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Second derivatives of solutions to the 3D incompressible Navier-Stokes equation in Lebesgue spaces

Igor Honoré*

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Abstract

We obtain new controls for the Leray solutions u of the incompressible Navier-Stokes equation in \mathbb{R}^3 . Specifically, we estimate u , ∇u , and $\nabla^2 u$ in suitable Lebesgue spaces $L_T^{\tilde{r}} L^r$, $r < +\infty$ with some constraints on $\tilde{r} > 0$. Our method is based on a Duhamel formula around a perturbed heat equation, allowing to thoroughly exploit the well-known energy estimates which balances the potential singularities. We also perform a new Bihari-LaSalle argument in this context.

Eventually, we adapt our strategy to prove that $\sup_{t \in [0, T]} \int_0^t (t-s)^{-\theta} \|\nabla^k u(s, \cdot)\|_{L^r} ds < +\infty$, for all $\theta < \frac{3-kr}{2r}$, $k \in \llbracket 0, 2 \rrbracket$, and $1 < r < \frac{3}{k}$.

Keywords: Incompressible Navier-Stokes equation, Leray solution, Integral inequality.

MSC: Primary: 35Q30, 76D03; Secondary: 76D05, 26D10,

1 Introduction

The goal of this article is to get *a priori* controls of the incompressible Navier-Stokes equation in dimension 3:

$$\begin{cases} \partial_t u(t, x) + u(t, x) \cdot \nabla u(t, x) = \nu \Delta u(t, x) - \nabla p(t, x) + f(t, x), \\ u(0, x) = u_0(x), \\ \nabla \cdot u(t, x) = 0, \end{cases} \quad (1.1)$$

exploiting at the most the regularisation by the heat kernel.

Projecting the above system on the space of divergence free functions gives

$$\begin{cases} \partial_t u(t, x) + \mathbb{P}[u \cdot \nabla u](t, x) = \nu \Delta u(t, x) + \mathbb{P}f(t, x), \\ u(0, x) = u_0(x), x \in \mathbb{R}^3, \end{cases} \quad (1.2)$$

where \mathbb{P} is the Leray operator which is defined by $\mathbb{P} = I - \Xi$, I stands for the identity operator and Ξ is a non-local operator deriving from a gradient.

Without smallness assumption on u_0 or T , establishing new controls on u is a long-stand problem. The main contribution over the last century is due to Leray [Ler34] implying the following estimates

$$\|u(t, \cdot)\|_{L^2} \leq \|u_0\|_{L^2} + \frac{1}{2} \int_0^T \|f(s, \cdot)\|_{L^2} ds, \quad (1.3)$$

and

$$\int_0^t \|\nabla u(s, \cdot)\|_{L^2}^2 ds \leq \nu^{-1} \left(\|u_0\|_{L^2}^2 + \|u(t, \cdot)\|_{L^2} \int_0^t \|f(s, \cdot)\|_{L^2} ds \right). \quad (1.4)$$

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The proof of the above inequalities relies on the energy technique for the system (1.1). There are some variations of this, implying some fluctuations of controls (1.3)-(1.4). We can mention for instance [LR07] where well-posedness is established in a Morrey-Campanato space, being a variation of the L^2 estimates of Leray.

Proving that the solution u is globally smooth (L^∞ with bounded derivatives) constitutes a Millennium Prize Problem, see [Fef06]. For additional information on equation (1.2), the reader can look at the books [Tem79], [Lio96], [Che98], [MB02], [LR02], [LR16], [BCD11].

In dimension 2, the Cauchy problem (1.1) is globally well-posed in a smooth space, see Ladyzhenskaya [Lad69].

Our principal tools are the regularisation by heat kernel and a new Bihari-LaSalle type inequality. Comparing with the usual energy method, ours allows us to consider more spatial regularity, in term of derivatives and of Lebesgue spaces.

Our analysis can be extended to other dimension than 3; one of the change is in the application of Gagliardo-Nirenberg interpolation inequality.

For the best of the author's knowledge, the first controls of the second derivatives are due to Constantin [Con90] and also Lions [Lio96], where the authors proved that $\nabla^2 u \in L^{\tilde{r}}([0, T], L^r(\mathbb{R}^3, \mathbb{R}^3))$, with

$$\frac{2}{\tilde{r}} + \frac{3}{r} = 4, \forall \tilde{r} \in (1, 2).$$

The constraint on \tilde{r} means equivalently that $r \in (1, \frac{3}{2})$, the goal of the article is to consider arbitrary large r .

In the particular case $r = \tilde{r}$ there are some stronger results in term of scaling, indeed Constantin [Con90] obtains $\nabla^2 u \in L^{\frac{4}{3}-\varepsilon}$, $\varepsilon > 0$, and similarly Lions shows the control in Lorentz space $L^{\frac{4}{3}, \infty}([0, T] \times \mathbb{R}^3, \mathbb{R}^3)$. Next to Vasseur et al. [Vas10] and [VY21], for each $k \in \llbracket 1, 2 \rrbracket$, under the constraint

$$r < \frac{4}{k+1}, \tag{1.5}$$

they show that $\|\nabla^k u\|_{L_T^r L^r(\Omega)} < +\infty$, for any Ω , bounded subset of $(t_0, T) \times \mathbb{R}^3$, and for any $t_0 \in (0, T)$.

It is already proved, see [Cha92], that in a periodic framework or in a bounded domain that $t^k \|\partial_t^k u(t, \cdot)\|_{L^2}$ lies in L_T^2 , for each $k \in \mathbb{N}$. Similarly in Section 5, we establish a similar control but with space derivatives and with a singular time integral.

In [FGT81], [DF02], [Duf90], the solution u is proved to be such that $\partial_t^n \nabla^k u \in L_T^{\frac{2}{4n+2k-1}} L^2$. This matches with the limit case for $n = 0$, $k = 2$ and $r \rightarrow 2^-$ in Theorem 1, see also Corollary 3 further.

For other details on already known *a priori* controls of u , we can refer to [Tao13]

Our first main result is stated below.

Theorem 1. *Let $u \in L^2([0, T], H^1(\mathbb{R}^3, \mathbb{R}^3)) \cap L^\infty([0, T], L^2(\mathbb{R}^3, \mathbb{R}^3))$ a Leray solution of (1.2), if for all $k \in \llbracket 0, 2 \rrbracket$, and $(r, \tilde{r}) \in (1, +\infty)$, $u_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3) \cap \dot{B}_{r, \tilde{r}}^{\frac{2(\tilde{r}-1)}{r}}(\mathbb{R}^3, \mathbb{R}^3)$ and $f \in L^\infty([0, T], L^2(\mathbb{R}^3, \mathbb{R}^3)) \cap L^\infty([0, T], \dot{B}_{r, \infty}^2(\mathbb{R}^3, \mathbb{R}^3))$ then $\|\nabla^k u\|_{L_T^{\tilde{r}} L^r} < +\infty$ as soon as*

★ for $k = 0$

$$\begin{cases} \frac{4}{\tilde{r}} + \frac{6}{r} = 3, & \text{if } 2 \leq r \leq 6, \\ \frac{1}{\tilde{r}} + \frac{3}{r} = 1, & \text{if } r \geq 6, \end{cases}$$

★ for $k = 1$

$$\begin{cases} \frac{1}{\tilde{r}} + \frac{3}{r} = 2, & \text{if } r \in [2, 3], \\ \tilde{r} = \frac{r(r-4)}{4r^2-13r+6}, & \text{if } r \in [3, 6], \end{cases}$$

★ for $k = 2$

$$\begin{cases} \frac{2}{\tilde{r}} + \frac{3}{r} = 4, & \text{if } r < \frac{3}{2}, \\ \tilde{r} = \frac{2r(3r-4)}{13r^2-26r+12} & \text{if } r \in [\frac{3}{2}, 2), \\ \frac{1}{\tilde{r}} + \frac{6}{r} = 9, & \text{if } 2 \leq r < +\infty. \end{cases}$$

Remark 1. Our results do not reach the Prodi-Serrin criterion, see [Pro59] [Ser62], $\|\nabla^k u\|_{L_T^{\tilde{r}} L^r} < +\infty$ with $\frac{2}{\tilde{r}} + \frac{3}{r} \leq 1 + k$, which implies that there is no blow-up of the solution u .

With our method, it is not possible to consider the norm $\|\nabla \omega\|_{L_T^{\tilde{r}} L^\infty}$, because we use a kind of Bihari-LaSalle argument which involves to get, in the upper-bound, a contribution of the same norm instead of the $\|\nabla^2 u(s, \cdot)\|_{L^\infty}$ (and because there is a priori no control in L^∞ of ∇u by ω).

For $k = 0$ and $r = +\infty$, the result matches with $u \in L^1([0, T], L^\infty(\mathbb{R}^3, \mathbb{R}^3))$ already provided by [Tar78].

The statement comes from many uses of the *a priori* controls (1.3)-(1.4) and a new Bihari-LaSalle result, the strategy of the proof changes according to the indexes of the Lebesgue space in the considered estimates. Therefore, there are many different cases, and so, by interpolation and by Sobolev embedding (or generally by Gagliardo-Nirenberg inequality), many possible outcomes. In Section 4 further, we aim to obtain the best result from Theorem 1 and also from the result (1.5) for $k = 2$ by Constantin [Con91] (also [Lio96] [Vas10]...).

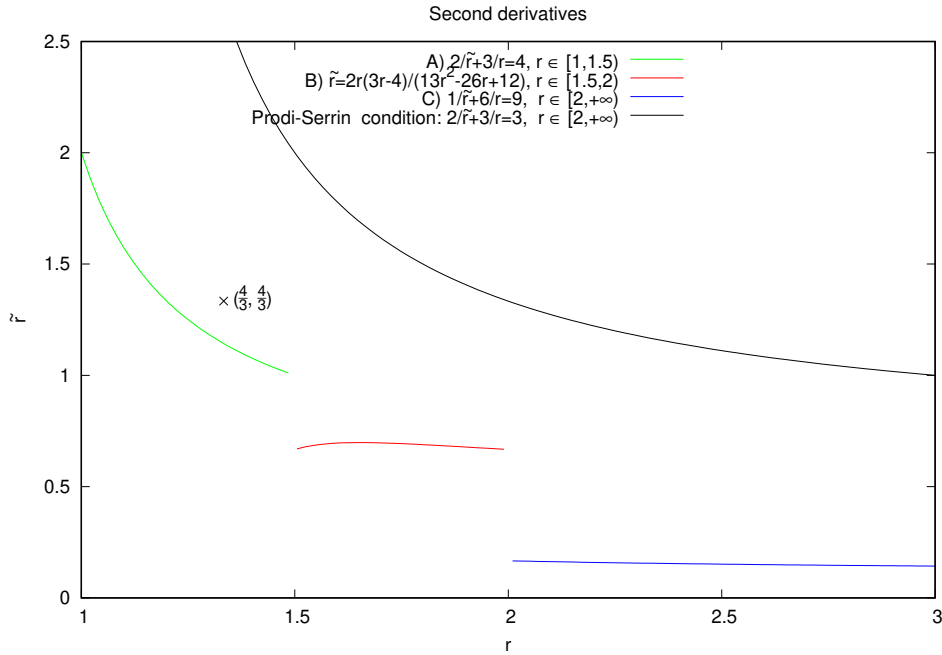


Figure 1: $\nabla^2 u \in L^{\tilde{r}}([0, T], L^r(\mathbb{R}^3, \mathbb{R}^3))$

The paper is organised as following. In Section 2, we set some useful notations and results for our analysis. The proof of Theorem 1 is developed in Section 3. We perform, in Section 4, interpolations between the already known controls and with our new results. We also provide, in Section 5, a counterpart in weighted Lebesgue space of Theorem 5.

2 Definitions and useful results

We denote by $C > 1$, depending only on the dimension d , a generic constant which can be different from line to line.

The Euclidean norm is denoted by $|\cdot|$, i.e. $|z| = (\sum_{k=1}^3 z_k^2)^{\frac{1}{2}}$, where we decompose $z = z_1 e_1 + z_2 e_2 + z_3 e_3$ with (e_1, e_2, e_3) the canonical base of \mathbb{R}^3 .

2.1 Functional notations

2.1.1 Differential operators

The derivative operator ∇ matches with the gradient or with the Jacobian matrix; the symbol $\nabla \cdot$ stands for the divergence, the Laplacian is as usual noted by Δ .

From these notations, we have for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$u(t, x) \cdot \nabla u(t, x) = \left(\sum_{j=1}^d u_j(t, x) \partial_{x_j} u_i(t, x) \right)_{i \in [1,3]}, \quad (2.1)$$

for $u = (u_i)_{i \in [1,3]}$.

The Leray projector \mathbb{P} is defined, for any function $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$, smooth enough, by

$$\forall x \in \mathbb{R}^3, \mathbb{P}\varphi(x) = \varphi(x) + \nabla(-\Delta)^{-1} \nabla \cdot \varphi(x) =: \varphi(x) - \Xi\varphi(x). \quad (2.2)$$

In case of Navier-Stokes equation, the operator Ξ matches with the pressure part which is indeed a gradient term in the Helmholtz-Hodge decomposition.

Let us also explicit the non-linear part in (1.2), for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, we write

$$u(t, x) \cdot \nabla u(t, x) = \left(\sum_{j=1}^3 u_j(t, x) \partial_{x_j} u_i(t, x) \right)_{i \in [1,3]}, \quad (2.3)$$

with $u = (u_i)_{i \in [1,3]}$.

Importantly, from the incompressible assumption, we have for any $z \in \mathbb{R}^3$,

$$\left([u(t, x) - u(t, z)] \cdot \nabla u(t, x) \right) = \nabla_x \cdot [u(t, x) - u(t, z)]^{\otimes 2} = \sum_{j=1}^3 \partial_{x_j} \sum_{i=1}^3 [u_j(t, x) - u_j(t, z)] [u_i(t, x) - u_i(t, z)]. \quad (2.4)$$

The symbol \otimes stands for the dyadic product, i.e. for any $u, v \in \mathbb{R}^3$

$$u \otimes v := \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}, \quad (2.5)$$

also, for the sake of simplicity, we take $u^{\otimes 2} := u \otimes u$.

Finally, we can rewrite equation (1.2) by

$$\partial_t u(t, x) + \mathbb{P}[\nabla \cdot u^{\otimes 2}](t, x) = \nu \Delta u(t, x) + \mathbb{P}f(t, x). \quad (2.6)$$

2.1.2 Symmetric increasing rearrangement

Our Bihari-LaSalle argument thoroughly depends on the non-decreasing rearrangement. We could use a non-increasing rearrangement, see *e.g.* [LL01], which seems less intuitive in our analysis.

Let E a one dimensional bounded* Euclidean space ($[0, T]$, $T > 0$, in the article). For any $\varphi : E \rightarrow \mathbb{R}$, we define distribution function

$$\mu_\varphi(s) = \lambda(\{x \in E : |\varphi(x)| > s\}),$$

where $\lambda(\cdot)$ stands for the Lebesgue measure.

The decreasing rearrangement is written by

$$\varphi_*(t) = \inf\{s \in [0, \infty] : \mu_\varphi(s) \leq t\}.$$

We are then able to state the associated increasing rearrangement by:

$$\varphi^*(t) = -(-\varphi)_*.$$

Let us remark that for any $p > 0$, we have the L^p invariance of the rearrangement,

$$\|\varphi\|_{L^p} = \|\varphi^*\|_{L^p} = \|\varphi_*\|_{L^p}. \quad (2.7)$$

The rearrangement keeps the order of inequalities, and we have $(\varphi^p)^* = (\varphi^*)^p$.

We can also refer to [Kaw85], [Car95], [Alb00] and [BLR04] for alternative increasing rearrangement transformation.

Eventually, in the paper we have to use Hardy–Littlewood inequality: for all functions φ, ψ non-negative measurable real functions,

$$\int_E \varphi(x)\psi(x)dx \leq \int_E \varphi^*(x)\psi^*(x)dx. \quad (2.8)$$

2.2 Heat kernel

In this short section, we recall some useful notations and inequalities for our analysis.

In the following, we denote for any $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$:

$$h_\nu(t, x) := \frac{e^{-\frac{|x|^2}{4\nu t}}}{(4\pi\nu t)^{\frac{3}{2}}}, \quad (2.9)$$

the heat kernel, satisfying the equation

$$\begin{cases} (\partial_t - \nu\Delta)h_\nu(t, \cdot) = 0, \forall t > 0, \\ h_\nu(0, \cdot) = \delta_0, \end{cases} \quad (2.10)$$

where δ_0 stands for the Dirac distribution.

In order to simplify some notations, we also write for all $0 \leq s \leq t \leq T$ and $(x, y) \in \mathbb{R}^3$

$$\tilde{p}(s, t, x, y) := h_\nu(t - s, x - y). \quad (2.11)$$

From this notation, we define \tilde{P} and \tilde{G} the semi-group and the Green operator associated with the heat equation, defined, for all $\varphi : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ and $\psi : [0, T] \times \mathbb{R}^3 \rightarrow \mathbb{R}^3$ smooth enough, and any $(t, x) \in [0, T] \times \mathbb{R}^3$ by:

$$\begin{aligned} \tilde{P}\varphi(t, x) &:= \int_{\mathbb{R}^3} h_\nu(t, x - y)\varphi(y)dy, \\ \tilde{G}\varphi(t, x) &:= \int_0^t \int_{\mathbb{R}^3} h_\nu(t - s, x - y)\psi(s, y)dy ds. \end{aligned} \quad (2.12)$$

*In this case, the properties of the symmetric increasing rearrangement clearly match with the symmetric decreasing rearrangement ones.

For any $\delta > 0$, there is a constant $1 < C$ such that:

$$\forall x \in \mathbb{R}^3, |x|^\delta e^{-|x|^2} \leq C e^{-C^{-1}|x|^2}. \quad (2.13)$$

Finally, for any $1 \leq q \leq +\infty$,

$$\|h_\nu(t, x - \cdot)\|_{L^q} = \left(\int_{\mathbb{R}^3} \frac{e^{-\frac{p|x-y|^2}{4\nu t}}}{(4\pi\nu t)^{\frac{3p}{2}}} dy \right)^{\frac{1}{p}} = \left(\frac{(p^{-1}4\pi\nu t)^{\frac{3}{2}}}{(4\pi\nu t)^{\frac{3p}{2}}} \right)^{\frac{1}{p}} = p^{-\frac{3}{2}} (4\pi\nu t)^{\frac{3}{2}(\frac{1-p}{p})}. \quad (2.14)$$

2.3 Interpolation results

2.3.1 In Lebesgue space

Simple interpolation

It is well-known, cf. [Bre99], that for all $1 \leq p \leq r \leq q \leq +\infty$, we have for any $\varphi : L^p(\mathbb{R}^3, \mathbb{R}^3) \cap L^q(\mathbb{R}^3, \mathbb{R}^3)$

$$\|\varphi\|_{L^r} \leq \|\varphi\|_{L^p}^\alpha \|\varphi\|_{L^q}^{1-\alpha}, \quad (2.15)$$

with

$$\frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q},$$

which is equivalent to

$$\alpha = \frac{p(q-r)}{r(q-p)}, \quad 1-\alpha = \frac{q(r-p)}{r(q-p)}. \quad (2.16)$$

Double interpolation

From (2.15), (2.16), we can perform another interpolation w.r.t. a second variable. For any function $\psi \in L^{\tilde{p}}([0, T], L^p(\mathbb{R}^3, \mathbb{R}^3)) \cap L^{\tilde{q}}([0, T], L^q(\mathbb{R}^3, \mathbb{R}^3))$

$$\|\psi(t, \cdot)\|_{L^r} \leq \|\psi(t, \cdot)\|_{L^p}^\alpha \|\psi(t, \cdot)\|_{L^q}^{1-\alpha}, \quad (2.17)$$

where α is defined in (2.16). Also, by Hölder inequality, with $1 \leq m, n \leq +\infty$ such that $m^{-1} + n^{-1} = 1$, for any $\tilde{r} \in [\min(\tilde{p}, \tilde{q}), \max(\tilde{p}, \tilde{q})]$,

$$\begin{aligned} \|\psi\|_{L^{\tilde{r}}L^r} &\leq \left(\int \|\psi(t, \cdot)\|_{L^p}^{\alpha\tilde{r}} \|\psi(t, \cdot)\|_{L^q}^{(1-\alpha)\tilde{r}} dt \right)^{\frac{1}{\tilde{r}}} \\ &\leq \left(\int \|\psi(t, \cdot)\|_{L^p}^{m\alpha\tilde{r}} dt \right)^{\frac{1}{m\tilde{r}}} \left(\int \|\psi(t, \cdot)\|_{L^q}^{n(1-\alpha)\tilde{r}} dt \right)^{\frac{1}{n\tilde{r}}} \\ &= \|\psi\|_{L^{m\alpha\tilde{r}}L^p}^\alpha \|\psi\|_{L^{n(1-\alpha)\tilde{r}}L^q}^{1-\alpha}. \end{aligned}$$

To match with the condition of integrability in t , we have to impose

$$m\alpha\tilde{r} = \tilde{p} \quad \text{and} \quad n(1-\alpha)\tilde{r} = \tilde{q}.$$

From the condition on the parameters of Hölder estimates we deduce $m = \frac{\tilde{p}}{\alpha\tilde{r}}$ and $n = \frac{\tilde{q}}{m(1-\alpha)} = \frac{\tilde{p}}{\tilde{p}-\alpha\tilde{r}}$, and

$$\tilde{r} = \frac{\tilde{q}}{n(1-\alpha)} = \frac{\tilde{q}\tilde{p}r(q-p)}{\tilde{p}q(r-p) + \tilde{q}p(q-r)}, \quad (2.18)$$

where we recall the final double interpolation

$$\|\psi\|_{L^{\tilde{r}}L^r} \leq C \|\psi\|_{L^{\tilde{p}}L^p}^\alpha \|\psi\|_{L^{\tilde{q}}L^q}^{1-\alpha}.$$

Hardy-Littlewood-Sobolev inequality

We also thoroughly use Hardy-Littlewood-Sobolev inequality, see Section V in [Ste70], that we recall below in dimension 1:

$$\forall \alpha \in (0, 1), (\tilde{r}, r') \in (0, +\infty)^2, \|\cdot\|^{-\alpha} \star f \|_{L^{\tilde{r}}} \leq C \|f\|_{L^{r'}}, \quad (2.19)$$

with $\frac{1}{\tilde{r}} + \alpha = 1 + \frac{1}{r'}$.

It is possible to extend (2.19) to the case $\tilde{r} = 1$, in the following case

$$\int_0^T \left| \int_0^t (t-s)^{-\alpha} f(s) ds \right| dt \leq \int_0^T \left| \int_0^t |t-s|^{-\alpha} f(s) ds \right| dt \leq C(1-\alpha)^{-1} T^{1-\alpha} \|f\|_{L^1}.$$

2.3.2 In Sobolev space

For any, $r \geq 1$ and all $1 \leq p, q \leq +\infty$, from interpolation inequality of Sobolev spaces by Brezis Mirunescu [BM18], we have for any $\varphi : L^p(\mathbb{R}^3, \mathbb{R}^3) \cap W^{1,q}(\mathbb{R}^3, \mathbb{R}^3)$

$$\|\varphi\|_{W^{\gamma,r}} \leq C \|\varphi\|_{W^{0,p}}^\alpha \|\varphi\|_{W^{1,q}}^{1-\alpha} = C \|\varphi\|_{L^p}^\alpha \|\nabla \varphi\|_{L^q}^{1-\alpha}, \quad (2.20)$$

such that

$$\begin{cases} \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}, \\ \gamma = (1-\alpha). \end{cases}$$

Similarly to the Lebesgue space

$$\alpha \frac{q-p}{pq} = \frac{q-r}{rq}, \quad (2.21)$$

then

$$\alpha = \frac{p(q-r)}{r(q-p)}, \quad 1-\alpha = \frac{q(r-p)}{r(q-p)}. \quad (2.22)$$

2.4 New Bihari-LaSalle lemma

Our analysis is strongly based on a Bihari-LaSalle type result, but with a time singularity, similar to the Grönwall Henry lemma see Lemma 7.1.1. [Hen81]. Let us mention similar results in Theorems 30 and 38 in [Dra87], or equivalently p 371 in [MPcF91]. But for the sake of completeness, we provide a suitable statement accompanied with its proof.

Lemma 1. *Let $T \in \mathbb{R}_+$. Assume that there are continuous non-negative functions $\varphi, \psi, a : [0, T] \mapsto \mathbb{R}_+$, where a is a non-decreasing function, such that*

$$\varphi(t) < a(t) + \int_0^t (t-s)^{-1+\gamma} \psi(s) \varphi^\beta(s) ds,$$

for a given $\beta \in [0, 1)$ and $\gamma \in (0, 1]$, then

$$\varphi^*(t) \leq K_t^*(t),$$

with

$$K_t(\tilde{s}) = \left(a^{1-\beta}(t) + \int_0^{\tilde{s}} (t-s)^{-1+\gamma} \psi(s) ds \right)^{\frac{1}{1-\beta}} \quad (2.23)$$

and where φ^* stands for the non-decreasing rearrangement of φ and $K_t^*(t)$ for the non-decreasing rearrangement of $t \mapsto K_t(t)$.

Proof of Lemma 1. The idea of the proof is to compare with the equality case, which allows to differentiate the corresponding non-linear Volterra equation.

First, it is clear that $\varphi(0) < a(0) = \tilde{K}_0(0)$.

Let us suppose that there is a point $\hat{s} \in [0, t)$ such that $\varphi(\hat{s}) > K_t(\hat{s})$, under this assumption we can define

$$t_0 := \inf\{\tilde{s} \in [0, t) : \varphi(\tilde{s}) > \tilde{K}_{\tilde{s}}(\tilde{s})\},$$

by the continuity of φ and \tilde{K} we have $\varphi(t_0) = \tilde{K}_{t_0}(t_0)$. We also get for any $t' \in [0, t_0]$ that $\varphi^*(t') \leq K_{t'}^*(t')$.

We derive from assumption on φ that for any $t' \in [0, t_0]$,

$$\begin{aligned} \varphi(t') &< a(t') + \int_0^{t'} (t' - s)^{-1+\gamma} \psi(s) \varphi^\beta(s) ds \\ &\leq a(t') + \int_0^{t'} ((t' - s)^{-1+\gamma} \psi(s))^* \varphi^{*\beta}(s) ds, \end{aligned}$$

by Hardy Littlewood inequality (2.8). We derive by definition of t_0 ,

$$\begin{aligned} \varphi(t') &< a(t') + \int_0^{t'} ((t' - s)^{-1+\gamma} \psi(s))^* \tilde{K}_s^{*\beta}(s) ds \\ &\leq a(t') + \tilde{K}_{t'}^{*\beta}(t') \int_0^{t'} (t' - s)^{-1+\gamma} \psi(s) ds, \end{aligned}$$

as $\tilde{K}_s^{*\beta}(s)$ is a non-decreasing function[†] and by the L^1 invariance of the non-decreasing rearrangement, see (2.7). We get from (2.23),

$$\begin{aligned} \varphi(t') &< a(t') + \tilde{K}_{t'}^{*\beta}(t') \left(K_{t'}^{1-\beta}(t') - a^{1-\beta}(t') \right) \\ &= a^{1-\beta}(t') (a^\beta(t') - \tilde{K}_{t'}^{*\beta}(t')) + \tilde{K}_{t'}^{*\beta}(t') K_{t'}^{1-\beta}(t') \\ &\leq K_{t_0}^{*\beta}(t_0) K_{t'}^{1-\beta}(t'). \end{aligned}$$

We then deduce by taking the Symmetric increasing rearrangement,

$$\varphi^*(t') < K_{t_0}^{*\beta}(t_0) K_{t'}^{*(1-\beta)}(t').$$

The above inequality is absurd by taking $t' = t_0$, in view of hypothesis[‡] on t_0 , the result then follows. \square

Remark 2. The Grönwall Henry type case, i.e. $\beta = 1$, is involved. Indeed adapting the above analysis leads to consider the upper-bound function

$$K_t(\tilde{s}) = a(t) \exp \left(\int_0^{\tilde{s}} (t - s)^{-1+\gamma} \psi(s) ds \right),$$

and the control of $\varphi(t')$ leads to

$$\varphi(t') < a(t') + K_{t'}^*(t') \left(\ln(K_{t'}(t')) - \ln(a(t')) \right) \leq K_{t_0}^*(t_0) \ln K_{t'}(t').$$

As a by-product, we have the following result.

[†]This is exactly this argument which imposes us to use the monotonous rearrangements.

[‡]Because the $(\eta^p)^* = (\eta^*)^p$ for any function positive η and any $p > 0$.

Corollary 2. Let $T \in \mathbb{R}_+$. Assume that there are continuous non-negative functions $\varphi, \psi, a : [0, T] \mapsto \mathbb{R}_+$, where a is a non-decreasing function, such that

$$\varphi(t) \leq a(t) + \int_0^t (t-s)^{-1+\gamma} \psi(s) \varphi^\beta(s) ds,$$

for a given $\beta \in [0, 1)$ and $\gamma \in (0, 1]$, then for any $r \geq 1 - \beta$ we have

$$\|\varphi\|_{L^r} \leq \|a\|_{L^r} + C \|\psi\|_{L^{\tilde{r}}}^{1-\beta}$$

with $\frac{1}{\tilde{r}} = \frac{1-\beta-\gamma r}{r}$.

Proof. The proof is direct by Hardy Littlewood Sobolev inequality (2.19), and because the norm in L^p , $p > 0$ is invariant under non-decreasing rearrangement, see (2.7). \square

3 Proof of Theorem 5

One of the main novelty of our approach is that we exploit the regularisation by the heat kernel implying some singularities in time, which is not provided by the energy techniques.

3.1 Case $k = 0$

In order to give some tools for the analysis, we begin with $k = 0$, whose result stated below is weaker than the one implied by the case $k = 1$, see Lemma 4 further.

Lemma 2. For all $T > 0$ and $r, \tilde{r} \in (1, +\infty)$, we have $\|u\|_{L^{\tilde{r}}_T L^r} < +\infty$, if

$$1 < \frac{1}{\tilde{r}} + \frac{3}{r}, \forall r > 6,$$

with equality if $r = 6$.

Proof of Lemma 2. By the usual Duhamel formula around the heat equation, we get, for any $(t, x) \in [0, T] \times \mathbb{R}^3$

$$u(t, x) = \tilde{P}u_0(t, x) + \tilde{G}\mathbb{P}f(t, x) - \tilde{G}\mathbb{P}(u \cdot \nabla u)(t, x),$$

see definitions (2.12).

Next, by Minkowski inequality, for any $r \geq 1$

$$\|u(t, \cdot)\|_{L^r} \leq \|u_0\|_{L^r} + \int_0^t \|\mathbb{P}f(s, \cdot)\|_{L^r} ds + \|\nabla \tilde{G}\mathbb{P}(u^{\otimes 2})(t, \cdot)\|_{L^r}. \quad (3.1)$$

The last term in the r.h.s comes from the incompressible property of u (2.4) (i.e. $u \cdot \nabla u = \nabla \cdot (u^{\otimes 2})$), and by the convolution property of the heat semi-group.

From Young inequality for the convolution, with $1 \leq p, q \leq +\infty$, such that

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \quad (3.2)$$

and from Minkowski inequality, we derive

$$\begin{aligned} \|\nabla \tilde{G}\mathbb{P}(u \otimes u)(t, \cdot)\|_{L^r} &\leq C \int_0^t \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \nabla \tilde{p}(s, t, x, y) \mathbb{P}u^{\otimes 2}(s, y) dy \right)^r dx \right)^{\frac{1}{r}} ds \\ &\leq C \int_0^t \|\nabla \tilde{p}(s, t, \cdot, 0)\|_{L^p} \|u^{\otimes 2}(s, \cdot)\|_{L^q} ds \\ &\leq C \int_0^t [\nu(t-s)]^{-\frac{1}{2} - \frac{3(p-1)}{2p}} \|u(s, \cdot)\|_{L^{2q}}^2 ds. \end{aligned} \quad (3.3)$$

The Leray operator disappears in the second inequality thanks to a Calderón-Zygmund estimate, for any $r \in (1, +\infty)$.

Importantly, we can already remark that the term $(t-s)^{-\frac{1}{2}-\frac{3(p-1)}{2p}}$ is integrable if

$$p < \frac{3}{2}. \quad (3.4)$$

Recalling that, from Gagliardo-Nirenberg inequality,

$$\int_0^T \|u(s, \cdot)\|_{L^6}^2 ds \leq C \int_0^T \|\nabla u(s, \cdot)\|_{L^2}^2 ds < +\infty. \quad (3.5)$$

From interpolation in Lebesgue space detailed in Section 2.3.1, for $\|u(s, \cdot)\|_{L^{2r'} L^{2q}}$ between $\|u(s, \cdot)\|_{L^2 L^6}$ and $\|u(s, \cdot)\|_{L^{r'} L^r}$

$$\|u(s, \cdot)\|_{L^{2q}} \leq \|u(s, \cdot)\|_{L^6}^\alpha \|u(s, \cdot)\|_{L^r}^{1-\alpha},$$

with, from (2.16),

$$\alpha = \frac{3(r-2q)}{q(r-6)}, \quad 1-\alpha = \frac{r(2q-6)}{2q(r-6)}. \quad (3.6)$$

Injecting this interpolation inequality into (3.3) yields

$$\|\nabla \tilde{G}\mathbb{P}(u^{\otimes 2})\|_{L^r} \leq C \int_0^t [\nu(t-s)]^{-\frac{1}{2}-\frac{3(p-1)}{2p}} \|u(s, \cdot)\|_{L^r}^{2(1-\alpha)} \|u(s, \cdot)\|_{L^6}^{2\alpha} ds. \quad (3.7)$$

With this inequality, we are tempted to use a kind of Bihari-LaSalle inequality. We need to state a new inequality adapted to the current situation stated in Lemma 1.

To use this lemma, we have to suppose that $\|u(s, \cdot)\|_{L^r}, \|u(s, \cdot)\|_{L^6}$ are continuous in s , which is guaranteed in the mollified equation case (3.8) stated further. In the future arguments based on Lemma 1, we keep using the same argument by regularisation.

In order to check the conditions, required for the use of this Bihari-LaSalle type lemma, we have to separate different possibilities:

If $r > 6$, in order to use Lemma 1, the contribution of $\|u(s, \cdot)\|_{L^r}$ has to satisfy

$$2(1-\alpha) = 2\frac{r(q-3)}{q(r-6)} < 1, \Leftrightarrow \frac{1}{r} < \frac{1}{q} - \frac{1}{6}.$$

Hence from (3.2), we have the condition

$$\frac{1}{r} < \left(1 + \frac{1}{r} - \frac{1}{p}\right) - \frac{1}{6} \Leftrightarrow p > \frac{6}{5}.$$

If $r < 6$, again from (3.2), we have to suppose $p < \frac{6}{5}$.

We can combine (3.1) and (3.7)

$$\|u(t, \cdot)\|_{L^r} \leq \|u_0\|_{L^r} + \int_0^t \|\mathbb{P}f(s, \cdot)\|_{L^r} + C \int_0^t [\nu(t-s)]^{-\frac{1}{2}-\frac{3(p-1)}{2p}} \|u(s, \cdot)\|_{L^r}^{2(1-\alpha)} \|u(s, \cdot)\|_{L^6}^{2\alpha} ds,$$

and from Lemma 1,

$$\|u(t, \cdot)\|_{L^r}^* \leq \left(\left(\|u_0\|_{L^r} + \int_0^t \|\mathbb{P}f(s, \cdot)\|_{L^r} + C \int_0^t [\nu(t-s)]^{-\frac{1}{2}-\frac{3(p-1)}{2p}} \|u(s, \cdot)\|_{L^6}^{2\alpha} ds \right)^{\frac{1}{1-2(1-\alpha)}} \right)^*.$$

We can freely remove the non-decreasing rearrangement symbol $*$ by equality of the Lebesgue norm (2.7). To permute the time Lebesgue norm, we use Hardy-Littlewood-Sobolev inequality (2.19).

That is to say, we derive from Corollary 2, for $\tilde{r} > 2\alpha - 1$,

$$\|u\|_{L^{\tilde{r}}_T L^r} \leq CT^{\frac{1}{\tilde{r}}} \|u_0\|_{L^r} + CT^{\frac{1}{\tilde{r}}} \int_0^T \|\mathbb{P}f(s, \cdot)\|_{L^r} ds + C\nu^{-2+\frac{3}{2p}} \|u\|_{L^{2\alpha r'} L^6}^2,$$

with $\frac{1}{r'} + \frac{1}{2} + \frac{3(p-1)}{2p} = 1 + \frac{1-2(1-\alpha)}{\tilde{r}} \Leftrightarrow \frac{1}{r'} - \frac{3}{2p} = -1 + \frac{6r-6q-qr}{q(r-6)\tilde{r}}$.

And we have to impose that $2\alpha r' = 2$ (needed in (1.4)), namely $r' = \frac{1}{\alpha} = \frac{q(r-6)}{3(r-2q)}$, hence

$$\frac{3(r-2q)}{q(r-6)} - \frac{3}{2p} = -1 + \frac{6r-6q-qr}{q(r-6)\tilde{r}},$$

and by (3.2), we obtain

$$3r\left(1 + \frac{1}{r} - \frac{1}{p}\right) - 6 - \frac{3(r-6)}{2p} = -(r-6) + \frac{6r\left(1 + \frac{1}{r} - \frac{1}{p}\right) - (6+r)}{\tilde{r}}.$$

This is equivalent, for $p \neq \frac{3(3r-6)}{2(4r-9)} = \frac{9(r-2)}{2(4r-9)}$, to

$$\tilde{r} = \frac{2r(5p-6)}{8rp-9r-18p+18} =: F_1(r, p).$$

Differentiate this function by p gives

$$\partial_p F_1(r, p) = \frac{3(r-6)r}{(8rp-9r-18p+18)^2}.$$

The sign of the above derivative then depends on the position of r relative to 6.

- If $r < 6$, then[§]

$$\tilde{r} \leq F_1(r, 1) = 2,$$

which is weaker than the result obtained by the direct interpolation between $L^\infty([0, T], L^2(\mathbb{R}^3, \mathbb{R}^3))$ and $L^2([0, T], H^1(\mathbb{R}^3, \mathbb{R}^3))$:

$$\|u(t, \cdot)\|_{L^r} \leq \|u(t, \cdot)\|_{L^2}^\alpha \|u(t, \cdot)\|_{L^6}^{1-\alpha},$$

with $\frac{1}{r} = \frac{\alpha}{2} + \frac{1-\alpha}{6} \Leftrightarrow \alpha = \frac{6-r}{2r}$. So $\|u(t, \cdot)\|_{L^r}$ lies in the Lebesgue space $L^{\tilde{r}}_T$ with $\tilde{r} = \frac{2}{1-\alpha} = \frac{4r}{3(r-2)}$, which equivalently write

$$\frac{4}{\tilde{r}} + \frac{6}{r} = 3.$$

- If $r = 6$, then, for any $p \geq 1$,

$$\tilde{r} = F_1(6, p) = 2,$$

which matches with the existing result (1.4).

- If $r > 6$, from (3.4), $\frac{6}{5} < p < \frac{3}{2}$, then

$$\tilde{r} < F_1\left(r, \frac{3}{2}\right) = \frac{r}{r-3},$$

namely

$$\frac{1}{\tilde{r}} + \frac{3}{r} > 1.$$

For the limit case $r = 6$, the above identity is weaker than in the known $L^2_T L^6$ control, still $r > 6$ is out of this usual scope. Lemma 4 below explains how to turn the above inequality into an equality. \square

[§]For $p = 1$, we indeed have $1 \neq \frac{9(r-2)}{2(4r-9)} \Leftrightarrow r \neq 0$.

3.2 Case $k = 1$

To perform the same analysis with an extra derivative, we develop an other representation of u based on a *proxy* taking account of the transport part of equation (1.2) *via* the associated flow of u .

To give a meaning of the flow, we first mollify Navier-Stokes equation (1.2), for each $n \in \mathbb{N}$,

$$\begin{cases} \partial_t u^n(t, x) + \mathbb{P}[u_n^n \cdot \nabla u^n](t, x) = \nu \Delta u^n(t, x) + \mathbb{P}f(t, x), \\ u^n(0, x) = u_0(x), x \in \mathbb{R}^3, \end{cases} \quad (3.8)$$

where, for a given $m \in \mathbb{N}$, u_n^m stands for a mollification of u_n such that $u_n^m \in C_b^\infty([0, T] \times \mathbb{R}^3, \mathbb{R}^3)$ and by Leray [Ler34] we know that u_n^m converges towards u solution to (1.2) in $L^\infty([0, T], L^2(\mathbb{R}^3, \mathbb{R}^3)) \cap L^2([0, T], L^2(\mathbb{R}^3, \mathbb{R}^3))$.

For the sake of simplicity, we omit the index n in the following analysis. All the considered upper-bounds in the current article do not depend on n , which *in fine* allows to pass to the limit as n tends to infinity.

Lemma 3. *For all $T > 0$, $(r, \tilde{r}) \in (1, +\infty)^2$, we obtain $\|\nabla u\|_{L_T^{\tilde{r}} L^r} < +\infty$, if*

$$\begin{cases} \tilde{r} = \frac{r(r-4)}{4r^2-13r+6}, \text{ if } r \in [3, 6], \\ \frac{1}{\tilde{r}} + \frac{3}{r} = 2, \text{ if } r \in [2, 3]. \end{cases} \quad (3.9)$$

Unexpected, we also see that the time integrable $\tilde{r} = \frac{r(r-4)}{4r^2-13r+6}$ is non-decreasing for $4 < r \leq 6$.

As a consequence of the above result, we derive the equality case for $k = 0$ by Gagliardo-Nirenberg.

Lemma 4. *For all $T > 0$, $(r, \tilde{r}) \in (1, +\infty)^2$, we have $\|u\|_{L_T^{\tilde{r}} L^r} < +\infty$, if*

$$\frac{1}{\tilde{r}} + \frac{3}{r} = 1, \text{ if } r \in [6, +\infty]. \quad (3.10)$$

Proof of Lemma 4. From Gagliardo-Nirenberg inequality, we have for any $r' \geq 1$,

$$\|u\|_{L_T^{\tilde{r}} L^{r'}} \leq C \|\nabla u\|_{L_T^{\tilde{r}} L^r},$$

with

$$\frac{1}{r'} = \frac{1}{r} - \frac{1}{3}.$$

Then the result of Lemma 3 for $r \in [2, 3]$ becomes

$$\frac{1}{\tilde{r}} + \frac{3}{r'} = 1, \text{ for } r' \in [6, +\infty].$$

□

The following subsections are dedicated to the proof of Lemma 3.

3.2.1 Proxy

For any *freezing* point $(\tau, \xi) \in [0, T] \times \mathbb{R}^3$, we define the flow associated to u by

$$\theta_{s,\tau}(x) := x + \int_s^\tau u(\tilde{s}, \theta_{\tilde{s},\tau}(x)) d\tilde{s}, \quad s \in [0, \tau], \quad (3.11)$$

which write equivalently, for any $t \in [0, \tau]$, by

$$\dot{\theta}_{t,\tau}(\xi) = -u(t, \theta_{t,\tau}(\xi)), \quad \theta_{\tau,\tau}(\xi) = \xi.$$

Let us carefully precise that there is *a priori* no unique solution to ODE (3.11), as no Cauchy-Lipschitz and even no Cauchy-Peano theorem can be straightly applied here. However, we can rewrite equation (3.11) with a mollified[¶] version of u , as done in [Ler34]. From Section IV.4 in [Bre99], we know that the mollified u converges forward to u in $L_T^\infty L^2$.

For the sake of simplicity, we get ride of this regularisation index which is only useful in the definition of the flow $\theta_{s,t}$. The remaining of the analysis does not depend on other estimates than those provided in (1.3) and (1.4). The limit of the mollification is direct after the following computations.

Thanks to this notation, we can rewrite equation (1.2) by

$$\partial_t u(t, x) + u(t, \theta_{t,\tau}(\xi)) \cdot \nabla u(t, x) - \nu \Delta u(t, x) = u_\Delta[\tau, \xi] \cdot \nabla u(t, x) + \Xi[u \cdot \nabla u](t, x) + \mathbb{P}f(t, x), \quad (3.12)$$

with

$$u_\Delta[\tau, \xi](t, x) := u(t, \theta_{t,\tau}(\xi)) - u(t, x). \quad (3.13)$$

We can deduce that the fundamental solution associated with the l.h.s. of (3.12) is the probability density,

$$\hat{p}^{\tau,\xi}(s, t, x, y) := \frac{1}{(4\pi\nu(t-s))^{\frac{3}{2}}} \exp\left(-\frac{\left|x + \int_s^t u(\tilde{s}, \theta_{\tilde{s},\tau}(\xi)) d\tilde{s} - y\right|^2}{4\nu(t-s)}\right). \quad (3.14)$$

From definition (3.11), if $\xi = x$, we identify

$$\hat{p}^{t,x}(s, t, x, y) = \frac{1}{(4\pi\nu(t-s))^{\frac{3}{2}}} \exp\left(-\frac{|\theta_{s,t}(x) - y|^2}{4\nu(t-s)}\right).$$

For each $\alpha \in \mathbb{N}^3$, there exists a constant $C_\alpha > 1$ such that, for each derivative of order α ,

$$\begin{aligned} |D^\alpha \hat{p}^{\tau,\xi}(s, t, x, y)| &\leq \frac{C_\alpha [\nu(s-t)]^{-\frac{|\alpha|}{2}}}{(4\pi\nu(t-s))^{\frac{3}{2}}} \exp\left(-C_\alpha^{-1} \frac{\left|x + \int_s^t u(\tilde{s}, \theta_{\tilde{s},\tau}(\xi)) d\tilde{s} - y\right|^2}{4\nu(t-s)}\right) \\ &=: C[\nu(s-t)]^{-\frac{|\alpha|}{2}} \bar{p}^{\tau,\xi}(s, t, x, y). \end{aligned} \quad (3.15)$$

Similarly, for any $\gamma \in [0, 1]$,

$$\begin{aligned} &|D^\alpha \hat{p}^{\tau,\xi}(s, t, x, y)| \times \left|y - x - \int_s^t u(\tilde{s}, \theta_{\tilde{s},\tau}(\xi)) d\tilde{s}\right|^\gamma \Big|_{(\tau,\xi)=(t,x)} \\ &= |D^\alpha \hat{p}^{\tau,\xi}(s, t, x, y)| \Big|_{(\tau,\xi)=(t,x)} \times |y - \theta_{s,t}(x)|^\gamma \\ &\leq C[\nu(s-t)]^{-\frac{|\alpha|}{2} + \frac{\gamma}{2}} \bar{p}^{t,x}(s, t, x, y). \end{aligned} \quad (3.16)$$

Also, for any $0 \leq s < t$, that $\hat{p}^{\tau,\xi}(s, t, x, y)$ is the fundamental solution of

$$\partial_t \hat{p}^{\tau,\xi}(s, t, x, y) = \nu \Delta \hat{p}^{\tau,\xi}(s, t, x, y) - \langle u(t, \theta_{t,\tau}(\xi)), \nabla \hat{p}^{\tau,\xi}(s, t, x, y) \rangle.$$

We can readily see the link with the classical heat kernel:

$$\tilde{p}(s, t, x, y) := \hat{p}^{\tau,\xi}\left(s, t, x - \int_s^t u(\tilde{s}, \theta_{\tilde{s},\tau}(\xi)) d\tilde{s}, y\right) = \frac{1}{(4\pi\nu(t-s))^{\frac{3}{2}}} \exp\left(-\frac{|x-y|^2}{4\nu(t-s)}\right). \quad (3.17)$$

We then write, the Duhamel formula corresponding to (3.12),

$$u(t, x) = \hat{P}^{\tau,\xi} u_0(t, x) + \hat{G}^{\tau,\xi} \mathbb{P}f(t, x) + \hat{G}^{\tau,\xi}(u_\Delta[\tau, \xi] \cdot \nabla u)(t, x) + \hat{G}^{\tau,\xi} \Xi[u \cdot \nabla u](t, x), \quad (3.18)$$

[¶]Of course we could suppose that the final time $T > 0$ is small enough to get a solution u smooth by [Ose11] [Ose12].

with, for all $0 \leq t \leq T$ and $x \in \mathbb{R}^3$,

$$\hat{G}^{\tau,\xi} f(t, x) := \int_0^t \int_{\mathbb{R}^3} \hat{p}^{\tau,\xi}(s, t, x, y) f(s, y) dy ds, \quad (3.19)$$

and the semi-group is defined by

$$\hat{P}^{\tau,\xi} g(t, x) := \int_{\mathbb{R}^3} \hat{p}^{\tau,\xi}(0, t, x, y) g(y) dy. \quad (3.20)$$

It is crucial to observe that u does not depend on the *freezing* point (τ, ξ) , which allows to pick the most suitable.

With these notations, we can also establish a new Duhamel formula for the vorticity defined by

$$\omega := \nabla \times u. \quad (3.21)$$

3.2.2 Duhamel formula of $\omega = \nabla \times u$

From the representation (3.18), from definitions (3.13), (3.19), and the incompressible property (2.4), we can write by integration by parts,

$$\begin{aligned} \hat{G}^{\tau,\xi}(u_\Delta[\tau, \xi] \cdot \nabla u)(t, x) &= \int_0^t \int_{\mathbb{R}^3} \hat{p}^{\tau,\xi}(s, t, x, y) [u(s, \theta_{s,\tau}(\xi)) - u(s, y)] \cdot \nabla u(s, y) dy ds \\ &= \int_0^t \int_{\mathbb{R}^3} \nabla_y \hat{p}^{\tau,\xi}(s, t, x, y)|_{(\tau,\xi)=(t,x)} [u(s, \theta_{s,\tau}(\xi)) - u(s, y)]^{\otimes 2} dy ds. \end{aligned} \quad (3.22)$$

Next, we can take the curl operator in the Duhamel formula implying the representation of the vorticity

$$\begin{aligned} \omega(t, x) &= \nabla \times u(t, x) \\ &= \nabla \times \hat{P}^{\tau,\xi} u_0(t, x) + \nabla \times \hat{G}^{\tau,\xi} f(t, x) + \nabla \times \hat{G}^{\tau,\xi} u_\Delta[\tau, \xi] \cdot \nabla u(t, x) + \nabla \times \hat{G}^{\tau,\xi} \Xi(u \cdot \nabla u)(t, x). \end{aligned}$$

But from the convolution property,

$$\nabla \times \hat{G}^{\tau,\xi} \Xi(u \cdot \nabla u)(t, x) = \int_0^t \int_{\mathbb{R}^3} \hat{p}^{\tau,\xi}(s, t, 0, y) \nabla_x \times \Xi(u \cdot \nabla u)(s, y + x) dy ds = 0,$$

recalling that $\nabla \times \Xi = -\nabla \times \nabla(-\Delta)^{-1} \nabla \cdot = 0$.

Therefore, by the property of the Green operator (3.22),

$$\begin{aligned} \omega(t, x) &= \hat{P}^{\tau,\xi} \nabla \times u_0(t, x) + \hat{G}^{\tau,\xi} \nabla \times f(t, x) \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \nabla_x \times \left(\nabla_y \hat{p}^{\tau,\xi}(s, t, x, y) [u(s, \theta_{s,\tau}(\xi)) - u(s, y)]^{\otimes 2} \right) dy ds. \end{aligned} \quad (3.23)$$

We first remark that

$$\begin{aligned} &\nabla_x \times \left(\nabla_y \hat{p}^{\tau,\xi}(s, t, x, y) [u(s, \theta_{s,\tau}(\xi)) - u(s, y)]^{\otimes 2} \right) \\ &= \left(\begin{aligned} &\partial_{x_2} \nabla_y \hat{p}^{\tau,\xi}(s, t, x, y) ([u(s, \theta_{s,\tau}(\xi)) - u(s, y)]^{\otimes 2})_3 - \partial_{x_3} \nabla_y \hat{p}^{\tau,\xi}(s, t, x, y) ([u(s, \theta_{s,\tau}(\xi)) - u(s, y)]^{\otimes 2})_2 \\ &\partial_{x_3} \nabla_y \hat{p}^{\tau,\xi}(s, t, x, y) ([u(s, \theta_{s,\tau}(\xi)) - u(s, y)]^{\otimes 2})_1 - \partial_{x_1} \nabla_y \hat{p}^{\tau,\xi}(s, t, x, y) ([u(s, \theta_{s,\tau}(\xi)) - u(s, y)]^{\otimes 2})_3 \\ &\partial_{x_1} \nabla_y \hat{p}^{\tau,\xi}(s, t, x, y) ([u(s, \theta_{s,\tau}(\xi)) - u(s, y)]^{\otimes 2})_2 - \partial_{x_2} \nabla_y \hat{p}^{\tau,\xi}(s, t, x, y) ([u(s, \theta_{s,\tau}(\xi)) - u(s, y)]^{\otimes 2})_1 \end{aligned} \right). \end{aligned} \quad (3.24)$$

Next, after differentiating by the curl, we choose naturally the *freezing* point $(\tau, \xi) = (t, x)$,

$$\begin{aligned} \omega(t, x) &= \hat{P}^{t,x} \nabla \times u_0(t, x) + \hat{G}^{t,x} \nabla \times f(t, x) \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \nabla_x \times \left(\nabla_y \hat{p}^{\tau,\xi}(s, t, x, y) [u(s, \theta_{s,\tau}(\xi)) - u(s, y)]^{\otimes 2} \right) \Big|_{(\tau,\xi)=(t,x)} dy ds. \end{aligned} \quad (3.25)$$

We insist on the fact that we do not differentiate with respect to the variable ξ , given that we pick $\xi = x$ after differentiation and that neither u nor ω depends on (τ, ξ) .

Remark 3. Another way to get identity (3.25) could be to directly handle with the vorticity equation

$$\partial_t \omega(t, x) + u(t, \theta_{t,\tau}(\xi)) \cdot \nabla \omega(t, x) = \nabla \times [u_\Delta[\tau, \xi] \cdot \nabla u](t, x) + \nu \Delta \omega(t, x) + \nabla \times f(t, x). \quad (3.26)$$

To obtain (3.25), it suffices to integrate by part the contribution $\nabla \times [u_\Delta[\tau, \xi] \cdot \nabla u](t, y)$, next to impose $(\tau, \xi) = (t, x)$.

3.2.3 Control in L^r of $\nabla \times u$

Finally, by property (3.24), we can readily derive

$$\begin{aligned} |\omega(t, x)| &\leq |\hat{P}^{t,x} \nabla \times u_0(t, x)| + |\hat{G}^{t,x} \nabla \times f(t, x)| \\ &\quad + 2 \int_0^t \int_{\mathbb{R}^3} |\nabla_y^2 \hat{p}^{t,x}(s, t, x, y)| |u(s, \theta_{s,t}(x)) - u(s, y)|^2 dy ds \\ &\leq |\hat{P}^{t,x} \nabla \times u_0(t, x)| + |\hat{G}^{t,x} \nabla \times f(t, x)| \\ &\quad + C \int_0^t [\nu(t-s)]^{-1} \int_{\mathbb{R}^3} \bar{p}^{t,x}(s, t, x, y) |u(s, \theta_{s,t}(x)) - u(s, y)|^2 dy ds. \end{aligned} \quad (3.27)$$

By taking the norm L^q , $q \geq 1$, the last term in the r.h.s. becomes by Minkowski inequality

$$\begin{aligned} &\left\| x \mapsto \int_0^t [\nu(t-s)]^{-1} \int_{\mathbb{R}^3} \bar{p}^{t,x}(s, t, x, y) |u(s, \theta_{s,t}(x)) - u(s, y)|^2 dy ds \right\|_{L^r} \\ &\leq C \int_0^t [\nu(t-s)]^{-1} \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \bar{p}^{t,x}(s, t, x, y) |u(s, \theta_{s,t}(x)) - u(s, y)|^2 dy \right)^r dx \right)^{\frac{1}{r}} ds \\ &\leq C \int_0^t [\nu(t-s)]^{-1} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \bar{p}^{t,x}(s, t, x, y) |u(s, \theta_{s,t}(x)) - u(s, y)|^{2r} dy dx \right)^{\frac{1}{r}} ds, \end{aligned}$$

by Jensen inequality (the function $\bar{p}^{t,x}(s, t, x, y)$ being a probability density).

By change of variable $x' = \theta_{s,t}(x)$, we also get

$$\begin{aligned} &\left\| x \mapsto \int_0^t [\nu(t-s)]^{-1} \int_{\mathbb{R}^3} \bar{p}^{t,x}(s, t, x, y) |u(s, \theta_{s,t}(x)) - u(s, y)|^2 dy ds \right\|_{L^r} \\ &\leq C \int_0^t [\nu(t-s)]^{-1} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \tilde{p}(s, t, x, y) |u(s, x) - u(s, y)|^{2r} dx dy \right)^{\frac{1}{r}} ds, \end{aligned} \quad (3.28)$$

recalling that \tilde{p} is the usual heat kernel (3.17), because the Jacobian associated is equal to 1, as u is incompressible, see for instance [Che98] Lemma 1.1.1.

To make integrable the time singularity, we make appearing the Sobolev–Slobodeckij norm, for $\gamma \in (0, 1)$, $\tilde{q} \in [1, +\infty)$, defined by

$$[u(s, \cdot)]_{W^{\gamma, \tilde{q}}} := \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(s, x) - u(s, y)|^{\tilde{q}}}{|x - y|^{3+\gamma \tilde{q}}} dy dx \right)^{\frac{1}{\tilde{q}}}, \quad (3.29)$$

see [Tri83] Section 2.2.2.

The norm

$$\|u(s, \cdot)\|_{W^{\gamma, \tilde{q}}} := \|u(s, \cdot)\|_{L^{\tilde{q}}} + [u(s, \cdot)]_{W^{\gamma, \tilde{q}}} \quad (3.30)$$

stands for the non-homogeneous Sobolev norm.

We then can write,

$$\begin{aligned} &\left\| x \mapsto \int_0^t [\nu(t-s)]^{-1} \int_{\mathbb{R}^3} \bar{p}^{t,x}(s, t, x, y) |u(s, \theta_{s,t}(x)) - u(s, y)|^2 dy ds \right\|_{L^r} \\ &\leq C \int_0^t [\nu(t-s)]^{-1} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \tilde{p}(s, t, x, y) |x - y|^{3+2r\gamma} \frac{|u(s, x) - u(s, y)|^{2r}}{|x - y|^{3+2r\gamma}} dy dx \right)^{\frac{1}{r}} ds \\ &\leq C \int_0^t [\nu(t-s)]^{-1+\gamma} \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(s, x) - u(s, y)|^{2r}}{|x - y|^{3+2r\gamma}} dy dx \right)^{\frac{1}{r}} ds \\ &\leq C \int_0^t [\nu(t-s)]^{-1+\gamma} [u(s, \cdot)]_{W^{\gamma, 2r}}^2 ds. \end{aligned} \quad (3.31)$$

The penultimate inequality comes from the exponential absorption (2.13) of the heat kernel, i.e.

$$\tilde{p}(s, t, x, y)|x - y|^{3+2r\gamma} \leq C\tilde{p}(s, t, Cx, Cy)[\nu(t - s)]^{\frac{3}{2}+r\gamma} \leq C[\nu(t - s)]^{r\gamma}.$$

In fine, we can deduce the result of Lemma 6 for ∇u in the following section.

3.2.4 Control of $\|\nabla u\|_{L_T^{\tilde{r}}L^r}$

We can handle with $\|\nabla u\|_{L_T^{\tilde{r}}L^r}$ instead of $\|\omega\|_{L_T^{\tilde{r}}L^r}$ by usual Gagliardo-Nirenberg inequality as soon as $r \in (1, +\infty)$.

For the Lebesgue space in time, we have to be more careful than in Section 5.1.1. Indeed, an *extra* interpolation argument is required, because there is no margin in the previous analysis in the time integrability as we upper-bound by $\|\nabla u(s, \cdot)\|_{L^2}^2$.

We have to rewrite the Sobolev norm $\|u(s, \cdot)\|_{W^{\gamma, 2r}}$ appearing in (3.31):

$$\begin{aligned} & \left\| (t, x) \mapsto \int_0^t [\nu(t - s)]^{-1} \int_{\mathbb{R}^3} \tilde{p}^{t,x}(s, t, x, y) |u(s, \theta_{s,t}(x)) - u(s, y)|^2 dy ds \right\|_{L^r} \\ & \leq C \int_0^t [\nu(t - s)]^{-1+\gamma} \|u(s, \cdot)\|_{W^{\gamma, 2r}}^2 ds. \end{aligned}$$

• If $r \leq 3$.

From interpolation inequality of Sobolev spaces by Brezis Mirunescu [BM18], we have from (2.20),

$$\|u(s, \cdot)\|_{W^{\gamma, 2r}} \leq C \|u(s, \cdot)\|_{W^{0,6}}^\alpha \|u(s, \cdot)\|_{W^{1,r}}^{1-\alpha} = C \|u(s, \cdot)\|_{L^6}^\alpha \|\nabla u(s, \cdot)\|_{L^r}^{1-\alpha},$$

such that from (2.22),

$$\alpha = \frac{3}{(6-r)} \quad \text{and} \quad 1 - \alpha = \frac{3-r}{(6-r)},$$

also $2\alpha - 1 = \frac{r}{6-r}$.

Let us remark that $1 - \alpha < \frac{1}{2}$. Therefore,

$$\begin{aligned} \|\nabla u(t, \cdot)\|_{L^r} & \leq \|\nabla \times u_0\|_{L^r} + \int_0^t [\nu(t - s)]^{-\frac{1}{2}} \|f(s, \cdot)\|_{L^r} ds \\ & \quad + \left\| x \mapsto \int_0^t [\nu(t - s)]^{-1} \int_{\mathbb{R}^3} \tilde{p}^{t,x}(s, t, x, y) |u(s, \theta_{s,t}(x)) - u(s, y)|^2 dy ds \right\|_{L^r} \\ & \leq \|\nabla \times u_0\|_{L^r} + \int_0^t [\nu(t - s)]^{-\frac{1}{2}} \|f(s, \cdot)\|_{L^r} ds \\ & \quad + C\nu^{-1+\gamma} \int_0^t (t - s)^{-1+\gamma} \|u(s, \cdot)\|_{L^6}^{2\alpha} \|\nabla u(t - \tilde{s}, \cdot)\|_{L^r}^{2(1-\alpha)} ds. \end{aligned}$$

By Lemma 1, we derive

$$\begin{aligned} & \|\nabla u(t, \cdot)\|_{L^r}^* \\ & \leq \left(\left(\|\nabla \times u_0\|_{L^r} + \int_0^t [\nu(t - s)]^{-\frac{1}{2}} \|f(s, \cdot)\|_{L^r} ds \right)^{2\alpha-1} + C\nu^{-1+\gamma} \int_0^t (t - s)^{-1+\gamma} \|u(s, \cdot)\|_{L^6}^{2\alpha} ds \right)^{\frac{1}{2\alpha-1}}. \end{aligned}$$

Hence, we get by Hardy-Littlewood-Sobolev inequality (2.19),

$$\|\nabla u\|_{L_T^{\tilde{r}}L^r} \leq CT^{\frac{1}{\tilde{r}}} \|\nabla \times u_0\|_{L^r} + CT^{\frac{1}{2}} \nu^{-\frac{1}{2}} \|f\|_{L_T^{\tilde{r}}L^r} + C\nu^{-1+\gamma} \|u(s, \cdot)\|_{L^{2\alpha r'}L^6}^{\frac{1}{2\alpha-1}}.$$

To use inequality of $\|\nabla u\|_{L_T^2L^2}$ in (1.4), we have to impose that

$$r' = \frac{1}{\alpha} = \frac{6-r}{3},$$

then from (2.19)

$$\frac{3}{6-r} + \frac{3}{6-r} = 1 + \frac{1}{\frac{\tilde{r}}{2\alpha-1}} \Leftrightarrow \frac{1}{\tilde{r}} = 1.$$

Namely, we get $\|\nabla u\|_{L_T^1 L^r} \leq C\|\omega\|_{L_T^1 L^r}$, $1 < r \leq 3$. In particular, the most powerful case is $r = 3$, i.e. $\|\nabla u\|_{L_T^1 L^3} < +\infty$ (implying, by Sobolev embedding, that $\|u\|_{L_T^1 L^\infty} < +\infty$).

To get a time norm $\frac{\tilde{r}}{2\alpha-1} \geq 1$, required in (2.19), we must have

$$\tilde{r} \geq 2\alpha - 1 = \frac{r}{6-r} \Leftrightarrow \frac{1}{\tilde{r}} \leq \frac{6}{r} - 1, \quad (3.32)$$

which is indeed true for $\tilde{r} = 1$ and $r = 3$.

Finally, by an extra interpolation, we get, for any $r \in [2, 3]$,

$$\|\nabla u(t, \cdot)\|_{L^r} \leq C\|\nabla u(t, \cdot)\|_{L^2}^\alpha \|\nabla u(t, \cdot)\|_{L^3}^{1-\alpha},$$

with

$$\frac{1}{r} = \frac{\alpha}{2} + \frac{1-\alpha}{3} \Leftrightarrow \frac{\alpha}{6} = \frac{3-r}{3r} \Leftrightarrow \alpha = \frac{2(3-r)}{r},$$

and by Hölder's inequality

$$\|\nabla u(t, \cdot)\|_{L_T^{\tilde{r}} L^r} \leq C\|\nabla u(t, \cdot)\|_{L_T^2 L^2}^{p\alpha} \|\nabla u(t, \cdot)\|_{L_T^1 L^3}^{q(1-\alpha)},$$

with $p^{-1} + q^{-1} = 1$, such that $p\alpha\tilde{r} = 2$ and $q(1-\alpha)\tilde{r} = 1$.

From double interpolation (2.18),

$$\tilde{r} = \frac{r}{2r-3},$$

That is to say

$$\frac{1}{\tilde{r}} + \frac{3}{r} = 2.$$

- If $r > 3$, and if $r \in (4, 6)$

From interpolation inequality of Sobolev spaces by Brezis Mirunescu (2.20), we can write

$$\|u(s, \cdot)\|_{W^{\gamma, 2r}} \leq C\|u(s, \cdot)\|_{W^{0, \infty}}^\alpha \|u(s, \cdot)\|_{W^{1, 2}}^{1-\alpha} = C\|u(s, \cdot)\|_{L^\infty}^\alpha \|\nabla u(s, \cdot)\|_{L^2}^{1-\alpha},$$

such that from (2.22),

$$\alpha = \frac{r-1}{r} \quad \text{and} \quad 1-\alpha = \frac{1}{r}.$$

Moreover, from Gagliardo-Nirenberg inequality,

$$\|u(t, \cdot)\|_{L^\infty} \leq C\|\nabla u(t, \cdot)\|_{L^r}^{\alpha'} \|u(t, \cdot)\|_{L^p}^{1-\alpha'},$$

with

$$0 = \left(\frac{1}{r} - \frac{1}{3}\right)\alpha' + \frac{1-\alpha'}{p},$$

which equivalently write

$$\alpha' = \frac{3r}{rp + 3r - 3p}.$$

If $p = 6$, we get $\alpha' = \frac{3r}{9r-18} = \frac{r}{3(r-2)}$.

To apply Lemma 1, we need to impose $2\alpha\alpha' < 1$, in other word

$$1 > 2\frac{r-1}{r} \frac{3r}{rp + 3r - 3p} \Leftrightarrow r > 3 + \frac{3}{p-3}. \quad (3.33)$$

In particular, if $p = 6$, then $r > 3\frac{6-2}{6-3} = 4$.

Next, we obtain

$$\begin{aligned} \|\nabla u(t, \cdot)\|_{L^r} &\leq \|\nabla \times u_0\|_{L^r} + C \int_0^t [\nu(t-s)]^{-\frac{1}{2}} \|f(s, \cdot)\|_{L^r} ds \\ &\quad + C\nu^{-1+\gamma} \int_0^t (t-s)^{-1+\gamma} \|u(s, \cdot)\|_{L^p}^{2\alpha(1-\alpha')} \|\nabla u(s, \cdot)\|_{L^2}^{2(1-\alpha)} \|\nabla u(s, \cdot)\|_{L^r}^{2\alpha\alpha'} ds. \end{aligned}$$

By Lemma 1, we derive for $r > 4$,

$$\begin{aligned} \|\nabla u(t, \cdot)\|_{L^r}^* &\leq \left(\left(\|\nabla \times u_0\|_{L^r} + C \int_0^t [\nu(t-s)]^{-\frac{1}{2}} \|f(s, \cdot)\|_{L^r} ds \right)^{2\alpha-1} \right. \\ &\quad \left. + C\nu^{-1+\gamma} \int_0^t (t-s)^{-1+\gamma} \|u(s, \cdot)\|_{L^p}^{2\alpha(1-\alpha')} \|\nabla u(s, \cdot)\|_{L^2}^{2(1-\alpha)} ds \right)^{\frac{1}{1-2\alpha\alpha'}}. \end{aligned} \quad (3.34)$$

Now, let us impose $p = 6$, such that, up to a Gagliardo-Nirenberg inequality, the contribution of the term $\|\nabla u(s, \cdot)\|_{L^2}$ is

$$2\alpha(1-\alpha') + 2(1-\alpha) = 2(1-\alpha\alpha') = 2\left(1 - \frac{r-1}{r} \frac{r}{3(r-2)}\right) = 2\frac{2r-5}{3(r-2)}.$$

Hence, we get by Corollary 2, using Hardy-Littlewood-Sobolev inequality (2.19) with the same notations, recalling that $\gamma = \frac{1}{r}$:

$$\|\nabla u\|_{L_{\tilde{r}}^{\tilde{r}} L^r} \leq CT^{\frac{1}{\tilde{r}}} \|\nabla \times u_0\|_{L^r} + CT^{\frac{1}{2}} \nu^{-\frac{1}{2}} \|f(s, \cdot)\|_{L_{\tilde{r}}^{\tilde{r}} L^r} ds + C\nu^{-1+\gamma} \|u(s, \cdot)\|_{L^{2(1-\alpha\alpha')}}^{2(1-\alpha\alpha')} \|u(s, \cdot)\|_{L^6}^{2\alpha\alpha'}.$$

From (1.4), we have to impose that

$$r' = \frac{1}{(1-\alpha\alpha')} = \frac{3(r-2)}{(2r-5)},$$

then from (2.19)

$$\frac{(2r-5)}{3(r-2)} - \frac{1}{r} = \frac{1-2\alpha\alpha'}{\tilde{r}} = \frac{\frac{r-4}{3r-6}}{\tilde{r}}.$$

Hence,

$$\frac{1}{\tilde{r}} = \frac{3r-6}{r-4} \frac{2r^2-5r-3r+6}{3r(r-2)} = \frac{2(r-1)(r-3)}{r(r-4)},$$

namely

$$\tilde{r} = \frac{r(r-4)}{4r^2-13r+6} > 0, \quad (3.35)$$

as soon as $r > 4$.

Furthermore, the time integration index in (3.34) is ≥ 1 if

$$\tilde{r} \geq 1 - 2\alpha\alpha' = 1 - \frac{2(r-1)}{r} \frac{3r}{rp+3r-3p} = \frac{r-4}{3r-6}.$$

This is compatible with (3.35), if $r \leq 6$.

- If $3 < r \leq 4$

Lets us show that, we can hope to obtain similar result with the same technique by choosing in (3.34) $p < 6$, such that

$$\|u(s, \cdot)\|_{L^p} \leq \|u(s, \cdot)\|_{L^6}^{\alpha''} \|u(s, \cdot)\|_{L^2}^{1-\alpha''},$$

with

$$\frac{1}{p} = \frac{\alpha''}{6} + \frac{1 - \alpha''}{2} \Leftrightarrow \alpha'' = \frac{3(p-2)}{2p}, \quad 1 - \alpha'' = \frac{6-p}{2p},$$

from (2.16).

The computations are the same except that we change the contribution of $\|\nabla u(s, \cdot)\|_{L^2}$ by

$$\begin{aligned} 2\alpha\alpha''(1 - \alpha') + 2(1 - \alpha) &= 2\frac{r-1}{r}\frac{3(p-2)}{2p}\left(1 - \frac{3r}{rp+3r-3p}\right) + \frac{2}{r} \\ &= \frac{(3p-6)r^2 + (-10p+30)r + 3p-18}{r(rp+3r-3p)} =: F_2(r, p). \end{aligned}$$

Hence, still by Hardy-Littlewood-Sobolev inequality (2.19), (recalling that $\gamma = \frac{1}{r}$):

$$\|\nabla u\|_{L_T^{\tilde{r}}L^r} \leq CT^{\frac{1}{\tilde{r}}}\|\nabla \times u_0\|_{L^r} + CT^{\frac{1}{2}}\nu^{-\frac{1}{2}}\|f(s, \cdot)\|_{L_T^{\tilde{r}}L^r} ds + C\nu^{-1+\gamma}\|\nabla u(s, \cdot)\|_{L^{\frac{F_2(r,p)}{1-2\alpha\alpha'}}}{}_{r' L^2},$$

where we have to impose that

$$r' = \frac{1}{F_2(r, p)} = \frac{r(rp+3r-3p)}{(3p-6)r^2 + (-10p+30)r + 3p-18},$$

and from inequality (2.19)

$$\left(\frac{1-2\alpha\alpha'}{\tilde{r}} = \right) \frac{\frac{r-4}{3r-6}}{\tilde{r}} = \frac{(3p-6)r^2 + (-10p+30)r + 3p-18}{r(rp+3r-3p)} - \frac{1}{r},$$

which is equivalent to,

$$\tilde{r} = \frac{r(r-4)(pr+3r-3p)}{3(r-2)(3pr^2-6r^2-11pr+27r+6p-18)} =: F_3(r, p).$$

We differentiate by p the function,

$$\partial_p F_3(r, p) = -\frac{(r-4)(r-3)(r-1)r(5r-6)}{(r-2)(3pr^2-6r^2-11pr+27r+6p-18)^2} \leq 0,$$

for $3 \leq r \leq 4$. Hence, the maximum is reached for $p = 6$

$$\tilde{r} = \frac{r(r-4)}{4r^2-13r+6} \geq 0.$$

3.3 Case $k = 2$

Lemma 5. For all $T > 0$ and $r \in (1, +\infty)$, we have $\|\nabla^2 u\|_{L_T^{\tilde{r}}L^r} < +\infty$ if

$$\begin{cases} \frac{2}{\tilde{r}} + \frac{3}{r} = 4, & \text{if } r < \frac{3}{2}, \\ \tilde{r} = \frac{2r(3r-4)}{13r^2-26r+12} & \text{if } r \in [\frac{3}{2}, 2), \\ \frac{1}{\tilde{r}} + \frac{6}{r} = 9, & \text{if } r \geq 2. \end{cases}$$

Proof of Lemma 5. Let us start again the analysis of ω , with an extra derivative. Specifically, taking the gradient of (3.23),

$$\begin{aligned} \nabla \omega(t, x) &= \hat{P}^{\tau, \xi} \nabla \nabla \times u_0(t, x) + \hat{G}^{\tau, \xi} \nabla \nabla \times f(t, x) \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \nabla_x \nabla_x \times \left(\nabla_y \hat{p}^{\tau, \xi}(s, t, x, y) [u(s, \theta_{s, \tau}(\xi)) - u(s, y)]^{\otimes 2} \right) dy ds. \end{aligned}$$

The definition, coefficient by coefficient, of $\nabla_x \times \left(\nabla_y \hat{p}^{\tau, \xi}(s, t, x, y)[u(s, \theta_{s, \tau}(\xi)) - u(s, y)]^{\otimes 2} \right)$ is described in (3.24). After applying the curl and the gradient, we choose the *freezing* point $(\tau, \xi) = (t, x)$,

$$\begin{aligned} \nabla \omega(t, x) &= \hat{P}^{t, x} \nabla \nabla \times u_0(t, x) + \hat{G}^{t, x} \nabla \nabla \times f(t, x) \\ &+ \int_0^t \int_{\mathbb{R}^3} \nabla_x \nabla_x \times \left(\nabla_y \hat{p}^{\tau, \xi}(s, t, x, y)[u(s, \theta_{s, \tau}(\xi)) - u(s, y)]^{\otimes 2} \right) \Big|_{(\tau, \xi)=(t, x)} dy ds. \end{aligned} \quad (3.36)$$

By integration by parts and by Fourier multipliers estimates, for any $r > 1$, we get from the thermic representation of Besov norm,

$$\left(\int_0^t \|\hat{P}^{t, \cdot} \nabla \nabla \times u_0(t, \cdot)\|_{L^r}^{\tilde{r}} dt \right)^{\frac{1}{\tilde{r}}} \leq \left(\int_0^t t^{-1} t^{(1-\alpha/2)\tilde{r}} \|\hat{P}^{t, \cdot} \nabla \nabla \times u_0(t, \cdot)\|_{L^r}^{\tilde{r}} dt \right)^{\frac{1}{\tilde{r}}} \leq C \|u_0\|_{\dot{B}_{r, \tilde{r}}^\alpha},$$

with

$$1 - \alpha/2 = \frac{1}{\tilde{r}} \Leftrightarrow \alpha = 2 \frac{\tilde{r} - 1}{\tilde{r}}.$$

The semi-group $\hat{P}^{t, \cdot}$ above becomes the usual heat semi-group thanks to the change of variables as performed in (3.28).

For the force function f , with the same argument leads to

$$\|\hat{G}^{t, \cdot} \nabla \nabla \times f(t, \cdot)\|_{L^r} \leq \|f\|_{L^\infty(\dot{B}_{r, \infty}^2)}. \quad (3.37)$$

- Cases $r < \frac{3}{2}$

Still by interpolation of Brezis Mirunescu and in the Lebesgue spaces (2.15), for $p \geq 2r$

$$\begin{aligned} \|u(s, \cdot)\|_{W^{\gamma, 2r}} &\leq C \|u(s, \cdot)\|_{L^p}^\alpha \|\nabla u(s, \cdot)\|_{L^2}^{1-\alpha} \\ &\leq C \|u(s, \cdot)\|_{L^2}^\alpha \|u(s, \cdot)\|_{L^6}^{\alpha(1-\tilde{\alpha})} \|\nabla u(s, \cdot)\|_{L^2}^{1-\alpha} \\ &\leq C \|u(s, \cdot)\|_{L^2}^{\alpha \tilde{\alpha}} \|\nabla u(s, \cdot)\|_{L^2}^{1-\alpha \tilde{\alpha}}, \end{aligned} \quad (3.38)$$

by Gagliardo-Nirenberg interpolation inequality, and by (2.22)

$$\alpha = \frac{p(2-2r)}{2r(2-p)} = \frac{p(r-1)}{r(p-2)}, \quad 1-\alpha = \gamma = \frac{2(2r-p)}{2r(2-p)} = \frac{(p-2r)}{r(p-2)},$$

and from (2.16)

$$\tilde{\alpha} = \frac{2(6-p)}{p(6-2)} = \frac{6-p}{2p}, \quad 1-\tilde{\alpha} = \frac{6(p-2)}{p(6-2)} = \frac{3(p-2)}{2p}.$$

Coming back to (3.36), with the same analysis performed in (3.31)

$$\begin{aligned} &\left\| x \mapsto \int_0^t \int_{\mathbb{R}^3} \nabla_x \nabla_x \times \left(\nabla_y \hat{p}^{\tau, \xi}(s, t, x, y)[u(s, \theta_{s, \tau}(\xi)) - u(s, y)]^{\otimes 2} \right) \Big|_{(\tau, \xi)=(t, x)} dy ds \right\|_{L^r} \\ &\leq C \int_0^t [\nu(t-s)]^{-\frac{3}{2}+\gamma} \|u(s, \cdot)\|_{W^{\gamma, 2r}}^2 ds. \end{aligned}$$

Importantly, the time singularity is integrable as soon as $\gamma > \frac{1}{2}$.

Next, by interpolation inequality (3.38), we derive

$$\begin{aligned} &\left\| x \mapsto \int_0^t \int_{\mathbb{R}^3} \nabla_x \nabla_x \times \left(\nabla_y \hat{p}^{\tau, \xi}(s, t, x, y)[u(s, \theta_{s, \tau}(\xi)) - u(s, y)]^{\otimes 2} \right) \Big|_{(\tau, \xi)=(t, x)} dy ds \right\|_{L^r} \\ &\leq C \|u\|_{L_T^\infty L^2}^{2\alpha \tilde{\alpha}} \int_0^t [\nu(t-s)]^{-\frac{3}{2}+\gamma} \|\nabla u(s, \cdot)\|_{L^2}^{2(1-\alpha \tilde{\alpha})} ds. \end{aligned} \quad (3.39)$$

We obtain, by Hardy-Littlewood-Sobolev inequality,

$$\begin{aligned} & \left\| (t, x) \mapsto \int_0^t \int_{\mathbb{R}^3} \nabla_x \nabla_x \times \left(\nabla_y \hat{p}^{\tau, \xi}(s, t, x, y) [u(s, \theta_{s, \tau}(\xi)) - u(s, y)]^{\otimes 2} \right) \Big|_{(\tau, \xi) = (t, x)} dy ds \right\|_{L_T^{\tilde{r}} L^r} \\ & \leq C \|u\|_{L_T^{\infty} L^2}^{2\alpha\tilde{\alpha}} \|\nabla u\|_{L_T^{(2(1-\alpha\tilde{\alpha})r')} L^2}^{2(1-\alpha\tilde{\alpha})}, \end{aligned}$$

with

$$\frac{1}{r'} + \frac{3}{2} - \gamma = 1 + \frac{1}{\tilde{r}}.$$

In order to use (1.4), we have to suppose that

$$r' = (1 - \alpha\tilde{\alpha})^{-1} = \frac{1}{1 - \frac{p(r-1)}{r(p-2)} \frac{6-p}{2p}} = \frac{2r(p-2)}{3rp - 10r + 6 - p}.$$

Hence,

$$\frac{1}{\tilde{r}} = \frac{3rp - 10r + 6 - p}{2r(p-2)} + \frac{1}{2} - \frac{(p-2r)}{r(p-2)} = \frac{4r-3}{2r},$$

namely

$$\frac{2}{\tilde{r}} + \frac{3}{r} = 4.$$

In fine, once again, to get an inequality with $\nabla^2 u$ instead of $\nabla \omega$, we can use a Calderòn-Zygmund inequality for the Biot and Savart representation in L^r , $r \in (1, +\infty)$.

The result for the case $(r, \tilde{r}) = (\frac{4}{3}, \frac{8}{7})$ is somehow weaker than the ones of [Con90], [Lio96], [Vas10], [VY21] where $(r, \tilde{r}) = (\frac{4}{3}^-, \frac{4}{3}^-)$. This last case $r = \tilde{r}$ allows to perform a specific analysis on the PDE which does not seem to work in general Lebesgue space $L_T^{\tilde{r}} L^r$. That is why, we perform interpolations, in Section 4 below, taking account of this case.

- Cases $2 > r \geq \frac{3}{2}$

Let us change the interpolation analysis (3.38),

$$\|u(s, \cdot)\|_{W^{\gamma, 2r}} \leq C \|u(s, \cdot)\|_{L^\infty}^\alpha \|\nabla u(s, \cdot)\|_{L^2}^{1-\alpha}, \quad (3.40)$$

where we define

$$\frac{1}{2r} = \frac{1-\alpha}{2} \Leftrightarrow \alpha = \frac{r-1}{r} \Leftrightarrow \gamma = \frac{1}{r}.$$

Let us recall the constraint $\gamma > \frac{1}{2}$ which means that $r < 2$.

Similarly to Section 5.1, we derive

$$\begin{aligned} & \|\nabla^2 u(t, \cdot)\|_{L^r} \\ & \leq \|\nabla^2 u_0\|_{L^r} + C \int_0^t [\nu(t-s)]^{-1+\frac{\varepsilon}{2}} \|f(s, \cdot)\|_{W^{\varepsilon, r}} ds + C \int_0^t [\nu(t-s)]^{-\frac{3}{2}+\gamma} \|\nabla u(s, \cdot)\|_{L^2}^{2(1-\alpha)} \|u(s, \cdot)\|_{L^\infty}^{2\alpha} ds. \end{aligned}$$

Since, by Sobolev embedding, we get

$$\|u\|_{L^\infty} \leq C \|\nabla^2 u\|_{L^r}^{\alpha'} \|u\|_{L^6}^{1-\alpha'},$$

with

$$0 = \left(\frac{1}{r} - \frac{2}{3}\right)\alpha' + \frac{1-\alpha'}{6} \Leftrightarrow \alpha' = \frac{r}{5r-6}.$$

In other words, in such a case, by inequality $\|u\|_{L^6} \leq C\|\nabla u\|_{L^2}$, there is an extra contribution of $\|\nabla u\|_{L^2}$. Therefore, to be able to use the $L_T^2 L^2$ control (1.4), we have to suppose that

$$2(1 - \alpha) + 2\alpha(1 - \alpha') = 2(1 - \alpha\alpha') \leq 2,$$

which is always true.

We deduce,

$$\begin{aligned} \|\nabla^2 u(t, \cdot)\|_{L^r} &\leq \|\nabla^2 u_0\|_{L^r} + C \int_0^t [\nu(t-s)]^{-1+\frac{\varepsilon}{2}} \|f(s, \cdot)\|_{W^{\varepsilon, r}} ds \\ &\quad + C \int_0^t [\nu(t-s)]^{-\frac{3}{2}+\gamma} \|\nabla u(s, \cdot)\|_{L^2}^{2(1-\alpha\alpha')} \|\nabla^2 u(s, \cdot)\|_{L^r}^{2\alpha\alpha'} ds. \end{aligned}$$

Consequently, from Lemma 1, for $2\alpha\alpha' = 2\frac{r-1}{5r-6} \in (0, 1)$ (true for $r > \frac{4}{3}$ or $r < 1$):

$$\begin{aligned} \|\nabla^2 u(t, \cdot)\|_{L^r}^* &\leq \left(\left(\|\nabla^2 u_0\|_{L^r} + C \int_0^t [\nu(t-s)]^{-1+\frac{\varepsilon}{2}} \|f(s, \cdot)\|_{W^{\varepsilon, r}} ds \right. \right. \\ &\quad \left. \left. + C \int_0^t [\nu(t-s)]^{-\frac{3}{2}+\gamma} \|\nabla u(s, \cdot)\|_{L^2}^{2(1-\alpha\alpha')} ds \right)^{\frac{1}{1-2\alpha\alpha'}} \right)^*. \end{aligned} \quad (3.41)$$

Hence, from Corollary 2 (based on Hardy-Littlewood-Sobolev inequality)

$$\|\nabla^2 u\|_{L_T^{\tilde{r}} L^r} \leq CT^{\frac{1}{\tilde{r}}} \|\nabla^2 u_0\|_{L^r} + CT^\varepsilon \|f\|_{L_T^{\tilde{r}} W^{\varepsilon, r}} + C\nu^{-\frac{3}{2}+\gamma} \|\nabla u\|_{L_T^{2(1-\alpha\alpha')r'} L^2}^{2(1-\alpha\alpha')(1-2\alpha\alpha')},$$

with, $\frac{1}{r'} + \frac{3}{2} - \gamma = 1 + \frac{1-2\alpha\alpha'}{\tilde{r}}$, to apply $L_T^2 L^2$ control (1.4), we suppose that

$$r' = \frac{1}{1 - \alpha\alpha'} = \frac{1}{1 - \frac{r-1}{5r-6}} = \frac{5r-6}{4r-5},$$

then

$$\frac{4r-5}{5r-6} + \frac{1}{2} - \frac{1}{r} = \frac{1 - 2\frac{r-1}{5r-6}}{\tilde{r}}. \quad (3.42)$$

This above identity is equivalent to

$$\tilde{r} = \frac{2r(3r-4)}{13r^2 - 26r + 12}.$$

Let us remark that, the indexes of the Lebesgue norms in (3.41) have to be greater than 1, namely

$$\tilde{r} = \frac{2r(3r-4)}{13r^2 - 26r + 12} \geq 1 - 2\alpha\alpha' = 1 - 2\frac{r-1}{5r-6} = \frac{3r-4}{5r-6}$$

in other words

$$(5r-6)2r = 10r^2 - 12r > 13r^2 - 26r + 12 \Leftrightarrow 0 > 3r^2 - 14r + 12$$

which is true for

$$r \in \left(\frac{7 - \sqrt{13}}{3}, \frac{7 + \sqrt{13}}{3} \right) \approx (1.1315; 3.5352) \supset \left[\frac{3}{2}, 2 \right).$$

Remark 4. We could imagine a strategy with another interpolation

$$\|u(s, \cdot)\|_{L^\infty} \leq C \|\nabla^2 u(s, \cdot)\|_{L^r}^{\alpha'} \|u(s, \cdot)\|_{L^2}^{1-\alpha'},$$

with

$$0 = \left(\frac{1}{r} - \frac{2}{3} \right) \alpha' + \frac{1-\alpha'}{2} \Leftrightarrow \alpha' = \frac{3r}{7r-6},$$

and $2\alpha\alpha' = 2\frac{3(r-1)}{7r-6} \in (0, 1)$, for $r \geq 1$.

With the same computations as performed, the time Lebesgue space in the estimates becomes $L_T^{1-2\alpha\alpha'} = L_T^{1-2\frac{3(r-1)}{7r-6}} = L_T^{\frac{r}{7r-6}}$ which is weaker than the previous analysis as soon as $r \geq \frac{3}{2}$ or $r \leq 1$ (in these cases $\frac{3r-4}{5r-6} \leq \frac{r}{7r-6}$).

- Cases $r \geq 2$

In order to consider such big r , we change the interpolation analysis (3.38) by

$$\|u(s, \cdot)\|_{W^{\gamma, 2r}} \leq C \|u(s, \cdot)\|_{L^\infty}^\alpha \|\nabla u(s, \cdot)\|_{L^p}^{1-\alpha}, \quad (3.43)$$

where we recall (2.22),

$$\alpha = \frac{2r-p}{2r}, \quad 1-\alpha = \gamma = \frac{p}{2r},$$

and by Gagliardo-Nirenberg inequality

$$\|\nabla u(s, \cdot)\|_{L^p} \leq \|\nabla^2 u(s, \cdot)\|_{L^r}^{\tilde{\alpha}} \|\nabla u(s, \cdot)\|_{L^2}^{1-\tilde{\alpha}},$$

with

$$\frac{1}{p} = \left(\frac{1}{r} - \frac{1}{3}\right)\tilde{\alpha} + \frac{1-\tilde{\alpha}}{2}.$$

This identity equivalently write

$$\tilde{\alpha} = \frac{3r}{5r-6} \frac{p-2}{p} \quad \text{and} \quad 1-\tilde{\alpha} = \frac{2(rp+3r-3p)}{p(5r-6)}.$$

Let us recall the constraint $1 \geq \gamma = \frac{p}{2r} > \frac{1}{2}$ meaning that

$$2r \geq p > r. \quad (3.44)$$

Similarly to Section 5.1, we derive

$$\begin{aligned} \|\nabla^2 u(t, \cdot)\|_{L^r} &\leq \|u_0\|_{\dot{B}_{r, \tilde{r}}^{2\frac{\tilde{r}-1}{r}}} + C \|f\|_{\dot{B}_{r, \infty}^2} \\ &\quad + C \int_0^t [\nu(t-s)]^{-\frac{3}{2}+\gamma} \|\nabla^2 u(s, \cdot)\|_{L^r}^{2(1-\alpha)\tilde{\alpha}} \|\nabla u(s, \cdot)\|_{L^2}^{2(1-\alpha)(1-\tilde{\alpha})} \|u(s, \cdot)\|_{L^\infty}^{2\alpha} ds. \end{aligned}$$

Again by Sobolev embedding, we get

$$\|u\|_{L^\infty} \leq C \|\nabla^2 u\|_{L^r}^{\alpha'} \|u\|_{L^6}^{1-\alpha'},$$

with

$$\alpha' = \frac{r}{5r-6}.$$

We deduce,

$$\begin{aligned} \|\nabla^2 u(t, \cdot)\|_{L^r} &\leq \|\nabla^2 u_0\|_{L^r} + C \|f\|_{\dot{B}_{r, \infty}^2} \\ &\quad + C \int_0^t [\nu(t-s)]^{-\frac{3}{2}+\gamma} \|\nabla^2 u(s, \cdot)\|_{L^r}^{2(1-\alpha)\tilde{\alpha}} \|\nabla u(s, \cdot)\|_{L^2}^{2(1-\alpha)(1-\tilde{\alpha})} \|\nabla^2 u\|_{L^r}^{2\alpha\alpha'} \|u\|_{L^6}^{2\alpha(1-\alpha')} ds \\ &\leq \|\nabla^2 u_0\|_{L^r} + C \|f\|_{\dot{B}_{r, \infty}^2} \\ &\quad + C \int_0^t [\nu(t-s)]^{-\frac{3}{2}+\gamma} \|\nabla^2 u(s, \cdot)\|_{L^r}^{2(1-\alpha)\tilde{\alpha}+2\alpha\alpha'} \|\nabla u(s, \cdot)\|_{L^2}^{2(1-\alpha)(1-\tilde{\alpha})+2\alpha(1-\alpha')} ds. \end{aligned}$$

We have to suppose, in view of Lemma 1, that

$$1 > 2(1-\alpha)\tilde{\alpha} + 2\alpha\alpha' = 2\left(1 - \frac{2r-p}{2r}\right) \frac{3r}{5r-6} \frac{p-2}{p} + 2 \frac{2r-p}{2r} \frac{r}{5r-6} = \frac{2(p+r-3)}{5r-6}.$$

This is equivalent to (for $r \geq \frac{6}{5}$)

$$5r-6 > 2(p+r-3) \Leftrightarrow 3r > 2p \Leftrightarrow \frac{3}{2}r > p. \quad (3.45)$$

Let us remark that the contribution of $\|\nabla u(s, \cdot)\|_{L^2}$ is

$$\begin{aligned} 2(1-\alpha)(1-\tilde{\alpha}) + 2\alpha(1-\alpha') &= 2\left(1 - \frac{2r-p}{2r}\right)\left(1 - \frac{3r}{5r-6} \frac{p-2}{p}\right) + 2\frac{2r-p}{2r}\left(1 - \frac{r}{5r-6}\right) \\ &= \frac{8r - (2p+6)}{5r-6}. \end{aligned}$$

Then, from Lemma 1, we derive

$$\begin{aligned} \|\nabla^2 u(t, \cdot)\|_{L^r}^* &\leq \left(\left(\|\nabla^2 u_0\|_{L^r} + C \int_0^t [\nu(t-s)]^{-1+\frac{\varepsilon}{2}} \|f(s, \cdot)\|_{W^{\varepsilon, r}} ds \right)^{1-2(1-\alpha)\tilde{\alpha}+2\alpha\alpha'} \right. \\ &\quad \left. + C \int_0^t [\nu(t-s)]^{-\frac{3}{2}+\gamma} \|\nabla u(s, \cdot)\|_{L^2}^{2(1-\alpha)(1-\tilde{\alpha})+2\alpha\alpha'} ds \right)^{\frac{1}{1-2(1-\alpha)\tilde{\alpha}+2\alpha\alpha'}}. \end{aligned}$$

By Corollary 2, using Hardy-Littlewood-Sobolev inequality (2.19),

$$\begin{aligned} \|\nabla^2 u\|_{L_T^{\tilde{r}} L^r} &\leq CT^{\frac{1}{\tilde{r}}} \left(\|\nabla^2 u_0\|_{L^r} + C \int_0^t [\nu(t-s)]^{-1+\frac{\varepsilon}{2}} \|f(s, \cdot)\|_{W^{\varepsilon, r}} ds \right) \\ &\quad + C\nu^{-\frac{3}{2}+\gamma} \|\nabla u(s, \cdot)\|_{L_T^{(2(1-\alpha)(1-\tilde{\alpha})+2\alpha(1-\alpha'))(1-2(1-\alpha)\tilde{\alpha}+2\alpha\alpha')}^{r'} L^2}, \end{aligned} \quad (3.46)$$

with

$$\frac{1}{r'} + \frac{1}{2} - \gamma = \frac{1 - 2(1-\alpha)\tilde{\alpha} + 2\alpha\alpha'}{\tilde{r}}.$$

Using (1.4) requires that

$$r' = \frac{1}{(1-\alpha)(1-\tilde{\alpha}) + \alpha(1-\alpha')} = \frac{5r-6}{8r-(2p+6)},$$

then

$$\frac{8r-(2p+6)}{5r-6} + \frac{1}{2} - \frac{p}{2r} = \frac{1 - \frac{2(p+r-3)}{5r-6}}{\tilde{r}},$$

which is equivalent to

$$\tilde{r} = \frac{2r(3r-2p)}{21r^2 - 18r - 9rp + 6p} =: \tilde{F}_3(r, p). \quad (3.47)$$

The denominator is non-negative if

$$p \leq \frac{r(7r-6)}{3r-2},$$

which is true because $\frac{r(7r-6)}{3r-2} \geq \frac{3}{2}$ since $r \geq \frac{\sqrt{105}+21}{28} \approx 1.12 < 2$. In other words, the above inequality is induced by (3.45).

Let us differentiate $\tilde{F}_3(r, p)$ w.r.t. p ,

$$\partial_p \tilde{F}_3(r, p) = -\frac{-2r^2(5r-6)}{3(7r^2 - 3pr - 6r + 2p)^2} < 0.$$

That is to say that we have, as $p > r$ (see (3.44)),

$$\tilde{r} < \tilde{F}_3(r, r) = \frac{2r(3r-2r)}{21r^2 - 18r - 9r^2 + 6r} = \frac{r}{6(r-1)} \Leftrightarrow \frac{1}{\tilde{r}} + \frac{6}{r} > 6.$$

Importantly, let us remark that the indexes of the Lebesgue norms in time in (3.46) have to be greater than 1, which means that we have to impose that

$$\frac{2r(3r-2p)}{21r^2 - 18r - 9rp + 6p} = \tilde{r} \geq 1 - 2(1-\alpha)\tilde{\alpha} + 2\alpha\alpha' = 1 - \frac{2(p+r-3)}{5r-6} = \frac{3r-2p}{5r-6},$$

which is compatible with (3.47) if

$$-11r^2 + (9p + 6)r - 6p \geq 0 \Leftrightarrow p \geq \frac{r(11r - 6)}{9r - 6}.$$

Because $p \mapsto \tilde{F}_3(r, p)$ is a non-increasing function, we pick $p = \frac{11r^2 - 6r}{9r - 6}$. This choice yields

$$\tilde{r} = \tilde{F}_3\left(r, \frac{11r^2 - 6r}{9r - 6}\right) = \frac{r}{9r - 6},$$

which is equivalent to

$$\frac{1}{\tilde{r}} + \frac{6}{r} = 9.$$

Let us observe that this choice of p is compatible with (3.45): $\frac{3}{2}r > p = \frac{11r^2 - 6r}{9r - 6}$, which is granted for $r > \frac{3}{2}$. □

4 Gathering interpolations

In this section, we enhance the result of Theorem 1. We perform interpolations and Sobolev embedding, based on with condition (1.5), from [Con91].

4.1 Second derivatives

Let us precise some interpolations for the second derivatives. Namely, we gather all of these results in the below graphic, we draw the links between r and \tilde{r} in the above statement: $\nabla^2 u \in L^{\tilde{r}}([0, T], L^r(\mathbb{R}^3, \mathbb{R}^3))$.

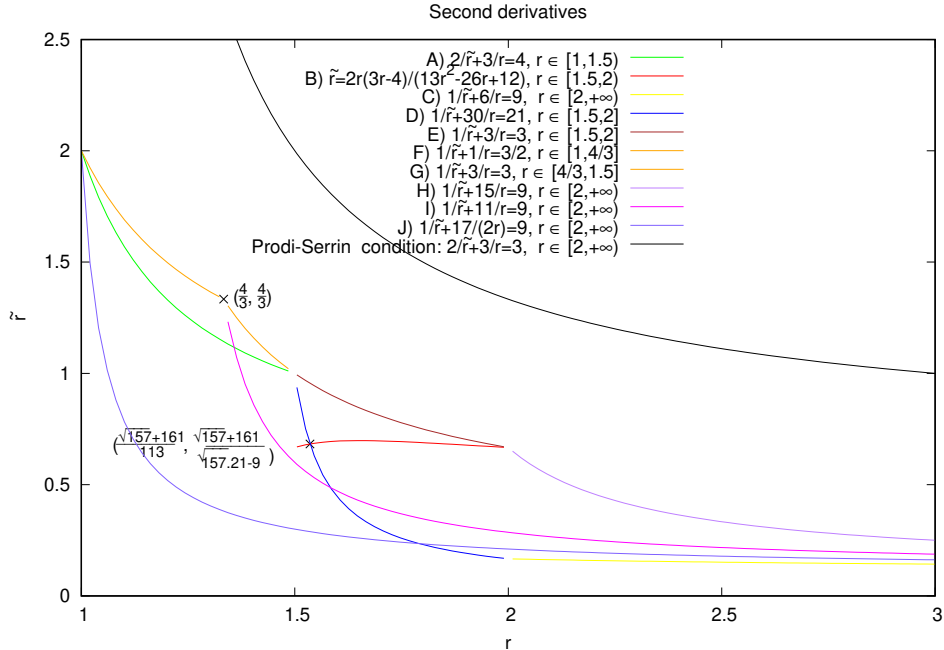


Figure 2: $\nabla^2 u \in L^{\tilde{r}}([0, T], L^r(\mathbb{R}^3, \mathbb{R}^3))$

Corollary 3. Let $u \in L^2([0, T], H^1(\mathbb{R}^3, \mathbb{R}^3)) \cap L^\infty([0, T], L^2(\mathbb{R}^3, \mathbb{R}^3))$ a Leray solution of (1.2), and if $(r, \tilde{r}) \in (1, +\infty)$, $u_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3) \cap \dot{B}_{r, \tilde{r}}^{\frac{2(\tilde{r}-1)}{r}}(\mathbb{R}^3, \mathbb{R}^3)$ and $f \in L^\infty([0, T], L^2(\mathbb{R}^3, \mathbb{R}^3)) \cap L^\infty([0, T], \dot{B}_{r, \infty}^2(\mathbb{R}^3, \mathbb{R}^3))$ then $\|\nabla^2 u\|_{L_T^{\tilde{r}} L^r} < +\infty$ as soon as

$$\begin{cases} \frac{2}{\tilde{r}} + \frac{2}{r} > 3, & \text{if } r < \frac{4}{3}, \text{ Curve F) } \\ \frac{1}{\tilde{r}} + \frac{3}{r} > 3, & \text{if } r \in (\frac{4}{3}, \frac{1}{2}) \text{ Curve G) } \\ \frac{1}{\tilde{r}} + \frac{3}{r} = 3, & \text{if } r \in [\frac{3}{2}, 2] \text{ Curve E) } \\ \frac{1}{\tilde{r}} + \frac{15}{r} = 9, & \text{if } 2 \leq r < +\infty \text{ Curve H) .} \end{cases}$$

The equality case $(r, \tilde{r}) = (\frac{4}{3}, \frac{4}{3})$ is not proved yet. The work of Lions [Lio96] and Vasseur et al. [Vas10], [VY21] get closer to this equality.

Proof of Corollary 3 and interpolation Figure 2. The curves A), B) and C) are already proved by Theorem 1.

★ Curve D) By interpolation, from $\nabla^2 u \in L^{1^+}([0, T], L^{\frac{3}{2}^-}(\mathbb{R}^3, \mathbb{R}^3))$ and $\nabla^2 u \in L^{\frac{1}{6}}([0, T], L^2(\mathbb{R}^3, \mathbb{R}^3))$ we compute[‡] that $\nabla^2 u \in L^{\tilde{r}}([0, T], L^r(\mathbb{R}^3, \mathbb{R}^3))$ with

$$\frac{1}{\tilde{r}} + \frac{30}{r} = 21, \quad r \in (\frac{3}{2}, 2],$$

which equivalently writes $\tilde{r} = \frac{r}{21r-30}$.

We have $\frac{r}{21r-30} \geq \frac{2r(3r-4)}{13r^2-26r+12} \Leftrightarrow -113r^2 + 322r - 228 \geq 0 \Leftrightarrow r \in [\frac{\sqrt{157}-161}{113}, \frac{\sqrt{157}+161}{113}] \approx [1.3139, 1.5357]$.

In the following we set

$$r_0 := \frac{\sqrt{157} + 161}{113}.$$

★ Curve E) We interpolate between the curve A) at the point (1.5, 1) and the end of the curve B) at the point (2, 2/3), thanks to identity (2.18)

$$\tilde{r} = \frac{r/3}{r/2 - 3 + 3r/2 + 2 - r} = \frac{r}{3(r-1)} \Leftrightarrow \frac{1}{\tilde{r}} + \frac{3}{r} = 3. \quad (4.1)$$

As we can see the best result is at the point $(\frac{4}{3}, \frac{4}{3})$ given by (1.5), let us interpolate from this point.

★ Curve F) We interpolate from (1, 2) to $(\frac{4}{3}, \frac{4}{3})$ thanks to identity (2.18):

$$\tilde{r} < \frac{2r}{3r-2} \Leftrightarrow \frac{1}{\tilde{r}} + \frac{1}{r} > \frac{3}{2}.$$

We surprisingly obtain the same identity as in the curve E) (4.1).

★ Curve G) We interpolate from $(\frac{4}{3}, \frac{4}{3})$ to $(\frac{3}{2}, 1)$ thanks to identity (2.18)

$$\tilde{r} < \frac{r}{3(r-1)} \Leftrightarrow \frac{1}{\tilde{r}} + \frac{3}{r} > 3.$$

★ Curve H) We interpolate from (2, 2/3) to the end of curve C) $(+\infty, 1/9)$ thanks to (2.18):

$$\tilde{r} = \frac{r}{9r-15} \Leftrightarrow \frac{1}{\tilde{r}} + \frac{15}{r} = 9.$$

[‡]See (2.18) further.

★ Curve I) We interpolate from $(4/3, 4/3)$ to the end of curve C) $(+\infty, 1/9)$ thanks to (2.18):

$$\tilde{r} = \frac{r}{9r-11} \Leftrightarrow \frac{1}{\tilde{r}} + \frac{11}{r} = 9.$$

★ Curve J) We interpolate from $(1, 2)$ to the end of curve C) $(+\infty, 1/9)$ thanks to (2.18):

$$\tilde{r} = \frac{2r}{18r-17} \Leftrightarrow \frac{1}{\tilde{r}} + \frac{17}{2r} = 9.$$

□

4.2 First derivatives

We can also perform the same reasoning for ∇u , directly from Corollary 3 and Sobolev embedding

Corollary 4. *Let $u \in L^2([0, T], H^1(\mathbb{R}^3, \mathbb{R}^3)) \cap L^\infty([0, T], L^2(\mathbb{R}^3, \mathbb{R}^3))$ a Leray solution of (1.2), if for any $(r, \tilde{r}) \in (1, +\infty)$, $u_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3) \cap W^{1,r}(\mathbb{R}^3, \mathbb{R}^3)$ and $f \in L^\infty([0, T], L^2(\mathbb{R}^3, \mathbb{R}^3))$ such that $\|\nabla f\|_{L_T^{\tilde{r}} L^r} < +\infty$ then $\|\nabla u\|_{L_T^{\tilde{r}} L^r} < +\infty$ as soon as*

$$\begin{cases} \frac{1}{\tilde{r}} + \frac{3}{r} = 2, & \text{if } r \in (2, 6], \\ \frac{1}{\tilde{r}} + \frac{15}{r} = 4, & \text{if } r \in (6, +\infty). \end{cases}$$

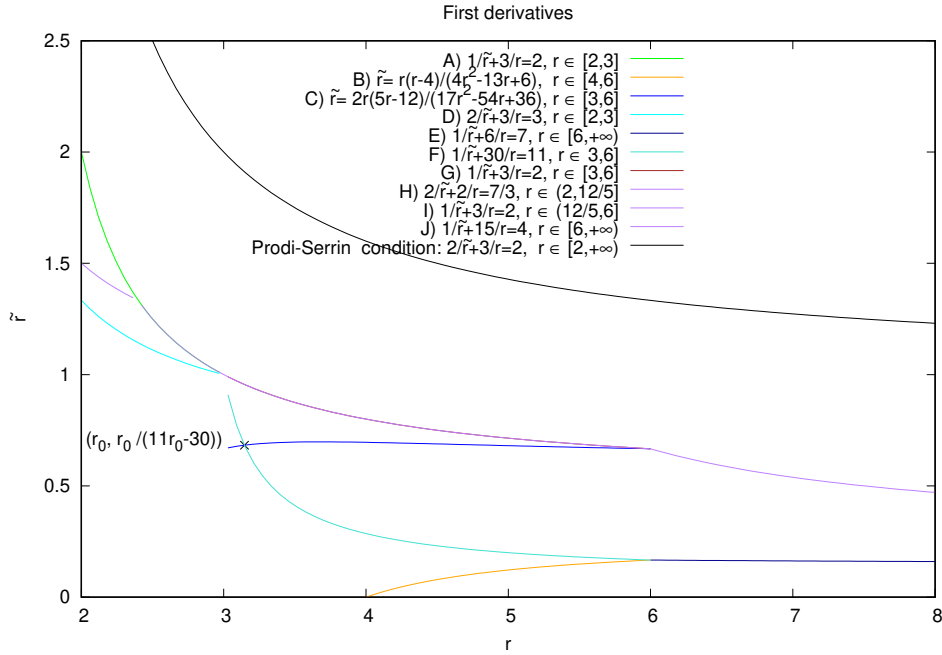


Figure 3: $\nabla u \in L^{\tilde{r}}([0, T], L^r(\mathbb{R}^3, \mathbb{R}^3))$

Proof of Corollary 4 and interpolation Figure 3. The curves A) and B) are already proved by Theorem 1.

By Gagliardo-Nirenberg interpolation, we can derive the case $k = 1$ from the case $k = 2$, indeed the corresponding Lebesgue space L^{r_1} for $k = 1$ matches with the Lebesgue space $L^{r_2} = L^{\frac{3r_1}{3+r_1}}$ for $k = 2$.

★ Curve C) We get for any $r_2 \in [\frac{3}{2}, 2) \Leftrightarrow r_1 \in [3, 6)$, the corresponding relation from Theorem 1,

$$\frac{2\frac{3r}{3+r}(3\frac{3r}{3+r}-4)}{13(\frac{3r}{3+r})^2-26\frac{3r}{3+r}+12} = \frac{2r(5r-12)}{17r^2-54r+36}.$$

★ Curve D) The corresponding case for $k = 2$ is $r_2 = (\frac{1}{r_1} + \frac{1}{3})^{-1} < \frac{3}{2} \Leftrightarrow r_1 < 3$ and,

$$\frac{2}{\tilde{r}} + 3(\frac{1}{r_1} + \frac{1}{3}) = 4 \Leftrightarrow \tilde{r} = \frac{2r_1}{3(r_1-1)}.$$

★ Curve E) If $r_1 > 6$, the corresponding case for $k = 2$ is $r_2 > 2$ and from Theorem 1,

$$\frac{1}{\tilde{r}} + 6(\frac{1}{r_1} + \frac{1}{3}) = 9 \Leftrightarrow \frac{1}{\tilde{r}} + \frac{6}{r_1} = 7.$$

★ Curve F) The corresponding case for $k = 2$ is $r_2 \in [\frac{3}{2}, 2) \Leftrightarrow r_1 \in [3, 6)$,

$$\frac{1}{\tilde{r}} + \frac{30}{r_2} = 21 \Leftrightarrow \tilde{r} = \frac{r_1}{11r_1-30}.$$

Let us remark that $r_2 \in [\frac{3}{2}, \frac{\sqrt{157}+161}{113}) \Leftrightarrow r_1 \in [3, \frac{3(\sqrt{157}+161)}{178-\sqrt{157}})$.

★ Curve G) For any $3 \leq r \leq 6$, we interpolate the curve A) at $(3, 1)$ to the curve C) at $(6, \frac{2}{3})$, and from (2.18) we get

$$\frac{1}{\tilde{r}} + \frac{3}{r} = 2.$$

Let us remark that we have $\frac{r_1}{11r_1-30} \geq \frac{r_1}{5r_1-9} \Leftrightarrow r_1 \leq \frac{7}{2} > \frac{3(\sqrt{157}+161)}{178-\sqrt{157}} (\approx 3.1461) = r_0$.

Next, we derive from Sobolev embedding and Corollary 3,

★ Curve H) $\frac{2}{\tilde{r}} + 2(\frac{1}{r} + \frac{1}{3}) = 3 \Leftrightarrow \frac{2}{\tilde{r}} + \frac{2}{r} = \frac{7}{3}$, if $\frac{1}{r} > \frac{3}{4} - \frac{1}{3} = \frac{5}{12}$

★ Curve I) $\frac{1}{\tilde{r}} + 3(\frac{1}{r} + \frac{1}{3}) = 3 \Leftrightarrow \frac{1}{\tilde{r}} + \frac{3}{r} = 2$, if $r \in (\frac{5}{12}, 6]$

★ Curve J) $\frac{1}{\tilde{r}} + 15(\frac{1}{r} + \frac{1}{3}) = 9 \Leftrightarrow \frac{1}{\tilde{r}} + \frac{15}{r} = 4$,

Gathering the upper curves in the current proof yields the corollary. □

Eventually, there is no need to perform again interpolation to extend the analysis to u . Indeed, the only possible case from Corollary 4 gives, Sobolev embedding, $\frac{1}{\tilde{r}} + 3(\frac{1}{r} + \frac{1}{3}) = 2 \Leftrightarrow \frac{1}{\tilde{r}} + \frac{3}{r} = 1$, for $r \in (6, +\infty)$. This exactly matches identity associated with $k = 0$, in Theorem 1.

5 Weighted Lebesgue space in time

In the analysis of the solution of (1.2), in a mild formulation, a control of time integral with singularity might be directly useful.

Theorem 5. *Let $u \in L^2([0, T], H^1(\mathbb{R}^3, \mathbb{R}^3)) \cap L^\infty([0, T], L^2(\mathbb{R}^3, \mathbb{R}^3))$ a Leray solution of (1.2), if for all $k \in \llbracket 0, 2 \rrbracket$, $\theta < 1$, and $r \in (1, +\infty)$, $u_0 \in L^2(\mathbb{R}^3, \mathbb{R}^3) \cap W^{k,r}(\mathbb{R}^3, \mathbb{R}^3)$ and $f \in L^\infty([0, T], L^2(\mathbb{R}^3, \mathbb{R}^3))$ such that*

$$\int_0^T (T-t)^{-\theta} \|\nabla^k f(t, \cdot)\|_{L^r} dt < +\infty,$$

then

$$\int_0^T (T-t)^{-\theta} \|\nabla^k u(t, \cdot)\|_{L^r} dt < +\infty, \tag{5.1}$$

if

$$\theta < \frac{3-kr}{2r} \tag{5.2}$$

and if $r < \frac{3}{k}$ (with $r < +\infty$ for $k = 0$).

Also, if $k = 0$ and $r \in [3, +\infty)$, or $k \in \{1, 2\}$, we can suppose that $\theta \leq \frac{3-kr}{2r}$.

Remark 5. However, by replacing suitably the parameters $k = 1$, $r = 2$, $\theta < \frac{1}{4}$, we get the constraint $\theta < \frac{1}{2}$ which is a poorer result than the Leray estimates (1.4). By a homogeneity argument, there is no hope to get substantially better estimates without getting ride of the bilinear contribution $\mathbb{P}[u \cdot \nabla u]$ (disappearing with the energy method).

We can postulate that for any $k \in \mathbb{N}$,

$$\int_0^T (T-t)^{-\theta} \|\nabla^k u(t, \cdot)\|_{L^r} dt < +\infty,$$

if $\theta < \min(1, \frac{3-2k}{2r})$, which is negative if $k \geq 3$.

The main problem to by-product our analysis in order to prove this result, is how to handle with a time singularity of order $-2 + \gamma$, with $\gamma \in [0, 1]$.

Another difficulty is to find the appropriate Lebesgue space $L^r(\mathbb{R}^3, \mathbb{R}^3)$ in a such case. For instance, it seems that we should consider $r < 1$, for $k = 3$, to obtain the same results as in Theorem 5 by Gagliardo-Nirenberg estimates in three dimensions.

The analysis is similar to the one performed for Theorem 5, the main change lies in the non-linear contribution $u \cdot \nabla u$. We do not use neither Lemma 1 nor Hardy-Littlewood-Sobolev inequality (2.19), but instead we take advantage of the double time integrals.

5.1 Case $k = 0$

Lemma 6. For all $T > 0$, $r \in (1, +\infty)$ and $\theta < 1$, $\int_0^T (T-t)^{-\theta} \|u(t, \cdot)\|_{L^r} dt < +\infty$, if $\theta < \frac{3}{2r}$.

Furthermore, if $r \in [3, +\infty)$, we can suppose that $\theta = \frac{3}{2r}$.

Remark 6. Then by Lemma 6, the condition is $\theta < \frac{1}{4}$, which is a weaker control than (1.4).

Still, this result allows to handle with the Lebesgue space L^r , with $r < +\infty$ arbitrary big.

Proof of Lemma 6. By the usual Duhamel formula around the heat equation, we get, for any $(t, x) \in [0, T] \times \mathbb{R}^3$

$$u(t, x) = \tilde{P}u_0(t, x) + \tilde{G}\mathbb{P}f(t, x) - \tilde{G}\mathbb{P}(u \cdot \nabla u)(t, x).$$

Next, by Minkowski inequality, for any $r \geq 1$,

$$\|u(t, \cdot)\|_{L^r} \leq \|u_0\|_{L^r} + \int_0^t \|\mathbb{P}f(s, \cdot)\|_{L^r} ds + \|\nabla \tilde{G}\mathbb{P}(u \otimes u)(t, \cdot)\|_{L^r}.$$

The last term in the r.h.s comes from the incompressible property of u (2.4) (i.e. $u \cdot \nabla u = \nabla \cdot (u^{\otimes 2})$), and by the convolution property of the heat semi-group.

From Young inequality for the convolution with $1 \leq p, q \leq +\infty$ such that

$$1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}, \tag{5.3}$$

and by Minkowski inequality,

$$\begin{aligned} \|\nabla \tilde{G}\mathbb{P}(u \otimes u)(t, \cdot)\|_{L^r} &\leq C \int_0^t \left(\int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} \nabla \tilde{p}(s, t, x, y) u^{\otimes 2}(s, y) dy \right)^r dx \right)^{\frac{1}{r}} ds \\ &\leq C \int_0^t \|\nabla \tilde{p}(s, t, \cdot, 0)\|_{L^p} \|u^{\otimes 2}(s, \cdot)\|_{L^q} ds \\ &\leq C \int_0^t [\nu(t-s)]^{-\frac{1}{2} - \frac{3(p-1)}{2p}} \|u(s, \cdot)\|_{L^{2q}}^2 ds. \end{aligned} \tag{5.4}$$

The Leray operator disappears in the first inequality thanks to a Calderón-Zygmund estimate.

Importantly, we can already remark that the term $(t-s)^{-\frac{1}{2}-\frac{3(p-1)}{2p}}$ is integrable if $p < \frac{3}{2}$.

Recalling that, from Gagliardo-Nirenberg inequality,

$$\int_0^T \|u(s, \cdot)\|_{L^6}^2 ds \leq C \int_0^T \|\nabla u(s, \cdot)\|_{L^2}^2 ds. \quad (5.5)$$

Without looking at the time singularity in the r.h.s. of (3.3), we can choose $q \leq 3$, and in order to get this control in L^6 of u , we can interpolate in the Lebesgue spaces,

$$\|u(s, \cdot)\|_{L^{2q}}^2 \leq C \|u(s, \cdot)\|_{L^2}^{2\alpha} \|u(s, \cdot)\|_{L^6}^{2(1-\alpha)}, \quad (5.6)$$

with $\alpha \in [0, 1]$ such that

$$\frac{1}{2q} = \frac{\alpha}{2} + \frac{1-\alpha}{6},$$

in other words,

$$\alpha = \frac{3-q}{2q}.$$

Let us also remark that $1-\alpha = \frac{3q-1}{2q}$, and $\frac{2\alpha}{1-\alpha} = \frac{2(3-q)}{3(q-1)}$.

That is to say, as $s \mapsto \|u(s, \cdot)\|_{L^6}$ lies in L_t^2 and that $s \mapsto \|u(s, \cdot)\|_{L^2}$ is in L_t^∞ , we can see from (5.6), that $\|u(s, \cdot)\|_{L^{2q}}^2$ is in $L^{\frac{1}{1-\alpha}} = L^{\frac{2q}{3(q-1)}}$. Specifically, for any $q \in [1, 3)$, from (5.6), we derive

$$\int_0^t \|u(s, \cdot)\|_{L^{2q}}^{\frac{2}{1-\alpha}} ds = \int_0^t \|u(s, \cdot)\|_{L^{2q}}^{\frac{4q}{3q-3}} ds \leq C \|u\|_{L_t^\infty L^2}^{\frac{2(3-q)}{3(q-1)}} \int_0^t \|u(s, \cdot)\|_{L^6}^2 ds. \quad (5.7)$$

Let us go back to (3.3), for $q \leq 3$, $p < \frac{3}{2}$, and for $\theta \in [0, 1)$ small enough (specified further),

$$\begin{aligned} & \int_0^T (T-t)^{-\theta} \|\nabla \tilde{G}\mathbb{P}(u \otimes u)(t, \cdot)\|_{L^r} dt \\ & \leq C \int_0^T (T-t)^{-\theta} \int_0^t [\nu(t-s)]^{-\frac{1}{2}-\frac{3(p-1)}{2p}} \|u(s, \cdot)\|_{L^{2q}}^2 ds dt \\ & = C \int_0^T \|u(s, \cdot)\|_{L^{2q}}^2 \int_s^T (T-t)^{-\theta} [\nu(t-s)]^{-\frac{1}{2}-\frac{3(p-1)}{2p}} ds dt. \end{aligned}$$

We have the useful integral result.

Proposition 6. For all $0 \leq s \leq T$ and $\theta, \beta < 1$

$$\int_s^T (T-t)^{-\theta} (t-s)^{-\beta} dt \leq C \left(\frac{T-s}{2}\right)^{1-\theta-\beta}.$$

Proof of Proposition 6.

$$\begin{aligned} \int_s^T (T-t)^{-\theta} (t-s)^{-\beta} dt &= \int_{\frac{s+T}{2}}^T (T-t)^{-\theta} (t-s)^{-\beta} dt + \int_s^{\frac{s+T}{2}} (T-t)^{-\theta} (t-s)^{-\beta} dt \\ &\leq \left(\frac{T-s}{2}\right)^{-\beta} \int_{\frac{s+T}{2}}^T (T-t)^{-\theta} dt + \left(\frac{T-s}{2}\right)^{-\theta} \int_s^{\frac{s+T}{2}} (t-s)^{-\beta} dt \\ &\leq [(1-\theta)^{-1} + (1-\beta)^{-1}] \left(\frac{T-s}{2}\right)^{1-\theta-\beta}. \end{aligned}$$

□

Hence, for all $\theta < 1$ and $p < \frac{3}{2}$,

$$\int_0^T (T-t)^{-\theta} \|\nabla \tilde{G}\mathbb{P}(u \otimes u)(t, \cdot)\|_{L^r} dt \leq C\nu^{-\frac{1}{2} - \frac{3(p-1)}{2p}} \int_0^T \|u(s, \cdot)\|_{L^{2q}}^2 (T-s)^{\frac{1}{2} - \theta - \frac{3(p-1)}{2p}} ds.$$

• If $q = 3$, then $\alpha = 0$ and

$$\int_0^T (T-t)^{-\theta} \|\nabla \tilde{G}\mathbb{P}(u \otimes u)(t, \cdot)\|_{L^r} dt \leq C\nu^{-\frac{1}{2} - \frac{3(p-1)}{2p}} \int_0^T \|u(s, \cdot)\|_{L^6}^2 (T-s)^{-1-\theta + \frac{3}{2p}} ds.$$

To use (5.5), we have to suppose that

$$\theta \leq \frac{3}{2p} - 1 = \frac{3}{2} \left(1 + \frac{1}{r} - \frac{1}{3}\right) - 1 = \frac{3}{2r},$$

from (5.3), which also yields that

$$r = \frac{3p}{3-2p} \in [3, +\infty),$$

for $1 \leq p < \frac{3}{2}$.

• If $q < 3$,

Next, by Hölder inequality, with the conjugate value $\left(\frac{1}{1-\alpha}\right)' = \left(\frac{2q}{3q-3}\right)' = \frac{2q}{3-q}$,

$$\begin{aligned} & \int_0^T (T-t)^{-\theta} \|\nabla \tilde{G}\mathbb{P}(u \otimes u)(t, \cdot)\|_{L^r} dt \\ & \leq C\nu^{-\frac{1}{2} - \frac{3(p-1)}{2p}} \left(\int_0^T (T-s)^{\left(\frac{1}{2} - \theta - \frac{3(p-1)}{2p}\right) \frac{2q}{3-q}} ds \right)^{\frac{3-q}{2q}} \left(\int_0^T \|u(s, \cdot)\|_{L^{2q}}^{\frac{4q}{3q-3}} ds \right)^{\frac{3q-3}{2q}}, \end{aligned} \quad (5.8)$$

which is finite, see (5.7), if

$$\left(\frac{1}{2} - \theta - \frac{3(p-1)}{2p}\right) \frac{2q}{3-q} > -1,$$

namely

$$\theta < -\frac{3}{2} + \frac{3}{2p} + \frac{3}{2q} = -\frac{3}{2} + \frac{3}{2} \left(1 + \frac{1}{r}\right) = \frac{3}{2r}.$$

In that case, we have $r \in (1, +\infty)$. □

Remark 7. We have to pay attention, if $r < \frac{3}{2}$, we cannot suppose that $\theta > 1$, implying a non integrable singularity in time, as we have to impose $\theta < 1$ to use Proposition 6.

5.1.1 Final control of $\int_0^T (T-t)^{-\theta} \|\nabla u(t, \cdot)\|_{L^r} dt$

Lemma 7. For all $T > 0$, $r \in (1, 3)$ and $\theta < 1$, $\int_0^T (T-t)^{-\theta} \|\nabla u(t, \cdot)\|_{L^r} dt < +\infty$, if $\theta \leq \frac{3-r}{2r}$.

Proof of Lemma 7. Let us recall interpolation inequality of Sobolev spaces by Brezis Mirunescu [BM18], for $6 \geq 2r$ and $\alpha \in [0, 1]$:

$$\|u(s, \cdot)\|_{W^{\gamma, 2r}} \leq C \|u(s, \cdot)\|_{W^{0,6}}^\alpha \|u(s, \cdot)\|_{W^{1,2}}^{1-\alpha} = C \|u(s, \cdot)\|_{L^6}^\alpha \|\nabla u(s, \cdot)\|_{L^2}^{1-\alpha},$$

such that

$$\begin{cases} \frac{1}{2r} = \frac{\alpha}{6} + \frac{1-\alpha}{2}, \\ \gamma = (1-\alpha). \end{cases}$$

That is to say, we obtain

$$1 = \frac{2r\alpha}{6} + (1 - \alpha)r \Leftrightarrow \alpha = \frac{3(r - 1)}{2r}, \quad (5.9)$$

and

$$\gamma = 1 - \alpha = \frac{3 - r}{2r}. \quad (5.10)$$

Hence, we get with the same integral permutation performed in Section 5.1,

$$\begin{aligned} & \int_0^T (T - t)^{-\theta} \left\| x \mapsto \int_0^t [\nu(t - s)]^{-1} \int_{\mathbb{R}^3} \bar{p}^{t,x}(s, t, x, y) |u(s, \theta_{s,t}(x)) - u(s, y)|^2 dy ds \right\|_{L^r} dt \\ & \leq C \int_0^T (T - t)^{-\theta} \int_0^t [\nu(t - s)]^{-1+\gamma} \|u(s, \cdot)\|_{W^{\gamma, 2r}}^2 ds dt \\ & \leq C \nu^{-1+\gamma} \int_0^T \|u(s, \cdot)\|_{L^6}^{2\alpha} \|\nabla u(t - \tilde{s}, \cdot)\|_{L^2}^{2(1-\alpha)} \int_s^T (t - s)^{-1+\gamma} (T - t)^{-\theta} dt d\tilde{s}, \end{aligned}$$

by interpolation inequality (2.20) and by Gagliardo-Nirenberg inequality $\|u(s, \cdot)\|_{L^6} \leq C \|\nabla u(s, \cdot)\|_{L^2}$, so

$$\begin{aligned} & \int_0^T (T - t)^{-\theta} \left\| x \mapsto \int_0^t [\nu(t - s)]^{-1} \int_{\mathbb{R}^3} \bar{p}^{t,x}(s, t, x, y) |u(s, \theta_{s,t}(x)) - u(s, y)|^2 dy ds \right\|_{L^r} dt \\ & \leq C \nu^{-1+\gamma} \int_0^T \|\nabla u(s, \cdot)\|_{L^2}^2 \int_s^T (t - s)^{-1+\gamma} (T - t)^{-\theta} dt ds \\ & \leq C \nu^{-1+\gamma} \int_0^T \|\nabla u(s, \cdot)\|_{L^2}^2 (T - s)^{-\theta+\gamma} ds, \end{aligned}$$

by Proposition 6 as soon as $\gamma > 0$ and $\theta < 1$.

To use inequality (1.4), we have to impose that

$$\theta \leq \gamma = \frac{6 - 2r}{r(6 - 2)} = \frac{3 - r}{2r},$$

and

$$r = \frac{1}{2} \left(\frac{\alpha}{6} + \frac{1 - \alpha}{2} \right)^{-1} = \frac{1}{2} \left(\frac{\alpha + 3(1 - \alpha)}{6} \right)^{-1} = \frac{3}{3 - 2\alpha} \in [1, 3),$$

for any $\alpha \in [0, 1)$.

Ultimately, to derive a inequality with ∇u from our analysis with ω , it suffices to use a Calderón-Zygmund estimate from the Biot and Savart representation $u = \nabla \times (-\Delta)^{-1} \omega$, available in the incompressible case.

Actually, we do not take into account $r = 1$ ($\alpha = 0$), because in such a case it is not direct to upper bound $\|\omega(t, \cdot)\|_{L^r}$ by $\|\nabla u(t, \cdot)\|_{L^r}$; unlike for $r \in (1, +\infty)$. Furthermore, we even have $\omega \in L^\infty([0, T], L^1(\mathbb{R}^3, \mathbb{R}^3))$.

Remark 8. We deduce from Lemma 7, and by Gagliardo-Nirenberg inequality,

$$\int_0^T (T - t)^{-\theta} \|u(t, \cdot)\|_{L^p} dt < +\infty,$$

with

$$\frac{1}{p} = \frac{1}{r} - \frac{1}{3} = \frac{3 - r}{3r} \Leftrightarrow p = \frac{3r}{3 - r},$$

and

$$\theta < \frac{3 - r}{2r} = \frac{3}{2r} - \frac{1}{2},$$

which is the same constraint as in Lemma 6.

As we already said in Remark 6, we have to exploit more the regularisation by the heat kernel in order to get controls involving more regularity than in (1.3) and (1.4). This is what is done in the next section, where we differentiate once more Duhamel formula (3.23); yet we have to be careful on the regularity of the Sobolev norm considered in a way that the time singularity remains integrable. \square

5.2 Control of $\int_0^T (T-t)^{-\theta} \|\nabla \omega(t, \cdot)\|_{L^r} dt$

Lemma 8. For all $T > 0$, $r \in [1, \frac{3}{2})$ and $\theta \in \mathbb{R}$, $\int_0^T (T-t)^{-\theta} \|\nabla^2 u(t, \cdot)\|_{L^r} dt < +\infty$ if $\theta < \min(1, \frac{3-2r}{2r})$.

Proof. Let us start again the analysis of ω from the proof of Lemma 5. We recall the representation of $\nabla \omega$,

$$\begin{aligned} \nabla \omega(t, x) &= \hat{P}^{t,x} \nabla \nabla \times u_0(t, x) + \hat{G}^{t,x} \nabla \nabla \times f(t, x) \\ &\quad + \int_0^t \int_{\mathbb{R}^3} \nabla_x \nabla_x \times \left(\nabla_y \hat{p}^{\tau, \xi}(s, t, x, y) [u(s, \theta_{s, \tau}(\xi)) - u(s, y)]^{\otimes 2} \right) \Big|_{(\tau, \xi)=(t, x)} dy ds. \end{aligned} \quad (5.11)$$

Still by interpolation of Brezis Mirunescu and in the Lebesgue spaces, for $p \geq 2r$

$$\|u(s, \cdot)\|_{W^{\gamma, 2r}} \leq C \|u(s, \cdot)\|_{L^2}^{\alpha \tilde{\alpha}} \|u(s, \cdot)\|_{L^6}^{\alpha(1-\tilde{\alpha})} \|\nabla u(s, \cdot)\|_{L^2}^{1-\alpha} \leq C \|u(s, \cdot)\|_{L^2}^{\alpha \tilde{\alpha}} \|\nabla u(s, \cdot)\|_{L^2}^{1-\alpha \tilde{\alpha}}, \quad (5.12)$$

by Gagliardo-Nirenberg interpolation inequality, and where we recall identities (5.9) and (5.10)

$$\begin{cases} \alpha = \frac{p(r-1)}{r(p-2)}, \\ \gamma = 1 - \alpha = \frac{p-2r}{r(p-2)}, \\ \tilde{\alpha} = \frac{6-p}{2p}. \end{cases}$$

With the same argument performed in (3.39), and from (5.11), but with the same integral permutation in time performed in Proposition 6, we derive

$$\begin{aligned} &\int_0^T (T-t)^{-\theta} \left\| x \mapsto \int_0^t \int_{\mathbb{R}^3} \nabla_x \nabla_x \times \left(\nabla_y \hat{p}^{\tau, \xi}(s, t, x, y) [u(s, \theta_{s, \tau}(\xi)) - u(s, y)]^{\otimes 2} \right) \Big|_{(\tau, \xi)=(t, x)} dy ds \right\|_{L^r} dt \\ &\leq C \int_0^T (T-t)^{-\theta} \left\| x \mapsto \int_0^t [\nu(t-s)]^{-\frac{3}{2}} \int_{\mathbb{R}^3} \hat{p}^{t,x}(s, t, x, y) |u(s, \theta_{s, t}(x)) - u(s, y)|^2 dy ds \right\|_{L^r} dt \\ &\leq C \int_0^T [\nu \tilde{s}]^{-\frac{3}{2} + \gamma} \int_{\tilde{s}}^T (T-t)^{-\theta} \|u(t - \tilde{s}, \cdot)\|_{W^{\gamma, 2r}}^2 dt d\tilde{s}. \end{aligned}$$

Next, by interpolation inequality (5.12),

$$\begin{aligned} &\int_0^T (T-t)^{-\theta} \left\| x \mapsto \int_0^t \int_{\mathbb{R}^3} \nabla_x \nabla_x \times \left(\nabla_y \hat{p}^{\tau, \xi}(s, t, x, y) [u(s, \theta_{s, \tau}(\xi)) - u(s, y)]^{\otimes 2} \right) \Big|_{(\tau, \xi)=(t, x)} dy ds \right\|_{L^r} dt \\ &\leq C \|u\|_{L_T^\infty L^2}^{2\alpha \tilde{\alpha}} \int_0^T [\nu \tilde{s}]^{-\frac{3}{2} + \gamma} \int_{\tilde{s}}^T (T-t)^{-\theta} \|\nabla u(t - \tilde{s}, \cdot)\|_{L^2}^{2(1-\alpha \tilde{\alpha})} dt d\tilde{s}. \end{aligned}$$

By Hölder inequality, we have

$$\begin{aligned} &\int_0^T (T-t)^{-\theta} \left\| x \mapsto \int_0^t \int_{\mathbb{R}^3} \nabla_x \nabla_x \times \left(\nabla_y \hat{p}^{\tau, \xi}(s, t, x, y) [u(s, \theta_{s, \tau}(\xi)) - u(s, y)]^{\otimes 2} \right) \Big|_{(\tau, \xi)=(t, x)} dy ds \right\|_{L^r} dt \\ &\leq C \left(\gamma - \frac{1}{2}\right)^{-1} \|u\|_{L_T^\infty L^2}^{2\alpha \tilde{\alpha}} \nu^{-\frac{3}{2} + \gamma} T^{\gamma - \frac{1}{2}} \left(\int_0^T (T-t)^{-\frac{\theta}{\alpha \tilde{\alpha}}} dt \right)^{\alpha \tilde{\alpha}} \left(\int_0^T \|u(t - \tilde{s}, \cdot)\|_{L^6}^2 dt \right)^{1-\alpha \tilde{\alpha}}, \end{aligned}$$

which is finite if $\gamma = 1 - \alpha = \frac{p-2r}{r(p-2)} > \frac{1}{2}$, implying that, for any $p \in [2, 6)$,

$$2(p-2r) > r(p-2) \Leftrightarrow r < \frac{2p}{p+2} \in [1, \frac{3}{2}) \Leftrightarrow p > \frac{2r}{2-r},$$

and if the following condition is satisfied,

$$1 > \frac{\theta}{\alpha\tilde{\alpha}} \Leftrightarrow \theta < \alpha\tilde{\alpha} = \frac{p(r-1)}{r(p-2)} \frac{6-p}{2p} < \frac{(r-1)(6 - \frac{2r}{2-r})}{2r(\frac{2r}{2-r} - 2)} = \frac{(3-2r)}{2r}.$$

as $p > \frac{2r}{2-r}$.

In fine, once again, to get an inequality with $\nabla^2 u$ instead of $\nabla\omega$, we can use a Calderón-Zygmund inequality for the Biot and Savart representation. \square

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