^{\epsilon, \eta} \in C^{1,\gamma}([0, a_i])$  for all  $\gamma < 1$ . It means that one can upgrade the regularity of  $f$  in (4.23): from Lemma 2.2 once again and since  $H_i$  is Lipschitz continuous, we get that  $f \in C^{0,1-\sigma}([0, a_i])$ . Thus  $u_{x_i x_i}^{\epsilon, \eta} \in C^{0,1-\sigma}([0, a_i])$  and we conclude  $u_i^{\epsilon, \eta} \in C^{2,1-\sigma}([0, a_i])$  as desired. Note that  $\|u_i^{\epsilon, \eta}\|_{C^{2,1-\sigma}(\bar{J}_i)}$  depends on the  $L^\infty$  bound (4.7) and the Lipschitz constant  $C_\epsilon$  of  $u_i^{\epsilon, \eta}$  (see (4.18)).

All the above estimates are independent of  $\eta$ , from which we can pass to the limit as  $\eta \rightarrow 0$  in problem (4.6) and conclude the existence of a solution  $u^\epsilon$  to (4.1) by stability. Reproducing the arguments above, we obtain that  $u_i^\epsilon \in C^{2,1-\sigma}([0, a_i])$ .

(v) *Continuous dependence with respect to  $\theta$ .* Let  $\theta_n \rightarrow \theta$  and denote  $u_i^{\epsilon, \eta}$  the associated solution of (4.1) in  $\bar{J}_i$ . By (ii), the family  $\{u_i^{\epsilon, \eta}\}_n$  is uniformly

bounded as  $n \rightarrow \infty$ . Since  $u_i^{\epsilon,n} \in C^{2,1-\sigma}(\bar{J}_i)$  with a  $C^{2,1-\sigma}$  norm depending only on  $|\theta_n|$  through the  $L^\infty$  bound and the Lipschitz constant of  $u_i^\epsilon$ , it follows that  $\|u_i^{\epsilon,n}\|_{C^{2,1-\sigma}(\bar{J}_i)}$  is bounded with respect to  $n$ . By Ascoli Theorem, we can extract a subsequence still denoted by  $\{u_i^{\epsilon,n}\}$ , such that  $u_i^{\epsilon,n} \rightarrow \bar{u}_i^\epsilon$  in  $C^{2,\alpha}(\bar{J}_i)$  for  $0 < \alpha < 1 - \sigma$ , and such a function  $\bar{u}_i^\epsilon$  must satisfy (4.1) with pointwise Dirichlet condition thanks to (i)-(ii). By uniqueness of the solution of (4.1), all the subsequences converge to the same limit from which we infer that the whole sequence  $u_i^{\epsilon,n}$  converges to the solution of (4.6) associated with  $\theta$ . This proves the result.  $\square$

The following is the existence result for the original Kirchhoff-type equation.

**Theorem 4.4.** *Under assumption (2.15), there exists a viscosity solution  $u \in C(\Gamma)$  for the Kirchhoff problem (2.12)-(2.14)-(2.13). Moreover,  $u \in C^{0,\gamma}(\Gamma)$  for any  $\gamma \in (0, 1)$ , and is locally Lipschitz continuous in  $\Gamma \setminus \{\bar{\mathbf{v}}_i\}_{1 \leq i \leq N}$ .*

*Proof.* We divide the proof in several steps.

*Step 1. Getting the Kirchhoff condition.* For  $\theta \in \mathbb{R}$ , denote by  $u^{\epsilon,\theta} \in C^{2,1-\sigma}(\Gamma)$  the unique solution of (4.1) associated with  $\theta$  given by Proposition 4.3. At this point we follow the arguments of Ohavi [34]. Let

$$F(\theta) = \sum_{i=1}^N -u_{x_i}^{\epsilon,\theta}(O) - B.$$

and  $p = (p_i)_{1 \leq i \leq N}$  be any vector in  $\mathbb{R}^N$  such that

$$\sum_{i=1}^N -p_i = B.$$

Define  $U \in C^2(\Gamma)$  by  $U_i(x) := \theta + p_i x_i$  for  $x \in \bar{E}_i$ ,  $1 \leq i \leq N$ . We have

$$(4.24) \quad G_i(U, U_{x_i}(x), x) = \lambda(\theta + p_i x) - \mathcal{I}_i U(x) + H_i(x, p_i),$$

with

$$\begin{aligned} |\mathcal{I}_i U(x)| &= \left| \int_{J_i} p_i(z - x_i) \nu_{ii}(x, \rho(x, \gamma_i(z))) dz \right. \\ &\quad \left. + \sum_{j \neq i} \int_{J_j} (p_j z - p_i x_i) \nu_{ij}(x, \rho(x, \gamma_j(z))) dz \right| \\ &\leq \Lambda |p| \left( \int_{J_i} |z - x_i|^{-\sigma} dz + \sum_{j \neq i} \int_{J_j} (x_i + z)^{-\sigma} dz \right) \\ &\leq \frac{(2\bar{a})^{1-\sigma} N \Lambda |p|}{1 - \sigma}, \end{aligned}$$

and  $|H_i(x, p_i)| \leq C_H(1 + |p_i|) \leq C_H(1 + |p|)$ . Here we have adopted the notation  $|p|$  for the Euclidean norm of  $p \in \mathbb{R}^N$  and  $\bar{a} = \max_i a_i$ .



We claim that, if  $\theta = \theta^+$  is large enough, then  $U$  is a supersolution of (4.1). Indeed, from (4.24), for  $x \in E_i$ , we have

$$G_i(U, U_{x_i}(x), x) \geq \lambda(\theta^+ - \bar{a}|p|) - \frac{(2\bar{a})^{1-\sigma} N \Lambda |p|}{1-\sigma} - C_H(1+|p|) \geq 0,$$

provided  $\theta^+ \geq \bar{a}|p| + \lambda^{-1}((2\bar{a})^{1-\sigma}(1-\sigma)^{-1} \Lambda N |p| + C_H(1+|p|))$ . If, in addition,  $\theta^+ \geq \bar{a}|p| + \max_i |h_i|$ , then  $U(O) = \theta^+$  and  $U_i(a_i) = \theta^+ + p_i a_i \geq h_i$ . Therefore the boundary conditions at the vertices are satisfied. Notice that  $\theta^+$  does not depend on  $\epsilon$ .

By Lemma B.1, we can compare the solution  $u^{\epsilon, \theta^+}$  to (4.1) associated with  $\theta^+$  with  $U$  to obtain that, for all  $1 \leq i \leq N$  and  $x \in \bar{J}_i$ ,

$$u_i^{\epsilon, \theta^+}(x) \leq U_i(x) = \theta^+ + p_i x = u_i^{\epsilon, \theta^+}(O) + p_i x,$$

from which we deduce

$$u_{x_i}^{\epsilon, \theta^+}(O) \leq p_i.$$

Then

$$F(\theta^+) = \sum_{i=1}^N -u_{x_i}^{\epsilon, \theta^+}(O) - B \geq \sum_{i=1}^N -p_i - B = 0.$$

In the same way, if we consider  $\theta = \theta^-$  adequate (say, negative and large in absolute value), then  $U$  is a subsolution of (4.1). Similarly, we obtain that the solution  $u^{\epsilon, \theta^-}$  to problem (4.1) satisfies

$$F(\theta^-) = \sum_{i=1}^N -u_{x_i}^{\epsilon, \theta^-}(O) - B \leq \sum_{i=1}^N -p_i - B = 0.$$

Then, by continuity of  $\theta \mapsto F(\theta)$  (Proposition 4.3 (v)), there exists  $\theta_\epsilon \in [\theta^-, \theta^+]$  such that the associated solution  $u^{\epsilon, \theta_\epsilon}$  to problem (4.1) satisfies

$$(4.25) \quad F(\theta_\epsilon) = 0,$$

which is the Kirchhoff condition (2.13).

*Step 2.* The family  $\{u^{\epsilon, \theta_\epsilon}\}_\epsilon$  is uniformly Lipschitz continuous on a neighborhood of  $O$ . We will take profit of the Kirchhoff condition. At first, since  $\theta^\pm$  do not depend on  $\epsilon$ , the value  $\theta_\epsilon$  is bounded from above and below uniformly with respect to  $\epsilon$ . Thus, it follows that  $u^\epsilon := u^{\epsilon, \theta_\epsilon}$  is uniformly bounded with respect to  $\epsilon$ , with  $L^\infty$ -bounds given by  $C_0$  in (4.3) with  $|\theta| = \max\{|\theta^+|, |\theta^-|\}$ .

Let  $\delta < \min_{1 \leq i \leq N} |a_i|$  and consider

$$(4.26) \quad \Gamma^\delta := \{x \in \Gamma : \rho(x, \bar{\mathbf{v}}_i) > \delta, 1 \leq i \leq N\} = \Gamma \setminus \bigcup_{1 \leq i \leq N} B(\bar{\mathbf{v}}_i, \delta).$$

Our goal is to prove that  $u^\epsilon$  is Lipschitz continuous in  $\Gamma^\delta$  uniformly with respect to  $\epsilon$ .

We proceed as in the proof of Proposition 4.3 (iii) borrowing some arguments of [7, Proposition 6.3]. Let  $x_0 \in \Gamma^\delta \cap \bar{E}_i$  and consider the maximum as in (4.19), that is

$$(4.27) \quad \sup_{x \in \Gamma} \{u^\epsilon(x) - u^\epsilon(x_0) - \varphi(x)\}$$

but this time with the modified function  $\varphi = \varphi^i$  defined by

$$\varphi^i(x) = \begin{cases} K\rho(x, x_0) & \text{if } x \in E_i \\ K\rho(O, x_0) + L\rho(x, O) & \text{if } x \in E_j, j \neq i, \end{cases}$$

for  $K, L \geq K_0$ , where  $K_0$  is defined in the proof of Proposition 4.3 (iii) and does not depend on  $\epsilon$ . With a straightforward adaptation of the proof of Proposition 4.3 (iii), we obtain that (4.27) is achieved at  $\bar{x} = x_0$  or at a vertex. On the one hand, if  $\bar{x} = x_0$ , then the maximum (4.27) is 0, from which we get the Lipschitz estimates on  $\Gamma^\delta$ .

On the other hand, up to enlarge  $K, L$  in order that  $K, L \geq 4\|u^\epsilon\|_\infty/\delta$ , we have  $\rho(\bar{x}, x_0) \leq \delta/2$  and therefore  $\bar{x}$  cannot be equal to any exterior vertex  $\bar{\mathbf{v}}_i$ ,  $1 \leq i \leq N$ .

It remains to prove that the case  $\bar{x} = O \neq x_0$  is not possible either. If  $\bar{x} = O$ , then, using that  $u^\epsilon$  is a classical (hence, viscosity) solution to the Kirchhoff problem, we have

$$\sum_{1 \leq j \leq N} -\varphi_{x_j}^i(O) \leq B.$$

A direct computation shows that

$$\varphi_{x_i}^i(O) = -K, \quad \varphi_{x_j}^i(O) = L, \quad \text{for all } j \neq i,$$

from which the Kirchhoff condition reads

$$K - (N - 1)L \leq B.$$

and from here, fixed  $L$  as above and enlarging  $K$ , we get a contradiction.

Finally, taking  $K, L \geq \max\{K_0, 4\|u^\epsilon\|_\infty/\delta\}$  and  $K > (N - 1)L + B$ , the maximum (4.27) is 0. Since, this is true for all  $x_0 \in \Gamma^\delta$ , we obtain that  $u^\epsilon$  is Lipschitz continuous in  $\Gamma^\delta$ , with a constant  $\max\{K, L\}$  depending on  $\delta$  but independent of  $\epsilon$ .

*Step 3. Sending  $\epsilon$  to 0.* In the previous steps, we have obtained that  $u^\epsilon := u^{\epsilon, \theta^\epsilon}$  is bounded independently of  $\epsilon$  on  $\Gamma$ , and uniformly Lipschitz continuous in  $\Gamma^\delta$  for every  $\delta > 0$  small enough. Denote  $\mathring{\Gamma} = \Gamma \setminus \{\bar{\mathbf{v}}_i\}_{1 \leq i \leq N}$ . By Ascoli Theorem and a diagonal process, we can extract a subsequence  $\{u^\epsilon\}_\epsilon$  (not relabeled) which converges locally uniformly to some  $u \in C(\mathring{\Gamma})$ , which is locally Lipschitz continuous in  $\mathring{\Gamma}$  (notice that we loose both the Lipschitz continuity and the barriers on  $\{\bar{\mathbf{v}}_i\}_{1 \leq i \leq N}$  when  $\epsilon \rightarrow 0$ ). Thanks to the coercivity assumption (2.6) (iii), from [15, Theorem 2.1], we have  $u \in C^{0, \gamma}(\mathring{\Gamma})$  for any  $\gamma \in (0, 1)$ , with finite Hölder seminorm in  $\mathring{\Gamma}$  ([15, Theorem 2.1] is applied with  $m = 1$  and  $\theta = 0$ , yielding  $\gamma_0 = 1$ ). It follows that  $u$  can be extended by continuity up to the ends of the edges into a function (still called

$u$ ) in  $C^{0,\gamma}(\Gamma)$ . By standard stability for viscosity solutions,  $u$  is a viscosity solution of (2.12) in each edge  $E_i$ . It remains to establish the boundary conditions.

We first prove that  $u$  satisfies the generalized Kirchhoff condition at  $O$ . We only check the subsolution property since the proof for the supersolution is similar. If  $\phi \in C^2(\Gamma)$  is such that  $u - \phi$  has a strict maximum point at  $O$ , by uniform convergence of  $u^\epsilon$  in a neighborhood of  $O$ , there exists a sequence  $x^\epsilon \rightarrow O$  of local maximum points of  $u^\epsilon - \phi$ . If  $x^\epsilon = O$  along a subsequence  $\epsilon \rightarrow 0$ , we use the classical Kirchhoff condition for  $u^\epsilon$  to conclude that

$$B = - \sum_{1 \leq i \leq N} u_{x_i}^\epsilon(O) \geq - \sum_{1 \leq i \leq N} \phi_{x_i}(O),$$

from which the Kirchhoff condition is satisfied by  $u$ . On the other hand, if  $x^\epsilon \in \Gamma \setminus \{O\}$  for all  $\epsilon$  small, then we can assume that there exists  $i$  such that  $x^\epsilon \in E_i$  for all  $\epsilon$  (up to subsequences). It follows  $G_i^\delta(u^\epsilon, \phi, \phi_{x_i}(x_\epsilon), x_\epsilon) \leq 0$  and, since  $\phi_i \in C^1(\bar{J}_i)$  and  $u^\epsilon \rightarrow u$  uniformly as  $\epsilon \rightarrow 0$ , we can pass to the limit in the inequality to get  $G_i^\delta(u, \phi, \phi_{x_i}(O), O) \leq 0$  and conclude that the generalized Kirchhoff condition holds.

We turn to the Dirichlet condition at  $\bar{\mathbf{v}}_i$ . Since the barrier (4.14) from above is independent of  $\epsilon$ , we can pass to the limit  $\epsilon \rightarrow 0$  in the equality  $u_i^\epsilon(x) \leq h_i + L(a_i - x)$  to obtain  $u_i(x) \leq h_i + L(a_i - x)$  for  $0 \leq x < a_i$ . Then, sending  $x \rightarrow a_i^-$  and recalling that  $u$  is extended by continuity at  $a_i$ , we get  $u_i(a_i) \leq h_i$ . Therefore, the boundary condition for subsolutions is satisfied pointwisely.

For the supersolution property at  $\bar{\mathbf{v}}_i$ , the barrier from below depend on  $\epsilon$  and we cannot conclude in the same way. Consider  $\phi \in C^2(\Gamma)$  such that  $u - \phi$  has a strict minimum point at  $\bar{\mathbf{v}}_i$ . For every  $\epsilon > 0$ , let  $x_\epsilon$  be a minimum of  $u^\epsilon - \phi$  on  $\Gamma$ , which by standard arguments is in  $\bar{E}_i$ , i.e.,

$$(4.28) \quad u_i^\epsilon(x) - \phi_i(x) \geq u_i^\epsilon(x_\epsilon) - \phi(x_\epsilon), \quad x \in \bar{J}_i.$$

By (4.7), we can extract a subsequence (still denoted  $\epsilon$ ) such that  $x_\epsilon \rightarrow x_0 \in \bar{J}_i$  and  $u_i^\epsilon(x_\epsilon) \rightarrow \ell \in [-C_0, C_0]$ . If  $x_0 \neq a_i$ , then, thanks to the local uniform convergence in  $[0, a_i)$ , passing to the limit in (4.28), we obtain  $u_i(x) - \phi_i(x) \geq u_i(x_0) - \phi(x_0)$  for all  $x \in \bar{J}_i$ , which is a contradiction with the strict minimum at  $a_i$ . Therefore  $x_\epsilon \rightarrow a_i$ . Sending  $\epsilon \rightarrow 0$  in (4.28) for  $0 \leq x < a_i$ , we have  $u_i(x) - \phi_i(x) \geq \ell - \phi_i(a_i)$ . Then, letting  $x \rightarrow a_i^-$ , we obtain

$$(4.29) \quad u_i(a_i) \geq \lim_{\epsilon \rightarrow 0} u_i^\epsilon(x_\epsilon) = \ell.$$

It remains to consider the case when  $x_\epsilon \rightarrow x_0 = a_i$ . If  $x_\epsilon = a_i$  along a subsequence, then, using that  $u^\epsilon$  satisfies the Dirichlet condition pointwisely at  $\bar{\mathbf{v}}_i$ , yields at the limit  $u_i(a_i) \geq \ell \geq h_i$ . In this case  $u$  satisfies also the condition pointwisely. Now we deal with the case when  $x_\epsilon < a_i$  for all  $\epsilon$ .

Writing the viscosity supersolution inequality for  $u^\epsilon$ , we get

$$\begin{aligned} & G_i^\delta(u^\epsilon, \phi, \phi_{x_i}(x_\epsilon), x_\epsilon) \\ &= \lambda u_i^\epsilon(x_\epsilon) - \mathcal{I}_i[B_\delta^\epsilon(x_\epsilon)]u^\epsilon(x_\epsilon) - \mathcal{I}_i[B_\delta(x_\epsilon)]\phi(x_\epsilon) + H_i(x_\epsilon, \phi_{x_i}(x_\epsilon)) \geq 0. \end{aligned}$$

We want to pass to the limit  $\epsilon \rightarrow 0$  in the above inequality. On the one hand, by Lebesgue Theorem, using (2.7) and (4.29), we have

$$\begin{aligned} & \lim_{\epsilon \rightarrow 0} \lambda u_i^\epsilon(x_\epsilon) - \mathcal{I}_i[B_\delta^\epsilon(x_\epsilon)]u^\epsilon(x_\epsilon) \\ &= \lambda \ell - \sum_{1 \leq j \leq N} \int_{\{z_j \in J_j: \rho(\bar{\mathbf{v}}_i, \gamma_j(z)) \geq \delta\}} [u_j(z) - \ell] \nu_{ij}(\bar{\mathbf{v}}_i, \rho(\bar{\mathbf{v}}_i, \gamma_j(z))) dz \\ &\leq \lambda u_i(a_i) - \sum_{1 \leq j \leq N} \int_{\{z_j \in J_j: \rho(\bar{\mathbf{v}}_i, \gamma_j(z)) \geq \delta\}} [u_j(z) - u_i(a_i)] \nu_{ij}(\bar{\mathbf{v}}_i, \rho(\bar{\mathbf{v}}_i, \gamma_j(z))) dz \\ &= \lambda u_i(a_i) - \mathcal{I}_i[B_\delta^c(\bar{\mathbf{v}}_i)]u(\bar{\mathbf{v}}_i). \end{aligned}$$

On the other hand, thanks to the smoothness of  $\phi$ , the convergence of the other terms is obvious, and we arrive finally at  $G_i^\delta(u, \phi, \phi_{x_i}(\bar{\mathbf{v}}_i), \bar{\mathbf{v}}_i) \geq 0$ , which proves that the generalized Dirichlet condition holds at  $\bar{\mathbf{v}}_i$ .

Hence, we have concluded the existence of  $u \in C^{0,\gamma}(\Gamma)$  for any  $\gamma \in (0, 1)$ , solution of the Kirchhoff problem (2.12)-(2.13)-(2.14).  $\square$

**Remark 4.5.** (i) Using the same global sub/supersolutions for the Kirchhoff-Dirichlet problem in Step 1 in the proof above, we can complement the standard Perron's method with the strong comparison principle given by Theorem 5.1 below, and directly obtain the existence and uniqueness of the generalized Kirchhoff problem (2.12)-(2.13)-(2.14) at once.

(ii) Since all the subsolutions we use to have the estimates from below, namely (4.15)-(4.16), depend on  $\epsilon$ , we may have a loss of boundary conditions at the vertices when passing to the limit  $\epsilon \rightarrow 0$ . This comes from the fact that there is not enough ellipticity in the nonlocal term to counterbalance the coercivity of the Hamiltonian in absence of classical diffusion. At the junction  $O$ , the Kirchhoff condition allows to recover some control from below and to prove that the solution  $u^\epsilon$  is Lipschitz continuous uniformly in  $\epsilon$ , providing a continuous limit  $u$  at  $O$ . In contrast, at the boundary point, the limit  $u$  may be discontinuous. But thanks to Hölder regularity results for coercive equations, we can extend it by continuity up to the boundary into a function satisfying the generalized Dirichlet boundary conditions (see [13] for related discussions).

## 5. UNIQUENESS FOR THE KIRCHHOFF-TYPE PROBLEM ON A JUNCTION.

The comparison result is the following

**Theorem 5.1.** *Assume (2.15). Let  $u \in USC(\Gamma)$  be a viscosity subsolution and  $v \in LSC(\Gamma)$  be a viscosity supersolution to the problem (2.12)-(2.13)-(2.14). Then,  $u \leq v$  on  $\Gamma$ .*

In particular, there exists a unique continuous viscosity solution  $u \in C^{0,1}(\Gamma_\delta)$  to the problem, which coincides with the solution found in Theorem 4.4 (recall that  $\Gamma^\delta$  is defined by (4.26)).

**Lemma 5.2.** *Let  $u \in USC(\Gamma)$  be a bounded viscosity subsolution of (2.12)-(2.13). Then  $u$  is Lipschitz continuous on  $\Gamma\delta$  for some  $\delta > 0$ .*

*Proof.* We follow the proof of the Lipschitz estimates in Proposition 4.3 (iii). Steps 1 and 2 work readily for any  $USC$  viscosity subsolution of (2.12)-(2.13)-(2.14); they rely only on the coercivity of the Hamiltonian and do not depend on the ellipticity constant in the equation.

Nevertheless, we cannot follow the proof of Step 3 since the barriers from below are not true anymore due to the degeneracy of the equation. To recover the Lipschitz estimate at the junction point, we revisit the proof of the uniform Lipschitz estimates near the junction in Theorem 4.4. It is enough to prove that the maximum in (4.27), with  $u^\epsilon = u$  subsolution to (2.12)-(2.13) and  $x_0 \in \Gamma_\delta \cap E_i$  for some  $i$ , cannot be achieved at  $\bar{x} = O$  for  $K, L$  large enough.

Indeed, we first notice, as a direct consequence of Step 1 in the proof of Proposition 4.3 (iii), and the choice of  $K \geq K_0$ , that the PDE (2.12) cannot hold at  $\bar{x} = O$  for subsolutions, i.e.,  $G_i^\delta(u, \varphi, \varphi_{x_i}(0), O) > 0$  for any  $1 \leq i \leq N$ . We then prove that the Kirchhoff condition at  $O$  does not hold neither for large  $K, L$  exactly as in the proof of Theorem 4.4, Step 2. Thus, the maximum in (4.27) holds in the interior of the branch, from which the result holds by the choice of  $K, L$ .  $\square$

In order to prove Theorem 5.1, we first have to examine the viscosity inequalities which are satisfied at the junction on each branch in terms of sub- and superdifferential of the solutions (see [32, 11]).

To do so, for each  $i$ , we define

$$\begin{aligned} \bar{p}_i &= \limsup_{x_i \rightarrow 0^+} \frac{u_i(x_i) - u_i(0)}{x_i}, & \underline{p}_i &= \liminf_{x_i \rightarrow 0^+} \frac{u_i(x_i) - u_i(0)}{x_i}, \\ \bar{q}_i &= \limsup_{x_i \rightarrow 0^+} \frac{v_i(x_i) - v_i(0)}{x_i}, & \underline{q}_i &= \liminf_{x_i \rightarrow 0^+} \frac{v_i(x_i) - v_i(0)}{x_i}. \end{aligned}$$

Notice that, if  $D^+u_i(0) \neq \emptyset$ , then  $\bar{p}_i < +\infty$ , meanwhile, if  $D^-v_i(0) \neq \emptyset$ , then  $\underline{q}_i > -\infty$ . And both cases happen if either  $D_\Gamma^+u(O) \neq \emptyset$  or  $D_\Gamma^-v(O) \neq \emptyset$ .

**Lemma 5.3.** *Let  $u \in USC(\Gamma)$  (respectively  $v \in LSC(\Gamma)$ ) be a bounded viscosity subsolution (respectively supersolution) of (2.12)-(2.13) such that  $D_\Gamma^+u(O) \neq \emptyset$  (respectively  $D_\Gamma^-v(O) \neq \emptyset$ ). Then*

(i) *If  $\bar{p}_i$  is finite, then  $D^+u_i(0) = [\bar{p}_i, +\infty)$ ,  $\mathcal{I}_i u(O) \in \mathbb{R}$  and*

$$G_i(u, \bar{p}_i, O) \leq 0.$$

Similarly, if  $\underline{q}_i$  is finite, then  $D^-v_i(0) = (-\infty, \underline{q}_i]$ ,  $\mathcal{I}_i v(O) \in \mathbb{R}$  and

$$G_i(v, \underline{q}_i, O) \geq 0.$$

(ii) If, in addition,  $u$  is  $C^{0, \sigma + \varepsilon}$  in an  $\Gamma$ -neighborhood of  $O$  for some  $\varepsilon > 0$ , then we have

$$(5.1) \quad G_i(u, p, O) \leq 0 \quad \text{for all } p \in (\underline{p}_i, \bar{p}_i),$$

and the inequality holds for  $p = \underline{p}_i$  and/or  $p = \bar{p}_i$  if some is finite. Similarly, if  $v$  is  $C^{0, \sigma + \varepsilon}$  in an  $\Gamma$ -neighborhood of  $O$  for some  $\varepsilon > 0$ , then we have

$$G_i(v, q, O) \geq 0 \quad \text{for all } q \in (\underline{q}_i, \bar{q}_i),$$

and the inequality holds for  $q = \underline{q}_i$  and/or  $q = \bar{q}_i$  if some is finite.

Part (i) of Lemma 5.3 is a basic result: it is the nonlocal analogue to [11, Proposition 2.10, p. 84] and it does not require any additional regularity. On the contrary, Part (ii) is a more sophisticated result inspired from Lions and Souganidis [31, 32] (see also [11, Lemma 15.1, p.258]) and it is also surprising because none of the  $p \in (\underline{p}_i, \bar{p}_i)$  is in  $D^+u_i(0)$  and none of the  $q \in (\underline{q}_i, \bar{q}_i)$  is in  $D^-v_i(0)$ . Unfortunately, we were unable obtain this stronger result without the additional regularity we impose; of course, the difficulty was to take into account the nonlocal term. This additional assumption is automatically satisfied by subsolutions since they are Lipschitz continuous by Lemma 5.2 but it is not the case for supersolutions. For this reason, in the comparison proof, we can use Result (ii) for subsolutions but only Result (i) for supersolutions.

*Proof.* We first prove (i). We provide the proof for supersolutions, the one for subsolutions follows from the same arguments.

Since  $D^-v(O) \neq \emptyset$ , there exists a function  $\varphi \in C^1(\Gamma)$  such that  $v - \varphi$  has a minimum at  $O$ . It is possible to choose  $\varphi$  in order that the minimum is global over  $\Gamma$  and strict at  $O$ . Using that  $\underline{q}_i$  is finite, we have that  $D^-v_i(0) = (-\infty, \underline{q}_i]$  (see [11, Proposition 2.10, p. 84]), and, up to a modification of  $\varphi$  in  $E_i$ , we can take  $\varphi$  such that  $\varphi_{x_i}(O) = \underline{q}_i$ .

Now, for  $\varepsilon > 0$ , we consider the perturbed test-function  $\varphi^\varepsilon$  given on  $E_j$  ( $j = 1, \dots, N$ ), by

$$\varphi_j^\varepsilon(x_j) = \varphi_j(x_j) + \delta_{ij} \varepsilon x_j,$$

where  $\delta_{ij}$  is the usual Kronecker symbol.

We claim that there exists a sequence  $x^\varepsilon \in E_i$  of global minimum points for  $v - \varphi^\varepsilon$ , and this sequence tends to  $O$ . The existence of a sequence of global minimum points  $x^\varepsilon \in \Gamma$ , which tends to  $O$ , is given by standard arguments since  $O$  is a global, strict minimum point of  $v - \varphi$  on  $\Gamma$ .

It remains to prove that  $x^\varepsilon \in E_i$ . In fact, if  $x^\varepsilon \in E_j$  with  $j \neq i$ , then we have  $v(x^\varepsilon) - \varphi^\varepsilon(x^\varepsilon) = v(x^\varepsilon) - \varphi(x^\varepsilon) > v(O) - \varphi(O) = v(O) - \varphi^\varepsilon(O)$  by the

strict minimum property. Thus,  $x^\varepsilon \in \bar{E}_i$ . If  $x^\varepsilon = O$  along a subsequence, for each  $x \in E_i$  we have

$$v_i(x_i) - \varphi_i^\varepsilon(x_i) = v_i(x_i) - \varphi_i(x_i) - \varepsilon x_i \geq v_i(0) - \varphi_i(0),$$

from which we deduce

$$v_i(x_i) \geq v_i(0) + (\varphi_i(x_i) + \varepsilon x_i - \varphi_i(0)) \geq v_i(0) + (\underline{q}_i + \varepsilon)x_i + o(x_i),$$

and  $\underline{q}_i + \varepsilon \in D^-v_i(0)$ , which contradicts the maximality of  $\underline{q}_i$  in the subdifferential of  $v_i$  at 0. It ends the proof of the claim.

It follows that we can write the viscosity inequality for the supersolution at  $x^\varepsilon \in E_i$ , and since the testing is global, for all  $\delta > 0$ , we have

$$G_i^\delta(v, \varphi^\varepsilon, \underline{q}_i + \varepsilon, x^\varepsilon) \geq 0.$$

Using that  $v \in LSC(\Gamma)$ , we have  $v(x^\varepsilon) \rightarrow v(O)$  by standard arguments. Adding that  $\varphi^\varepsilon$  is Lipschitz continuous with a constant bounded uniformly with respect to  $\varepsilon$ , we can control the nonlocal terms  $\mathcal{I}_i[B_\delta(x^\varepsilon)]\varphi^\varepsilon(x^\varepsilon)$  and  $\mathcal{I}_i[B_\delta^\varepsilon(x^\varepsilon)]v(x^\varepsilon)$  with Dominated Convergence Theorem, for all fixed  $\delta > 0$ . It allows to pass to the limit  $x^\varepsilon \rightarrow 0$  in the previous inequality to obtain

$$G_i^\delta(v, \varphi, \underline{q}_i, O) \geq 0.$$

By Lemma 3.2,  $v \in \mathcal{F}_O$  and we can send  $\delta \downarrow 0$  to conclude that  $G_i(v, \underline{q}_i, O) \geq 0$  holds. As a by-product, we have that  $\mathcal{I}_i v(O)$  is finite.

Now we turn to the proof of (ii) which is strongly inspired by [11, Lemma 15.1, p.258]. We concentrate on the subsolution's case, the proof for supersolution being similar. Since  $D_\Gamma^+ u(O) \neq \emptyset$ , we know, by Lemma 3.2, that  $u \in \mathcal{F}_O$  and we can find  $\varphi \in C^1(\Gamma)$  such that  $u - \varphi$  achieves a maximum at  $O$ . We can even assume that  $u(O) = \varphi(O)$ .

We fix  $1 \leq i \leq N$ . We first assume that  $\underline{p}_i < \bar{p}_i$  and we take any  $p \in (\underline{p}_i, \bar{p}_i)$ . By definition of  $\underline{p}_i, \bar{p}_i$ , there exists sequences  $0 < a_k < b_k$  converging to 0 such that

$$(5.2) \quad u_i(b_k) - u_i(0) \leq (\underline{p}_i + k^{-1})b_k, \quad u_i(a_k) - u_i(0) \geq (\bar{p}_i - k^{-1})a_k.$$

For  $k$  large enough, in order to have  $\underline{p}_i + k^{-1} < p < \bar{p}_i - k^{-1}$ , we consider the function

$$y_i \in [0, b_k] \mapsto \chi(y_i) := u_i(y_i) - u_i(0) - py_i.$$

We have  $\chi(0) = 0$ ,  $\chi(b_k) < 0$  and  $\chi(a_k) > 0$  and therefore the function  $\chi$  has a maximum points  $y_k \in (0, b_k)$ . Of course,  $y_k \rightarrow 0$  and, by classical arguments,  $u_i(y_k) \rightarrow u_i(0)$ .

Modifying the function  $\varphi$  by setting  $\varphi_i(y) = u_i(0) + py$  on the branch  $E_i$ , we obtain that  $u - \varphi$  has a local maximum inside the branch  $E_i$  at  $\gamma_i(y_k)$ . Since  $u$  is  $(\sigma + \varepsilon)$ -Hölder continuous, we use Lemma 2.2 and Lemma 3.3 to arrive at

$$G_i(u, p, \gamma_i(y_k)) \leq 0,$$

and since  $G_i$  is continuous, using Lemma 2.2 again we conclude that

$$G_i(u, p, O) \leq 0.$$

When  $\underline{p}_i$  is finite (respectively  $\bar{p}_i$  is finite) with  $\underline{p}_i < \bar{p}_i$ , the same inequality holds for  $p = \underline{p}_i$  (respectively  $p = \bar{p}_i$ ) by continuity, letting  $p$  tend to  $\underline{p}_i$  (respectively to  $\bar{p}_i$ ).

It remains to treat the case when both  $\underline{p}_i$  and  $\bar{p}_i$  are finite with  $p = \underline{p}_i = \bar{p}_i$ . In this case, we perturb  $u$  in the branch  $E_i$  by setting

$$u_i^\epsilon(x) = u_i(x) + \epsilon x \sin(\log(x)), \quad x \in J_i, \quad u_i^\epsilon(0) = u_i(0),$$

which converges uniformly to  $u_i$  in  $\bar{J}_i$  as  $\epsilon \rightarrow 0$ , and satisfies

$$\liminf_{x \rightarrow 0^+} \frac{u_i^\epsilon(x) - u_i^\epsilon(O)}{x} =: \underline{p}_i^\epsilon = \underline{p}_i - \epsilon < \bar{p}_i + \epsilon = \bar{p}_i^\epsilon := \limsup_{x \rightarrow 0^+} \frac{u_i^\epsilon(x) - u_i^\epsilon(O)}{x}.$$

Denote  $\phi(x) = x \sin(\log(x))$ . As in the first step above, since  $\underline{p}_i^\epsilon < p < \bar{p}_i^\epsilon$ , the function

$$y \in \bar{J}_i \mapsto u_i^\epsilon(y) - u_i^\epsilon(O) - py = u_i(y) - u_i(O) - py + \epsilon\phi(y)$$

has a sequence of local maximum points  $y_k \rightarrow 0^+$ . We modify the test function  $\varphi$  in  $E_i$  by setting  $\varphi_i(x) = px + \epsilon\phi(x)$ . Noticing that  $\phi$  is Lipschitz continuous. Writing the viscosity inequality for  $u$  at  $y_k$ , and using the regularity of  $u$ , we obtain  $G_i(u, p + \epsilon\phi_x(y_k), \gamma_i(y_k)) \leq 0$ . Using (2.6) (ii), it follows  $G_i(u, p, \gamma(y_k)) \leq O(\epsilon)$ , where  $O(\epsilon)$  is independent of  $k$ . Sending  $y_k \rightarrow 0$ , we infer  $G_i(u, p, O) \leq O(\epsilon)$ . We then send  $\epsilon \rightarrow 0$  and finally  $\delta \downarrow 0$  to conclude.  $\square$

*Proof of Theorem 5.1.* By contradiction, we assume that

$$\sup_{\Gamma} \{u - v\} > 0.$$

Since  $\Gamma$  is compact, the supremum is attained at some point  $x_0 \in \Gamma$ .

If  $x_0 \in \Gamma \setminus \mathbf{V}$ , where  $\mathbf{V}$  is the set of vertices, then the proof follows standard arguments in the viscosity theory. If  $x_0 = \bar{\mathbf{v}}_i$  for some  $i$ , then the proof follows the boundary analysis presented in [38, 37], which is possible since  $u \in C^\gamma(\Gamma)$  for all  $\gamma \in (0, 1)$ . So, we consider only the case  $x_0 = O$ , that is

$$\max_{\Gamma} \{u - v\} = (u - v)(O) =: M > 0,$$

which concentrates the main difficulties. We follow closely the lines of [32], see also [11, Section 15.3].

At first, notice that  $u$  is Lipschitz continuous on  $\Gamma_\delta$  for some  $\delta > 0$  by Lemma 5.2. From Lemma 2.2, we obtain that  $\mathcal{I}_i u(O) \in \mathbb{R}$  for all  $i$ . Next, since  $u$  is touching  $v$  from below at  $O$ , from Lemma 3.2, we obtain that  $v \in \mathcal{F}_O$  and

$$(5.3) \quad -\infty < \mathcal{I}_i u(O) \leq \mathcal{I}_i v(O) \leq +\infty, \quad \text{for all } i.$$

Notice that  $u \notin C^1(\Gamma_\delta)$  as required in the statement of Lemma 3.2 but  $u \in C^{0,1}(\Gamma_\delta)$  is enough in the proof. Using the Lipschitz continuity of  $u$ , we



can apply Lemma 5.3 (ii) to  $u$  and then Lemma 5.3 (i) to  $v$ . Since  $O$  is a maximum point of  $u - v$ , we also have

$$-\infty < \underline{p}_i \leq \bar{p}_i < +\infty \quad \text{and} \quad \underline{p}_i \leq \underline{q}_i \leq +\infty \quad \text{for all } i.$$

We divide the analysis in two cases.

*Case 1.* There exists  $i$  such that  $\underline{q}_i \leq \bar{p}_i$ . We have  $\underline{q}_i$  is finite and we can choose  $\underline{q}_i$  in the viscosity inequality of  $u$  and  $v$ , from which we get

$$\begin{aligned} G_i(u, \underline{q}_i, O) &= \lambda u(O) - \sum_{1 \leq j \leq N} \int_{J_j} [u_j(z) - u(O)] \nu_{ij}(O, z) dz + H_i(O, \underline{q}_i) \leq 0, \\ G_i(v, \underline{q}_i, O) &= \lambda v(O) - \sum_{1 \leq j \leq N} \int_{J_j} [v_j(z) - v(O)] \nu_{ij}(O, z) dz + H_i(O, \underline{q}_i) \geq 0, \end{aligned}$$

where both nonlocal terms are finite.

Subtracting both inequalities, we arrive at

$$\lambda M \leq \sum_{1 \leq j \leq N} \int_{J_j} [(u_j - v_j)(z) - M] \nu_{ij}(O, z) dz,$$

and since  $O$  is the maximum point of  $u - v$  on  $\Gamma$ , we arrive at  $0 < \lambda M \leq 0$ , which is a contradiction. This concludes this case.

*Case 2.* For all  $i$ ,  $\bar{p}_i < \underline{q}_i \leq +\infty$ . In this case, for each  $i$ , we define

$$r \in [\bar{p}_i, \underline{q}_i] \mapsto U_i(r) := \lambda \frac{u(O) + v(O)}{2} - \mathcal{I}_i u(O) + H_i(O, r).$$

By Lemma 5.3 (ii), we have

$$U_i(\bar{p}_i) = G_i(u, \bar{p}_i, O) - \lambda \frac{M}{2} < 0.$$

Meanwhile, if  $\underline{q}_i < +\infty$ , then, by Lemma 5.3 (i), we have that  $\mathcal{I}_i v(O)$  is finite, and

$$U_i(\underline{q}_i) = G_i(u, \underline{q}_i, O) - \lambda \frac{M}{2} \geq G_i(v, \underline{q}_i, O) + \lambda \frac{M}{2} > 0.$$

If  $\underline{q}_i = +\infty$ , then, thanks to the coercivity of  $H_i$  at  $+\infty$ , we can find  $\hat{q}_i > \bar{p}_i$  such that

$$U_i(\hat{q}_i) = G_i(u, \hat{q}_i, O) - \lambda \frac{M}{2} > 0.$$

By continuity of  $U_i$ , there exists  $r_i \in (\bar{p}_i, \underline{q}_i)$  (or in  $(\bar{p}_i, \hat{q}_i)$  in the case  $\underline{q}_i = +\infty$ ) such that

$$U_i(r_i) = 0 \quad \text{for all } i,$$

hence

$$G_i(u, r_i, O) = U_i(r_i) + \lambda \frac{M}{2} > 0, \quad \text{for all } 1 \leq i \leq N.$$

Using the continuity of  $H_i$ , it follows that there exists  $\varrho > 0$  small enough such that  $r_i - \varrho > \bar{p}_i$  and

$$(5.4) \quad G_i(u, r_i - \varrho, O) > 0 \quad \text{for all } i.$$

For all  $1 \leq i \leq N$  and  $x \in \bar{J}_i$ , by definition of  $\bar{p}_i$ , we have

$$u_i(x) - u(O) \leq \bar{p}_i x + o(x),$$

thus

$$u_i(x) - u(O) - (r_i - \varrho)x \leq (\bar{p}_i - (r_i - \varrho))x + o(x).$$

Since  $\bar{p}_i - (r_i - \varrho) < 0$ , we obtain that the function  $x \in \Gamma \mapsto u(x) - \phi(x)$ , where  $\phi \in C^1(\Gamma)$  is defined by

$$\phi_i(x) = u(O) + (r_i - \varrho)x, \quad x \in \bar{J}_i,$$

has a local maximum at  $x = O$ .

Then, recalling (5.4), the Kirchhoff condition at  $x = O$  is activated in the viscosity inequality of Lemma 3.3, from which

$$(5.5) \quad \sum_i -(r_i - \varrho) = \sum_i -r_i + N\varrho \leq B.$$

On the other hand, by (5.3) we have  $-\infty \leq -\mathcal{I}_i v(O) \leq -\mathcal{I}_i v(O) < +\infty$ , hence

$$(5.6) \quad G_i(v, r_i, O) \leq G_i(u, r_i, O) - \lambda M = U_i(r_i) - \lambda \frac{M}{2} < 0 \quad \text{for all } i.$$

Since  $r_i < \underline{q}_i$ , arguing as above, we have that the function  $x \in \Gamma \mapsto v(x) - \psi(x)$ , where  $\psi \in C^1(\Gamma)$  is defined by

$$\psi_i(x) = v(O) + r_i x, \quad x \in \bar{J}_i,$$

has a local minimum at  $x = O$ .

From (5.6), the viscosity inequality of Lemma 3.3 at  $x = O$  for supersolution implies again that the Kirchhoff condition holds, that is

$$\sum_i -r_i \geq B,$$

which contradicts (5.5). This concludes Case 2 and the proof of the comparison.  $\square$

## 6. OTHER CLASSES OF BOUNDARY CONDITIONS.

In this section we briefly discuss some possible extensions of our results for other boundary conditions on the exterior vertices  $\{\bar{\mathbf{v}}_i\}_{1 \leq i \leq N}$ .

The key point here is that the arguments around the junction point  $O$  can be isolated from the boundary conditions imposed on the exterior vertices  $\{\bar{\mathbf{v}}_i\}_{1 \leq i \leq N}$ . For instance, Lipschitz regularity of subsolutions and/or local barriers around at the junction point  $O$  can be performed in the same way by localizing the analysis on the set  $\Gamma^\delta$  in (4.26). Analogously, the local analysis on each of these boundary points  $\{\bar{\mathbf{v}}_i\}_{1 \leq i \leq N}$  when other type of

boundary conditions are imposed is not substantially modified by the Kirchhoff condition on the junction point  $O$ . This remark is valid both for the arguments presented in Sections 4 and (5).

Since a precise presentation of the result would be very demanding and would enlarge too much the manuscript, we provide the main ideas of other boundary conditions, referring to the articles where the main ideas to address the problems can be found.

1.- *Unbounded edges.* It is possible to assume that for some  $1 \leq i \leq N$ ,  $a_i = +\infty$ , and conclude the existence and uniqueness of a bounded solution for the problem, provided the data are bounded, that is,  $\sup_{x \in \bar{E}_i} |H_i(x, 0)| < +\infty$ , following the approach in [14, Theorem 3].

2.- *Exterior data.* For a given subset  $I \subset \{1, \dots, N\}$  and  $i \in I$ , let  $0 < A_i \leq +\infty$  with  $a_i < A_i$ ,  $E_i^{out} = \{x e_i : x \in [a_i, A_i]\}$  where  $e_i := \bar{\mathbf{v}}_i / a_i$ , and  $g_i \in C_b(E_i^{out})$  (the ‘‘exterior data’’). Then, we define a new nonlocal operator with the formula (2.9) with an extended domain of integration, by replacing  $J_i$  with  $J_i^* = (0, A_i)$  for  $i \in I$ . We complemented (2.12)-(2.13) with (2.14) for  $i \in \{1, \dots, N\} \setminus I$  and the exterior condition

$$u(x) = g_i(x), \quad x \in E_i^c, i \in I.$$

In this case, the arguments in [38, Theorem 2.1] can be adapted to conclude the existence and uniqueness of a viscosity solution.

3.- *State-constraint conditions.* Instead of (2.14), we can impose the state-constraint boundary condition

$$\lambda u(\bar{\mathbf{v}}_i) - \mathcal{I}_i u(\bar{\mathbf{v}}_i) + H_i(\bar{\mathbf{v}}_i, u_{x_i}(\bar{\mathbf{v}}_i)) \geq 0,$$

where the inequality is understood in the viscosity sense: for each  $\phi \in C^1(\Gamma)$  such that  $u - \phi$  has a minimum point constrained to  $\bar{E}_i$  at  $\bar{\mathbf{v}}_i$ , then  $G_i^\delta(u, \phi, \phi_{x_i}(\bar{\mathbf{v}}_i), \bar{\mathbf{v}}_i) \geq 0$  (see (2.3) and Definition (3.1) for the notations). Here, it is possible to follow the arguments discussed in [16, Proposition 4.1] to provide a strong comparison principle for problems with this type of boundary conditions.

4.- *Neumann boundary conditions.* Instead of (2.14), we can impose the Neumann boundary condition

$$u_{x_i}(\bar{\mathbf{v}}_i) = 0,$$

where the boundary condition is understood in the generalized (viscosity) sense, see [21]. The current setting fits into the assumptions considered by Ghilli [24, Theorem 3.1], to conclude the well-posedness of the problem.

## 7. THE PROBLEM ON A NETWORK.

7.1. **Problem set-up.** In this section we address the problem on a general connected network, that is, in the presence of more than one junction point. Most of the basic definitions and notations were already presented in the introduction, and we complement it here for sake of clarity.

We consider a nonempty, finite set of vertices  $\mathbf{V} \subset \mathbb{R}^d$  and a finite set of edges  $E \in \mathbf{E}$ , which are straight lines joining a pair of vertices. More specifically, for each edge  $E \in \mathbf{E}$ , there exists exactly two vertices  $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2 \in \mathbf{V}$  such that  $\bar{E} \cap \mathbf{V} = \{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2\}$  and

$$(7.1) \quad E = \left\{ \bar{\mathbf{v}}_1 + t \frac{\bar{\mathbf{v}}_2 - \bar{\mathbf{v}}_1}{|\bar{\mathbf{v}}_2 - \bar{\mathbf{v}}_1|} : t \in J_E \right\},$$

where  $J_E = (0, a_E) \subset \mathbb{R}$ ,  $a_E = |\bar{\mathbf{v}}_2 - \bar{\mathbf{v}}_1|$  is the length of the edge  $E$  and  $\bar{E}$  for the closure of the subset  $E$  in  $\mathbb{R}^d$ . We assume that, for any  $E_1, E_2 \in \mathbf{E}$ , if  $E_1 \neq E_2$ , then  $E_1 \cap E_2 = \emptyset$ ; this means that, if  $E_1, E_2$  have a common point, this is necessarily a vertex.

A network  $\Gamma \subset \mathbb{R}^d$  is the set

$$\Gamma = \bigcup_{E \in \mathbf{E}} \bar{E}.$$

We parametrize the network  $\Gamma$  by arc length in the following way (Notice that it is not exclusive). We label the vertices, namely  $\mathbf{V} = \{\bar{\mathbf{v}}_i\}_{i=1}^{\text{card}(\mathbf{V})}$ . Therefore, for each  $E \in \mathbf{E}$ , there exist a unique pair of indices  $i, j$  with  $i < j$  such that

$$\bar{E} = \left\{ \bar{\mathbf{v}}_i + t \frac{\bar{\mathbf{v}}_j - \bar{\mathbf{v}}_i}{a_E} : t \in \bar{J}_E \right\},$$

and we denote  $\gamma_E(t) = \bar{\mathbf{v}}_i + t \frac{\bar{\mathbf{v}}_j - \bar{\mathbf{v}}_i}{a_E}$ , the parametrization of the (closure) of the edge  $E$ .

We treat the case of connected networks, that is, for each pair of vertices  $\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2 \in \mathbf{V}$ , there exists a finite set of edges  $\{E_i\}_{i=1}^k$  connecting them, i.e., such that  $\bar{E}_1$  contains  $\bar{\mathbf{v}}_1$ ,  $\bar{E}_k$  contains  $\bar{\mathbf{v}}_2$ , and  $\bar{E}_i \cap \bar{E}_{i+1} \neq \emptyset$ . Therefore  $\alpha := \bar{E}_1 \cup \dots \cup \bar{E}_k$  is connected with respect to the usual topology in  $\mathbb{R}^d$ . Such an  $\alpha$  is called a path between  $\bar{\mathbf{v}}_1$  and  $\bar{\mathbf{v}}_2$ .

Given two points  $x, y \in \Gamma$ , the geodesic distance  $\rho(x, y)$  between  $x$  and  $y$  is defined by

$$\rho(x, y) = \inf \{ \text{length}(\alpha) : \alpha \text{ is a path between } x \text{ and } y \}.$$

For each  $\bar{\mathbf{v}} \in \mathbf{V}$ , we denote by  $\text{Inc}(\bar{\mathbf{v}})$  the set of the *incident* edges to  $\bar{\mathbf{v}}$ , that is the edges  $E \in \mathbf{E}$  such that  $\gamma_E(0) = \bar{\mathbf{v}}$  or  $\gamma_E(a_E) = \bar{\mathbf{v}}$ . The vertices are divided according to their nature in terms of its role in the equation. We write  $\mathbf{V} = \mathbf{V}_i \cup \mathbf{V}_b$ , where  $\bar{\mathbf{v}} \in \mathbf{V}_i$  are like junction points (where we will impose a Kirchhoff condition), and  $\bar{\mathbf{v}} \in \mathbf{V}_b$  are like boundary vertices (where we will impose Dirichlet boundary conditions, or possibly, another boundary condition, see Section 6).

We define the function spaces as in Section 2.2 with straightforward adaptations. In particular, given  $E \in \mathbf{E}$ , for each  $x \in \bar{E}$  we denote  $x_E \in \bar{J}_E$  such that  $\gamma_E(x_E) = x$ . In this setting, for  $u : \Gamma \rightarrow \mathbb{R}$ , we denote  $u_E : \bar{J}_E \rightarrow \mathbb{R}$  as  $u_E(t) = u(\gamma_E(t))$ , which can be equivalently written as  $u(x) = u_E(x_E)$ . When  $u \in C^1(\Gamma)$ , we write

$$u_x(x) = u'_E(x_E) \quad \text{for } x \in E, E \in \mathbf{E},$$

and we extend the derivative at the vertices by setting, if  $E \in \text{Inc}(\bar{\mathbf{v}})$ ,

$$u_{x_E}(\bar{\mathbf{v}}) = \lim_{x \rightarrow \bar{\mathbf{v}}, x \in E} u_x(x).$$

Then we define the inward derivative of  $u$  at  $\bar{\mathbf{v}}$  with respect to  $E$  as

$$(7.2) \quad \partial_E u(\bar{\mathbf{v}}) = \lim_{x \rightarrow \bar{\mathbf{v}}, x \in E} \frac{u(x) - u(\bar{\mathbf{v}})}{\rho(x, \bar{\mathbf{v}})}.$$

Be careful,  $\partial_E u(\bar{\mathbf{v}})$  and  $u_{x_E}(\bar{\mathbf{v}})$  do not always coincide as it was the case for junctions when  $\bar{\mathbf{v}} = O$ . More precisely, defining the *index of incidence* of  $\bar{\mathbf{v}}$  with respect to  $E \in \text{Inc}(\bar{\mathbf{v}})$  as

$$i_E(\bar{\mathbf{v}}) = \begin{cases} +1 & \text{if } \bar{\mathbf{v}} = \gamma_E(0), \\ -1 & \text{if } \bar{\mathbf{v}} = \gamma_E(a_E), \end{cases}$$

we have

$$(7.3) \quad u_{x_E}(\bar{\mathbf{v}}) = i_E(\bar{\mathbf{v}}) \partial_E u(\bar{\mathbf{v}}).$$

Next, we introduce the terms involved in (1.1) and the assumptions of each one. These conditions, together with the previous definitions and assumptions, are going to be the standing ones throughout this section.

We start with the Hamiltonian  $H$ . We intend it as a collection  $\{H_E\}_{E \in \mathbf{E}}$  with  $H_E \in C(\bar{E} \times \mathbb{R})$ , satisfying (2.6), which reads here,

$$(7.4) \quad \begin{cases} C_H^{-1}|p| - C_H \leq H_E(x, p) \leq C_H(1 + |p|), \\ |H_E(x, p) - H_E(y, q)| \leq C_H \left( (1 + |p| \wedge |q|) |x - y| + |p - q| \right), \end{cases}$$

for all  $x, y \in \bar{E}$ ,  $E \in \mathbf{E}$ ,  $p, q \in \mathbb{R}$ .

For  $u \in SC(\Gamma)$ ,  $E \in \mathbf{E}$  and  $x \in \bar{E}$ , we set

$$(7.5) \quad \mathcal{I}_E u(x) = \sum_{F \in \mathbf{E}} \int_{J_F} [u_F(z) - u(x)] \nu_{EF}(x, \rho(x, \gamma_F(z))) dz.$$

The kernels  $\nu_{EF}$  satisfy condition (2.7), that is there exist  $\Lambda > 0$  and  $\sigma \in (0, 1)$  such that

$$(7.6) \quad \begin{cases} 0 \leq \nu_{EF}(x, t) \leq \Lambda t^{-(1+\sigma)}, \\ |\nu_{EF}(x, t) - \nu_{EF}(y, t)| \leq \Lambda |x - y| t^{-(1+\sigma)}, \end{cases}$$

for each  $E, F \in \mathbf{E}$ ,  $x, y \in \bar{E}$ , and  $t > 0$ .

We then fix some  $B_v \in \mathbb{R}$  for each  $v \in \mathbf{V}_i$  and  $h_v \in \mathbb{R}$  for each  $v \in \mathbf{V}_b$  to state the Kirchhoff and Dirichlet boundary conditions in (1.1).

Before defining the solutions of (1.1), we extend the notations of the beginning to Section 3 to the network case.

Given  $\varphi \in SC(\Gamma)$ ,  $E \in \mathbf{E}$ ,  $A \subseteq \Gamma$  measurable, and  $x \in \bar{E}$ , we write

$$\mathcal{I}_E[A] \varphi(x) := \sum_{F \in \mathbf{E}} \int_{\{z \in F \cap A\}} [\varphi(z) - \varphi(x)] \nu_{EF}(x, \rho(x, z)) dz,$$

which is the natural extension of the nonlocal operator defined in (2.9)-(2.10)-(2.11).

Then, for viscosity evaluation, given  $u \in SC(\Gamma)$ ,  $\varphi \in C^1(\Gamma)$ ,  $\delta > 0$ ,  $E \in \mathbf{E}$ ,  $x \in \bar{E}$  and  $p \in \mathbb{R}$ , we write

$$G_E^\delta(u, \varphi, p, x) := \lambda u(x) - \mathcal{I}_E[B_\delta^c(x)]u(x) - \mathcal{I}_E[B_\delta(x)]\varphi(x) + H_E(x, p).$$

**Definition 7.1.** We say that  $u \in USC(\Gamma)$  is a viscosity subsolution to Problem (1.1) if, for each  $x \in \Gamma$ , each  $\delta > 0$  and each  $\varphi \in C^1(\Gamma)$  such that  $x$  is a maximum point of  $u - \varphi$  in  $B_\delta(x)$ , we have

$$\begin{aligned} & G_E^\delta(u, \varphi, \varphi_x(x), x) \leq 0 \quad \text{if } x \in E, \\ \min \left\{ \min_{E \in \text{Inc}(x)} G_E^\delta(u, \varphi, \varphi_{x_E}(x), x), \sum_{E \in \text{Inc}(x)} -\partial_E \varphi(x) - B_x \right\} & \leq 0 \quad \text{if } x \in \mathbf{V}_i, \\ \min \left\{ \min_{E \in \text{Inc}(x)} G_E^\delta(u, \varphi, \varphi_{x_E}(x), x), u(x) - h_x \right\} & \leq 0 \quad \text{if } x \in \mathbf{V}_b. \end{aligned}$$

We say that  $u \in LSC(\Gamma)$  is a viscosity supersolution to Problem (1.1) if, for each  $x \in \Gamma$ , each  $\delta > 0$  and each  $\varphi \in C^1(\Gamma)$  such that  $x$  is a minimum point of  $u - \varphi$  in  $B_\delta(x)$ , we have

$$\begin{aligned} & G_E^\delta(u, \varphi, \varphi_x(x), x) \geq 0 \quad \text{if } x \in E, \\ \max \left\{ \max_{E \in \text{Inc}(x)} G_E^\delta(u, \varphi, \varphi_{x_E}(x), x), \sum_{E \in \text{Inc}(x)} -\partial_E \varphi(x) - B_x \right\} & \geq 0 \quad \text{if } x \in \mathbf{V}_i, \\ \max \left\{ \max_{E \in \text{Inc}(x)} G_E^\delta(u, \varphi, \varphi_{x_E}(x), x), u(x) - h_x \right\} & \geq 0 \quad \text{if } x \in \mathbf{V}_b. \end{aligned}$$

A viscosity solution is a continuous function on  $\Gamma$  which is simultaneously a viscosity sub and supersolution in the sense above.

**Remark 7.2.** The definition depends on the parametrization we choose at the beginning, which is close to what is done in Siconolfi [36]. Notice that, even in the case of a junction, the parametrization is not canonical, see for instance [27] and [32].

Finally, since we also use a vanishing viscosity approach for the existence in networks, for functions  $u \in C^2(\Gamma)$ , we define  $u_{xx}(x) := u''_E(x_E)$  for  $x \in E$ ,  $E \in \mathbf{E}$ . For the second-derivative at a vertex  $\bar{\mathbf{v}} \in \mathbf{V}$ , we denote

$$u_{x_E x_E}(\bar{\mathbf{v}}) = \lim_{x \rightarrow \bar{\mathbf{v}}, x \in E} u_{xx}(x).$$

**7.2. Well-posedness on a general network.** We have the following result, analogous to Proposition 4.3.

**Proposition 7.3.** Let  $\lambda > 0$ ,  $\{H_E\}_{E \in \mathbf{E}}$  satisfies (7.4),  $\{\mathcal{I}_E\}_{E \in \mathbf{E}}$  satisfies (7.6), and let  $\{h_{\bar{\mathbf{v}}}\}_{\bar{\mathbf{v}} \in \mathbf{V}_b}$ ,  $\{\theta_{\bar{\mathbf{v}}}\}_{\bar{\mathbf{v}} \in \mathbf{V}_i} \subset \mathbb{R}$  be given. Then, for each  $\epsilon > 0$ , there exists a unique viscosity solution  $u^\epsilon \in C(\Gamma)$  to the problem

$$(7.7) \quad \lambda u - \epsilon u_{xx} - \mathcal{I}_E u(x) + H_E(x, u_x) = 0 \quad \text{on } E, \quad E \in \mathbf{E},$$

complemented by Dirichlet conditions

$$(7.8) \quad u(\bar{\mathbf{v}}) = h_{\bar{\mathbf{v}}} \text{ for } \bar{\mathbf{v}} \in \mathbf{V}_b, \quad u(\bar{\mathbf{v}}) = \theta_{\bar{\mathbf{v}}} \text{ for } \bar{\mathbf{v}} \in \mathbf{V}_i.$$

The solution enjoys analogous estimates as the ones presented in Proposition 4.3.

We stress on the fact that the construction of our solution on a junction in Lemma 4.2 is based on the well-posedness of the PDE in an interval with Dirichlet condition at the two end points (see Appendix B). It follows that the proof of Proposition 7.3 follow same lines as the proof of Lemma 4.2 and Proposition 4.3 with minor changes.

The main result of this section is the following

**Theorem 7.4.** *Assume  $\lambda > 0$ ,  $\{H_E\}_{E \in \mathbf{E}}$  satisfies (7.4),  $\{\mathcal{I}_E\}_{E \in \mathbf{E}}$  satisfies (7.6),  $\{h_{\bar{\mathbf{v}}}\}_{\bar{\mathbf{v}} \in \mathbf{V}_b}$ ,  $\{B_{\bar{\mathbf{v}}}\}_{\bar{\mathbf{v}} \in \mathbf{V}_i} \subset \mathbb{R}$  are given.*

*Then, there exists a unique viscosity solution  $u \in C(\Gamma)$  for the general Kirchhoff problem (1.1) on  $\Gamma$ . This solution is Lipschitz continuous in  $\Gamma \setminus$*

$$\bigcup_{\bar{\mathbf{v}} \in \mathbf{V}_b} B(\bar{\mathbf{v}}, \delta) \text{ for all } \delta > 0.$$

*Proof.* We follow here the ideas of Theorem 4.4, adapted to the general network setting. The main difference here is to prove the existence of a family  $\Theta := \{\theta_{\bar{\mathbf{v}}}\}_{\bar{\mathbf{v}} \in \mathbf{V}_i}$  for which the Kirchhoff condition

$$(7.9) \quad \sum_{E \in \text{Inc}(\bar{\mathbf{v}})} -\partial_E u^{\epsilon, \Theta}(\bar{\mathbf{v}}) = B_{\bar{\mathbf{v}}}$$

holds pointwisely for the solution  $u^{\epsilon, \Theta}$  of (7.7)-(7.8).

We set  $\bar{B} = \max_{\bar{\mathbf{v}} \in \mathbf{V}_i} |B_{\bar{\mathbf{v}}}| + 1$  and define  $\psi_0 \in C^2[0, 1]$  such that  $\psi_0(x) = \psi_0(1-x)$  for all  $x \in [0, 1]$ , non-increasing in  $[0, 1/2]$ , with  $\psi_0(x) = -\bar{B}x$  if  $x \in [0, 1/8]$  and  $\psi_0(x) = -3\bar{B}/16$  if  $x \in [1/4, 1/2]$ . Notice this function is such that  $\|\psi_0\|_{C^2[0,1]} \leq C_0 \bar{B}$  for some universal constant  $C_0 > 0$ .

Then, we define the function  $\Psi \in C^2(\Gamma)$  by

$$\Psi_E(x_E) = a_E \psi_0(a_E^{-1} x_E), \quad \text{for } x_E \in \bar{J}_E, E \in \mathbf{E}.$$

Notice that  $\|\Psi\|_{C^2(\Gamma)} \leq \underline{a}^{-1} C_0 \bar{B}$ , where  $\underline{a} = \min_{E \in \mathbf{E}} |a_E|$ , and, for each  $\bar{\mathbf{v}} \in \mathbf{V}$ , we have  $\Psi(\bar{\mathbf{v}}) = 0$ .

With similar computations as in the proof of Theorem 4.4, we obtain that, for each  $E \in \mathbf{E}$  and  $x \in E$ ,

$$\begin{aligned} |H_E(x, \Psi_x(x))| &\leq C_H(1 + C_0 \bar{B}), \\ |\mathcal{I}_E \Psi(x)| &\leq \frac{2\Lambda \text{card}(\mathbf{E})^2 C_0 \bar{B}}{1 - \sigma}, \\ |\epsilon \Psi_{xx}(x)| &\leq \underline{a}^{-1} C_0 \bar{B}, \quad \epsilon \in (0, 1), \end{aligned}$$

and, for  $\theta^+$  large enough in terms of the above estimates,  $\bar{h} = \max_{\bar{\mathbf{v}} \in \mathbf{V}_i} |h_{\bar{\mathbf{v}}}|$  and  $\lambda$ , the function

$$\Psi^+(x) = \theta^+ + \Psi(x), \quad x \in \Gamma$$

is a viscosity supersolution of (7.7)-(7.8), for all  $\theta_{\bar{\mathbf{v}}} \in [-\theta^+, \theta^+]$ .

Now, we enumerate the set of interior vertices, writing  $\mathbf{V}_i = \{\bar{\mathbf{v}}_1, \bar{\mathbf{v}}_2, \dots, \bar{\mathbf{v}}_{N_i}\}$  (assuming for instance that the interior vertices are labelled first in the starting parametrization) and  $\Theta = (\theta_1, \dots, \theta_{N_i})$ , where  $N_i = \text{card}(\mathbf{V}_i) \geq 1$ . For  $\Theta \in [-\theta^+, \theta^+]^{N_i}$ , let  $u^{\epsilon, \Theta} \in C^{2,1-\sigma}(\Gamma)$  be the unique solution of (7.7)-(7.8) given by Proposition 7.3, and define the functions

$$\Theta \in [-\theta^+, \theta^+]^{N_i} \mapsto F_j(\Theta) = \sum_{E \in \text{Inc}(\bar{\mathbf{v}}_j)} -\partial_E u^{\epsilon, \Theta}(\bar{\mathbf{v}}_j) - B_{v_j}, \quad 1 \leq j \leq N_i.$$

For  $\Theta \in [-\theta^+, \theta^+]^{N_i}$ , we denote  $\Theta^{j,+}$  the vector  $\Theta$  where the coordinate  $\theta_j$  is replaced by  $\theta^+$ . By comparison (Lemma B.1 and Remark B.2), we have  $u^{\epsilon, \Theta^{j,+}} \leq \Psi^+$  on  $\Gamma$ . Since  $\Psi^+(\bar{\mathbf{v}}_j) = \theta^+ = u^{\epsilon, \Theta^{j,+}}(\bar{\mathbf{v}}_j)$ , it follows

$$\partial_E u^{\epsilon, \Theta^{j,+}}(\bar{\mathbf{v}}_j) \leq \partial_E \Psi^+(\bar{\mathbf{v}}_j) = \psi'_0(0) = -\bar{B}, \quad \text{for all } E \in \text{Inc}(\bar{\mathbf{v}}_j),$$

leading to

$$F_j(\Theta^{j,+}) = \sum_{E \in \text{Inc}(\bar{\mathbf{v}}_j)} -\partial_E u^{\epsilon, \Theta^{j,+}}(\bar{\mathbf{v}}_j) - B_{\bar{\mathbf{v}}_j} \geq \text{card}(\text{Inc}(\bar{\mathbf{v}}_j))|\bar{B}| - B_{\bar{\mathbf{v}}_j} > 0,$$

for all  $j$  and all  $\Theta$ .

Similarly,  $\Psi^-(x) = \theta^- - \Psi(x)$  with  $\theta^- = -\theta^+$  is a viscosity subsolution for the same problem, from which we conclude that  $F_j(\Theta^{j,-}) < 0$  for all  $j$  and all  $\Theta$ .

We are in position to apply Poincaré-Miranda theorem (see [30]) to the family  $F_j$ ,  $1 \leq j \leq N_i$ , in  $[-\theta^+, \theta^+]^{N_i}$ . It yields the existence of  $\Theta^* \in [-\theta^+, \theta^+]^{N_i}$  such that  $F_j(\Theta^*) = 0$  for all  $j$ .

Then, the Kirchhoff condition (7.9) is satisfied for the solution  $u^{\epsilon, \Theta^*}$  of (7.7)-(7.8), from which we conclude that  $u^{\epsilon, \Theta^*}$  is a solution to (1.1) with the vanishing viscosity term. By similar arguments as in the proof of Theorem 4.4, we can send  $\epsilon \rightarrow 0$  in the vanishing viscosity problem and obtain a solution  $u \in C(\Gamma)$  to (1.1), which is locally Lipschitz continuous in  $\Gamma \setminus \mathbf{V}_b$ .

The proof of uniqueness follows readily the one of Theorem 5.1 since, arguing by contradiction, we are brought back to the case when a positive maximum is achieved at a vertex  $\bar{\mathbf{v}} \in \mathbf{V}_i$  and we can argue locally as in the junction case.  $\square$

Collecting the results in the paper, we obtain the following well-posedness for viscous nonlocal Hamilton-Jacobi with Kirchhoff conditions on general networks. This result has its own interest.

**Theorem 7.5.** *Assume hypotheses of Theorem 7.4 hold, and consider families  $\{f_E\}_{E \in \mathbf{E}}, \{\mu_E\}_{E \in \mathbf{E}}$  such that*

$$\begin{cases} 0 \leq \underline{\mu} \leq \mu_E \in C(\bar{E}), & |\sqrt{\mu_E(x)} - \sqrt{\mu_E(y)}| \leq C_\mu |x - y|, \quad x, y \in \bar{E}, \\ f_E \in C(\bar{E}). \end{cases}$$



Then, there exists a viscosity solution  $u \in C(\Gamma)$  to the problem

$$(7.10) \quad \begin{cases} \lambda u - \mu(x)u_{xx} - \mathcal{I}u(x) + H(x, u_x) = f(x), & x \in \Gamma \setminus \mathbf{V}, \\ \sum_{E \in \text{Inc}(\bar{\mathbf{v}})} -\partial_E u(\bar{\mathbf{v}}) = B_{\bar{\mathbf{v}}}, & \bar{\mathbf{v}} \in \mathbf{V}_i, \\ u(\bar{\mathbf{v}}) = h_{\bar{\mathbf{v}}}, & \bar{\mathbf{v}} \in \mathbf{V}_b, \end{cases}$$

which is Lipschitz continuous in a neighborhood of each vertex  $v \in \mathbf{V}_i$ .

Moreover, if the equation is strictly elliptic (i.e.,  $\underline{\mu} > 0$  in (B.2)), then there exists a unique classical solution  $u \in C^{2,1-\sigma}(\Gamma)$ , which satisfies the conditions at the vertices pointwisely.

*Proof.* The first part of the Theorem is a straightforward adaptation of Proposition 4.3 and Theorem 4.4 to the case of general networks. The only difference is that we argue on (4.1) by replacing the diffusion  $-\varepsilon u_{x_E x_E}$  by  $-(\varepsilon + \mu_E)u_{x_E x_E}$  in the edges  $E$ ,  $E \in \mathbf{E}$ .

When the equation is strictly elliptic, the existence of a  $C^{2,1-\sigma}(\Gamma)$  solution is directly given by Proposition 4.3 and Steps 1-2 of the proof of Theorem 4.4, arguing directly on the PDE in (7.10) without adding  $\varepsilon$  to the diffusion.

The proof of the uniqueness of the classical solution is an easy adaptation of [2, Lemma 3.6] or [20]. We provide it to be self-contained.

Consider two solutions  $u, v \in C^{2,1-\sigma}(\Gamma)$  of (7.10). Assume by contradiction that  $\delta := \max_{\Gamma} u - v > 0$ .

If the maximum is achieved inside a branch at  $x_0 \in \mathbf{E} \setminus \mathbf{V}$ , then we perform a classical proof of the maximum principle. More precisely, the PDE in (7.10) holds pointwisely both for  $u$  and  $v$  at  $x_0$ . Using that  $(u - v)(x_0) = \delta > 0$ ,  $u_x(x_0) = v_x(x_0)$ ,  $u_{xx}(x_0) \leq v_{xx}(x_0)$  and  $\mathcal{I}u(x_0) \leq \mathcal{I}v(x_0)$ , by subtracting the inequality, we reach a contradiction.

It follows that the maximum is necessarily achieved at a vertex. Since the Dirichlet condition holds pointwisely,  $x_0$  cannot be a boundary vertex, thus  $x_0 \in \mathbf{V}_i$ .

By maximality, we have  $(u - v)(x) \leq (u - v)(x_0)$  for all  $x \in \Gamma$ . Hence,  $\partial_E u(x_0) \leq \partial_E v(x_0)$  for all  $E \in \text{Inc}(x_0)$ . From the Kirchhoff conditions at  $x_0$ , we get  $\sum_{E \in \text{Inc}(x_0)} (\partial_E u(x_0) - \partial_E v(x_0)) = 0$ . We finally obtain

$$(7.11) \quad \partial_E u(x_0) = \partial_E v(x_0) \quad \text{for all } E \in \text{Inc}(x_0).$$

Let  $E \in \text{Inc}(x_0)$  and  $x \in E$ . Writing the PDEs at  $x$  for  $u$  and  $v$ , we infer

$$\begin{aligned} & \mu(x)(u_{xx}(x) - v_{xx}(x)) \\ &= \lambda(u - v)(x) - \mathcal{I}_E u(x) + \mathcal{I}_E v(x) + H_E(x, u_x(x)) - H_E(x, v_x(x)). \end{aligned}$$

Sending  $x \rightarrow x_0$ , we have  $\lambda(u - v)(x) \rightarrow \lambda\delta > 0$ ,  $-\mathcal{I}_E u(x) + \mathcal{I}_E v(x) \rightarrow -\mathcal{I}_E u(x_0) + \mathcal{I}_E v(x_0) \geq 0$  by Lemma 2.2 and maximality of  $x_0$ , and

$$H_E(x, u_x(x)) - H_E(x, v_x(x)) \rightarrow 0$$

since

$$\begin{aligned} \lim_{x \rightarrow x_0, x \in E} u_x(x) = u_{x_E}(x_0) &= i_E(x_0) \partial_E u(x_0) \\ &= i_E(x_0) \partial_E v(x_0) = v_{x_E}(x_0) = \lim_{x \rightarrow x_0, x \in E} v_x(x) \end{aligned}$$

by (7.3) and (7.11). We finally obtain that  $\mu(x_0)(u_{x_E x_E}(x_0) - v_{x_E x_E}(x_0)) > 0$  with  $u_{x_E}(x_0) = v_{x_E}(x_0)$ . Therefore, the function  $u - v$  is strictly convex in a neighborhood of  $x_0$  in the edge  $E$ , with a minimum at  $x_0$ , which contradicts the fact that  $x_0$  is a maximum of  $u - v$ . This ends the proof.  $\square$

## 8. ON FLUX LIMITED SOLUTIONS.

We want to make the link between the solutions of the Kirchhoff problem (2.12)-(2.13) and flux limited solutions introduced in [27, 32, 11]. For the sake of simplicity of notations, we come back to the simpler case of a junction, see Section 2 for the setting and Section 3 for the definition of viscosity solutions.

We start by giving a precise definition of flux limited solutions in our context.

At first, we need to add an extra assumption on  $H$ . In [27], they assume quasiconvexity, namely that all  $H_i(x, \cdot)$  are nonincreasing on some interval  $(-\infty, p_i^0(x)]$  and then nondecreasing on  $[p_i^0(x), +\infty)$ . Here, to simplify the presentation, we follow the assumptions of [32]. Throughout this section, we assume

- (i)  $H$  satisfies (2.6)
- (8.1) (ii) For all  $1 \leq i \leq N$ ,  $x \in \bar{E}_i$ ,  $p \in \mathbb{R} \mapsto H_i(x, p)$  is convex with a unique minimum  $p_i^0(x)$ .

Under this assumption, we can define the nondecreasing and nonincreasing parts of  $H_i$ , respectively, by

$$(8.2) \quad H_i^-(x, p) = \begin{cases} H_i(x, p_i^0(x)) & \text{if } p \leq p_i^0(x), \\ H_i(x, p) & \text{if } p \geq p_i^0(x), \end{cases}$$

$$(8.3) \quad H_i^+(x, p) = \begin{cases} H_i(x, p) & \text{if } p \leq p_i^0(x), \\ H_i(x, p_i^0(x)) & \text{if } p \geq p_i^0(x). \end{cases}$$

Notice that

$$(8.4) \quad H_i(x, p) = \max\{H_i^-(x, p), H_i^+(x, p)\}.$$

Following the notations (3.1) and (3.3) and recalling the definition of  $\mathcal{F}_x$  introduced in (3.2), for every  $u \in SC(\Gamma)$ ,  $\varphi \in C^1(\Gamma)$ ,  $\delta \in (0, 1)$ ,  $x \in \bar{E}_i$  and  $p \in \mathbb{R}$ , we denote

$$(8.5) \quad G_i^{\pm, \delta}(u, \varphi, p, x) := \lambda u(x) - \mathcal{I}_i[B_\delta^c(x)]u(x) - \mathcal{I}_i[B_\delta(x)]\varphi(x) + H_i^\pm(x, p),$$

and, for  $u \in \mathcal{F}_x$ ,

$$(8.6) \quad G_i^\pm(u, p, x) := \lim_{\delta \downarrow 0} G_i^{\pm, \delta}(u, \varphi, p, x) = \lambda u(x) - \mathcal{I}_i u(x) + H_i^\pm(x, p).$$

As for (3.3), this latter quantity is well-defined in  $[-\infty, +\infty]$  and does not depend on the function  $\varphi$  chosen for computing the limit. When we can touch an *USC* function from above or a *LSC* function from below, we have more precise estimates, see Lemma 3.2. When  $u$  is Hölder continuous,  $\mathcal{I}_i u(x)$  is well-defined in  $\mathbb{R}$  (see Lemma 3.3) and so  $G_i^\pm(u, p, x)$ .

Next we introduce the so-called flux limiter. Here we follow the approach presented in [11, Section 16.4] but, due to the nonlocal nature of our problem, the definition is more involved.

**Definition 8.1** (Flux limiter). *Let  $B \in \mathbb{R}$  be given. For every  $u \in SC(\Gamma)$ ,  $\varphi \in C^1(\Gamma)$  and  $\delta \in (0, 1)$ ,*

$$FL^{-, \delta}(u, \varphi, O) := \min_{p_1, \dots, p_N \in \mathbb{R}} \max \left\{ \max_{1 \leq i \leq N} G_i^{-, \delta}(u, \varphi, p_i, O), \sum_{i=1}^N -p_i - B \right\}.$$

For  $u \in \mathcal{F}_O$ , the flux limiter operator is defined by

$$FL^-(u, O) = \min_{p_1, \dots, p_N \in \mathbb{R}} \max \left\{ \max_{1 \leq i \leq N} G_i^-(u, p_i, O), \sum_{i=1}^N -p_i - B \right\} \in [-\infty, +\infty].$$

These definitions make sense. Indeed,  $\sum_{j=1}^N -p_j - B \rightarrow -\infty$  whereas, when they are finite,  $G_i^{-, \delta}(u, \varphi, p_i, O), G_i^-(u, p_i, O) \rightarrow +\infty$  as  $p_i \rightarrow +\infty$ , thus the minima are well-defined. See the related [11, Lemma 5.3.1] for details.

We are now in position to define the viscosity flux limited (FL) solutions.

**Definition 8.2.** *We say that  $u \in USC(\Gamma)$  is a viscosity FL subsolution to problem (2.12)-(2.14) if, for each function  $\varphi \in C^1(\Gamma)$  such that  $u - \varphi$  has a local maximum at  $x \in \Gamma$ , for each  $\delta > 0$ , we have*

$$\begin{aligned} G_i^\delta(u, \varphi, \varphi_{x_i}(x), x) &\leq 0 \quad \text{if } x \in E_i, \\ \max \left\{ \max_{1 \leq i \leq N} G_i^{+, \delta}(u, \varphi, \varphi_{x_i}(O), O), FL^{-, \delta}(u, \varphi, O) \right\} &\leq 0 \quad \text{if } x = O, \\ \min \left\{ G_i^\delta(u, \varphi, \varphi_{x_i}(x), x), u(x) - h_i \right\} &\leq 0 \quad \text{if } x = \bar{\mathbf{v}}_i. \end{aligned}$$

*We say that  $u \in LSC(\Gamma)$  is a viscosity FL supersolution to problem (2.12)-(2.14) if, for each function  $\varphi \in C^1(\Gamma)$  such that  $u - \varphi$  has a local minimum at  $x \in \Gamma$ , for each  $\delta > 0$ , we have*

$$\begin{aligned} G_i^\delta(u, \varphi, \varphi_{x_i}(x), x) &\geq 0 \quad \text{if } x \in E_i, \\ \max \left\{ \max_{1 \leq i \leq N} G_i^{+, \delta}(u, \varphi, \varphi_{x_i}(O), O), FL^{-, \delta}(u, \varphi, O) \right\} &\geq 0 \quad \text{if } x = O, \\ \max \left\{ G_i^\delta(u, \varphi, \varphi_{x_i}(x), x), u(x) - h_i \right\} &\geq 0 \quad \text{if } x = \bar{\mathbf{v}}_i. \end{aligned}$$

A viscosity solution is a continuous function on  $\Gamma$  which is simultaneously a viscosity sub and supersolution in the sense above.

We give an equivalent definition for the FL solutions. We concentrate in the case of a junction point, the other cases being similar.

**Proposition 8.3.** *An  $u \in USC(\Gamma)$  (resp.  $LSC(\Gamma)$ ) is a viscosity FL subsolution (resp. supersolution) to problem (2.12)-(2.14) if and only if the viscosity inequality at  $x = O$  is replaced with*

$$(8.7) \quad \max \left\{ \max_{1 \leq i \leq N} G_i^+(u, p_i, O), FL^-(u, O) \right\} \leq 0 \text{ (resp. } \geq 0),$$

for every  $\mathbf{p} = (p_1, \dots, p_N) \in D_\Gamma^+ u(O)$  (resp.  $p \in D_\Gamma^- u(O)$ ).

*Proof.* We focus on the proof for subsolutions. Let  $\mathbf{p} \in D_\Gamma^+ u(O)$ . There exists  $\varphi \in C^1(\Gamma)$  such that  $O$  is a local maximum point of  $u - \varphi$ , and  $\varphi_{x_i}(O) = p_i$  for each  $i$ .

Since  $u$  is a subsolution in the sense of Definition 8.2, we have for all  $\delta > 0$  small enough that

$$G_i^{+, \delta}(u, \varphi, p_i, O) \leq 0, \quad \text{for all } i,$$

and

$$(8.8) \quad FL^{-, \delta}(u, \varphi, O) \leq 0.$$

From the first set of inequalities and Lemma 3.2, we infer  $\mathcal{I}_i u(O) = \lim_{\delta \rightarrow 0} \mathcal{I}_i[B_\delta(O)^c]u(O) \in \mathbb{R}$  for all  $i$ , and taking limit as  $\delta \rightarrow 0$  we arrive at

$$G_i^+(u, p_i, O) \leq 0, \quad \text{for all } i.$$

On the other hand, from (8.8) and the definition of  $FL^{-, \delta}(u, \varphi, O)$ , we see that for all  $\delta > 0$ , there exists  $\mathbf{q}^\delta \in \mathbb{R}^N$  such that

$$\max \left\{ \max_{1 \leq i \leq N} \{G_i^{-, \delta}(u, \varphi, q_i^\delta, O), \sum_{i=1}^N -q_i^\delta - B\} \right\} \leq 0.$$

Using that  $H_i^-(O, p) \rightarrow +\infty$  as  $p \rightarrow +\infty$ , and since we already know that  $\mathcal{I}_i u(O)$  is finite, we infer that there exists  $C > 0$  not depending on  $\delta$  such that  $q_i^\delta \leq C$  for all  $i$  and all  $\delta$  small. Using this and the fact that  $\sum_{i=1}^N -q_i^\delta \leq B$ , we conclude that, enlarging  $C$  if necessary, we have  $|q_i^\delta| \leq C$  for all  $i$  and  $\delta$  small. Thus, for each  $i$ , there exists  $\tilde{q}_i \in \mathbb{R}$  such that, taking subsequences, we have  $q_i^\delta \rightarrow \tilde{q}_i$  as  $\delta \rightarrow 0$ . Taking limit as  $\delta \rightarrow 0$ , we arrive at  $FL^-(u, O) \leq 0$ , which concludes this case.

Now we deal with the converse. Assume  $u$  is a subsolution in the sense of (8.7). Let  $\varphi \in C^1(\Gamma)$  be a test function such that  $u - \varphi$  has a local maximum at  $O$ . Then,  $\mathbf{p} \in \mathbb{R}^N$  such that  $p_i = \varphi_{x_i}(O)$ ,  $1 \leq i \leq N$ , belongs to  $D_\Gamma^+ u(O)$ . By (8.7), we have

$$(8.9) \quad G_i^+(u, p_i, O) \leq 0 \text{ for } 1 \leq i \leq N \quad \text{and} \quad FL^-(u, O) \leq 0.$$

Since  $u$  is touched from above, we have  $\mathcal{I}_i u(O) \in [-\infty, +\infty)$  by Lemma 3.2, and thus, the above inequalities imply that  $\mathcal{I}_i u(O) \in \mathbb{R}$ .

Using that  $u - \varphi$  has a local maximum at  $O$ , we readily have

$$\begin{aligned} \mathcal{I}_i u(O) &= \mathcal{I}_i[B_\delta^c(O)]u(O) + \mathcal{I}_i[B_\delta(O)]u(O) \\ &\leq \mathcal{I}_i[B_\delta^c(O)]u(O) + \mathcal{I}_i[B_\delta(O)]\varphi(O), \end{aligned}$$

and using these inequalities into (8.9) we conclude the viscosity inequality for subsolutions in the sense of Definition 8.2.

For supersolution, the proof is similar to the one above complemented with Lemma 3.2 for functions touched from below by  $C^1$  functions. The arguments are slightly simpler since we require that only one expression inside the maximum is nonnegative to get the viscosity inequality.  $\square$

### 8.1. Kirchhoff solutions are flux limiter solutions.

**Theorem 8.4.** *For any  $B \in \mathbb{R}$ , a viscosity subsolution (resp. supersolution) to the Kirchhoff problem (2.12)-(2.14)-(2.13) is a viscosity FL subsolution (resp. supersolution) for the flux limiter introduced in Definition 8.1.*

*Proof.* The only difference in Definitions 3.1 and 8.2 is at the junction point  $O$ .

1.- *Subsolution case.* In this case we can assume that  $u$  is Lipschitz continuous at the junction, see Lemma 5.2.

Let  $\mathbf{p} = (p_1, \dots, p_N) \in D_\Gamma^+ u(O)$ . We want to prove that

$$\max \left\{ \max_i G_i^+(u, p_i, O), FL^-(u, O) \right\} \leq 0.$$

We start by proving that for all  $i$ , we have  $G_i^+(u, p_i, O) \leq 0$ . Since  $\mathbf{p} \in D_\Gamma^+ u(O)$ , there exists  $\varphi \in C^1(\Gamma)$  such that  $O$  is a strict maximum point of  $u - \varphi$  and  $\varphi_{x_i}(O) = p_i$ . Using that  $O$  is a strict maximum point of  $u - \varphi$ , for all  $\epsilon \in (0, 1)$  small enough, we have that for all  $i$ , the function

$$x_i \mapsto u_i(x_i) - (\varphi_i(x_i) + \epsilon x_i^{-\beta}), \quad x_i \in J_i,$$

attains its maximum at some point  $x_\epsilon \in J_i$ , and this point is such that  $x_\epsilon \rightarrow 0^+$ , and  $u_i(x_\epsilon) - \varphi_i(x_\epsilon) \rightarrow u(O) - \varphi(O)$  as  $\epsilon \rightarrow 0$ . Hence, we can use the viscosity inequality for subsolutions on  $J_i$ , and since  $u$  is Lipschitz, from Lemma 3.3, we have

$$G_i(u, p_i - \beta \epsilon x_\epsilon^{-\beta-1}, \gamma_i(x_\epsilon)) \leq 0.$$

From (8.4), it follows

$$G_i^+(u, p_i - \beta \epsilon x_\epsilon^{-\beta-1}, \gamma_i(x_\epsilon)) \leq 0,$$

and since  $p \mapsto G_i^+(u, p, x)$  is nonincreasing, we arrive at

$$G_i^+(u, p_i, \gamma_i(x_\epsilon)) \leq 0.$$

Taking the limit  $\epsilon \rightarrow 0$ , by continuity of the map  $x \mapsto G_i^+(u, p, x)$  (see Lemma 2.2), we conclude the asserted inequality.

Thus, it remains to prove that  $FL^-(u, O) \leq 0$ . For this, a simple generalization of [11, Lemma 5.3.1] implies the existence of a  $(\tilde{p}_1, \dots, \tilde{p}_N) \in \mathbb{R}^N$  such that, for all  $i$  we have

$$(8.10) \quad FL^-(u, O) = G_i^-(u, \tilde{p}_i, O) = \sum_{i=1}^N -\tilde{p}_i - B.$$

Now, for each  $\epsilon \in (0, 1)$ , consider the function  $\psi \in C^1(\Gamma)$  given by  $\psi_i(x_i) = \tilde{p}_i x_i + \epsilon^{-1} x_i^2$ ,  $1 \leq i \leq N$ . By boundedness of  $u$ , for all  $\epsilon$  small enough we have the existence of a maximum point  $x_\epsilon \in \Gamma$  to  $u - \psi$  such that  $x_\epsilon \rightarrow O$  as  $\epsilon \rightarrow 0$ .

If there is a subsequence  $\epsilon \rightarrow 0$  such that  $x_\epsilon \in E_i$  for some  $i$ , then by the viscosity inequality inside the edge we have

$$G_i(u, \tilde{p}_i + 2\epsilon^{-1}x_\epsilon, x_\epsilon) \leq 0,$$

which implies, thanks to (8.4),

$$G_i^-(u, \tilde{p}_i + 2\epsilon^{-1}x_\epsilon, x_\epsilon) \leq 0,$$

and, since  $p \mapsto G_i^-(u, p, x)$  is nondecreasing, we obtain

$$G_i^-(u, \tilde{p}_i, x_\epsilon) \leq 0.$$

Sending  $\epsilon \rightarrow 0$  and using (8.10), we arrive at

$$FL^-(u, O) = G_i^-(u, \tilde{p}_i, O) \leq 0.$$

If  $x_\epsilon = O$  for all  $\epsilon$ , writing the subsolution inequality at  $x_\epsilon = O$  in Lemma 3.3, we obtain that, either  $G_i(u, \tilde{p}_i, O) \leq 0$  for some  $i$ , or  $\sum_{j=1}^N -\tilde{p}_j - B \leq 0$ . In both cases, using (8.4) and (8.10), we obtain  $FL^-(u, O) \leq 0$ , which concludes the proof of the subsolution case.

2.- *Supersolution case.* Here we cannot assume  $u$  is Lipschitz continuous.

Thanks to Proposition 8.3, it is enough to prove that, for all  $\mathbf{p} = (p_1, \dots, p_N) \in D_\Gamma^- u(O)$ , we have

$$(8.11) \quad \max\left\{\max_{1 \leq i \leq N} G_i^+(u, p_i, O), FL^-(u, O)\right\} \geq 0,$$

If  $D_\Gamma^- u(O) = \emptyset$  or  $FL^-(u, O) \geq 0$ , then we are done. Therefore, in the rest of the proof, we assume that  $D_\Gamma^- u(O) \neq \emptyset$  (and a fortiori  $D_{j_i}^\pm u_i(O) \neq \emptyset$  for all  $i$ ) and

$$(8.12) \quad FL^-(u, O) < 0.$$

From the fact that the subdifferential is non empty, we can deduce two things.

At first, from Lemma 3.2,  $-\mathcal{I}_i u(O)$ , hence  $G_i(u, p_i, O)$  and  $G_i^\pm(u, p_i, O)$ , is in  $[-\infty, \infty)$  for all  $i$ . We define the subset  $\mathcal{A}$  of ‘‘active indices’’  $i$ , for which they are finite.

Secondly, from [11, Proposition 2.5.4], the set  $D_{\bar{J}_i}^- u_i(0)$  is a nonempty interval, either  $D_{\bar{J}_i}^- u_i(0) = \mathbb{R}$ , or  $\underline{p}_i = \liminf_{x \rightarrow 0^+} \frac{u_i(x) - u_i(0)}{x}$  is finite and  $D_{\bar{J}_i}^- u_i(0) = (-\infty, \underline{p}_i]$ .

Now, we divide the proof in two cases showing that the first one yields (8.11) as desired, whereas the second one leads to a contradiction.

*Case 1.* There exists  $i$  such that, for all  $p \in D_{\bar{J}_i}^- u_i(0)$ ,  $G_i(u, p, O) \geq 0$ . In this case, we have that  $i \in \mathcal{A}$ .

If  $p_i^0(O) \in D_{\bar{J}_i}^- u_i(0)$  (see (8.1)), then  $G_i(u, p_i^0(O), O) = G_i^+(u, p_i^0(O), O) \geq 0$ . Since  $G_i^+(u, p_i^0(O), O) = \min_{p \in \mathbb{R}} G_i^+(u, p, O)$ , we conclude that (8.11) holds true.

If  $p_i^0(O) \notin D_{\bar{J}_i}^- u_i(0)$ , then  $p_i^0(O) > \underline{p}_i \in D_{\bar{J}_i}^- u_i(0)$ , thus  $G_i(u, \underline{p}_i, O) = G_i^+(u, \underline{p}_i, O) \geq 0$ . Using that  $H_i^+(O, \cdot)$  is nonincreasing, we obtain that  $G_i^+(u, p, O) \geq 0$  for all  $p \in D_{\bar{J}_i}^- u_i(0)$ , and (8.11) also holds true. The proof is done in this case.

*Case 2.* For all  $i$ , there exists  $p_i \in D_{\bar{J}_i}^- u_i(O)$  such that

$$(8.13) \quad G_i(u, p_i, O) < 0.$$

We claim that, in this case, all indices  $i$  are active. To prove the claim, let us fix  $\mathbf{p} = (p_1, \dots, p_N) \in D_{\Gamma}^- u(O)$  such that the inequalities (8.13) hold.

At first, if there exists  $i_0$  such that  $G_{i_0}(u, p_{i_0}, O) = -\infty$ , i.e.,  $i_0 \notin \mathcal{A}$ , and  $D_{\bar{J}_{i_0}}^- u_{i_0}(O) = \mathbb{R}$  then we can increase  $p_{i_0}$  in order that we still have  $G_{i_0}(u, p_{i_0}, O) = -\infty$  and moreover the Kirchhoff condition is negative, i.e.,  $\sum_{i=1}^N -p_i - B < 0$ . Putting this together with (8.13), we obtain that  $u$  cannot be a Kirchhoff subsolution at  $O$ , which is a contradiction.

Secondly, if there exists  $i_0$  such that  $G_{i_0}(u, p_{i_0}, O) = -\infty$  and  $D_{\bar{J}_{i_0}}^- u_{i_0}(O) = (-\infty, \underline{p}_{i_0}]$ , then we also reach a contradiction since, by Lemma 5.3 we should have  $G_{i_0}(u, p_{i_0}, O) \geq 0$ .

Therefore, all the indices are active. Taking into account (8.12), from Definition 8.1 of the flux limiter and [11, Lemma 5.3.1], we infer the existence of  $\tilde{\mathbf{p}} = (\tilde{p}_1, \dots, \tilde{p}_N) \in \mathbb{R}^N$  such that

$$(8.14) \quad FL^-(u, O) = G_i^-(u, \tilde{p}_i, O) = \sum_{i=1}^N -\tilde{p}_i - B < 0 \quad \text{for all } i.$$

From (8.4) and (8.13),  $G_i^+(u, p_i, O) < 0$  for all  $i$ . Since  $p_i \leq \underline{p}_i$  and  $p \mapsto H_i^+(O, p)$  is nonincreasing, we get  $G_i^+(u, \underline{p}_i, O) < 0$ . Since  $G_i(u, \underline{p}_i, O) \geq 0$  from Lemma 5.3, necessarily

$$G_i^-(u, \underline{p}_i, O) \geq 0 \quad \text{for all } i.$$

Using that  $G_i^-(u, \tilde{p}_i, O) < 0$  and  $p \mapsto H_i^-(O, p)$  is nondecreasing, we infer that  $\underline{p}_i$  is in the increasing part of  $H_i^-$ , and therefore  $\tilde{p}_i < \underline{p}_i$ . It follows that

we can find  $p_i^* \in (\tilde{p}_i, \underline{p}_i)$ , thus  $p_i^* \in D_{\tilde{J}_i}^- u_i(0)$ , such that  $p_i^*$  is in the increasing part of  $H_i^-$  and

$$(8.15) \quad G_i(u, p_i^*, O) = G_i^-(u, p_i^*, O) < 0 \quad \text{for all } i.$$

At the end, we have constructed  $\mathbf{p}^* = (p_1^*, \dots, p_N^*) \in D_{\Gamma}^- u(O)$  such that  $G_i^-(u, p_i^*, O) < 0$  for all  $i$ . In addition, since  $p_i^* > \tilde{p}_i$ , (8.14) implies

$$(8.16) \quad \sum_{i=1}^N -p_i^* - B < \sum_{i=1}^N -\tilde{p}_i - B < 0.$$

Finally, (8.15) and (8.16) contradict the fact that  $u$  is a Kirchhoff supersolution at  $O$ . This concludes the proof.  $\square$

## 8.2. Flux limiter solutions are Kirchhoff solutions.

**Theorem 8.5.** *For any  $B \in \mathbb{R}$ , a viscosity FL subsolution (resp. supersolution) is a viscosity subsolution (resp. supersolution) to the Kirchhoff problem (2.12)-(2.14)-(2.13).*

*Proof.* We again concentrate on the junction point  $O$ .

1.- *Subsolution case.* Let  $\mathbf{p} = (p_1, \dots, p_N) \in D_{\Gamma}^+ u(O)$ . We want to prove that

$$(8.17) \quad \min\left\{\min_{1 \leq i \leq N} G_i(u, p_i, O), \sum_{i=1}^N -p_i - B\right\} \leq 0,$$

knowing that

$$\max\left\{\max_{1 \leq i \leq N} G_i^+(u, p_i, O), FL^-(u, O)\right\} \leq 0.$$

In particular, taking into account Lemma 3.2, all the nonlocal terms are finite.

Assume that  $\sum_{i=1}^N -p_i > B$ . The flux limited condition  $FL^-(u, O) \leq 0$  together with [11, Lemma 5.3.1] implies the existence of  $(\tilde{p}_1, \dots, \tilde{p}_N)$  such that

$$FL^-(u, O) = G_i^-(u, \tilde{p}_i, O) = \sum_i -\tilde{p}_i - B \leq 0 \quad \text{for all } i.$$

In particular, we have  $\sum_i -p_i > \sum_i -\tilde{p}_i$ , from which there exists an index  $i_0$  such that  $p_{i_0} < \tilde{p}_{i_0}$ . Since  $p \mapsto H_i^-(O, p)$  is nondecreasing, we have that  $G_{i_0}^-(u, p_{i_0}, O) \leq 0$ . The inequality for the FL subsolution also provides the inequality  $G_{i_0}^+(u, p_{i_0}, O) \leq 0$ . Finally, we obtain  $G_{i_0}(u, p_{i_0}, O) \leq 0$ , from which we conclude that (8.17) holds.

2.- *Supersolution case.* Let  $\mathbf{p} = (p_1, \dots, p_N) \in D_{\Gamma}^+ u(O)$ . We want to prove that

$$\max\left\{\max_{1 \leq i \leq N} G_i(u, p_i, O), \sum_{i=1}^N -p_i - B\right\} \geq 0,$$



knowing that

$$\max\left\{\max_{1 \leq i \leq N} G_i^+(u, p_i, O), FL^-(u, O)\right\} \geq 0.$$

The above inequality means that, either there exists an index  $i$  such that

$$G_i^+(u, p_i, O) \geq 0,$$

or  $FL^-(u, O) \geq 0$ . In the first case, we get the conclusion since  $G_i \geq G_i^+$ . In the second case, [11, Lemma 5.3.1] implies the existence of  $(\tilde{p}_1, \dots, \tilde{p}_N)$  such that

$$FL^-(u, O) = G_i^-(u, \tilde{p}_i, O) = \sum_i -\tilde{p}_i - B \geq 0 \quad \text{for all } i.$$

If  $\sum_{i=1}^N -p_i < B$ , then, as before, there exists  $i_0$  such that  $\tilde{p}_{i_0} < p_{i_0}$ , from which

$$0 \leq G_{i_0}^-(u, \tilde{p}_{i_0}, O) \leq G_{i_0}^-(u, p_{i_0}, O) \leq G_{i_0}(u, p_{i_0}, O),$$

and the result follows.  $\square$

#### APPENDIX A. PROOF OF LEMMA 2.2.

This subsection is completely devoted to the proof of Lemma 2.2.

Let  $x, y \in \bar{E}_i$ . We have that

$$\mathcal{I}_i u(x) - \mathcal{I}_i u(y) = \sum_{j=1}^N \mathcal{I}_{ij} u(x) - \mathcal{I}_{ij} u(y).$$

In what follows, we identify  $x, y, z \in \bar{E}_i$  and  $x_i, y_i, z_i \in [0, a_i]$ . Assume  $x \leq y$ . Then, we see that

$$\begin{aligned} \mathcal{I}_{ii} u(x) - \mathcal{I}_{ii} u(y) &= - \int_{-y}^{-x} [u_i(x+z) - u_i(x)] \nu_{ii}(x, |z|) dz \\ &\quad + \int_{-x}^{a_i-y} [u_i(x+z) - u_i(x)] \nu_{ii}(x, |z|) dz \\ &\quad - \int_{-x}^{a_i-y} [u_i(y+z) - u_i(y)] \nu_{ii}(y, |z|) dz \\ &\quad + \int_{a_i-y}^{a_i-x} [u_i(y+z) - u_i(y)] \nu_{ii}(y, |z|) dz \\ &=: I_1 + I_2 + I_3 + I_4. \end{aligned}$$

The terms  $I_1$  and  $I_4$  can be estimated similarly. For the first term, using the Hölder continuity of  $u_i$  and the inequality

$$y^\alpha - x^\alpha \leq (y-x)^\alpha \quad \text{for } 0 \leq x \leq y \text{ and } \alpha \in [0, 1],$$

we have

$$\begin{aligned} - \int_{-y}^{-x} [u_i(x+z) - u_i(x)] \nu_{ii}(x, |z|) dz &\leq \Lambda [u_i]_{C^{0,\gamma}(\bar{J}_i)} \int_{-y}^{-x} |z|^{\gamma-1-\sigma} dz \\ &\leq \frac{\Lambda}{\gamma-\sigma} [u]_{C^{0,\gamma}(\Gamma)} |x-y|^{\gamma-\sigma}. \end{aligned}$$

We obtain the same estimate for  $I_4$ .

Next, we estimate

$$\begin{aligned} I_2 + I_3 &= \int_{-x}^{a_i-y} [u_i(x+z) - u_i(x)] (\nu_{ii}(x, |z|) - \nu_{ii}(y, |z|)) dz \\ &\quad + \int_{-x}^{a_i-y} [u_i(x+z) - u_i(x) - (u_i(y+z) - u_i(y))] \nu_{ii}(y, |z|) dz \\ &=: I_5 + I_6. \end{aligned}$$

For  $I_5$ , using Hölder estimates of  $u_i$  and the Lipschitz assumption on  $\nu$ , we see that

$$I_5 \leq 2\Lambda |x-y| [u_i]_{C^{0,\gamma}(\bar{J}_i)} \int_{-x}^{a_i-y} |z|^{\gamma-1-\sigma} dz \leq C\Lambda \|u\|_{C^\gamma} |x-y|,$$

where  $C > 0$  is a constant just depending on  $\sigma, \gamma$  and  $a_i$ .

For  $I_6$ , we write  $[-x, a_i-y] = A \cup B$  with  $A = [-x, a_i-y] \cap [-|x-y|, |x-y|]$  and  $B = [-x, a_i-y] \setminus A$ . We take profit of the Hölder regularity of  $u$ , that is,

(A.1)

$$\begin{aligned} u_i(x+z) - u_i(x) - (u_i(y+z) - u_i(y)) &\leq 2[u]_{C^{0,\gamma}(\Gamma)} |z|^\gamma \quad \text{for } z \in A, \\ u_i(x+z) - u_i(x) - (u_i(y+z) - u_i(y)) &\leq 2[u]_{C^{0,\gamma}(\Gamma)} |x-y|^\gamma \quad \text{for } z \in B. \end{aligned}$$

to obtain, thanks to the assumptions on  $\nu$ ,

$$\begin{aligned} I_6 &\leq 2\Lambda [u]_{C^{0,\gamma}(\Gamma)} \left( \int_A |z|^{\gamma-1-\sigma} dz + |x-y|^\gamma \int_B |z|^{-1-\sigma} dz \right) \\ &\leq C\Lambda [u]_{C^{0,\gamma}(\Gamma)} |x-y|^{\gamma-\sigma}, \end{aligned}$$

for some  $C > 0$  just depending on  $\sigma, \gamma$  and  $a_i$ .

Since the argument are symmetric in  $x$  and  $y$ , we conclude that

$$|\mathcal{I}_{ii}u(x) - \mathcal{I}_{ii}u(y)| \leq C\Lambda [u]_{C^{0,\gamma}(\Gamma)} |x-y|^{\gamma-\sigma}.$$

For  $j \neq i$ , since for  $x, y \in E_i$  and  $z \in E_j$ , we have  $\rho(x, z) = x_i + z_j$ ,  $\rho(y, z) = y_i + z_j$ , by definition we obtain

$$\begin{aligned} \text{(A.2) } \mathcal{I}_{ij}u(x) - \mathcal{I}_{ij}u(y) &= \int_{x_i}^{a_j+x_i} [u_j(z_j - x_i) - u_i(x_i)] \nu_{ij}(x_i, z_j) dz_j \\ &\quad - \int_{y_i}^{a_j+y_i} [u_j(z_j - y_i) - u_i(y_i)] \nu_{ij}(y_i, z_j) dz_j, \end{aligned}$$

and from here we follow the same estimates as above, noticing that now, thanks to (2.4) and (2.5), the estimate (A.1) reads

$$u_i(z_j - x_i) - u_i(x_i) - (u_i(z_j - y_i) - u_i(y_i)) \leq 2^{2-\gamma} [u]_{C^{0,\gamma}(\Gamma)} |z_j|^\gamma \quad \text{for } z \in A.$$

□

## APPENDIX B. WELL-POSEDNESS FOR CENSORED PROBLEMS

We prove here the comparison principle for the Dirichlet problem

$$(B.1) \quad \begin{cases} \lambda u - \mu_i(x) u_{x_i x_i} - \mathcal{I}_i u + H_i(x, u_{x_i}) = f_i(x) & \text{on } E_i, 1 \leq i \leq N, \\ u(O) = \theta, \\ u_i(a_i) = h_i, \quad 1 \leq i \leq N, \end{cases}$$

on a junction  $\Gamma$  when the sub and supersolution are ordered at the vertices.

The PDE (B.1) is a little more general than (4.1). We will assume that the datas satisfy (2.15) and, in addition, for all  $1 \leq i \leq N$ ,

$$(B.2) \quad \begin{cases} 0 \leq \underline{\mu} \leq \mu_i \in C(\bar{J}_i), \quad |\sqrt{\mu_i(x)} - \sqrt{\mu_i(y)}| \leq C_\mu |x - y|, \quad x, y \in \bar{J}_i, \\ f_i \in C(\bar{J}_i). \end{cases}$$

**Lemma B.1.** *Assume  $\Gamma$  is a junction with  $N \geq 1$  edges. Assume that (2.15) and (B.2) hold and  $\theta \in \mathbb{R}$ . Let  $u \in USC(\Gamma)$  be a viscosity subsolution and  $v \in LSC(\Gamma)$  be a viscosity supersolution of (B.1) such that  $u \leq v$  on  $\mathbf{V} = \{\bar{\mathbf{v}}_i\}_{1 \leq i \leq N} \cup \{O\}$ . Then  $u \leq v$  on  $\Gamma$ .*

**Remark B.2.** For the sake of notations, we present the proof in the case of a junction. But it applies readily to the case of a general network (up to use the tedious notations of Section 7). Indeed, since  $u \leq v$  at the vertices, a positive maximum of  $u - v$  cannot be achieved at the vertices and we can localize the following proof by contradiction inside one particular edge. Moreover, all the nonlocal estimates are done edge by edge, and do not depend on the special structure of the network.

*Proof.* By contradiction, we assume

$$M := \max_{\Gamma} \{u - v\} > 0.$$

Then, doubling variables, by standard arguments in the viscosity solution's theory, we have the function

$$\Phi(x, y) = u(x) - v(y) - \alpha^{-2} \rho(x, y)^2, \quad x, y \in \Gamma,$$

attains its maximum at some point  $(\bar{x}, \bar{y})$  (which depends upon  $\alpha$ ) such that

$$(B.3) \quad \alpha^{-2} \rho(\bar{x}, \bar{y})^2 \rightarrow 0, \quad u(\bar{x}) - v(\bar{y}) \rightarrow M, \quad \text{as } \alpha \rightarrow 0.$$

In particular, up to subsequences (not relabeled), we have  $\bar{x}, \bar{y} \rightarrow \hat{x} \in \Gamma$  as  $\alpha \rightarrow 0$ . Since  $(x, y) \mapsto u(x) - v(y)$  is upper semi continuous on  $\Gamma \times \Gamma$  and  $u - v \leq 0$  on the vertices  $\mathbf{V}$ , there exists  $\delta_0 > 0$  such that  $\rho(\bar{x}, \mathbf{V}), \rho(\bar{y}, \mathbf{V}) \geq \delta_0$  for all  $\alpha$  small enough. Thus, without loss of generality we may assume that  $\bar{x}, \bar{y} \in E_i$  for some fixed index  $i$ , for all  $\alpha$  small.

Now, denoting  $\phi(x, y) = \alpha^{-2}|x - y|^2$  and  $\bar{p} = D_x\phi(\bar{x}, \bar{y}) = -D_y\phi(\bar{x}, \bar{y})$ , we use [14, Corollary 1] to infer the existence of  $\bar{\varrho} > 0$  such that, for all  $\varrho < \bar{\varrho}$ , there exists  $X_\varrho, Y_\varrho \in \mathbb{R}$  such that, for all  $\delta > 0$  small enough, we have the viscosity inequalities

$$\begin{aligned} \lambda u(\bar{x}) - \mu_i(\bar{x})X_\varrho - \mathcal{I}_i[B_\delta(\bar{x})]\phi(\cdot, \bar{y})(\bar{x}) - \mathcal{I}_i[B_\delta^c(\bar{x})]u(\bar{x}) + H_i(\bar{x}, \bar{p}) \\ \leq f_i(\bar{x}) + \alpha^{-2}o_\varrho(1), \\ \lambda v(\bar{y}) - \mu_i(\bar{y})Y_\varrho + \mathcal{I}_i[B_\delta(\bar{y})]\phi(\bar{x}, \cdot)(\bar{y}) - \mathcal{I}_i[B_\delta^c(\bar{y})]v(\bar{y}) + H_i(\bar{y}, \bar{p}) \\ \geq f_i(\bar{y}) + \alpha^{-2}o_\varrho(1), \end{aligned}$$

where  $X_\varrho, Y_\varrho$  are such that

$$\begin{bmatrix} X_\varrho & 0 \\ 0 & -Y_\varrho \end{bmatrix} \leq D_{(x,y)}^2\phi(\bar{x}, \bar{y}) + o_\varrho(1),$$

and where  $o_\varrho(1) \rightarrow 0$  as  $\varrho \rightarrow 0$  uniformly with respect to the other parameters. Taking into account (B.2) for  $a_i$  and following the classical arguments in [21] on the matrix inequality above, we see that

$$\mu_i(\bar{x})X_\varrho - \mu_i(\bar{y})Y_\varrho \leq 3C_\mu^2\alpha^{-2}\rho(\bar{x}, \bar{y}) + o_\varrho(1) \leq o_\alpha(1) + o_\varrho(1)$$

for all  $\varrho$ .

On the other hand, by the assumptions (2.6), (B.2) on  $H_i, f_i$ , and the asymptotic properties of the maximum point  $(\bar{x}, \bar{y})$ , we have

$$-H_i(\bar{x}, \bar{p}) + H_i(\bar{y}, \bar{p}) + f_i(\bar{x}) - f_i(\bar{y}) \leq o_\alpha(1).$$

To estimate the nonlocal terms, we first assume  $0 < \delta < \delta_0/2$ , from which  $B_\delta(\bar{x}), B_\delta(\bar{y}) \subset E_i$  for all  $\alpha$  small. Using the smoothness of  $\phi$  and (2.7), We have

$$|\mathcal{I}_i[B_\delta]\phi(\cdot, \bar{y})(\bar{x})|, |\mathcal{I}_i[B_\delta]\phi(\bar{x}, \cdot)(\bar{y})| \leq C\Lambda\alpha^{-2}\delta^{1-\sigma},$$

for some  $C > 0$  just depending on  $\sigma$ .

Subtracting the viscosity inequalities and using the above estimates, we arrive at

$$(B.4) \quad \lambda M - \alpha^{-2}O(\delta^{1-\sigma}) - o_\alpha(1) - \alpha^{-2}o_\varrho(1) \leq I,$$

where we have denoted

$$I := \mathcal{I}_i[B_\delta^c(\bar{x})]u(\bar{x}) - \mathcal{I}_i[B_\delta^c(\bar{y})]v(\bar{y}).$$

It remains to estimate this term. If  $\bar{x} = \bar{y}$  along a subsequence  $\alpha \rightarrow 0$ , we have

$$\begin{aligned} I &= \sum_{j \neq i} \int_{J_j} [u_j(z) - v_j(z) - (u_i(\bar{x}) - v_i(\bar{x}))]\nu_{ij}(\bar{x}, \rho(\bar{x}, z))dz \\ &\quad + \int_{J_i \setminus B_\delta(\bar{x})} [u_i(z) - v_i(z) - (u_i(\bar{x}) - v_i(\bar{x}))]\nu_{ii}(\bar{x}, |\bar{x} - z|)dz. \end{aligned}$$

Since  $\Phi(\bar{x}, \bar{x}) \geq \Phi(z, z)$  for all  $z \in \Gamma$ , we obtain that  $I \leq 0$ . Replacing it into (B.4) and sending  $\varrho \rightarrow 0, \delta \rightarrow 0$  and  $\alpha \rightarrow 0$ , we arrive at a contradiction with the fact that  $M > 0$ .

In the case  $\bar{x} \neq \bar{y}$ , we divide the analysis depending on which edge we integrate.

For  $j \neq i$ , following (A.2), we write

$$\begin{aligned} \mathcal{I}_{ij}u(\bar{x}) - \mathcal{I}_{ij}v(\bar{y}) &= \int_{\bar{x}}^{a_j+\bar{x}} [u_j(z-\bar{x}) - u_i(\bar{x})]\nu_{ij}(\bar{x}, z)dz \\ &\quad - \int_{\bar{y}}^{a_j+\bar{y}} [v_j(z-\bar{y}) - v_i(\bar{x})]\nu_{ij}(\bar{y}, z)dz. \end{aligned}$$

Without loss of generality, we can assume  $\bar{x} < \bar{y}$ . As we already mentioned, there exists  $\delta_0 > 0$  such that  $\delta_0 < \bar{x}$  for all  $\alpha$ . Hence, by the boundedness of  $u, v$ , we have

$$(B.5) \quad \begin{aligned} \left| \int_{\bar{x}}^{\bar{y}} [u_j(z-\bar{x}) - u_i(\bar{x})]\nu_{ij}(\bar{x}, z)dz \right| &\leq C\Lambda\|u\|_\infty\delta_0^{-\sigma}|\bar{x}-\bar{y}| = o_\alpha(1), \\ \left| \int_{a_j+\bar{x}}^{a_j+\bar{y}} [v_j(z-\bar{y}) - v_i(\bar{y})]\nu_{ij}(\bar{y}, z)dz \right| &\leq C\Lambda\|v\|_\infty\delta_0^{-\sigma}|\bar{x}-\bar{y}| = o_\alpha(1), \end{aligned}$$

since  $|\bar{x} - \bar{y}| \rightarrow 0$  as  $\alpha \rightarrow 0$ . It follows

$$\begin{aligned} \mathcal{I}_{ij}u(\bar{x}) - \mathcal{I}_{ij}v(\bar{y}) &= \int_{\bar{y}}^{a_j+\bar{x}} [u_j(z-\bar{x}) - v_j(z-\bar{y}) - (u_i(\bar{x}) - v_i(\bar{y}))]\nu_{ij}(\bar{x}, z)dz \\ &\quad + \int_{\bar{y}}^{a_j+\bar{x}} [v_j(z-\bar{y}) - v_i(\bar{y})](\nu_{ij}(\bar{x}, z) - \nu_{ij}(\bar{y}, z))dz + o_\alpha(1). \end{aligned}$$

Using that  $(\bar{x}, \bar{y})$  is maximum point of  $\Phi$ , we see that

$$\mathcal{I}_{ij}u(\bar{x}) - \mathcal{I}_{ij}v(\bar{y}) \leq \int_{\bar{y}}^{a_j+\bar{x}} [v_j(z-\bar{y}) - v_i(\bar{y})](\nu_{ij}(\bar{x}, z) - \nu_{ij}(\bar{y}, z))dz + o_\alpha(1),$$

and by the continuity assumption on the kernels and the fact that we are integrating away the origin, we conclude that

$$\mathcal{I}_{ij}u(\bar{x}) - \mathcal{I}_{ij}v(\bar{y}) \leq C\Lambda\|v\|_\infty|\bar{x}-\bar{y}| + o_\alpha(1) = o_\alpha(1).$$

In order to estimate the same term for  $j = i$ , we need to be careful since we require an estimate independent of  $\delta$ . Recalling  $\delta < \delta_0/2$ , we can write

$$\begin{aligned} &\mathcal{I}_{ii}[B_\delta^c(\bar{x})]u(\bar{x}) - \mathcal{I}_{ii}[B_\delta^c(\bar{y})]v(\bar{y}) \\ &= \left( \int_{-\bar{x}}^{-\delta} + \int_{\delta}^{a_i-\bar{x}} \right) (u_i(\bar{x}+z) - u_i(\bar{x}))\nu_{ii}(\bar{x}, z)dz \\ &\quad - \left( \int_{-\bar{y}}^{-\delta} + \int_{\delta}^{a_i-\bar{y}} \right) (v_i(\bar{y}+z) - v_i(\bar{x}))\nu_{ii}(\bar{y}, z)dz. \end{aligned}$$

We then perform the same measure decomposition as in [12]. For  $t > 0$ , we consider the kernels

$$\begin{aligned}\tilde{\nu}_{ii}(t) &= \min\{\nu_{ii}(\bar{x}, t), \nu_{ii}(\bar{y}, t)\}, \\ \nu_{ii}^+(t) &= \nu_{ii}(\bar{x}, t) - \tilde{\nu}_{ii}(t), \\ \nu_{ii}^-(t) &= \tilde{\nu}_{ii}(t) - \nu_{ii}(\bar{y}, t),\end{aligned}$$

from which, assuming without loss of generality  $\bar{x} < \bar{y}$ , denoting  $A = (-\bar{x}, -\delta) \cup (\delta, a_i - \bar{y})$ , and using similar estimates as in (B.5), we have

$$\begin{aligned}& \mathcal{I}_{ii}[B_\delta^c(\bar{x})]u(\bar{x}) - \mathcal{I}_{ii}[B_\delta^c(\bar{y})]v(\bar{y}) \\ &= o_\alpha(1) + \int_A [u_i(\bar{x} + z) - v_i(\bar{y} + z) - (u_i(\bar{x}) - v_i(\bar{y}))]\tilde{\nu}_{ii}(z)dz \\ & \quad + \int_A [u_i(\bar{x} + z) - u_i(\bar{x})]\nu_{ii}^+(z)dz - \int_A [v_i(\bar{y} + z) - v_i(\bar{y})]\nu_{ii}^-(z)dz.\end{aligned}$$

Using that  $\tilde{\nu}_{ii}$  is nonnegative and the fact that  $(\bar{x}, \bar{y})$  is maximum point of  $\Phi$ , we get

$$\begin{aligned}& \mathcal{I}_{ii}[B_\delta^c(\bar{x})]u(\bar{x}) - \mathcal{I}_{ii}[B_\delta^c(\bar{y})]v(\bar{y}) \\ & \leq o_\alpha(1) + \int_A [u_i(\bar{x} + z) - u_i(\bar{x})]\nu_{ii}^+(z)dz - \int_A [v_i(\bar{y} + z) - v_i(\bar{y})]\nu_{ii}^-(z)dz \\ & =: o_\alpha(1) + I_1 + I_2.\end{aligned}$$

The estimates of  $I_1$  and  $I_2$  are similar, thus we concentrate on  $I_1$ . Taking  $0 < \delta < \min\{|\bar{x} - \bar{y}|, \delta_0/2\}$ , we use the fact that, by maximality of  $(\bar{x}, \bar{y})$  the following inequality holds

$$u_i(\bar{x} + z) - u_i(\bar{x}) \leq \alpha^{-2}(2|\bar{x} - \bar{y}||z| + |z|^2), \quad z \in A \subset (-\bar{x}, a_i - \bar{x}).$$

Denote  $A_1 = \{z \in A : |z| \leq |\bar{x} - \bar{y}|\}$  and  $A_2 = A \setminus A_1$ . Since  $\nu_{ii}^+$  is nonnegative, we can write

$$I_1 \leq \alpha^{-2} \int_{A_1} [2|\bar{x} - \bar{y}||z| + |z|^2]\nu_{ii}^+(z)dz + 2\|u\|_\infty \int_{A_2} \nu_{ii}^+(z)dz.$$

At this point, we notice that by the assumptions (2.7) on  $\nu$  we have

$$\nu_{ii}^+(z) \leq \Lambda|\bar{x} - \bar{y}||z|^{-(1+\sigma)},$$

from which we conclude that

$$I_1 \leq \Lambda\alpha^{-2}|\bar{x} - \bar{y}|^{3-\sigma}(2(1-\sigma)^{-1} + (2-\sigma)^{-1}) + 4\Lambda\sigma^{-1}\|u\|_\infty|\bar{x} - \bar{y}|^{1-\sigma}.$$

Thus, in view of (B.3), we obtain  $I_1 \leq o_\alpha(1)$ , and similarly,  $I_2 \leq o_\alpha(1)$ .

Putting these estimates into (B.4) yields

$$\lambda M - \alpha^{-2}\delta^{1-\sigma} - o_\alpha(1) - \alpha^{-2}o_\rho(1) \leq 0.$$

Letting  $\rho \rightarrow 0, \delta \rightarrow 0$  and finally  $\alpha \rightarrow 0$ , we reach a contradiction. This concludes the result.  $\square$

**Theorem B.3.** *Assume  $\Gamma$  is a junction with  $N \geq 1$  edges. Assume that (2.15) and (B.2) hold with  $\underline{\mu} > 0$ , and  $\theta \in \mathbb{R}$ . Then, there exists a unique continuous viscosity solution  $u \in C(\Gamma)$  to the problem (B.1) satisfying the Dirichlet boundary conditions pointwisely.*

*Proof.* From Lemma 4.1, there exist continuous sub- and supersolution  $\psi^-$ ,  $\psi^+$  to (B.1) satisfying the boundary condition pointwisely (the adaptation of Lemma 4.1 to (B.1) when the equation is strictly elliptic is straightforward). By Perron's method, we infer the existence of a (possibly discontinuous) viscosity solution  $u$  to (B.1) satisfying  $\psi^- \leq u \leq \psi^+$  on  $\Gamma$ . Applying Lemma B.1 with the subsolution  $u^*$  and the supersolution  $u_*$  satisfying  $u^* = u_*$  on  $\mathbf{V}$ , we conclude that  $u^* \leq u_*$ . Thus  $u$  is a continuous viscosity solution to (B.1), which is unique, thanks again to Lemma B.1.  $\square$

#### REFERENCES

- [1] Yves Achdou, Fabio Camilli, Alessandra Cutrì, and Nicoletta Tchou. Hamilton-Jacobi equations constrained on networks. *NoDEA Nonlinear Differential Equations Appl.*, 20(3):413–445, 2013.
- [2] Yves Achdou, Manh-Khang Dao, Olivier Ley, and Nicoletta Tchou. A class of infinite horizon mean field games on networks. *Netw. Heterog. Media*, 14(3):537–566, 2019.
- [3] Yves Achdou, Manh-Khang Dao, Olivier Ley, and Nicoletta Tchou. Finite horizon mean field games on networks. *Calc. Var. Partial Differential Equations*, 59(5):Paper No. 157, 34, 2020.
- [4] Yves Achdou, Salomé Oudet, and Nicoletta Tchou. Hamilton-Jacobi equations for optimal control on junctions and networks. *ESAIM Control Optim. Calc. Var.*, 21(3):876–899, 2015.
- [5] G. Barles, A. Briani, and E. Chasseigne. A Bellman approach for two-domains optimal control problems in  $\mathbb{R}^N$ . *ESAIM Control Optim. Calc. Var.*, 19(3):710–739, 2013.
- [6] G. Barles, A. Briani, and E. Chasseigne. A Bellman approach for regional optimal control problems in  $\mathbb{R}^N$ . *SIAM J. Control Optim.*, 52(3):1712–1744, 2014.
- [7] G. Barles, A. Briani, E. Chasseigne, and C. Imbert. Flux-limited and classical viscosity solutions for regional control problems. *ESAIM Control Optim. Calc. Var.*, 24(4):1881–1906, 2018.
- [8] Guy Barles. An introduction to the theory of viscosity solutions for first-order Hamilton-Jacobi equations and applications. In *Hamilton-Jacobi equations: approximations, numerical analysis and applications*, volume 2074 of *Lecture Notes in Math.*, pages 49–109. Springer, Heidelberg, 2013.
- [9] Guy Barles, Rainer Buckdahn, and Etienne Pardoux. Backward stochastic differential equations and integral-partial differential equations. *Stochastics Stochastics Rep.*, 60(1-2):57–83, 1997.
- [10] Guy Barles and Emmanuel Chasseigne. (Almost) everything you always wanted to know about deterministic control problems in stratified domains. *Netw. Heterog. Media*, 10(4):809–836, 2015.
- [11] Guy Barles and Emmanuel Chasseigne. *On Modern Approaches of Hamilton-Jacobi Equations and Control Problems with Discontinuities*, volume 104 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser/Springer, Cham, 2024. A Guide to Theory, Applications, and Some Open Problems, PNLDE Subseries in Control, 104.
- [12] Guy Barles, Emmanuel Chasseigne, and Cyril Imbert. Hölder continuity of solutions of second-order non-linear elliptic integro-differential equations. *J. Eur. Math. Soc. (JEMS)*, 13(1):1–26, 2011.

- [13] Guy Barles and Francesca Da Lio. On the generalized Dirichlet problem for viscous Hamilton-Jacobi equations. *J. Math. Pures Appl. (9)*, 83(1):53–75, 2004.
- [14] Guy Barles and Cyril Imbert. Second-order elliptic integro-differential equations: viscosity solutions’ theory revisited. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 25(3):567–585, 2008.
- [15] Guy Barles, Shigeaki Koike, Olivier Ley, and Erwin Topp. Regularity results and large time behavior for integro-differential equations with coercive Hamiltonians. *Calc. Var. Partial Differential Equations*, 54(1):539–572, 2015.
- [16] Guy Barles and Erwin Topp. Lipschitz regularity for censored subdiffusive integro-differential equations with superfractional gradient terms. *Nonlinear Anal.*, 131:3–31, 2016.
- [17] Alberto Bressan and Yunho Hong. Optimal control problems on stratified domains. *Netw. Heterog. Media*, 2(2):313–331, 2007.
- [18] Luis Caffarelli and Luis Silvestre. Regularity theory for fully nonlinear integro-differential equations. *Comm. Pure Appl. Math.*, 62(5):597–638, 2009.
- [19] Fabio Camilli and Claudio Marchi. Stationary mean field games systems defined on networks. *SIAM J. Control Optim.*, 54(2):1085–1103, 2016.
- [20] Fabio Camilli, Claudio Marchi, and Dirk Schieborn. The vanishing viscosity limit for Hamilton-Jacobi equations on networks. *J. Differential Equations*, 254(10):4122–4143, 2013.
- [21] M. G. Crandall, H. Ishii, and P.-L. Lions. User’s guide to viscosity solutions of second order partial differential equations. *Bull. Amer. Math. Soc. (N.S.)*, 27(1):1–67, 1992.
- [22] L. C. Evans. *Partial differential equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 1998.
- [23] Nicolas Forcadel and Wilfredo Salazar. Homogenization of a discrete model for a bifurcation and application to traffic flow. *J. Math. Pures Appl. (9)*, 136:356–414, 2020.
- [24] Daria Ghilli. On Neumann problems for nonlocal Hamilton-Jacobi equations with dominating gradient terms. *Calc. Var. Partial Differential Equations*, 56(5):Paper No. 139, 41, 2017.
- [25] Qing-Yang Guan and Zhi-Ming Ma. Reflected symmetric  $\alpha$ -stable processes and regional fractional Laplacian. *Probab. Theory Related Fields*, 134(4):649–694, 2006.
- [26] Qing Han and Fanghua Lin. *Elliptic partial differential equations*, volume 1 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.
- [27] Cyril Imbert and Régis Monneau. Flux-limited solutions for quasi-convex Hamilton-Jacobi equations on networks. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(2):357–448, 2017.
- [28] Cyril Imbert, Régis Monneau, and Hasnaa Zidani. A Hamilton-Jacobi approach to junction problems and application to traffic flows. *ESAIM Control Optim. Calc. Var.*, 19(1):129–166, 2013.
- [29] Cyril Imbert and Vinh Duc Nguyen. Effective junction conditions for degenerate parabolic equations. *Calc. Var. Partial Differential Equations*, 56(6):Paper No. 157, 27, 2017.
- [30] Wladyslaw Kulpa. The Poincaré-Miranda theorem. *Amer. Math. Monthly*, 104(6):545–550, 1997.
- [31] Pierre-Louis Lions and Panagiotis Souganidis. Viscosity solutions for junctions: well posedness and stability. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 27(4):535–545, 2016.
- [32] Pierre-Louis Lions and Panagiotis Souganidis. Well-posedness for multi-dimensional junction problems with Kirchoff-type conditions. *Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl.*, 28(4):807–816, 2017.



- [33] Peter S. Morfe. Convergence & rates for Hamilton-Jacobi equations with Kirchoff junction conditions. *NoDEA Nonlinear Differential Equations Appl.*, 27(1):Paper No. 10, 69, 2020.
- [34] Isaac Ohavi. Quasi linear parabolic pde posed on a network with non linear Neumann boundary condition at vertices. *J. Math. Anal. Appl.*, 500(1):Paper No. 125154, 29, 2021.
- [35] Dirk Schieborn and Fabio Camilli. Viscosity solutions of Eikonal equations on topological networks. *Calc. Var. Partial Differential Equations*, 46(3-4):671–686, 2013.
- [36] Antonio Siconolfi. Time-dependent Hamilton-Jacobi equations on networks. *J. Math. Pures Appl. (9)*, 163:702–738, 2022.
- [37] Halil Mete Soner. Optimal control with state-space constraint. I. *SIAM J. Control Optim.*, 24(3):552–561, 1986.
- [38] Erwin Topp. Existence and uniqueness for integro-differential equations with dominating drift terms. *Comm. Partial Differential Equations*, 39(8):1523–1554, 2014.
- [39] Joachim von Below. Classical solvability of linear parabolic equations on networks. *J. Differential Equations*, 72(2):316–337, 1988.

GUY BARLES: INSTITUT DENIS POISSON (UMR CNRS 7013) UNIVERSITÉ DE TOURS, UNIVERSITÉ D'ORLÉANS, CNRS. PARC DE GRANDMONT. 37200 TOURS, FRANCE. [guy.barles@idpoisson.fr](mailto:guy.barles@idpoisson.fr)

OLIVIER LEY: UNIV RENNES, INSA RENNES, CNRS, IRMAR - UMR 6625, F-35000 RENNES, FRANCE. [olivier.ley@insa-rennes.fr](mailto:olivier.ley@insa-rennes.fr)

ERWIN TOPP: INSTITUTO DE MATEMÁTICAS, UNIVERSIDADE FEDERAL DO RIO DE JANEIRO, RIO DE JANEIRO - RJ, 21941-909, BRAZIL; AND DEPARTAMENTO DE MATEMÁTICA Y C.C., UNIVERSIDAD DE SANTIAGO DE CHILE, CASILLA 307, SANTIAGO, CHILE. [etopp@im.ufrj.br](mailto:etopp@im.ufrj.br); [erwin.topp@usach.cl](mailto:erwin.topp@usach.cl).