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LONG TIME BEHAVIOR OF KILLED FEYNMAN-KAC SEMIGROUPS WITH SINGULAR SCHRÖDINGER POTENTIALS

ARNAUD GUILLIN[†], DI LU[†], BORIS NECTOUX[†], AND LIMING WU[†]

ABSTRACT. In this work, we investigate the compactness and the long time behavior of killed Feynman-Kac semigroups of various processes arising from statistical physics with very general singular Schrödinger potentials. The processes we consider cover a large class of processes used in statistical physics, with strong links with quantum mechanics and (local or not) Schrödinger operators (including e.g. fractional Laplacians). For instance we consider solutions to elliptic differential equations, Lévy processes, the kinetic Langevin process with locally Lipschitz gradient fields, and systems of interacting Lévy particles. Our analysis relies on a Perron-Frobenius type theorem derived in a previous work [A. Guillin, B. Nectoux, L. Wu, 2020 J. Eur. Math. Soc.] for Feller kernels and on the tools introduced in [L. Wu, 2004, Probab. Theory Relat. Fields] to compute bounds on the essential spectral radius of a bounded nonnegative kernel.

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Key words. Feynman-Kac, singular potentials, Schrödinger, quasi-stationary distributions, Brownian particle, Lévy processes, kinetic Langevin.

1. SETTING AND RESULTS

1.1. Feynman-Kac semigroups and models.

1.1.1. *Setting and purpose of this work.* The purpose of this work is to study the basic properties and above all, the long time behavior of killed Feynman-Kac semigroups of several models with very general singular Schrödinger potentials (see Section 1.1.2 and more precisely [S1], [S2], and (1.22)). The killing occurs when the process exists a (position) subdomain¹. We have no assumption on the regularity of the boundary of this subdomain which can be bounded or not and whose boundary can intersect the points where the Schrödinger potential is infinite (which is the case of interest). For the first three considered processes, the singular potential arises from fixed a point located at 0 whereas for the fourth one, it is created by n interacting Lévy particles (see the fourth model below in Section 1.1.7). As explained in Section 1.3, more general singular potentials can be considered. This work is also motivated by the strong link between Feynman-Kac semigroups and solution of evolution equations associated with a Schrödinger type operator, see (1.29).

The introduction and the paper are organized as follows. We first introduce the class of singular Schrödinger potentials we will treat in details in this work and we then give the notation which will be used throughout this work. Next, we introduce the four models and we give preliminary results on their behaviors. These are the purposes of Section 1.1. The main results, which are Theorems 1, 2, 3, and 4 (see also Theorem 5), are given in Section 1.2. We provide also extensions of these results in Section 1.3. The related literature is given in Section 1.4. Section 2 is dedicated to the proofs of the main results.

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¹A domain is by definition a non-empty, open, and connected set.

In all this work, the set $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ is a filtered probability space, where the filtration satisfies the usual condition, $(B_s, s \geq 0)$ is a \mathbb{R}^d -standard Brownian motion, and²

$$d \geq 2.$$

1.1.2. *Singular Schrödinger potential.* As already mentioned above, we will consider the killed Feynman-Kac semigroups of four models used in statistical mechanics (which are also related to quantum mechanics through their generators) though many other processes can also be treated with our techniques (see Section 1.3). For the first three models we will work with a very general singular (at 0) Schrödinger potential $\mathbf{V}_{\mathbf{S}}$. More precisely, $\mathbf{V}_{\mathbf{S}} : \mathbb{R}^d \setminus \{0\} \rightarrow \mathbb{R}$ is assumed to be continuous, lower bounded (say by $-k_{\mathbf{S}}, k_{\mathbf{S}} \geq 0$) and satisfies one of the two conditions:

- [S1] $\mathbf{V}_{\mathbf{S}} \rightarrow +\infty$ if and only if $|x| \rightarrow 0^+$.
- [S2] $\mathbf{V}_{\mathbf{S}} \rightarrow +\infty$ if (and only if) $|x| \rightarrow 0^+$ or $|x| \rightarrow +\infty$.

Assumption [S2] differs from [S1] because the potentials satisfying [S2] confine also at $+\infty$. Notice that we can consider any kind of singularities at 0, e.g. the Coulomb potential, the Riesz potential, the Lennard-Jones potential, and the log-potential. Note that near 0 the potential $\mathbf{V}_{\mathbf{S}}$ is repulsive. In the fourth model below, the singularities in the Schrödinger potentials are created when the interacting (moving) Lévy particles collide, see more precisely (1.22). We also mention that our techniques allow to consider more general singularities, see Section 1.3 for examples.

1.1.3. *Notation.* Let \mathcal{S} be a polish space and denote by $\mathcal{B}(\mathcal{S})$ the Borel σ -algebra over \mathcal{S} . In the following, $b\mathcal{B}(\mathcal{S})$ (resp. $\mathcal{C}_b(\mathcal{S})$) is the set of bounded measurable (resp. bounded and continuous) functions $f : \mathcal{S} \rightarrow \mathbb{R}$. For a measurable function $\mathbf{W} : \mathcal{S} \rightarrow [1, +\infty]$, we denote by $b\mathcal{B}_{\mathbf{W}}(\mathcal{S})$ (resp. $\mathcal{C}_{b\mathbf{W}}(\mathcal{S})$) the space of measurable (resp. continuous) functions $f : \mathcal{S} \rightarrow \mathbb{R}$ such that f/\mathbf{W} is bounded over \mathcal{S} . These spaces are endowed with the norm $\|f\|_{b\mathcal{B}_{\mathbf{W}}(\mathcal{S})} := \sup_{\mathcal{S}} |f/\mathbf{W}|$. The function $\mathbf{1}$ denotes the constant function over \mathcal{S} which equals 1. The set $\mathcal{P}(\mathcal{S})$ is the space of probability measures over \mathcal{S} and $\mathcal{P}_{\mathbf{W}}(\mathcal{S}) := \{\nu \in \mathcal{P}(\mathcal{S}), \nu(\mathbf{W}) < +\infty\}$. Note that $b\mathcal{B}_{\mathbf{1}}(\mathcal{S}) = b\mathcal{B}(\mathcal{S})$ and $\mathcal{P}_{\mathbf{1}}(\mathcal{S}) = \mathcal{P}(\mathcal{S})$. The space $\mathcal{C}([0, T], \mathcal{S})$ denotes the space of continuous \mathcal{S} -valued functions over $[0, T]$ and the (Skorokhod) space of \mathcal{S} -valued functions that are right-continuous and have left-hand limits (say *càdlàg*) over $[0, T]$ is denoted by $\mathcal{D}([0, T], \mathcal{S})$. For a bounded linear operator T over a Banach space, we denote by $r_{sp}(T)$ its spectral radius, and by $r_{ess}(T)$ its essential spectral radius (see e.g. [36, Section 3.1] for a definition). For a stochastic process $(\eta_t, t \geq 0)$ and $T \geq 0$, we denote by $\eta_{[0, T]} = (\eta_t, t \in [0, T])$ the process up to time T .

1.1.4. *First model, a laboratory Brownian elliptic model.* In this section, we introduce the overdamped Langevin process (1.1), which is an elliptic diffusion driven by a Brownian noise. This is our first model. The analysis of its killed Feynman-Kac semigroup will turn out to be very instructive, which explains our choice to start with this prototypical model.

Model 1. For $x \in \mathbb{R}^d$, consider a single Brownian particle $(X_t, t \geq 0)$ solution to the elliptic stochastic differential equation in \mathbb{R}^d

$$dX_t = \mathbf{b}_{\mathbf{c}}(X_t)dt + dB_t, \quad X_0 = x. \quad (1.1)$$

²Except in Section 1.2.3 where, only in this section, $d \geq 1$.

In all this work, $\mathbf{b}_c : \mathcal{R}^d \rightarrow \mathcal{R}^d$ is a locally Lipschitz vector field. We define the following two assumptions:

$$[\mathbf{c1}] \lim_{|x| \rightarrow +\infty} \mathbf{b}_c(x) \cdot \frac{x}{|x|} = -\infty.$$

$$[\mathbf{c2}] |\mathbf{b}_c| \text{ has at most linear growth over } \mathcal{R}^d.$$

We mention that the case when the potential \mathbf{b}_c is singular has already been treated in [37, 35]³ and thus will not be considered in this work.

Proposition 1. *Assume [c1] or [c2]. For all $x \in \mathcal{R}^d$ there exists a unique strong solution $(X_t(x), t \geq 0)$ to (1.1) over \mathcal{R}^d whose sample paths are a.s. continuous. In addition, for all $T > 0$, the following Girsanov's formula holds:*

$$\left. \frac{d\mathbf{P}_x}{d\mathbf{P}_x^0} \right|_{\mathcal{F}_T} = m_T^0(x), \quad (1.2)$$

where \mathbf{P}_x (resp. \mathbf{P}_x^0) is the law of $(X_t(x), t \geq 0)$ (resp. of $(B_s(x) = x + B_s, s \geq 0)$), and for $t \geq 0$, $m_t^0(x) = \exp \left[\int_0^t \mathbf{b}_c(B_s(x)) \cdot dB_s - \frac{1}{2} \int_0^t |\mathbf{b}_c(B_s(x))|^2 ds \right]$ is the Doléans-Dade exponential (true) martingale associated with the process $(X_t, t \geq 0)$.

Proof. Assume [c2]. Then, there exists a unique pathwise solution to (1.1), see e.g. [33, Theorem 2.2 in Section 5]. The Girsanov formula then follows from [33, Theorem 3.1 in Section 7] and [33, Theorem 1.1 in Section 7]. Note also that in this case there exists $C > 0$ such that for all $T > 0$, it holds a.s. $|X_t(x)| \leq Ce^T(|x| + T + \sup_{u \in [0, T]} |B_u|)$ and therefore

$$\mathbb{P}_x[\sigma_{B(0, R)}^{oL} \leq t] \rightarrow 0 \text{ as } R \rightarrow +\infty \text{ uniformly in } x \text{ in the compact sets,} \quad (1.3)$$

where $\sigma_{B(x, R)}^{oL}$ is the first exit time from the open ball $B(x, R)$ for the process $(X_t, t \geq 0)$.

Assume now [c1]. For all $x \in \mathcal{R}^d$, define $\mathbf{L}(x) = |x| \times (1 - \chi(x))$ where $\chi \in \mathcal{C}^\infty(B(0, 1), [0, 1])$ and $\chi = 1$ on $B(0, 1/2)$, and

$$\mathbf{W}(x) = e^{\epsilon \mathbf{L}(x)}, \quad \epsilon > 0. \quad (1.4)$$

Denote by $\mathcal{L}^{oL} = \mathbf{b}_c \cdot \nabla + \frac{1}{2} \Delta$ the infinitesimal generator of the diffusion (1.1). We have for every $x \in \mathcal{R}^d$,

$$\mathcal{L}^{oL} \mathbf{W}(x) = \left[\epsilon \mathbf{b}_c(x) \cdot \nabla \mathbf{L}(x) + \frac{1}{2} \epsilon^2 |\nabla \mathbf{L}(x)|^2 + \frac{1}{2} \epsilon \Delta \mathbf{L}(x) \right] \mathbf{W}(x).$$

Since $\nabla \mathbf{L}$ and $\Delta \mathbf{L}$ are bounded over \mathcal{R}^d , we have $\mathcal{L}^{oL} \mathbf{W}(x) / \mathbf{W}(x) \rightarrow -\infty$ thanks to [c1]. In particular $\mathcal{L}^{oL} \mathbf{W} \leq c \mathbf{W}$ over \mathcal{R}^d . Together with the fact that \mathbf{b}_c is locally Lipschitz and using well known arguments, this implies the first statement in Proposition 1 and also that for some $c > 0$ and all $x \in \mathcal{W}_R$,

$$\mathbb{P}_x[\sigma_{\mathcal{W}_R}^{oL} \leq t] \leq \frac{e^{ct}}{R} \mathbf{W}(x), \quad (1.5)$$

where for $R > 0$, $\mathcal{W}_R := \{y \in \mathcal{R}^d, \mathbf{W}(y) < R\}$ is an open bounded subset of \mathcal{R}^d and $\sigma_{\mathcal{W}_R}^{oL} := \inf\{t \geq 0, X_t \notin \mathcal{W}_R\}$. Using the fact that for all $x \in \mathcal{R}^d$, a.s. both $\sigma_{\mathcal{W}_R}^{oL}(x) \nearrow +\infty$ and $\sigma_{\mathcal{W}_R}^0 \nearrow +\infty$ (where $\sigma_{\mathcal{W}_R}^0 := \inf\{t \geq 0, B_t \notin \mathcal{W}_R\}$) as $R \rightarrow +\infty$, one proves the Girsanov formula with the same arguments as those used in the proof of [9, Proposition 2.2] (see also the proof of [74, Lemma 1.1]). \square

³For both elliptic and hypoelliptic processes.

Define the space

$$\mathcal{R}_0^d := \mathcal{R}^d \setminus \{0\}. \quad (1.6)$$

The space \mathcal{R}_0^d is Polish (it is equipped with a metric generating the original topology and making \mathcal{R}_0^d a complete space⁴).

Lemma 1. *Assume [c1] or [c2]. Then, for all $x \in \mathcal{R}_0^d$ and $T > 0$, $\mathbb{P}_x[X_{[0,T]} \in \mathcal{C}([0,T], \mathcal{R}_0^d)] = 1$, and, if [S1] or [S2] holds, for all $t \geq 0$, $\int_0^t \mathbf{V}_s(X_s(x)) ds$ is a.s. finite.*

Proof. Since $d \geq 2$, it is well known, see e.g. [34, Theorem 4.1 in Section 11] that any point is *nonattainable* for a \mathcal{R}^d -standard Brownian motion ($W_s, s \geq 0$) and therefore it holds for all $y \in \mathcal{R}^d$, $\mathbb{P}[W_s = y, \text{ for some } s > 0] = 0$. Thanks to the Girsanov formula (1.2), we have for all $T > 0$, $\mathbb{P}[|X_s(x)| = 0, \text{ for some } s \in [0, T]] = 0, \forall x \in \mathcal{R}_0^d$. Thus, $\mathbb{P}_x[\forall s \geq 0, X_s \in \mathcal{R}_0^d] = 1, \forall x \in \mathcal{R}_0^d$. By continuity of the trajectories of $(X_s(x), s \geq 0)$ in \mathcal{R}^d , for all $x \in \mathcal{R}_0^d$, we deduce that

$$\mathbb{P}_x[X_{[0,t]} \in \mathcal{C}([0,t], \mathcal{R}_0^d)] = 1. \quad (1.8)$$

This ends the proof of the Lemma. \square

In view of Lemma 1, \mathcal{R}_0^d is the natural state space to study the Feynman-Kac semigroup associated with \mathbf{V}_s for the process (1.1) (see also Remark 1). In all this work, \mathcal{O} is a subdomain of \mathcal{R}_0^d , and

$$\sigma_{\mathcal{O}}^{\text{oL}} := \inf\{t \geq 0, X_t \notin \mathcal{O}\}$$

is the first exit time from \mathcal{O} for the process $(X_t, t \geq 0)$. The killed (outside \mathcal{O}) Feynman-Kac semigroup $(Q_t^{\text{oL}}, t \geq 0)$ over \mathcal{R}_0^d associated with \mathbf{V}_s and the process (1.1) is then defined by

$$Q_t^{\text{oL}} f(x) = \mathbb{E}_x \left[f(X_t) e^{-\int_0^t \mathbf{V}_s(X_s) ds} \mathbf{1}_{t < \sigma_{\mathcal{O}}^{\text{oL}}} \right], \quad t \geq 0, x \in \mathcal{O}, \text{ and } f \in b\mathcal{B}(\mathcal{O}). \quad (1.9)$$

Note that the confinement at $+\infty$ comes from the dynamics itself (i.e. from \mathbf{b}_c) or from the Schrödinger potential \mathbf{V}_s . The confinement at 0 comes from \mathbf{V}_s . Note also that there is no killing when $\mathcal{O} = \mathcal{R}^d$. The superscript oL in the previous notation stands for the fact that we consider the overdamped Langevin process (1.1).

For $t \geq 0$, the transition kernel $Q_t^{\text{oL}}(x, dy)$ at time $t \geq 0$ defines, through a natural pairing, an (adjoint) operator on the set $\mathcal{M}_b(\mathcal{O})$ of all σ -additive measures ν of bounded variations over \mathcal{O} :

$$\nu Q_t^{\text{oL}}(\mathcal{A}) = \int_{\mathcal{R}_0^d} Q_t^{\text{oL}}(x, \mathcal{A}) \nu(dx) = \mathbb{E}_{\nu} \left[\mathbf{1}_{\mathcal{A}}(X_t) e^{-\int_0^t \mathbf{V}_s(X_s) ds} \mathbf{1}_{t < \sigma_{\mathcal{O}}^{\text{oL}}} \right], \quad \mathcal{A} \in \mathcal{B}(\mathcal{O}).$$

The killed renormalized Feynman-Kac semigroup is defined by:

$$\nu P_t^{\text{oL}}(\mathcal{A}) = \frac{\nu Q_t^{\text{oL}}(\mathcal{A})}{\nu Q_t^{\text{oL}}(\mathcal{O})} = \frac{\mathbb{E}_{\nu} \left[\mathbf{1}_{\mathcal{A}}(X_t) e^{-\int_0^t \mathbf{V}_s(X_s) ds} \mathbf{1}_{t < \sigma_{\mathcal{O}}^{\text{oL}}} \right]}{\mathbb{E}_{\nu} \left[e^{-\int_0^t \mathbf{V}_s(X_s) ds} \mathbf{1}_{t < \sigma_{\mathcal{O}}^{\text{oL}}} \right]}. \quad (1.10)$$

The main result for this model is Theorem 1.

⁴In our setting, it is the metric

$$d_{\mathcal{R}_0^d} : (x, y) \in (\mathcal{R}_0^d)^2 \mapsto |x - y| + ||x|^{-1} - |y|^{-1}| \quad (1.7)$$

which makes the boundary of \mathcal{R}_0^d (namely $\{0\}$) looks like $+\infty$. Note that for $x \in \mathcal{R}_0^d$ and a sequence $(x_n) \subset \mathcal{R}_0^d$, $x_n \rightarrow x$ in \mathcal{R}_0^d if and only if $x_n \rightarrow x$ in the original space \mathcal{R}^d , i.e. in \mathcal{R}_0^d the topology induced by $d_{\mathcal{R}_0^d}$ coincides with the one of \mathcal{R}^d .

Remark 1. *The first thing to establish was obviously the space to work in.*

Lemma 1 suggests indeed to work outside the singularity of $\mathbf{V}_{\mathbf{S}}$, i.e. on \mathcal{R}_0^d . Anticipating a little bit, another reason to work on \mathcal{R}_0^d is the following. Compactness properties of the non-conservative semigroup $(Q_t^{oL}, t \geq 0)$ is (with our techniques) obtained by constructing a Lyapunov function \mathbf{W} over the chosen space \mathcal{S} such that $(\mathcal{L}^{oL} - \mathbf{V}_{\mathbf{S}})\mathbf{W}(x)/\mathbf{W}(x) \rightarrow -\infty$ when $x \rightarrow \{\infty\} \cup \{\partial\mathcal{S}\}$ ($\mathcal{L}^{oL} - \mathbf{V}_{\mathbf{S}}$ being the infinitesimal generator of the Feynman-Kac semigroup). This definitely suggests to work outside the singularity of $\mathbf{V}_{\mathbf{S}}$ since we have for free that $\mathbf{V}_{\mathbf{S}}(x) \rightarrow +\infty$ as $|x| \rightarrow 0^+$. Apart from that, note that there exist a large class of singular potentials⁵ $\mathbf{V}_{\mathbf{S}}$ such that $\mathbb{P}_0[\int_0^t \mathbf{V}_{\mathbf{S}}(B_s)ds = +\infty] = 1$ and in this case the associated killed Feynman-Kac semigroup of the Brownian motion will not be topologically irreducible over \mathcal{R}^d .

Let us mention that working with Schrödinger potentials in the Kato class⁶ allows, with different techniques, to work in a state space including the singularity, see e.g. [17, Section 3.2] and [8, 43] where the spectral analysis of some $L^2(\mathcal{R}^d)$ -symmetric Feynman-Kac semigroups is carried out in $L^p(\mathcal{R}^d)$ when the Schrödinger potential belongs to the (local) Kato class (which however allows some singular attractive potentials). Note that the Kato class depends on the underlying process and does not include all kinds of singularities as it is a subset of $L_{loc}^1(\mathcal{R}^d)$.

1.1.5. *Second model: the case of Lévy processes.* The second model we consider is the killed Feynman-Kac semigroup of Lévy processes with a singular Schrödinger potential.

Model 2. Let $(L_s, s \geq 0)$ be Lévy process over \mathcal{R}^d and

$$\sigma_{\mathcal{O}}^{Le} := \inf\{t \geq 0, X_t \notin \mathcal{O}\},$$

where we recall \mathcal{O} is any subdomain of \mathcal{R}_0^d (see (1.6)). We make the following assumptions over the Lévy process $(L_s, s \geq 0)$:

- [L1] For all $t > 0$, L_t admits a density w.r.t. the Lebesgue measure over \mathcal{R}^d .
- [L2] The killed semigroup $(L_t, t \in [0, \sigma_{\mathcal{O}}^{Le}))$ is topologically irreducible over \mathcal{O} , i.e. for all $t > 0$, all $x \in \mathcal{O}$, and all non-empty open subset O of \mathcal{O} , $\mathbb{P}_x(L_t \in O, t < \sigma_{\mathcal{O}}^{Le}) > 0$.
- [L3] If $\mathcal{R}_0^d \setminus \overline{\mathcal{O}}$ is nonempty, then $\mathbb{P}_x[\sigma_{\mathcal{O}}^{Le} < +\infty] > 0$ for some $x \in \mathcal{O}$.

We give in the appendix below a non exhaustive list of important examples of Lévy processes over \mathcal{R}^d ($d \geq 2$) satisfy [L1], and also [L2]-[L3] for any subdomain \mathcal{O} of \mathcal{R}^d . For $x \in \mathcal{R}^d$, we will denote by $(L_s(x), s \geq 0)$ the process $(x + L_s, s \geq 0)$. We start with the following lemma.

Lemma 2. *Assume [L1]. Then, for all $z \in \mathcal{R}_0^d$,*

$$\mathbb{P}_z[\forall t \geq 0, L_t = 0 \text{ or } L_{t-} = 0] = 0. \tag{1.12}$$

In addition, for all $x \in \mathcal{R}_0^d$ and $t \geq 0$, $\mathbb{P}_x[L_{[0,t]} \in \mathcal{D}([0, t], \mathcal{R}_0^d)] = 1$, and, if [S1] or [S2] holds, $\int_0^t \mathbf{V}_{\mathbf{S}}(L_s(x))ds$ is a.s. finite.

⁵E.g. $|x|^{-b}$, $b > 2$. Use indeed the L.I.L $\mathbb{P}[\limsup_{s \rightarrow 0} |B_s|/(2s \log_2(1/s))^{1/2} = 1] = 1$, see e.g. [63, Chapter II].

⁶This is the class of functions $\mathbf{V}_{\mathbf{S}} \in L_{loc}^1(\mathcal{R}^d)$ such that (see e.g. [8]),

$$\lim_{t \rightarrow 0^+} \sup_{x \in \mathcal{R}^d} \mathbb{E}_x \left[\int_0^t |\mathbf{V}_{\mathbf{S}}|(X_s)ds \right] = 0. \tag{1.11}$$

Proof. Since for all $s > 0$, L_s has a density w.r.t. the Lebesgue measure, (1.12) is a consequence of [46, Theorem 3] (recall that we assume $d \geq 2$). Moreover, (1.12) implies that for all $z \in \mathcal{R}_0^d$ and $T > 0$,

$$\text{there exists a.s. } \epsilon > 0 \text{ such that } |L_t(z)| \geq \epsilon \text{ for all } t \in [0, T]. \quad (1.13)$$

Indeed, assume that there exist two sequences $\epsilon_n \rightarrow 0^+$ and $t_n \in [0, T]$ such that $|L_{t_n}(z)| \leq \epsilon_n$. Up to extracting a subsequence, we assume that $t_n \rightarrow t_* \in [0, T]$. One of the two sets $S_-^* = \{n \geq 1, t_n < t_*\}$ and $S_+^* = \{n \geq 1, t_* \leq t_n\}$ has an infinite cardinality. Since the trajectories of $(L_s(x), s \geq 0)$ are right-continuous and have left limits everywhere, if S_-^* has an infinite cardinality, then, $L_{t_*^-}(z) = 0$ whereas if S_+^* has an infinite cardinality, $L_{t_*^+}(z) = 0$, but this occurs with null probability. This proves (1.13). The proof of the lemma is complete. \square

The killed Feynman-Kac semigroup $(Q_t^{Le}, t \geq 0)$ over the space \mathcal{O} associated with the potential \mathbf{V}_s and the process $(L_s, s \geq 0)$ is defined by

$$Q_t^{Le} f(x) = \mathbb{E}_x \left[f(L_t) e^{-\int_0^t \mathbf{V}_s(L_s) ds} \mathbf{1}_{t < \sigma_{\mathcal{O}}^{Le}} \right], \quad t \geq 0, x \in \mathcal{O}, \text{ and } f \in b\mathcal{B}(\mathcal{O}). \quad (1.14)$$

Note also that there is no killing when $\mathcal{O} = \mathcal{R}_0^d$ since in this case, due to Lemma 2, for all $x \in \mathcal{R}_0^d$, a.s. $\inf\{t \geq 0, X_t \notin \mathcal{R}_0^d\} = +\infty$. The associated killed renormalized Feynman-Kac semigroup is then defined by, for $\nu \in \mathcal{M}_b(\mathcal{O})$ and $\mathcal{A} \in \mathcal{B}(\mathcal{O})$, $\nu Q_t^{Le}(\mathcal{A}) = \nu Q_t^{Le}(\mathcal{A}) / \nu Q_t^{Le}(\mathcal{O})$. The main result for this killed semigroup is Theorem 2.

1.1.6. *The third model: a prototypical hypoelliptic process.* The third model we consider is the Feynman-Kac semigroup associated with the kinetic Langevin process (1.15) and with the singular Schrödinger potential \mathbf{V}_s .

Model 3. Let $\gamma > 0$ and $(Y_t = (x_t, v_t), t \geq 0)$ be the solution to the stochastic differential equation in $\mathcal{R}^d \times \mathcal{R}^d$:

$$dx_t = v_t dt, \quad dv_t = -\nabla \mathbf{V}_c(x_t) dt - \gamma v_t dt + dB_t, \quad (1.15)$$

where $\mathbf{V}_c : \mathcal{R}^d \rightarrow [1, +\infty)$ is a differentiable function such that $\nabla \mathbf{V}_c$ is locally Lipschitz. The kinetic Langevin process (also called the underdamped Langevin process) is a prototypical kinetic diffusion which is widely used in statistical physics and in molecular dynamics [55]. The long time behavior of such a process as well as its ergodic properties are now well known, see for instance [71, 74, 57, 40, 41, 56, 7]. In all this work, we write $\mathbf{y} = (x, v)$ for an element in $\mathcal{R}^d \times \mathcal{R}^d$. The Hamiltonian of the process (1.15) is, for $\mathbf{y} = (x, v) \in \mathcal{R}^{2d}$, $\mathbf{H}(x, v) = \mathbf{V}_c(x) + \frac{1}{2}|v|^2$. We introduce the process $(Y_t^0 = (x_t^0, v_t^0), t \geq 0)$ solution to the stochastic differential equation in $\mathcal{R}^d \times \mathcal{R}^d$:

$$dx_t = v_t dt, \quad dv_t = dB_t. \quad (1.16)$$

Since \mathbf{V}_c is lower bounded, we have the following result from [74, Lemma 1.1].

Proposition 2. *Then, for all $\mathbf{y} \in \mathcal{R}^{2d}$, there exists a unique strong solution $(Y_t(\mathbf{y}), t \geq 0)$ to (1.15) over \mathcal{R}^{2d} whose sample paths are a.s. continuous. Finally, for all $T > 0$, the following Girsanov's formula holds for all $T > 0$,*

$$\frac{d\mathbf{P}_{\mathbf{y}}}{d\mathbf{P}_{\mathbf{y}}^0} \Big|_{\mathcal{F}_T} = \mathbf{m}_T^0(\mathbf{y}), \quad (1.17)$$

where $\mathbf{P}_{\mathbf{y}}$ (resp. $\mathbf{P}_{\mathbf{y}}^0$) is the law of $(Y_t(\mathbf{y}), t \geq 0)$ (resp. of $(Y_t^0(\mathbf{y}), t \geq 0)$, see (1.16)), and $(\mathbf{m}_t^0, t \geq 0)$ is the Doléans-Dade exponential (true) martingale defined by $\mathbf{m}_t^0 = \exp \left[-\int_0^t (\gamma v_s^0 + \nabla \mathbf{V}_c(x_s^0)) dB_s - \frac{1}{2} \int_0^t |\gamma v_s^0 + \nabla \mathbf{V}_c(x_s^0)|^2 ds \right]$.

In all this work, when we will deal with the process (1.15), the state space we will consider is

$$\mathcal{R}_0^{2d} := (\mathcal{R}^d \setminus \{0\}) \times \mathcal{R}^d = \mathcal{R}_0^d \times \mathcal{R}^d \quad (\text{see (1.6)}). \quad (1.18)$$

Consider a metastable set for the dynamics (1.15), where we recall that such sets are of the form $\mathcal{D} = \mathcal{O} \times \mathcal{R}^d$ (\mathcal{O} being a subdomain of \mathcal{R}_0^d), see [36, 54]. Set

$$\sigma_{\mathcal{D}}^{kL} := \inf\{t \geq 0, Y_t \notin \mathcal{D}\} = \inf\{t \geq 0, x_t \notin \mathcal{O}\}.$$

We have the following result.

Lemma 3. *For all $\mathbf{y} \in \mathcal{R}_0^{2d}$,*

$$\mathbb{P}_{\mathbf{y}}[\forall t \geq 0, |x_t| > 0] = 1. \quad (1.19)$$

In particular, for all $\mathbf{y} \in \mathcal{R}_0^{2d}$ and $t \geq 0$, $\mathbb{P}_{\mathbf{y}}[Y_{[0,t]} \in \mathcal{C}([0,t], \mathcal{R}_0^{2d})] = 1$, and, if [S1] or [S2] holds, $\int_0^t \mathbf{V}_{\mathbf{S}}(x_s(\mathbf{y}))ds$ is a.s. finite.

Lemma 3 shows that the unbounded closed set $M = \{0\} \times \mathcal{R}^d$ is, using the terminology introduced in [34], nonattainable. The full degeneracy of the noise on the orthogonal of M as well as the fact that M is not the boundary of a \mathcal{C}^3 bounded domain prevent us from using directly the results of [34]. We will rather rely on direct arguments as well as nonattainability results for Gaussian processes. The proof of Lemma 3 is given in Section 2.3.1.

The killed (outside \mathcal{D}) Feynman-Kac semigroup $(Q_t^{kL}, t \geq 0)$ over \mathcal{D} associated with $\mathbf{V}_{\mathbf{S}}$ and the process $(Y_t, t \geq 0)$ solution to (1.15) is defined by:

$$Q_t^{kL} f(\mathbf{y}) = \mathbb{E}_{\mathbf{y}} \left[f(Y_t) e^{-\int_0^t \mathbf{V}_{\mathbf{S}}(x_s) ds} \mathbf{1}_{t < \sigma_{\mathcal{D}}^{kL}} \right], \quad t \geq 0, \mathbf{y} \in \mathcal{D}, \text{ and } f \in b\mathcal{B}(\mathcal{D}). \quad (1.20)$$

Its associated killed renormalized Feynman-Kac semigroup is then defined by, for $\nu \in \mathcal{M}_b(\mathcal{D})$ and $\mathcal{A} \in \mathcal{B}(\mathcal{D})$, $\nu P_t^{kL}(\mathcal{A}) = \nu Q_t^{kL}(\mathcal{A}) / \nu Q_t^{kL}(\mathcal{D})$. The main result for this killed semigroup is Theorem 3.

1.1.7. The last model: interacting Lévy particles. So far, in the three previous models, the potential is created by a fixed point (located at 0). We now consider the situation when it is no longer the case. More precisely, we consider the following archetypical model of interacting particles.

Model 4. Let $(L_t^j, t \geq 0)_{j=1, \dots, n}$ be $n \geq 2$ independent \mathcal{R}^d -copies ($d \geq 2$) of a Lévy process $(L_t, t \geq 0)$ satisfying [L1]. Denote by

$$\mathcal{E} := \{\mathbf{x} = (x_1, \dots, x_n) \in (\mathcal{R}^d)^n \text{ s.t. } \forall i \neq j, x_i \neq x_j\}. \quad (1.21)$$

The set \mathcal{E} is a non-empty open (unbounded) connected subset of \mathcal{R}^{2d} (recall that $d \geq 2$) and its boundary is $\partial \mathcal{E} := \cup_{i < j} \{\mathbf{x} \in (\mathcal{R}^d)^n, x_i = x_j\}$. Let $\mathbf{U}_{\mathbf{S}}$ be the Schrödinger potential defined by

$$\mathbf{U}_{\mathbf{S}} : \mathbf{x} = (x_1, \dots, x_n) \in \mathcal{E} \mapsto \sum_{i=1}^n \mathbf{V}_{\infty}(x_i) + \sum_{i < j}^n \mathbf{v}_{\mathbf{S}}(|x_i - x_j|), \quad (1.22)$$

where and for some $k_{\mathbf{S}} \geq 0$:

$$\mathbf{V}_{\infty} : \mathcal{R}^d \rightarrow \mathcal{R} \text{ is a continuous coercive potential,} \quad (1.23)$$

and for some $k_{\mathbf{S}} \geq 0$:

$$\mathbf{v}_{\mathbf{S}} : \mathcal{R}_+^* \rightarrow \mathcal{R} \text{ is continuous, } \mathbf{v}_{\mathbf{S}} \geq -k_{\mathbf{S}}, \text{ and } \mathbf{v}_{\mathbf{S}}(u) \rightarrow +\infty \text{ iff } u \rightarrow 0^+. \quad (1.24)$$

By coercive we mean here: $\mathbf{V}_\infty(x) \rightarrow +\infty$ when $|x| \rightarrow +\infty$. Note that $\mathbf{U}_\mathbf{S}$ is continuous and lower bounded over $\{\mathbf{U}_\mathbf{S} < +\infty\} = \mathcal{E}$, and $\mathbf{U}_\mathbf{S}(\mathbf{x}) \rightarrow +\infty$ if and only if $|\mathbf{x}| \rightarrow \infty$ or $\mathbf{x} \rightarrow \partial\mathcal{E}$. For $t \geq 0$ and $\mathbf{x} = (x_1, \dots, x_n) \in (\mathcal{R}^d)^n$, consider the process $(\Theta_t(\mathbf{x}), t \geq 0)$ defined by:

$$\Theta_t(\mathbf{x}) := (x_1 + L_t^1, \dots, x_n + L_t^n). \quad (1.25)$$

Let \mathcal{U} be a subdomain of \mathcal{E} and set $\sigma_{\mathcal{U}}^\Theta := \inf\{t \geq 0, \Theta_t \notin \mathcal{U}\}$. The killed (outside \mathcal{U}) Feynman-Kac semigroup $(Q_t^\Theta, t \geq 0)$ over \mathcal{U} associated with $\mathbf{U}_\mathbf{S}$ and the Lévy process $(\Theta_t, t \geq 0)$ is defined by:

$$Q_t^\Theta f(\mathbf{x}) = \mathbb{E}_\mathbf{x} \left[f(\Theta_t) e^{-\int_0^t \mathbf{U}_\mathbf{S}(\Theta_s) ds} \mathbf{1}_{t < \sigma_{\mathcal{U}}^\Theta} \right], \quad t \geq 0, \mathbf{x} \in \mathcal{U}, \text{ and } f \in b\mathcal{B}(\mathcal{U}), \quad (1.26)$$

and its killed renormalized semigroup is $\nu P_t^\Theta(\mathcal{A}) = \nu Q_t^\Theta(\mathcal{A}) / \nu Q_t^\Theta(\mathcal{U})$, $\mathcal{A} \in \mathcal{B}(\mathcal{U})$ and $\nu \in \mathcal{M}_b(\mathcal{U})$.

Lemma 4. *For all $\mathbf{x} \in \mathcal{E}$ and $T \geq 0$, $\mathbb{P}_\mathbf{x}[\Theta_{[0,T]} \in \mathcal{D}([0,T], \mathcal{E})] = 1$ and $\int_0^T \mathbf{U}_\mathbf{S}(\Theta_s(\mathbf{x})) ds$ is a.s. finite.*

Proof. For all $i \neq j$ and $t \geq 0$, set $\Upsilon_t^{i,j} = L_t^j - L_t^i$. Then, $(\Upsilon_t^{i,j}, t \geq 0)$ is a Lévy process and for $t > 0$, $\Upsilon_t^{i,j}$ has a density w.r.t. the Lebesgue measure. Therefore, from [46, Theorem 3], we have for all $\mathbf{x} \in \mathcal{E}$ and for all $i \neq j$, $\mathbb{P}_\mathbf{x}[\exists t \geq 0, \Upsilon_t^{i,j} = x_i - x_j \text{ or } \Upsilon_t^{i,j} = x_i - x_j] = 0$. Fix $\mathbf{x} \in \mathcal{E}$, $i \neq j$, and $t \geq 0$. Then there exists a.s. $\epsilon > 0$ such that $\text{dist}(\Theta_s(\mathbf{x}), \partial\mathcal{E}) \geq \epsilon$, for all $s \in [0, t]$. Else there exists $t_* \in [0, T]$ such that either $\text{dist}(\Theta_{t_*}(\mathbf{x}), \partial\mathcal{E}) = 0$ or $\text{dist}(\Theta_{t_*}(\mathbf{x}), \partial\mathcal{E}) = 0$, but this event has null probability. \square

This fourth model is motivated, when considering e.g. Brownian motions, by its strong relation with the Schrödinger operator $-\Delta + \frac{1}{2} \sum_{i < j} |x_j - x_i|^{-1}$ with Dirichlet boundary conditions over \mathcal{U} . Given a subdomain \mathcal{U} of \mathcal{E} , we define the assumptions

[L4] $(\Theta_t(\mathbf{x}), t \in [0, \sigma_{\mathcal{U}}^\Theta])$ is topologically irreducible over \mathcal{U} , i.e. for all $t > 0$, all $x \in \mathcal{U}$, and all non-empty open subset O of \mathcal{U} , $\mathbb{P}_x(L_t \in O, t < \sigma_{\mathcal{U}}^\Theta) > 0$.

[L5] If $\mathcal{E} \setminus \overline{\mathcal{U}}$ is nonempty, then $\mathbb{P}_\mathbf{x}[\sigma_{\mathcal{U}}^\Theta < +\infty] > 0$ for some $\mathbf{x} \in \mathcal{U}$.

Inter alia, processes satisfying **[L4]** and **[L5]** include the cases when $(L_t, t \geq 0)$ is: (i) a standard Brownian motion, (ii) a jump diffusion process, or (iii) rotationally invariant α -stable processes, see the appendix. The main result for the killed semigroup $(Q_t^\Theta, t \geq 0)$ is Theorem 4.

1.2. Main results.

1.2.1. *Some definitions.* Let \mathcal{M} be an open subset of \mathcal{R}^k , $k \geq 2$, and $(\mathfrak{X}_t, t \geq 0)$ be a strong Markov process with càdlàg sample paths in \mathcal{M} . Consider a lower bounded and continuous Schrödinger potential $\mathbf{V} : \mathcal{M} \rightarrow \mathcal{R}$ and a subdomain \mathcal{V} of \mathcal{M} . Let $\sigma_{\mathcal{V}}^{\mathfrak{X}} := \inf\{t \geq 0, \mathfrak{X}_t \notin \mathcal{V}\}$ be the first exit time from \mathcal{V} for the process $(\mathfrak{X}_t, t \geq 0)$. Assume that for all $\mathfrak{z} \in \mathcal{M}$ and $t \geq 0$,

$$\int_0^t \mathbf{V}(\mathfrak{X}_s) ds \text{ is a.s. finite.} \quad (1.27)$$

The associated killed Feynman-Kac semigroup is then defined by

$$Q_t^{\mathfrak{X}} f(\mathfrak{z}) = \mathbb{E}_{\mathfrak{z}} [f(\mathfrak{X}_t) e^{-\int_0^t \mathbf{V}(\mathfrak{X}_s) ds} \mathbf{1}_{t < \sigma_{\mathcal{V}}^{\mathfrak{X}}}], \quad (1.28)$$

for $\mathfrak{z} \in \mathcal{V}$ and $f \in b\mathcal{B}(\mathcal{V})$, and the killed renormalized Feynman-Kac semigroup is:

$$\nu P_t^{\mathfrak{x}}(\mathcal{A}) = \frac{\nu Q_t^{\mathfrak{x}}(\mathcal{A})}{\nu Q_t^{\mathfrak{x}}(\mathcal{V})}, \text{ for } \mathcal{A} \in \mathcal{B}(\mathcal{V}) \text{ and } \nu \in \mathcal{M}_b(\mathcal{V}).$$

Note that the killed Feynman-Kac semigroup is strongly related to the solution of the evolution equation

$$\partial_t g = \mathcal{L}g - \mathbf{V}g, \quad (1.29)$$

where $\mathcal{L} - \mathbf{V}$ is the so-called Schrödinger operator and \mathcal{L} the infinitesimal generator of the process $(\mathfrak{x}_t, t \geq 0)$.

We recall the notion of quasi-stationary distribution (q.s.d.) of absorbed Markov chains introduced in the context of population processes [18, 58, 10], an object which is also at the heart of the analysis of metastable processes [29, 31, 30].

Definition 1. A measure $\mu \in \mathcal{P}(\mathcal{V})$ is a q.s.d. for the killed renormalized Feynman-Kac semigroup $(P_t, t \geq 0)$ over \mathcal{V} if $\mu P_t^{\mathfrak{x}}(\mathcal{A}) = \mu(\mathcal{A})$, $\forall t \geq 0$ and $\forall \mathcal{A} \in \mathcal{B}(\mathcal{V})$.

In order to easily state our result, we introduce the notion of *compact-ergodic* operator.

Definition 2. Let $\mathbf{F} : \mathcal{V} \rightarrow [1, +\infty]$ be a measurable function. We say that $(Q_t^{\mathfrak{x}}, t \geq 0)$ is \mathbf{F} compact-ergodic over \mathcal{V} if:

1. There exists a unique q.s.d. ρ for $(P_t^{\mathfrak{x}}, t \geq 0)$ in $\mathcal{P}_{\mathbf{F}}(\mathcal{V})$.
2. For all $t > 0$, $Q_t^{\mathfrak{x}} : b\mathcal{B}_{\mathbf{F}}(\mathcal{V}) \rightarrow b\mathcal{B}_{\mathbf{F}}(\mathcal{V})$ is compact and there exists $\lambda \geq \inf_{\mathcal{V}} \mathbf{V}$ such that $r_{sp}(Q_t^{\mathfrak{x}}|_{b\mathcal{B}_{\mathbf{F}}(\mathcal{V})}) = e^{-\lambda t}$, $\forall t > 0$. Furthermore, $\rho Q_t^{\mathfrak{x}} = e^{-\lambda t} \rho$, for all $t \geq 0$, and $\rho(O) > 0$ for all nonempty open subsets O of \mathcal{V} . In addition, there is a unique function $\varphi \in \mathcal{C}_{b\mathbf{F}}(\mathcal{V})$ such that $\rho(\varphi) = 1$ and $Q_t^{\mathfrak{x}}\varphi = e^{-\lambda t}\varphi$ on \mathcal{V} , $\forall t \geq 0$. Moreover, $\varphi > 0$ everywhere on \mathcal{V} .
3. There exist $\mathbf{m}_1 > 0$, and $\mathbf{m}_2 \geq 1$, s.t. for all $t > 0$ and all $\nu \in \mathcal{P}_{\mathbf{F}}(\mathcal{V})$:

$$\sup_{\mathcal{A} \in \mathcal{B}(\mathcal{V})} |\nu P_t^{\mathfrak{x}}(\mathcal{A}) - \rho(\mathcal{A})| \leq \mathbf{m}_2 e^{-\mathbf{m}_1 t} \frac{\nu(\mathbf{F})}{\nu(\varphi)}.$$

The real number λ (resp. the function φ) in the above definition is usually called the *principal eigenvalue* (resp. the *principal eigenfunction*) of $(Q_t^{\mathfrak{x}}, t \geq 0)$ in $b\mathcal{B}_{\mathbf{F}}(\mathcal{V})$. When $\mathbf{F} = \mathbf{W}^{1/p}$, $p \in (1, +\infty)$, we write these two quantities λ_p and ρ_p . We will simply say that $(Q_t^{\mathfrak{x}}, t \geq 0)$ is *compact-ergodic over \mathcal{V}* when $(Q_t^{\mathfrak{x}}, t \geq 0)$ is $\mathbf{1}$ compact-ergodic over \mathcal{V} .

1.2.2. *Main result for the first model.* The first main result of this section concerns the long time behavior of the killed semigroup defined in (1.9). Recall (1.6), Lemma 1, and that \mathcal{O} is a subdomain of \mathcal{R}_0^d .

Theorem 1. Let $(Q_t^{oL}, t \geq 0)$ be defined in (1.9). Then:

Case 1. Assume [c1]. Assume also [S1] or [S2]. Consider the function \mathbf{W} defined in (1.4). Then, for all $p \in (1, +\infty)$, $(Q_t^{oL}, t \geq 0)$ is $\mathbf{W}^{1/p}$ compact-ergodic over \mathcal{O} . If moreover $\mathcal{R}_0^d \setminus \overline{\mathcal{O}}$ is non-empty or if $\mathcal{O} = \mathcal{R}_0^d$, then $\lambda_p > \inf_{\mathcal{O}} \mathbf{V}_{\mathbf{S}}$ where λ_p is the principal eigenvalue of $(Q_t^{oL}, t \geq 0)$ in $b\mathcal{B}_{\mathbf{W}^{1/p}}(\mathcal{O})$.

Actually when [S2] holds we have the following stronger result.

Case 2. Assume [S2]. Assume also [c1] or [c2]. Then, $(Q_t^{oL}, t \geq 0)$ is compact-ergodic over \mathcal{O} . If moreover $\mathcal{R}_0^d \setminus \overline{\mathcal{O}}$ is non-empty or if $\mathcal{O} = \mathcal{R}_0^d$, then $\lambda > \inf_{\mathcal{O}} \mathbf{V}_{\mathbf{S}}$ where λ is the principal eigenvalue of $(Q_t^{oL}, t \geq 0)$ in $b\mathcal{B}(\mathcal{O})$.

In both cases the confinement at 0 comes from \mathbf{V}_S . In the first case the confinement at $+\infty$ comes from \mathbf{V}_c (and also from \mathbf{V}_S if [S2] holds). In the second case the confinement at $+\infty$ comes from \mathbf{V}_S (and also from \mathbf{V}_c if [c1] holds).

Note that in the first case above, any deterministic initial conditions $x \in \mathcal{O}$ belongs to $\mathcal{P}_{\mathbf{W}^{1/p}}(\mathcal{O})$ and thus, the q.s.d. $\rho_p \in \mathcal{P}_{\mathbf{W}^{1/p}}(\mathcal{O})$ and the principal eigenvalue λ_p of $(Q_t^{oL}, t \geq 0)$ in $b\mathcal{B}_{\mathbf{W}^{1/p}}(\mathcal{O})$ are independent of $p > 1$. We also mention that in the first case above, Theorem 1 also holds with a smaller Lyapunov function than (1.4) when \mathbf{V}_c grows sufficiently fast at $+\infty$, see (2.12). Theorem 1 is proved in Section 2.1.

1.2.3. *Non singular Schrödinger potential.* Assume in this section (and only in this section) that $d \geq 1$. Let \mathcal{O} be any subdomain of \mathcal{R}^d . Here, we consider the much simpler case when the Schrödinger potential is not singular. Let $\mathbf{J}_S : \mathcal{R}^d \rightarrow \mathcal{R}$ be a continuous lower bounded and coercive potential. The killed Feynman-Kac semigroup over \mathcal{O} associated with \mathbf{J}_S and (1.1) is defined by:

$$S_t^{oL} f(x) = \mathbb{E}_x \left[f(X_t) e^{-\int_0^t \mathbf{J}_S(X_s) ds} \mathbf{1}_{t < \sigma_{\mathcal{O}}^L} \right], \quad t \geq 0, x \in \mathcal{O}, \text{ and } f \in b\mathcal{B}(\mathcal{O}). \quad (1.30)$$

Proposition 3. *Assume [S2]. Assume also [c1] or [c2]. Then, the semigroup $(S_t^{oL}, t \geq 0)$ given by (1.30) is compact-ergodic over \mathcal{R}^d (see Definition 2). Moreover, $\lambda > \inf_{\mathcal{R}^d} \mathbf{J}_S$ where λ is the principal eigenvalue of $(S_t^{oL}, t \geq 0)$ in $b\mathcal{B}(\mathcal{R}^d)$.*

The proof of Proposition 3 is a direct adaptation of Theorem 1 in a much simpler setting, and is therefore omitted. Note that if \mathbf{J}_S is only continuous and lower bounded, then when \mathbf{b}_c satisfies [c1] (in this case the confinement at ∞ comes from the dynamics itself), the semigroup $(S_t^{oL}, t \geq 0)$ given by (1.30) is $\mathbf{W}^{1/p}$ compact-ergodic over \mathcal{R}^d (see (1.4) and (2.12)), for all $p \in (1, +\infty)$.

1.2.4. *Main result for the second model: a killed Lévy particle in a singular potential.* The second main result of this section concerns the killed Feynman-Kac semigroup $(Q_t^{Le}, t \geq 0)$ defined in (1.14), where we recall (1.6), (1.12), (1.13), and that \mathcal{O} is a subdomain of \mathcal{R}_0^d .

Theorem 2. *Assume [L1], [L2], and [S2]. Then, $(Q_t^{Le}, t \geq 0)$ is compact-ergodic over \mathcal{O} (see Definition 2). If $\mathcal{R}_0^d \setminus \overline{\mathcal{O}}$ is non-empty (in this case we assume in addition [L3]) or if $\mathcal{O} = \mathcal{R}_0^d$, then $\lambda > \inf_{\mathcal{O}} \mathbf{V}_S$ where λ is the principal eigenvalue of $(Q_t^{Le}, t \geq 0)$ in $b\mathcal{B}(\mathcal{O})$.*

The proof of Theorem 2 is made in Section 2.2.

1.2.5. *Main result for the third model: a killed kinetic particle in a singular potential.* The third main result concerns the killed Feynman-Kac semigroup (1.20) of the kinetic Langevin process (1.15) with singular potentials, where we recall that $\mathcal{D} = \mathcal{O} \times \mathcal{R}^d$ and that \mathcal{O} is a subdomain of \mathcal{R}_0^d (see (1.6)). Recall also Lemma 3.

Theorem 3. *Assume that $-\nabla \mathbf{V}_c$ satisfies [c1]. Assume also [S1] or [S2]. Let \mathbf{W} be defined in (2.36) below. Then, for all $p \in (1, +\infty)$, $(Q_t^{kL}, t \geq 0)$ is $\mathbf{W}^{1/p}$ compact-ergodic over \mathcal{D} . If in addition $\mathcal{R}_0^d \setminus \overline{\mathcal{O}}$ is non-empty or if $\mathcal{O} = \mathcal{R}_0^d$, then $\lambda_p > \inf_{\mathcal{O}} \mathbf{V}_S$ where λ_p is the principal eigenvalue of $(Q_t^{kL}, t \geq 0)$ in $b\mathcal{B}_{\mathbf{W}^{1/p}}(\mathcal{D})$.*

A new, the q.s.d. ρ_p and the principal eigenvalue λ_p of $(Q_t^{kL}, t \geq 0)$ in $\mathcal{P}_{\mathbf{W}^{1/p}}(\mathcal{D})$ do not depend on $p > 1$. The proof of Theorem 3 is made in Section 2.3.

1.2.6. *Main result for the last model: killed Brownian particles interacting through singular potential.* Let $(L_t^j, t \geq 0)_{j=1, \dots, n}$ be $n \geq 2$ independent \mathcal{R}^d -copies of a Lévy process satisfying [L1]. Let \mathcal{U} be a subdomain of \mathcal{E} and assume that the process (1.25) satisfies [L4]. Recall also Lemma 4. We then have the following result.

Theorem 4. *Assume (1.23) and (1.24). Then, $(Q_t^\ominus, t \geq 0)$, see (1.26), is compact-ergodic over \mathcal{U} (see Definition 2). If $\mathcal{E} \setminus \overline{\mathcal{U}}$ is non-empty (in this case we assume [L5]) or if $\mathcal{U} = \mathcal{E}$, then $\lambda > \inf_{\mathcal{U}} \mathbf{U}_{\mathbf{S}}$ where λ is the principal eigenvalue of $(Q_t^\ominus, t \geq 0)$ in $b\mathcal{B}(\mathcal{U})$.*

The proof of Theorem 4 is made in Section 2.4.

1.3. **More general Schrödinger potentials.** We chose for the three first models to deal with a Schrödinger potential $\mathbf{V}_{\mathbf{S}}$ with a unique singularity located at 0. As it is clear from the proofs of the main theorems above, one can easily adapt our techniques to Schrödinger potential $\mathbf{V}_{\mathbf{S}}$ having much more singularities (say a closed set \mathcal{C}) as soon these singularities are nonattainable for the càdlàg process $(\mathfrak{X}_t, t \geq 0)$ we consider (namely $\mathbb{P}_{\mathfrak{z}}[\exists t \geq 0, \mathfrak{X}_t \in \mathcal{C} \text{ or } \mathfrak{X}_{t-} \in \mathcal{C}] = 0$ for all $\mathfrak{z} \notin \mathcal{C}$), see e.g. [34, 46, 59] for examples of such sets \mathcal{C} .

For instance, consider $\mathcal{C} = \{(0, 0, t), t \in \mathcal{R}\}$ be the vertical axis in \mathcal{R}^3 and the potential (see (1.24)):

$$\mathbf{C}_{\mathbf{S}} : x \in \mathcal{R}^3 \setminus \mathcal{C} \mapsto \mathbf{v}_{\mathbf{S}}(\text{dist}(x, \mathcal{C})),$$

where $\text{dist}(x, \mathcal{C}) = (|x_1|^2 + |x_2|^2)^{1/2}$, $x = (x_1, x_2, x_3)$. When $\mathbf{v}_{\mathbf{S}}(y) = -\log y$ for $0 < y < r_0$, $\mathbf{C}_{\mathbf{S}}$ is the potential generated by the infinite line \mathcal{C} with a uniform charge density per unit length. Let $(B_t^j, t \geq 0)_{j=1,2,3}$ be three independent standard Brownian motions in \mathcal{R} and set $B_t = (B_t^1, B_t^2, B_t^3)$. Note that $\mathbb{P}_x[\exists s \geq 0, |x_1 + B_s^1| + |x_2 + B_s^2| = 0] = \mathbb{P}_x[\exists s \geq 0, -(B_s^1, B_s^2) = (x_1, x_2)] = 0$ for all $x \in \mathcal{R}^3 \setminus \mathcal{C}$. Then, if moreover (1.23) holds, we have the following result, whose proof is an easy adaptation of the proof of Theorem 1, and is therefore omitted.

Theorem 5. *The (non-killed) semigroup*

$$Q_t^{\mathcal{C}} f(x) := \mathbb{E}_x \left[f(B_t) e^{-\int_0^t (\mathbf{C}_{\mathbf{S}} + \mathbf{V}_{\infty})(B_s) ds} \right], \quad t \geq 0, \quad x \in \mathcal{R}^3 \setminus \mathcal{C}, \quad \text{and } f \in b\mathcal{B}(\mathcal{R}^3 \setminus \mathcal{C}),$$

is compact-ergodic over $\mathcal{R}^3 \setminus \mathcal{C}$ and $\lambda > \inf_{\mathcal{R}^3 \setminus \mathcal{C}} (\mathbf{C}_{\mathbf{S}} + \mathbf{V}_{\infty})$.

One can also add a killing (outside some subdomains of $\mathcal{R}^3 \setminus \mathcal{C}$) in the previous theorem. We finally mention that the previous result is just an example and one can also choose more complicated singular potentials.

1.4. **Related results.** Feynman-Kac semigroups appear in many fields of science ranging e.g. from statistical physics, engineering science, nonlinear filtering, and genealogical tree models, see the classical textbook [23], and the related mathematical literature is very rich. Such semigroups have also a strong connection with Schrödinger operators and non linear Fokker–Planck equations [61, 28, 23]. To discuss the related literature, let us consider a potential \mathbf{V}_{\star} and a process $(Z_t, t \geq 0)$, we denote by $Q_t f(x) = \mathbb{E}_x[f(Z_t) \exp(-\int_0^t \mathbf{V}_{\star}(Z_s) ds)]$ its associated (non-killed) Feynman-Kac semigroup. When $(Z_s, s \geq 0)$ is a (Lévy) rotationally invariant α -stable process with $\alpha \in (0, 2)$, the generator of the non-killed Feynman-Kac semigroup Q is $-(-\Delta)^{\alpha/2} - \mathbf{V}_{\star}$ where $(-\Delta)^{\alpha/2}$ is the fractional Laplacian. When \mathbf{V}_{\star} is non negative and locally bounded, the spectral properties in $L^2(\mathcal{R}^d)$ of such a semigroup have been investigated in [42] (see also the references therein) and the same investigation has been carried out in [50] in the spaces $L^p(\mathcal{R}^d)$ when $(Z_s, s \geq 0)$

is a relativistic stable process (see also [15, 12, 13, 1]). We refer to [43] for a spectral analysis in $L^p(\mathcal{R}^d)$ of $L^2(\mathcal{R}^d)$ -symmetric Feynman-Kac semigroups associated with some symmetric Lévy processes⁷ and when the Schrödinger potential belongs to the Kato class (see (1.11)), see also [21, 8, 38]. We mention that in the papers quoted just above, the authors are concerned with the precise study of the eigenfunctions and to contractivity-type properties. In this direction, we also refer to [68, 6, 14] and references therein. When $(Z_s, s \geq 0)$ is a Brownian motion (this is our second model when $\alpha = 2$), the generator of the non-killed Feynman-Kac semigroup is the Schrödinger operator $\Delta - \mathbf{V}_*$, and we refer to the classical textbooks [45, 20, 60, 69, 76, 17] for the spectral study of this operator.

In [64], the interacting particle model introduced in [26] is used to approximate, via Feynman-Kac formulas, the principal eigenvalue of Schrödinger type operators as well as its associated eigenfunction. The stability of non-killed Feynman Kac semigroups and their large time behavior have also been investigated in [27, 23, 16] and very recently in [32, 19] (see also [25]).

In [19], the authors studied in a one-dimensional setting the long time properties of a non-killed Feynman-Kac semigroup related to linear functionals of branching diffusion processes when the potential \mathbf{V}_* is non singular. Note that Proposition 3 provides a similar result as [19, Theorem 2.1] in any dimension and without assumption on the growth of the Schrödinger potential \mathbf{J}_S at infinity.

In [32], the authors give several conditions to obtain the long time convergence of non-killed Feynman-Kac semigroups and apply them to the solutions to elliptic diffusions with smooth coefficients and with non singular Schrödinger potential. Among these conditions is the Lyapunov criterion [32, Assumption 4: Eqs (31) and (32)]. When working on the set \mathcal{M} where the singular Schrödinger potential is finite, [32, Assumption 4] has now to be satisfied when $\mathfrak{z} \rightarrow \partial\mathcal{M} \cup \{\infty\}$, $\mathfrak{z} \in \mathcal{M}$. It turns out that [32, Equation (32)] is actually too challenging for $\mathfrak{z} \rightarrow \partial\mathcal{M}$. For that reason, and inspired by our previous work [36, 37], we prove using the tools introduced in [75], that is enough to only work with the Lyapunov condition [32, Equation (31)] adapted to the singular setting, see indeed Theorems 7 and 8 below. We mention that in [32], the authors are also concerned by the study of discretized (in time) Feynman-Kac semigroups.

In this context, we mention [11, 3] for other general criteria to study the long-time behavior of non-conservative semigroups with applications to quasi-stationary distributions and growth-fragmentation semigroups, see also [48, 24]. These different techniques can also be used to study Feynman-Kac semigroups. Finally, we mention [44] for the study of the quasi-ergodicity in $L^2(\mathcal{M})$ of ultracontractive semigroups on locally compact Polish space \mathcal{M} with various applications to (non killed) Feynman-Kac semigroups with locally bounded, lower bounded, and confining Schrödinger potential.

2. PROOF OF THE MAIN RESULTS

2.1. Preliminary regularity results on Q_t^{oL} and proof of Theorem 1. In this section we perform the proof of Theorem 1.

In all this section we consider the process $(X_t, t \geq 0)$ solution to (1.1) (see Proposition 1). We start this section by giving some regularity and some spectral properties of the transition kernel Q_t^{oL} .

2.1.1. Properties of the transition kernel Q_t^{oL} .

⁷Satisfying in particular [L1] and other regularity conditions on the Green function.

Proposition 4. *Assume [c1] or [c2]. Then, for all $t \geq 0$ and all sequence $(x_n)_n$ in \mathcal{R}^d such that $x_n \rightarrow x \in \mathcal{R}^d$ as $n \rightarrow +\infty$, it holds for all $\epsilon > 0$, as $n \rightarrow +\infty$*

$$\mathbb{P}\left[\sup_{s \in [0, t]} |X_s(x_n) - X_s(x)| \geq \epsilon\right] \rightarrow 0.$$

Proof. Since the coefficients in (1.1) are locally Lipschitz, using in addition (1.5) and (1.3), the proof of the proposition is the same as the one made for Proposition 2.2 in [35]. \square

Let \mathcal{O} be a subdomain of \mathcal{R}_0^d .

Theorem 6. *Assume [c1] or [c2] and assume also [S1] or [S2]. Then, Q_t^{oL} is strongly Feller over \mathcal{O} for all $t > 0$, i.e. $Q_t^{oL}(b\mathcal{B}(\mathcal{O})) \subset \mathcal{C}_b(\mathcal{O})$.*

Proof. Let $t > 0$ and $f \in b\mathcal{B}(\mathcal{O})$. The goal is to show that

$$z \in \mathcal{O} \mapsto Q_t^{oL} f(z) = \mathbb{E}_z \left[f(X_t) e^{-\int_0^t \mathbf{v}_s(X_s) ds} \mathbf{1}_{t < \sigma_{\mathcal{O}}^{oL}} \right] \text{ is continuous.} \quad (2.1)$$

Step A. The purpose of this step is to prove that the non-killed Feynman-Kac semigroup $(T_t^{oL}, t \geq 0)$ is strongly Feller⁸ over \mathcal{R}_0^d , i.e. for all $t > 0$ and $f \in b\mathcal{B}(\mathcal{R}_0^d)$,

$$z \in \mathcal{R}_0^d \mapsto T_t^{oL} f(z) := \mathbb{E}_z \left[f(X_t) e^{-\int_0^t \mathbf{v}_s(X_s) ds} \right] \text{ is continuous.} \quad (2.2)$$

Step A.1. In this step we prove that for all sequence $(x_n)_n$ in \mathcal{R}^d such that $x_n \rightarrow x \in \mathcal{R}^d$ as $n \rightarrow +\infty$, it holds for all $f \in b\mathcal{B}(\mathcal{R}^d)$,

$$\mathbb{E}_{x_n}[f(X_t)] \rightarrow \mathbb{E}_x[f(X_t)], \quad (2.3)$$

namely U_t^{oL} is strongly Feller over \mathcal{R}^d , where $U_t^{oL} f(x) := \mathbb{E}_x[f(X_t)]$. Note that if \mathbf{b}_c is smooth, (2.3) follows from the standard elliptic theory. Since it is not *a priori* the case, we argue differently. There are probably many ways to prove (2.3) in the weak setting where the drift \mathbf{b}_c is only locally Lipschitz, and we use the *energy splitting method* introduced in [35, Section 2.1] whose starting point is the simple equality

$$\mathbb{E}_y[f(X_t)] = \mathbb{E}_y[f(X_t) \mathbf{1}_{\sigma_{\mathcal{W}_R}^{oL} \leq t}] + \mathbb{E}_y[f(X_t) \mathbf{1}_{t < \sigma_{\mathcal{W}_R}^{oL}}], \quad y \in \mathcal{R}^d. \quad (2.4)$$

Let K be a compact subset of \mathcal{R}^d containing the x_n 's and x and such that $K \subset \mathcal{W}_R$ for all $R > R_x$, for some $R_x > 0$ (where we recall that $\mathcal{W}_R = \{y \in \mathcal{R}^d, \mathbf{W}(y) < R\}$, see just after (1.5)). By (1.5),

$$\lim_{R \rightarrow +\infty} \sup_{x \in K} \mathbb{E}_x[f(X_t) \mathbf{1}_{\sigma_{\mathcal{W}_R}^{oL} \leq t}] = 0. \quad (2.5)$$

Let us now prove that for any fixed $R > R_x$,

$$x \in \mathcal{W}_R \mapsto \mathbb{E}_x[f(X_t) \mathbf{1}_{t < \sigma_{\mathcal{W}_R}^{oL}}] \text{ is continuous.} \quad (2.6)$$

Let us now consider a fixed $R > R_x$ and a globally Lipschitz and bounded vector field $\mathbf{a} : \mathcal{R}^d \rightarrow \mathcal{R}$ such that $\mathbf{a} = \mathbf{b}_c$ on the closure of \mathcal{W}_R . Consider the solution $(Z_t, t \geq 0)$ to $dZ_t = \mathbf{a}(Z_t) + dB_t$. Since the laws of the two processes $(X_t, t \geq 0)$ and $(Z_t, t \geq 0)$ coincide up to their first exit time from \mathcal{W}_R , we then have for all $y \in \mathcal{W}_R$,

$$\mathbb{E}_y[f(X_t) \mathbf{1}_{t < \sigma_{\mathcal{W}_R}^{oL}}] = \mathbb{E}_y[f(Z_t) \mathbf{1}_{t < \sigma_{\mathcal{W}_R}^Z}], \quad (2.7)$$

where $\sigma_{\mathcal{W}_R}^Z = \inf\{t \geq 0, Z_t \notin \mathcal{W}_R\}$. Note that since \mathbf{a} is globally Lipschitz, Proposition 4 also holds for the process $(Z_t, t \geq 0)$. Moreover, using again that \mathbf{a} is globally Lipschitz, the transition density $p_t^*(x, y) dy$ of the process $(Z_t, t \geq 0)$ satisfies a gaussian upper

⁸Note that, as the product of two strong Feller kernels, (2.2) implies that T_t^{oL} is strong Feller in the strict sense [62].

bound [2] and hence, the sequence $(p_t^*(x_n, y))_n$ is uniformly integrable w.r.t. a fixed probability measure. Using [73, Lemma 3.2], we finally deduce that for all $f \in b\mathcal{B}(\mathcal{R}^d)$:

$$f(Z_t(x_n)) \rightarrow f(Z_t(x)) \text{ in } \mathbb{P}\text{-probability.} \quad (2.8)$$

In addition, using the same arguments as those used in the proof of [35, Lemma 2.5], it holds for all $\delta > 0$ and all compact subset K of \mathcal{R}^d ,

$$\lim_{s \rightarrow 0^+} \sup_{y \in K} \mathbb{P}_y[\sigma_{B(y, \delta)}^Z \leq s] = 0,$$

where $\sigma_{B(x, \delta)}^Z$ is the first exit time from the open ball $B(x, \delta)$ for the process $(Z_t, t \geq 0)$. Together with (2.8), one therefore finally gets, with the same arguments as those used at the end of the proof of [35, Theorem 2.6], that the mapping

$$y \in \mathcal{R}^d \mapsto \mathbb{E}_y[f(Z_t)\mathbf{1}_{t < \sigma_{\mathcal{R}}^Z}] \text{ is continuous.}$$

Thanks to (2.7), we deduce that (2.6) holds. Finally, Equations (2.4), (2.5), and (2.6) imply (2.3).

Step A.2. In this step we prove that for all $f \in b\mathcal{B}(\mathcal{R}^d)$, and all sequence $(x_n)_n \subset \mathcal{R}^d$ such that $\lim_n x_n = x$, it holds:

$$Y_n := f(X_t(x_n)) \rightarrow Y := f(X_t(x)) \text{ in } \mathbb{P}\text{-probability.} \quad (2.9)$$

By the Girsanov formula (1.2), for every $z \in \mathcal{R}^d$, $X_t(z)$ admits for every $t > 0$ a density $p_t(z, y)$ w.r.t. the Lebesgue measure dy over \mathcal{R}^d . Let us now consider an arbitrary fixed smooth function $\psi : \mathcal{R}^d \rightarrow \mathcal{R}_+^*$ such that $\int_{\mathcal{R}^d} \psi(y) dy = 1$. Define the probability measure $\rho(dy) := \psi(y) dy$ and set $q_t(z, y) := p_t(z, y)\psi^{-1}(y)$ the density of $X_t(z)$ w.r.t. the probability measure $\rho(dy)$. We have thanks to (2.3), for all $\phi \in L^\infty(\rho)$,

$$\lim_n \int_{\mathcal{R}^d} \phi(y) q_t(x_n, y) \rho(dy) = \int_{\mathcal{R}^d} \phi(y) q_t(x, y) \rho(dy).$$

Hence, $\lim_n q_t(x_n, \cdot) = q_t(x, \cdot)$ in $\sigma(L^1(\rho), L^\infty(\rho))$. This implies that for all $\mathcal{A} \in \mathcal{B}(\mathcal{R}^d)$, $\int_{\mathcal{A}} q_t(x_n, y) \rho(dy)$ has a finite limit. Consequently the sequence $(q_t(x_n, y))_n$ is uniformly integrable in $L^1(\rho)$ (see [5, Theorem 4.5.6]). Therefore, (2.9) follows from [73, Lemma 3.2] and the fact that $X_t(x_n) \rightarrow X_t(x)$ in \mathbb{P} -probability (see Proposition 4).

Step A.3. We now conclude the proof of (2.2). To this end, consider a sequence $(x_n)_n \subset \mathcal{R}_0^d$ such that $\lim_n x_n = x \in \mathcal{R}_0^d$, where we recall that $\mathcal{R}_0^d = \mathcal{R}^d \setminus \{0\}$. Introduce the following function

$$F : \gamma \in \mathcal{C}([0, t], \mathcal{R}^d) \mapsto \int_0^t \mathbf{V}_S(\gamma_s) ds,$$

where $\mathcal{C}([0, t], \mathcal{R}^d)$ is endowed with the topology of the uniform convergence. For this topology, the function F is continuous at any $\gamma \in \mathcal{C}([0, t], \mathcal{R}_0^d)$. On the other hand, recall that by Lemma 1, $\mathbb{P}_x[X_{[0, t]} \in \mathcal{C}([0, t], \mathcal{R}_0^d)] = 1$. Hence, by the continuous mapping theorem and Proposition 4, it holds $F(X_n) \rightarrow F(X)$ in \mathbb{P} -probability (where $X_n := X_{[0, t]}(x_n)$ and $X := X_{[0, t]}(x)$). Hence as $n \rightarrow +\infty$ and in \mathbb{P} -probability,

$$Z_n = \exp \left[- \int_0^t \mathbf{V}_S(X_s(x_n)) ds \right] \rightarrow Z = \exp \left[- \int_0^t \mathbf{V}_S(X_s(x)) ds \right].$$

In conclusion the bounded sequence of random variables $(Z_n Y_n)_n$ converges towards ZY as $n \rightarrow +\infty$ in \mathbb{P} -probability, and hence, also in L^1 . Note here that $b\mathcal{B}(\mathcal{R}_0^d) \subset b\mathcal{B}(\mathcal{R}^d)$ if

we extend any element of $b\mathcal{B}(\mathcal{R}_0^d)$ by 0 at the point 0. This shows (2.2) and concludes the proof of (2.2).

Step B. We now conclude the proof of (2.1). Let $t > 0$ and $f \in b\mathcal{B}(\mathcal{O})$. By the Markov property we have for $0 < s < t$ and $z \in \mathcal{R}_0^d$,

$$Q_t^{oL} f(z) = \mathbb{E}_z \left[\mathbf{1}_{s < \sigma_{\mathcal{O}}^{oL}} e^{-\int_0^s \mathbf{V}_s(X_u) du} \mathbb{E}_{X_s} \left[f(X_{t-s}) e^{-\int_0^{t-s} \mathbf{V}_s(X_u) du} \mathbf{1}_{t-s < \sigma_{\mathcal{O}}^{oL}} \right] \right].$$

Set for $0 < s < t$ and $x \in \mathcal{R}_0^d$,

$$\varphi_s(x) = \mathbb{E}_x \left[f(X_{t-s}) e^{-\int_0^{t-s} \mathbf{V}_s(X_u) du} \mathbf{1}_{t-s < \sigma_{\mathcal{O}}^{oL}} \right].$$

The function φ_s is bounded over \mathcal{R}_0^d (by $\|f\|_{b\mathcal{B}(\mathcal{R}_0^d)} e^{ks^t}$) and thus, by (2.2), $T_s^{oL} \varphi_s$ is continuous over \mathcal{R}_0^d . We then have for $z \in \mathcal{O}$:

$$\begin{aligned} |T_s^{oL} \varphi_s(z) - Q_t^{oL} f(z)| &\leq \|f\|_{b\mathcal{B}(\mathcal{R}_0^d)} e^{ks^t} \mathbb{E}_z \left[\mathbf{1}_{\sigma_{\mathcal{O}}^{oL} \leq s} e^{-\int_0^s \mathbf{V}_s(X_s) ds} \right] \\ &\leq \|f\|_{b\mathcal{B}(\mathcal{R}_0^d)} e^{ks^{2t}} \mathbb{P}_z[\sigma_{\mathcal{O}}^{oL} \leq s]. \end{aligned}$$

Arguing as in the proof of [35, Lemma 2.5] (the three ingredients are Equation (1.5), the strong Markov property of the process, and fact that the coefficients in (1.1) are locally Lipschitz), we have that for all compact subset K of \mathcal{O} , setting $\delta := \text{dist}(K, \partial\mathcal{O})/2$,

$$\sup_{z \in K} \mathbb{P}_z[\sigma_{\mathcal{O}}^{oL} \leq s] \leq \sup_{z \in K} \mathbb{P}_z[\sigma_{B(z, \delta)}^{oL} \leq s] \rightarrow 0,$$

as $s \rightarrow 0^+$. This ends the proof of (2.1). \square

Let us mention that there is an alternative proof of (2.3) based on the Girsanov formula (1.2). Indeed, using e.g. [53, Proposition 5.8], one can show that as $x_n \rightarrow x \in \mathcal{R}^d$,

$$m_t^0(x_n) \rightarrow m_t^0(x) \text{ in } \mathbb{P}\text{-probability.} \quad (2.10)$$

Since $\mathbb{E}[m_t^0(x_n)] = \mathbb{E}[m_t^0(x)] = 1$, one then deduces with the Vitali convergence theorem, that $m_t^0(x_n) \rightarrow m_t^0(x)$ in L^1 . On the other hand, for any $f \in b\mathcal{B}(\mathcal{R}^d)$, $f(x_n + B_t) \rightarrow f(x + B_t)$ in \mathbb{P} -probability (one can use for instance [73, Lemma 3.2] to prove it). Using (1.2) then proves (2.3).

Proposition 5. *Assume [c1] or [c2] and assume also [S1] or [S2]. Then, for all $t > 0$, Q_t^{oL} is topologically irreducible on \mathcal{O} , i.e. for all $x \in \mathcal{R}_0^d$ and all non-empty open subset O of \mathcal{R}_0^d , $Q_t^{oL}(x, O) > 0$.*

Proof. Since \mathbf{b}_c is locally Lipschitz and \mathcal{O} is a subdomain of \mathcal{R}^d , with the same arguments as those used to prove [35, Proposition 2.10], which are based on the knowledge of the support of the law of the trajectories of the Brownian motion, one shows that for all $t > 0$, all $x \in \mathcal{R}_0^d$ and all non-empty open subset O of \mathcal{R}_0^d , $\mathbb{E}_x[\mathbf{1}_O(X_t) \mathbf{1}_{t < \sigma_{\mathcal{O}}^{oL}}] > 0$. Note that this property can also be obtained with the Girsanov formula (1.2). The desired result follows from the fact that for all $x \in \mathcal{O}$ and $t > 0$, $\exp[-\int_0^t \mathbf{V}_s(X_s(x)) ds] > 0$ a.s. by Lemma 1. \square

Recall that the infinitesimal generator of the diffusion (1.1) is $\mathcal{L}^{oL} = \mathbf{b}_c \cdot \nabla + \frac{1}{2} \Delta$.

Proposition 6. *Assume [c1] and assume also [S1] or [S2]. Let \mathbf{W} be defined by (1.4). Then, when $|x| \rightarrow +\infty$, $\mathcal{L}^{oL} \mathbf{W}(x) / \mathbf{W}(x) \rightarrow -\infty$. In particular, for any $c > 0$, when $x \rightarrow \partial\mathcal{R}_0^d \cup \{\infty\}$ and $x \in \mathcal{R}_0^d$ (where $\partial\mathcal{R}_0^d = \{0\}$),*

$$\frac{\mathcal{L}^{oL} \mathbf{W}(x)}{\mathbf{W}(x)} - c \mathbf{V}_s(x) \rightarrow -\infty. \quad (2.11)$$

Assume that **[S2]** holds and that either **[c1]** or **[c2]** holds. Then, (2.11) is satisfied with the Lyapunov function **1**.

Proof. See the proof of Proposition 1. \square

Note that when there exists $k > 2$ and $c_1, c_2 > 0$ such that $-\mathbf{b}_c(x) \cdot x \geq c_2|x|^k - c_1$ for all $x \in \mathcal{R}^d$, one can choose the smaller Lyapunov function than (1.4) defined by:

$$\mathbf{W} = \ell + 1, \quad \ell = |\cdot|^k \times (1 - \chi). \quad (2.12)$$

We then indeed have for $|x|$ sufficiently large and some $c > 0$ that $\mathcal{L}^{oL}\mathbf{W}(x)/\mathbf{W}(x) \leq -c|x|^{2(k-1)-k} \rightarrow -\infty$ as $|x| \rightarrow \infty$.

2.1.2. Proof of Theorem 1. In this section we prove Theorem 1. We will only prove it when **[c1]** holds and either **[S1]** or **[S2]** holds (the other case is proved with the same arguments but with the Lyapunov function **1**).

To prove Theorem 1, we will rely on [36, Theorem 4.1] with the Lyapunov function defined by, for $p > 1$,

$$\mathbf{W}_\star := \mathbf{W}^{1/p}, \quad (2.13)$$

where \mathbf{W} is defined in (1.4). Moreover, we will use the tools developed in [75]. To this end, we recall the two measures of non compactness β_w and β_τ introduced there as well as the key assumption [75, **(A1)**]. For a bounded non negative transition kernel Q on a Polish space \mathcal{S} , set

$$\beta_w(Q) := \inf_{K \subset \mathcal{S}} \sup_{x \in \mathcal{S}} Q(x, \mathcal{S} \setminus K) \quad \text{and} \quad \beta_\tau(Q) := \sup_{(A_n)_n} \lim_{n \rightarrow +\infty} \sup_{x \in \mathcal{S}} Q(x, A_n), \quad (2.14)$$

where the infimum is taken over all compact subset K of \mathcal{S} and supremum is taken over all the sequences $(A_n)_n$ of elements of $\mathcal{B}(\mathcal{S})$ decreasing towards \emptyset . The kernel Q satisfies [75, **(A1)**] if by definition, for any compact subset K of \mathcal{S} ,

$$\mathbf{(A1)} \quad \beta_w(\mathbf{1}_K Q) = 0 \quad \text{and} \quad \beta_\tau(\mathbf{1}_K Q^N) = 0,$$

for some $N \geq 1$ independent of K . Let us finally mention that under this previous assumption, the last author proves in [75, Theorem 3.5] the following Gelfand-Nussbaum type formula (2.24)

$$r_{ess}(Q|_{b\mathcal{B}(\mathcal{S})}) = \lim_{n \rightarrow +\infty} [\beta_w((Q|_{b\mathcal{B}(\mathcal{S})})^n)]^{1/n}. \quad (2.15)$$

for the essential spectral radius of the transition kernel Q .

Let \mathcal{O} be a subdomain of \mathcal{R}_0^d .

Step 1. We start by checking that Q_1^{oL} satisfies all the conditions of [36, Theorem 4.1] with the Lyapunov function \mathbf{W}_\star (see (2.13)) on the state space $\mathbf{S} = \mathcal{O}$. First of all, the transition kernel Q_1^{oL} is Feller over \mathcal{O} and $Q_1^{oL}(x, \mathcal{O}) > 0$ for all $x \in \mathcal{O}$ and all non-empty open subset O of \mathcal{O} (see Theorem 6 and Proposition 5). We now prove that Q_1^{oL} has a spectral gap on $b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{O})$.

Step 1.a. In this step we check that the non-killed Feynman-Kac operator T_1^{oL} is a bounded operator on $b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{R}_0^d)$ and that $T_{1, \mathbf{W}_\star}^{oL}$ (see (2.2) and (2.23)), which is then well defined, satisfies (see (2.14))

$$\beta_w(T_{1, \mathbf{W}_\star}^{oL}|_{b\mathcal{B}(\mathcal{R}_0^d)}) = 0. \quad (2.16)$$

To this end, we apply Theorem 7. By (2.11), there exist two sequences of positive constants (\mathbf{r}_n) and (\mathbf{b}_n) where $\mathbf{r}_n \rightarrow +\infty$, and an increasing sequence of compact subsets (\mathcal{K}_n) of \mathcal{R}_0^d , such that for all $x \in \mathcal{R}_0^d$,

$$\frac{\mathcal{L}^{oL}\mathbf{W}(x)}{\mathbf{W}(x)} - p\mathbf{V}_S(x) \leq -\mathbf{r}_n + \mathbf{b}_n \mathbf{1}_{\mathcal{K}_n}(x).$$

Hence, over \mathcal{R}_0^d , it holds (see (2.13)):

$$\begin{aligned} (\mathcal{L}^{oL} - \mathbf{V}_S)\mathbf{W}_\star &\leq \frac{1}{p}\mathbf{W}^{1/p-1}(\mathcal{L}^{oL} - p\mathbf{V}_S)\mathbf{W} \leq \left[-\frac{\mathbf{r}_n}{p} + \frac{\mathbf{b}_n}{p}\mathbf{1}_{\mathcal{K}_n}\right]\mathbf{W}^{1/p} \\ &= \left[-\frac{\mathbf{r}_n}{p} + \frac{\mathbf{b}_n}{p}\mathbf{1}_{\mathcal{K}_n}\right]\mathbf{W}_\star. \end{aligned}$$

Note that the first inequality above can be obtained by a direct computations, but it is actually a consequence of a more general result [36, Proposition 5.1]. On the other hand, by (2.11), there exists $\mathbf{m}_0 > 0$ such that on \mathcal{R}_0^d ,

$$(\mathcal{L}^{oL} - \mathbf{V}_S)\mathbf{W}_\star^p = (\mathcal{L}^{oL} - \mathbf{V}_S)\mathbf{W} \leq \mathbf{m}_0\mathbf{W} = \mathbf{m}_0\mathbf{W}_\star^p.$$

Applying Theorem 7, we then have that T_1^{oL} is a bounded operator on $b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{R}_0^d)$ (see (2.22)), and moreover, we also have that (2.16) holds and $r_{ess}(T_1^{oL}|_{b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{R}_0^d)}) = 0$.

Step 1.b. In this step we prove the following spectral gap:

$$0 = r_{ess}(Q_1^{oL}|_{b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{O})}) < r_{sp}(Q_1^{oL}|_{b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{O})}).$$

Note first that Q_1^{oL} is a bounded kernel over $b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{O})$ (because so is T_1^{oL} , see (2.22), and $Q_1^{oL} \leq T_1^{oL}$). Then, one can consider the bounded kernel $Q_{1,\mathbf{W}_\star}^{oL}$ on $b\mathcal{B}(\mathcal{O})$ defined by

$$Q_{1,\mathbf{W}_\star}^{oL}(x, dy) = \frac{\mathbf{W}_\star(y)}{\mathbf{W}_\star(x)} Q_1^{oL}(x, dy),$$

which has the same essential spectral radius as $Q_1^{oL}|_{b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{O})}$. Because $Q_{1,\mathbf{W}_\star}^{oL} \leq T_{1,\mathbf{W}_\star}^{oL}$ it follows by Theorem 7 that $\beta_w(Q_{1,\mathbf{W}_\star}^{oL}) = 0$.

On the other hand, $Q_{1,\mathbf{W}_\star}^{oL}$ is strongly Feller as Q_1^{oL} . Indeed, to see it, one can argue as follows. Recall \mathbf{W}_\star is continuous. For all compact set K of \mathcal{O} , $f \in b\mathcal{B}(\mathcal{O})$, we have, setting $f_n = ([\mathbf{W}_\star \wedge n]/\mathbf{W}_\star)f \in b\mathcal{B}(\mathcal{O})$ ($n \geq 1$), $\sup_{x \in K} |Q_1^{oL}(\mathbf{W}_\star f)(x) - Q_1^{oL}(\mathbf{W}_\star f_n)(x)| \leq \sup_{x \in K} |T_1^{oL}\mathbf{W}_\star^p(x)|^{1/p} \sup_{x \in K} |T_1^{oL}|f - f_n|^q(x)|^{1/q}$ which goes to 0 as $n \rightarrow +\infty$ by (2.21) and Dini's theorem. Hence $Q_1^{oL}(\mathbf{W}_\star f)$ is continuous over \mathcal{O} . This proves that $Q_{1,\mathbf{W}_\star}^{oL}$ is strongly Feller. In particular, the operator $Q_{1,\mathbf{W}_\star}^{oL}$ satisfies **(A1)**.

Hence, we have thanks to (2.15),

$$r_{ess}(Q_{1,\mathbf{W}_\star}^{oL}) = 0 \text{ and consequently } r_{ess}(Q_1^{oL}|_{b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{O})}) = 0.$$

Finally, arguing as in the first step of the proof of [36, Theorem 5.3], the strong Feller property as well as the topological irreducibility property of Q_1^{oL} imply that

$$r_{sp}(Q_1^{oL}|_{b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{O})}) > 0.$$

The desired spectral gap of $Q_1^{oL}|_{b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{O})}$ is thus proved.

Step 1.c. Thanks to the previous steps, we can apply [36, Theorem 4.1] with the operator Q_1^{oL} on $b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{O})$. We then deduce, setting

$$\lambda_p := -\log r_{sp}(Q_1^{oL}|_{b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{O})}) \in \mathcal{R},$$

that there is a unique couple (ρ_p, φ_p) such that all the following conditions hold:

- i) The measure ρ_p is a probability measure on \mathcal{O} , $\rho_p(\mathbf{W}_*) < +\infty$, $\varphi_p \in \mathcal{C}_{b\mathbf{W}_*}(\mathcal{O})$, and $\rho_p(\varphi_p) = 1$.
- ii) It holds $\rho_p Q_1^{\circ L} = e^{-\lambda_p} \rho_p$ and $Q_1^{\circ L} \varphi_p = e^{-\lambda_p} \varphi_p$.
- iii) There exist $C \geq 1$ and $\delta > 0$, such that for all $f \in b\mathcal{B}_{\mathbf{W}_*}(\mathcal{O})$ and $n \geq 1$,

$$\|e^{n\lambda_p} Q_n^{\circ L} f - \rho_p(f) \varphi_p\|_{b\mathcal{B}_{\mathbf{W}_*}(\mathcal{O})} \leq C e^{-\delta n} \|f\|_{b\mathcal{B}_{\mathbf{W}_*}(\mathcal{O})}.$$

In addition $\rho_p(O) > 0$ for all nonempty open subsets O of \mathcal{O} and φ_p is positive everywhere on \mathcal{O} . In particular, one has:

- If $\nu \in \mathcal{M}_{b\mathbf{W}_*}(\mathcal{O})$ satisfies for some $\eta \in \mathbb{R}$, $\nu Q_1^{\circ L} = \eta \nu$ and $\nu(\varphi_p) \neq 0$, then $\eta = e^{-\lambda_p}$ and $\nu = c \rho_p$ for some constant c .
- If $g \in b\mathcal{B}_{\mathbf{W}_*}(\mathcal{O})$ satisfies for some $\eta \in \mathbb{R}$, $Q_1^{\circ L} g = \eta g$ and $\rho_p(g) \neq 0$, then $\eta = e^{-\lambda_p}$ and $g = c \varphi_p$ for some constant c .

The space $\mathcal{M}_{b\mathbf{W}_*}(\mathcal{O})$ is defined by $\mathcal{M}_{b\mathbf{W}_*}(\mathcal{O}) = \{\nu \in \mathcal{M}_b(\mathcal{O}), \mathbf{W}_*(x)\nu(dx) \in \mathcal{M}_b(\mathcal{O})\}$.

Step 2. End of the proof of Theorem 1. Arguing as in the second step of the proof of [36, Theorem 5.3], one proves all the statements in Theorem 1 except that it remains to prove:

- The compactness of $Q_t^{\circ L} : b\mathcal{B}_{\mathbf{W}^{1/p}}(\mathcal{O}) \rightarrow b\mathcal{B}_{\mathbf{W}^{1/p}}(\mathcal{O})$, for $t > 0$.
- The fact that $\lambda_p \in [i_{\mathbf{V}_s, \mathcal{O}}, +\infty)$ (where we set $i_{\mathbf{V}_s, \mathcal{O}} := \inf_{\mathcal{O}} \mathbf{V}_s$).
- If moreover $\mathcal{R}_0^d \setminus \overline{\mathcal{O}}$ is non-empty or if $\mathcal{O} = \mathcal{R}_0^d$, then $\lambda_p > i_{\mathbf{V}_s, \mathcal{O}}$.

Note that the second point is obvious (i.e. $\lambda_p \geq i_{\mathbf{V}_s, \mathcal{O}}$) since

$$\mathbb{E}_{\rho_p} \left[e^{-\int_0^t \mathbf{V}_s(X_s) ds} \mathbf{1}_{t < \sigma_{\mathcal{O}}} \right] = \rho_p(Q_t \mathbf{1}) = e^{-\lambda_p t} \rho_p(\mathbf{1}) = e^{-\lambda_p t}.$$

Step 2.a. Let us prove that for all $t > 0$, $Q_t^{\circ L} : b\mathcal{B}_{\mathbf{W}^{1/p}}(\mathcal{O}) \rightarrow b\mathcal{B}_{\mathbf{W}^{1/p}}(\mathcal{O})$ (or equivalently, $Q_{t, \mathbf{W}_*}^{\circ L}$ over $b\mathcal{B}(\mathcal{O})$) is compact. By (2.25) below and Proposition 6, $\beta_w(Q_{t, \mathbf{W}_*}^{\circ L}) = 0$ for all $t > 0$. We also have that (see the lines just above (2.24)), $\beta_{\tau}(\mathbf{1}_K Q_{t, \mathbf{W}_*}^{\circ L}) = 0$ for all compact subset K of \mathcal{O} , $t > 0$. For any $t > 0$, write $t = 3s$, with $s > 0$. Then, using [75, Proposition 3.2 (f)], we have $\beta_{\tau}(Q_{2s, \mathbf{W}_*}^{\circ L}) = 0$. Consequently, thanks to [75, Proposition 3.2 (g)], $Q_{3s, \mathbf{W}_*}^{\circ L} = Q_{s, \mathbf{W}_*}^{\circ L} Q_{2s, \mathbf{W}_*}^{\circ L}$ is compact.

Step 2.b. Let us prove that $\lambda_p > i_{\mathbf{V}_s, \mathcal{O}}$ when $\mathcal{R}_0^d \setminus \overline{\mathcal{O}}$ is non-empty or $\mathcal{O} = \mathcal{R}_0^d$. Assume that it is not the case, i.e. that $\lambda_p = i_{\mathbf{V}_s, \mathcal{O}}$. Then, since for all $t \geq 0$, $\rho_p(Q_t^{\circ L} \mathbf{1}) = e^{-\lambda_p t} \mathbf{1} = e^{-i_{\mathbf{V}_s, \mathcal{O}} t} \mathbf{1}$, we get that $\rho_p(e^{-i_{\mathbf{V}_s, \mathcal{O}} t} \mathbf{1} - Q_t^{\circ L} \mathbf{1}) = 0$. Since the function $e^{-i_{\mathbf{V}_s, \mathcal{O}} t} \mathbf{1} - Q_t^{\circ L} \mathbf{1}$ is non negative and continuous over \mathcal{O} together with the fact that ρ_p charges all non-empty open subsets of \mathcal{O} , we deduce that $e^{-i_{\mathbf{V}_s, \mathcal{O}} t} \mathbf{1} = Q_t^{\circ L} \mathbf{1}$ on \mathcal{O} , i.e. for all $t \geq 0$ and $x \in \mathcal{O}$,

$$\mathbb{E}_x \left[e^{-\int_0^t \mathbf{V}_s(X_s) ds} \mathbf{1}_{t < \sigma_{\mathcal{O}}^{\circ L}} \right] = e^{-i_{\mathbf{V}_s, \mathcal{O}} t}. \quad (2.17)$$

Let us first consider the case when $\mathcal{R}_0^d \setminus \overline{\mathcal{O}}$ is non-empty. Then, using (2.17), we have $e^{-i_{\mathbf{V}_s, \mathcal{O}} t} \leq e^{-i_{\mathbf{V}_s, \mathcal{O}} t} \mathbb{P}_x[t < \sigma_{\mathcal{O}}^{\circ L}]$ so that for all $t > 0$ and $x \in \mathcal{O}$, $\mathbb{P}_x[t < \sigma_{\mathcal{O}}^{\circ L}] = 1$ and thus

$$\mathbb{P}_x[\sigma_{\mathcal{O}}^{\circ L} = +\infty] = 1, \forall x \in \mathcal{O}.$$

However, as $\mathcal{R}_0^d \setminus \overline{\mathcal{O}}$ is open, there exists a non-empty open ball $B \subset \mathcal{R}_0^d \setminus \overline{\mathcal{O}}$. Moreover, there exist $x_0 \in \mathcal{O}$ and $t_0 > 0$ (actually this is true for all $x_0 \in \mathcal{O}$ and $t_0 > 0$) such that

$$\mathbb{P}_{x_0}[X_{t_0} \in B] > 0.$$

Hence, $0 < \mathbb{P}_{x_0}[X_{t_0} \in B] \leq \mathbb{P}_{x_0}[\sigma_{\mathcal{O}}^{\circ L} \leq t_0]$. A contradiction.

Assume now that $\mathcal{O} = \mathcal{R}_0^d$ so that by Lemma 1, for all $x \in \mathcal{R}_0^d$ a.s.

$$\sigma_{\mathcal{R}_0^d}^{\circ L}(x) = +\infty.$$

Using (2.17), we then have for all $t \geq 0$,

$$\mathbb{E}_x[e^{-\int_0^t \mathbf{V}_*(X_s) ds}] = 1,$$

where $\mathbf{V}_* = \mathbf{V}_S - i_{\mathbf{V}_S, \mathcal{R}_0^d} \geq 0$. Consequently, for all $t \geq 0$ and $x \in \mathcal{O}$, $\mathbb{P}_x[\int_0^t \mathbf{V}_*(X_s) ds = 0] = 1$. Then, $\mathbb{P}_x[\mathbf{V}_*(X_s) = 0, \text{ for almost every } s \in [0, t]] = 1$. Due to the fact that \mathbf{V}_* is continuous over \mathcal{R}_0^d and since a.s. the trajectories of the process $(X_s, s \geq 0)$ are also continuous, for all $t \geq 0$ and all $x \in \mathcal{O}$ (actually càdlàg sample paths would be enough), it holds $\mathbb{P}_x[\mathbf{V}_*(X_s) = 0, \text{ for all } s \in [0, t]] = 1$. In particular, for all $s > 0$ and all $x \in \mathcal{O}$,

$$\mathbb{P}_x[\mathbf{V}_*(X_s) = 0] = 1. \quad (2.18)$$

On the other hand, since $\mathbf{V}_S \neq i_{\mathbf{V}_S, \mathcal{R}_0^d}$ and \mathbf{V}_S is continuous, there exist $c > 0$ and a nonempty ball $B \subset \mathcal{R}_0^d$ such that $\mathbf{V}_S \geq i_{\mathbf{V}_S, \mathcal{R}_0^d} + c$ over B . Furthermore, there exist $x_0 \in \mathcal{O}$ and $s_0 > 0$ (actually this is true for all x_0 and $s_0 > 0$) such that,

$$\mathbb{P}_{x_0}[X_{s_0} \in B] > 0.$$

This contradicts (2.18) and shows that $\lambda_p > i_{\mathbf{V}_S, \mathcal{O}}$. The proof of Theorem 1 is thus complete.

2.1.3. On the essential spectral radius of the non killed Feynman-Kac semigroup. Recall (2.2) for the definition of the non killed Feynman-Kac semigroup $(T_t^{\circ L}, t \geq 0)$.

Theorem 7. *Assume [c1] or [c2] and assume also [S1] or [S2]. Moreover, assume that there exist constants $\mathbf{m}_0, \mathbf{r}_0, \mathbf{b}_0 > 0$, a compact subset \mathcal{K}_0 of $\mathcal{R}_0^d = \mathcal{R}^d \setminus \{0\}$, and a \mathcal{C}^2 function $\mathbf{W}_* : \mathcal{R}^d \rightarrow [1, +\infty)$ such that*

$$\frac{\mathcal{L}^{\circ L} \mathbf{W}_*}{\mathbf{W}_*} - \mathbf{V}_S \leq -\mathbf{r}_0 \mathbf{1}_{\mathcal{R}_0^d \setminus \mathcal{K}_0} + \mathbf{b}_0 \mathbf{1}_{\mathcal{K}_0} \text{ on } \mathcal{R}_0^d, \quad (2.19)$$

and for some $p > 1$,

$$(\mathcal{L}^{\circ L} - \mathbf{V}_S) \mathbf{W}_*^p \leq \mathbf{m}_0 \mathbf{W}_*^p \text{ on } \mathcal{R}_0^d. \quad (2.20)$$

Then, for all $t \geq 0$, $T_t^{\circ L}$ is a bounded operator on $b\mathcal{B}_{\mathbf{W}_*}(\mathcal{R}_0^d)$. In addition (see (2.14) and (2.23) for the definitions of β_w and $T_{t, \mathbf{W}_*}^{\circ L}$),

$$\beta_w(T_{t, \mathbf{W}_*}^{\circ L} |_{b\mathcal{B}(\mathcal{R}_0^d)}) \leq e^{-\mathbf{r}_0 t} \text{ and } r_{\text{ess}}(T_t^{\circ L} |_{b\mathcal{B}_{\mathbf{W}_*}(\mathcal{R}_0^d)}) \leq e^{-\mathbf{r}_0 t}.$$

Proof. In all this proof, $k_S \geq 0$ is such that $\mathbf{V}_S \geq -k_S$.

Step 1. In this step we prove that $T_t^{\circ L}$ is a bounded operator on $b\mathcal{B}_{\mathbf{W}_*}(\mathcal{R}_0^d)$. The proof is rather standard, we make it for sake of completeness. Differentiating the function $t \in \mathcal{R}_+ \mapsto e^{-\mathbf{m}_0 t} T_t^{\circ L} \mathbf{W}_*^p(x)$ for $x \in \mathcal{R}_0^d$ and using (2.20) yields:

$$\frac{d}{dt} (e^{-\mathbf{m}_0 t} T_t^{\circ L} \mathbf{W}_*^p) = e^{-\mathbf{m}_0 t} T_t^{\circ L} (-\mathbf{m}_0 \mathbf{W}_*^p + (\mathcal{L}^{\circ L} - \mathbf{V}_S) \mathbf{W}_*^p) \leq 0.$$

Therefore, for all $t \geq 0$,

$$T_t^{\circ L} \mathbf{W}_*^p \leq e^{\mathbf{m}_0 t} \mathbf{W}_*^p \text{ on } \mathcal{R}_0^d. \quad (2.21)$$

In addition, by Hölder's inequality, we have on \mathcal{R}_0^d ,

$$T_t^{\circ L} \mathbf{W}_* \leq [T_t^{\circ L} \mathbf{1}]^{1/q} [T_t^{\circ L} \mathbf{W}_*^p]^{1/p},$$

where $q = p/(p-1)$. On the other hand, it holds for all $x \in \mathcal{R}_0^d$, $T_t^{oL} \mathbf{1}(x) = \mathbb{E}_x[e^{-\int_0^t \mathbf{V}_s(X_s) ds}] \leq e^{ks^t}$. Consequently, on \mathcal{R}_0^d ,

$$T_t^{oL} \mathbf{W}_\star \leq e^{(ks/q+m_0/p)t} \mathbf{W}_\star. \quad (2.22)$$

Hence, $T_t^{oL} : b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{R}_0^d) \rightarrow b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{R}_0^d)$ is well defined and is a bounded operator. We can then define the operator $T_{t, \mathbf{W}_\star}^{oL}$ on $b\mathcal{B}(\mathcal{R}_0^d)$ by

$$T_{t, \mathbf{W}_\star}^{oL}(x, dy) = \frac{\mathbf{W}_\star(y)}{\mathbf{W}_\star(x)} T_t^{oL}(x, dy), \quad (2.23)$$

which is similar to $T_t^{oL}|_{b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{R}_0^d)}$. In particular, $T_{t, \mathbf{W}_\star}^{oL}|_{b\mathcal{B}(\mathcal{R}_0^d)}$ and $T_t^{oL}|_{b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{R}_0^d)}$ have the same essential spectral radius.

Step 2. We now prove that $T_{t, \mathbf{W}_\star}^{oL}|_{b\mathcal{B}(\mathcal{R}_0^d)}$ satisfies the assumption **(A1)**. Since T_t^{oL} is strongly Feller for all $t > 0$, see Theorem 6, T_t^{oL} satisfies **(A1)** with $N = 1$ (see [36, Remark 3.3]). Let K_1 be a compact subset of \mathcal{R}_0^d . Applying Hölder's inequality, we have that since $\beta_w(\mathbf{1}_{K_1} T_t^{oL}) = 0$,

$$\begin{aligned} \beta_w(\mathbf{1}_{K_1} T_{t, \mathbf{W}_\star}^{oL}) &= \inf_{K \subset \mathcal{R}_0^d} \sup_{x \in K_1} T_{t, \mathbf{W}_\star}^{oL}(x, \mathcal{R}_0^d \setminus K) \\ &= \inf_{K \subset \mathcal{R}_0^d} \sup_{x \in K_1} \frac{1}{\mathbf{W}_\star(x)} \int_{\mathcal{R}_0^d \setminus K} \mathbf{W}_\star(y) T_t^{oL}(x, dy) \leq e^{m_0 t/p} [\beta_w(\mathbf{1}_{K_1} T_t^{oL})]^{1/q} = 0. \end{aligned}$$

With the same arguments,

$$\beta_\tau(\mathbf{1}_{K_1} T_{t, \mathbf{W}_\star}^{oL}) = \sup_{(A_n)} \lim_{n \rightarrow +\infty} \sup_{x \in K_1} T_{t, \mathbf{W}_\star}^{oL}(x, A_n) \leq e^{m_0 t/p} [\beta_\tau(\mathbf{1}_{K_1} T_t^{oL})]^{1/q} = 0.$$

Hence $T_{t, \mathbf{W}_\star}^{oL}|_{b\mathcal{B}(\mathcal{R}_0^d)}$ satisfies **(A1)**. This allows us to use (2.15) to get that:

$$r_{ess}(T_{t, \mathbf{W}_\star}^{oL}|_{b\mathcal{B}(\mathcal{R}_0^d)}) = \lim_{n \rightarrow +\infty} [\beta_w([T_{t, \mathbf{W}_\star}^{oL}]^n)]^{1/n}. \quad (2.24)$$

Let us now prove that

$$\beta_w(T_{t, \mathbf{W}_\star}^{oL}) \leq e^{-r_0 t}. \quad (2.25)$$

Note that (2.25) implies that $r_{ess}(T_{t, \mathbf{W}_\star}^{oL}|_{b\mathcal{B}_{\mathbf{W}_\star}(\mathcal{R}_0^d)}) \leq e^{-r_0 t}$. Indeed, by [75, Proposition 3.2.(e)], we have $\beta_w([T_{t, \mathbf{W}_\star}^{oL}]^n) \leq \beta_w(T_{t, \mathbf{W}_\star}^{oL})^n$.

Step 3. Let us now prove (2.25). We have by definition that

$$\beta_w(T_{t, \mathbf{W}_\star}^{oL}) = \inf_{K \subset \mathcal{R}_0^d} \sup_{x \in \mathcal{R}_0^d} \frac{1}{\mathbf{W}_\star(x)} \mathbb{E}_x \left[\mathbf{W}_\star(X_t) \mathbf{1}_{\mathcal{R}_0^d \setminus K}(X_t) e^{-\int_0^t \mathbf{V}_s(X_s) ds} \right]. \quad (2.26)$$

Denote by $\tau_{\mathcal{K}_0}$ the first hitting time of the compact \mathcal{K}_0 for the process (where \mathcal{K}_0 is the compact set such that (2.19) holds), that is to say $\tau_{\mathcal{K}_0} := \inf\{s \geq 0, X_s \in \mathcal{K}_0\}$. We have for all $x \in \mathcal{R}_0^d$ and all compact subset K of \mathcal{R}_0^d :

$$\begin{aligned} & \frac{1}{\mathbf{W}_\star(x)} \mathbb{E}_x \left[\mathbf{W}_\star(X_t) \mathbf{1}_{\mathcal{R}_0^d \setminus K}(X_t) e^{-\int_0^t \mathbf{V}_s(X_s) ds} \right] \\ &= \frac{1}{\mathbf{W}_\star(x)} \mathbb{E}_x \left[\mathbf{W}_\star(X_t) \mathbf{1}_{\tau_{\mathcal{K}_0} \leq t} \mathbf{1}_{\mathcal{R}_0^d \setminus K}(X_t) e^{-\int_0^t \mathbf{V}_s(X_s) ds} \right] \\ &+ \frac{1}{\mathbf{W}_\star(x)} \mathbb{E}_x \left[\mathbf{W}_\star(X_t) \mathbf{1}_{t < \tau_{\mathcal{K}_0}} \mathbf{1}_{\mathcal{R}_0^d \setminus K}(X_t) e^{-\int_0^t \mathbf{V}_s(X_s) ds} \right]. \end{aligned} \quad (2.27)$$

Step 3a. In this step, we deal with the first term in the r.h.s. of (2.27). Let K be a compact subset of \mathcal{R}_0^d . Using the strong Markov property of the process $(X_t, t \geq 0)$, for all $x \in \mathcal{R}_0^d$:

$$\begin{aligned} & \frac{1}{\mathbf{W}_*(x)} \mathbb{E}_x \left[\mathbf{W}_*(X_t) \mathbf{1}_{\tau_{\mathcal{X}_0} \leq t} \mathbf{1}_{\mathcal{R}_0^d \setminus K}(X_t) e^{-\int_0^t \mathbf{v}_s(X_s) ds} \right] \\ & \leq \frac{\mathbb{E}_x \left[\mathbf{W}_*^p(X_t) e^{-\int_0^t \mathbf{v}_s(X_s) ds} \right]^{1/p}}{\mathbf{W}_*(x)} \mathbb{E}_x \left[\mathbf{1}_{\tau_{\mathcal{X}_0} \leq t} \mathbf{1}_{\mathcal{R}_0^d \setminus K}(X_t) e^{-\int_0^t \mathbf{v}_s(X_s) ds} \right]^{1/q} \\ & \leq e^{(ks/q+m_0/p)t} \mathbb{E}_x \left[\mathbf{1}_{\tau_{\mathcal{X}_0} \leq t} \mathbf{1}_{\mathcal{R}_0^d \setminus K}(X_t) \right]^{1/q} \\ & = e^{(ks/q+m_0/p)t} \mathbb{E}_x \left[\mathbf{1}_{\tau_{\mathcal{X}_0} \leq t} \mathbb{P}_{X_{\tau_{\mathcal{X}_0}}} [X_{t-\tau_{\mathcal{X}_0}} \in \mathcal{R}_0^d \setminus K] \right]^{1/q}. \end{aligned}$$

In addition, we have for all $x \in \mathcal{R}_0^d$,

$$\begin{aligned} \mathbb{E}_x \left[\mathbf{1}_{\tau_{\mathcal{X}_0} \leq t} \mathbb{P}_{X_{\tau_{\mathcal{X}_0}}} [X_{t-\tau_{\mathcal{X}_0}} \in \mathcal{R}_0^d \setminus K] \right] & \leq \mathbb{E}_x \left[\mathbf{1}_{\tau_{\mathcal{X}_0} \leq t} \mathbb{P}_{X_{\tau_{\mathcal{X}_0}}} [\exists s \in [0, t], X_s \in \mathcal{R}_0^d \setminus K] \right] \\ & \leq \sup_{z \in \mathcal{X}_0} \mathbb{P}_z [\exists s \in [0, t], X_s \in \mathcal{R}_0^d \setminus K]. \end{aligned}$$

Let us now prove that

$$\inf_{K \subset \mathcal{R}_0^d} \sup_{z \in \mathcal{X}_0} \mathbb{P}_z [\exists s \in [0, t], X_s \in \mathcal{R}_0^d \setminus K] = 0. \quad (2.28)$$

First of all, note that by (1.8) and Proposition 4, the mapping

$$z \in \mathcal{R}_0^d \mapsto \mathbb{P}_z [X_{[0,t]} \in \cdot] \in \mathcal{P}(\mathcal{C}([0, t], \mathcal{R}_0^d))$$

is continuous for the weak topology. Note indeed that (1.8) and Proposition 4 imply that

$$\mathbb{P} \left[\sup_{s \in [0, t]} \mathbf{d}_{\mathcal{R}_0^d}(X_s(x_n), X_s(x)) \geq \epsilon \right] \rightarrow 0,$$

as $x_n \rightarrow x \in \mathcal{R}_0^d$ and for all $\epsilon > 0$ (see (1.7)). Hence, the family of probability measures $\{z \in \mathcal{X}_0, \mathbb{P}_z [X_{[0,t]} \in \cdot]\}$ over $\mathcal{C}([0, t], \mathcal{R}_0^d)$ is relatively compact for the weak convergence topology and thus, is tight. Consequently, for all $\epsilon > 0$, there exists a compact set A_ϵ of $\mathcal{C}([0, t], \mathcal{R}_0^d)$ such that $\sup_{z \in \mathcal{X}_0} \mathbb{P}_z [X_{[0,t]} \notin A_\epsilon] < \epsilon$. Then, introduce $K_\epsilon = \{\gamma_s, s \in [0, t], \gamma \in A_\epsilon\}$ which is a compact subset of \mathcal{R}_0^d . Therefore, we have

$$\begin{aligned} \inf_{K \subset \mathcal{R}_0^d} \sup_{z \in \mathcal{X}_0} \mathbb{P}_z [\exists s \in [0, t], X_s \in \mathcal{R}_0^d \setminus K] & \leq \sup_{z \in \mathcal{X}_0} \mathbb{P}_z [\exists s \in [0, t], X_s \in \mathcal{R}_0^d \setminus K_\epsilon] \\ & \leq \sup_{z \in \mathcal{X}_0} \mathbb{P}_z [X_{[0,t]} \notin A_\epsilon] < \epsilon. \end{aligned}$$

This ends the proof of (2.28) and shows that the first term in the r.h.s. of (2.27) vanishes.

Step 3b. In this step, we deal with the second term in the r.h.s. of (2.27). We have that

$$\mathbb{E}_x \left[\mathbf{W}_*(X_t) \mathbf{1}_{t < \tau_{\mathcal{X}_0}} \mathbf{1}_{\mathcal{R}_0^d \setminus K}(X_t) e^{-\int_0^t \mathbf{v}_s(X_s) ds} \right] \leq \mathbb{E}_x \left[\mathbf{W}_*(X_t) \mathbf{1}_{t < \tau_{\mathcal{X}_0}} e^{-\int_0^t \mathbf{v}_s(X_s) ds} \right].$$

Let $(\mathbf{m}_t, t \geq 0)$ be defined by:

$$\mathbf{m}_t := \frac{\mathbf{W}_*(X_t)}{\mathbf{W}_*(X_0)} \exp \left(- \int_0^t \frac{\mathcal{L}^{oL} \mathbf{W}_*}{\mathbf{W}_*}(X_s) ds \right).$$

This is a local martingale $(\mathbf{m}_t, t \geq 0)$, which turns out to be actually a supermartingale. Hence, it holds:

$$\text{for all } x \in \mathcal{R}_0^d, \mathbb{E}_x [\mathbf{m}_t] \leq 1.$$

Thus, using (2.19), we have for $x \in \mathcal{R}_0^d$,

$$\frac{1}{\mathbf{W}_*(x)} \mathbb{E}_x \left[\mathbf{W}_*(X_t) \mathbf{1}_{t < \tau_{\mathcal{X}_0}} e^{-\int_0^t \mathbf{V}_s(X_s) ds} \right] \leq e^{-\tau_0 t} \mathbb{E}_x [\mathbf{1}_{t < \tau_{\mathcal{X}_0}} \mathbf{m}_t] \leq e^{-\tau_0 t}.$$

Consequently, the second term in the r.h.s. of (2.27) is upper bounded by $e^{-\tau_0 t}$. In conclusion, coming back to (2.26) and (2.27), we have proved (2.25). This achieves the proof of Theorem 7. \square

2.2. A general result and proof of Theorem 2.

2.2.1. *A general result.* In view of the proof of Theorem 1, we have the following result, where we use the notation and the assumptions of Section 1.2.1 (recall in particular (1.27) and (1.28)). Assume that the process $(\mathfrak{X}_t, t \geq 0)$ satisfies the following conditions (below $(U_t^{\mathfrak{X}}, t \geq 0)$ denotes the semigroup⁹ of the process $(\mathfrak{X}_t, t \geq 0)$):

[SF₁] For all $t > 0$ and $f \in b\mathcal{B}(\mathcal{M})$, the mapping $\mathfrak{z} \in \mathcal{M} \mapsto T_t^{\mathfrak{X}} f(\mathfrak{z}) := \mathbb{E}_{\mathfrak{z}}[f(\mathfrak{X}_t) e^{-\int_0^t \mathbf{V}(\mathfrak{X}_s) ds}]$ is continuous.

[SF₂] For all $t > 0$ and $f \in b\mathcal{B}(\mathcal{V})$, $\mathfrak{z} \in \mathcal{V} \mapsto Q_t^{\mathfrak{X}} f(\mathfrak{z})$ is continuous.

[Ti] For all $t > 0$, $\mathfrak{z} \in \mathcal{V}$ and all nonempty open subset O of \mathcal{V} , $Q_t^{\mathfrak{X}}(\mathfrak{z}, O) > 0$.

[L_{yap}] There exists a continuous function $\mathbf{W} : \mathcal{M} \rightarrow [1, +\infty)$ in the extended domain $\mathbb{D}_e(\mathcal{L})$ of the generator¹⁰ \mathcal{L} of $(U_t^{\mathfrak{X}}, t \geq 0)$ such that for all $p > 1$, there exist positive constants (\mathfrak{r}_n) and (\mathfrak{b}_n) where $\mathfrak{r}_n \rightarrow +\infty$, and an increasing sequence of compact subsets (\mathcal{K}_n) of \mathcal{M} , such that for all $\mathfrak{z} \in \mathcal{M}$,

$$\frac{\mathcal{L}\mathbf{W}(\mathfrak{z})}{\mathbf{W}(\mathfrak{z})} - p\mathbf{V}(\mathfrak{z}) \leq -\mathfrak{r}_n + \mathfrak{b}_n \mathbf{1}_{\mathcal{K}_n}(\mathfrak{z}).$$

[P_{traj}] For all $t \geq 0$, the mapping $\mathfrak{z} \in \mathcal{M} \mapsto \mathbb{P}[\mathfrak{X}_{[0,t]}(\mathfrak{z}) \in \cdot] \in \mathcal{P}(\mathcal{D}([0,t], \mathcal{M}))$ is continuous for the weak topology.

Note also that thanks to (1.27), [Ti] is equivalent to the fact that $(\mathfrak{X}_t, t \in [0, \sigma_{\mathcal{V}}^{\mathfrak{X}}))$ is topologically irreducible over \mathcal{V} . Then, under the five assumptions above, we have the following result whose proof is left to the reader since it is a direct generalization of the proof of Theorem 1.

Theorem 8. *Assume [SF₁], [SF₂], [Ti], [L_{yap}], and [P_{traj}]. Then, for all $p > 1$, the killed Feynman-Kac semigroup $(Q_t^{\mathfrak{X}}, t \geq 0)$ is $\mathbf{W}^{1/p}$ compact-ergodic over \mathcal{V} . In addition, if:*

- [O1] $\mathbb{P}_{\mathfrak{z}^*}[\sigma_{\mathcal{V}}^{\mathfrak{X}} < \infty] > 0$ for some $\mathfrak{z}^* \in \mathcal{V}$, or,
- [O2] if $\mathcal{V} = \mathcal{M}$ and \mathbf{V} is a non constant function over \mathcal{M} ,

then $\lambda_p > \inf_{\mathcal{V}} \mathbf{V}$ where λ is the principal eigenvalue of $(Q_t^{\mathfrak{X}}, t \geq 0)$ in $b\mathcal{B}_{\mathbf{W}^{1/p}}(\mathcal{V})$.

Let us just give some indications and remarks on the proof of Theorem 1. On the one hand, for $p > 1$, $\mathbf{W}^{1/p} \in \mathbb{D}_e(\mathcal{L})$, and hence

$$t \mapsto \mathbf{M}_t := \frac{\mathbf{W}^{1/p}(\mathfrak{X}_t)}{\mathbf{W}^{1/p}(\mathfrak{X}_0)} \exp\left(-\int_0^t \frac{\mathcal{L}\mathbf{W}^{1/p}}{\mathbf{W}^{1/p}}(\mathfrak{X}_s) ds\right)$$

is a local martingale, which is a supermartingale by Fatou's lemma. This is a key argument in the proof of Theorem 8. Note also that [L_{yap}] implies that $\mathcal{L}\mathbf{W}^{1/p}/\mathbf{W}^{1/p} - \mathbf{V} \leq \mathfrak{m} = \mathfrak{b}_0/p$. Hence, using $(\mathbf{M}_t, t \geq 0)$, $T_t^{\mathfrak{X}} \mathbf{W}^{1/p}(\mathfrak{z}) = \mathbb{E}_{\mathfrak{z}}[\mathbf{W}^{1/p}(\mathfrak{X}_t) e^{-\int_0^t \mathbf{V}(\mathfrak{X}_s) ds}] \leq e^{\mathfrak{m}t} \mathbf{W}^{1/p}(\mathfrak{z})$,

⁹Defined by $U_t^{\mathfrak{X}} f(\mathfrak{z}) = \mathbb{E}_{\mathfrak{z}}[f(\mathfrak{X}_t)]$, $\mathfrak{z} \in \mathcal{M}$ and $f \in b\mathcal{B}(\mathcal{M})$.

¹⁰See [63, Chapter VII], [22], or [36] and references therein for a definition.

which proves (2.22) in this general setting. Moreover, let us recall that $[\mathbf{P}_{\text{traj}}]$ was indeed used in the third step of the proof of Theorem 7.

In view of Theorem 6, one way to check $[\mathbf{SF}_1]$ and $[\mathbf{SF}_2]$ is the following.

Theorem 9. *Assume that:*

- $[\mathbf{SF}_0]$ For every $t > 0$ and $f \in b\mathcal{B}(\mathcal{M})$, $\mathfrak{z} \in \mathcal{M} \mapsto U_t^{\mathfrak{z}} f(\mathfrak{z})$ is continuous.
- $[\mathbf{P}_{t=s}]$ The mapping $\mathfrak{z} \in \mathcal{M} \mapsto \mathfrak{X}_t(\mathfrak{z})$ is continuous in \mathbb{P} -probability.
- $[\mathbf{D}_e]$ For all $t > 0$ and $\mathfrak{z} \in \mathcal{M}$, $\mathfrak{X}_t(\mathfrak{z})$ has a density w.r.t. the Lebesgue measure.
- $[\mathbf{IV}_S]$ For all $t \geq 0$ and all $\mathfrak{z}_n \rightarrow \mathfrak{z} \in \mathcal{M}$, $\int_0^t \mathbf{V}(\mathfrak{X}_s(\mathfrak{z}_n)) ds \rightarrow \int_0^t \mathbf{V}(\mathfrak{X}_s(\mathfrak{z})) ds$ in \mathbb{P} -probability.

Then, $[\mathbf{SF}_1]$ is satisfied. Assume in addition that

- $[\mathbf{B}]$ For all compact subset K of \mathcal{V} , $\lim_{s \rightarrow 0^+} \sup_{\mathfrak{z} \in K} \mathbb{P}_{\mathfrak{z}}[\sigma_{\mathcal{V}}^{\mathfrak{z}} \leq s] = 0$.

Then $[\mathbf{SF}_2]$ holds.

Proof. Using $[\mathbf{SF}_0]$ together with $[\mathbf{P}_{t=s}]$ and $[\mathbf{D}_e]$, we deduce as in Step A.2 in the proof of Theorem 6 that for every $f \in b\mathcal{B}(\mathcal{M})$, and for all $\mathfrak{z}_n \rightarrow \mathfrak{z} \in \mathcal{M}$, it holds as $n \rightarrow +\infty$:

$$f(\mathfrak{X}_s(\mathfrak{z}_n)) \rightarrow f(\mathfrak{X}_s(\mathfrak{z})) \text{ in } \mathbb{P}\text{-probability.}$$

Using $[\mathbf{IV}_S]$ and the fact that \mathbf{V} is lower bounded, we deduce as in Step A.3 in the proof of Theorem 6 that for every $f \in b\mathcal{B}(\mathcal{M})$, and for all $\mathfrak{z}_n \rightarrow \mathfrak{z} \in \mathcal{M}$, one has:

$$\mathbb{E}_{\mathfrak{z}_n}[f(\mathfrak{X}_t) e^{-\int_0^t \mathbf{V}(\mathfrak{X}_s) ds}] \rightarrow \mathbb{E}_{\mathfrak{z}}[f(\mathfrak{X}_t) e^{-\int_0^t \mathbf{V}(\mathfrak{X}_s) ds}],$$

which proves $[\mathbf{SF}_1]$. We then conclude the proof of $[\mathbf{SF}_2]$ using $[\mathbf{B}]$ and the same arguments as those used in Step B in the proof of Theorem 6. \square

2.2.2. Proof of Theorem 2. Let $(L_t, t \geq 0)$ be a Lévy process over \mathcal{R}^d satisfying $[\mathbf{L1}]$ and recall Lemma 2. Let \mathcal{O} be a subdomain of \mathcal{R}_0^d (see (1.6)). Assume that $(L_t, t \geq 0)$ satisfies $[\mathbf{L2}]$, and assume also $[\mathbf{S2}]$. To prove Theorem 2 we apply Theorems 8 and 9 to the Lévy process $(L_t, t \geq 0)$ over \mathcal{R}_0^d . Let us denote by \mathcal{L}^{Le} the generator of the Lévy process $(L_s, s \geq 0)$, see [67, Theorem 6.8]. The non-killed Feynman-Kac semigroup $(T_t^{Le}, t \geq 0)$ over the space \mathcal{R}_0^d associated with the potential \mathbf{V}_S and the process $(L_s, s \geq 0)$ is defined by

$$T_t^{Le} f(x) = \mathbb{E}_x \left[f(L_t) e^{-\int_0^t \mathbf{V}_S(L_s) ds} \right], \quad t \geq 0, x \in \mathcal{R}_0^d, \text{ and } f \in b\mathcal{B}(\mathcal{R}_0^d). \quad (2.29)$$

Let us first check Assumptions $[\mathbf{SF}_1]$ and $[\mathbf{SF}_2]$ using Theorem 9 for the process $(L_t, t \geq 0)$ over $\mathcal{M} = \mathcal{R}_0^d$ and with $\mathbf{V} = \mathbf{V}_S$. By assumption, for all $x \in \mathcal{R}^d$, $L_s(x)$ has a density w.r.t. the Lebesgue measure dy (and more precisely, $\mathbb{E}_z[f(L_s)] = \int p_s(z-y)f(y)dy$, for every $z \in \mathcal{R}^d$ and $f \in b\mathcal{B}(\mathcal{R}^d)$). In particular it is well-known that this implies that the semigroup of $(L_s, s \geq 0)$ is strongly Feller (see e.g. [67, Lemma 4.9]). Moreover, for all $y, z \in \mathcal{R}^d$,

$$\sup_{s \geq 0} |L_s(y) - L_s(z)| = |y - z|. \quad (2.30)$$

Thus, for all $t \geq 0$ and all $(x_n)_n \subset \mathcal{R}_0^d$ such that $x_n \rightarrow x \in \mathcal{R}_0^d$, we have that almost surely:

$$\int_0^t \mathbf{V}_S(L_s(x_n)) ds \rightarrow \int_0^t \mathbf{V}_S(L_s(x)) ds.$$

This follows from a dominated convergence theorem together with (2.30). Indeed, recall that by (1.13), $\inf_{s \in [0, t]} |L_s(x)| > 0$ almost surely. Then, for $\epsilon \in (0, \inf_{s \in [0, t]} |L_s(x)|/2)$, there exists $n_\epsilon \geq 1$ such that for all $n \geq n_\epsilon$ and all $s \in [0, t]$, $|L_s(x_n)| \geq |L_s(x)| - \epsilon \geq$

$\inf_{s \in [0, t]} |L_s(x)|/2 > 0$. Hence there exists $c > 0$ such that for all $n \geq n_\epsilon$ and all $s \in [0, t]$, $|\mathbf{V}_s(L_s(x_n))| \leq c$. This proves [IV \mathbf{S}]. Moreover, [SF $_1$]. By Theorem 9, T_t^{Le} is strongly Feller for $t > 0$. On the other hand, for all compact subset K of \mathcal{R}^d and $\delta > 0$ it holds

$$\lim_{s \rightarrow 0^+} \sup_{x \in K} \mathbb{P}_x[\sigma_{B(x, \delta)}^{Le} \leq s] = 0,$$

where $\sigma_{B(x, \delta)}^{Le}(x) := \inf\{t \geq 0, L_t(x) \notin B(x, \delta)\}$. Note indeed that as $s \rightarrow 0^+$

$$\mathbb{P}_x[\sigma_{B(x, \delta)}^{Le} \leq s] = \mathbb{P}[\sigma_{B(0, \delta)}^{Le} \leq s] \rightarrow 0$$

since a.s. $L_t \rightarrow L_0 = 0$ when $t \rightarrow 0^+$. This proves [B] and then, [SF $_2$] by Theorem 9.

We now check the other assumptions in Theorem 8. First the fact that [Ti] is satisfied (i.e. for all $t > 0$, all $x \in \mathcal{O}$, and all non-empty open subset O of \mathcal{O} , $Q_t^{Le}(x, O) > 0$) follows from [L2] together with the fact that $\int_0^t \mathbf{V}_s(L_s(x)) ds$ is a.s. finite for all $x \neq 0$ (see Lemma 2). In addition, the following Lyapunov condition is satisfied thanks to [S2]. When $x \rightarrow \partial \mathcal{R}_0^d \cup \{\infty\}$ ($x \in \mathcal{R}_0^d$), where we recall that $\partial \mathcal{R}_0^d = \{0\}$,

$$\frac{\mathcal{L}^{Le} \mathbf{1}(x)}{\mathbf{1}(x)} - p \mathbf{V}_s(x) = -p \mathbf{V}_s(x) \rightarrow -\infty, \quad \forall p > 1.$$

Finally, the mapping $z \in \mathcal{R}_0^d \mapsto \mathbb{P}[L_{[0, t]}(z) \in \cdot] \in \mathcal{P}(\mathcal{D}([0, t], \mathcal{R}_0^d))$ is continuous for the weak topology (this is [P $_{\text{traj}}$]). This function is well-defined and continuous because we first have for all $z \in \mathcal{R}_0^d$ and $t \geq 0$, $\mathbb{P}_x[L_{[0, t]} \in \mathcal{D}([0, t], \mathcal{R}_0^d)] = 1$ and we also have, thanks to (2.30), that when $x_n \rightarrow x \in \mathcal{R}_0^d$ (see (1.7)):

$$\mathbb{P}[\sup_{s \in [0, t]} \mathbf{d}_{\mathcal{R}_0^d}(L_s(x_n), L_s(x)) \geq \epsilon] \rightarrow 0.$$

It thus remains to check [O1] and [O2]. First assume that $\overline{\mathcal{R}_0^d} \setminus \overline{\mathcal{O}}$ is non-empty. Then, we assume [L3] which is precisely [O1]. Moreover when $\mathcal{O} = \mathcal{R}_0^d$, [O2] is clearly satisfied since, by [S2], \mathbf{V}_s is non constant over \mathcal{R}_0^d .

2.3. Proofs of Lemma 3 and of Theorem 3. In this section, we prove Lemma 3 and Theorem 3.

2.3.1. Proof of Lemma 3. Recall that because \mathbf{V}_c is lower bounded over \mathcal{R}^d , thanks to Proposition 2 we have a unique strong solution $(Y_t = (x_t, v_t), t \geq 0)$ to (1.15) over $\mathcal{R}^d \times \mathcal{R}^d$ which satisfies the Girsanov formula (1.17). Let us prove (1.19) for all $y \in \mathcal{R}_0^{2d}$ where we recall that $\mathcal{R}_0^{2d} = (\mathcal{R}^d \setminus \{0\}) \times \mathcal{R}^d$, see (1.18). Note that (1.19) is equivalent to the fact that for all $T \geq 0$ and $y \in \mathcal{R}_0^{2d}$,

$$\mathbb{P}_y[\forall t \in [0, T], |x_t| > 0] = 1. \quad (2.31)$$

By the Girsanov formula (see Proposition 2), (2.31) is equivalent to: for all $T \geq 0$ and $y \in \mathcal{R}_0^{2d}$,

$$\mathbb{P}_y[\forall t \in [0, T], |x_t^0| > 0] = 1, \quad (2.32)$$

where $(Y_t^0 = (x_t^0, v_t^0), t \geq 0)$ is the solution to the stochastic differential equation (1.16). Let us prove (2.32). Introduce to this end

$$\tau_{\{0\}}^0 := \inf\{t \geq 0, x_t^0 = 0\}.$$

Let us show that $\mathbb{P}_y[\tau_{\{0\}}^0 \leq T] = 0$ (this is exactly (2.32)).

Step 1. Let $T > 0$ (otherwise the result is obvious). First of all we claim that for all $\eta \in (0, T]$,

$$\mathbb{P}_y[\forall t \in [\eta, T], |x_t^0| > 0] = 1. \quad (2.33)$$

To prove (2.33), we use [70, Theorem 1] with $I = [\eta, T]$ and $N = 1$ there and for the Gaussian process $x_t(\mathbf{y}) = x + vt + \int_0^t B_u du$, $\mathbf{y} = (x, v) \in \mathcal{R}^{2d}$. To this end we check that **Conditions 1** and **2** in [70] are satisfied for the process $(x_t(\mathbf{y}), t \geq 0)$ on I . These assumptions are easy to check and we check them for sake of completeness. On the one hand, we have for all $0 \leq s \leq t$,

$$|x_t(\mathbf{y}) - x_s(\mathbf{y})| \leq |v||t - s| + \left| \int_s^t B_u du \right| \leq |v||t - s| + |t - s|^{1/2} \left(\int_s^t |B_u|^2 du \right)^{1/2},$$

so that for some $C > 0$,

$$|\mathbb{E}[|x_t(\mathbf{y}) - x_s(\mathbf{y})|^2]|^{1/2} \leq C|t - s|.$$

This is **Condition 1** in [70]. Adopting the terminology of [70, p. 845], this implies that $H_1 = 1$ and the parameter Q is equal to 1. Introduce now $e = (e_1, \dots, e_d) \in \mathcal{R}^d$ such that $\sum_j e_j^2 = 1$. Write $x = (x_1, \dots, x_d)$, $v = (v_1, \dots, v_d)$, and $B_u = (B_u^1, \dots, B_u^d)$ where the B^j 's are independent standard real Brownian motions. Then, one has for all $t \in I = [\eta, T]$:

$$\begin{aligned} \mathbb{E} \left[\left(\sum_{j=1}^d e_j \left(x_j + v_j t + \int_0^t B_u^j du \right) \right)^2 \right] &= \left| \sum_{j=1}^d e_j (x_j + v_j t) \right|^2 + \mathbb{E} \left[\left(\sum_{j=1}^d e_j \int_0^t B_u^j du \right)^2 \right] \\ &= |e \cdot (x + vt)|^2 + \sum_{j=1}^d |e_j|^2 \mathbb{E} \left[\left| \int_0^t B_u^j du \right|^2 \right] \\ &= |e \cdot (x + vt)|^2 + t^3/3 \geq \eta^3/3. \end{aligned}$$

This proves **Condition 2** in [70] when $I = [\eta, T]$. Since $Q = 1 < d$, by [70, Theorem 1] and since the Hausdorff measure $\dim_{d-1}(\{0\})$ of the set $\{0\}$ in dimension $d - 1$ (also called the $d - 1$ -Hausdorff measure of the set $\{0\}$. See for instance [59, Definition 4.7] for a definition) is 0, we have that for all $\eta \in (0, T]$,

$$\mathbb{P}[\exists s \in [\eta, T], x_s(\mathbf{y}) \in \{0\}] = 0.$$

This achieves the proof of (2.33). Note that we have not used yet that the initial condition satisfies $x \neq 0$.

Step 2. We now conclude the proof of the fact that $\mathbb{P}_y[\tau_{\{0\}}^0 \leq T] = 0$. Fix $\mathbf{y} \in \mathcal{R}_0^{2d}$ and $T > 0$. Let $\epsilon > 0$. Consider $M_\epsilon > 0$ large enough such that

$$\mathbb{P} \left[\sup_{u \in [0, T]} |B_u| > M_\epsilon \right] \leq \epsilon.$$

Then, one has for all $\eta \in (0, T]$, using (2.33), $\mathbb{P}_y[\tau_{\{0\}}^0 \in [\eta, T]] \leq \mathbb{P}_y[\exists s \in [\eta, T], |x_s| = 0] = 0$. Hence, for all $\eta \in (0, T]$,

$$\mathbb{P}_y[\tau_{\{0\}}^0 \leq T] \leq \mathbb{P}_y[\tau_{\{0\}}^0 \in (0, \eta), \sup_{u \in [0, T]} |B_u| \leq M_\epsilon] + \epsilon.$$

Note that when $\tau_{\{0\}}^0 \in (0, \eta)$ and $\sup_{u \in [0, T]} |B_u| \leq M_\epsilon$, we have, since $x_{\tau_{\{0\}}^0}(\mathbf{y}) = 0$,

$$0 < |x| \leq |v|\tau_{\{0\}}^0 + \int_0^{\tau_{\{0\}}^0} |B_u| du \leq \eta(|v| + M_\epsilon).$$

Choose $\eta_\epsilon \in (0, T]$ such that $\eta_\epsilon(|v| + M_\epsilon) < |x|$. For such a $\eta_\epsilon > 0$, one has

$$\mathbb{P}_y \left[\tau_{\{0\}}^0 \in (0, \eta_\epsilon), \sup_{u \in [0, T]} |B_u| \leq M_\epsilon \right] = 0$$

and thus $\mathbb{P}_y[\tau_{\{0\}}^0 \leq T] \leq \epsilon$. This proves that $\mathbb{P}_y[\tau_{\{0\}}^0 \leq T] = 0$. The proof of (2.32), and hence also the one of (1.19), are complete. The last statements in Lemma 3 are consequences of (1.19).

2.3.2. Proof of Theorem 3. Recall that we work here on the space \mathcal{R}_0^{2d} , see (1.18). Assume that $-\nabla \mathbf{V}_c$ (resp. \mathbf{V}_s) satisfies **[c1]** (resp. **[S1]** or **[S2]**). Let $(Y_t, t \geq 0)$ be the strong solution to (1.15) (see Proposition 2). Let us denote by $\mathcal{L}^{kL} = \frac{1}{2}\Delta_v + v \cdot \nabla_x - \nabla \mathbf{V}_c \cdot \nabla_v - \gamma v \cdot \nabla_v$ the infinitesimal generator of the diffusion $(Y_t, t \geq 0)$. Since $\mathcal{L}^{kL} \mathbf{H} \leq c\mathbf{H}$ over \mathcal{R}^{2d} , it holds for all $\mathbf{y} \in \mathcal{H}_R$ and $t \geq 0$,

$$\mathbb{P}_y[\sigma_{\mathcal{H}_R}^{kL} \leq t] \leq \frac{e^{ct}}{R} \mathbf{H}(\mathbf{y}), \quad (2.34)$$

where for $R > 0$, $\mathcal{H}_R := \{\mathbf{y} \in \mathcal{R}^{2d}, \mathbf{H}(\mathbf{y}) < R\}$ is an open bounded subset of \mathcal{R}^d and $\sigma_{\mathcal{H}_R}^{kL} := \inf\{t \geq 0, Y_t \notin \mathcal{H}_R\}$.

To prove Theorem 3, and in view of Theorem 8 and Theorem 9, it is enough to check the conditions **[SF₀]**, **[P_{t=s}]**, **[D_e]**, **[IV_s]**, **[B]**, **[Ti]**, **[L_{yap}]**, and **[P_{traj}]** that we rewrite for the kinetic Langevin process (1.15):

- [SF₀]** The semigroup $(U_t^{kL}, t \geq 0)$ of the process $(Y_t, t \geq 0)$ is strongly Feller.
- [P_{t=s}]** For all $t \geq 0$, the mapping $\mathbf{y} \in \mathcal{R}^{2d} \mapsto Y_t(\mathbf{y})$ is continuous in probability.
- [D_e]** For all $t > 0$ and $\mathbf{y} \in \mathcal{R}^{2d}$, $Y_t(\mathbf{y})$ has density w.r.t. the Lebesgue measure dy over \mathcal{R}^{2d} .
- [IV_s]** As $\mathbf{y}_n \rightarrow \mathbf{y} \in \mathcal{R}_0^{2d}$, for all $t \geq 0$, $\int_0^t \mathbf{V}_s(x_s(\mathbf{y}_n)) ds \xrightarrow{\mathbb{P}} \int_0^t \mathbf{V}_s(x_s(\mathbf{y})) ds$.
- [B]** For all $\delta > 0$ and any compact subset K of \mathcal{R}^{2d} , $\lim_{s \rightarrow 0^+} \sup_{\mathbf{y}=(x,v) \in K} \mathbb{P}_y[\sigma_{B(x,\delta)}^{kL} \leq s] = 0$, where $\sigma_{B(x,\delta)}^{kL}(\mathbf{y}) := \inf\{t \geq 0, x_t(\mathbf{y}) \notin B(x, \delta)\}$.
- [Ti]** The semigroup $(Q_t^{kL}, t \geq 0)$ is topologically irreducible over $\mathcal{D} = \mathcal{O} \times \mathcal{R}^d$.
- [L_{yap}]** There exists a $\mathcal{C}^{1,2}$ function $\mathbf{W} : \mathcal{R}^{2d} \rightarrow [1, +\infty)$ (which thus belongs to the extended domain of the generator of $(U_t^{kL}, t \geq 0)$) such that for all $\mathbf{y} = (x, v) \in \mathcal{R}_0^{2d}$ satisfying either $|\mathbf{y}| \rightarrow +\infty$ or $\mathbf{y} \rightarrow \partial \mathcal{R}_0^{2d} = \{0\} \times \mathcal{R}^d$ (i.e. when $x \rightarrow 0$),

$$\mathcal{L}^{kL} \mathbf{W}(\mathbf{y}) / \mathbf{W}(\mathbf{y}) - p \mathbf{V}_s(x) \rightarrow -\infty, \quad \forall p > 1.$$

- [P_{traj}]** The mapping $\mathbf{y} \in \mathcal{R}_0^{2d} \mapsto \mathbb{P}[Y_{[0,t]}(\mathbf{y}) \in \cdot] \in \mathcal{P}(\mathcal{C}([0, t], \mathcal{R}_0^{2d}))$ is continuous for the weak topology.

We check these conditions. First of all, using the same arguments as those used in the proof of Theorem 6 (see more precisely Step A.1 there), one proves **[SF₀]** for the kinetic Langevin process (1.15). Note that an alternative proof consists in using the Girsanov formula (1.17) as we did for the overdamped Langevin process, see (2.10) and the lines below.

Condition **[P_{traj}]** is a consequence of the fact that for all $\mathbf{y} \in \mathcal{R}_0^{2d}$, $\mathbb{P}_y[Y_{[0,t]} \in \mathcal{C}([0, t], \mathcal{R}_0^{2d})] = 1$ (see Lemma 3), together with the fact that for all $t \geq 0$ and all sequence $(\mathbf{y}_n)_n$ in \mathcal{R}^{2d} such that $\mathbf{y}_n \rightarrow \mathbf{y} \in \mathcal{R}^{2d}$ as $n \rightarrow +\infty$, for all $\epsilon > 0$,

$$\mathbb{P}[\sup_{s \in [0, t]} |Y_s(\mathbf{y}_n) - Y_s(\mathbf{y})| \geq \epsilon] \rightarrow 0 \quad \text{as } n \rightarrow +\infty. \quad (2.35)$$

Equation (2.35) is a consequence of (2.34) together with the fact that the coefficients of the stochastic differential equation (1.15) are locally Lipschitz (this is proved e.g. by adapting the proof of [35, Proposition 2.2]). Note that (2.35) implies **[P_{t=s}]**. Condition **[D_e]** is a consequence of the Girsanov formula stated in Proposition 2 (see (1.17)). Condition **[IV_s]** is proved as in Step A.3 in the proof of Theorem 6 using that $\mathbb{P}_y[Y_{[0,T]} \in \mathcal{C}([0, T], \mathcal{R}_0^{2d})] = 1$,

(2.35), and the continuous mapping theorem. Condition **[B]** is proved as in Lemma 2.4 in [35]. Condition **[Ti]** is a consequence of the fact that the killed semigroup of the process $(Y_s, s \in [0, \sigma_{\mathcal{D}}^{kL}))$ is topologically irreducible over \mathcal{D} (to see this, use the Girsanov formula and the fact that the killed semigroup $(Y_s^0, s \in [0, \sigma_{\mathcal{D}}^{Y^0}))$ is topologically irreducible over \mathcal{D} , see e.g. [36]) together with the fact that $\int_0^t \mathbf{V}_{\mathbf{S}}(x_s(\mathbf{y})) ds < +\infty$ a.s. for all $\mathbf{y} \in \mathcal{R}_0^{2d}$ (see Lemma 3). Let us now check condition **[L_{Yap}]**. We choose the Lyapunov function constructed in [74]. Define $\mathbf{F}(x, v) = a \mathbf{H}(x, v) + b v \cdot \mathbf{G}(x)$ ($a, b > 0$), where \mathbf{G} is the \mathcal{C}^1 function

$$x \in \mathcal{R}^d \mapsto \mathbf{G}(x) = \frac{x}{|x|} (1 - \chi(x)),$$

where χ is defined just before (1.4). Note that both \mathbf{G} and its gradient are bounded over \mathcal{R}^d . In particular \mathbf{F} is lower bounded. For all $\mathbf{y} = (x, v) \in \mathcal{R}^{2d}$, we then define as in [74]:

$$\mathbf{W}(x, v) = \exp \left[\mathbf{F}(x, v) - \inf_{\mathcal{R}^{2d}} \mathbf{F} \right]. \quad (2.36)$$

We then have for all $\mathbf{y} \in \mathcal{R}^{2d}$,

$$\begin{aligned} & -\mathcal{L}^{kL} \mathbf{W}(\mathbf{y}) / \mathbf{W}(\mathbf{y}) \\ &= -\mathcal{L}^{kL} \mathbf{F}(\mathbf{y}) - \frac{1}{2} |\nabla_v \mathbf{F}(\mathbf{y})|^2 \\ &= -\frac{ad}{2} + a\gamma |v|^2 - b v \cdot \nabla \mathbf{G}(x) v + b\gamma v \cdot \mathbf{G}(x) + b \mathbf{G}(x) \cdot \nabla \mathbf{V}_{\mathbf{c}}(x) - \frac{1}{2} |av + b\mathbf{G}(x)|^2. \end{aligned}$$

Hence, for some constants $c_1, c_2 > 0$, such that $-\mathcal{L}^{kL} \mathbf{W} / \mathbf{W}$ is lower bounded by

$$|v|^2 \left[a \left[\gamma - \frac{a}{2} \right] - b |\nabla \mathbf{G}|_{\infty} \right] - c_1 b (a + 1) |v| + b \mathbf{G}(x) \cdot \nabla \mathbf{V}_{\mathbf{c}}(x) - c_2.$$

Choose $a > 0$ such that $a < 2\gamma$ and then $b > 0$ such that $a \left[\gamma - \frac{a}{2} \right] - b > 0$. Then, when $|\mathbf{y}| \rightarrow +\infty$, $-\mathcal{L}^{kL} \mathbf{W}(\mathbf{y}) / \mathbf{W}(\mathbf{y}) \rightarrow +\infty$. Hence, for $\mathbf{y} = (x, v) \in \mathcal{R}_0^{2d}$, when $|\mathbf{y}| \rightarrow +\infty$ or $\mathbf{y} \rightarrow \partial \mathcal{R}_0^{2d} = \{0\} \times \mathcal{R}^d$ (i.e. when $|x| \rightarrow 0$), $\mathcal{L}^{kL} \mathbf{W}(\mathbf{y}) / \mathbf{W}(\mathbf{y}) - p \mathbf{V}_{\mathbf{S}}(x) \rightarrow -\infty$, for all $p > 1$.

Theorem 3 is a consequence of Theorems 8 and 9. Note that **[O1]** is a consequence of the fact that when $\mathcal{R}_0^d \setminus \overline{\mathcal{O}}$ is non-empty, there exist a non empty open ball $B \subset \mathcal{R}_0^d \setminus \overline{\mathcal{O}}$, $\mathbf{y}_0 \in \mathcal{D} = \mathcal{O} \times \mathcal{R}^d$, and $t_0 > 0$ (actually this is true for all $\mathbf{y}_0 \in \mathcal{D}$ and $t_0 > 0$) such that

$$\mathbb{P}_{\mathbf{y}_0} [Y_{t_0} \in B \times \mathcal{R}^d] > 0.$$

Indeed, this implies that $0 < \mathbb{P}_{\mathbf{y}_0} [Y_{t_0} \in B \times \mathcal{R}^d] \leq \mathbb{P}_{\mathbf{y}_0} [\sigma_{\mathcal{D}}^{kL} \leq t_0]$.

2.4. Proof of Theorem 4. Recall the definition of the state space \mathcal{E} and the Lévy process $(\Theta_t, t \geq 0)$ over \mathcal{E} , see (1.21) and (1.25). Recall also (1.24) and Lemma 4. To prove Theorem 4 we use Theorems 8 and 9. Recall that $\mathbf{U}_{\mathbf{S}}(\mathbf{x}) \rightarrow +\infty$ if and only if $|\mathbf{x}| \rightarrow +\infty$ or $\mathbf{x} \rightarrow \partial \mathcal{E}$ ($\mathbf{x} \in \mathcal{E}$). By Assumption **[L1]**, for any $t > 0$, Θ_t has density w.r.t. the Lebesgue measure over \mathcal{R}^{dn} (this is **[D_e]**) and its semigroup is thus strong Feller (this is **[SF₀]**). Let $t \geq 0$. Moreover, as $\mathbf{x}_n \rightarrow \mathbf{x} \in \mathcal{E}$, it holds in probability (and actually a.s.)

$$\int_0^t \mathbf{U}_{\mathbf{S}}(\Theta_s(\mathbf{x}_n)) ds \rightarrow \int_0^t \mathbf{U}_{\mathbf{S}}(\Theta_s(\mathbf{x})) ds \quad (\text{this is } [IV_{\mathbf{S}}]).$$

Indeed, on the one hand, it holds a.s. $\sup_{s \geq 0} |\Theta_s(\mathbf{x}_n) - \Theta_s(\mathbf{x})| = |\mathbf{x}_n - \mathbf{x}|$. On the other hand, recall (see the proof of Lemma 4), that there exist a.s. $\epsilon > 0$ and $M > 0$ such that $|\Theta_s(\mathbf{x})| \leq M$ and $\text{dist}(\Theta_s(\mathbf{x}), \partial \mathcal{E}) \geq \epsilon$, for all $s \in [0, t]$. Hence, there exists a.s. $n_0 \geq 1$,

for all $n \geq n_0$ and $s \geq 0$, $|\Theta_s(\mathbf{x}_n)| \leq 2M$ and $\text{dist}(\Theta_s(\mathbf{x}_n), \partial\mathcal{E}) \geq \epsilon/2$. The result follows from the dominated convergence theorem. Note also that the previous analysis implies $[\mathbf{P}_{\text{traj}}]$. Consequently, using Theorem 9, the non killed semigroup is strongly Feller, i.e. for all $t > 0$ and $f \in b\mathcal{B}(\mathcal{E})$, the function

$$\mathbf{x} \in \mathcal{E} \mapsto T_t^B f(\mathbf{x}) = \mathbb{E}_{\mathbf{x}} \left[f(\Theta_t) e^{-\int_0^t \mathbf{U}_S(\Theta_s) ds} \right] \text{ is continuous (this is } [\mathbf{SF}_1]).$$

On the other hand, for all compact subset K of \mathcal{E} and all $\delta > 0$, we have that (recalling that $B(\mathbf{x}, \delta)$ is the open ball in \mathcal{R}^{dn} of radius $\delta > 0$ and centered at \mathbf{x}):

$$\lim_{s \rightarrow 0^+} \sup_{\mathbf{x} \in K} \mathbb{P}_{\mathbf{x}}[\sigma_{B(\mathbf{x}, \delta)}^{\Theta} \leq s] = \lim_{s \rightarrow 0^+} \mathbb{P}_0[\sigma_{B(0, \delta)}^{\Theta} \leq s] = 0,$$

which implies $[\mathbf{B}]$. We thus deduce $[\mathbf{SF}_2]$ using Theorem 9. Furthermore $[\mathbf{L}_{\text{yap}}]$ is satisfied with the constant function $\mathbf{1}$. Condition $[\mathbf{Ti}]$ is a consequence of $[\mathbf{L4}]$ together with the fact that for all $\mathbf{x} \in \mathcal{E}$, $\int_0^t \mathbf{U}_S(\Theta_s(\mathbf{x})) ds$ is finite almost surely (see Lemma 4). Finally, note that $[\mathbf{O1}]$ is precisely $[\mathbf{L5}]$ and that \mathbf{U}_S is non constant over \mathcal{E} . The proof of Theorem 4 is complete using Theorem 8.

Appendix: on Assumptions $[\mathbf{L1}] \rightarrow [\mathbf{L5}]$

In this section we give examples of Lévy processes satisfying $[\mathbf{L1}]$, $[\mathbf{L2}]$, and $[\mathbf{L3}]$ (see the second model). We also give examples of such processes such that the process Θ (see (1.25)) satisfies $[\mathbf{L4}]$ and $[\mathbf{L5}]$ (see the fourth model).

On Assumptions $[\mathbf{L1}]$. In the mathematical literature, several conditions exist ensuring $[\mathbf{L1}]$ for a Lévy process, see e.g. [39, 72, 66, 47, 49] and references therein. For instance, the following simple well-known conditions imply that the Lévy process has a density w.r.t. the Lebesgue measure for all $t > 0$:

- Its Gaussian covariance matrix has full rank, or if its Lévy measure ν is absolutely continuous w.r.t. the Lebesgue measure and $\nu(\mathcal{R}^d \setminus \{0\}) = +\infty$, see e.g. [66, Theorem 27.7].
- An isotropic Lévy process in \mathcal{R}^d ($d \geq 2$) which is not a compound Poisson process, see [77, Eq. (4.6)].
- A Lévy process without Gaussian component such that $\lim_{|\xi| \rightarrow +\infty} \frac{\Re(\Psi(\xi))}{\ln(1+|\xi|)} = +\infty$, where Ψ is the characteristic exponent and $\Re(z)$ the real part of a complex number z , see [39, 47].
- A subordinate Brownian motion $(B_{\ell_t}, t \geq 0)$ where $(\ell_t, t \geq 0)$ is a subordinator with infinite lifetime (independent from $(B_t, t \geq 0)$), see [52, Lemma 3.1].

On Assumptions $[\mathbf{L2}]$ and $[\mathbf{L3}]$. The following proposition is a way to check $[\mathbf{L2}]$ and $[\mathbf{L3}]$ for a Lévy process.

Proposition 7. *Let $(L_t, t \geq 0)$ be a purely jump Lévy process over \mathcal{R}^d such that its Lévy measure ν has full topological support over \mathcal{R}_0^d , i.e. for all $y \in \mathcal{R}_0^d$ and all $r \in (0, |y|)$, $\nu(B(y, r)) > 0$, and such that for some $\beta > 0$, $\int_{|u| \leq 1} |u|^\beta \nu(du) < +\infty$. Let \mathcal{O} be a subdomain of \mathcal{R}^d . Let $T > 0$ and $x, z \in \mathcal{O}$. Then, for all $\epsilon > 0$,*

$$\mathbb{P}_x[|L_T - z| < \epsilon, T < \sigma_{\mathcal{O}}] > 0, \quad (2.37)$$

where $\sigma_{\mathcal{O}} := \inf\{t \geq 0, L_t \notin \mathcal{O}\}$. In addition, if $\mathcal{R}^d \setminus \overline{\mathcal{O}}$ is nonempty, for all $x \in \mathcal{O}$, $\mathbb{P}_x[\sigma_{\mathcal{O}} < +\infty] > 0$.

Proof. We will use [51, Theorem 2.1] and for this reason, we adopt the notation of [51, Sections 2.1 and 2.2]. Note that Assumptions \mathbf{H}_1 and \mathbf{H}_2 there are satisfied here for the process $(L_t, t \geq 0)$ with $r \equiv 0$, $b \equiv 0$, $\sigma \equiv 1$, and $c(x, u) \equiv u$. Fix $\epsilon, T > 0$ and $x, z \in \mathcal{O}$. In view of [51, Theorem 2.1], we aim at constructing $\phi \in \mathbf{S}_{0,T,x}^{\text{const}}$ such that $\overline{\text{Ran}} \phi \subset \mathcal{O}$ with $\phi_0 = x$, $|\phi_T - z| < \epsilon/2$. By [51, Theorem 2.1], one then has in particular that:

$$\mathbb{P}_x[d_T(L, \phi) < \epsilon'] > 0, \quad \forall \epsilon' > 0, \quad (2.38)$$

where d_T is the Skorokhod metric of $\mathcal{D}([0, T], \mathcal{R}^d)$, see e.g. [4, Section 12]. First note that for any $r_0 > 0$ and $a \neq b \in \mathcal{R}^d$, $J(a, B(b, r_0)) = \nu(B(b - a, r_0)) \in (0, +\infty)$ if $r_0 < |b - a|$. Therefore $b \neq a \Rightarrow b \in \text{supp}(J(a, \cdot))$ (i.e. the jump from a to b is admissible). In our setting, we have $\tilde{b} = -\mathbf{b}$ is a constant where $\mathbf{b} = \int_{|u| \leq 1} u_L \nu(du)$ and where u_L is the orthogonal projection of u on the *integrability subvector space*

$$L = \left\{ \ell \in \mathcal{R}^d, \int_{|u| \leq 1} |u \cdot \ell| \nu(du) < +\infty \right\}.$$

It is then not difficult¹¹ to construct a curve $\phi \in \mathbf{S}_{0,T,x}^{\text{const}}$ such that $\overline{\text{Ran}} \phi \subset \mathcal{O}$, $\phi_0 = x$, and $|\phi_T - z| < \epsilon/2$ (ϕ is usually called a *control curve*). Assume now that $\epsilon' > 0$ is small enough (say $\epsilon' \in (0, \epsilon_\phi)$, $\epsilon_\phi \in (0, \epsilon)$) such that $d_T(f, \phi) < \epsilon'/2$ implies that $\overline{\text{Ran}} f \subset \mathcal{O}$ ¹² (in particular $|f_T - z| \leq |f_T - \phi_T| + |\phi_T - z| \leq d_T(f, \phi) + \epsilon/2 < \epsilon$). Then, using (2.38) with such a small $\epsilon' > 0$, $\mathbb{P}_x[\{|L_T - z| < \epsilon\} \cap \{\overline{\text{Ran}} L_{[0,T]} \subset \mathcal{O}\}] > 0$. Therefore, (2.37) holds. The second statement in Proposition 7 is easy to obtain with the same arguments. \square

Examples. As a conclusion, due to their importance both in theory and in applications, one can easily check with the discussion above that the following examples of (isotropic) Lévy processes over \mathcal{R}^d ($d \geq 2$) satisfy **[L1]**, and also **[L2]**-**[L3]** for any subdomain \mathcal{O} of \mathcal{R}^d :

- The standard Brownian motion.
- The rotationally invariant α -stable processes, $\alpha \in (0, 2)$. If $\alpha \in (0, 2)$, these are purely jump Lévy processes with characteristic exponent $\Psi(u) = |u|^\alpha$, see [65, 1, 6].
- The rotationally invariant relativistic α -stable processes, $\alpha \in (0, 2)$, see [65, 1, 6]. For $m > 0$, these are purely jump Lévy processes with characteristic exponent $\Psi(u) = (|u|^\alpha + m^{2/\alpha})^{\alpha/2} - m$.
- The rotationally invariant geometric α -stable processes, $\alpha \in (0, 2)$. These are purely jump Lévy processes with characteristic exponent $\Psi(u) = \log(1 + |u|^\alpha)$, see [68, 6].
- The rotationally symmetric geometric 2-stable process (also called the *variance gamma process* in some finance literature). Its characteristic exponent is $\Psi(u) = \log(1 + |u|^2)$, see [43, Example 4.7].
- The jump-diffusion processes. These are Lévy processes with characteristic exponent $\Psi(u) = |u|^2 + |u|^\alpha$, $\alpha \in (0, 2)$, see [14].

¹¹There are many ways to do it, choosing e.g. $f_t \equiv 0$ in [51, Eq. (7)].

¹²Indeed, use the distance $d(A, B) = \sup_{a \in A} d(a, B) = d(\bar{A}, \bar{B})$, $A, B \subset \mathcal{R}^d$. Since $d_T(f, \phi) < \epsilon'/2$, there exists a strictly increasing and continuous curve λ from $[0, T]$ onto itself such that

$$\sup_{u \in [0, T]} |f_u - \phi_{\lambda_u}| < \epsilon'/2.$$

Thus, we have $d(\text{Ran } f, \text{Ran } \phi) = \sup_{t \in [0, T]} \inf_{s \in [0, T]} |f_t - \phi_s| \leq \sup_{t \in [0, T]} |f_t - \phi_{\lambda_t}| < \epsilon'/2$.

Note that the first five processes above are subordinate Brownian motions. This list is clearly a non exhaustive list and there are many other Lévy processes satisfying Assumptions [L1], [L2], and [L3].

On Assumptions [L4]-[L5]. We consider here the fourth model (see (1.26) and Theorem 4). Conditions [L4]- [L5] hold when $(L_t^i, t \geq 0)$'s are independent copies of a standard Brownian motion since in this case $\Theta(0)$ is also a standard Brownian motion (see (1.25)). Let us mention that when $(L_t^i, t \geq 0)$'s are independent copies of one of the examples given just above in the appendix, [L4]- [L5] are satisfied (one can use e.g. [51] and explicit constructions of controls). Let us for instance prove that [L4]- [L5] hold when $(L_t^i, t \geq 0)$'s are independent copies of a jump-diffusion process. More precisely, let $(B^1, \dots, B^n, Z^1, \dots, Z^n)$ be $2n$ ($n \geq 2$) independent \mathcal{R}^d -processes such that each B^i is a \mathcal{R}^d -standard Brownian motion and each Z^i is a \mathcal{R}^d -rotationally invariant α -stable processes ($\alpha \in (0, 2)$). Then $\Theta = (B^1 + Z^1, \dots, B^n + Z^n) = \mathfrak{B} + \mathfrak{Z}$ where $\mathfrak{B} = (B^1, \dots, B^n)$ is a \mathcal{R}^{dn} -standard Brownian motion and $\mathfrak{Z} = (Z^1, \dots, Z^n)$. Let \mathcal{U} be a subdomain of \mathcal{E} (see (1.21)), $T > 0$, and $\mathbf{x}, \mathbf{z} \in \mathcal{U}$. Pick a smooth curve $\Phi : [0, T] \rightarrow \mathcal{U}$ joining \mathbf{x} to \mathbf{z} . Note that for any $\epsilon > 0$,

$$\mathbb{P}\left[\sup_{t \in [0, T]} |\mathbf{x} + \mathfrak{B}_t - \Phi_t| \leq \epsilon\right] > 0.$$

Let now $\epsilon_0 \in (0, 1)$ be such that: for any $\epsilon \in (0, \epsilon_0)$ and for any càdlàg curve $f : [0, T] \rightarrow \mathcal{R}^{dn}$, $\sup_{t \in [0, T]} |f_t - \Phi_t| \leq \epsilon \Rightarrow \overline{\text{Ran}} f \subset \mathcal{U}$. On the other hand, for any $\epsilon > 0$, $\mathbb{P}[\sup_{t \in [0, T]} |Z_t^1|^2 < \epsilon] > 0$. Thus, one has:

$$\begin{aligned} 0 &< \mathbb{P}\left[\sup_{t \in [0, T]} |\mathbf{x} + \mathfrak{B}_t - \Phi_t| \leq \epsilon_0/2\right] \cdot \prod_{i=1}^n \mathbb{P}\left[\sup_{t \in [0, T]} |Z_t^i|^2 < \epsilon_0^2/(16n)\right] \\ &= \mathbb{P}\left[\forall i, \sup_{t \in [0, T]} |Z_t^i|^2 < \epsilon_0^2/(16n), \sup_{t \in [0, T]} |\mathbf{x} + \mathfrak{B}_t - \Phi_t| \leq \epsilon_0/2\right] \\ &\leq \mathbb{P}_0\left[\sup_{t \in [0, T]} \underbrace{|\mathbf{x} + \mathfrak{B}_t + \mathfrak{Z}_t - \Phi_t|}_{\Theta_t(\mathbf{x})} \leq 3\epsilon_0/4\right]. \end{aligned}$$

In conclusion this shows that [L4] is satisfied. A similar argument proves that [L5] is also satisfied for this process.

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